UNIVERSAL PROPERTIES OF SPACES OF GENERALIZED FUNCTIONS

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Abstract. By means of several examples, we motivate that universal properties are the simplest way to solve a given mathematical problem, explaining in this way why they appear everywhere in mathematics. In particular, we present the co-universal property of Schwartz distributions, as the simplest way to have derivatives of continuous functions, Colombeau algebra as the simplest quotient algebra where representatives of zero are infinitesimal, and generalized smooth functions as the universal way to associate set-theoretical maps of non-Archimedean numbers defined by nets of smooth functions (e.g. regularizations of distributions) and having arbitrary derivatives. Each one of these properties yields a characterization up to isomorphisms of the corresponding space. The paper requires only the notions of category, functor, natural transformation and Schwartz's distributions, and introduces the notion of universal solution using a simple and non-abstract language.

1. Introduction

Mathematicians eventually try to solve a problem in the best possible way. For example, we can consider a geometrically intrinsic solution, or the best computational algorithm, or the most general answer. Frequently motivated by the searching of beauty, [24], we can also require that the solution is the "simplest" one, i.e. it has to use the minimal amount of conventional constructions and data other than the given ones from which the problem must depend on. At a first sight, a non-trivial possible mathematical formalization of the idea of *simplest solution* should involve information theory (see e.g. [36] and references therein) or mathematical logic. In this paper, using only a minimal amount of category theory, we see a common informal interpretation of *universal solution* as the simplest way to solve a given problem. It is well-known that universal constructions appear everywhere in mathematics, [31], and hence this interpretation justifies why this happens. We list several examples justifying this interpretation, in particular for spaces of generalized functions (GF) both in linear and nonlinear frameworks.

We will see that a universal solution not only candidates itself as the simplest way to solve a given problem, but its universal property is able to highlight what are the data of the problem and the conventional choices in any other possible construction. Frequently, this paves the way for generalizations, and it always directly yields an axiomatic characterization of these universal solutions. In the

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point of view of several mathematicians, universal properties are so important that they take them as a starting point: "it is not important how you solve this problem, because the key point is that you have a universal solution, which is unique up to isomorphisms".

Concerning spaces of GF, in this paper we clearly consider Schwartz's distributions as "the simplest way to have derivatives of continuous functions" (see [38]), and hence we show the corresponding (not-well known) universal property in Sec. 3 (see also [34], but where the universal property mistakenly lacks condition Thm. 7.(iv)). Following the algebraic construction of Sebastiao e Silva ([39]) of distributions, we see how to obtain a similar universal construction for distributions on Hilbert spaces. Any other solution of the same problem will reasonably satisfy the same minimal and meaningful universal property, and hence it will be isomorphic to our solution, see Sec. 3.1.

Among nonlinear settings for GF extending Schwartz's distributions, Colombeau's special algebra (see e.g. [2, 3, 4, 23]) is frequently perceived as the simplest one. In Sec. 4, we prove that it is actually the most simple quotient algebra. We will also consider the recent *generalized smooth functions* (see [16, 35, 17] and references therein) because it is even more general than Colombeau's algebras, but with several improved properties such as more general domains, the closure with respect to composition, a better integration theory and Hadamard's well-posedness for every PDE (in infinitesimal neighborhoods), see Sec. 5. As a secondary result, we hence have an axiomatic description up to isomorphisms of Colombeau's special algebra and of generalized smooth functions. In particular, the ring of Colombeau's generalized numbers reveals to be the simplest quotient ring $\mathcal{M}/\sim$ containing the infinite numbers $[\varepsilon^{-q}]_\sim$, $q \in \mathbb{N}$, and where every zero-net $[x_\varepsilon]_\sim = 0$ is generated by an infinitesimal function: $\lim_{\varepsilon\to 0^+} x_\varepsilon = 0$ (see Sec. 4.2.1; see also [43] for another axiomatic description in the framework of nonstandard analysis and where the latter property does not hold).

In the following, we will use the conventions:

- universal = terminal = limit = projective: the unique arrow arrives to the universal object.
- co-universal = initial = co-limit = injective: the unique arrow starts from the universal object.

The paper is self-contained, in the sense that it requires only the notions of category, functor, natural transformation and Schwartz's distributions.
We start by introducing the notion of universal solution using a simple and non-abstract language.

2. General definition of (co-) universal property

We start by defining in general terms what a universal property is. We will use only basic notions of category theory, and will give a definition near to the common use of universal properties.

Definition 1. Let $\mathbf{C}$ be a category. Let $\mathcal{P}(C)$ and $\mathcal{Q}(f, A, B)$ be two properties of A, B, C and f, where A, B, C are objects of $\mathbf{C}$ and f is an arrow of $\mathbf{C}$. Assume that:

$$\mathcal{Q}(f, A, B),\ \mathcal{Q}(g, B, C)\ \Rightarrow\ \mathcal{Q}(g \circ f, A, C), \tag{2.1}$$

$$\mathcal{Q}(1_A, A, A). \tag{2.2}$$

Then we say that C *is a universal solution of* $\mathcal{P}$ *with respect to* $\mathcal{Q}$ if

(i) $\mathcal{P}(C)$, i.e. the object C solves the problem $\mathcal{P}(-)$.

(ii) $\forall D \in \mathbf{C} : \mathcal{P}(D) \Rightarrow \exists! \varphi : D \longrightarrow C : \mathcal{Q}(\varphi, D, C)$, i.e. for any other solution D of the same problem $\mathcal{P}(-)$, we can find one and only one morphism $\varphi : D \longrightarrow C$ that satisfies the property $\mathcal{Q}$.

Similarly, we say that C *is a co-universal solution of* $\mathcal{P}$ *with respect to* $\mathcal{Q}$ if

(iii) $\mathcal{P}(C)$,

(iv) $\forall D \in \mathbf{C} : \mathcal{P}(D) \Rightarrow \exists! \varphi : C \longrightarrow D : \mathcal{Q}(\varphi, C, D)$.

The proof of the following theorem trivially generalizes the classical proofs concerning the uniqueness of universal objects up to isomorphisms:

Theorem 2. *Suppose that C_1 and C_2 are two (co-)universal solutions of $\mathcal{P}$ with respect to $\mathcal{Q}$. Then C_1 is isomorphic to C_2 in $\mathbf{C}$.*

Proof. Since C_1 is a universal solution of $\mathcal{P}$ with respect to $\mathcal{Q}$, using (ii) of Def. 1 for $D = C_2$, there exists a unique $\varphi_1 : C_2 \longrightarrow C_1$ such that the property $\mathcal{Q}(\varphi_1, C_2, C_1)$ holds. In a similar way, there exists a unique φ_2 such that $\varphi_2 : C_1 \longrightarrow C_2$ so we have $\mathcal{Q}(\varphi_2, C_1, C_2)$. By assumption (2.1) on $\mathcal{Q}$, the property $\mathcal{Q}(\varphi_2 \circ \varphi_1, C_2, C_2)$ holds. Using again Def. 1(ii) with $D = C_2$, we get that only one arrow φ satisfies $\mathcal{Q}(\varphi, C_2, C_2)$. Since $\mathcal{Q}(1_{C_2}, C_2, C_2)$ also holds by (2.2), then $\varphi_2 \circ \varphi_1 = 1_{C_2}$. In a similar way, we have $\varphi_1 \circ \varphi_2 = 1_{C_1}$, which proves the theorem. □

Starting from the properties $\mathcal{P}$ and $\mathcal{Q}$, we can define a new category $\mathbf{C}(\mathcal{P}, \mathcal{Q})$. Its objects are the objects of the category $\mathbf{C}$ that satisfy the property $\mathcal{P}$ (i.e. all the solutions of our problem $\mathcal{P}(-)$), and its arrows are the arrows φ of the category $\mathbf{C}$ such that $\mathcal{Q}(f, C, D)$ holds (so that the property $\mathcal{Q}$ links all these solutions), i.e.:

- $C \in \mathbf{C}(\mathcal{P}, \mathcal{Q}) :\iff \mathcal{P}(C)$,
- $D \xrightarrow{\varphi} C$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$: $\iff \mathcal{Q}(\varphi, D, C)$, $D \xrightarrow{\varphi} C$ in $\mathbf{C}$,
- $\theta = \psi \circ \varphi$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q}) :\iff \theta = \psi \circ \varphi$ in $\mathbf{C}$.

Then, we have that C is a universal solution of $\mathcal{P}$ with respect to $\mathcal{Q}$ if and only if C is terminal in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$ (i.e. for all $D \in \mathbf{C}(\mathcal{P}, \mathcal{Q})$ there exists one and only one $\varphi : D \longrightarrow C$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$), and C is a co-universal solution of $\mathcal{P}$ with respect to $\mathcal{Q}$ if and only if C is initial in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$ (i.e. for all $D \in \mathbf{C}(\mathcal{P}, \mathcal{Q})$ there exists one and only one $\varphi : C \longrightarrow D$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$).

As we mentioned above, a (co-)universal solution of $\mathcal{P}$ is considered as the (co-)simplest or (co-)most natural solution of that problem. Even considering only the following elementary examples, we can start to justify this interpretation:

Example 3.

(i) Let's consider the problem to specify a topology on a set $X \in \mathbf{Set}$. The category $\mathbf{C}$ in this example is the category of all the topologies on X viewed as a poset, i.e. "$\subseteq$" is the unique arrow of $\mathbf{C}$, and we write $\tau \subseteq \sigma$ if the topology τ is coarser than the topology σ. The properties $\mathcal{P}$ and $\mathcal{Q}$ are defined as follow.

$$\begin{aligned} \mathcal{P}(\tau) &:\iff \tau \text{ is a topology on } X, \\ \mathcal{Q}(i, \tau, \sigma) &:\iff i = \subseteq, \ \tau \subseteq \sigma. \end{aligned}$$

The trivial topology $(\{\emptyset\}, X)$ is the co-universal solution of the property $\mathcal{P}$ with respect to the property $\mathcal{Q}$ and the discrete topology is the universal solution. Clearly, these also appear as trivial solutions; on the other hand, note that they are also the simplest/non-conventional solutions starting from the unique data $X \in \mathbf{Set}$ and with respect to the problem "set a topology on X": any other solution would necessarily introduce (in the case of the trivial topology) or delete (in the case of the discrete topology) something which is not related to the problem or the data itself. This example also shows that the notion of *simplest solution* can be implemented in two ways: from "below" (co-universal) or from "above" (universal).

(ii) Let R be a ring and let $x \notin R$. What would be the smallest/simplest ring containing both x and R? Any ring that contains x and R must contain also sums of terms of the form $r \cdot x^n$ for any integer n and any element $r \in R$. Intuitively, the simplest solution is therefore the ring of polynomials $R[x]$. The co-universal property can be highlighted as follow: Let $S \in \mathbf{Ring}$ be a ring, then we can consider the property $\mathcal{P}(S, s)$ whenever $x \in S$ and $s : R \longrightarrow S$ is a ring homomorphism, and the property $\mathcal{Q}(f, (S, s), (L, l))$ if $S \xrightarrow{f} L$ in $\mathbf{Ring}$ (i.e. it is a morphism of rings) and $f \circ s = l$. The ring of polynomials $R[x]$ is the co-universal solution of $\mathcal{P}$ with respect to $\mathcal{Q}$, i.e. the simplest way to extend the ring R by adding a new element $x \notin R$. Clearly, we have $\mathcal{P}(R[x], i)$, where $i : R \longrightarrow R[x]$ is the inclusion. Let $S \in \mathbf{Ring}$, and let $s : R \longrightarrow S$ be a ring homomorphism, i.e. $\mathcal{P}(S, s)$ holds, then the unique $\varphi : R[x] \longrightarrow S$ of Def. 1.(ii) is given by $\varphi\left(\sum_i r_i x^i\right) = \sum_i s(r_i) x^i$.

(iii) Let (X, d) be a metric space and (X^*, d^*) be its completion as the usual quotient of Cauchy sequences: $(x_n)_n \sim (y_n)_n$ if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Define an isometry $\varphi : X \longrightarrow X^*$ by setting $\varphi(x) := [x]_\sim$, where $[x]_\sim$ is the equivalent class generated by the constant sequence $x_n = x \in X$; we have that $\varphi(X)$ is dense in X^* (see e.g. [42]). The triple (X^*, d^*, φ) is co-universal among all the triples (Y, δ, ψ), where (Y, δ) is a complete metric space and $\psi : X \longrightarrow Y$ is an isometry such that $\psi(X)$ is dense in Y. There is therefore a unique map $\iota : X^* \longrightarrow Y$ such that $\iota \circ \varphi = \psi$, which is defined as follows: Let $x^* \in X^*$. Since $\varphi(X)$ is dense in X^*, there exists a sequence $(x_n)_n$ of X such that $(\varphi(x_n))_n$ converges to x^*. The sequence $(\varphi(x_n))_n$ is a Cauchy sequence and since φ and ψ are isometries, the sequence $(\psi(x_n))_n$ is also a Cauchy sequence in Y which converges because Y is Cauchy complete. We can thus set $\iota(x^*) := \lim_{n\to\infty}(\psi(x_n))$ which is well defined because φ and ψ are isometries.

