PARITY OF THE COEFFICIENTS OF CERTAIN ETA-QUOTIENTS, III: THE CASE OF PURE ETA-POWERS

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Abstract. We continue a program of investigating the parity of the coefficients of eta-quotients, with the goal of elucidating the parity of the partition function. In this paper, we consider positive integer powers t of the Dedekind eta-function. Previous work and conjectures suggest that arithmetic progressions in which the Fourier coefficients of these functions are even should be numerous. We explicitly identify infinite classes of such progressions modulo prime squares for several values of t, and we offer two broad conjectures concerning their existence in general.

1. Introduction and Statements of the Results and Conjectures

The problem of the parity of the partition function $p(n)$ is one of long standing. The Parkin-Shanks Conjecture [11], that the density of the odd values of $p(n)$ exists and equals 1/2, is widely believed but little progress has been made. To this day, it has not even been established that a positive proportion of the values of $p(n)$ is odd. The current series of papers by the present and other authors, including [3, 5, 6, 7, 8, 9, 14], seeks to shed light on the problem by producing results concerning the parity of closely related functions, such as the densities of the m-regular partitions $b_m(n)$, and relations between their densities and those of $p(n)$.

In this note, we focus on a fundamental class of these functions: pure eta-powers, or series of the form f_1^t, $t \geq 0$, where f_1 is the Dedekind eta-function, shifted to have integer powers in q:

$$f_k := \prod_{i=1}^{\infty}(1 - q^{ki}).$$

An *eta-quotient* is a quotient of products of such powers. (Properly speaking, the Dedekind eta-function is $\eta(q) = q^{1/24} f_1$. However, for our purposes the shift is ignorable.)

In order to give context to our results, we begin by restating our "master conjecture" on the parity of eta-quotients presented in [8], which has motivated substantial portions of this

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line of work. First, we introduce notation for the arithmetic density of the odd coefficients of a series.

Definition 1. The power series $f(q) = \sum_{n=0}^{\infty} a(n)q^n$ is said to have *(odd) density* δ if the limit

$$\lim_{x\to\infty} \frac{1}{x}\#\{a(n) : n \leq x,\ a(n) \equiv 1 \pmod 2\}$$

exists and equals δ.

When $\delta = 0$, we may also say that f (or its coefficients) is *lacunary modulo 2*.

We are now ready for our "master conjecture."

Conjecture 2 ([8], Conjecture 4)**.** *Let $F(q) = \sum_{n\geq 0} c(n)q^n$ be an eta-quotient, shifted by a suitable power of q so powers are integral, and let δ_F be the odd density of the $c(n)$. Then:*

i) For any F, δ_F exists and satisfies $\delta_F \leq 1/2$.

ii) If $\delta_F = 1/2$, then for any nonnegative integer-valued polynomial P of positive degree, the odd density of $c(P(n))$ is $1/2$. (In particular, for all nonconstant subprogressions $An + B$, $c(An + B)$ has odd density $1/2$.)

iii) If $\delta_F < 1/2$, then the coefficients of F are identically even on some *nonconstant subprogression.*

iv) If the coefficients of F are not identically even on any *nonconstant subprogression, then they have odd density $1/2$ on* every *nonconstant subprogression; in particular, $\delta_F = 1/2$.*

(Note that i), ii), and iii) together imply iv), and iv) implies iii).)

In the case of positive integer powers of f_1, it is already known that these are all lacunary modulo 2 by work of Cotron, Michaelsen, Stamm, and Zhu [4]:

Theorem 3 ([4], Theorem 1.1)**.** *Suppose $u, w \geq 0$. Let $F(q) = \frac{\prod_{i=1}^{u} f_{\alpha_i}^{r_i}}{\prod_{i=1}^{w} f_{\gamma_i}^{s_i}}$, and assume that*

$$\sum_{i=1}^{u} \frac{r_i}{\alpha_i} \geq \sum_{i=1}^{w} s_i\gamma_i.$$

Then the coefficients of F are lacunary modulo 2.

Therefore, for any positive power t, by the "master conjecture" we expect to find arithmetic progressions $An+B$ for which the coefficients of $f_1^t := \sum_{j=0}^{\infty} c_t(j)q^j$ are identically even, i.e.,

$$c_t(An + B) \equiv 0 \pmod 2.$$

Here and in all following claims, when we use n without qualifier in discussing an arithmetic progression, we mean "for all $n \geq 0$."

