Convolutional Coded Poisson Receivers

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Abstract-In this paper, we present a framework for convolutional coded Poisson receivers (CCPRs) that incorporates spatially coupled methods into the architecture of coded Poisson receivers (CPRs). We use density evolution equations to track the packet decoding process with the successive interference cancellation (SIC) technique. We derive outer bounds for the stability region of CPRs when the underlying channel can be modeled by a ϕ -ALOHA receiver. The stability region is the set of loads that every packet can be successfully received with a probability of 1. Our outer bounds extend those of the spatiallycoupled Irregular Repetition Slotted ALOHA (IRSA) protocol and apply to channel models with multiple traffic classes. For CCPRs with a single class of users, the stability region is reduced to an interval. Therefore, it can be characterized by a percolation threshold. We study the potential threshold by the potential function of the base CPR used for constructing a CCPR. In addition, we prove that the CCPR is stable under a technical condition for the window size. For the multiclass scenario, we recursively evaluate the density evolution equations to determine the boundaries of the stability region. Numerical results demonstrate that the stability region of CCPRs can be enlarged compared to that of CPRs by leveraging the spatiallycoupled method. Moreover, the stability region of CCPRs is close to our outer bounds when the window size is large.

Index Terms—coded Poisson receivers, Irregular Repetition Slotted ALOHA, density evolution, potential function, successive interference cancellation

I. INTRODUCTION

The fifth-generation networks (5G) and beyond aim to support various connectivity classes of users that have different quality-of-service (QoS) requirements, including (i) enhanced mobile broadband (eMBB), (ii) ultra-reliable lowlatency communications (URLLC), and (iii) massive machinetype communications (mMTC) (see, e.g., [3]–[6] and references therein). *Network slicing* [7] that partitions and allocates available radio resources to provide differentiated QoS in a *single* radio network has received much attention lately. For downlink transmissions, network slicing could be tackled by centralized scheduling algorithms [8]. On the other hand, various schemes of coded multiple access (see, e.g., [9]–[23]) have been proposed in the literature to address the problem of uplink transmissions. The framework of *Poisson receivers* [24] is an abstraction of these coded multiple access schemes to hide the encoding/decoding complexity of the physical layer from the Medium Access Control (MAC) layer.

A Poisson receiver specifies the probability that a packet is successfully received (decoded) when the number of packets transmitted simultaneously to the receiver follows a Poisson distribution (Poisson offered load). Inspired by Irregular Repetition Slotted ALOHA (IRSA) [10] and coded slotted ALOHA (CSA) [11]-[16], a Poisson receiver can be used as a building block to construct a system of coded Poisson receivers (CPR) with multiple classes of users and multiple classes of receivers. In [25], the stability region of a system of CPR is defined as the region of Poisson offered loads in which all the packets can be successfully received (with probability 1). The stability regions of various network slicing policies, including the complete sharing and the receiver reservation policies, were computed numerically in [25]. The numerical results in [25] showed that different network slicing policies lead to different stability regions. Given a targeted stability region in a network with various classes of traffic, one engineering problem is to design a system of CPR associated with a network slicing policy to cover the targeted stability region.

One objective of this paper is to explore the boundaries of the stability region of a system of CPRs and to devise schemes to approach the boundaries. The contributions of the paper are as follows:

(i) Outer bounds for the stability region (see Theorem 9): For IRSA and CSA, there is only a single class of users and a single class of receivers. Thus, the stability region is reduced to an interval and its boundary is limited to one packet per time slot (as the underlying channel is the slotted ALOHA channel). When the underlying channel is replaced by the Dfold ALOHA channel (in which, at most, D packets can be successfully received in a time slot), the boundary is limited to D packets per time slot. These boundaries are simply the capacities of the underlying channels. As an application of the Area Theorem [26], tighter bounds for the stability regions for IRSA and coded D-fold ALOHA were derived in [13] and [27], respectively. In Theorem 9, we derive outer bounds for the stability region of a system of CPR when the underlying channel is modeled by a ϕ -ALOHA receiver [28]. Such a model is a generalization of both the slotted ALOHA channel and the D-fold ALOHA channel. As such, our outer bounds recover the upper bounds for IRSA and coded D-fold ALOHA in [13], [27] as special cases. Our outer bounds can also be applied to the setting with multiple traffic classes.

(ii) *Threshold saturation of Convolutional Coded Poisson Receivers* (see Theorem 13): Motivated by the remarkable performance of convolutional Low-Density Parity-Check (LDPC) codes (see, e.g., [29]–[31]) and the spatially coupled IRSA

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[27], [32], we propose using the construction method for convolutional LDPC codes and spatially coupled IRSA to construct Convolutional Coded Poisson Receivers (CCPRs) with multiple classes of users and multiple classes of receivers. To construct a CCPR, one first constructs L stages of random bipartite graphs from a base CPR. The L stages of bipartite graphs are concatenated to form a single bipartite graph. Then, we rewire the receiver ends of the ℓ -th copy, where $1 \leq \ell \leq L$, to a randomly selected copy in a window w. We refer the reader to Section IV-A for more details on the constructions. By using the *puncturing* technique in LDPC codes, we show in Theorem 13 that the stability region with L stages is not larger than that with L-1 stages, and thus the stability region converges as $L \to \infty$. Such a phenomenon is analogous to *threshold saturation* for convolutional LDPC codes and spatially coupled IRSA (see, e.g., [29], [31]-[33]). (iii) Potential functions and percolation thresholds for CCPRs with a single class of users (see Theorem 17 and Theorem 20): Even though we know that the stability region converges as $L \to \infty$, characterizing such a stability region is difficult for CCPR with multiple classes of users. For CCPRs with a single class of users, we use a remarkable mathematical tool called potential functions by Yedle et al. [30], [34] and Schlegel et al. [35] in their study of LDPC codes. In Theorem 17, we derive the potential function of a (base) CPR. Based on such a potential function, we define three thresholds: the single-system threshold G_s^* , the potential threshold G_{conv}^* , and the potential bound G_{up}^* . We show the threshold saturation theorem in Theorem 20 that for all $G \leq G^*_{conv}$, a CCPR with L stages is stable if the window size w satisfies a technical condition. The three thresholds are related as follows:

$$G_s^* < G_{conv}^* < G_{uv}^*. \tag{1}$$

The single-system threshold G_s^* recovers the percolation threshold of a (base) CPR in [25]. When the underlying channel is modeled by the *D*-fold ALOHA, the potential bound G_{up}^* is the same as the upper bound for stability in [27].

(iv) Numerical results (see Section VI): We provide numerical results for various systems of convolutional coded Poisson receivers. For CCPRs with one class of users and one class of receivers, our numerical results for G^*_{conv} and G^*_{up} in various parameter settings match very well with those in Table 1 of [32] and Table I of [27]. We also study the convolutional IRSA with two classes of users and two classes of receivers. We consider the two network slicing policies in [25]: the complete sharing policy and the receiver reservation policy. For L = 40, our numerical results show that the convolutional effect (due to spatial coupling) can indeed enlarge the stability region. Such enlargement appears to be monotone in the window size w.

The structure of this paper is organized as follows: In Section II, we provide a review of the coded Poisson receiver framework. In Section III, we show the outer bounds of the stability region of CPRs. In Section IV, we introduce the concept of CCPRs and present the development of density evolution equations for their decoding process, as well as the threshold saturation phenomena. In Section V, we delve into the application of the potential function to our CCPR framework, focusing on a single class of users. We utilize potential functions to characterize three significant thresholds. We establish the saturation theorem in this section as well. Some numerical results for both single and multiple traffic classes are presented in Section VI. We conclude the paper and discuss potential extensions of our work in Section VII. Several proofs of theorems are included in Appendices A, B, and C of the supplemental material. We provide a list of notations in Appendix D.

II. REVIEW OF THE FRAMEWORK OF CODED POISSON RECEIVERS

For the paper to be self-contained, we briefly review the framework of coded Poisson receivers in [24], [25], [28].

A. Poisson receivers and ALOHA receivers

A Poisson receiver is an *abstract* receiver proposed in [24]. The key insight of a Poisson receiver is to specify the packet success probability when the system is subject to Poisson arrivals. Specifically, a system with K classes of input traffic is said to have a Poisson offered load $\rho = (\rho_1, \rho_2, \dots, \rho_K)$ if these K classes of input traffic are *independent*, and the number of class k packets arriving at the system follows a Poisson distribution with mean ρ_k , for $k = 1, 2, \dots, K$.

Definition 1: (Poisson receiver with multiple classes of input traffic [24]) An abstract receiver is called a $(P_{\text{suc},1}(\rho), P_{\text{suc},2}(\rho), \ldots, P_{\text{suc},K}(\rho))$ -Poisson receiver with K classes of input traffic if the receiver is subject to a Poisson offered load $\rho = (\rho_1, \rho_2, \ldots, \rho_K)$, a tagged (randomly selected) class k packet is successfully received with probability $P_{\text{suc},k}(\rho)$, for $k = 1, 2, \ldots, K$.

The throughput of class k packets (defined as the expected number of class k packets that are successfully received) for a $(P_{\text{suc},1}(\rho), P_{\text{suc},2}(\rho), \ldots, P_{\text{suc},K}(\rho))$ -Poisson receiver subject to a Poisson offered load ρ is thus

$$\Theta_k = \rho_k \cdot P_{\mathrm{suc},k}(\rho),\tag{2}$$

 $k = 1, 2, \ldots, K.$

By viewing each time slot as a Poisson receiver, various well-known systems can be modeled by Poisson receivers in [24], [25], [28]. These include Slotted ALOHA (SA) [36], SA with multiple cooperative receivers, and Rayleigh block fading channel with capture [16], [37], [38]. It can also be applied to the setting with multiple packet reception in a time slot (see, e.g., [21], [22], [27], [39]–[41]).

Like a Poisson receiver, an ALOHA receiver [28] is also an abstract receiver. Such an abstract receiver treats the physical layer with a *deterministic* input-output function. Denote by \mathcal{Z}^+ the set of nonnegative integers. We say a system with K classes of input traffic is subject to a *deterministic* load $n = (n_1, n_2, \ldots, n_K) \in \mathcal{Z}^{+K}$ if the number of class k packets arriving at the system is n_k .

Definition 2: (ALOHA receiver with multiple classes of input traffic [28]) Consider a deterministic function

$$\phi: \mathcal{Z}^+{}^K \to \mathcal{Z}^+{}^K$$

that maps a K-vector $n = (n_1, n_2, ..., n_K)$ to the K-vector $(\phi_1(n), \phi_2(n), ..., \phi_K(n))$. An abstract receiver is called a ϕ -ALOHA receiver (with K classes of input traffic) if the number of class k packets that are successfully received is exactly $\phi_k(n)$, k = 1, 2, ..., K, when the receiver is subject to a deterministic load $n = (n_1, n_2, ..., n_K)$.

One typical example of the ϕ -ALOHA receiver is a time slot in the slotted ALOHA (SA) system, where at most one packet can be received. The *D*-fold ALOHA system proposed in [21] is a generalization of the SA system. If there are less than or equal to *D* packets transmitted in a time slot, then all these packets can be successfully received. On the other hand, if there are more than *D* packets transmitted in a time slot, then all these packets are lost. Thus, a time slot in the *D*-fold ALOHA system is a ϕ -ALOHA receiver (with a single class of input traffic), where

$$\phi(n) = \begin{cases} n & \text{if } n \le D \\ 0 & \text{otherwise} \end{cases}$$
(3)

Another example of a ϕ -ALOHA receiver is the model for the near-far [21] decoding scheme. In such a model, there are two classes of input traffic and the power from one class (the near users) is much stronger than that of the other class (the far users). By decoding the near users first and using the SIC technique, a time slot in the near-far SIC decoding scheme can be modeled by a ϕ -ALOHA with

$$\phi(n_1, n_2) = \begin{cases} (n_1, n_2) & \text{if } (n_1, n_2) \leq (1, 1) \\ (0, 0) & \text{otherwise} \end{cases} .$$
(4)

It was shown in Theorem 14 of [28] that for every ALOHA receiver, there is an induced Poisson receiver. This is done by computing the throughput of the ϕ -ALOHA receiver when it is subject to the Poisson offered load ρ and then using (2) to find the success probability function. For the *D*-fold ALOHA system, the throughput for a Poisson offered load ρ is

$$\sum_{t=1}^{D} t \frac{e^{-\rho} \rho^t}{t!} = \rho \sum_{t=0}^{D-1} \frac{e^{-\rho} \rho^t}{t!}.$$

From (2), it is a Poisson receiver with the following success probability function:

$$P_{\rm suc}(\rho) = \sum_{t=0}^{D-1} \frac{e^{-\rho} \rho^t}{t!}.$$
 (5)

B. Coded Poisson receivers with multiple classes of users and receivers

The idea of using the SIC technique in the Contention Resolution Diversity Slotted ALOHA (CRDSA) protocol [9] and the Irregular Repetition Slotted ALOHA (IRSA) protocol leads to the development of coded Poisson receivers (CPR) with multiple classes of users and receivers in [25]. As in [25], let us consider a system with G_kT class k users, k = 1, 2, ..., K, and F_jT class j Poisson receivers, j = 1, 2, ..., J. Class j Poisson receivers have the success probability functions $P_{\text{suc},1,j}(\rho), P_{\text{suc},2,j}(\rho), ..., P_{\text{suc},K,j}(\rho)$ for the K classes of input traffic. Each class k user transmits its packet for (a random number of) $L_k \ge 1$ times (copies). There are two assumptions made in [25]:

- (i) Perfect SIC: as long as one copy of a packet is successfully received by one of the receivers, then it can be used to remove the other copies of that packet from the other receivers.
- (ii) Independent Poisson receivers: the event that a packet is successfully received by a Poisson receiver is independent of the outcomes of the other Poisson receivers as long as their input traffic is independent of each other.

As in [10], [25], let $\Lambda_{k,d}$ be the probability that a class k packet is transmitted d times, i.e.,

$$P(L_k = d) = \Lambda_{k,d}, \ d = 1, 2, \dots$$
 (6)

Define the generating function

$$\Lambda_k(x) = \sum_{d=0}^{\infty} \Lambda_{k,d} \cdot x^d \tag{7}$$

of the *degree distribution* of a class k user node, and the generating function

$$\lambda_k(x) = \sum_{d=0}^{\infty} \lambda_{k,d} \cdot x^d \tag{8}$$

of the excess degree distribution of a class k user node, where

$$\Lambda_{k,d} = \frac{\Lambda_{k,d+1} \cdot (d+1)}{\sum_{d=0}^{\infty} \Lambda_{k,d+1} \cdot (d+1)}$$
(9)

is the probability that the user end of a randomly selected class k edge has additional d edges, excluding the randomly selected class k edge. Note that the mean degree of a class k user node is

$$\Lambda_k'(1) = \sum_{d=0}^{\infty} d \cdot \Lambda_{k,d}, \tag{10}$$

and that

$$\lambda_k(x) = \frac{\Lambda'_k(x)}{\Lambda'_k(1)}.$$
(11)

Let

$$G = (G_1, G_2, \dots, G_K),$$

$$\Lambda'(x) = (\Lambda'_1(x), \Lambda'_2(x), \dots, \Lambda'_K(x)),$$

$$R_j = (\frac{r_{1,j}}{F_j}, \frac{r_{2,j}}{F_j}, \dots, \frac{r_{K,j}}{F_j}),$$
(12)

and

$$\tilde{\rho}_j = G \circ \Lambda'(1) \circ R_j, \tag{13}$$

where \circ denotes the element-wise multiplication of two vectors.

Using the tree evaluation (density evolution) method in [10], [12], [42], [43] and the reduced Poisson offered load argument in [24], the following result was derived in Theorem 1 of [24].

Theorem 3: (Theorem 1 of [24]) As $T \to \infty$, the system of CPRs after the *i*-th SIC iteration converges to a Poisson receiver with the success probability function for the class k traffic

$$\tilde{P}_{\mathrm{suc},k}^{(i)}(G) = 1 - \Lambda_k \Big(1 - \sum_{j=1}^J r_{k,j} P_{\mathrm{suc},k,j} (q^{(i-1)} \circ \tilde{\rho}_j) \Big),$$
(14)

k = 1, 2, ..., K, where $q^{(i)} = (q_1^{(i)}, q_2^{(i)}, ..., q_K^{(i)})$ can be computed recursively from the following equation:

$$q_{k}^{(i+1)} = \lambda_{k} \left(1 - \sum_{j=1}^{J} r_{k,j} P_{\mathrm{suc},k,j} (q^{(i)} \circ \tilde{\rho}_{j}) \right), \quad (15)$$

with $q^{(0)} = (1, 1, \dots, 1)$.

If the number of copies $L_k \ge 2$ for all k, i.e, $\Lambda_{k,1} = 0$, then the generating function $\lambda_k(x)$ is strictly increasing for $0 \le x \le 1$, and thus invertible. By letting $p_k^{(i+1)} = \lambda_k^{-1}(q_k^{(i+1)})$, one can rewrite (15) as follows:

$$p_k^{(i+1)} = 1 - \sum_{j=1}^J r_{k,j} P_{\text{suc},k,j} (G \circ \Lambda'(p^{(i)}) \circ R_j), \quad (16)$$

where $p^{(i)} = (p_1^{(i)}, p_2^{(i)}, \dots, p_K^{(i)})$ and $p^{(0)} = (1, 1, \dots, 1)$.

Note that a system of CPRs with K classes of users and J classes of receivers is characterized by the following parameters: (i) the total number of Poisson receivers T, (ii) the offered load vector $G = (G_1, G_2, \ldots, G_K)$, (iii) the degree distribution vector $\Lambda(x) = (\Lambda_1(x), \Lambda_2(x), \ldots, \Lambda_K(x))$, (iv) the $K \times J$ routing matrix $R = (r_{k,j})$, and (v) the partition vector $F = (F_1, F_2, \ldots, F_J)$. In [25], the routing matrix and the partition vector correspond to network slicing policies. To ease the presentation, we denote a system of CPRs with K classes of users and J classes of receivers as the $(T, G, \Lambda(x), R, F)$ -CPR.