(iv) Let U, $V \in \mathbf{Vect}$ be vector spaces. The simplest way to obtain a bilinear map $U \times V \xrightarrow{b} T$ into another vector space T is the tensor product $T = U \otimes V$, with $b(u, v) = u \otimes v$. This construction is indeed co-universal with respect to the properties:

$$
\begin{aligned}
\mathcal{P}(b, T) \quad &:\Longleftrightarrow \quad T \in \mathbf{Vect},\ U \times V \xrightarrow{b} T \text{ is bilinear} \\
\mathcal{Q}(\varphi, (b, T), (w, W)) \quad &:\Longleftrightarrow \quad \varphi : T \longrightarrow W \text{ in } \mathbf{Vect}.
\end{aligned}
$$

It is well-known that there are several ways to define the tensor product $U \otimes V$, even if they all satisfy this co-universal property (and hence, by Thm. 2, they are all isomorphic as vector spaces). Note that the category $\mathbf{C}$ of Def. 1 is the category of all the pairs (b, T) satisfying $\mathcal{P}(b, T)$.

We underscore that in all these universal solutions (as well as in products, sums, quotients, etc. of spaces) there are no conventional choices and they are the most natural solutions: any other (non-isomorphic) solution would appear as less natural, e.g. by adding (to the co-universal solution) or subtracting (to the universal solution) anything that does not strictly depend on the data of the problem.

2.1. **Preliminary notions: presheaf and sheaf.** For the sake of completeness, and also to specify all our notations, in this section we briefly recall the notions of presheaf and sheaf, because they are used in our universal characterization of spaces of GF.

In the following, we denote by **Set** the category of sets and functions, by $\mathbf{Mod}_R$ the category of modules over the ring R, so that $\mathbf{Vect}_K := \mathbf{Mod}_K$ is the category of vector spaces over a given field K, $\mathcal{O}\mathbb{R}^\infty$ is the category having as objects open sets $U \subseteq \mathbb{R}^u$ of any dimension $u \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and smooth functions as arrows, and finally **Ring** is the category of rings and ring-homomorphisms. If $\mathbb{T} = (|\mathbb{T}|, \tau)$ is a topological space, we use the same symbol to also denote the category induced by its open sets as a preorder, i.e. the category of open sets $A \in \tau$ of the given topology and only one arrow "$\subseteq$", i.e. we write $A \xrightarrow{\subseteq} B$ in $\mathbb{T}$ if $A \subseteq B$. We finally denote by $\mathbf{C}^{\mathrm{op}}$ the opposite of any category $\mathbf{C}$; for example, we write $f \in (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}}(A, B)$ if $f \in \mathcal{C}^\infty(B, A)$ is a smooth function from $B \subseteq \mathbb{R}^b$ into $A \subseteq \mathbb{R}^a$, and $\mathbb{T}^{\mathrm{op}}(A, B)$ is non empty if and only if $B \subseteq A$.

Definition 4.

(i) Let R be a ring. A presheaf P of $\mathbf{Mod}_R$ is a functor $P : \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_R$. We denote by $P(U) \in \mathbf{Mod}_R$ its evaluation at $U \in \mathbb{T}^{\mathrm{op}}$ and by $P_{U,V} := P(U \leq V) : P(U) \longrightarrow P(V)$ its evaluation on the arrow $U \supseteq V$. The map $P_{U,V}$ is called *restriction* from U to V.

(ii) If $(U_j)_{j\in J}$ is a covering in $\mathbb{T}$ of $U \in \mathbb{T}^{\mathrm{op}}$, then we say that $(f_j)_{j\in J}$ is a P-*compatible family* if and only if
  (i) $\forall j \in J : \ f_j \in P(U_j)$.
  (ii) $\forall j, h \in J : \ P_{U_j, U_j \cap U_h}(f_j) = P_{U_h, U_h \cap U_j}(f_h)$.

(iii) Moreover, we say that $P : \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_R$ is a *sheaf* if it is a presheaf satisfying the following conditions; for any $U \in \mathbb{T}^{\mathrm{op}}$, for any covering $(U_j)_{j\in J}$ of U in $\mathbb{T}$ and for any P-compatible family $(f_j)_{j\in J}$:
  (i) If $f, g \in P(U)$ and $P_{U,U_j}(f) = P_{U,U_j}(g)$ for all $j \in J$, then $f = g$ (*locality condition*); if P satisfies only this condition, it is called a *separated presheaf* or a *monopresheaf*.
  (ii) $\exists f \in P(U)\, \forall j \in J : \ P_{U,U_j}(f) = f_j$ (*gluing condition*).

(iv) Finally, if P, $Q : \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_R$ are sheaves, we say that $\varphi : P \longrightarrow Q$ is a *sheaf morphism* if φ is a natural transformation from P to Q, i.e. it is a family $(\varphi_U)_{U\in\mathbb{T}}$ such that $Q_{U,V} \circ \varphi_U = \varphi_V \circ P_{U,V}$ in $\mathbf{Mod}_R$ for all U, $V \in \mathbb{T}$ such that $U \supseteq V$.

Clearly, conditions (i), (ii) imply $\exists! f \in P(U)\, \forall j \in J : \ P_{UU_j}(f) = f_j$; we set $P_U\left[(f_j)_{j\in J}\right] := f$ and call it the P-*gluing of the family* $(f_j)_{j\in J}$. For example, it is not hard to prove (see e.g. [33]) that

$$P_{UV}\left(P_U\left[(f_j)_{j\in J}\right]\right)=P_V\left[\left(P_{U_j,V\cap U_j}(f_j)\right)_{j\in J}\right], \tag{2.3}$$

$$\psi_U\left(P_U\left[(f_j)_{j\in J}\right]\right)=Q_U\left[\left(\psi_{U_j}(f_j)\right)_{j\in J}\right]\quad \text{if } \psi:P\longrightarrow Q \text{ is a sheaf morphism.} \tag{2.4}$$

3. Co-universal property of Schwartz distributions

In this section, we want to show a co-universal property of the space of Schwartz's distributions. More precisely, as stated in [38], in this section we formalize the idea that the sheaf $\mathcal{D}'$ of Schwartz distributions is the simplest sheaf where we can take derivatives of continuous functions *and* preserving partial derivatives $\partial_k f$ of functions f which are continuously differentiable in the k-th variable. A similar statement can be found in [25]: "*In differential calculus one encounters immediately the unpleasant fact that not every function is differentiable. The purpose of distribution theory is to remedy this flaw; indeed, the space of distributions is essentially* the smallest extension *of the space of continuous functions where differentiability is always well defined*". Co-universal properties correspond to this informal notion of "smallest extension". This formalization also allows us to understand the importance of preservation of partial derivative of sufficiently regular functions, which is not explicitly included in the previous statement. For this reason, we define

Definition 5. Let $U\subseteq\mathbb{R}^n$ be an open set, $\alpha\in\mathbb{N}^n$ be a multi-index, and $k=1,\dots,n$, then:

(i) For all $x\in U$, we set $U_k(x):=\{t\in\mathbb{R}\mid(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)\in U\}$. We hence have a map $j_k:t\in U_k(x)\mapsto(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)\in U$.

(ii) Let $\alpha=(0,\overset{k-1}{\ldots},0,1,0,\dots,0)=:e_k$, and $f\in\mathcal{C}^0(U)$. Then, we write $f\in\mathcal{C}^\alpha(U)$ if f is of class 1 in the k-th variable, i.e.

$$\forall x\in U:\ f\circ j_k\in\mathcal{C}^1\left(U_k(x)\right),$$

and $\partial_k f:=(f\circ j_k)'\in\mathcal{C}^0(U)$. The space $\mathcal{C}^{e_k}(U)$ is also denoted by $\mathcal{C}^1_k(U)$.

(iii) If $\alpha\in\mathbb{N}^n$, the set of all the functions of class α_k in the k-th variable ($k=1,\dots,n$) is

$$\mathcal{C}^\alpha(U):=\left\{f\in\mathcal{C}^0(U)\mid\forall k=1,\dots,n:\ \alpha_k\neq0\ \Rightarrow\ f\in\mathcal{C}^{e_k}(U),\ \partial_k f\in\mathcal{C}^{\alpha-e_k}(U)\right\}.$$

In the usual way, it is possible to prove that $\mathcal{C}^\alpha$ is a sheaf. In case $\alpha=je_k$, the space $\mathcal{C}^\alpha(U)$ is also denoted by $\mathcal{C}^j_k(U)$. Note that if $f\in\mathcal{C}^\alpha(U)$ and k, j are such that α_k, $\alpha_j\neq0$, then by Schwarz's theorem we have $\partial_k\partial_j f=\partial_j\partial_k f$ on U.

(iv) We say that U is an n-dimensional interval if $U=(c_1-r,c_1+r)\times\overset{n}{\ldots}\times(c_n-r,c_n+r)$ for some $c\in\mathbb{R}^n$ and $r\in\mathbb{R}_{>0}$.

In what follows, the notations $\mathcal{C}^1_k \underset{\partial_k}{\overset{\iota_k}{\rightrightarrows}} \mathcal{C}^0$, are used to denote the inclusion and the partial derivatives of $\mathcal{C}^1_k$-functions (thought of as sheaves morphisms, e.g. we think $\iota_{kU}:\mathcal{C}^1_k(U)\hookrightarrow\mathcal{C}^0(U)$ as a natural transformation).

Remark 6. Schwartz's solution leads to the following objects:

(i) $\mathcal{D}':(\mathbb{R}^n)^{\mathrm{op}}\longrightarrow\mathbf{Vect}_{\mathbb{R}}$ is the sheaf of real valued distributions on $\mathbb{R}^n$.

(ii) $\mathcal{C}^0 \xrightarrow{\lambda} \mathcal{D}'$ is the inclusion of the space of continuous functions into the space of distributions. The map λ is a sheaf morphism, i.e. it is a natural transformation: $\lambda_U : \mathcal{C}^0(U) \longrightarrow \mathcal{D}'(U)$ for all open sets U, $V \subseteq \mathbb{R}^n$ with $V \subseteq U$, such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C}^0(U) \arrow[r, "\lambda_U"] \arrow[d, "\mathcal{C}^0_{U,V}"'] & \mathcal{D}'(U) \arrow[d, "\mathcal{D}'_{U,V}"] \\ \mathcal{C}^0(V) \arrow[r, "\lambda_V"'] & \mathcal{D}'(V) \end{tikzcd}$$

Therefore, $\mathcal{C}^0_{U,V}(f) := f|_V$ and $\mathcal{D}'_{U,V}(T) := T|_V$ are the corresponding restrictions.

(iii) $\mathcal{D}' \xrightarrow{D_k} \mathcal{D}'$, for $k = 1, \ldots, n$, are the partial derivatives of distributions. Once again, each D_k is a sheaf morphism because $D_{kU} : \mathcal{D}'(U) \longrightarrow \mathcal{D}'(U)$ for all open sets $U \subseteq \mathbb{R}^n$, and they commute with restrictions of distributions: $D_{kV}(\mathcal{D}'_{UV}(T)) = D_k(T|_V) = \mathcal{D}'_{UV}(D_{kU}(T)) = D_k(T)|_V$ if $V \subseteq \mathbb{R}^n$ is open and $V \subseteq U$. Moreover, D_{kU} is compatible with partial derivatives of $\mathcal{C}^1_k$ functions, i.e. $\lambda(\partial_k f) = D_k(\lambda(f))$ or, specifying all the domains and inclusions $\lambda_U(\partial_{kU} f) = D_{kU}(\lambda_U(\iota_{kU}(f)))$. In the following, we use the notations $D^j_{kU} := D_{kU} \circ \overset{j}{\ldots} \circ D_{kU}$ and $D^\alpha_U := D^{\alpha_1}_{1U} \circ \ldots \circ D^{\alpha_n}_{nU}$ for any multi-index $\alpha \in \mathbb{N}^n$ and any open set $U \subseteq \mathbb{R}^n$. Note explicitly that $D^\alpha_U(\lambda_U(f)) = \lambda_U(\partial^\alpha f)$ if $f \in \mathcal{C}^\alpha(U)$.

(iv) If $\alpha \in \mathbb{N}^n$, $f, g \in \mathcal{C}^0(U)$ and U is an n-dimensional interval, then $D^\alpha_U(\lambda_U(f)) = D^\alpha_U(\lambda_U(g))$ holds if and only if we can write $f - g = \theta_1 + \ldots + \theta_n$, where each θ_k is a polynomial in x_k of degree $< \alpha_k$ whose coefficients are continuous functions on U independent by x_k.

(v) $D_h \circ D_k = D_k \circ D_h$ for all h, $k = 1, \ldots, n$.

Theorem 7. *$(\mathcal{D}', \lambda, (D_k)_k)$ is a co-universal solution of the problem $\mathcal{P}(H, j, (\delta_k)_k)$ given by:*

(i) $H : (\mathbb{R}^n)^{op} \longrightarrow \mathbf{Vect}_\mathbb{R}$ *is a sheaf of real vector spaces.*

(ii) $j : \mathcal{C}^0 \longrightarrow H$ *is a sheaf morphism.*

(iii) $\delta_k : H \longrightarrow H$, $k = 1, \ldots, n$, *are compatible with partial derivatives of* $\mathcal{C}^1_k$ *functions:* $\delta_k \circ j \circ \iota_k = j \circ \partial_k$, *i.e. the following diagram of sheaves morphisms commutes for all* $k = 1, \ldots, n$*:*

$$\begin{tikzcd} \mathcal{C}^1_k \arrow[r, hook, "\iota_k"] \arrow[dr, "\partial_k"'] & \mathcal{C}^0 \arrow[r, "j"] & H \arrow[d, "\delta_k"] \\ & \mathcal{C}^0 \arrow[r, "j"] & H \end{tikzcd}$$

(iv) *Let* $\alpha \in \mathbb{N}^n$, $f \in \mathcal{C}^0(U)$ *and* U *be an* n*-dimensional interval. Assume that* $f = \theta_1 + \ldots + \theta_n$, *where each* θ_k *is a polynomial in* x_k *of degree* $< \alpha_k$ *whose coefficients are continuous functions on* U *independent by* x_k*, then* $\delta^\alpha_U(j_U(f)) = 0$.