The goal of this note is to identify large classes of such progressions for many powers t, and investigate the structure of the progressions appearing. For the remainder of the paper, to avoid heavy repetition, all congruences are modulo 2 unless otherwise specified. We say that two power series $\sum a(n)q^n$ and $\sum b(n)q^n$ are *congruent modulo* m if $a(n) \equiv b(n) \pmod{m}$ for all n.

Remark 4. We note that the existence of such progressions in an arbitrary power series is not implied solely by lacunarity modulo 2, as may be seen by an example to the contrary: simply obstruct the countably infinite list of progressions at rapidly-increasing intervals.

In the current literature, results similar to those in this paper may be obtained by reinterpreting theorems for partition functions of different kinds. For instance, the number of k-tuples of partitions of n wherein even parts are distinct has generating function

$$\sum_{n=0}^{\infty} ped_k(n)q^n \equiv f_1^{3k}.$$

Then a theorem of Chen [2] gives us that for $f_1^{3k} = \sum_{n=0}^{\infty} c_{3k}(n)q^n$, we have:

Theorem 5 (Chen)**.** *Let $k = 2^r s$ with s odd, $s = \sum_{i=0}^{\infty} \beta_i 2^i$ with $\beta_i \in \{0, 1\}$. Let $g_s = 1 + \sum_{i=0}^{\infty} \beta_{2i+1} 2^i + \sum_{j=0}^{\infty} \beta_{2j+2} 2^j$ and let l be an integer such that $l \geq g$. Then for any distinct odd primes $\ell_1, \ldots, \ell_l$,*

$$c_{3k}\left(\frac{2^r \ell_1 \ell_2 \cdots \ell_k n - k}{8}\right) \equiv 0.$$

Our results cover arithmetic progressions modulo p^2 for primes p, so the moduli of the progressions involved are typically much smaller, but concern fewer possible powers of f_1.

By Lemma 17, we have that ${f_j}^2 \equiv f_{2j}$. Hence, for even $t = 2^k m$ with m odd, all coefficients are even outside of the subprogression $c_t(2^m n)$, and within this progression they coincide with the coefficients of f_1^m. Therefore, we completely understand all positive integer values of t by focusing on the case of t odd.

Our first two theorems are simple consequences of progressions forbidden by the pentagonal or triangular numbers. We record them explicitly for completeness.

Theorem 6. *For m coprime to 6, we have $c_1(mn + B) \equiv 0$ whenever $2 \cdot 3^{-1}B + 36^{-1}$ is not a quadratic residue modulo m.*

Theorem 7. *For m coprime to 6, we have $c_3(mn + B) \equiv 0$ whenever $2^{1-d}B + 4^{-1}$ is not a quadratic residue modulo m.*

Definition 8. We say that f_1^t is p^2-*even* at a prime p with base $r \in \{0, \ldots, p^2 - 1\}$ if it is the case that $c_t(p^2 n + kp + r) \equiv 0$ for all $k \in \{1, \ldots, p - 1\}$.

The following are the two main conjectures of this paper.

Conjecture A. *For any given $t \geq 1$ odd, f_1^t is p^2-even for a positive proportion of primes p, for some base r depending on t and p.*

Conjecture B. *For any given prime p, there exist infinitely many $t \geq 1$ odd such that f_1^t is p^2-even, for some base r depending on t and p.*

We will establish that Conjecture A is true for t a sum of two quadratic terms, and Conjecture B for all primes other than 2 or those congruent to 1 mod 24.

Theorem 9. *Let $t = a + b \cdot 2^e$, $a, b \in \{1, 3\}$, $e > 0$. Then f_1^t is p^2-even at p for some base r, when -2^e is a quadratic nonresidue modulo p if $a = b$, and when $-3 \cdot 2^e$ is a quadratic nonresidue modulo p if $a \neq b$. In particular, f_1^t is p^2-even for a set of relative density at least 1/2 in the primes, if such density exists.*

Theorem 10. *We have the following cases of t as the sum of two quadratic terms.*