C. Stability

The stability of a system of coded Poisson receivers in [25] is motivated by the stability of a queueing system, where the system is (rate) stable if the departure rate is the same as the arrival rate. In that regard, a system of coded Poisson receivers is stable if all its packets can be successfully received. The precise definition is given below.

Definition 4: (Stability of coded Poisson receivers with multiple classes of input traffic [25]) Consider the $(T, G, \Lambda(x), R, F)$ -CPR described in Section II-B. A Poisson offered load $G = (G_1, G_2, \ldots, G_K)$ to a system of coded Poisson receivers is said to be *stable* if, as $T \to \infty$, the probability that a packet is successfully received approaches 1 when the number of iterations goes to infinity, i.e.,

$$\lim_{i \to \infty} \tilde{P}_{\text{suc},k}^{(i)}(G) = 1, \quad k = 1, 2, \dots, K,$$
(17)

where $\tilde{P}^{(i)}_{{
m suc},k}(G)$ is defined in (14).

As in [25], we make the following four assumptions for the stability analysis:

(A1) For all k = 1, ..., K and j = 1, 2, ..., J, the success probability function $P_{\text{suc},k,j}(\rho)$ is a continuous and decreasing function of ρ , and $P_{\text{suc},k,j}(\mathbf{0}) = 1$, where **0** is the vector with all its elements being

0. Furthermore, for the analysis in Section V, we require $P_{\text{suc},k,j}(\rho)$ to have a continuous second derivative.

- (A2) If $\rho \neq \mathbf{0}$, then $P_{\operatorname{suc},k,j}(\rho) < 1$ for all $k = 1, \dots, K$, and $j = 1, 2, \dots, J$.
- (A3) $r_{k,j} > 0$ for all k = 1, ..., K, and j = 1, 2, ..., J.
- (A4) Every packet is transmitted at least twice, i.e., $\Lambda_{k,1} = 0$ for all k = 1, 2, ..., K.

Under these assumptions, a necessary and sufficient condition for a Poisson offered load G to be stable is presented in Theorem 5. A monotonicity result is presented in Theorem 6. These two theorems appear in Theorems 2 and 3 in [25]. They are duplicated here for their importance and for the completeness of this paper.

Theorem 5: (Theorem 2 of [25]) Under (A1), a Poisson offered load G is stable if q = 0 is the unique solution in $[0, 1]^K$ of the following K equations:

$$q_k = \lambda_k \Big(1 - \sum_{j=1}^J r_{k,j} P_{\operatorname{suc},k,j}(q \circ \tilde{\rho}_j) \Big), \qquad (18)$$

k = 1, 2, ..., K, where $\tilde{\rho}_j$ is defined in (13) and $q = (q_1, q_2, ..., q_K)$. On the other hand, under (A1), (A2), and (A3), a positive Poisson offered load G (with $G_k > 0$ for all k) is stable only if q = 0 is the unique solution in $[0, 1]^K$ of the K equations in (18). Moreover, by (16), under (A1), (A2), (A3) and (A4), a positive Poisson offered load G (with $G_k > 0$ for all k) is stable if and only if p = 0 is the unique solution in $[0, 1]^K$ of the following K equations:

$$p_k = 1 - \sum_{j=1}^J r_{k,j} P_{\operatorname{suc},k,j} (G \circ \Lambda'(p) \circ R_j), \qquad (19)$$

 $k = 1, 2, \ldots, K$ and $p = (p_1, p_2, \ldots, p_K)$.

Theorem 6: (Theorem 3 in [25]) Suppose that (A1), (A2), and (A3) hold. If a positive Poisson offered load $\hat{G} = (\hat{G}_1, \hat{G}_2, \dots, \hat{G}_K)$ (with $\hat{G}_k > 0$ for all k) is stable, then any Poisson offered load G with $G \leq \hat{G}$ is also stable.

The monotonicity results in Theorem 6 leads to the notion of the stability region in [25].

Definition 7: Under (A1), (A2), and (A3), the stability region S is defined as the maximal stable set such that (i) any $G \in S$ is stable, and (ii) any $G \notin S$ is not stable.

III. OUTER BOUNDS FOR THE STABILITY REGION

In this section, we derive an outer bound for the stability region for a system of coded Poisson receivers when the Poisson receivers are induced from ϕ -ALOHA receivers. For this, we need to define the concept of "capacity" for ϕ -ALOHA receivers. Analogous to the terminology used in queueing theory, we use "capacity" for systems with deterministic loads and "stability" for systems with stochastic loads.

For a ϕ -ALOHA receiver, the *failure* function ϕ^c [28] is defined by

$$\phi^c(n) = n - \phi(n). \tag{20}$$

A ϕ -ALOHA receiver is called *monotone* if the *failure* function ϕ^c is *increasing* in the deterministic load n, i.e., for any $n' \leq n''$,

$$\phi^c(n') \leqslant \phi^c(n''). \tag{21}$$

The failure function represents the number of packets remaining to be decoded. With the monotonicity, we now define the capacity region.

Definition 8: (Capacity of a ϕ -ALOHA receiver) For a monotone ϕ -ALOHA receiver, its capacity region S is defined to be the set of deterministic loads such that all the packets are successfully received, i.e.,

$$S = \{n : \phi(n) = n\}.$$
 (22)

An K-vector $(b_1, b_2, ..., b_K)$ is called an *affine capacity* envelope with the bound B if for $n \in S$,

$$\sum_{k=1}^{K} b_k \phi_k(n) \leqslant B.$$
(23)

For instance, for the *D*-fold ALOHA system with *K* classes of users, it has the capacity region $S = \{n : \sum_{k=1}^{K} n_k \leq D\}$ as the total number of packets that can be successfully received in a time slot is at most *D*. As such, the *K*-vector (1, 1, ..., 1)is also an affine capacity envelope with the bound *D* for the *D*-fold ALOHA system with *K* classes of users.

We will use affine capacity envelopes to derive outer bounds for the stability region for a system of CPRs. For this, we need two properties of ϕ -ALOHA receivers in [28]: the closure property and the all-or-nothing property. As the failure function represents the number of packets that remain to be decoded, we can decode the remaining packets for the second time by removing those successfully decoded packets. This corresponds to the SIC technique in the literature. The number of packets remaining to be decoded after the second time of decoding is $\phi^c(\phi^c(n))$. Intuitively, we can carry out the iterative decoding approach (for an infinite number of times) until no more packets can be decoded. A ϕ -ALOHA receiver that always does the iterative decoding approach until no more packets can be decoded is said to satisfy the *closure* property, i.e.,

$$\phi^c(\phi^c(n)) = \phi^c(n), \tag{24}$$

for all *n*. In addition to this, we also need the all-or-nothing property. A ϕ -ALOHA receiver with *K* classes of input traffic is said to be an *all-or-nothing* receiver if it is *monotone* and satisfies the *all-or-nothing* property, i.e., either $\phi_k(n) = n_k$ or $\phi_k(n) = 0$ for all $n = (n_1, n_2, \dots, n_K)$ and $k = 1, 2, \dots, K$. The all-or-nothing property implies if class k packets (for some k) are successfully decoded, then they are all decoded during the same iteration.

In the following theorem, we derive an outer bound for the stability region of a system of coded Poisson receivers by using the concept of the affine capacity envelope.

Theorem 9: Consider the $(T, G, \Lambda(x), R, F)$ -CPR with T Poisson receivers being induced from a ϕ -ALOHA receiver. Suppose that the ϕ -ALOHA receiver that satisfies the closure property and the all-or-nothing property, and it has an affine capacity envelope (b_1, b_2, \ldots, b_K) with the bound B and that b_k 's are binary. Then for any stable offered load G,

$$\sum_{k=1}^{K} b_k G_k$$

$$\leq \sum_{j=1}^{J} F_j \Big(\sum_{\tau=0}^{B-1} \tau e^{-\mu_j} \frac{\mu_j^{\tau}}{\tau!} + B (1 - \sum_{\tau=0}^{B-1} e^{-\mu_j} \frac{\mu_j^{\tau}}{\tau!}) \Big),$$
(25)

where

$$\mu_j = \sum_{k=1}^{K} b_k G_k \Lambda'_k(1) r_{k,j} / F_j.$$
 (26)

For the proof of Theorem 9, we need the following lemma. The proof is presented in Appendix A.

Lemma 10: Under the assumptions in Theorem 9, the number of packets that are *actually* decoded by any one of the T Poisson receivers is within the capacity region of the ϕ -ALOHA receiver.

Proof. (Theorem 9) Let $X_k(t)$ be the number of class k packets sent to the t-th Poisson receiver and $Y_k(t)$ be the number of class k packets that are *actually* decoded by the t-th Poisson receiver. In view of Lemma 10 and the definition of the affine capacity envelope in (23), we have

$$\sum_{k=1}^{K} b_k Y_k(t) \leqslant B.$$
(27)

Note that (27) holds trivially when nothing is decoded in the *t*-th Poisson receiver, i.e., $Y_k(t) = 0$ for all *k*. Since $Y_k(t) \leq X_k(t)$, we have from (27) that

$$\sum_{k=1}^{K} b_k Y_k(t) \le \min[\sum_{k=1}^{K} b_k X_k(t), B].$$
 (28)

Taking expectations on both sides of (28) yields

$$\sum_{k=1}^{K} b_k \mathsf{E}[Y_k(t)] \le \mathsf{E}[\min[\sum_{k=1}^{K} b_k X_k(t), B]].$$
(29)

If the *t*-th Poisson receiver is a class *j* receiver, then we have from (13) that $X_k(t)$, k = 1, 2, ..., K, are independent Poisson random variables with mean

$$\rho_{k,j} = G_k \Lambda'_k(1) r_{k,j} / F_j. \tag{30}$$

Since $b'_k s$ are binary and the sum of independent Poisson random variables is still a Poisson random variable, we know that $\sum_{k=1}^{K} b_k X_k(t)$ is a Poisson random variable with mean μ_j as

$$\sum_{k=1}^{K} b_k \rho_{k,j} = \sum_{k=1}^{K} b_k G_k \Lambda'_k(1) r_{k,j} / F_j = \mu_j.$$
(31)

As there are F_jT class *j* Poisson receivers, the probability that the *t*-th Poisson receiver is a class *j* Poisson receiver is F_j . Thus, the expectation in the right-hand side of (29) can be computed by using the Poisson distribution, and we have

$$\sum_{k=1}^{K} b_{k} \mathsf{E}[Y_{k}(t)] \\ \leqslant \sum_{j=1}^{J} F_{j} \Big(\sum_{\tau=1}^{B-1} \tau e^{-\mu_{j}} \frac{\mu_{j}^{\tau}}{\tau!} + B \big(1 - \sum_{\tau=1}^{B-1} e^{-\mu_{j}} \frac{\mu_{j}^{\tau}}{\tau!} \big) \Big).$$
(32)

For a system of CPRs to be stable (the departure rate must be the same as the arrival rate), we must have $G_k = \mathsf{E}[Y_k(t)]$. Using this in (32) leads to the upper bound for the stable region in (25).

Example 1: (Spatially coupled IRSA) For the spatially coupled IRSA with *D*-multipacket reception capability in [27], every time slot is a *D*-fold ALOHA and thus has the affine capacity envelope (1, 1, ..., 1) with the bound *D*. Since there is only one class of users (K = 1) and one class of receivers (J = 1) in [27], the upper bound in (25) can be further simplified as follows:

$$G \leqslant \sum_{\tau=0}^{D-1} \tau e^{-G\Lambda'(1)} \frac{G\Lambda'(1)^{\tau}}{\tau!} + D\left(1 - \sum_{\tau=0}^{D-1} e^{-G\Lambda'(1)} \frac{G\Lambda'(1)^{\tau}}{\tau!}\right) = D - \sum_{\tau=0}^{D-1} (D - \tau) e^{-G\Lambda'(1)} \frac{G\Lambda'(1)^{\tau}}{\tau!}.$$
 (33)

This recovers the result in Theorem 1 of [27].

One straightforward extension of the upper bound in (33) is to consider a mixture of *D*-fold ALOHA receivers in [28]. Specifically, with probability π_D (that satisfies $\sum_{D=1}^{D_{\text{max}}} \pi_D = 1$ for some positive integer D_{max}), the (induced) Poisson receiver is selected from a *D*-fold ALOHA. For such a Poisson receiver, the success probability function (cf. (5)) is

$$P_{\rm suc}(\rho) = \sum_{D=1}^{D_{\rm max}} \pi_D \sum_{t=0}^{D-1} \frac{e^{-\rho} \rho^t}{t!}.$$
 (34)

Now the bound B in Theorem 9 is a random variable with the probability mass function

$$\mathsf{P}(B=D)=\pi_D,$$

for $D = 1, ..., D_{\text{max}}$. Following the same argument in the proof of Theorem 9, one can show that the upper bound for the stable offered load is

$$G \leqslant \sum_{D=1}^{D_{\max}} \pi_D \Big(D - \sum_{\tau=0}^{D-1} (D-\tau) e^{-G\Lambda'(1)} \frac{G\Lambda'(1)^{\tau}}{\tau!} \Big).$$
(35)

Example 2: (Near-far SIC decoding) For the setting with the capacity of near-far SIC decoding, every time slot can be modeled by the ϕ -ALOHA receiver with

$$\phi(n_1, n_2) = \begin{cases} (n_1, n_2) & \text{if } (n_1, n_2) \leq (1, 1) \\ (0, 0) & \text{otheriwse} \end{cases}$$
(36)

The near-far SIC decoding is commonly used for modeling power domain NOMA [44], [45]. There are three affine capacity envelopes for the ϕ -ALOHA receiver: (i) the vector (1,0) with the bound 1, and (ii) the vector (0,1) with the bound 1, and (iii) the vector (1,1) with the bound 2. Suppose that there is only one class of receivers (J = 1). Then the first affine capacity envelope leads to the bound

$$G_1 \le 1 - e^{-G_1 \Lambda_1'(1)},$$
 (37)

and the second affine capacity envelope leads to the bound

$$G_2 \leqslant 1 - e^{-G_2 \Lambda_2'(1)},$$
 (38)

and the third affine capacity envelope leads to the bound

$$G_{1} + G_{2}$$

$$\leq 1 \cdot e^{-(G_{1}\Lambda'_{1}(1) + G_{2}\Lambda'_{2}(1))} \frac{(G_{1}\Lambda'_{1}(1) + G_{2}\Lambda'_{2}(1))}{1!}$$

$$+ 2\Big(1 - e^{-(G_{1}\Lambda'_{1}(1) + G_{2}\Lambda'_{2}(1))}(1 + \frac{(G_{1}\Lambda'_{1}(1) + G_{2}\Lambda'_{2}(1))}{1!})\Big).$$
(39)

Example 3: (IRSA system with two classes of users and two classes of receivers) We consider a system of coded Poisson receivers with two classes of users (K = 2), two classes of receivers (J = 2), and the success probability function $P_{suc}(\rho) = e^{-\rho}$. This system is referred to as the *IRSA system with two classes of users and two classes of receivers* in [25]. As in [25], we set $F_1 = F_2 = 0.5$. We consider the following two packet routing policies (that correspond to two network slicing policies for resource allocation in uplink grant-free transmissions):

- 1) Complete sharing: every packet has an equal probability to be routed to the two classes of receivers, i.e., $r_{11} = r_{22} = r_{12} = r_{21} = 0.5$.
- 2) Receiver reservation: class 1 packets are routed to the two classes of receivers with an equal probability, i.e., $r_{11} = r_{12} = 0.5$, and class 2 packets are routed to the class 2 receivers, i.e., $r_{21} = 0$, $r_{22} = 1$.

Such a system is a CPR having three capacity envelopes: (i) the vector (1,0) with the bound 1, and (ii) the vector (0,1) with the bound 1, and (iii) the vector (1,1) with the bound 1.

For the complete sharing policy and the affine capacity envelope (1,1) with the bound 1, we have from (26) that

$$\mu_1 = \mu_2 = G_1 \Lambda'_1(1) + G_2 \Lambda'_2(1). \tag{40}$$

Hence, by (25), we have the outer bound

$$G_1 + G_2 \leqslant 1 - e^{-G_1 \Lambda'_1(1) - G_2 \Lambda'_2(1)}.$$
 (41)

Similarly, for the receiver reservation policy and the affine capacity envelope (1, 1) with the bound 1, we have from (26) that

$$\mu_1 = G_1 \Lambda'_1(1), \mu_2 = G_1 \Lambda'_1(1) + 2G_2 \Lambda'_2(1).$$
(42)

Hence, by (25), we have the outer bound

$$G_1 + G_2 \leqslant 1 - \frac{1}{2}e^{-G_1\Lambda'_1(1)} - \frac{1}{2}e^{-G_1\Lambda'_1(1) - 2G_2\Lambda'_2(1)}.$$
 (43)

On the other hand, for the receiver reservation policy and the affine capacity envelope (0, 1) with the bound 1, we have from (26) and (25) that $\mu_1 = 0$, $\mu_2 = 2G_2\Lambda'_2(1)$ and

$$G_2 \leqslant \frac{1}{2} - \frac{1}{2}e^{-2G_2\Lambda'_2(1)}.$$
 (44)

We present the visualization of the bounds (41), (43), and (44) in this example in Section VI-B.

IV. CONVOLUTIONAL CODED POISSON RECEIVERS

Inspired by the great performance of convolutional LDPC codes (see, e.g., [29]–[31]) and the spatially coupled IRSA [27], [32], in this section, we propose using the construction method for convolutional LDPC codes and spatially coupled IRSA to construct convolutional CPRs.