(v) $\delta_h \circ \delta_k = \delta_k \circ \delta_h$ *for all* h, $k = 1, \ldots, n$.

The problem is solvable with respect to the property $\mathcal{Q}(\psi, H, j, (\delta_k)_k, \overline{H}, \overline{\jmath}, (\overline{\delta}_k)_k)$ *to preserve embeddings and derivatives given by*

$$\psi : H \longrightarrow \overline{H}, \qquad \psi \circ j = \overline{\jmath}, \qquad \psi \circ \delta_k = \overline{\delta}_k \circ \psi \quad \forall k = 1, \ldots, n,$$

i.e. when the following diagrams of sheaves morphisms commute

$$\begin{tikzcd} H \arrow[r, "\delta_k"] \arrow[d, "\psi"'] & H \arrow[d, "\psi"] \\ \overline{H} \arrow[r, "\overline{\delta}_k"'] & \overline{H} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^0 \arrow[r, "j"] \arrow[dr, "\overline{\jmath}"] & H \arrow[d, "\psi"] \\ & \overline{H} \end{tikzcd}$$

Therefore, if $(H, j, (\delta_k)_k)$ *is any solution of (i)-(iii), then*

$$\exists ! \psi : \mathcal{D}' \longrightarrow H : \ j = \psi \circ \lambda, \ \psi \circ D_k = \delta_k \circ \psi \quad \forall k = 1, \ldots, n. \tag{3.1}$$

Proof. We only have to prove (3.1), because it is clear from [38] that $(\mathcal{D}', \lambda, (D_k)_k)$ is a solution of (i)-(iii).

Let $U \subseteq \mathbb{R}^n$ and let $T \in \mathcal{D}'(U)$. The key idea to define $\psi_U(T)$ is to use the local structure of distributions to define $\psi_C(T|_C)$ for any $C \subseteq U$ with $\overline{C} \Subset U$, and then to use the gluing property to define $\psi_U(T)$ as the gluing of the compatible family $(\psi_C(T|_C))_C$.

The local structure theorem of distributions (see [38]) yields

$$T|_C = D_C^\alpha(\lambda_C(f)) \tag{3.2}$$

for some multi-index $\alpha \in \mathbb{N}^n$ and some continuous function $f \in \mathcal{C}^0(C)$. We necessarily have to define

$$\psi_C(T|_C) := \delta_C^\alpha\left(j_C(f)\right) \in H(C), \tag{3.3}$$

but we clearly have to prove that this definition does not depend on α and f in (3.2). Assume that

$$T|_C = D_C^{\alpha'}\left(\lambda_C(g)\right), \tag{3.4}$$

for another $\alpha' \in \mathbb{N}^n$ and another $g \in \mathcal{C}^0(C)$. We claim that

$$\delta_C^\alpha\left(j_C(f)\right) = \delta_C^{\alpha'}\left(j_C(g)\right). \tag{3.5}$$

Indeed, since $\delta^\alpha \circ j$ is a sheaf of morphisms, and since the set of all the n-dimensional intervals included in C is a covering of C, it is sufficient to show that (3.5) holds for any n-dimensional interval $C = (c_1 - r, c_1 + r) \times \overset{n}{\ldots} \times (c_n - r, c_n + r)$ of center $c \in \mathbb{R}^n$ and sides $2r \in \mathbb{R}_{>0}$. We first prove that we can change the functions f and g so that to have the same multi-index $\alpha = \alpha'$. Assume e.g. that $\alpha'_k > \alpha_k$, set $a_k := \alpha'_k - \alpha_k$, and integrate f in the variable x_k for a_k times:

$$\bar{f}(x) := \int_{c_k}^{x_k} \ldots \int_{c_k}^{t_2} \int_{c_k}^{t_1} f(x_1, \ldots, x_{k-1}, t_0, x_{k+1}, \ldots, x_n)\, \mathrm{d}t_0\, \mathrm{d}t_1 \ldots \mathrm{d}t_{a_k - 1} \quad \forall x \in C. \tag{3.6}$$

The function $\overline{f}$ is well-defined because C is an n-dimensional interval of center c and we have $\bar{f} \in \mathcal{C}_k^{a_k}(C)$ and $\partial_k^{a_k} \bar{f} = f$. Therefore, using the compatibility of D_k and ∂_k, we get

$$D_C^\alpha(\lambda_C(f)) = D_C^\alpha(\lambda_C(\partial_k^{a_k} \bar{f})) = D_C^\alpha(D_{kC}^{a_k}(\lambda_C \bar{f})) = D_C^{\alpha + a_k e_k}(\lambda_C(\bar{f})), \tag{3.7}$$

and

$$\alpha + a_k e_k = (\alpha_1, \dots, \alpha_{k-1}, \alpha'_k, \alpha_{k+1}, \dots, \alpha_k).$$

If $\alpha'_k < \alpha_k$, we can proceed similarly using (3.4) instead of (3.2). Therefore, for $\bar{\alpha}_k := \max(\alpha_k, \alpha'_k)$, $\alpha^f_k := \max(\alpha'_k - \alpha_k, 0)$, $\alpha^g_k := \max(\alpha_k - \alpha'_k, 0)$ and for suitable $\bar{f} \in \mathcal{C}^{\alpha^f}(C)$, $\bar{g} \in \mathcal{C}^{\alpha^g}(C)$, we have

$$T|_C = D^{\bar{\alpha}}_C \left(\lambda_C(\bar{f})\right) = D^{\bar{\alpha}}_C \left(\lambda_C(\bar{g})\right),$$

i.e. $D^{\bar{\alpha}}_C \left(\lambda_C(\bar{f} - \bar{g})\right) = 0$. Therefore, Rem. 6.(iv) (the necessary condition part) yields that $\bar{f} - \bar{g}$ can be written (on C) as $\bar{f} - \bar{g} = \theta_1 + \ldots + \theta_n$, where each θ_k is a polynomial in x_k of degree $< \bar{\alpha}_k$ whose coefficients are continuous functions on C independent by x_k. Property (iv) for $(H, j, (\delta_k)_k)$ (note explicitly that this condition only states the sufficient part of Rem. 6.(iv)) implies $\delta^{\bar{\alpha}}_C(j_C(\bar{f} - \bar{g})) = 0$, and hence $\delta^{\bar{\alpha}}_C(j_C(\bar{f})) = \delta^{\bar{\alpha}}_C(j_C(\bar{g}))$ because we are considering sheaves of vector spaces. Exactly as we increased α_k by a_k (if $\alpha'_k > \alpha_k$) in (3.7), we can now proceed backward to return to the old multi-index: since $\bar{f} \in \mathcal{C}^{a_k}_k(C)$

$$\delta^{\bar{\alpha}}_C(j_C(\bar{f})) = \delta^{\bar{\alpha} - a_k e_k}_C(\delta^{a_k}_{kC}(j_C \bar{f})) = \delta^{\bar{\alpha} - a_k e_k}_C(j_C(\partial^{a_k}_k \bar{f})),$$

and by induction we get $\delta^{\bar{\alpha}}_C(j_C(\bar{f})) = \delta^{\alpha}_C(j_C(\partial^{\overline{\alpha} - \alpha}\overline{f})) = \delta^{\alpha}_C(j_C(f))$. This proves that $\delta^{\alpha}_C(j_C(f)) = \delta^{\alpha'}_C(j_C(g))$, and hence our claim is proved.

We denote by $B(U)$ the set of all the relatively compact sets of U, which is, by the local structure of distributions, a covering of U. The family $(\psi_C(T|_C))_{C \in B(U)}$ is a compatible one. In fact $H_{C', C' \cap C}(\psi_{C'}(T|_{C'})) = H_{C', C' \cap C}(\delta^{\alpha}_{C'}(j_{C'}(f))) = \delta^{\alpha}_{C' \cap C}(j_{C' \cap C}(f)) = H_{C, C \cap C'}(\psi_C(T|_C))$, and we can hence set

$$\psi_U(T) := H_U \left[(\psi_C(T|_C))_{C \in B(U)} \right] \quad \forall T \in \mathcal{D}'(U).$$

We claim that if $T := D^{\alpha}_U(\lambda_U(f))$ for some $\alpha \in \mathbb{N}^n$ and for some $f \in \mathcal{C}^0(U)$, then $\psi_U(T) = \delta^{\alpha}_U(j_U(f))$. Indeed, for any $V \in B(U)$ we have

$$\begin{aligned} H_{U,V}(\psi_U(T)) &= H_{U,V}\left(H_U \left[(\delta^{\alpha}_C(j_C(f)))_{C \in B(U)} \right] \right) \\ &= H_V \left[(\delta^{\alpha}_{C \cap V}(j_{C \cap V}(f)))_{C \in B(U))} \right] = \delta^{\alpha}_V(j_V(f)) = H_{U,V}(\delta^{\alpha}_U(j_U(f))). \end{aligned}$$

where we used (2.3) in the second equality. Thus, by the locally condition of H (see Def. 4.(iii).((i))), our claim is proved. It follows in particular that $\psi_U(\lambda_U(f)) = \delta^0_U(j_U(f)) = j_U(f)$ for all $f \in \mathcal{C}^0(U)$.

If C, $C' \in B(U)$ are such that $C' \subseteq C$ then for any $T \in \mathcal{D}'(U)$

$$\begin{aligned} H_{C,C'}(\psi_C(T|_C)) &= H_{C,C'}(\delta^{\alpha}_C(j_C(f))) = \delta^{\alpha}_{C'}(j_{C'}(f)) = \psi_{C'}(D^{\alpha}_{C'}(\lambda_{C'}(f))) \quad (3.8) \\ &= \psi_{C'}(T|_{C'}) \quad (3.9) \end{aligned}$$

because $\delta^{\alpha} \circ j$ is sheaf morphism. Thus, for any $V \subseteq U$, (3.8) together with (2.3) imply that

$$\begin{aligned} H_{U,V}(\psi_U(T)) &= H_{U,V}\left(H_U \left[(\psi_C(T|_C))_{C \in B(U)} \right] \right) \\ &= H_V \left[(H_{C, V \cap C}(\psi_C(T|_C)))_{C \in B(U)} \right] \\ &= H_V \left[(\psi_{V \cap C}(T|_{V \cap C}))_{C \in B(U)} \right] = H_V \left[(\psi_D(T|_D))_{D \in B(V)} \right]. \end{aligned}$$

where the latter equality follows from the fact that H is sheaf morphism, and the fact that the families $(\psi_{C\cap V}(T|_{C\cap V}))_{C\in B(U)}$, $(\psi_D(T|_D))_{D\in B(V)}$ are compatible and locally equal. Therefore $\psi:\mathcal{D}'\longrightarrow H$ is a sheaf morphism. To prove the equality $\psi\circ D_k=\delta_k\circ\psi$, we have

$$\begin{aligned}\psi_U(D_{kU}(T)) &= H_U\left[(\psi_C(D_{kU}T)|_C)_{C\in B(U)}\right]=H_U\left[(\psi_C(D_{kC}(T|_C)))_{C\in B(U)}\right]\\ &= H_U\left[(\delta_{kC}(\psi_C(T|_C))_{C\in B(U)}\right]=\delta_{kU}(\psi_U(T)),\end{aligned}$$

where we used the equality

$$\psi_C(D_{kC}(T|_C)=\psi_C(D_C^{e_k+\alpha}(\lambda_C(f)))=\delta_{kC}\delta_C^\alpha(j_C(f))=\delta_{kC}\psi_C(T|_C)$$

for some continuous function $f\in\mathcal{C}(U)$ and multi-index $\alpha\in\mathbb{N}^n$. Note explicitly that in the step $\delta_C^{\alpha+e_k}=\delta_{kC}\circ\delta_C^\alpha$ above we need the commutativity property (v).