- *If $t = 2^d+3$, $p \equiv 23 \pmod{24}$ prime, then f_1^t is p^2-even with base $r \equiv -\left(2^{d-3}3^{-1} + 2^{-3}\right) \pmod{p^2}$.*
- *If d is even in the previous clause, we may take $p \equiv 5 \pmod 6$. If d is odd we may instead take $p \equiv 13 \pmod{24}$.*
- *If $t = 2^d + 1$, $p \equiv 7 \pmod 8$ prime, then f_1^t is p^2-even with base $r \equiv -3\left(2^{d-3} + 2^{-3}\right) \pmod{p^2}$.*
- *If d is even in the previous clause, we may take $p \equiv 3 \pmod 4$, $p \geq 7$.*
- *If $t = 3 \cdot 2^d + 1$, $p \equiv 23 \pmod{24}$ prime, then f_1^t is p^2-even with base $r \equiv -\left(2^{d-3} + 2^{-3} \cdot 3^{-1}\right) \pmod{p^2}$.*
- *If d is even in the previous clause, we may take $p \equiv 5 \pmod 6$. If d is odd we may instead take $p \equiv 13 \pmod{24}$.*
- *If $t = 3 \cdot 2^d + 3$, $p \equiv 7 \pmod 8$ prime, then f_1^t is p^2-even with base $r \equiv -\left(2^{d-3} + 2^{-3}\right) \pmod{p^2}$.*
- *If d is even in the previous clause, we may take $p \equiv 3 \pmod 4$.*

A brief inspection shows that the clauses above cover all odd primes other than those congruent to 1 mod 24. Hence we have the following corollary.

Corollary 11. *For any prime $p \geq 3$, $p \not\equiv 1 \pmod{24}$, there exist infinitely many values of t for which f_1^t is p^2-even at p for some base r.*

Remark 12. Numerical computation certainly suggests many additional progressions exist, but simply are not provable directly by the arguments of this paper. A notable example is

f_1^{13}, which also seems to be p^2-even for primes 1 mod 6 in addition to the 5 mod 6 proved above; however, it cannot be so for the same reasons as in the proof of Theorem 10.

Two corollaries of the previous results are:

Corollary 13. *For $p = 5$, $t = 4^d+3$, $d \geq 1$, we have $r \equiv t \pmod{25}$. For $p = 3$, $t = 3\cdot 4^d+3$, $d \geq 1$, we have $r \equiv t/3 \pmod 9$ (and the latter will always be 2 mod 3).*

The first example of this is that $c_{15}(9n+2)$ and $c_{15}(9n+8)$ are always even.

Example 14. As a consequence of the proofs of the above theorems, we have a large class of known specific congruences, of which the following is the barest sample. Here t is the power, p a relevant prime from Theorem 10, and r the base given in the theorem, so that $c_t(p^2n + kp + r)$ is even for all $k \in \{1, \ldots, p-1\}$. For instance, the first line says that $c_7(25n + B)$ is even for $B \in \{2, 12, 17, 22\}$.

t	p	r		t	p	r
7	5	7		7	23	154
17	7	34		17	11	85
193	5	18		193	47	84
195	3	2		195	71	622

Note that theorems regarding powers t that require more terms seem to rapidly increase in difficulty. The next two results will rely on known facts from the theory of modular forms.

Theorem 15. *There exists no progression $An + B$ for which $c_{2^d-1}(An + B) \equiv 0$ for all d.*

Theorem 16. *We have that $c_{21}(49n + k) \equiv 0 \pmod 2$ for $k \in \{14, 28, 35\}$.*

2. Background and Preliminary Notions

The unreduced forms of the following two identities we employ throughout are well known (see for instance [1], equations (1.3.1) and (2.2.13)):

$$f_1 \equiv \sum_{n\in\mathbb{Z}} q^{\frac{n}{2}(3n-1)} \qquad \text{and} \qquad f_1^3 \equiv \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}.$$

The powers appearing in the expressions above are the (generalized) pentagonal and the triangular numbers, respectively. Since the integers represented by either are given by quadratic forms, after completing the square an equivalent title for this paper might have been, "Parity of the number of representations of integers in arithmetic progressions by certain quadratic forms in two or more variables."

By completing squares, the exponents appearing in the two expressions can be given the following forms modulo m for m coprime to 6 and 2, respectively.

$$\frac{n}{2}(3n-1) \equiv 2^{-1} \cdot 3 \left(n - 6^{-1}\right)^2 - 24^{-1} \pmod{m};$$
$$\binom{n+1}{2} \equiv 2^{-1} \left(n + 2^{-1}\right)^2 - 8^{-1} \pmod{m}.$$

Here inverses are being taken modulo m, hence the necessity for m to be coprime to 6 in the first line, and 2 in the second.