A. Circular convolutional coded Poisson receivers

Recall that the ensemble of bipartite graphs in the $(T, G, \Lambda(x), R, F)$ -CPR is constructed with the following parameters:

- 1) The total number of Poisson receivers T.
- 2) The offered load vector $G = (G_1, G_2, \dots, G_K)$.
- 3) The degree distribution vector $\Lambda(x) = (\Lambda_1(x), \Lambda_2(x), \dots, \Lambda_K(x)).$
- 4) The $K \times J$ routing matrix $R = (r_{k,j})$.
- 5) The partition vector $F = (F_1, F_2, \ldots, F_J)$.

Following the construction of convolutional LDPC codes, we first take L independent copies of $(T, G^{(\ell)}, \Lambda(x), R, F)$ -CPR with different offered load vectors, $G^{(\ell)} = (G_1^{(\ell)}, G_2^{(\ell)}, \ldots, G_K^{(\ell)})$, $\ell = 1, 2, \ldots, L$, and concatenate the L bipartite graphs to form a single bipartite graph. Then we rewire the edges in the concatenated bipartite graph. The receiver end of each edge in the ℓ^{th} copy is rewired to the corresponding receiver node in the ℓ^{th} copy, where ℓ is chosen uniformly in $[\ell, \ell \oplus (w - 1)]$, where $1 \leq w \leq L$ is known as the "smoothing" window size in [29], and the \oplus operator is the usual addition in a circular manner, i.e., for $1 \leq \ell \leq L$,

$$\ell \oplus (w-1) = \begin{cases} \ell + (w-1) & \text{if } \ell + (w-1) \leq L \\ \ell + (w-1) - L & \text{if } \ell + (w-1) > L \end{cases}$$
(45)

Also, we define the \ominus operator as the usual subtraction in a circular manner, i.e.,

$$\ell \ominus (w-1) = \begin{cases} \ell - (w-1) & \text{if } \ell - (w-1) \ge 1\\ \ell - (w-1) + L & \text{if } \ell - (w-1) < 1 \end{cases}$$
(46)

We call such a system the circular convolutional CPR with L stages. By viewing the class k user nodes (resp. class j receiver nodes) at the ℓ stage as the class (k, ℓ) users (resp. class (j, ℓ) receivers), the circular convolutional CPR with L stages is a CPR with KL classes of users and JL classes of receivers. Moreover, the routing probability from a (k, ℓ) user node to a (j, ℓ) receiver node is $r_{k,j}/w$ if $\ell \in [\ell, \ell \oplus (w - 1)]$ and 0 otherwise. Thus, the density evolution analysis in Theorem 3 can still be applied and we have the following corollary for the circular convolutional CPR with L stages.

Corollary 11: Consider the circular convolutional CPR with L stages described in this section. Let $q_{k,\ell}^{(i)}$ be the probability that the *user end* of a randomly selected class k edge in the ℓ^{th} stage has not been successfully received after the i^{th} SIC iteration. Also, let

$$\rho_{k,j,\ell}^{(i)} = \sum_{\hat{\ell}=\ell \ominus (w-1)}^{\ell} q_{k,\hat{\ell}}^{(i)} G_k^{(\hat{\ell})} \Lambda_k'(1) \frac{1}{w} r_{k,j} / F_j, \qquad (47)$$

be the offered load of class k users to a class j receiver in the ℓ^{th} stage after the i^{th} SIC iteration, and

$$\tilde{\rho}_{j,\ell}^{(i)} = (\rho_{1,j,\ell}^{(i)}, \rho_{2,j,\ell}^{(i)}, \dots, \rho_{K,j,\ell}^{(i)}), \tag{48}$$

be the offered load vector to a class j receiver in the ℓ^{th} stage after the i^{th} SIC iteration.

As $T \to \infty$, the success probability for a class k user in the ℓ^{th} stage after the i^{th} SIC iteration converges to

$$\tilde{P}_{\mathrm{suc},k,\ell}^{(i)}(G^{(1)}, G^{(2)}, \dots, G^{(L)}) = 1 - \Lambda_k \Big(1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^J \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i)}) \Big),$$
(49)

 $k=1,2,\ldots,K,\ \ell=1,2,\ldots,L,$ and $q_\ell^{(i)}=(q_{1,\ell}^{(i)},q_{2,\ell}^{(i)},\ldots,q_{K,\ell}^{(i)})$ can be computed recursively from the following equation:

$$q_{k,\ell}^{(i+1)} = \lambda_k \Big(1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^J \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i)})) \Big), \quad (50)$$

with $q_{\ell}^{(0)} = (1, 1, ..., 1)$. Also, under (A4), we can let $p_{k,\ell}^{(i+1)} = \lambda_k^{-1}(q_{k,\ell}^{(i+1)})$ and $p_{\ell}^{(i)} = (p_{1,\ell}^{(i)}, p_{2,\ell}^{(i)}, ..., p_{K,\ell}^{(i)})$. Then we can rewrite (50) as follows:

$$p_{k,\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^{J} \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i)}),$$
(51)

with $p_{\ell}^{(0)} = (1, 1, \dots, 1)$.

One may refer to Appendix B in the supplemental material for a detailed derivation. Similarly, the stability results in Section II-C can also be applied.

B. Stability and threshold saturation

Now we construct the convolutional CPR from the circular convolutional CPR by setting the offered load vectors in the last w - 1 stages to be the zero vector **0**. This is known as "puncturing" for convolutional LDPC codes.

Definition 12: The convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages is the circular convolutional CPR with L stages when $G^{(\ell)} = G$ for $\ell = 1, 2, ..., L - w + 1$, and $G^{(\ell)} = \mathbf{0}$ for $\ell = L - w + 2, ..., L$.

As the circular convolutional CPR with L stages is a CPR with KL classes of users and JL classes of receivers, the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages is a CPR with K classes of users and JL classes of receivers (by grouping the (k, ℓ) user nodes at the L stages into a single class of user nodes, i.e., class k user nodes). The notion of

stability in Section II-C can be defined the same way for the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages, i.e., it is stable if the probability that a packet is successfully received approaches 1 when the number of iterations goes to infinity.

Note that the normalized offered load for the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages is

$$\frac{\sum_{\ell=1}^{L} G^{(\ell)} T}{LT} = \frac{L - w + 1}{L} G.$$
 (52)

Thus, as $L \to \infty$, the normalized offered load approaches G.

One of the most interesting phenomena for convolutional LDPC codes is threshold saturation (see, e.g., [29], [31]-[33]). In the following theorem, we show an analogous result for the convolutional coded Poisson receivers.

Theorem 13: (Threshold saturation) Let S be the stability region of the $(T, G, \Lambda(x), R, F)$ -CPR and S_L be the stability region of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages. Then for any positive integer L,

$$S \subset S_L \subset S_{L-1}.$$
 (53)

In view of Theorem 13, the stability region S_L of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages "saturates" when $L \to \infty$.

Proof. Recall that the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages is the circular convolutional Poisson receiver with L stages when the last w - 1 stages are "punctured," i.e., $G^{(\ell)} = G$ for $\ell = 1, 2, ..., L - w + 1$, and $G^{(\ell)} = \mathbf{0}$ for $\ell = L - w + 2, \dots, L$. If the last w stages are "punctured," i.e., $G^{(\ell)} = G$ for $\ell = 1, 2, \dots, L - w$, and $G^{(\ell)} = \mathbf{0}$ for $\ell = L - w + 1, \dots, L$, then no packets are sent to the last stage and it can be removed. Thus, it reduces to the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L-1stages. As the success probability functions are decreasing in the offered load in (A1), the success probability in the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages is not larger than that of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L-1 stages. This shows that $S_L \subset S_{L-1}$.

Now consider the circular convolutional Poisson receiver with L stages when no stages are "punctured," i.e., $G^{(\ell)} = G$ for $\ell = 1, 2, \dots, L$. From the monotonicity, it is clear that the success probability of this system is not larger than that of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages. We will argue that the success probability of this system is exactly the same as that of the $(T, G, \Lambda(x), R, F)$ -CPR. As such, we have $S \subset S_L$. When $G^{(\ell)} = G$ for $\ell = 1, 2, ..., L$, $\rho_{k,j,\ell}^{(i)}$ in (47) can be

simplified as follows:

$$\rho_{k,j,\ell}^{(i)} = G_k \Lambda'_k(1) r_{k,j} / F_j \sum_{\hat{\ell} = \ell \ominus (w-1)}^{\ell} q_{k,\hat{\ell}}^{(i)}.$$
 (54)

We now use induction to show that $q_{k,\ell}^{(i)} = q_k^{(i)}$ for all *i*. Since $q_{k,\ell}^{(0)} = 1$ and $q_k^{(0)} = 1$, we have from (54) that

$$\rho_{k,j,\ell}^{(1)} = G_k \Lambda'_k(1) r_{k,j} / F_j.$$
(55)

In view of (13) and (48), we have

$$\tilde{\rho}_{j,\ell}^{(1)} = q_\ell^{(1)} \circ \tilde{\rho}_j.$$

It is easy to see from (50) and (15) that $q_{k,\ell}^{(1)} = q_k^{(1)}$ for all $\ell = 1, 2, \ldots, L$. By inducting on the number of iterations *i*, we then have $q_{k,\ell}^{(i)} = q_k^{(i)}$ for all *i*. Because of (49) and (14), we conclude that the success probability of the circular convolutional Poisson receiver without puncturing is the same as that of the $(T, G, \Lambda(x), R, F)$ -CPR.

In the case where there is only one class of users and one class of receivers (the single-class system), the stability region of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages can be reduced to a set in \mathbb{R} . The supremum of this set represents the percolation threshold of the CCPR. A direct finding from Theorem 13 is the monotonic decrease of the percolation threshold as the number of stages L increases.

Example 4: (Convolutional IRSA) For slotted ALOHA, the success probability function $P_{\rm suc}(\rho) = e^{-\rho}$. Suppose that the degree distribution $\Lambda(x) = x^d$, i.e., each packet is transmitted exactly d times. For such a degree distribution, we have $\lambda(x) = x^{d-1}$. Assume w = 2, K = 1, and J = 1. It follows from (50) in Corollary 11 that for $\ell = 2, \ldots, L-2$,

$$q_{1}^{(i+1)} = \left(1 - \frac{1}{2}e^{-\frac{1}{2}Gdq_{1}^{(i)}} - \frac{1}{2}e^{-\frac{1}{2}Gd(q_{1}^{(i)} + q_{2}^{(i)})}\right)^{d-1},$$

$$q_{\ell}^{(i+1)}$$
(56)

$$= \left(1 - \frac{1}{2}e^{-\frac{1}{2}Gd(q_{\ell-1}^{(i)} + q_{\ell}^{(i)})} - \frac{1}{2}e^{-\frac{1}{2}Gd(q_{\ell}^{(i)} + q_{\ell+1}^{(i)})}\right)^{d-1},$$
(57)

and

$$q_{L-1}^{(i+1)} = \left(1 - \frac{1}{2}e^{-\frac{1}{2}Gd(q_{L-2}^{(i)} + q_{L-1}^{(i)})} - \frac{1}{2}e^{-\frac{1}{2}Gdq_{L-1}^{(i)}}\right)^{d-1},$$
(58)

with the initial condition $q_k^{(0)} = 1, k = 1, 2, \dots, L-1$. It can be shown that from Example 5 of [25] that the percolation thresholds are G = 0.9179, 0.9767, 0.9924, 0.9973 for d =3, 4, 5, 6. We evaluate the thresholds for the system described in this example with w = 2, 3, 4 in Section VI-A.

V. PERCOLATION THRESHOLDS OF SYSTEMS WITH A SINGLE CLASS OF USERS

In this section, we focus on CPR systems with only one class of users and one class of receivers. We derive the potential function of such a system of CPR and the associated percolation thresholds of both CPRs and CCPRs.

A. The single-system threshold

Since there is only one class of users and one class of receivers, (15) and (16) reduce to the following form:

$$p^{(i)} = 1 - P_{\text{suc}}(q^{(i)}G\Lambda'(1)),$$
 (59)

$$q^{(i+1)} = \lambda(p^{(i)}).$$
 (60)

We combine the two equations in (59) and (60) to form the following recursive equation:

$$p^{(i+1)} = 1 - P_{\text{suc}}(\lambda(p^{(i)})G\Lambda'(1)), \tag{61}$$

with $p^{(0)} = 1$. For every packet to be decoded successfully, we need to ensure that

$$\lim_{i \to \infty} p^{(i)} = 0. \tag{62}$$

We are interested in finding out the maximum offered load G such that (62) is satisfied. Let

$$f(p;G) = 1 - P_{\text{suc}}(pG\Lambda'(1)),$$
 (63)

$$h(p) = \lambda(p). \tag{64}$$

Then (61) can be written as follows:

$$p^{(i+1)} = f(h(p^{(i)}); G).$$
(65)

In [30], the recursion of the form in (65) is said to be a *scalar admissible system* characterized by a pair of functions (f, h) that satisfy the four properties in Definition 14 below.

Definition 14: (cf. Def. 1 in [30]) The scalar admissible system (f, h) parameterized by $G \ge 0$ is defined by the recursion $p^{(i+1)} = f(h(p^{(i)}); G)$, where f and h satisfy the following four properties:

- (P1) $f: [0,1] \times [0,\infty) \to [0,1]$ is strictly increasing in both p and G.
- (P2) $h: [0,1] \to [0,1]$ satisfies h'(p) > 0 for $p \in (0,1]$.
- (P3) f(0;G) = f(p;0) = h(0) = 0.
- (P4) f has continuous second derivatives on $[0,1] \times [0,\infty)$ w.r.t. all arguments, so is h on [0,1].

In the following lemma, we show that the $(T, G, \Lambda(x), R, F)$ -CPR with one class of users and one class of receivers is indeed a scalar admissible system under the assumptions in (A1)-(A4).

Lemma 15: Under the assumptions in (A1) and (A4), a system of CPR described by the density evolution equation in (61) is the scalar admissible system (f, h) with f and h specified in (63) and (64), respectively.

Proof. It suffices to show that f in (63) and h in (64) satisfy (P1)-(P4) of Definition 14.

(P1) Since P_{suc} is (strictly) decreasing in (A1), we have for $p \in [0, 1]$ and $G \in [0, \infty)$,

$$\begin{array}{ll} \displaystyle \frac{\partial f(p;G)}{\partial p} & = & \displaystyle -\frac{\partial P_{\rm suc}(pG\Lambda'(1))}{\partial p} \cdot G\Lambda'(1) > 0, \\ \displaystyle \frac{\partial f(p;G)}{\partial G} & = & \displaystyle -\frac{\partial P_{\rm suc}(pG\Lambda'(1))}{\partial G} \cdot p\Lambda'(1) > 0. \end{array}$$

(P2) Since every packet is transmitted at least twice in (A4), we have $\Lambda''(p) > 0$, $\forall p \in [0, 1]$. Also, since $\Lambda'(1)$ is positive, $h'(p) = \lambda''(p) = \Lambda''(p)/\Lambda'(1) > 0$.

(P3) Since $P_{\rm suc}(0) = 1$, and λ is a polynomial without constant terms,

$$\begin{aligned} f(0,G) &= 1 - P_{\rm suc}(0) = 0, \\ f(p,0) &= 1 - P_{\rm suc}(0) = 0, \\ h(0) &= \lambda(0) = 0. \end{aligned}$$

(P4) is obvious from (A1).

One of the most powerful tools for analyzing the stability of the scalar admissible system (f, h) is the *potential function* in [30].

Definition 16: The function U(p;G) is called the *potential* function of the scalar admissible system (f,h) in [30] if it satisfies

$$p^{(i+1)} - p^{(i)} = f(h(p^{(i)}); G) - p^{(i)}$$
$$= -\frac{1}{h'(p^{(i)})} \frac{\partial U(p; G)}{\partial p} \Big|_{p=p^{(i)}}.$$
 (66)

To ease the representation, for a function U(p;G) with two variables, we use U'(p;G) to denote the partial differentiation with respect to the *first* variable, given by

$$U'(p;G) = \frac{\partial U(p;G)}{\partial p}$$

Rewrite (66) as follows:

$$U'(p;G)\Big|_{p=p^{(i)}} = -h'(p^{(i)})(f(h(p^{(i)});G) - p^{(i)}).$$
(67)

Integrating both sides of (67) yields

$$U(p;G) = \int_{0}^{p} h'(z)(z - f(h(z);G))dz$$

= $ph(p) - H(p) - F(h(p);G),$ (68)

where

$$F(p;G) = \int_0^p f(z;G)dz,$$
(69)

and

$$H(p) = \int_0^p h(z)dz.$$
 (70)

To see how the potential function can be used for analyzing the stability of a scalar admissible system, let us consider the *single-system threshold* [30] defined below:

$$G_s^* = \sup\{G \in [0,1] | U'(p;G) > 0, \forall p \in (0,1]\}.$$
 (71)

Then for all $G \leq G_s^*$, the system is stable. To see this, note from (P1) of Definition 14 that U'(p;G) > 0 for all $p \in (0,1]$ and $G \leq G_s^*$. Also, note from (P3) of Definition 14, that is, f(0;G) = f(p;0) = h(0) = 0, that p = 0 is a solution of p = f(h(p);G). Since h'(p) > 0 for $p \in (0,1]$ ((P2) in Definition 14), one can see from (66) that $p^{(i+1)} < p^{(i)}$ for all *i*. Thus, $p^{(i)}$ (with $p^{(0)} = 1$) converges to 0 as $i \to \infty$ for all $G \leq G_s^*$.

Since we have shown in Lemma 15 that the $(T, G, \Lambda(x), R, F)$ -CPR with one class of users and one class of receivers is a scalar admissible system with f and h specified in (63) and (64), we can use (68) to derive the potential function U(p; G) for $(T, G, \Lambda(x), R, F)$ -CPRs, and then use that to show the following stability result.