It remains to prove the uniqueness. Assume that also $\bar\psi$ satisfies (3.1); let $C\in B(U)$ and let f and α be such that $T|_C=D_C^\alpha(\lambda_C(f))$, then

$$\bar\psi_C(T|_C)=\bar\psi_C(D_C^\alpha(\lambda_C(f)))=\delta_C^\alpha(\bar\psi_C(\lambda_C(f)))=\delta_C^\alpha(j_C(f))=\psi_C(T|_C).$$

Therefore, property (2.4) yields

$$\begin{aligned}\bar\psi_U(T)=\bar\psi_U\left(\mathcal{D}'_U\left[(T|_C)_{C\in B(U)}\right]\right)=H_U\left[(\bar\psi_C(T|_C))_{C\in B(U)}\right]=\\ H_U\left[(\psi_C(T|_C))_{C\in B(U)}\right]=\psi_U(T).\end{aligned}$$

□

Using a categorical language, the universal property Thm. 7 (and the general Thm. 2) corresponds to the axiomatic characterization of distributions given by Sebastiao e Silva in [39, 40]. However, note that Thm. 7 yields a characterization up to isomorphisms of the whole sheaf of distributions, not only those defined locally as in [39, 40]. Moreover, note that the universal property allows one to avoid both the axiom of local structure of distributions [40, Axiom 3], and the necessary condition of Rem. 6.(iv) (see [40, Axiom 4]). In fact, we have the following

Corollary 8. *If $(H,j,(\delta_k)_k)$ is a co-universal solution of the problem stated in Thm. 7, then*

(i) *If $U\subseteq\mathbb{R}^n$ and C is a relatively compact set of U, then $\forall w\in H(U)\,\exists\alpha\in\mathbb{N}^n\,\exists f\in\mathcal{C}^0(C):\ w|_C=\delta_C^\alpha(j_C(f))$.*

(ii) *If $\alpha\in\mathbb{N}^n$, f, $g\in\mathcal{C}^0(U)$, U is an n-dimensional interval, and $\delta_U^\alpha(j_U(f))=\delta_U^\alpha(j_U(g))$ then we can write $f-g=\theta_1+\ldots+\theta_n$, where each θ_k is a polynomial in x_k of degree $<\alpha_k$ whose coefficients are continuous functions on U independent by x_k.*

Proof. In fact, Thm. 2 yields an isomorphism $\psi:\mathcal{D}'\longrightarrow H$ which preserves derivatives $\psi\circ D_k=\delta_k\circ\psi$ and embeddings $j=\psi\circ\lambda$. Therefore, the equalities $\psi^{-1}(w|_C)=D_C^\alpha(\lambda_C(f))$ and $\delta_U^\alpha(j_U(f))=\delta_U^\alpha(j_U(g))$ are equivalent to $w|_C=\delta_C^\alpha(j_C(f))$ and $D_U^\alpha(\lambda_U(f))=D_U^\alpha(\lambda_U(g))$. The claims then follow from similar properties of $(\mathcal{D}',\lambda,(D_k)_k)$. □

3.1. **Application: Sebastiao e Silva algebraic definition of distributions.** In the study of universal properties, it frequently happens that this characterization (up to isomorphisms) suggests possible generalizations. For distributions, these ideas are actually already contained in the proof of Thm. 7, but we prefer to explain them using the thoughts of Sebastiao e Silva, see [39, 40]. Assume that the open set I is an n-dimensional interval $I = (c_1 - r, c_1 + r) \times \overset{n}{\ldots\ldots} \times (c_n - r, c_n + r)$. For each continuous function $f \in \mathcal{C}^0(I)$ and each $k = 1, \ldots, n$, we can consider any primitive of f with respect to the variable x_k, e.g. setting

$$\mathfrak{I}_k f(x) := \int_{c_k}^{x_k} f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)\, \mathrm{d}t, \tag{3.10}$$

and, more generally, $\mathfrak{I}^\alpha := \mathfrak{I}_1^{\alpha_1} \circ \ldots \circ \mathfrak{I}_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$, so that $\partial^\beta \mathfrak{I}^\alpha f = \mathfrak{I}^{\alpha-\beta} f$ if $\alpha \ge \beta$ and $f \in \mathcal{C}^0(I)$. Assume that f, $g \in \mathcal{C}^0(I)$, r, $s \in \mathbb{N}^n$, and $D^r f = D^s g$ (in the sense of distributions; for simplicity, here we omit the dependence on the open set I and we identify $\lambda_I(f)$ with f). Set $m := \max(r, s)$, then the compatibility property Thm. 7.(iii) yields $D^r f = D^r(\partial^{m-r}(\mathfrak{I}^{m-r} f)) = D^m(\mathfrak{I}^{m-r} f) = D^m(\mathfrak{I}^{m-s} g) = D^s g$. Therefore, Cor. 8.(ii) yields $\mathfrak{I}^{m-r} f - \mathfrak{I}^{m-s} g = \theta_1 + \ldots + \theta_n$, where each θ_k is a polynomial in x_k of degree $< m_k$ whose coefficients are continuous functions on I independent by x_k. Denote by $\mathcal{P}_m$ the set of all the functions θ of this form $\theta = \theta_1 + \ldots + \theta_n$. Therefore, we proved that

$$D^r f = D^s g \iff \mathfrak{I}^{m-r} f - \mathfrak{I}^{m-s} g \in \mathcal{P}_m, \text{ where } m := \max(r, s). \tag{3.11}$$

The main idea of [39, 40] is that a condition such as the right hand side of (3.11) can be stated for pair of continuous functions without any need to use methods of functional analysis, but only using a *formal* algebraic approach: we can say that the derivative $D^r f$ of a continuous function $f \in \mathcal{C}^0(I)$ is simply a formal operation corresponding to the pair (r, f), and two pairs are equivalent if the right hand side of (3.11) holds. Therefore, if I is an n-dimensional interval, we can define: $(r, f) \sim (s, g)$ if r, $s \in \mathbb{N}^n$, f, $g \in \mathcal{C}^0(I)$ and $\mathfrak{I}^{m-r} f - \mathfrak{I}^{m-s} g \in \mathcal{P}_m$, where $m := \max(r, s)$; $\mathcal{D}'_{\mathrm{f}}(I) := (\mathbb{N}^n \times \mathcal{C}^0(I))/\sim$; $\lambda_I(f) := [(0, f)]_\sim$; $D_k([(r, f)]_\sim) := [(r + e_k, f)]_\sim$, so that $D^r f = [(r, f)]_\sim \in \mathcal{D}'_{\mathrm{f}}(I)$; finally, the vector space operations are defined as $D^r f + D^s g := D^m(\mathfrak{I}^{m-r} f + \mathfrak{I}^{m-s} g)$ ($m := \max(r, s)$) and $\mu \cdot D^r f := D^r(\mu f)$ for all $\mu \in \mathbb{R}$; the restriction to another n-dimensional interval $J \subseteq I$ is defined by $(D^r f)|_J := D^r(f|_J)$. With these definitions we obtain a functor

$$\mathcal{D}'_{\mathrm{f}} : \mathcal{I}(\mathbb{R}^n)^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}, \tag{3.12}$$

where $\mathcal{I}(U)$ is the poset of all the n-dimensional intervals contained in the open set $U \subseteq \mathbb{R}^n$. Clearly, $\mathcal{I}(\mathbb{R}^n)$ is not a topological space, but it is a base for the Euclidean topology of $\mathbb{R}^n$, and this suffices to apply a general co-universal method (called *sheafification*, see [1, 26]) to associate a sheaf $\mathcal{D}' : (\mathbb{R}^n)^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ to $\mathcal{D}'_{\mathrm{f}}$: this corresponds to the intuitive idea that any distribution is obtained by gluing a compatible family, where each element of the family is the (distributional) derivative of a continuous function. We first use distribution theory as an example to motivate sheafification in this case, but then we introduce this construction in general terms as another example to solve a problem in the simplest way.

For an arbitrary $T \in \mathcal{D}'(U)$, $U \subseteq \mathbb{R}^n$ being an open set, we can consider all the possible intervals $I \in \mathcal{I}(U)$ such that $T|_I$ is in $\mathcal{D}'_{\mathrm{f}}(I)$:

$$\mathcal{B}(T) := \{I \in \mathcal{I}(U) \mid T|_I \in \mathcal{D}'_{\mathrm{f}}(I)\}. \tag{3.13}$$

By the local structure of distributions, and the fact that $\mathcal{I}(U)$ is a base, we have that $\mathcal{B}(T)$ is a covering of U. Intuitively, among all the possible coverings of U made of intervals, $\mathcal{B}(T)$ is the largest one (e.g. it surely contains all the $I \in \mathcal{I}(U)$ such that $\bar{I} \subseteq U$ where the local structure theorem applies). We start by understanding how to formalize this idea that $\mathcal{B}(T)$ is "the largest one" because this would allow us to use only the separateness of $\mathcal{D}'_{\mathrm{f}}(-)$ and an arbitrary $\mathcal{B}(T)$-indexed compatible family such as $(T|_I)_{I\in\mathcal{B}(T)}$.

Remark 9. Separateness and being a compatible family can clearly be formulated also for a functor of the type (3.12):

(i) We say that $\mathcal{D}'_{\mathrm{f}}$ is *separated* because if T, $S \in \mathcal{D}'_{\mathrm{f}}(I)$, $I \in \mathcal{I}(\mathbb{R}^n)$, and $(I_j)_{j\in J}$ is a covering of I made of intervals such that $T|_{I_j} = S|_{I_j}$ for all $j \in J$, then $T = S$.

(ii) For all $I \in \mathcal{B}(T)$, we have $T|_I \in \mathcal{D}'_{\mathrm{f}}(I)$; moreover, $(T|_I)\,|_K = (T|_J)\,|_K$ for all I, $J \in \mathcal{B}(T)$ and all $K \in \mathcal{I}(\mathbb{R}^n)$ such that $K \subseteq I \cap J$, i.e. $(T|_I)_{I\in\mathcal{B}(T)}$ is a *compatible family*.

Now, let $S \in \mathcal{D}'_{\mathrm{f}}(J)$, $J \in \mathcal{I}(U)$; assume that S is *locally equal to* $(T|_I)_{I\in\mathcal{B}(T)}$, i.e. it satisfies

$$\forall I \in \mathcal{B}(T)\,\forall K \in \mathcal{I}(\mathbb{R}^n):\ K \subseteq I \cap J\ \Rightarrow\ S|_K = T|_K, \tag{3.14}$$

then by the sheaf property of $\mathcal{D}'$, we have $S = T|_J$ and hence $J \in \mathcal{B}(T)$: in these general sheaf-theoretical terms the covering $\mathcal{B}(T)$ is the largest one. It clearly also holds the opposite implication: if $J \in \mathcal{B}(T)$, then $S := T|_J$ satisfy (3.14). We write $S =_J (T|_I)_{I\in\mathcal{B}(T)}$ if (3.14) holds, so that

$$\mathcal{B}(T) = \left\{ J \in \mathcal{I}(U) \mid \exists S \in \mathcal{D}'_{\mathrm{f}}(J):\ S =_J (T|_I)_{I\in\mathcal{B}(T)} \right\}.$$

Intuitively, we can say that the distribution T can be identified with the family $(T|_I)_{I\in\mathcal{B}(T)}$ defined on the largest possible domain (in this sense, we expect it is co-universal).

All this motivates the following general

Definition 10. Let $\mathcal{I}$ be a base for the topological space $\mathbb{T}$, $P : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ a functor, $U \in \mathbb{T}$, $\mathcal{B} \subseteq \mathcal{I}$ be a covering of U, $J \in \mathcal{I}$, $S \in P(J)$, and $(T_I)_{I\in\mathcal{B}}$ a P-compatible family. Then, we write $S =_J (T_I)_{I\in\mathcal{B}}$ and we say S *locally equals* $(T_I)_{I\in\mathcal{B}}$ *on* J if and only if

$$\forall I \in \mathcal{B}\,\forall K \in \mathcal{I}:\ K \subseteq I \cap J\ \Rightarrow\ P_{J,K}(S) = P_{I,K}(T_I).$$

Moreover, we say that $(T_I)_{I\in\mathcal{B}}$ *is a maximal family on* U if and only if

(i) $(T_I)_{I\in\mathcal{B}}$ is a compatible family

(ii) $\forall J \in \mathcal{I}\,\forall S \in P(J):\ S =_J (T_I)_{I\in\mathcal{B}}\ \Rightarrow\ J \in \mathcal{B},\ S = T_J$.

The separateness of P is used in the following result, that allows us to consider the maximal family generated by a given compatible family. The idea is to consider all the section $S \in P(J)$ of the presheaf that locally equals the given family.

Theorem 11. *Let $\mathcal{I}$ be a base for the topological space $\mathbb{T}$, $P : \mathcal{I}^{op} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ a separated functor, $\mathcal{B} \subseteq \mathcal{I}$ be a covering of $U \in \mathbb{T}$ and let $(T_I)_{I\in\mathcal{B}}$ be a compatible family. Set*

$$\bar{\mathcal{B}} := \{J \in \mathcal{I} \mid \exists S \in P(J):\ S =_J (T_I)_{I\in\mathcal{B}}\} \tag{3.15}$$

then, we have

(i) $\forall J \in \bar{\mathcal{B}}\, \exists!\, \bar{T} \in P(J) : \ \bar{T} =_J (T_I)_{I \in \mathcal{B}}$. *We denote by* $\bar{T}_J$ *this unique* $\overline{T}$.
(ii) $\forall I \in \mathcal{B} : \ I \in \bar{\mathcal{B}}$ *and* $\bar{T}_I = T_I$.
(iii) $(\bar{T}_I)_{I \in \bar{\mathcal{B}}}$ *is a maximal family on* U.

Proof. To prove (i) simply use (3.15) and the separateness of P. To prove (ii) use the assumption that $(T_I)_{I \in \mathcal{B}}$ is a compatible family. To prove (iii) use (3.15) and Def. 10. □

We use the notation $\max[(T_I)_{I \in \mathcal{B}}] := (\bar{T}_I)_{I \in \bar{\mathcal{B}}}$, and we can now define the sheaf $\overline{P}$ on objects:

Definition 12. If $U \in \mathbb{T}$, set $(T_I)_{I \in \mathcal{B}} \in \overline{P}(U)$ if and only if

(i) $\mathcal{B} \subseteq \mathcal{I}$ is a covering of U;
(ii) $(T_I)_{I \in \mathcal{B}}$ is a maximal family on U.