We will employ the following fact so frequently that we will avoid further explicit comment:

Lemma 17. *For any power series $f(q) = \sum_{n=0}^{\infty} a(n)q^n$, it holds that*

$$(f(q))^2 \equiv \sum_{n=0}^{\infty} a(n)q^{2n}.$$

In particular, ${f_j}^2 \equiv f_{2j}$.

The following results concerning quadratic residues can be found in any standard number theory text (see, e.g., [13]).

Lemma 18. *Let p be an odd prime. Then:*

- *2 is a quadratic residue modulo p if and only if $p \equiv 1, 7 \pmod{8}$.*
- *3 is a quadratic residue modulo p if and only if $p \equiv 1, 11 \pmod{12}$.*
- *-1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod{4}$.*

3. Proofs

Proof of Theorem 6. We have

$$f_1 = \sum_{n=0}^{\infty} c_1(n)q^n \equiv \sum_{n\in\mathbb{Z}} q^{\frac{n}{2}(3n-1)}.$$

Since for m coprime to 6 we may write

$$\frac{n}{2}(3k-1) \equiv 2^{-1} \cdot 3 \left(k - 6^{-1}\right)^2 - 24^{-1} \pmod{m},$$

should $mn + B$ not be of this form for any n, it will then hold that $c_1(mn + B)$ is identically 0 mod 2 on the progression. Solving throughout for the square, we have:

$$mn + B \equiv 2^{-1} \cdot 3 \left(k - 6^{-1}\right)^2 - 24^{-1} \pmod{m};$$
$$2 \cdot 3^{-1}B + 36^{-1} \equiv \left(k - 6^{-1}\right)^2 \pmod{m}.$$

Therefore, if $2 \cdot 3^{-1}B + 36^{-1}$ is not a quadratic residue modulo m, f_1 is identically even on $mn + B$.

Conversely, if $2 \cdot 3^{-1}B + 36^{-1}$ is a quadratic residue mod m, then there exists some $k - 6^{-1}$ for which the congruence holds. Thus, $c_1(mn+B)$ is not identically 0 on the progression. $\square$

Proof of Theorem 7. We have

$$f_3 = \sum_{n=0}^{\infty} c_3(n) q^n \equiv \sum_{n \geq 0} q^{\binom{n+1}{2}}.$$

Since for m odd we may write

$$\binom{k+1}{2} \equiv 2^{-1} \left(k + 2^{-1}\right)^2 - 8^{-1} \pmod{m},$$

should $mn + B$ not be of this form for any n, it will then hold that $c_3(mn + B)$ is identically 0 mod 2 on the progression. Solving throughout for the square, we have:

$$\begin{aligned} mn + B &\equiv 2^{-1} \left(k + 2^{-1}\right)^2 - 8^{-1} \pmod{m}; \\ 2B + 4^{-1} &\equiv \left(k + 2^{-1}\right)^2 \pmod{m}. \end{aligned}$$

Thus, if $2B + 4^{-1}$ is not a quadratic residue modulo m, f_3 is identically even on $mn + B$.

Conversely, if $2B + 4^{-1}$ is a quadratic residue mod m, then there exists some $k - 6^{-1}$ for which the congruence holds, and hence $c_1(mn+B)$ is not identically 0 on the progression. $\square$

For the case when two quadratics are required to write the power, we have Theorem 9.

Proof of Theorem 9. The numbers $t = a + b \cdot 2^e$, $a, b \in \{1, 3\}$, are precisely the odd values of t for which we may write

$$f_1^t \equiv f_{1 \text{ or } 3} \cdot f_{1 \text{ or } 3}^{2^e}.$$

Therefore, t can only be odd if it is representable as the sum of two quadratics, one either the pentagonal or the triangular numbers, and the other (independently) the 2^e-magnified pentagonal or triangular numbers. (Or an odd number of the unmagnified quadratic, but here we focus on those t that can be written with one of each.)