Theorem 17: Under the assumptions in (A1)-(A4), the potential function of the $(T, G, \Lambda(x), R, F)$ -CPR with one class of users and one class of receivers is given by

$$U(p;G) = \lambda(p)(p-1) - \frac{\Lambda(p)}{\Lambda'(1)} + \frac{1}{G\Lambda'(1)} \int_0^{G\Lambda'(1)\lambda(p)} P_{\text{suc}}(\rho)d\rho.$$
(72)

Moreover, if the inverse function of $P_{suc}(\cdot)$ exists and is continuous and decreasing on [0, 1], then for all $G \leq G_s^*$, the system is stable, where G_s^* in (71) has the following representation:

$$G_s^* = \inf_{p \in [0,1]} \frac{P_{\text{suc}}^{-1}(1-p)}{\Lambda'(p)}.$$
(73)

Proof. In view of (63) and(69), integrating the function f w.r.t. z from 0 to p yields

$$F(p;G) = \int_0^p 1 - P_{\text{suc}}(zG\Lambda'(1))dz$$
$$= p - \frac{1}{G\Lambda'(1)} \int_0^{G\Lambda'(1)p} P_{\text{suc}}(\rho)d\rho. \quad (74)$$

Also, note that $h(p) = \lambda(p) = \Lambda'(p)/\Lambda'(1)$, $H(p) = \Lambda(p)/\Lambda'(1)$. Using (68), we obtain (72).

From (66), the partial derivative of the potential function w.r.t. its first variable, known as the *balance function* has the following form:

$$U'(p;G) = \lambda'(p)(p-1+P_{suc}(G\Lambda'(p))),$$
 (75)

where $\Lambda'(p) = \Lambda'(1)\lambda(p)$. Since we assume that P_{suc}^{-1} exists, from (71) and (75), we have that

$$G_{s}^{*} = \sup\{G \in [0,\infty) | U'(p;G) > 0, \forall p \in (0,1]\}$$

=
$$\sup\{G \in [0,\infty) | P_{suc}(G\Lambda'(p)) > 1 - p,$$

$$\forall p \in (0,1]\}$$

=
$$\inf_{p \in [0,1]} \frac{P_{suc}^{-1}(1-p)}{\Lambda'(p)}.$$
(76)

The following example shows the potential functions of the *D*-fold ALOHA system.

Example 5: (*D*-fold ALOHA) For the *D*-fold ALOHA system described by (3) and (5) with $\Lambda(x) = x^d$, we use (72) and integration by parts (as in Theorem 3 of [46]) to evaluate the potential function as follows:

$$U(p;G) = \frac{1}{d} \Big((d-1)p^d - dp^{d-1} + \frac{1}{G} \Big(D - e^{-Gdp^{d-1}} \sum_{\tau=0}^{D-1} (D-\tau) \frac{(Gdp^{d-1})^{\tau}}{\tau!} \Big) \Big).$$
(77)

B. The saturation theorem

In this section, we study the stability of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages. The density evolution equation for the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR is given by (51) in Corollary 11. For K = J = 1, by (11), (47), and the relation $q_{\ell}^{(i)} = \lambda(p_{\ell}^{(i)})$, (51) can be simplified as

$$p_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\ell}^{\ell \bigoplus (w-1)} \frac{1}{w} P_{\text{suc}} \left(\frac{1}{w} \sum_{\tilde{\ell}=\hat{\ell} \ominus (w-1)}^{\hat{\ell}} \lambda(p_{\tilde{\ell}}^{(i)}) G^{(\tilde{\ell})} \Lambda'(1) \right),$$
(78)

for $\ell = 1, \ldots, L$. Moreover, since $G^{(\tilde{\ell})} = G$ for $\tilde{\ell} = 1, 2, \ldots, L - w + 1$ and $G^{(\tilde{\ell})} = 0$ for $\tilde{\ell} = L - w + 2, \ldots, L$, (78) can be further simplified as follows:

$$p_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\ell}^{\ell+(w-1)} \frac{1}{w} P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\max[1,\hat{\ell}-(w-1)]}^{\min[L-w+1,\hat{\ell}]} \lambda(p_{\tilde{\ell}}^{(i)}) \right)$$
(79)

for $\ell = 1, 2, \dots, L - w + 1$.

Denote by $G^*_{conv}(L, w)$ the percolation threshold of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system with Lstages. The only way to know the exact value of $G^*_{conv}(L, w)$ is to evaluate the density evolution equations in (78). In this section, we prove a lower bound of $G^*_{conv}(L, w)$ by using the potential function of the corresponding $(T, G, \Lambda(x), R, F)$ -CPR (under certain conditions of the window size w). Such a stability result for the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system with L stages is called the *saturation theorem* in this paper.

In addition to the single-system threshold G_s^* in (71), we define the *potential threshold* of the $(T, G, \Lambda(x), R, F)$ -CPR. We will show that the potential threshold is a lower bound of $G_{conv}^*(L, w)$ in the saturation theorem.

Definition 18: (cf. Def. 6 in [30]) Consider the scalar admissible system described in Definition 14 with the potential function U(p; G) in (68). The potential threshold of the scalar admissible system, denoted by G^*_{conv} , is defined below:

$$G_{conv}^* = \sup\{G \in [0,1] | \min_{p \in [0,1]} U(p;G) \ge 0\}.$$
 (80)

To give the specific conditions for the window size of the saturation theorem, we define the following terms.

Definition 19: (cf. Def. 5, and 6 in [30]) Consider the scalar admissible system (f, h) in Definition 14 with the potential function U(p; G) in (68).

(i) For $G > G_s^*$, the minimum unstable fixed point is the number

$$u(G) = \sup\{\tilde{p} \in [0,1] | U'(p;G) \ge 0, p \in [0,\tilde{p}]\}.$$
(81)

(ii) The energy gap of the scalar admissible system for $G \in (G_s^*, 1]$ is the number

$$\Delta E(G) = \min_{p \in [u(G), 1]} U(p; G).$$
(82)

(iii) A constant $K_{f,h} = ||h'||_{\infty} + ||h'||_{\infty}^{2} ||f'||_{\infty} + ||h''||_{\infty}$, where

$$||h||_{\infty} = \sup_{x \in [0,1]} |h(x)|$$

for functions $h: [0,1] \to \mathbb{R}$.

Now we state the saturation theorem.

Theorem 20: (The saturation theorem for the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR) Consider the single-class (K = J = 1) convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system with L stages governed by the density evolution recursion in (79). If $G < G^*_{conv}$ and

$$w > \frac{K_{f,h}}{\Delta E(G)},\tag{83}$$

then the only fixed point of (79) is $\mathbf{p} = \mathbf{0}$. As such, the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system with L stages is stable for all $G < G_{conv}^*$.

Our proof is analogous to that for the stability of spatially coupled LDPC codes (Theorem 1 in [30]), and it requires a sequence of lemmas and definitions of terms. Here we outline the basic steps of the proof. The detailed proof can be found in Appendix B of the supplemental material.

(i) Let $\hat{L} = L - w + 1$. We reverse all the indices of the recursion relation (79). Then, the recursion (79) becomes

$$p_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell} = \tilde{L} + 1 - \ell}^{\tilde{L} + w - \ell} \frac{1}{w} P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell} = \max[1, \tilde{L} - \hat{\ell} + 1]}^{\min[\tilde{L}, \tilde{L} - \hat{\ell} + w]} \lambda(p_{\tilde{\ell}}^{(i)}) \right)$$
(84)

(ii) If h is a function with one variable and $\mathbf{p} = (p_1, \dots, p_{\tilde{L}}) \in \mathbb{R}^{\tilde{L}}$ is a vector, we define

$$\mathbf{h}(\mathbf{p}) = (h(p_1), h(p_2), \dots, h(p_{\tilde{L}}))$$

and

$$\mathbf{f}(\mathbf{p};G) = (f(p_1;G), f(p_2;G), \dots, f(p_{\tilde{L}};G))$$

for the ease of representation. Let $\mathbf{p}^{(i)} = (p_1^{(i)}, p_2^{(i)}, \dots, p_{\tilde{L}}^{(i)})$, then (84) can be written as the following matrix form:

$$\mathbf{p}^{(i+1)} = \mathbf{A}_2 \mathbf{f}(\mathbf{A}_2^T \mathbf{h}(\mathbf{p}^{(i)}); G), \tag{85}$$

where \mathbf{A}_2 is an $\tilde{L} \times L$ matrix defined as:

$$\mathbf{A}_{2} = \frac{1}{w} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \end{bmatrix}, \quad (86)$$

where all the rows of A_2 contain w ones, and the functions f and h are specified in (63) and (64), respectively. Such a system of recursive equations is called the *basic spatially-coupled* system in [30], up to that the matrix A_2 is transposed. Like the scalar admissible system, the basic spatially-coupled system is also characterized by the two functions f and h. (iii) Consider the recursive equations in the following matrix

(iii) Consider the recursive equations in the following matrix form:

$$\mathbf{p}^{(i+1)} = \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p}^{(i)}); G), \tag{87}$$

where A is an $L \times L$ matrix defined as:

$$\mathbf{A} = \frac{1}{w} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix},$$
(88)

where the 1st, 2nd, ..., and the (L - w + 1)-th rows of **A** contain w ones, and the functions f and h are specified in (63) and (64), respectively. Such a system of recursive equations is called the *vector one-sided spatially-coupled recursion* system

in [30] and [34]. We will briefly call this kind of system a *one-sided system* in this paper. Like the scalar admissible system, the one-sided system is also characterized by the two functions f and h.

One may illustrate that the one-sided system gives a component-wise upper bound of the basic spatially-coupled system for $\ell = 1, \ldots, \tilde{L}$. Hence, the percolation threshold of the basic spatially-coupled system is not smaller than that of the one-sided system. Once we prove the saturation theorem for the one-sided system, the results will also apply to the basic spatially-coupled system.

(iv) The *potential function of the one-sided system* has the following representation:

$$U(\mathbf{p};G) = \mathbf{h}(\mathbf{p})^T \mathbf{p} - H(\mathbf{p}) - F(\mathbf{A}\mathbf{h}(\mathbf{p});G), \qquad (89)$$

where

and

$$H(\mathbf{p}) = \sum_{\ell} H(p_{\ell}),\tag{90}$$

$$F(\mathbf{p};G) = \sum_{\ell} F(p_{\ell};G), \qquad (91)$$

with the functions F and H being defined in (69) and (70), respectively. This shows how the potential function of the one-sided system is coupled with that of the scalar admissible system.

(v) Reproduce the properties of the potential function in Lemma 3 and Lemma 4 of [30] to show a specific relation between the potential function of the one-sided system and that of the scalar admissible system. Specifically, for a non-decreasing vector $\mathbf{p} = (p_1, \ldots, p_L) \in [0, 1]^L$, define the shift operator $\mathbf{S} : \mathbb{R}^L \to \mathbb{R}^L$ so that $[\mathbf{Sp}] = (0, p_1, p_2, \ldots, p_{L-1})$. Then

$$U(\mathbf{Sp};G) - U(\mathbf{p};G) = -U(p_L;G).$$
(92)

(vi) Reproduce the bound for the norm of the Hessian $U''(\mathbf{p}; G)$ in Lemma 5 of [30], i.e.,

$$||U''(\mathbf{p};G)||_{\infty} \leqslant K_{f,h},\tag{93}$$

where the constant $K_{f,h}$ is specified in Definition 19 (iii).

(vii) Suppose $\mathbf{p} \neq \mathbf{0}$ is a fixed point of (87). As in the proof of Theorem 1 of [30], one can use the Taylor series (with the bound for the norm of the Hessian $U''(\mathbf{p}; G)$) to show that the inner product of the gradient $U'(\mathbf{p}; G)$ with a specific direction $\mathbf{Sp} - \mathbf{p}$ is smaller than 0 for $G < G_{conv}^*$ and w satisfying the condition in (83). Since each component of the vector $\mathbf{Sp} - \mathbf{p}$ is not greater than 0, there must exist a positive component, say the ℓ_0 -th component, of the gradient $U'(\mathbf{p}; G)$. As such, one more iteration further reduces the value of p_{ℓ_0} . This then leads to a contradiction and thus shows that $\mathbf{p} = \mathbf{0}$ is the only fixed point of (78).

Note that if L is large enough, one can always choose a window size w large enough to satisfy (83). An immediate consequence of Theorem 20 is that $G^*_{conv}(L,w) \ge$ G^*_{conv} if $w > K_{f,h}/\Delta E(G)$, as Theorem 20 provides a sufficient condition for the stability of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system with L stages. However, it is also possible that $G^*_{conv}(L,w)$ is larger than G^*_{conv} even when the condition in (83) is not satisfied. This is due to the puncturing effect that reduces the offered load. In addition, the condition for w in Theorem 20 often results in large window size. However, our numerical results in Section VI-A demonstrate that $|G^*_{conv}(L,w) - G^*_{conv}| < 0.0001$ can be achieved in some cases.

C. The upper bound by the area theorem

In this section, we consider another threshold G_{up}^* that is defined as the unique positive solution of the equation U(1;G) = 0. For the potential function of a system of CPR in (72), G_{up}^* is the unique positive solution to the following equation:

$$G = \int_0^{G\Lambda'(1)} P_{\rm suc}(\rho) d\rho. \tag{94}$$

Now we show by the area theorem [26] that G_{up}^* is an upper bound for the stable offered load of a system of CPR. For $p^{(i)}$ and $q^{(i)}$ in the two recursive mappings in (59) and (60) to converge to 0 as $i \to \infty$, the area theorem in [26] provides a necessary condition that requires the sum of the two areas under the two mappings to be not greater than 1, i.e.,

$$\int_{0}^{1} (1 - P_{\text{suc}}(qG\Lambda'(1))dq + \int_{0}^{1} \lambda(p)dp \leq 1.$$
 (95)

Since $\lambda(p) = \Lambda'(p)/\Lambda'(1)$, $\int_0^1 \lambda(p)dp = 1/\Lambda'(1)$. Let $\rho = qG\Lambda'(1)$. We can write the inequality in (95) as follows:

$$G \leqslant \int_{0}^{G\Lambda'(1)} P_{\rm suc}(\rho) d\rho.$$
(96)

For the *D*-fold ALOHA system described by (3) and (5), G_{up}^* is the solution to the following equation:

$$G = D - \sum_{\tau=0}^{D-1} (D-\tau) e^{-Gd} \frac{(Gd)^{\tau}}{\tau!}.$$
 (97)

This is the same as the upper bound in Theorem 1 of [27] for the spatially coupled *D*-fold ALOHA system. It is also the same as the upper bound in Section III when the underlying channel is modeled by a ϕ -ALOHA receiver [28]. In view of the extended upper bound for a mixture of *D*-fold ALOHA in Example 1, we have the following corollary.

Corollary 21: Suppose that the T Poisson receivers of the $(T, G, \Lambda(x), R, F)$ -CPR with one class of users and one class of receivers in (72) are induced from a mixture of D-fold ALOHA in Example 1. Then the stable offered load of such a system is bounded above by G_{up}^* , where G_{up}^* is the solution to (94). As such, G_{up}^* is an upper bound for the percolation threshold of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with an infinite number of stages, i.e.,

$$\lim_{L \to \infty} G^*_{conv}(L, w) < G^*_{up}$$

regardless of the choice of the degree distribution $\Lambda(x)$ and the window size w.

In the following theorem, we show the single-system threshold G_s^* in (71) is smaller than the potential threshold G_{conv}^* , and the potential threshold G_{conv}^* is smaller than the upper bound G_{uv}^* .

Theorem 22: Consider the scalar admissible system (f, h) in Definition 14 with the potential function U(p; G). Then

$$G_s^* < G_{conv}^* < G_{up}^*.$$
 (98)

Proof. (i) First, we demonstrate that $G_s^* < G_{conv}^*$. From the integral representation in (68) and the fact that the function f is strictly increasing in G from (P1), we deduce that U(0;G) = 0 for all G and U(p;G) is strictly decreasing in G. From the definition given in (71) of G_s^* , it is clear that U'(p;G) > 0 when $G \leq G_s^*$. Therefore, the function U(p;G) is strictly increasing in p when $G \leq G_s^*$. This implies that U(p;G) > 0 for all $p \in (0,1)$ and $G \leq G_s^*$. Since U(p;G) decreases for G, when U(p;G) becomes 0 for some value of G and p is not 0, such a G must be greater than G_s^* . The supremum of all such possible G values results in $G_s^* < G_{conv}^*$.

(ii) Now we show that $G_{conv}^* < G_{up}^*$. By (A4), we have $\Lambda'(1) > 0$ and $\lambda'(1) > 0$. Since $P_{suc}(G\Lambda'(1)) > 0$ from (A1), we have

$$U'(1;G) = \lambda'(1)P_{\text{suc}}(G\Lambda'(1)) > 0.$$

Thus, p = 1 cannot be a local minimum of U(p;G) for G > 0. As such,

$$U(1; G_{conv}^*) > \min_{p \in [0,1]} U(p; G_{conv}^*) = 0.$$
(99)

As argued in (i), U(1; G) is strictly decreasing in G. Thus, we have from (99) that U(1; G) > 0 for all $G \leq G_{conv}^*$. This shows that $G_{conv}^* < G_{up}^*$ if G_{up}^* exists. It remains to show the existence of the unique positive solution G_{up}^* of the equation U(1; G) = 0. Since U(1; G) is continuous and decreasing for G > 0, and f(p; G) is strictly increasing in both p and G, $U(1; G) \to -\infty$ when $G \to \infty$ by (68). Therefore, combining with (99), the unique positive solution to the equation U(1; G) = 0 exists on (G_{conv}^*, ∞) .

Given Theorem 20 and Theorem 22, one can improve the percolation threshold of a $(T, G, \Lambda(x), R, F)$ -CPR system from G_s^* to at least G_{conv}^* by adopting the spatially-coupled method. Though the window size w needs to be large to satisfy the condition in (83), our numerical results in Section VI-A show that adopting w = 2 significantly improves the percolation threshold.

Theorem 22 also reveals the existence of a numerical gap between G^*_{conv} and G^*_{up} . However, as shown in Section VI, this gap is small in most cases, with $|G^*_{up} - G^*_{conv}|$ typically being smaller than 10^{-2} .