To eventually get an R module (which is the case of real-valued distributions), we also have to define module operations:

Definition 13. Let $U \in \mathbb{T}$, $r \in R$ and let $(T_I)_{I \in \mathcal{B}}$, $(S_J)_{J \in \mathcal{C}} \in \overline{P}(U)$. Then

(i) $(T_I)_{I \in \mathcal{B}} + (S_J)_{J \in \mathcal{C}} := \max[(T_A + S_A)_{A \in \mathcal{B} \cap \mathcal{C}}]$, where $\mathcal{B} \cap \mathcal{C} := \{I \cap J \mid I \subseteq \mathcal{B},\ J \subseteq \mathcal{C}\}$ which is clearly a covering of U; clearly the family $(T_A + S_A)_{A \in \mathcal{B} \cap \mathcal{C}}$ is a compatible one.
(ii) $r \cdot (T_I)_{I \in \mathcal{B}} := \max[(r \cdot T_I)_{I \in \mathcal{B}}]$.

Using these operations, it is possible to prove that $(\overline{P}(U), +, \cdot) \in \mathbf{Mod}_R$. We still use the symbol $\overline{P}(U)$ to denote this R-module. We finally define $\overline{P}$ on arrows.

Definition 14. Let $U, V \in \mathbb{T}$, $V \subseteq U$. Then

(i) $\mathcal{C}_{\subseteq V} := \{J \subseteq V \mid J \in \mathcal{C}\}$ where $\mathcal{C} \subseteq \mathcal{I}$ is a covering of U.
(ii) $\overline{P}_{UV} : (T_I)_{I \in \mathcal{C}} \in \overline{P}(U) \longmapsto (P_{IJ}(T_I))_{J \in \mathcal{C}_{\subseteq V}} \in \overline{P}(V)$, where $I \in \mathcal{C}$ is any open set such that $I \supseteq J$ (two different of these I yield the same value of $P_{IJ}(T_I)$ by the compatibility property of $(T_I)_{I \in \mathcal{C}} \in \bar{P}(U)$). It is not hard to prove that the family $(P_{IJ}(T_I))_{J \in \mathcal{C}_{\subseteq V}}$ is already a maximal one.

The link between P and $\overline{P}$ is given by the following natural transformation

$$\eta_I : T \in P(I) \mapsto \max\left[(P_{IJ}(T))_{J \in \mathcal{I}_{\subseteq I}}\right] \in \overline{P}(I). \tag{3.16}$$

With these definitions, we have the following universal property, whose proof easily follows from our definitions and from Thm. 11:

Theorem 15. *If* $P : \mathcal{I}^{op} \longrightarrow \boldsymbol{Mod}_R$ *is separated then*

(i) $\overline{P} : \mathbb{T}^{op} \longrightarrow \boldsymbol{Mod}_R$ *is a sheaf*
(ii) (3.16) *defines a natural transformation*
(iii) $(\overline{P}, \eta)$ *is co-universal among all* $(\overline{P}, \eta)$ *that satisfy (i), (ii), i.e. if* $(\tilde{P}, \mu)$ *also satisfies (i), (ii), then there exists one and only one natural transformation* ψ *such that* $\psi_I \circ \eta_I = \mu_I$ *for all* $I \in \mathcal{I}$.

The general construction of sheafification of a presheaf can be found e.g. in [26, 33]. All this formalizes the intuitive idea that distributions on an arbitrary open set U are obtained by gluing together *in the simplest way* distributions on relatively compact n-dimensional intervals of U.

3.2. **Generalization: distributions on Hilbert spaces.** The previous construction leads naturally to consider several kind of potential generalizations:

(i) We can think about vector spaces over the complex field $\mathbb{C}$. Note explicitly that even in $\mathbb{C}^n$, the usual construction of distributions as continuous functionals on compactly supported smooth functions cannot be generalized to holomorphic maps because of the identity theorem.

(ii) The integrals (3.10) represent a way to construct a primitive in the direction e_k and can hence be generalized to suitable infinite dimensional spaces.

(iii) Definition (3.10) leads us to consider an at most countable orthonormal family $(e_k)_{k\in\Lambda}$, $\Lambda\subseteq\mathbb{N}$, in a Hilbert space, so that orthogonal complement $H=\operatorname{span}(e_k)\oplus\operatorname{span}(e_k)^\perp$ always exists.

(iv) Multidimensional intervals are used above as a base of the Euclidean topology, but in more abstract normed spaces balls can be more conveniently used.

On the other hand, the definition of a non-trivial space of generalized functions of a complex variable that allows one to consider derivatives of continuous functions is a non-obvious task. In fact, if we want that these generalized functions embed ordinary continuous maps and, at the same time, satisfy the Cauchy theorem, then the continuous functions would also be path-independent and, from Morera's theorem, they would actually be holomorphic functions, see e.g. [46]. Likewise, if we want that these generalized functions satisfy the Cauchy-Riemann equations (even with respect to distributional derivatives), then necessarily the embedded continuous ones will be ordinary holomorphic functions, see [22]. Using the language of the present paper, we could say that the co-universal solution of the problem to have derivatives of continuous functions of a complex variable which are path-independent or satisfy the Cauchy-Riemann equation is the sheaf of holomorphic functions, and a larger space is not possible.

In the following, we therefore consider a Hilbert space H with inner product (x,y). The field of scalars is denoted by $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$. In this space, we fix an orthonormal Schauder basis $(e_k)_{k\in\Lambda}$, $\Lambda\subseteq\mathbb{N}$ of H. An interesting example is the space $\mathcal{C}^0(K,\mathbb{R}^d)$ of all the $\mathbb{R}^d$ valued continuous functions on a compact set.

For simplicity, we deal with the case $\mathbb{F}=\mathbb{R}$, and the case $\mathbb{F}=\mathbb{C}$ can be treated in a very similar way. Let x, $c\in H$, $k\in\Lambda$, $J\subseteq\Lambda$ a finite subset. Under our assumptions, $x=x_k+x_k^\perp$, where $x_k=(x,e_k)e_k=:\hat{x}_ke_k\in\operatorname{span}(e_k)$ and $x_k^\perp\in\operatorname{span}(e_k)^\perp$; more generally, $x=x_J+x_J^\perp$, where $x_J:=\sum_{j\in J}x_j\in\operatorname{span}(\{e_j\}_{j\in J})$, and $x_J^\perp\in\operatorname{span}(\{e_j\}_{j\in J})^\perp$. We set $\widehat{[c,x]}_j:=[\min(\hat{c}_j,\hat{x}_j),\max(\hat{c}_j,\hat{x}_j)]\subseteq\mathbb{R}$, and $[c,x]_J=\{\sum_{j\in J}t_je_j\mid t_j\in\widehat{[c,x]}_j\ \forall j\in J\}$.

Let $f\in\mathcal{C}^0(B_r(c),H)$ a continuous function defined in the ball $B_r(c)\subseteq H$ of radius $r>0$ and center $c\in H$. We have that

$$\forall x\in B_r(c):\ f(x)=\sum_{k\in\Lambda}\hat{f}_k(x)e_k.$$

Using the orthogonality property and the continuity of f, one can see that each $\hat{f}_k$ is also continuous. Hence, for any $x\in B_r(c)$, for any j, $k\in\Lambda$, the function $\widehat{[c,x]}_j\longrightarrow\mathbb{R}$, $t\mapsto\hat{f}_k(x_j^\perp+te_j)$ is continuous. Therefore, the integral

$$\int_{\hat{c}_j}^{\hat{x}_j}\hat{f}_k(x_j^\perp+te_j)\,\mathrm{d}t$$

is well defined. We assume that the following assumption holds

$$\forall x \in B_r(c)\,\forall J \subseteq \Lambda\,\text{finite}: \sum_{k\in\Lambda} \sup_{y\in[c,x]_J} |\hat{f}_k(x_J^\perp + y)|^2 < \infty. \tag{3.17}$$

The sheaf of continuous functions $f \in \mathcal{C}^0(B_r(c), H)$ satisfying (3.17) is denoted by $\mathcal{C}^0_{\mathrm{p}}(B_r(c), H)$. Then, we clearly have

$$\left\|\sum_{k\in\Lambda} \int_{\hat{c}_j}^{\hat{x}_j} \hat{f}_k(x_j^\perp + te_j)\,\mathrm{d}t \cdot e_k\right\|^2 \le |\hat{x}_j - \hat{c}_j|^2 \sum_{k\in\Lambda} \sup_{y\in[c,x]_j} |\hat{f}_k(x_j^\perp + y)|^2 < \infty.$$

Therefore, we can set

$$\mathfrak{I}_j(f)(x) := \int_{\hat{c}_j}^{\hat{x}_j} f(x)\,\mathrm{d}e_j := \sum_{k\in\Lambda} \int_{\hat{c}_j}^{\hat{x}_j} \hat{f}_k(x_j^\perp + te_j)\,\mathrm{d}t \cdot e_k \tag{3.18}$$

and is called the primitive of f in the direction e_j. Indeed, (3.18) is a generalization of (3.10). Moreover, we have

$$\forall x \in B_r(c)\,\forall J \subseteq \Lambda\,\text{finite}\,\forall y \in [c,x]_J:\ \widehat{\mathfrak{I}_j(f)}_k(x_J^\perp + y) := \int_{\hat{c}_j}^{\hat{x}_j} \hat{f}_k((x_J^\perp + y)_j^\perp + te_j)\,\mathrm{d}t.$$

We can easily see that

$$(x_J^\perp + y)_j^\perp := \begin{cases} x_J^\perp + y_j^\perp & j \in J \\ x_{J\cup\{j\}}^\perp + y & j \notin J \end{cases}$$

In former situation we have

$$\begin{aligned}\sup_{y\in[c,x]_J} \left|\widehat{\mathfrak{I}_j(f)}_k(x_J^\perp + y)\right| &\le |\hat{x}_j - \hat{c}_j| \sup_{z\in[c,x]_{J\setminus\{j\}}} \sup_{t\in\widehat{[c,x]}_j} \left|\hat{f}_k(x_J^\perp + z + te_j)\right| \\ &= |\hat{x}_j - \hat{c}_j| \sup_{y\in[c,x]_J} \left|\hat{f}_k(x_J^\perp + y)\right|,\end{aligned}$$

and in the latter situation we have

$$\begin{aligned}\sup_{y\in[c,x]_J} \left|\widehat{\mathfrak{I}_j(f)}_k(x_J^\perp + y)\right| &\le |\hat{x}_j - \hat{c}_j| \sup_{y\in[c,x]_J} \sup_{t\in\widehat{[c,x]}_j} \left|\hat{f}_k(x_{J\cup\{j\}}^\perp + y + te_j)\right| \\ & |\hat{x}_j - \hat{c}_j| \sup_{z\in[c,x]_{J\cup\{j\}}} \left|\hat{f}_k(x_{J\cup\{j\}}^\perp + z)\right|.\end{aligned}$$

Thus, $\mathfrak{I}_j(f)$ also satisfies assumption (3.17). Therefore, for any continuous function $f \in \mathcal{C}^0(B_r(c), H)$ satisfying (3.17), and for any finite family $(j_1, ..., j_m) \in \Lambda^m$, we can consider the function $\mathfrak{I}_{j_m} \circ \ldots \circ \mathfrak{I}_{j_1}(f)$.

One can ask now whether the equality

$$\forall j,\, l \in \Lambda:\ \mathfrak{I}_l \circ \mathfrak{I}_j(f) = \mathfrak{I}_j \circ \mathfrak{I}_l(f) \tag{3.19}$$

holds or not. Indeed, using Fubini's theorem (to the continuous function $\widehat{[c,x]}_j \times \widehat{[c,x]}_l \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(t,s) \mapsto g_{x,k}(s,t) := \hat{f}_k(x^{\perp}_{\{l\}\cup\{j\}} + te_j + se_l))$ we obtain

$$\begin{aligned}\int_{\hat{c}_l}^{\hat{x}_l} \widehat{\mathfrak{I}_j(f)}_k(x_l^{\perp} + se_l)\mathrm{d}s &= \int_{\hat{c}_l}^{\hat{x}_l}\int_{\hat{c}_j}^{\hat{x}_j} \hat{f}_k(x^{\perp}_{\{l\}\cup\{j\}} + te_j + se_l)\mathrm{d}t\mathrm{d}s.\\ &= \int_{\hat{c}_j}^{\hat{x}_j}\int_{\hat{c}_l}^{\hat{x}_l} \hat{f}_k(x^{\perp}_{\{l\}\cup\{j\}} + te_j + se_l)\mathrm{d}s\mathrm{d}t\\ &= \int_{\hat{c}_j}^{\hat{x}_j} \widehat{\mathfrak{I}_l(f)}_k(x_j^{\perp} + te_j)\mathrm{d}t\end{aligned}$$

which shows that (3.19) holds.

Furthermore, for any j, $k \in \Lambda$, $x \in H$, we have that the function $(-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$, ($\varepsilon > 0$ is sufficiently small) $t \mapsto \widehat{\mathfrak{I}_j(f)}_k(x + te_j)$ is derivable and

$$\forall\, k \in \Lambda\, \forall x \in B_r(c): \lim_{t\to 0} \frac{\widehat{\mathfrak{I}_j(f)}_k(x+te_j) - \widehat{\mathfrak{I}_j(f)}_k(x)}{t} = f_k(x),$$

which implies that

$$\frac{\partial \mathfrak{I}_j f}{\partial e_j}(x) := \lim_{t\to 0} \frac{\mathfrak{I}_j f(x+te_j) - \mathfrak{I}_j f(x)}{t} = f(x) \quad \forall j \in \Lambda\, \forall x \in B_r(c)$$

where the limit is with respect to the weak topology.

Remark 16.

(i) A possible generalization is to consider continuous functions with values in another Hilbert space B having an orthogonal Schauder basis.