Suppose $a = b = 1$. Then we have two pentagonal progressions, one magnified, and the terms N appearing with nonzero coefficient in their product must satisfy, for some $k_1, k_2 \in \mathbb{Z}$,

$$2^{-1} \cdot 3 \left(k_1 - 6^{-1}\right)^2 - 24^{-1} + (2^e) \left(2^{-1} \cdot 3(k_2 - 6^{-1})^2 - 24^{-1}\right) \equiv N \pmod{p^2}.$$

Let N be of the form $p^2 + kp - 24^{-1} - 2^e 24^{-1}$, $0 < k < p$. Simplify the notation by letting $x = k_1 - 6^{-1}$ and $y = k_2 - 6^{-1}$, where 6^{-1} is some suitable integer representing 6^{-1} modulo

p. Then we have:

$$2^{-1} \cdot 3x^2 - 24^{-1} + (2^e)\left(2^{-1} \cdot 3y^2 - 24^{-1}\right) \equiv kp - 24^{-1} - 2^e 24^{-1} \pmod{p^2};$$
$$2^{-1} \cdot 3x^2 + \left(2^{e-1}\right) \cdot 3y^2 \equiv kp \pmod{p^2}.$$

Now $2^{-1} \cdot 3x^2 + (2^{e-1}) \cdot 3y^2 \equiv 0 \pmod{p}$, so if x or y is divisible by p, the other must be as well, but then $2^{-1} \cdot 3x^2 + (2^{e-1}) \cdot 3y^2 \equiv 0 \pmod{p^2}$, which is false since $p \nmid k$. Hence $x, y \not\equiv 0 \pmod{p}$, and we may write

$$\left(\frac{x}{y}\right)^2 \equiv -2^e \pmod{p}.$$

It follows that if -2^e is not a quadratic residue modulo p^2, the chosen arithmetic progression $p^2 + kp - 24^{-1} - 2^e 24^{-1}$ with $0 < k < p$ cannot have nonzero coefficients mod 2 in f_1^t.

The other cases are similar. If $a = b = 3$, there is no factor of 3 to cancel in the first place; if $a \neq b$, then we have one, which by properties of quadratic residues may be 3 or 3^{-1}, without loss of generality. Only the resulting r is different. $\square$

We can now say that, given any prime $p \geq 3$, $p \not\equiv 1 \pmod{24}$, there are infinitely many t for which f_1^t is p^2-even at p, for some base r.

Proof of Theorem 10. We make note of Lemma 18.

If $p \equiv 3, 5 \pmod{8}$, then 2 is a quadratic nonresidue modulo p, and so among -2^e and $-3 \cdot 2^e$, half of the values will be quadratic nonresidues modulo p and the hypotheses of Theorem 9 will be satisfied.

If $p \equiv 7 \pmod{8}$, then -1 is a quadratic nonresidue mod p and 2 is a quadratic residue, so -2^e is always a quadratic nonresidue mod p.

If $p \equiv 1 \pmod{8}$, then -1 and 2 are quadratic residues mod p, but if $p \equiv 17 \pmod{24}$ then 3 is a quadratic nonresidue and so $-3 \cdot 2^e$ is always a quadratic nonresidue as well.

The only remaining case is $p \equiv 1 \pmod{24}$, as stated. $\square$

Remark 19. The above claim is not an "if and only if" statement. For instance, numerical calculation suggests, though we cannot establish with these arguments, that f_1^5 is 73^2-even with base 1110, even though $73 \equiv 1 \pmod{24}$.

Proof of Corollary 11. We prove the first two clauses. The arguments for the remaining cases are similar.

Let $t = 2^d + 3$. We can easily calculate that $-\left(2^{d-3}3^{-1} + 2^{-3}\right) \pmod{p^2}$ is the necessary value of r. We are concerned with when $-3 \cdot 2^d$ is a quadratic nonresidue modulo p.

If $p \equiv 23 \pmod{24}$, then 2 is a quadratic residue modulo p, as is 3, but -1 is a quadratic nonresidue, so $-3 \cdot 2^d$ is always a quadratic nonresidue and the claim holds.

If d is even, then 2^d is always a quadratic residue, and so we are only concerned with when -3 is a quadratic nonresidue. Since -1 is a quadratic nonresidue mod p if and only if $p \equiv -1 \pmod 4$, and 3 is a quadratic residue if and only if $p \equiv 1, 11 \pmod{12}$, then for $p \equiv 5 \pmod 6$, either -1 is a nonresidue and 3 a residue ($p \equiv 11 \pmod{12}$) or -1 is a residue and 3 a nonresidue ($p \equiv 5 \pmod{12}$).