Example 6: (**IRSA with a regular degree**) In this example, we illustrate how G_s^* , G_{conv}^* and G_{up}^* are related for IRSA with a regular degree d, i.e., $\Lambda(x) = x^d$ and $P_{suc}(\rho) = e^{-\rho}$. Then we have from (77) (with D = 1) that

$$U(p;G) = \frac{1}{d} \left((d-1)p^d - dp^{d-1} + \frac{1 - e^{-Gdp^{d-1}}}{G} \right),$$
(100)

and

$$U'(p;G) = (d-1)p^{d-2}(p-1+e^{-Gdp^{d-1}}).$$
 (101)



Fig. 1. The plot of the potential function U(p; G) as a function of p for various values of G in Example 6 with d = 3.

For d = 3, we use a computer search to find G_s^* , G_{conv}^* and G_{up}^* as follows:

$$\begin{array}{lll} G_s^* &=& \sup\{G \in [0,1]|\\ && d(d-1)p^{d-2}(p-1+e^{-Gdp^{d-1}}) > 0, \forall p \in (0,1]\}\\ &\approx& 0.8184,\\ G_{conv}^* &=& \sup\{G \in [0,1]|\\ && \min_{p \in [0,1]} (d-1)p^d - dp^{d-1} + \frac{1-e^{-Gdp^{d-1}}}{G} \geqslant 0\}\\ &\approx& 0.9179,\\ G_{up}^* &=& \{G:G=1-e^{-dG}\} \approx 0.9405. \end{array}$$

In Figure 1, we plot the potential function U(p;G) as a function of p for various values of G. This plot illustrates how the potential function changes for the parameter p under various values of the parameter G.

VI. NUMERICAL RESULTS

In this section, we provide numerical results for various systems of convolutional coded Poisson receivers. The convergence criteria are defined as $|q_{k,\ell}^{(i)} - q_{k,\ell}^{(i-1)}| < 10^{-8}$ (or $|p_{k,\ell}^{(i)} - p_{k,\ell}^{(i-1)}| < 10^{-8}$) for all ℓ . The percolation threshold is determined by identifying the value of G such that $\lim_{i\to\infty} p^{(i)}$ exhibits the first significant jump. The numerical results presented here are obtained using a step size of $\delta = 0.0001$ for G and L = 40. All numerical values are rounded to four decimal places.

A. Convolutional coded Poisson receivers with one class of users and one class of receivers

In this section, we focus on the single-class system (convolutional coded Poisson Receivers with K = J = 1). This section aims to validate the results obtained in Section V.

In Table I, we provide percolation thresholds for the scalar admissible system (w = 1) with $P_{suc}(\rho) = e^{-\rho}$, i.e., IRSA.

TABLE I PERCOLATION THRESHOLDS FOR CONVOLUTIONAL IRSA.

Convolutional IRSA							
d	w = 1	w = 2	w = 3	w = 4	G_s^*	G^*_{conv}	G_{up}^*
3	0.8184	0.9177	0.9179	0.9179	0.8184	0.9179	0.9405
4	0.7722	0.9708	0.9767	0.9767	0.7722	0.9767	0.9802
5	0.7017	0.9625	0.9914	0.9924	0.7017	0.9924	0.9930
6	0.6370	0.9258	0.9917	0.9970	0.6370	0.9973	0.9975

TABLE II PERCOLATION THRESHOLDS FOR THE CONVOLUTIONAL 2-FOLD ALOHA.

Convolutional 2-fold ALOHA							
d	w = 1	w = 2	w = 3	w = 4	G_s^*	G^*_{conv}	G_{up}^*
3	1.5528	1.9560	1.9760	1.9763	1.5528	1.9764	1.9790
4	1.3336	1.8966	1.9894	1.9961	1.3336	1.9964	1.9966
5	1.1577	1.7722	1.9639	1.9955	1.1577	1.9994	1.9995
6	1.0216	1.6326	1.9071	1.9833	1.0216	1.9999	1.9999

TABLE III Percolation thresholds for the convolutional 3-fold ALOHA.

Convolutional 3-fold ALOHA							
d	w = 1	w = 2	w = 3	w = 4	G_s^*	G^*_{conv}	G_{up}^*
3	2.1744	2.8990	2.9874	2.9916	2.1744	2.9918	2.9923
4	1.8108	2.7127	2.9577	2.9950	1.8108	2.9993	2.9994
5	1.5456	2.4744	2.8636	2.9739	1.5456	2.9999	3.0000
6	1.3487	2.2425	2.7240	2.9213	1.3487	2.9999	3.0000

These percolation thresholds are evaluated using the density evolution equation in (15). The columns w = 2, 3, 4 present the percolation thresholds $G^*_{conv}(40, w)$ for w = 2, 3, 4, respectively, evaluated using the equation in (50). Additionally, we numerically evaluate G^*_s , G^*_{conv} , and G^*_{up} by their definitions. As for the degree distribution, we set $\Lambda(x) = x^5$ to reproduce the numerical results presented in [27]. Table II and Table III show the corresponding results for the 2-fold ALOHA and the 3-fold ALOHA, respectively, with $P_{suc}(\rho)$ being defined in (5)).

From these tables, we observe that $G_s^* < G_{conv}^* < G_{up}^*$ holds. Also, though the condition of the saturation theorem is not satisfied, the gap between $G_{conv}^*(L,w)$ and G_{conv}^* is small. Specifically, the gap ranges from 0.0001 to 0.1. Note that $G_{conv}^*(L,w)$ and G_{up}^* in the first two rows of each table match very well with those in Table I of [32] and Table I of [27].

B. IRSA with two classes of users and two classes of receivers

In this section, we consider the IRSA system with two classes of users and two classes of receivers Example 3 with the spatially coupled applied. As in Example 3 and [25], we consider the following two packet routing policies:

1) Complete sharing: every packet has an equal probability to be routed to the two classes of receivers, i.e., $r_{11} = r_{22} = r_{12} = r_{21} = 0.5$.



Fig. 2. The boundaries of the stability region of the convolutional IRSA system with two classes of users and two classes of receivers under the complete sharing policy. The gray area presents the outer bound (41) for the complete sharing policy.



The stability region under the receiver reservation policy

Fig. 3. The boundaries of the stability region of the convolutional IRSA system with two classes of users and two classes of receivers under the receiver reservation policy. The gray area is the intersection of the outer bounds (43) and (44) for the receiver reservation policy.

2) Receiver reservation: class 1 packets are routed to the two classes of receivers with an equal probability, i.e., $r_{11} = r_{12} = 0.5$, and class 2 packets are routed to the class 2 receivers, i.e., $r_{21} = 0$, $r_{22} = 1$.

For our numerical validations, we set the number of iterations of (51) to 10,000. We set the number of stages L = 40and the window size w is set from 2 to 4. These numerical results are obtained from a grid search with a step size of $\delta = 0.01$ for both G_1 and G_2 . For the sake of numerical stability in our computation, we round up $p_{k,L/2}^{(10000)}(G_1,G_2)$ to 0 if its computed value is smaller than 10^{-5} .

In Figure 2 and Figure 3, we depict the stability region boundaries for two classes of users with two different degree distributions for w = 2, 3, 4. We choose $\Lambda_1(x) = x^5$ and $\Lambda_2(x) = 0.5102x^2 + 0.4898x^4$, where $\Lambda_2(x)$ is selected

from Table 1 of [10] to achieve a high percolation threshold of 0.868 in IRSA with a single class of users. The legend w = 1 denotes the conventional coded Poisson receiver (without convolution), reproducing the numerical results in [25]. The colored dashed lines represent the boundary of stability regions for different values of w. In both figures, the boundaries of the stability regions almost overlap for w = 2, 3, 4.

Furthermore, the shaded regions in both figures represent the outer bounds of the stability region evaluated in Example 3. Moreover, in both figures, we observe a significant enlargement of the stability region from w = 1 to w = 2, consistent with the findings of Theorem 13. Therefore, the spatial coupling effect can enlarge the stability region. This expansion continues as w monotonically increases from 2 to 4. Eventually, the boundaries of the expanded stability regions are close to their outer bounds. These empirical results align with the single-class scenario discussed in Section VI-A. For Figure 3, the reservation policy ($r_{21} = 0$) notably constrains the outer bound (44) for G_2 . Thus, the stability region is significantly limited in G_2 compared to that of G_1 .

VII. CONCLUSION

In this paper, we introduced a probabilistic framework known as convolutional coded Poisson receivers, extending the concept of coded Poisson receivers by drawing inspiration from convolutional LDPC codes. The main contributions of this paper are summarized as follows:

- (i) We established the outer bounds for the stability region of a system of CPRs in Theorem 9, applicable to both CPRs and CCPRs. These outer bounds are extended to encompass multiple traffic classes.
- (ii) We established the saturation theorem (Theorem 20) as a sufficient condition for CCPRs to demonstrate a higher percolation threshold compared to conventional CPRs. We validated the numerical results in Section VI-A.
- (iii) For the single-class user and receiver scenario of CCPRs, we employed the potential function to characterize three important thresholds, namely $G_s^* < G_{conv}^* < G_{up}^*$, as presented in Theorem 22.
- (iv) Our numerical results showed that the stability region of CCPRs might approach its outer bounds under finite iterations of the density evolution equations.

As a future direction, we are interested in exploring the parallel statement to Theorem 10 in [29]. Specifically, we aim to investigate whether

$$\lim_{w \to \infty} \lim_{L \to \infty} G^*_{conv}(L, w) = G^*_{conv}.$$

REFERENCES

- C.-E. Lee, H.-W. Yu, R. Zhang, C.-S. Chang, and D.-S. Lee, "Upper bounds for the stability regions of coded Poisson receivers," in 2023 IEEE International Symposium on Information Theory (ISIT). IEEE, 2023.
- [2] C.-E. Lee, K.-Y. Liao, Cheng-Shang, and D.-S. Lee, "Potential functions and percolation thresholds of coded Poisson receivers," in 2024 IEEE International Symposium on Information Theory (ISIT). IEEE, 2024.

- [3] C.-P. Li, J. Jiang, W. Chen, T. Ji, and J. Smee, "5G ultra-reliable and low-latency systems design," in *Networks and Communications* (*EuCNC*), 2017 European Conference on. IEEE, 2017, pp. 1–5.
- [4] M. Bennis, M. Debbah, and H. V. Poor, "Ultrareliable and low-latency wireless communication: Tail, risk, and scale," *Proceedings of the IEEE*, vol. 106, no. 10, pp. 1834–1853, 2018.
- [5] P. Popovski, Č. Stefanović, J. J. Nielsen, E. De Carvalho, M. Angjelichinoski, K. F. Trillingsgaard, and A.-S. Bana, "Wireless access in ultrareliable low-latency communication (URLLC)," *IEEE Transactions on Communications*, vol. 67, no. 8, pp. 5783–5801, 2019.
- [6] T.-K. Le, U. Salim, and F. Kaltenberger, "An overview of physical layer design for Ultra-Reliable Low-Latency Communications in 3GPP Releases 15, 16, and 17," *IEEE Access*, 2020.
- [7] P. Popovski, K. F. Trillingsgaard, O. Simeone, and G. Durisi, "5G wireless network slicing for eMBB, URLLC, and mMTC: A communication-theoretic view," *Ieee Access*, vol. 6, pp. 55765–55779, 2018.
- [8] A. Anand, G. De Veciana, and S. Shakkottai, "Joint scheduling of URLLC and eMBB traffic in 5G wireless networks," *IEEE/ACM Transactions on Networking*, vol. 28, no. 2, pp. 477–490, 2020.
- [9] E. Casini, R. De Gaudenzi, and O. D. R. Herrero, "Contention resolution diversity slotted ALOHA (CRDSA): An enhanced random access schemefor satellite access packet networks," *IEEE Transactions on Wireless Communications*, vol. 6, no. 4, 2007.
- [10] G. Liva, "Graph-based analysis and optimization of contention resolution diversity slotted ALOHA," *IEEE Transactions on Communications*, vol. 59, no. 2, pp. 477–487, 2011.
- [11] K. R. Narayanan and H. D. Pfister, "Iterative collision resolution for slotted ALOHA: An optimal uncoordinated transmission policy," in *Turbo Codes and Iterative Information Processing (ISTC), 2012 7th International Symposium on.* IEEE, 2012, pp. 136–139.
- [12] E. Paolini, G. Liva, and M. Chiani, "Random access on graphs: A survey and new results," in Signals, Systems and Computers (ASILOMAR), 2012 Conference Record of the Forty Sixth Asilomar Conference on. IEEE, 2012, pp. 1743–1747.
- [13] E. Paolini, C. Stefanović, G. Liva, and P. Popovski, "Coded random access: applying codes on graphs to design random access protocols," *IEEE Communications Magazine*, vol. 53, no. 6, pp. 144–150, 2015.
- [14] D. Jakovetić, D. Bajović, D. Vukobratović, and V. Crnojević, "Cooperative slotted aloha for multi-base station systems," *IEEE Transactions* on Communications, vol. 63, no. 4, pp. 1443–1456, 2015.
- [15] Z. Sun, Y. Xie, J. Yuan, and T. Yang, "Coded slotted ALOHA for erasure channels: Design and throughput analysis," *IEEE Transactions* on *Communications*, vol. 65, no. 11, pp. 4817–4830, 2017.
- [16] Č. Stefanović and D. Vukobratović, "Coded random access," in *Network Coding and Subspace Designs*. Springer, 2018, pp. 339–359.
- [17] R. Hoshyar, F. P. Wathan, and R. Tafazolli, "Novel low-density signature for synchronous CDMA systems over AWGN channel," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1616–1626, 2008.
- [18] H. Nikopour and H. Baligh, "Sparse code multiple access," in *Personal Indoor and Mobile Radio Communications (PIMRC), 2013 IEEE 24th International Symposium on.* IEEE, 2013, pp. 332–336.
- [19] Z. Yuan, G. Yu, W. Li, Y. Yuan, X. Wang, and J. Xu, "Multi-user shared access for Internet of Things," in *Vehicular Technology Conference* (*VTC Spring*), 2016 IEEE 83rd. IEEE, 2016, pp. 1–5.
- [20] S. Chen, B. Ren, Q. Gao, S. Kang, S. Sun, and K. Niu, "Pattern division multiple access—a novel nonorthogonal multiple access for fifth-generation radio networks," *IEEE Transactions on Vehicular Technology*, vol. 66, no. 4, pp. 3185–3196, 2017.
- [21] O. Ordentlich and Y. Polyanskiy, "Low complexity schemes for the random access Gaussian channel," in 2017 IEEE International Symposium on Information Theory (ISIT). IEEE, 2017, pp. 2528–2532.
- [22] A. Vem, K. R. Narayanan, J.-F. Chamberland, and J. Cheng, "A userindependent successive interference cancellation based coding scheme for the unsourced random access gaussian channel," *IEEE Transactions* on Communications, vol. 67, no. 12, pp. 8258–8272, 2019.
- [23] K. Andreev, E. Marshakov, and A. Frolov, "A polar code based TIN-SIC scheme for the unsourced random access in the quasi-static fading MAC," in 2020 IEEE International Symposium on Information Theory (ISIT). IEEE, 2020, pp. 3019–3024.
- [24] C.-H. Yu, L. Huang, C.-S. Chang, and D.-S. Lee, "Poisson receivers: a probabilistic framework for analyzing coded random access," *IEEE/ACM Transactions on Networking*, vol. 29, no. 2, pp. 862–875, 2021.
- [25] C.-M. Chang, Y.-J. Lin, C.-S. Chang, and D.-S. Lee, "On the stability regions of coded poisson receivers with multiple classes of users and receivers," *IEEE/ACM Transactions on Networking*, 2022.