(ii) The case where $\mathbb{F} = \mathbb{C}$ can be treated in a very similar way, and one can consider the primitive and the derivative with respect to the real part and the imaginary part of e_k. Of course, in this way we do not get complex differentiability but a trivially isomorphic construction of $\mathcal{D}'(\mathbb{R}^2)$.

(iii) In the finite dimensional case $\mathcal{D}'(\mathbb{R}^n)$, both continuity and differentiability of a function $f : U \longrightarrow \mathbb{R}^n$ can be equivalently formulated considering only the projections $f_k : U \longrightarrow \mathbb{R}$, as we did e.g. in Thm. 7.

We can now proceed as in the classical case:

Definition 17.

(i) $\mathfrak{I}^0 f := f$ and $\mathfrak{I}^r := \mathfrak{I}^{r_1} \circ \ldots \circ \mathfrak{I}^{r_n}$ for all $r \in \Lambda^n$, $n \in \mathbb{N}$;

(ii) If B is a ball in H and $m \in \Lambda^n$, then we define $\theta \in \mathcal{P}_m(B)$ if $\theta \in \mathcal{C}^0(B,H)$ with $\hat{\theta}_k := \sum_{j\in\Lambda,\, m_j\neq 0} \hat{\theta}_{kj}$ where $\hat{\theta}_{kj}$ is a polynomial function in $\hat{x}_j$ of order $< m_j$ whose coefficients are continuous functions on B independent by $\hat{x}_j$.

We can now proceed by following Sebastiao e Silva's idea: in each ball B in H we define $(r,f) \sim (s,g)$ if there exists $n \in \mathbb{N}$ such that r, $s \in \Lambda^n$, f, $g \in \mathcal{C}^0_{\mathrm{p}}(B)$ and $\mathfrak{I}^{m-r}f - \mathfrak{I}^{m-s}g \in \mathcal{P}_m(B)$, where $m := \max(r,s)$; $\mathcal{D}'_{\mathrm{f}}(B) := (\bigcup_{n\in\mathbb{N}} \Lambda^n \times \mathcal{C}^0_{\mathrm{p}}(B))/\sim$; $\lambda_B(f) := [(0,f)]_\sim$; $D_k([(r,f)]_\sim) := [(r+e_k,f)]_\sim$, so that $D^r f = [(r,f)]_\sim \in \mathcal{D}'_{\mathrm{f}}(B)$; finally, the vector space operations are defined as $D^r f + D^s g := D^m(\mathfrak{I}^{m-r}f + \mathfrak{I}^{m-s}g)$ ($m := \max(r,s)$) and $\mu \cdot D^r f := D^r(\mu f)$ for all $\mu \in \mathbb{R}$; the restriction to another ball $B' \subseteq B$ is defined by $(D^r f)|_{B'} := D^r(f|_{B'})$. With these definitions we obtain a separated functor

$$\mathcal{D}'_{\mathrm{f}} : \mathcal{B}(H)^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{F}}, \tag{3.20}$$

where $\mathcal{B}(U)$ is the poset of all the balls contained in the open set $U \subseteq H$. Using sheafification of this functor, as explained above in general terms, we obtain a co-universal solution of this problem.

4. Co-Universal properties of Colombeau algebras

A quotient space $A/\sim$ is the simplest way to get a new space (in the same category) and a morphism $p : A \longrightarrow A/\sim$ such that the new notion of equality $a_1 \sim a_2$, for elements $a_k \in A$, implies the standard one: $p(a_1) = p(a_2)$. The corresponding well-known universal property formalizes exactly this idea. Therefore, whenever we have a quotient space, we can use this general property to get a first simple characterization of $A/\sim$ *starting from the data* A *and* $\sim$. The defect of this general approach is clearly that we are not justifying neither the choice of A nor the equivalence relation $\sim$ as the simplest solution of a clarified problem. In this section, we first introduce Colombeau special algebra using the universal property of a quotient, but later, using another universal property, we make clear why we are using that space and that equivalence relation.

4.1. **Co-universal property as quotient of moderate nets.** In this section, we want to formulate the co-universal property of Colombeau algebras by formulating the classical co-universal property of a quotient at a "higher level", i.e. talking of functors of $\mathbb{R}$-algebras and natural transformations instead of algebras and their morphisms. In the following, we set $I := (0, 1]$, functions $f \in X^I$ are simply called *nets* and denoted as $f = (f_\varepsilon)$, any net $\rho = (\rho_\varepsilon) \in \mathbb{R}^I_{>0}$ such that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0^+$ will be called a *gauge*, and the set $AG(\rho^{-1}) := \{(\rho_\varepsilon^{-a}) \in \mathbb{R}^I \mid a \in \mathbb{R}_{>0}\}$ will be called the *asymptotic gauge* generated by ρ. If $\mathcal{P}\{\varepsilon\}$ is any property of $\varepsilon \in I$, we write $\forall^0\varepsilon :\ \mathcal{P}\{\varepsilon\}$ if the property holds for all ε sufficiently small, i.e. $\exists\varepsilon_0 \in I\, \forall\varepsilon \in (0, \varepsilon_0] :\ \mathcal{P}\{\varepsilon\}$.

Definition 18. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The *Colombeau algebras* is defined by the quotient ${}^\rho\mathcal{G}^{\mathrm{s}}(\Omega) := {}^\rho\mathcal{E}_{\mathrm{M}}(\Omega)/{}^\rho\mathcal{N}(\Omega)$, where

$$ {}^\rho\mathcal{E}_{\mathrm{M}}(\Omega) := \left\{ (u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall\alpha\, \exists N \in \mathbb{N} :\ \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\rho_\varepsilon^{-N}\right) \right\} $$

$$ {}^\rho\mathcal{N}(\Omega) := \left\{ (u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall\alpha\, \forall n \in \mathbb{N} :\ \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\rho_\varepsilon^{n}\right) \right\} $$

are resp. called *moderate* and *negligible* nets ($O(-)$ is the Landau symbol for $\varepsilon \to 0^+$). The equivalence class defined by the net $(u_\varepsilon) \in {}^\rho\mathcal{E}_{\mathrm{M}}(\Omega)$ is denoted by $[u_\varepsilon]_\rho$ or simply by $[u_\varepsilon]$ when we are considering only one gauge.

It is easy to prove that ${}^\rho\mathcal{G}^{\mathrm{s}}(\Omega)$ is a quotient $\mathbb{R}$-algebra with pointwise operations $[u_\varepsilon] + [v_\varepsilon] = [u_\varepsilon + v_\varepsilon]$ and $[u_\varepsilon] \cdot [v_\varepsilon] = [u_\varepsilon \cdot v_\varepsilon]$. Let $\mathcal{O}\mathbb{R}^\infty$ be the category having as objects open sets $U \subseteq \mathbb{R}^u$ of any dimension $u \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and smooth functions as arrows. If we extend ${}^\rho\mathcal{G}^{\mathrm{s}}(-)$ on the arrows of $(\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}}$ by ${}^\rho\mathcal{G}^{\mathrm{s}}(f)([u_\varepsilon]) := [u_\varepsilon \circ f]$, we get a functor ${}^\rho\mathcal{G}^{\mathrm{s}}(-) : (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}} \longrightarrow \mathbf{ALG}_\mathbb{R}$, where $\mathbf{ALG}_\mathbb{R}$ denotes the category of $\mathbb{R}$-algebras.

Definition 19. We denote by **Col** the *category of Colombeau algebras* and we write $(G, \pi) \in \mathbf{Col}$ if

(i) $G : (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}} \longrightarrow \mathbf{ALG}_\mathbb{R}$ is a functor;

(ii) $\pi : {}^\rho\mathcal{E}_{\mathrm{M}}(-) \longrightarrow G$ is a natural transformation such that ${}^\rho\mathcal{N}(\Omega) \subseteq \mathrm{Ker}(\pi_\Omega)$ for all $\Omega \in \mathcal{O}\mathbb{R}^\infty$. We simply write $\pi_\Omega(u_\varepsilon) := \pi_\Omega((u_\varepsilon))$ for all $(u_\varepsilon) \in {}^\rho\mathcal{E}_{\mathrm{M}}(\Omega)$.

Moreover, we write $(G, \pi) \xrightarrow{\tau} (F, \alpha)$ in **Col** if and only if the following diagram (of natural transformations) commutes

$$\begin{tikzcd} {}^\rho\mathcal{E}_{\mathrm{M}}(-) \arrow[r, "\alpha"] \arrow[d, "\pi"] & F \\ G \arrow[ur, "\tau"'] & \end{tikzcd}$$

Theorem 20. *For every $(G, \pi) \in$ **Col**, there exist a unique $\tau : ({}^\rho\mathcal{G}^s(-), [-]) \longrightarrow (G, \pi)$ in **Col**, i.e. $({}^\rho\mathcal{G}^s(-), [-])$ is co-universal in **Col**, i.e. is the simplest way to associate an algebra to any open set $\Omega \subseteq \mathbb{R}^d$ and saying that two moderate nets (u_ε), $(v_\varepsilon) \in {}^\rho\mathcal{E}_M(\Omega)$ are equal if they differ by a negligible net: $(u_\varepsilon - v_\varepsilon) \in {}^\rho\mathcal{N}(\Omega)$.*

Proof. We should find τ such that the following diagram commutes

$$\begin{tikzcd} {}^\rho\mathcal{E}_{\mathrm{M}}(-) \arrow[r, "\pi"] \arrow[d, "{[-]}"] & G \\ {}^\rho\mathcal{G}^{\mathrm{s}}(-) \arrow[ur, "\tau"'] & \end{tikzcd}$$

The only way τ can be defined is by setting $\tau_\Omega([u_\varepsilon]) := \pi_\Omega(u_\varepsilon)$ for all $\Omega \in \mathcal{O}\mathbb{R}^\infty$. In order to prove that τ_Ω is well defined, take two moderate nets (u_ε) and (v_ε) such that $[u_\varepsilon] = [v_\varepsilon]$, then we have $\tau_\Omega([u_\varepsilon]) = \pi_\Omega(u_\varepsilon) = \pi_\Omega(v_\varepsilon + (u_\varepsilon - v_\varepsilon)) = \pi_\Omega(v_\varepsilon) + \pi_\Omega(u_\varepsilon - v_\varepsilon)$ because for every Ω, π_Ω is an algebra-homomorphism. Since ${}^\rho\mathcal{N}(\Omega) \subseteq \mathrm{Ker}(\pi_\Omega)$, it follows that $\tau_\Omega([u_\varepsilon]) = \pi_\Omega(u_\varepsilon) = \pi_\Omega(v_\varepsilon) = \tau_\Omega([v_\varepsilon])$. □

Even this simple co-universal property highlights the following possible generalizations: instead of the category $\mathcal{O}\mathbb{R}^\infty$ we could take any category with a notion of smooth function with respect to a ring of scalars (e.g. in the field of hyperreals of nonstandard analysis, see e.g. [5, 43] and references therein; in the ring of Fermat reals, see [10, 11, 12, 21]; in the Levi-Civita field, see e.g. [41], etc. Note that the use of supremum in Def. 18 can be easily avoided using an upper bound inequality, and this can be useful if the ring of scalars is not Dedekind complete). Instead of the sheaf of smooth functions $\mathcal{C}^\infty(-)$, we can consider any sheaf of smooth functions in more general spaces, such as diffeological or Frölicher or convenient spaces, see e.g. [13] and references therein. Instead of the asymptotic gauge $AG(\rho^{-1})$, we can take more general structures, as proved in [18, 20].

4.2. **Co-universal properties as the simplest quotient algebras.** We now want to show another co-universal property of Colombeau algebras by completing the idea that *a Colombeau algebra is a quotient of a subalgebra of $\mathcal{C}^\infty(-)^I$, and moderate and negligible nets are the simplest choices in order to have non trivial representatives of zero.* We first define in general what is a quotient subalgebra of $\mathcal{C}^\infty(-)^I$ as an object of the category $\mathrm{QA{\scriptsize LG}}(\mathcal{C}^{\infty I})$:

Definition 21. We say that (G, π) is a *quotient subalgebra of* $\mathcal{C}^\infty(-)^I$, and we write $(G, \pi) \in \mathrm{QA{\scriptsize LG}}(\mathcal{C}^{\infty I})$, if:

(i) $G : (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}} \longrightarrow \mathbf{ALG}_\mathbb{R}$ is a functor;

(ii) $\pi : M \longrightarrow G$ is a natural transformation such that $M(\Omega)$ is a subalgebra of $\mathcal{C}^\infty(\Omega)^I$ and $\pi_\Omega : M(\Omega) \longrightarrow G(\Omega)$ is an epimorphism of $\mathbb{R}$-algebras for all $\Omega \in \mathcal{O}\mathbb{R}^\infty$.

Let us justify why this is related to quotient algebras. Since for every $\Omega \in \mathcal{O}\mathbb{R}^\infty$, π_Ω is an algebra homomorphism, for any (u_ε), $(v_\varepsilon) \in M(\Omega) \subseteq \mathcal{C}^\infty(\Omega)^I$ and for any $r \in \mathbb{R}$ we have

1) $\pi_\Omega(u_\varepsilon) + \pi_\Omega(v_\varepsilon) = \pi_\Omega(u_\varepsilon + v_\varepsilon)$
2) $\pi_\Omega(u_\varepsilon) \cdot \pi_\Omega(v_\varepsilon) = \pi_\Omega(u_\varepsilon \cdot v_\varepsilon)$
3) $r \cdot \pi_\Omega(u_\varepsilon) = \pi_\Omega(r \cdot u_\varepsilon)$.