If d is odd then we wish -6 to be a quadratic nonresidue, and in addition to the case $p \equiv 23 \pmod{24}$ given above, we may also take $p \equiv 13 \pmod{24}$, for which 2 is a quadratic nonresidue while -1 and 3 are residues. This concludes the proof. □

Proof of Corollary 13. Here d is even and we are considering $p = 5$, so the hypotheses of the second clause of Theorem 10 are satisfied. We have $t = 2^{2k} + 3$, and seek the value of $-\left(2^{2k-3}3^{-1} + 2^{-3}\right) \pmod{25}$. But observe that $3^{-1} \equiv -8 \pmod{25}$. Hence

$$r \equiv -\left(2^{2k-3}3^{-1} + 2^{-3}\right) \equiv -\left(2^{2k-3}(-1)2^3 - 3\right) \equiv 2^{2k} + 3 \equiv t \pmod{25},$$

as desired.

For $p = 3$, simply note that $-1 \equiv 2^3 \pmod 9$ and the calculations follow. □

Proof of Theorem 15. It is known by a result of Radu [12], completing work by Subbarao, Ono and other authors, that there exists no progression $An+B$, $A \neq 0$, for which $p(An+B) \equiv 0 \pmod 2$ for all n.

We have that

$$f_1^{2^d-1} = \frac{f_1^{2^d}}{f_1} \equiv \frac{f_{2^d}}{f_1}.$$

But then all coefficients $c_{2^d-1}(n)$ of $f_1^{2^d-1}$ up to $n = 2^d - 1$ must match the parity of the partition number $p(n)$. Now consider any progression $An + B$, $A \neq 0$, as d grows. If this is always even, then an indefinitely long initial segment of $c_{2^d-1}(An + B)$ must remain even, but by Radu, eventually the progression $p(An + B)$ will contain an odd entry, and hence so must $c_{2^d-1}(An + B)$. □

We observe that $t = 21 = 1 + 4 + 16$ is the smallest number not representable by two terms as in Theorem 10, and so a different proof is required.

Proof of Theorem 16. The actual proof is a standard exercise in the basic theory of modular forms, so we omit the details. For all relevant machinery, see the background section of [8] and employ the $U(7)$ operator from [10] after multiplication of $q^7 f_8^{21}$ by necessary powers of $q^7 f_7^{24}$. □

4. Concluding Remarks

We wrap up with several observations and suggestions for further research.

The remaining cases of Conjecture B not covered by Theorem 10, those of primes $p \equiv 1 \pmod{24}$, certainly bear investigation. The first such prime is 73, and experimental calculation suggests several values of t are indeed 73^2-even with various bases, from f_1^5 with base 1110 to f_1^{69} with base 4660.

There are also numerous cases where the full behavior of being p^2-even does not hold because exactly half of the required progressions are identically even. For example, $c_7(49n + B)$ appears to be even for $B \in \{21, 28, 42\}$, which is three instead of six of the required progressions. Similarly, $c_{195}\left(73^2n + B\right)$ seems even in 36 of the necessary 72 progressions. A different approach would be necessary to prove these behaviors, but possibly there is a unifying underlying reason that could be employed to prove an infinite class of such cases.

We pose the following questions.

Question 20. Fixing a prime p not congruent to 1 mod 24, Theorem 10 yields an infinite class of t for which f_1^t is p^2-even, but the class is of exponential growth. Could the behavior hold for a positive proportion of odd t, and hold for $p \equiv 1 \pmod{24}$ as well?

Question 21. Fixing t, Theorem 10 gives a positive proportion of primes p for which f_1^t is p^2-even as long as t is of the classes covered, but this is a density-zero set in the integers. Hence we ask: is f_1^t p^2-even for a positive proportion of primes p, for *all* t?

Question 22. Chen's theorem in [2] yields even progressions for all c_{3k}, but the moduli are large (i.e., they have many prime factors). Our theorems in this paper apply to fewer c_t, but have relatively small moduli p^2. Can there ever be an even arithmetic progression, other than for c_1 or c_3, of the form $pn + B$ for some prime p?

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