- [26] A. Ashikhmin, G. Kramer, and S. ten Brink, "Extrinsic information transfer functions: model and erasure channel properties," *IEEE Transactions on Information Theory*, vol. 50, no. 11, pp. 2657–2673, 2004.
- [27] Č. Stefanović, E. Paolini, and G. Liva, "Asymptotic performance of coded slotted ALOHA with multipacket reception," *IEEE Communications Letters*, vol. 22, no. 1, pp. 105–108, 2017.
- [28] T.-H. Liu, C.-H. Yu, Y.-J. Lin, C.-M. Chang, C.-S. Chang, and D.-S. Lee, "ALOHA receivers: a network calculus approach for analyzing coded multiple access with SIC," *IEEE/ACM Transactions on Networking*, vol. 29, no. 2, pp. 862–875, 2021.
- [29] S. Kudekar, T. J. Richardson, and R. L. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Transactions on Information Theory*, vol. 57, no. 2, pp. 803–834, 2011.
- [30] A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, "A simple proof of threshold saturation for coupled scalar recursions," in 2012 7th International Symposium on Turbo Codes and Iterative Information Processing (ISTC). IEEE, 2012, pp. 51–55.
- [31] D. G. Mitchell, M. Lentmaier, and D. J. Costello, "Spatially coupled LDPC codes constructed from protographs," *IEEE Transactions on Information Theory*, vol. 61, no. 9, pp. 4866–4889, 2015.
- [32] G. Liva, E. Paolini, M. Lentmaier, and M. Chiani, "Spatially-coupled random access on graphs," in 2012 IEEE International Symposium on Information Theory Proceedings. IEEE, 2012, pp. 478–482.
- [33] M. Xu, D. Zeng, Z. Sheng, Z. Zhang, and C. Xu, "Some results on density evolution of nonbinary sc-ldpc ensembles over the bec," *IEEE International Symposium on Information Theory (ISIT)*, vol. 90, pp. 90–95, 2021.
- [34] P. S. N. A. Yedla, Y. Y. Jian and H. D. Pfister, "A simple proof of threshold saturation for coupled vector recursions," 2012 IEEE Information Theory Workshop, pp. 25–29, 2012.
- [35] C. Schlegel and M. Burnashev, "Thresholds of spatially coupled systems via lyapunov's method," 2013 IEEE Information Theory Workshop (ITW), pp. 1–5, 2013.
- [36] N. Abramson, "THE ALOHA SYSTEM: another alternative for computer communications," in *Proceedings of the November 17-19, 1970*, *fall joint computer conference.* ACM, 1970, pp. 281–285.
- [37] Č. Stefanović, M. Momoda, and P. Popovski, "Exploiting capture effect in frameless ALOHA for massive wireless random access," in 2014 *IEEE Wireless Communications and Networking Conference (WCNC)*. IEEE, 2014, pp. 1762–1767.
- [38] F. Clazzer, E. Paolini, I. Mambelli, and Č. Stefanović, "Irregular repetition slotted ALOHA over the Rayleigh block fading channel with capture," in 2017 IEEE International Conference on Communications (ICC). IEEE, 2017, pp. 1–6.
- [39] M. Ghanbarinejad and C. Schlegel, "Irregular repetition slotted ALOHA with multiuser detection," in 2013 10th Annual Conference on Wireless On-demand Network Systems and Services (WONS). IEEE, 2013, pp. 201–205.
- [40] A. Glebov, N. Matveev, K. Andreev, A. Frolov, and A. Turlikov, "Achievability bounds for T-fold irregular repetition slotted ALOHA scheme in the Gaussian MAC," in 2019 IEEE Wireless Communications and Networking Conference (WCNC). IEEE, 2019, pp. 1–6.
- [41] Z. Chen, Y. Feng, C. Feng, L. Liang, Y. Jia, and T. Q. Quek, "Analytic distribution design for irregular repetition slotted ALOHA with multipacket reception," *IEEE Transactions on Vehicular Technology*, 2022.
- [42] M. Luby, M. Mitzenmacher, and M. A. Shokrollahi, "Analysis of random processes via and-or tree evaluation," in SODA, vol. 98, 1998, pp. 364–373.
- [43] E. Paolini, G. Liva, and M. Chiani, "Graph-based random access for the collision channel without feedback: Capacity bound," in 2011 IEEE Global Telecommunications Conference-GLOBECOM 2011. IEEE, 2011, pp. 1–5.
- [44] X. Shao, Z. Sun, M. Yang, S. Gu, and Q. Guo, "NOMA-based irregular repetition slotted ALOHA for satellite networks," *IEEE Communications Letters*, vol. 23, no. 4, pp. 624–627, 2019.
- [45] Y.-C. Huang, S.-L. Shieh, Y.-P. Hsu, and H.-P. Cheng, "Iterative collision resolution for slotted ALOHA with NOMA for heterogeneous devices," *IEEE Transactions on Communications*, vol. 69, no. 5, pp. 2948–2961, 2021.
- [46] M. Fernández-Veiga, M. Sousa-Vieira, A. Fernández-Vilas, and R. P. Dáz-Redondo, "Irregular repetition slotted aloha with multiuser detection: A density evolution analysis," *Computer Networks*, 2023.
- [47] M. Luby, M. Mitzenmacher, A. Shokrollah, and D. Spielman, "Analysis of low density codes and improved designs using irregular graphs," in *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, 1998, pp. 249–258.

- [48] T. J. Richardson and R. L. Urbanke, "The capacity of low-density paritycheck codes under message-passing decoding," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 599–618, 2001.
- [49] F. P. Kelly, *Reversibility and stochastic networks*. Cambridge University Press, 2011.
- [50] J. Walrand, "A probabilistic look at networks of quasi-reversible queues," *IEEE Transactions on Information Theory*, no. 6, pp. 825– 831, 1983.
- [51] F. P. Kelly, "Loss networks," *The annals of applied probability*, pp. 319–378, 1991.



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APPENDIX A Proof of Lemma 10

Proof. Consider a particular Poisson receiver. During the SIC decoding process, we assume that there are $n_{k\ell} > 0$ class k_{ℓ} packets that are decoded during the i_{ℓ} -th iteration, $\ell = 1, 2, \ldots, M$ for some M. Without loss of generality, we assume that $i_1 \leq i_2 \leq \ldots \leq i_M$. Let \mathbf{e}_k be the $1 \times K$ vector with its k-th element being 1 and 0 otherwise. Thus, the number of packets decoded by this receiver can be represented by the vector $\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}$. To prove this lemma, we need to show that the decoded packets by this Poisson receiver is within the capacity region of the ϕ -ALOHA receiver, i.e.,

$$\phi(\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}) = \sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}$$
(102)

This is equivalent to showing that

$$\phi_{k_{\ell}}^{c}(\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}) = 0, \qquad (103)$$

for $\ell = 1, 2, \dots, M$. From the all-or-nothing property, we know that $M \leq K$ as packets of the same class are decoded in the same iteration.

We prove (103) by induction. Suppose that there are $n^{(i)} = (n_1^{(i)}, n_2^{(i)}, \dots, n_K^{(i)})$ remaining packets at the receiver during the *i*-th iteration. As the number of decoded class *k* packets cannot be larger than the number of remaining class *k* packets, we have

$$n_{k_1} \mathbf{e}_{k_1} \leqslant n^{(i_1)}.$$
 (104)

As there are $n_{k_1} > 0$ class k_1 packets that are decoded during the i_1 -th iteration, we have from the monotone property and the all-or-nothing property that

$$\phi_{k_1}^c(\sum_{\ell=1}^M n_{k_\ell} \mathbf{e}_{k_\ell}) \leqslant \phi_{k_1}^c(n^{(i_1)}) = 0.$$
 (105)

Now assume that (103) holds for $\ell = 1, 2, ..., m$ as the induction hypothesis. From the induction hypothesis, we have

$$\phi^{c}(\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}) \leq \sum_{\ell=m+1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}.$$
 (106)

From the closure property, the monotone property and (106), we know that

$$\phi_{k_{m+1}}^{c} \left(\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}\right) = \phi_{k_{m+1}}^{c} \left(\phi^{c} \left(\sum_{\ell=1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}\right)\right)$$
$$\leqslant \phi_{k_{m+1}}^{c} \left(\sum_{\ell=m+1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}\right).$$
(107)

As the number of decoded class k packets cannot be larger than the number of remaining class k packets, we have at the i_{m+1} -th iteration that

$$\sum_{\ell=m+1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}} \leqslant n^{(i_{m+1})}. \tag{108}$$

It follows from the monotone property and the all-or-nothing property that

$$\phi_{k_{m+1}}^{c} \left(\sum_{\ell=m+1}^{M} n_{k_{\ell}} \mathbf{e}_{k_{\ell}}\right) \leqslant \phi_{k_{m+1}}^{c} \left(n^{(i_{m+1})}\right) = 0.$$
(109)

Thus, $\phi_{k_{m+1}}^c(\sum_{\ell=1}^M n_{k_\ell} \mathbf{e}_{k_\ell}) = 0$, and we complete the induction.

APPENDIX B DENSITY EVOLUTION FOR THE CIRCULAR CONVOLUTIONAL CPR IN COROLLARY 11

In this appendix, we conduct the density evolution analysis for the circular convolutional CPR with L stages. The analysis is similar to that in [25] that uses the density evolution method in [10], [12], [42], [43], [47], [48] and the reduced Poisson offered load argument in [24], [49]–[51]. For our analysis, we call an edge a class k edge if the user end of the edge is connected to a class k user. Also, we call an edge a class (k, j)-edge if the receiver (resp. user) end of the edge is connected to a class j receiver (resp. class k user).

The density evolution analysis consists of the following steps:

(i) The initial offered load of class k packets to a class j Poisson receiver in the ℓ^{th} stage, defined as the expected number of class k packets transmitted to that receiver, is

$$\rho_{k,j,\ell} = \sum_{\hat{\ell}=\ell \ominus (w-1)}^{\ell} G_k^{(\hat{\ell})} \Lambda_k'(1) \frac{1}{w} r_{k,j} / F_j.$$
(110)

To see (110), note that (a) there are $G_k^{(\hat{\ell})}T$ class k users in the $\hat{\ell}^{th}$ stage, (b) each class k user transmits on average $\Lambda'_k(1)$ copies, (c) each copy in the $\hat{\ell}^{th}$ stage (with $\hat{\ell} \in [\ell \ominus (w-1), \ell]$), is sent to class j Poisson receivers in the ℓ^{th} stage with the routing probability $r_{k,j}/w$, and (d) a copy sent to class j Poisson receivers is uniformly distributed among the F_jT class j Poisson receivers. When T goes to infinity, the number of class k packets at a class j receiver in the ℓ^{th} stage converges (from a binomial random variable) to a Poisson random variable with mean $\rho_{k,j,\ell}$, and the degree distribution of class k packets at a class j receiver node is a Poisson distribution with mean $\rho_{k,j,\ell}$.

(ii) Let $q_{k,\ell}^{(i)}$ be the probability that the *user end* of a randomly selected class k edge in the ℓ^{th} stage has not been successfully received after the i^{th} SIC iteration. The offered load of class k packets to a class j Poisson receiver in the ℓ^{th} stage after the i^{th} SIC iteration has a Poisson distribution with mean $\rho_{k,j,\ell}^{(i)}$ (from the reduced offered load argument), where

$$\rho_{k,j,\ell}^{(i)} = \sum_{\hat{\ell}=\ell \ominus (w-1)}^{\ell} q_{k,\hat{\ell}}^{(i)} G_k^{(\hat{\ell})} \Lambda_k'(1) \frac{1}{w} r_{k,j} / F_j.$$
(111)

Let

$$\tilde{\rho}_{j,\ell}^{(i)} = (\rho_{1,j,\ell}^{(i)}, \rho_{2,j,\ell}^{(i)}, \dots, \rho_{K,j,\ell}^{(i)}).$$
(112)

Note that one can represent the offered load at a class j Poisson receiver in the ℓ^{th} stage after the i^{th} SIC iteration by the vector $\tilde{\rho}_{j,\ell}^{(i)}$.

(iii) Let $p_{k,j,\ell}^{(i+1)}$ be the probability that the *receiver end* of a randomly selected class (k,j)-edge in the ℓ^{th} stage has not been successfully received after the $(i+1)^{th}$ SIC iteration. Then

$$p_{k,j,\ell}^{(i+1)} = 1 - P_{\text{suc},k,j}(\tilde{\rho}_{j,\ell}^{(i)}).$$
(113)

That (113) holds follows directly from the definition of a Poisson receiver in Definition 1 as the offered load at a class j Poisson receiver in the ℓ^{th} stage after the i^{th} SIC iteration is $\tilde{\rho}_{j,\ell}^{(i)}$.

(iv) Let $p_{k,\ell}^{(i+1)}$ be the probability that the *receiver end* of a randomly selected class k edge in the ℓ^{th} stage has not been successfully received after the $(i+1)^{th}$ SIC iteration. Since a class k edge in the ℓ^{th} stage is a class (k, j)-edge in the ℓ^{th} stage (for $\hat{\ell} \in [\ell, \ell \oplus (w-1)]$) with probability $r_{k,j}/w$, it follows that

$$p_{k,\ell}^{(i+1)} = \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^{J} \frac{1}{w} r_{k,j} p_{k,j,\hat{\ell}}^{(i+1)}$$
$$= 1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^{J} \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i)}).$$
(114)

(v) The probability $q_{k,\ell}^{(i)}$ can be computed recursively from the following equation:

$$q_{k,\ell}^{(i+1)} = \lambda_k (1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^J \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i)})), \quad (115)$$

with $q_{k,\ell}^{(0)} = 1$. To see this, note that a packet sent from a user (the user end of the bipartite graph) can be successfully received if at least one of its copies is successfully received at the *receiver* end. Since the probability that the user end of a randomly selected class k edge has additional d edges is $\lambda_{k,d}$, the probability that the *user* end of a randomly selected class k edge cannot be successfully received after the $(i+1)^{th}$ iteration is

$$q_{k,\ell}^{(i+1)} = 1 - \sum_{d=0}^{\infty} \lambda_{k,d} \cdot \left(1 - (p_{k,\ell}^{(i+1)})^d\right) \\ = \lambda_k (p_{k,\ell}^{(i+1)}).$$
(116)

Using (114) in (116) yields (115).

(vi) Let $\tilde{P}_{\text{suc},k,\ell}^{(i)}$ be the probability that a packet sent from a randomly selected *class k user* in the ℓ^{th} stage can be successfully received after the i^{th} iteration. Such a probability is the probability that at least one copy of the packet has been successfully received after the i^{th} iteration. Since the probability that a randomly selected *class* k user has d edges is $\Lambda_{k,d}$, we have from (114) that

$$\tilde{P}_{\mathrm{suc},k,\ell}^{(i)} = \sum_{d=0}^{\infty} \Lambda_{k,d} \cdot \left(1 - (p_{k,\ell}^{(i)})^d\right) \\
= 1 - \Lambda_k \left(1 - \sum_{\hat{\ell}=\ell}^{\ell \oplus (w-1)} \sum_{j=1}^J \frac{1}{w} r_{k,j} P_{\mathrm{suc},k,j}(\tilde{\rho}_{j,\hat{\ell}}^{(i-1)})\right).$$
(117)

APPENDIX C Proof of Theorem 20

The proof of Theorem 20 requires several lemmas similar to those in [30] using the notations employed in this paper. Though these lemmas may not have a direct physical interpretation, they are crucial for establishing the proof of Theorem 20.

Let $\tilde{L} = L - w + 1$. We first reverse the indices of the recursion equation (79). Let $\tilde{p}_{\ell}^{(i)} = p_{\tilde{L}-\ell+1}^{(i)}$, (79) becomes

$$\tilde{p}_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\tilde{L}-\ell+1}^{\tilde{L}-\ell+w} \frac{1}{w} P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \right)$$
$$\sum_{\tilde{\ell}=\max[1,\hat{\ell}-w+1]}^{\min[\tilde{L},\hat{\ell}]} \lambda(\tilde{p}_{\tilde{L}-\tilde{\ell}+1}^{(i)}) .$$
(118)

Let $\tilde{\ell}' = \tilde{L} - \tilde{\ell} + 1$. The upper and lower indices of the second summation could be changed into

and

$$\tilde{\ell}' = \max[1, \tilde{L} - \hat{\ell} + 1].$$

 $\tilde{\ell}' = \min[\tilde{L}, \tilde{L} - \hat{\ell} + w]$

Note that the latter becomes not larger than the former in this case. Thus, (118) further becomes

$$\tilde{p}_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} \frac{1}{w} P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}'=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+w]} \lambda(\tilde{p}_{\tilde{\ell}'}^{(i)}) \right)$$
(119)

Hence, reversing the indices, the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR with L stages with one class of users is governed by the following recursion equation:

$$p_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} \frac{1}{w} P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+w]} \lambda(p_{\tilde{\ell}}^{(i)}) \right)$$
(120)

where the initial condition is $\mathbf{p}^{(0)} = (1, 1, \dots, 1)$.

A $(T, G, \Lambda(x), R, F)$ -CPR is a kind of scalar admissible system characterized by a pair of functions (f, h), which can be represented as a bipartite graph. If we construct a large bipartite graph as the construction of convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR systems in Section IV, we obtain a basic spatially-coupled system in [30], which is also parametrized by a pair of functions (f, h).

Definition 23: (The basic spatially-coupled system (cf. Def. 10 in [30]) The basic spatially-coupled system is defined by concatenating L bipartite graphs of a scalar admissible system. The edges in the concatenated bipartite graphs are rewired to form a single bipartite graph as follows: The receiver end of each edge in the ℓ -th copy is rewired to the corresponding receiver node in the $\hat{\ell}$ -th copy, where $\hat{\ell}$ is chosen uniformly in $[\ell, \ell \oplus (w-1)]$. Then, we reverse the index. Additionally, the recursion is defined as:

$$p_{\ell}^{(i+1)} = \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\bigg(\frac{1}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+w]} h(p_{\tilde{\ell}}^{(i)});G\bigg),$$
(121)

for $\ell = 1, 2, \dots, L - w + 1$. Also, conditions (P1)-(P4) in Definition 14 should be satisfied.

Define $\mathbf{h}(\mathbf{p}) = (h(p_1), h(p_2), \dots, h(p_{\tilde{L}}))$ and $\mathbf{f}(\mathbf{p}; G) =$ $(f(p_1;G), f(p_2;G), \ldots, f(p_{\tilde{L}};G))$, then the vector recursion of (121) is given by:

$$\mathbf{p}^{(i+1)} = \mathbf{A}_2 \mathbf{f}(\mathbf{A}_2^T \mathbf{h}(\mathbf{p}^{(i)}); G), \qquad (122)$$

where \mathbf{A}_2 is an $\tilde{L} \times L$ matrix defined as

$$\mathbf{A}_{2} = \frac{1}{w} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \end{bmatrix}, \quad (123)$$

and all the columns of \mathbf{A} contain w ones.

Next, by examining the conditions in Definition 23, we prove that CCPRs are one kind of basic spatially-coupled system.

Lemma 24: The convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR system is a basic spatially-coupled system. That is, for $\ell =$ $1, 2, \ldots, L - w + 1,$

$$p_{\ell}^{(i+1)} = \frac{1}{w} \sum_{\hat{\ell} = \tilde{L} + 1 - \ell}^{\tilde{L} + w - \ell} \left(1 - P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell} = \max[1, \tilde{L} - \hat{\ell} + 1]}^{\min[\tilde{L}, \tilde{L} - \hat{\ell} + w]} \lambda(p_{\tilde{\ell}}^{(i)}) \right)$$

Proof. First, rearranging (120),

$$p_{\ell}^{(i+1)} = 1 - \sum_{\hat{\ell}=\hat{L}+1-\ell}^{\tilde{L}+w-\ell} \frac{1}{w} P_{\text{suc}} \\ \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+w]} \lambda(p_{\tilde{\ell}}^{(i)})\right).$$
(125)

Then, insert 1 into the summation,

$$p_{\ell}^{(i+1)} = \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} \left(\frac{1}{w} - \frac{1}{w} P_{\text{suc}}\left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+w]} \lambda(p_{\tilde{\ell}}^{(i)})\right)\right).$$
(126)

Taking 1/w out from the summation,

$$p_{\ell}^{(i+1)} = \frac{1}{w} \sum_{\hat{\ell} = \tilde{L} + 1 - \ell}^{\tilde{L} + w - \ell} \left(1 - P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell} = \max[1, \tilde{L} - \hat{\ell} + 1]}^{\min[\tilde{L}, \tilde{L} - \hat{\ell} + w]} \lambda(p_{\tilde{\ell}}^{(i)}) \right) \right).$$
(127)

Assign

$$f(p;G) = 1 - P_{\text{suc}}(pG\Lambda'(1)),$$
 (128)

$$h(p) = \lambda(p). \tag{129}$$

By Lemma 15, conditions (1)-(4) of Definition 14 are satisfied.