Moreover, the epimorphism condition Def. 21.(ii) means that for every $g \in G(\Omega)$, there exists $(u_\varepsilon) \in M(\Omega)$ such that $\pi_\Omega(u_\varepsilon) = g$. This implies

$$G(\Omega) \simeq M(\Omega)/\mathrm{Ker}(\pi_\Omega) \text{ in } \mathbf{ALG}_\mathbb{R}. \tag{4.1}$$

Why are moderate nets ${}^\rho\mathcal{E}_\mathrm{M}(\Omega)$ the simplest subalgebra in order to have nontrivial representatives of zero, and what does this "nontrivial" mean? Let $(z_\varepsilon) \in M(\Omega)$ be such a representative, i.e. $\pi_\Omega(z_\varepsilon) = 0 \in G(\Omega)$, and assume we can take a constant net $(J_\varepsilon) \in M(\Omega) \cap \mathbb{R}^I$ such that $\lim_{\varepsilon\longrightarrow 0^+} |J_\varepsilon| = +\infty$. Then, we have

$$\pi_\Omega(z_\varepsilon) \cdot \pi_\Omega(J_\varepsilon) = 0 \cdot \pi_\Omega(J_\varepsilon) = \pi_\Omega(z_\varepsilon \cdot J_\varepsilon), \tag{4.2}$$

and hence also $(z_\varepsilon \cdot J_\varepsilon)$ is another representative of zero, and this holds for all possible infinite constant nets (J_ε). On the other hand, we would like to have that "representatives of zero" are, in some sense, "small". This intuitive idea of being small is formalized in the following condition:

Definition 22. We say that *every representative of zero in* (G, π) *is infinitesimal* if for all representatives of zero, $(z_\varepsilon) \in M(\Omega) \subseteq \mathcal{C}^\infty(\Omega)^I$ such that $\pi_\Omega(z_\varepsilon) = 0 \in G(\Omega)$, each compact set $K \Subset \Omega$ and each multi-index $\alpha \in \mathbb{N}^d$, we have

$$\sup_{x\in K} |\partial^\alpha z_\varepsilon(x)| := p_{K,\alpha}(z_\varepsilon) \to 0 \text{ as } \varepsilon \to 0^+. \tag{4.3}$$

For example, this property does not hold in nonstandard analysis, see [5]. If this condition holds, equation (4.2) implies that for each $K \Subset \Omega$ and each multi-index $\alpha \in \mathbb{N}^d$, we have $p_{K,\alpha}(z_\varepsilon \cdot J_\varepsilon) = p_{K,\alpha}(z_\varepsilon) \cdot |J_\varepsilon| \longrightarrow 0$, which implies $p_{K,\alpha}(z_\varepsilon) \le |J_\varepsilon|^{-1}$ for ε sufficiently small.

For $\mathcal{R} \subseteq \mathbb{R}^I$, let

$$\infty(\mathcal{R}) := \left\{ (J_\varepsilon) \in \mathcal{R} \mid \lim_{\varepsilon\longrightarrow 0^+} |J_\varepsilon| = +\infty \right\} \tag{4.4}$$

be the set of all the infinite nets in $\mathcal{R}$. We then have two possibilities, which link property (4.3) with the intuitive idea of trivial representatives of zero:

(i) $\infty(M(\Omega) \cap \mathbb{R}^I)$ contains all the infinite nets. This implies that for all K and for all α, $p_{K,\alpha}(z_\varepsilon) = 0$ for all ε small (proceed by contradiction by taking $J_\varepsilon := r \cdot |p_{K,\alpha}(z_\varepsilon)|^{-1}$ for all ε such that $p_{K,\alpha}(z_\varepsilon) \neq 0$ and where $r \in \mathbb{R}_{>0}$). In this case, the quotient must be trivial and this situation corresponds to the Schmieden-Laugwitz-Egorov model, see [37, 7].

(ii) $\infty(M(\Omega) \cap \mathbb{R}^I)$ does not contain all the infinite nets.

We now define morphisms of $\mathrm{QAlg}(\mathcal{C}^{\infty I})$:

Definition 23. Let (G,π), $(H,\eta)\in\mathrm{QAlg}(\mathcal{C}^\infty(-)^I)$. *A morphism of quotient algebras* $i:(G,\pi)\longrightarrow(H,\eta)$ is given by an inclusion

$$i:\infty(\pi_\Omega)\hookrightarrow\infty(\eta_\Omega),\quad\forall\Omega\in\mathcal{O}\mathbb{R}^\infty \tag{4.5}$$

where $\infty(\pi_\Omega):=\infty(M(\Omega)\cap\mathbb{R}^I)$ for all $\Omega\in\mathcal{O}\mathbb{R}^\infty$.

We have the following

Lemma 24. *Quotient algebras of $\mathcal{C}^\infty(-)^I$ and their morphisms form a category $\mathrm{QAlg}(\mathcal{C}^{\infty I})$.*

Therefore, a co-universal quotient algebra (G,π) (when it exists) has the smallest class of infinities. We will see that it necessarily follows it has the largest kernel as well.

In the following theorem, we use the notation $[-]_\Omega:\,(x_\varepsilon)\in{}^\rho\mathcal{E}_\mathrm{M}(\Omega)\longmapsto[x_\varepsilon]_\Omega\in{}^\rho\mathcal{G}^\mathrm{s}(\Omega)$ for all $\Omega\in\mathcal{O}\mathbb{R}^\infty$.

Theorem 25. *Assume that:*

(i) $(G,\pi)\in\mathrm{QAlg}(\mathcal{C}^{\infty I})$ *is a quotient algebra;*
(ii) *Every representative of* 0 *in* (G,π) *is infinitesimal, i.e. Def. 22 holds;*
(iii) *If* $(u_\varepsilon)\in M(\mathbb{R})\cap\mathbb{R}^I$ *then* $\exists(v_\varepsilon)\in\infty(\pi_\mathbb{R})\,\forall^0\varepsilon:\,|u_\varepsilon|\le v_\varepsilon$ *(constant nets are bounded by infinities).*

For all open set $\Omega\subseteq\mathbb{R}^n$, we also assume that:

(iv) $(\rho_\varepsilon^{-1})\in M(\Omega)$*;*
(v) $\forall(u_\varepsilon)\in M(\Omega)\,\forall K\Subset\Omega\,\forall\alpha\in\mathbb{N}^n:\,p_{K\alpha}(u_\varepsilon)\in M(\mathbb{R})\cap\mathbb{R}^I$*;*
(vi) *Let* $(u_\varepsilon)\in\mathcal{C}^\infty(\Omega)^I$*. If for any* $K\Subset\Omega$*,* $\alpha\in\mathbb{N}^n$ *there exists* $(v_\varepsilon)\in\infty(M(\Omega)\cap\mathbb{R}^I)$ *such that* $\forall^0\varepsilon:\,p_{K,\alpha}(u_\varepsilon)\le v_\varepsilon$*, then* $(u_\varepsilon)\in M(\Omega)$ *(infinities determine* $M(\Omega)$*);*
(vii) (G,π) *is co-universal among all the quotient algebras verifying the previous conditions.*

Then $G(\Omega)\simeq{}^\rho\mathcal{G}^s(\Omega)$ *as* $\mathbb{R}$*-algebras, i.e. in the category* $\mathbf{ALG}_\mathbb{R}$*. Moreover,* $\infty(\pi_\mathbb{R})=\infty([-]_\mathbb{R})$*,* $\mathrm{Ker}(\pi_\Omega)=\mathrm{Ker}([-]_\Omega)$*.*

Proof. Conditions (iii), (v), (vi) are equivalent to

$$\begin{aligned}M(\Omega)&=\mathcal{E}_\mathrm{M}(\infty(\pi_\mathbb{R}),\Omega)\\&:=\left\{(u_\varepsilon)\in\mathcal{C}^\infty(\Omega)^I\mid\forall K,\,\alpha\,\exists(v_\varepsilon)\in\infty(\pi_\mathbb{R})\,\forall^0\varepsilon:\,p_{K\alpha}(u_\varepsilon)\le v_\varepsilon\right\}.\end{aligned}$$

In fact assumptions (iii), (v) yield $M(\Omega)\subseteq\mathcal{E}_\mathrm{M}(\infty(\pi_\mathbb{R}),\Omega)$, whereas (vi) gives the opposite inclusion.

The Colombeau algebra ${}^\rho\mathcal{G}^\mathrm{s}(\Omega)$ satisfies conditions (i)-(vi). Now, we prove that it also satisfies condition (vii). Let (G,π) be another quotient algebra satisfying conditions (i)-(vi), and take $(x_\varepsilon)\in\infty([-]_\mathbb{R})$, so that (x_ε) is an infinite but moderate net:

$$\exists N\in\mathbb{N}\,\forall^0\varepsilon:\,|x_\varepsilon|\le\rho_\varepsilon^{-N}.$$

From assumption (iv), we have that $(\rho_\varepsilon^{-1})\in M(\Omega)$, and hence (ρ_ε^{-N}) as well, since $M(\Omega)$ is a subalgebra of $\mathcal{C}^\infty(\Omega)^I$. Thereby $(\rho_\varepsilon^{-N})\in\infty(\pi_\mathbb{R})$. By using condition (vi) with $u_\varepsilon(-)\equiv x_\varepsilon$ and $v_\varepsilon(-)\equiv\rho_\varepsilon^{-N}$, we get that $(x_\varepsilon)\in\infty(\pi_\mathbb{R})$. This shows that $\infty([-]_\mathbb{R})\subseteq\infty(\pi_\mathbb{R})$, i.e. ${}^\rho\mathcal{G}^\mathrm{s}(\Omega)$ is co-universal. Note that this only implies (from

Thm. 2) that $G \simeq {}^\rho\mathcal{G}^{\mathrm{s}}$ in $\mathrm{QA{\scriptsize LG}}(\mathcal{C}^{\infty I})$, which is not our final claim. Moreover, we never used condition (ii) so far.

We now prove that $\mathrm{Ker}(\pi_\Omega) \subseteq \mathrm{Ker}([-]_\Omega)$: Let $\pi_\Omega(z_\varepsilon) = 0$. We have already seen that this and (ii) imply

$$\forall J_\varepsilon \in \infty(\pi_{\mathbb{R}})\,\forall K, \alpha\,\forall^0\varepsilon : \, p_{K\alpha}(z_\varepsilon) \le J_\varepsilon^{-1}. \tag{4.6}$$

Since we have already proved that $\infty([-]_{\mathbb{R}}) \subseteq \infty(\pi_{\mathbb{R}})$, (4.6) yields $[z_\varepsilon]_{\mathbb{R}} = 0$. Therefore, if (G, π) is also a co-universal solution, formula (4.1) gives

$$\begin{aligned} G(\Omega) \simeq M(\Omega)/\mathrm{Ker}(\pi_\Omega) &= \mathcal{E}_{\mathrm{M}}(\infty(\pi_{\mathbb{R}}), \Omega)/\mathrm{Ker}(\pi_\Omega) \\ &= \mathcal{E}_{\mathrm{M}}([-]_{\mathbb{R}}, \Omega)/\mathrm{Ker}([-]_\Omega) = {}^\rho\mathcal{G}^{\mathrm{s}}(\Omega). \end{aligned}$$

□

4.2.1. *A particular case: co-universal property of Robinson-Colombeau generalized numbers.* Proceeding as in Sec. 4.2, we obtain a co-universal property of the ring of Robinson-Colombeau generalized numbers, i.e. the ring of scalars of Colombeau theory with an arbitrary gauge ρ.

Definition 26.

(i) $\mathbb{R}_\rho := \left\{(x_\varepsilon) \in \mathbb{R}^I \mid \exists N \in \mathbb{N} : \, x_\varepsilon = O(\rho_\varepsilon^{-N}) \text{ as } \varepsilon \to 0^+\right\}$ is called the set of *ρ-moderate nets* of numbers.

(ii) Let $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}_\rho$. We write $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if $\forall n \in \mathbb{N} : \, x_\varepsilon - y_\varepsilon = O(\rho_\varepsilon^{-n})$ as $\varepsilon \to 0^+$. It is easy to prove that $\sim_\rho$ is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations.

(iii) The *Robinson-Colombeau ring of generalized numbers* is defined as ${}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho/\sim_\rho$. The equivalence class defined by $(x_\varepsilon) \in \mathbb{R}_\rho$ is simply denoted as $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

Clearly, the ring of Robinson-Colombeau is isomorphic to the subring of Colombeau generalized functions $f \in {}^\rho\mathcal{G}^{\mathrm{s}}(\mathbb{R})$ whose derivative is zero $f' = [f'_\varepsilon] = 0$, see e.g. [23].

Similarly to the category $\mathrm{QA{\scriptsize LG}}(\mathcal{C}^{\infty I})$ we can now introduce the category of quotient subrings of $\mathbb{R}^I$:

Definition 27. We say that (G, π) is a *quotient subring of* $\mathbb{R}^I$, and we write $(G, \pi) \in \mathrm{QR{\scriptsize ING}}(\mathbb{R}^I)$, if

(i) G is a ring;

(ii) $\pi : \mathcal{R} \longrightarrow G$ is an epimorphism of rings, where the domain $\mathcal{R} \subseteq \mathbb{R}^I$ is a subring of $\mathbb{R}^I$.