Now we introduce the vector one-sided spatially-coupled recursion system (briefly, a one-sided system in this paper) in [30] and [34]. It is also parametrized by a pair of functions (f, h). Hence, there exists a correspondence between onesided systems, basic spatially-coupled systems, and scalar admissible systems if they are characterized by the same pair of functions (f, h).

Definition 25: (The one-sided system (cf. Def. 10 in [30]) The one-sided system is defined by the recursion system:

$$p_{\ell}^{(i+1)} = \frac{1}{w} \sum_{\hat{\ell}=\max[1,\ell-w+1]}^{\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} h(p_{\tilde{\ell}}^{(i)});G\right),$$
(130)

where $\mathbf{p} \in [0, 1]^L$. Conditions (P1)-(P4) in Definition 14 should be satisfied. The vector recursion form of (130) is given by:

(124)
$$\mathbf{p}^{(i+1)} = \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p}^{(i)}); G), \qquad (131)$$

where **A** is an $L \times L$ matrix defined as:

$$\mathbf{A} = \frac{1}{w} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix},$$
(132)

where the 1st, 2nd, ..., $(\tilde{L} - w + 1)$ -th rows of A contain w ones.

Next, we give two lemmas for the one-sided system. The first one illustrates that the one-sided system gives a component-wise upper bound of the basic spatially-coupled system with a boundary condition $s_{\ell}^{(i)} = s_{\tilde{L}}^{(i)}$ for $\ell = \tilde{L}, \tilde{L} + 1, \ldots, L$ after each iteration. The second one shows that $\mathbf{p}^{(i)}$ in (131) is a non-decreasing vector.

Lemma 26: Consider a column vector $\mathbf{p}^{(i)} = (p_1^{(i)}, p_2^{(i)}, \dots, p_{\tilde{L}}^{(i)})$ and a basic spatially-coupled system

$$\mathbf{p}^{(i+1)} = \mathbf{A}_2 \mathbf{f} (\mathbf{A}_2^T \mathbf{h} (\mathbf{p}^{(i)}); G)$$

with the initial condition $\mathbf{p}^{(0)} = (1, 1, \dots, 1)$. Let $\mathbf{s}^{(i)}$ be a vector of length L defined by the one-sided system

$$\mathbf{s}^{(i+1)} = \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{s}^{(i)}); G)$$

with the initial condition

$$\mathbf{s}^{(0)} = (1, 1, \dots, 1).$$
 (133)

If we enforce $s_{\ell}^{(i)} = s_{\tilde{L}}^{(i)}$ for $\ell = \tilde{L}, \tilde{L} + 1, \dots, L$ after each iteration, we have

$$p_{\ell}^{(i)} \leq s_{\tilde{L}+w-\ell}^{(i)}, \quad \forall \ell = 1, 2, \dots, \tilde{L}.$$
 (134)

Thus, the percolation threshold of the basic spatially-coupled system is not smaller than that of the one-sided system.

Proof. We prove this lemma by induction. The initial condition (133) shows that $p_{\ell}^{(0)} = s_{\tilde{L}+w-\ell}^{(0)} = 1$ for $\ell = 1, \ldots, \tilde{L}$. Thus, (134) holds for i = 0. Next, by (P1), (P2), and the induction hypothesis, we have

$$p_{\ell}^{(i+1)} = \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[L,L-\hat{\ell}+w]} h(p_{\tilde{\ell}}^{(i)});G\right)$$

$$\leqslant \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\max[1,\tilde{L}-\hat{\ell}+1]}^{\min[\tilde{L},\tilde{L}-\hat{\ell}+1]} h(s_{\tilde{L}+w-\tilde{\ell}}^{(i)});G\right).$$
(135)

If we change the subscript by letting $\tilde{\ell}' = \tilde{L} + w - \tilde{\ell}$, then the upper bound of (135) could be further evaluated

$$p_{\ell}^{(i+1)} \leqslant \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}'=\max[\hat{\ell},w]}^{\min[\hat{\ell}+w-1,L]} h(s_{\tilde{\ell}'}^{(i)});G\right)$$

$$= \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\max[\hat{\ell},w]}^{\min[\hat{\ell}+w-1,L]} h(s_{\tilde{\ell}}^{(i)});G\right)$$

$$\leqslant \frac{1}{w} \sum_{\hat{\ell}=\tilde{L}+1-\ell}^{\tilde{L}+w-\ell} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} h(s_{\tilde{\ell}}^{(i)});G\right).$$

$$= s_{\tilde{L}+w-\ell}^{(i+1)}.$$
(136)

The first equality holds since we change the indices of the second summation. The second inequality follows from (P1) and the fact that more terms are included. The last equality follows from (130).

Hence, for $\ell = 1, 2, ..., \tilde{L}$, (134) also holds for the (i + 1)-th iteration. This completes the proof.

Lemma 26 implies that once the saturation theorem holds for the one-sided system, it automatically holds for the basic spatially-coupled system as well.

Lemma 27: Consider the one-sided system

$$\mathbf{p}^{(i+1)} = \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p}^{(i)}); G)$$

with the initial condition

$$\mathbf{p}^{(0)} = (1, 1, \dots, 1). \tag{137}$$

If we enforce $p_{\ell}^{(i)} = p_{\tilde{L}}^{(i)}$ for $\ell = \tilde{L}, \tilde{L} + 1, \dots, L$ after each iteration, then the vector $\mathbf{p}^{(i)}$ is non-decreasing, say

$$p_0^{(i)} \leqslant p_1^{(i)} \leqslant p_2^{(i)} \leqslant \ldots \leqslant p_{\tilde{L}}^{(i)} = p_{\tilde{L}+1}^{(i)} = \ldots = p_L^{(i)} \quad (138)$$

for all i.

Proof. For i = 0, (138) holds by the initial condition (137). Suppose that (138) holds for the *i*-th iteration, we shall examine (138) for the (i + 1)-th iteration. First, by (130), we have

$$p_{\ell+1}^{(i+1)} - p_{\ell}^{(i+1)} = \frac{1}{w} \left(f\left(\frac{1}{w} \sum_{\tilde{\ell} = \ell+1}^{\min[L,\ell+w]} h(p_{\tilde{\ell}}^{(i)}); G\right) - f\left(\frac{1}{w} \sum_{\tilde{\ell} = \ell-w+1}^{\min[\ell,L]} h(p_{\tilde{\ell}}^{(i)}); G\right) \right).$$
(139)

Case I. $\ell < w$: By (P1) and (P2)

$$p_{\ell+1}^{(i+1)} - p_{\ell}^{(i+1)} = \frac{1}{w} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\ell+1}^{\ell+w} h(p_{\tilde{\ell}}^{(i)}); G\right) > 0.$$
(140)

(135) Case II. $w < \ell < \tilde{L}$:

$$p_{\ell+1}^{(i+1)} - p_{\ell}^{(i+1)} = \frac{1}{w} \left(f\left(\frac{1}{w} \sum_{\tilde{\ell}=\ell+1}^{\ell+w} h(p_{\tilde{\ell}}^{(i)}); G\right) - f\left(\frac{1}{w} \sum_{\tilde{\ell}=\ell-w+1}^{\ell} h(p_{\tilde{\ell}}^{(i)}); G\right) \right) > 0.$$
(141)

The inequality follows from (P1), (P2), and the induction hypothesis.

Case III. $\ell \ge \tilde{L}$: By the enforcement we make, $p_{\ell+1}^{(i+1)} - p_{\ell}^{(i+1)} = 0$

Thus, (138) holds for all i.

Following Lemma 2 in [30] and Lemma 14 in [29], we introduce a lemma that serves as the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPR version of Lemma 5 in [24].

Lemma 28: Consider the recursion equation (130) with the condition that $p_{\ell}^{(i)} = p_{\tilde{L}}^{(i)}$ is enforced for $\ell = \tilde{L}, \tilde{L} + 1, \ldots, L$ after each iteration. If f and h is given by (63), then $p_{\ell}^{(i+1)} \leq p_{\ell}^{(i)}$ for all $\ell = 1, \ldots L$ and all positive integer *i*. Also, (130) converges to a well-defined fixed point, say

$$\lim_{i \to \infty} p^{(i)} := p^{(\infty)} = (p_1^{(\infty)}, \dots, p_L^{(\infty)}).$$

Moreover, $p^{(\infty)}$ represents the (element-wise) largest solution among all solutions in $[0,1]^L$. In other words, if $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_L)$ is another solution, then $\hat{p}_\ell = p_\ell^{(\infty)}$, for $\ell = 1, \ldots, L$. Furthermore, as a result of this and Lemma 26, the density evolution equation (120) of CCPRs also converges to a well-defined fixed point.

Proof. We begin by showing that $p_{\ell}^{(i)}$ is a decreasing sequence. Since $p_{\ell}^{(0)} = 1$, then

$$\begin{split} p_{\ell}^{(1)} &= \frac{1}{w} \sum_{\hat{\ell} = \max[1, \ell - w + 1]}^{\ell} \left(1 - \right. \\ & P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell} = \hat{\ell}}^{\min[\hat{\ell} + w - 1, L]} \lambda(1) \right) \right) \\ & \leqslant \frac{1}{w} \sum_{\hat{\ell} = \max[1, \ell - w + 1]}^{\ell} \left(1 - P_{\text{suc}}(G\Lambda'(1)) \right) \\ & = 1 - P_{\text{suc}}(G\Lambda'(1)) \\ & < p_{\ell}^{(0)} = 1. \end{split}$$

Suppose that $p_\ell^{(i+1)} < p_\ell^{(i)}$, by (P2) and that $P_{\rm suc}$ is decreasing,

$$\begin{split} &P_{\mathrm{suc}}\bigg(\frac{G\Lambda'(1)}{w}\sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]}\lambda(p_{\tilde{\ell}}^{(i+1)})\bigg) \\ &> P_{\mathrm{suc}}\bigg(\frac{G\Lambda'(1)}{w}\sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]}\lambda(p_{\tilde{\ell}}^{(i)})\bigg), \end{split}$$

for all $\hat{\ell} = 1, \ldots, L$.

Multiplying the above inequality by -1 and adding 1 to each side, then taking the summation, we have

$$\sum_{\hat{\ell}=\max[1,\ell-w+1]}^{\ell} \left(1 - P_{\text{suc}}\left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} \lambda(p_{\tilde{\ell}}^{(i+1)})\right)\right)$$
$$< \sum_{\hat{\ell}=\max[1,\ell-w+1]}^{\ell} \left(1 - P_{\text{suc}}\left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} \lambda(p_{\tilde{\ell}}^{(i)})\right)\right)$$

Multiplying by 1/w gives that $p_{\ell}^{(i+2)} < p_{\ell}^{(i+1)}$. Hence, by induction, $p_{\ell}^{(i)}$ is a decreasing sequence. Moreover, since $p_{\ell}^{(i)}$ itself is a probability, it is bounded below by 0. Hence $p^{(\infty)}$ exists and is the fixed point of (130).

Suppose $\hat{p} = (\hat{p}_1, \dots, \hat{p}_L)$ is another solution of (130). It is obvious that $\hat{p} \leq 1 = p_\ell^{(0)}$ for all ℓ . Suppose $\hat{p}_\ell \leq p_\ell^{(i)}$

for some $i \ge 1$, using the fact that λ is increasing and the assumption that P_{suc} is decreasing, from (130), we have

$$\begin{split} \hat{p}_{\ell} &= \frac{1}{w} \sum_{\hat{\ell}=\max[1,\ell-w+1]}^{\ell} \left(1 \\ &- P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} \lambda(\hat{p}_{\ell}) \right) \right) \\ &\leqslant \frac{1}{w} \sum_{\hat{\ell}=\max[1,\ell-w+1]}^{\ell} \left(1 \\ &- P_{\text{suc}} \left(\frac{G\Lambda'(1)}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\hat{\ell}+w-1,L]} \lambda(p_{\tilde{\ell}}^{(i)}) \right) \right) \\ &= p_{\ell}^{(i+1)} \end{split}$$

Hence, $\hat{p} \leq p_{\ell}^{(i)}, \forall i$. Taking $i \to \infty$, we have $\hat{p}_{\ell} \leq p_{\ell}^{(\infty)}$ for all ℓ .

The next lemma further illustrates the proposition of the fixed point of the one-sided system.

Lemma 29: Let $\mathbf{p} = (p_1, p_2, \dots, p_L)$ be a fixed point of (130). Suppose $\mathbf{p} \neq \mathbf{0}$ for $\ell = 1, 2, \dots, L$, then the system has no fixed point with $p_L < u(G)$.

Proof. By Lemma 27 and Lemma 28, such fixed point is non-decreasing, say $p_1 \leq p_2 \leq \ldots \leq p_L$. Thus, (130) gives that,

$$p_L = \frac{1}{w} \sum_{\hat{\ell}=L-w+1}^{L} f\left(\frac{1}{w} \sum_{\tilde{\ell}=\hat{\ell}}^{\min[\ell+w-1,L]} h(p_{\tilde{\ell}});G\right)$$
$$\leq \frac{1}{w} \sum_{\hat{\ell}=L-w+1}^{L} f\left(\frac{1}{w} \cdot w \cdot h(p_L);G\right)$$
$$= \frac{1}{w} \cdot w \cdot f(h(p_L);G)$$
(142)

Hence, $f(h(p_L); G) - p_L \ge 0$. By (75), we have that

$$U'(p_L;G) = \lambda'(p_L)(p_L - 1 + P_{suc}(G\Lambda'(p_L))) \le 0.$$
 (143)

Thus, by Definition 19, $p_L > u(G)$.

Additionally, similar to the potential function of the scalar admissible system (f, h) in (68), we define the potential function of the one-sided system below.

Definition 30: (The potential function of one-sided systems (cf. Def. 11 in [30]) The potential function of the onesided system describe in Definition 25 is the line integral along a curve C in $\mathbb{R}^{\tilde{L}}$ joining 0 and $\mathbf{p} = (p_1, p_2, \dots, p_{\tilde{L}})$,

$$U(\mathbf{p};G) = \int_{C} \mathbf{h}'(\mathbf{z})(\mathbf{z} - \mathbf{A}^{T}\mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{z});G))d\mathbf{z}, \qquad (144)$$

where $\mathbf{h}'(\mathbf{p}) = \text{diag}([h'(p_{\ell})])$. Let

$$H(\mathbf{p}) = \int_C \mathbf{h}(\mathbf{z}) d\mathbf{z} = \sum_{\ell=1}^L H(p_\ell)$$

and

$$F(\mathbf{p};G) = \int_C \mathbf{f}(\mathbf{p};G) d\mathbf{p} = \sum_{\ell=1}^L F(p_\ell;G).$$

Then, (144) could also be written in the form

$$U(\mathbf{p};G) = \mathbf{h}(\mathbf{p})^T \mathbf{p} - H(\mathbf{p}) - F(\mathbf{Ah}(\mathbf{p});G).$$
(145)

The key insight is that the potential functions of the onesided system and the scalar admissible system are connected through a shift operator. The advantage of this approach is that it eliminates the need for evaluating the line integral in (145) for exploring the properties of the stability region of the convolutional $(T, G, \Lambda(x), R, F, w)$ -CPRs.

Lemma 31: (Lemma 3, in [30]) Let $\mathbf{p} = (p_1, \ldots, p_L) \in [0, 1]^L$ be a non-decreasing vector generated by averaging $\mathbf{q} \in [0, 1]^L$ over a sliding window of size w. Let the shift operator $\mathbf{S} : \mathbb{R}^L \to \mathbb{R}^L$ be defined by $[\mathbf{Sp}] = (0, p_1, p_2, \ldots, p_{L-1})$. Then,

$$||\mathbf{Sp} - \mathbf{p}||_{\infty} < \frac{1}{w}$$

and

$$||\mathbf{Sp} - \mathbf{p}||_1 = p_L = ||\mathbf{p}||_{\infty}$$

Proof. Since

$$\mathbf{Sp} - \mathbf{p} = (-p_1, p_1 - p_2, p_2 - p_3, \dots, p_{L-1} - p_L),$$

and p is non-decreasing,

$$||\mathbf{Sp} - \mathbf{p}||_1 = |p_1| + \sum_{\ell=2}^{L} |p_{\ell-1} - p_{\ell}| = p_L = ||\mathbf{p}||_{\infty}$$

Furthermore,

$$|p_{\ell} - p_{\ell-1}| = \left| \frac{1}{w} \sum_{\hat{\ell}=0}^{w-1} (q_{\ell-\hat{\ell}} - q_{\ell-\hat{\ell}-1}) \right|$$
$$= \frac{1}{w} |q_0 - q_L|$$
$$\leqslant \frac{1}{w}$$

for any $\ell = 2, \ldots, L$. So $||\mathbf{Sp} - \mathbf{p}||_{\infty} \leq \frac{1}{w}$.

Lemma 32: (cf. Lemma 4 in [30]) For the one-sided system defined in (130) and the potential defined in Definition 30, a shift changes the potential by $U(\mathbf{Sp};G) - U(\mathbf{p};G) = -U(p_L;G)$.

Proof. First, we claim that

$$F([\mathbf{Ah}(\mathbf{Sp})]_{\ell}; G) = F([\mathbf{Ah}(\mathbf{p})]_{\ell-1}; G).$$

This can be done by careful inspection: Since $h(Sp) = (0, h(p_1), \dots, h(p_{L-1}))$, by the definition of A,

$$\mathbf{Ah}(\mathbf{Sp}) = \frac{1}{w} \left(0, \\ h(p_1) \\ h(p_1) + h(p_2) \\ \dots, \\ h(p_1) + h(p_2) + \dots + h(p_{w-1}) \\ h(p_2) + h(p_3) \dots + h(p_w), \\ \dots, \\ h(p_{L+w-2}) + \dots + h(p_{L-1}) \right).$$

This proves the claim. Next, write the potential functions in the form of (145), i.e.,

$$U(\mathbf{p};G) = \sum_{\ell=1}^{L} \left(h(p_{\ell})p_{\ell} - H(p_{\ell}) - F([\mathbf{Ah}(\mathbf{p})]_{\ell};G) \right)$$

and

$$U(\mathbf{Sp};G) = \sum_{\ell=1}^{L} (h(p_{\ell})p_{\ell} - H(p_{\ell}) - F([\mathbf{Ah}(\mathbf{Sp})]_{\ell};G))$$

= $\sum_{\ell=1}^{L-1} (h(p_{\ell})p_{\ell} - H(p_{\ell})) - \sum_{\ell=1}^{L} F([\mathbf{Ah}(\mathbf{Sp})]_{\ell};G).$

This gives that

$$U(\mathbf{Sp}; G) - U(\mathbf{p}; G)$$

= $-h(p_L)p_L + H(p_L)$
+ $\sum_{\ell=1}^{L} \left(F([\mathbf{Ah}(\mathbf{Sp})]_{\ell}; G) - F([\mathbf{Ah}(\mathbf{p})]_{\ell}; G) \right)$
= $-h(p_L)p_L + H(p_L) + F(0; G)$
 $-F(h(p_{L+w-1}) + \dots + h(p_L); G)$
= $-h(p_L)p_L + H(p_L) + F([\mathbf{Ah}(\mathbf{p})]_L; G))$
= $-U(p_L; G),$

where that F(0;G) comes from (69).

Lemma 33: (cf. Lemma 5 in [30]) For the potential function of the one-sided system in Definition 25, the norm of the Hessian $U''(\mathbf{p}; G)$ is independent of L and w. It satisfies

$$||U''(\mathbf{p};G)||_{\infty} \leq K_{f,h} := ||h'||_{\infty} + ||h'||_{\infty}^{2} ||f'||_{\infty} + ||h''||_{\infty},$$
where

where

$$||h||_{\infty} = \sup_{x \in [0,1]} |h(x)|$$

for functions $h: [0,1] \to \mathbb{R}$, and

$$||\mathbf{A}||_{\infty} = \max_{1 \le i \le L} \sum_{j=1}^{L} |a_{ij}|$$

for the matrix A.

Proof. The Hessian is given by

$$U''(\mathbf{p}; G) = \mathbf{h}'(\mathbf{p}) - (\mathbf{A}\mathbf{h}'(\mathbf{p}))^T \mathbf{f}'(\mathbf{A}\mathbf{h}(\mathbf{p})G)\mathbf{A}\mathbf{h}'(\mathbf{p}) + \mathbf{h}''(\mathbf{p})\operatorname{diag}(\mathbf{p} - \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p}); G)).$$

Here, $\mathbf{h}''(x) = \operatorname{diag}([h''(x_i)])$. Taking the norms, $||U''(\mathbf{p}; G)||_{\infty} = ||\mathbf{h}'(\mathbf{p})||_{\infty}$ $+ ||\mathbf{Ah}'(\mathbf{p})||_{\infty}||\mathbf{f}'(\mathbf{Ah}(\mathbf{p})G)||_{\infty}||\mathbf{Ah}'(\mathbf{p})||_{\infty}$ $+ ||\mathbf{h}''(\mathbf{p})\operatorname{diag}(\mathbf{p} - \mathbf{A}^T\mathbf{f}(\mathbf{Ah}(\mathbf{p}); G))||_{\infty}.$

Since $||\mathbf{A}||_{\infty} = 1$, $||\mathbf{h}'(\mathbf{p})||_{\infty} \leq ||h'||_{\infty}$, and all the elements inside $(\mathbf{p} - \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p}); G))$ are in [0, 1]. Hence,

$$||U''(\mathbf{p};G)||_{\infty} \leq K_{f,h} := ||h'||_{\infty} + ||h'||_{\infty}^{2} ||f'||_{\infty} + ||h''||_{\infty}.$$

Now we prove Theorem 20.

Proof. First, we claim that this theorem is true for the one-sided system (130) with the condition that $p_{\ell}^{(i)} = p_{\tilde{L}}^{(i)}$ is enforced for $\ell = \tilde{L}, \tilde{L} + 1, \ldots, L$ after each iteration. characterized by (f, h) in (63).

Consider a load $G < G_{conv}^*$. This suffices to show that **0** is the only fixed point of (130), given that the iteration times $i \to \infty$. Suppose that $\mathbf{p} \neq \mathbf{0}$ is a fixed point of (130). By Lemma 29, $p_L > u(G)$. By Lemma 32, expand $U(\mathbf{Sp}; G)$ in a Taylor series around $U(\mathbf{p}; G)$ with remainder.

$$U'(\mathbf{p}; G) \cdot (\mathbf{Sp} - \mathbf{p})$$

= $U(\mathbf{Sp}; G) - U(\mathbf{p}; G)$
 $-\int_{0}^{1} (1 - t)(\mathbf{Sp} - \mathbf{p})^{T} U''(\mathbf{p}(t); G)(\mathbf{Sp} - \mathbf{p}) dt$
 $\leq -U(p_{L}; G) + \left|\int_{0}^{1} (1 - t)(\mathbf{Sp} - \mathbf{p})^{T} U''(\mathbf{p}(t); G)(\mathbf{Sp} - \mathbf{p}) dt\right|$
 $\leq -U(p_{L}; G) + ||\mathbf{Sp} - \mathbf{p}||_{1}||\mathbf{Sp} - \mathbf{p}||_{\infty} \max_{t \in [0, 1]} ||U''(\mathbf{p}(t); G)||_{\infty}$

Lemma 27 indicates that **p** is non-decreasing. Thus, we may put Lemma 31 and Lemma 33 into use. The last inequality could be further estimated:

$$U'(\mathbf{p};G) \cdot (\mathbf{Sp} - \mathbf{p}) \leq -U(p_L;G) + \frac{1}{w} \cdot p_L \cdot K_{f,h}$$

Next, since $0 \leq p_L \leq 1$, using the condition $w > K_{f,h}/\Delta E(G)$ yields

$$U'(\mathbf{p};G) \cdot (\mathbf{Sp} - \mathbf{p}) \leq -U(p_L;G) + \frac{1}{w} K_{f,h}$$
$$\leq -U(p_L;G) + \Delta E(G)$$
$$\leq 0.$$

The inequality follows from the definition of $\Delta E(G)$ and $p_L > u(G)$.

By Lemma 27, **p** is non-decreasing. Therefore, each component of $\mathbf{Sp} - \mathbf{p}$ is not greater than 0. Hence, there exists a component, say ℓ_0 -th of $U'(\mathbf{p}; G)$, is greater than 0. Moreover, by (P2),

$$[U'(\mathbf{p};G)]_{\ell_0} = [h'(p_k)(\mathbf{p} - \mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p});G))]_{\ell_0}$$

gives that

$$[\mathbf{A}^T \mathbf{f}(\mathbf{A}\mathbf{h}(\mathbf{p});G))]_{\ell_0} < p_{\ell_0}.$$

This shows that one more iteration reduces the value of p_{ℓ_0} , for some ℓ_0 in $1, \ldots, L$, and it contradicts with such $\mathbf{p} \neq \mathbf{0}$ is a fix point. Therefore, the only fixed point of (130) is $\mathbf{p} = \mathbf{0}$ under the load $G < G_{conv}^*$. This completes the proof of the claim.

Since the CCPR system with K = J = 1 governed by (120) is the basic spatially-coupled system by Lemma 24, Lemma 28 demonstrates that (120) also converges to a well-defined fixed point. Moreover, since Lemma 26 shows that **p** elementwisely upper bounds the fixed point of (120), (120) converges to **0** as $i \rightarrow \infty$. Hence, by (19), Theorem 5, and Theorem 6, all the loads $G < G^*_{conv}$ are stable for (120). Finally, as (120) is a reversed-index version of (79), all the loads $G < G^*_{conv}$ are stable for (79).

APPENDIX D LIST OF NOTATIONS

We provide a list of notations used in this paper on the next page.

•	
A	An $L \times L$ matrix defined in (132)
\mathbf{A}_2	An $L \times L$ matrix defined in (123)
a_{ij}	The elements of the matrix A
B	The bound of a capacity envelope
(b_1,\ldots,b_K)	The capacity envelope
D	The maximum number of packets that can
	be successfully received in D-fold ALOHA
d	The degree of the regular degree distribution
E	The expectation operator
e	Euler's number
F	The partition vector $F = (F_1, \ldots, F_J)$
F_i	The fraction of Poisson receivers assigned to
3	class j
F(p;G)	The integral of $f(p;G)$ w.r.t. its first variable
$F(\mathbf{p}; G)$	The integral of $f(\mathbf{p}; G)$ w.r.t. its first variable
f	One of the two functions that parameterize a
5	one-sided system or a scalar admissible
	system. f is a real-valued function with
	two variables
f'	The partial derivative of f w.r.t. its first
5	variable
$\mathbf{f}(\mathbf{p};G)$	A vector with L components.
(1) -)	$\mathbf{f}(\mathbf{p};G) = (f(p_1;G), \dots, f(p_L;G))$
G	A vector of normalized offered load.
	$G = (G_1, G_2, \dots, G_K)$
G^*	The potential threshold
$G^*_{a}(L,w)$	The percolation threshold of a CCPR with L
conv	stages and window size w
G^*	The solution to the equation $U(1;G) = 0$
G_{k}	The normalized offered load of class k
$G^{(\ell)}$	The normalized offered load vector of ℓ^{th}
C.	stage $C^{(\ell)} = (C^{(\ell)} - C^{(\ell)})$
$C^{(\ell)}$	Stage, $G^{(k)} = (G_1^{(k)}, \dots, G_K^{(k)})$
G_k	The normalized offered foad of class κ off ℓ^{th} stage
<i>C</i> *	the single system threshold
G_s	The single-system uneshold The integral of $h(n)$
$H(\mathbf{p})$	The integral of $h(\mathbf{p})$
$\frac{\mu(\mathbf{p})}{h}$	One of the two functions that parameterize a
n	One of the two functions that parameterize a
	one-sided system of a scalar admissible
$\mathbf{h}(\mathbf{n})$	system. n is a real-valued function A vector with L components
$\mathbf{n}(\mathbf{p})$	A vector with L components. l(x) = l(x) if a second state x
	$\mathbf{n}(\mathbf{p}) = (n(p_1), \dots, n(p_L))$ with one variable
1 T	The number of slopped of receiver
J	The alars of a maximum with the last of the
) V	The class of a receiver or the index of stages
K	The number of classes of users
ĸ	I ne class of a packet or user
$\kappa_{f,h}$	A number defined in Lemma 33 related to
Ŧ	J and h
L	The number of stages of CCPRs

$$\begin{split} \tilde{L} & L-w+1 \\ L_k & \text{The number of copies of a class k packet } \\ \ell & \text{The index of stages of a CCPR} \\ m(j) & m(j) = \min\{1, \ell \ominus (w-1)\} \\ n & \text{The deterministic load $n = (n_1, n_2, \dots, n_K)$} \\ P_{suc}(\rho) & \text{The probability that a packet is successfully received if the receiver is subject to a Poisson offered load ρ \\ P_{suc,D}(\rho) & \text{The inverse of P_{suc}} \\ P_{suc,k}(\rho) & \text{The probability that a class k packet is successfully received if the receiver is subject to a Poisson offered load ρ \\ P_{suc,k}(\rho) & \text{The probability that a class k packet is successfully received if the receiver is subject to a Poisson offered load ρ \\ P_{suc,k,j}(\rho) & \text{The probability that a class k packet is successfully received by a class j receiver if the receiver is subject to a Poisson offered load ρ \\ P_{suc,k,i}(\rho) & \text{The probability that a packet sent from a randomly selected class k user can be successfully received after the i^{th} iteration \\ P_{suc,k,\ell}(\rho) & \text{The probability that a packet sent from a randomly selected class k user in the ℓ^{th} stage can be successfully received after the i^{th} iteration \\ P & A vector. $\mathbf{p} = (p_1, \dots, p_L)$ \\ $\tilde{p}_{\ell}^{(i)} & \text{The probability that a packet sent from a randomly selected class k user in the ℓ^{th} stage can be successfully received after the i^{th} iteration \\ P & A vector. $\mathbf{p} = (p_1, \dots, p_L)$ \\ $\tilde{p}_{\ell}^{(i)} & \text{The probability that the receiver end of a randomly selected class k degh has not been successfully received after the i^{th} SIC iteration \\ $p^{(i)} & \text{The probability that the receiver end of a randomly selected class k edge has not been successfully received after the i^{th} SIC iteration \\ $p_{k,j}^{(i)} & \text{The probability that the receiver end of a randomly selected class k edge has not been successfully received after the i^{th} SIC iteration \\ $p_{k,j}^{(i)} & \text{The probability that the receiver end of a randomly selected class k edge in the ℓ^{th} stage has not been successfully re$$

$p_{k,j,\ell}^{(i)}$	The probability that the receiver end of a randomly selected class (k, j) -edge in the ℓ^{th} stage has not
(i)	been successfully received after the i Sic fieldion
$q^{(0)}$	A vector of $q_k^{(i)}$, $q^{(i)} = (q_1^{(i)}, q_2^{(i)}, \dots, q_K^{(i)})$ The initial vector of $g^{(i)}$
$q^{(\alpha)}$	The limiting vector of $a^{(i)}$ with the initial vector $a^{(0)} = 1$
q (i)	The match bility that the user and of a matched should also be not been successfully associated
q_k	The probability that the user end of a randomly selected class k edge has not been successfully received often the i^{th} SIC iteration
(i)	after the i SIC increation A sector \mathbf{c} (i) (i) (i) (i) (i)
$q_{\ell}^{(i)}$	A vector of $q_{k,\ell}^{\vee}, q_{\ell}^{\vee} = (q_{1,\ell}^{\vee}, q_{2,\ell}^{\vee}, \dots, q_{K,\ell}^{\vee})$
$q_{k,\ell}^{(l)}$	The probability that the user end of a randomly selected class k edge in the ℓ^{th} stage has not been
	successfully received after the i^{th} SIC iteration
R	The $K \times J$ routing matrix $R = (r_{k,j})$
\mathbb{R}	The set of real numbers
R_{j}	A vector of parameters for class j receiver, $R_j = (\frac{r_{1,j}}{F_j}, \frac{r_{2,j}}{F_j}, \dots, \frac{r_{K,j}}{F_j})$
$r_{k,j}$	The routing probability that a class k packet transmitted to a class j receiver
r_{j}	$r_j = r_{1,j} + \ldots + r_{K,j}$
S	The stability region or the capacity region
\mathbf{S}	The shift operator
S_L	The stability region of a CCPR with L stages
T	The number of Poisson receivers
U(p;G)	The potential function of a scalar admissible system
$U(\mathbf{p};G)$	The potential function of a one-sided system
u(G)	The minimum unstable fixed point
w	The smooth window size of CCPRs
$X_k(t)$	The number of class k packets sent to the t^{th} receiver
$Y_k(t)$	The number of class k packets that are actually decoded by the t^{th} receiver
\mathcal{Z}^+	The set of nonnegative integers
$\Delta E(G)$	The energy gap
δ	The step size
Θ_k	The throughput for a $(P_{suc,1}(\rho), \ldots, P_{suc,K}(\rho))$ -Poisson receiver subject to a Poisson offered load ρ
$\Lambda_{k,d}$	The probability that a class k packet is transmitted d times $d = \frac{d}{d} \left(\frac{d}{d} \right)$
$\Lambda'(x)$	A vector of $\Lambda'_k(x)$, $\Lambda'(x) = (\Lambda'_1(x), \Lambda'_2(x), \dots, \Lambda'_K(x))$
$\Lambda_k(x)$	The generating function of the degree distribution of a class k user $(k = 1)$
$\Lambda_k(x)$	The derivative of $\Lambda_k(x)$
$\Lambda_k(1)$	The mean degree of a class k user node
$\lambda_{k,d}$	The probability that the user end of a randomly selected class κ edge has additional a edges excluding the
λ (m)	The generating function of the excess degree distribution of a close <i>k</i> user
$\lambda_k(x)$	The generating function of the excess degree distribution of a class κ user
μ_j	The Decision efforted load $a_{k=1} o_k A_k(t)$
ρ	The Poisson offered load $\rho = (\rho_1, \dots, \rho_K)$
$ ho_k$	The Poisson offered lead of class k The Poisson offered lead at a class i Poisson receiver $\tilde{a} = (a_1, a_2, \dots, a_{n-1})$
$ ho_j$	The Poisson offered load of class <i>k</i> packets to a class <i>i</i> Poisson receiver
$\rho_{k,j}$	The vector of α_{1} is $\alpha_{2} = (\alpha_{1} + \alpha_{2})^{\alpha_{1}} (\alpha_{2} + \alpha_{3})^{\alpha_{1}}$
$\rho_{j,\ell}$	The Poisson offered load of class k packets to a class i Poisson receiver in the l^{th} stage
$\rho_{k,j,\ell}$	The $\phi_{-}\Delta I O H \Delta$ receiver $\phi(n) = (\phi_{+}(n))$ $\phi_{+}(n)$
φ	$\prod_{i=1}^{n} \psi_{i}^{-1} = (\psi_{1}(i_{i}), \dots, \psi_{K}(i_{i}))$