Let $(H, \eta) \in \mathrm{QR{\scriptsize ING}}(\mathbb{R}^I)$. Then a *morphism of quotient rings* $i : (G, \pi) \longrightarrow (H, \eta)$ is given by an inclusion

$$i : \infty(\pi) \hookrightarrow \infty(\eta), \tag{4.7}$$

where $\infty(\pi) := \infty(\mathcal{R})$. Similarly to Def. 22, we say that *every representative of zero in (G, π) is infinitesimal* if for all representatives of zero, i.e. $(z_\varepsilon) \in \mathcal{R}$ such that $\pi(z_\varepsilon) = 0 \in G$, we have $\lim_{\varepsilon \to 0^+} z_\varepsilon = 0$.

The ring ${}^\rho\widetilde{\mathbb{R}}$ of Robinson-Colombeau is, up to isomorphisms of rings, the simplest quotient ring where every representative of zero is infinitesimal. This implies to have the smallest class of infinities, and consequently, the largest kernel. The proof is simply a particular case of that of Thm. 25:

Theorem 28. *Assume that:*

(i) (G, π) *is a* quotient subring of $\mathbb{R}^I$*;*

(ii) *Every representative of* 0 *in* (G,π) *is infinitesimal;*
(iii) *If* $(x_\varepsilon)\in\mathcal{R}$*, then* $\exists(v_\varepsilon)\in\infty(\mathcal{R})\,\forall^0\varepsilon:\ |x_\varepsilon|\le v_\varepsilon$ *(nets are bounded by infinities);*
(iv) $(\rho_\varepsilon^{-1})\in\mathcal{R}$*;*
(v) *Let* $(x_\varepsilon)\in\mathbb{R}^I$*. If there exist* $(v_\varepsilon)\in\infty(\mathcal{R})$ *such that* $\forall^0\varepsilon:\ |u_\varepsilon|\le v_\varepsilon$*, then* $(u_\varepsilon)\in\mathcal{R}$ *(infinities determine* $\mathcal{R}$*).*
(vi) (G,π) *is co-universal among all the quotient rings satisfying the previous conditions.*

Then, $G\simeq{}^{\rho}\widetilde{\mathbb{R}}$ *as rings. Moreover,* $\infty(\mathcal{R})=\infty(\mathbb{R}_\rho)$ *is the smallest class of infinities and* $\mathrm{Ker}(\pi)=\mathrm{Ker}([-])$ *is the largest kernel.*

See [43, 44] for a characterization up to isomorphisms of the *field* of scalars one has in the nonstandard approach to Colombeau theory.

5. Universal property of spaces of generalized smooth functions

Generalized smooth functions (GSF) are the simplest way to deal with a very large class of generalized functions and singular problems, by working directly with all their ρ-moderate smooth regularizations. GSF are close to the historically original conception of generalized function, [6, 29, 27]: in essence, the idea of authors such as Dirac, Cauchy, Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside (who informally worked with "numbers" which also comprise infinitesimals and infinite scalars) was to view generalized functions as certain types of smooth set-theoretical maps obtained from ordinary smooth maps by introducing a dependence on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, [27]. More generally, in the GSF approach, generalized functions are seen as set-theoretical functions defined on, and attaining values in, the non-Archimedean ring of scalars ${}^{\rho}\widetilde{\mathbb{R}}$. The calculus of GSF is closely related to classical analysis sharing several properties of ordinary smooth functions. On the other hand, GSF include all Colombeau generalized functions and hence also all Schwartz distributions [16, 23, 17]. They allow nonlinear operations on generalized functions and unrestricted composition [16, 17]. They enable to prove a number of analogues of theorems of classical analysis for generalized functions: e.g., mean value theorem, intermediate value theorem, extreme value theorem, Taylor's theorems, local and global inverse function theorems, integrals via primitives, and multidimensional integrals [15, 14, 17]. With GSF we can develop calculus of variations and optimal control for generalized functions, with applications e.g. in collision mechanics, singular optics, quantum mechanics and general relativity, see [30, 9] and [8, 28] for a comparison with CGF. We have new existence results for nonlinear singular ODE and PDE (e.g. a Picard-Lindelöf theorem for PDE), [32, 19], and with the notion of *hyperfinite Fourier transform* we can consider the Fourier transform of any GSF, without restriction to tempered type, [35]. GSF with their particular sheaf property define a Grothendieck topos, [17].

Definition 29. Let $X\subseteq{}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y\subseteq{}^{\rho}\widetilde{\mathbb{R}}^d$. We say that $f:X\to Y$ is a GSF ($f\in{}^{\rho}\mathcal{GC}^\infty(X,Y)$), if

(i) $f:X\to Y$ is a set-theoretical function

(ii) There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ such that for all $[x_\varepsilon] \in X$ (i.e. for all representatives (x_ε) of any point $x = [x_\varepsilon] \in X$):
   (i) $f(x) = [f_\varepsilon(x_\varepsilon)]$ (we say that f *is defined by the net* (f_ε))
   (ii) $\forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is ρ-moderate.

Following [45, Def. 3.1] it is possible to give an equivalent definition of GSF as a quotient set: therefore a co-universal characterization similar to the previous ones given for CGF (Sec. 4) is possible. However, in the present section we want to present a universal property of spaces of GSF as the simplest way to have set-theoretical functions defined on generalized numbers and having arbitrary derivatives. As we will see below, this property is important because it formalizes the idea that GSF contains all the possible ρ-moderate regularizations, e.g. as obtained by convolution with a mollifier of the form $\frac{1}{\rho_\varepsilon}\mu\left(\frac{x}{\rho_\varepsilon}\right)$, see e.g. [17].

The ring of scalars ${}^\rho\widetilde{\mathbb{R}}$ is hence the basic building block in the definition of GSF. Using the results of Sec. 4.2.1, in this section we could use any co-universal solution of Thm. 28, but that would only result into a useless abstract language, so that we work directly with ${}^\rho\widetilde{\mathbb{R}}$, as defined in Def. 26.

First of all, derivatives of GSF are well-defined on so-called sharply open sets: Let $x,\ y \in {}^\rho\widetilde{\mathbb{R}}$, we write $x \le y$ if for all representative $[x_\varepsilon] = x$, there exists $[y_\varepsilon] = y$ such that $\forall^0\varepsilon : \ x_\varepsilon \le y_\varepsilon$; on ${}^\rho\widetilde{\mathbb{R}}^n$, we consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$. Even if this generalized norm takes value in ${}^\rho\widetilde{\mathbb{R}}$, it shares essential properties with classical norms, like the triangle inequality and absolute homogeneity. It is therefore natural to consider on ${}^\rho\widetilde{\mathbb{R}}^n$ the topology generated by balls $B_r(x) := \left\{ y \in {}^\rho\widetilde{\mathbb{R}} \mid |x-y| < r \right\}$, for $r \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$ and invertible, which is called *sharp topology*, and its elements *sharply open sets*.

Theorem 30. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set and $\alpha \in \mathbb{N}^n$, then the map given by*

$$\partial^\alpha : [f_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d) \mapsto [\partial^\alpha f_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d)$$

is well-defined, i.e. it does not depend on the net of smooth functions (f_ε) that defines the GSF $[f_\varepsilon(-)] : x = [x_\varepsilon] \in U \mapsto [f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$.

For any sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, we set

$$\mathcal{M}^d_U := \left\{ (f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I \mid \forall [x_\varepsilon] \in U \, \forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}_\rho \right\}.$$

We hence have a map $[-]_{\mathrm{f}}$ that allows us to construct GSF starting from nets of smooth functions:

$$[-]_{\mathrm{f}} : (f_\varepsilon) \in \mathcal{M}^d_U \mapsto [f_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d).$$

By Def. 29 of GSF, this map is onto. Thanks to Thm. 30, derivatives $\partial^\alpha : {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d) \longrightarrow {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d) \subseteq \mathbf{Set}(U, {}^\rho\widetilde{\mathbb{R}}^d)$ are ε-wise well-defined, i.e. using the ring epimorphism $[-] : \mathbb{R}_\rho \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, the map $\pi^\alpha(f_\varepsilon) : [x_\varepsilon] \in U \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$ is defined for all nets $(f_\varepsilon) \in \mathcal{M}^d_U$ and makes this diagram commute

$$\begin{tikzcd}
\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d) \arrow[r, "\partial^\alpha"] & \mathbf{Set}(U, {}^\rho\widetilde{\mathbb{R}}^d) \\
\mathcal{M}^d_U \arrow[u, "{[-]_{\mathrm{f}}}"] \arrow[ur, "\pi^\alpha"'] &
\end{tikzcd}$$

The space ${}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}}^d)$ and the maps ∂^α, $[-]_\mathrm{f}$ are the simplest way to make this diagram commute, i.e. we have the following

Theorem 31. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set and $d \in \mathbb{N}$. If $G \in \mathbf{Set}$, q and $(D^\alpha)_{\alpha\in\mathbb{N}^n}$ are such that $D^0 : G \hookrightarrow \mathbf{Set}(U,{}^\rho\widetilde{\mathbb{R}}^d)$ is the inclusion and for all $\alpha \in \mathbb{N}^n$:*

$$\begin{tikzcd} G \arrow[r, "D^\alpha"] & \mathbf{Set}(U,{}^\rho\widetilde{\mathbb{R}}^d) \\ \mathcal{M}_U^d \arrow[u, "q"] \arrow[ur, "\pi^\alpha"'] & \end{tikzcd}$$

where q is surjective, then there exists one and only one $\varphi : G \longrightarrow {}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}}^d)$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] & \mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}}^d) \\ \mathcal{M}_U^d \arrow[u, "q"] \arrow[ur, "{[-]_f}"'] & \end{tikzcd}$$

and which preserves derivatives, i.e. $\partial^\alpha\varphi(F) = D^\alpha F$ for all $F \in G$ and all $\alpha \in \mathbb{N}^n$.

Proof. Since q is surjective, if $F \in G$, we can find $(f_\varepsilon) \in \mathcal{M}_U^d$ such that $F = q(f_\varepsilon)$. We necessarily have to define $\varphi(F) := \varphi(q(f_\varepsilon)) = [f_\varepsilon]_\mathrm{f} = [f_\varepsilon(-)]$. The map φ is well-defined: if $F = q(\bar{f}_\varepsilon)$, then $D^0(q(\bar{f}_\varepsilon)) = q(\bar{f}_\varepsilon) = \pi^0(\bar{f}_\varepsilon) = [f_\varepsilon]_\mathrm{f}$. It remains only to prove the preservation of derivatives. We have $\partial^\alpha\varphi(F) = \partial^\alpha[f_\varepsilon(-)] = [\partial^\alpha f_\varepsilon(-)] = [\partial^\alpha f_\varepsilon]_\mathrm{f}$, and $D^\alpha F = D^\alpha\left(q(f_\varepsilon)\right) = \pi^\alpha(f_\varepsilon) = [\partial^\alpha f_\varepsilon]_\mathrm{f}$. $\square$

We close this section by noting that trivially $\mathbf{Set}(U,{}^\rho\widetilde{\mathbb{R}}^d)$ is not a universal solution of the same problem because we cannot have a surjection like $[-]_\mathrm{f}$. Finally, the universal solution ${}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}}^d)$ is in a certain sense minimal because it is possible to prove that ${}^\rho\mathcal{GC}^\infty(\widetilde{\Omega}_c,{}^\rho\widetilde{\mathbb{R}}^d) \simeq {}^\rho\mathcal{G}^\mathrm{s}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is any open set and $\widetilde{\Omega}_c := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists K \Subset \Omega\, \forall^0\varepsilon : \ x_\varepsilon \in K\right\}$, see e.g. [17, 16], i.e. up to isomorphism we get exactly the Colombeau algebra on Ω.

6. Conclusions

The aim of the present paper is not to compare, in some sense, different spaces of generalized functions, but to characterize them using suitable universal properties. However, we want to briefly close the paper trying to clarify several frequent misunderstandings concerning Colombeau theory and nonlinear operations on distributions.

If we are interested only to linear operations, we showed in Sec. 3 that the space of Schwartz distributions is the simplest solutions, and any other solution would be less optimal. Schwartz impossibility theorem, see e.g. [23] and references therein, states that nonlinear operations are seriously problematic for distributions.

In Sec. 4, we presented Colombeau theory as the simplest solution of this problem among quotient algebras. A common mainstream objection to Colombeau construction is that distributions are not intrinsically embedded in the corresponding algebra. Besides the fact that this is false, because a different index set instead of $I = (0, 1]$ would enable to have the searched intrinsic embedding (and essentially with the same notations and with the same basic ideas, see [18]; see also [23]), one could argue that if Colombeau algebra had historically appeared before Schwartz distributions, now some people would not accept the latter because they do not intrinsically embed into the former. On the contrary, Colombeau theory is the formalization of the method of regularizations: a convenient setting, sharing several properties with ordinary smooth functions, and containing convolutions with any mollifier of the form $\frac{1}{\rho_\varepsilon}\mu\left(\frac{x}{\rho_\varepsilon}\right)$. We motivate the universal properties of GSF exactly in this way.

The real technical drawbacks of Colombeau algebras are the lacking of closure with respect to composition (see e.g. [23]), not good properties of Fourier transform as well as multidimensional integration on infinite sets (see e.g. [35]), and the lacking of general existence results for differential equations, such as the Picard-Lindelöf or the Nash-Moser theorems. GSF solve almost all these problems.

On the other hand, an important open problem of GSF is a clear link between the natural notion of pointwise solution for GSF (i.e. regularized) differential equations, and the notion of weak solution in Sobolev spaces.

In the present paper, we showed that any other idea to formalize the method of regularizing a singular problem would necessarily be less simple than Colombeau algebras or spaces of GSF.

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