ISOMONODROMY AND PAINLEVÉ TYPE EQUATIONS, CASE STUDIES

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Abstract. There is an abundance of equations of Painlevé type besides the classical Painlevé equations. Classifications have been computed by the Japanese school. Here we consider Painlevé type equations induced by isomonodromic families of linear ODE's having at most $z=0$ and $z=\infty$ as singularities. Requiring that the formal data at the singularities produce isomonodromic families parametrized by a single variable t leads to a small list of hierarchies of cases. The study of these cases involves Stokes matrices and moduli for linear ODE's on the projective line.

Case studies reveal interesting families of linear ODE's and Painlevé type equations. However, rather often the complexity (especially of the Lax pair) is too high for either the computations or for the output. Apart from classical Painlevé equations one rediscovers work of M. Mazzocco, M. Noumi and Y. Yamada. A hierarchy, probably new, related to the classical $P_3(D_8)$, is discovered. Finally, an amusing "companion" of P_1 is presented.

Introduction and Summary

We study families $\mathcal{M}$ of connections on a vector bundle on the complex projective line, defined by the data of a finite set of singular points and for each singular point the type of the singularity. These data give rise to a monodromy space $\mathcal{R}$ which is built from the possibilities for the topological monodromy, the Stokes matrices and the links.

The Riemann–Hilbert map $RH : \mathcal{M} \to \mathcal{R}$ sends a connection to its monodromy data. The fibres of RH are the *isomonodromic* families, i.e., the (maximal) subspaces of $\mathcal{M}$ with constant monodromy. We are interested in the cases where the fibres are locally parametrized by one complex variable t, called here *the time variable*. In general there are more "time variables".

A family $\mathcal{M}$ can be represented by a matrix differential operator $\frac{d}{dz}+A$ where the entries of the matrix $A=A(z,v_1,\dots,v_r)$ are rational functions in z and depend on a certain number of variables $v_1,\dots,v_r$.

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If the fibres of RH are parametrized by a time variable t, then the $v_1, \ldots, v_r$ are, restricted to the fibre, functions of t. By assumption, the monodromy, the links and the Stokes matrices of the operator $\frac{d}{dz} + A(z, v_1(t), \ldots, v_r(t))$ are constant. Let $Y(z,t)$ denote a fundamental matrix. Then $\frac{d}{dt}Y(z,t)$ has the same monodromy, links and Stokes matrices as $Y(z,t)$ and thus $B(z,t) := -\frac{d}{dt}Y(z,t) \cdot Y(z,t)^{-1}$ has trivial monodromy, links and Stokes matrices. Therefore $B(z,t)$ extends to a matrix whose entries are rational in z and locally analytic in t. It follows that $\frac{d}{dt} + B(z,t)$ commutes with the operator $z\frac{d}{dz} + A(z, v_1(t), \ldots, v_r(t))$. This pair of operators is called a *Lax pair*. The knowledge that $z\frac{d}{dz} + A(z, v_1(t), \ldots, v_r(t))$ is part of a Lax pair suffices to obtain an explicit system of differential equations

$$\frac{d}{dt}v_i(t) = R_i(v_1(t), \ldots, v_r(t)),\ i = 1, \ldots r,\ \text{with rational functions } R_i.$$

This is usually called a vector field of Painlevé type. This system has the Painlevé property, i.e., "local solutions extend to global multivalued meromorphic solutions". For more details, see [Sib].

The classical Painlevé equations $P_1 - P_6$ arise from isomonodromy. There are in fact many isomonodromic families. The Japanese school has an extensive literature on equations of Painlevé type and also developed classifications (see for instance [J-M-U, K-N-S, H-K-N-S, K2, K3, K4, Mi, N-Y, O1, O-O]).

Here, we modestly restrict ourselves to classifying and studying rather special cases, namely assuming that at most $z = 0$ and $z = \infty$ are singular and assuming that there is only one time variable. One reason for this restriction is that the quantum differential equations associated to algebraic varieties have two singularities $z = 0$ (regular singular) and $z = \infty$ (irregular singular). A further reason is that the theory and algorithm of Stokes matrices provide most of the information for $\mathcal{R}$ and $\mathcal{M}$. Furthermore, we are more interested in *hierarchies* of families than in individual families. A hierarchy can be obtained by putting a "natural" condition on the eigenvalues at the singular points $z = 0$ and $z = \infty$.

Now we describe the data for $\mathcal{M}$ and $\mathcal{R}$. The point $z = \infty$ is supposed to be irregular singular and $z = 0$ can be *regular* or *regular singular* or *irregular singular*.

For the convenience of the reader we describe the formal classification at $z=\infty$ *of differential modules in terms of tuples* $(V,\{V_q\}_q,\gamma)$*; details may be found in* [vdP-Si].

A tuple consists of a complex vector space V of dimension n with additional structure. The q's denote elements of $\cup_{r\geq 1} z^{1/r}\mathbb{C}[\![z^{1/r}]\!]$. For each q there is given a linear subspace $V_q\subseteq V$ and $V=\oplus V_q$. The *eigenvalues* are the finitely many $q_1,\dots,q_r$ with $V_q\neq 0$. The multiplicity $m=m(q)$ is the dimension of V_q. The dimension of V is therefore equal to $\sum m(q_j)$. One writes $(q)_m$ to denote eigenvalue q with multiplicity m.

The *ramification index* e is the smallest positive integer such that $q_j\in z^{1/e}\mathbb{C}[\![z^{1/e}]\!]$ for all j. The degree of q_j is the highest (rational) power of z occurring in q_j. The *Katz invariant* κ is the maximum of the degrees of the q_j's. The Galois group of $\cup_{r\geq 1}\mathbb{C}(\!(z^{-1/r})\!)/\mathbb{C}(\!(z^{-1})\!)$ has a topological generator σ which acts by $\sigma(z^\lambda)=e^{2\pi i\lambda}z^\lambda$ for $\lambda\in\mathbb{Q}$.

Further, γ, the *formal monodromy*, is an automorphism of V and has the property $\gamma(V_q)=V_{\sigma(q)}$ for all q.

The classification of differential modules M over $\mathbb{C}(\!(z^{-1})\!)$ by the tuples $(V,\{V_q\}_q,\gamma)$ is based upon the fact that the solutions of M can be written as a sum of expressions $\exp(\int q\frac{dz}{z})\cdot F$ with q as above and F a combination of formal power series in roots of z and $\log(z)$. The V associated to M is the space of these formal or symbolic expressions. It has a natural decomposition as $\oplus V_q$. Further, the formal monodromy γ, is given by σ applied to the above expressions. The functor $M\mapsto (V,\{V_q\}_q,\gamma)$ is an equivalence of Tannakian categories.

Let a family of data for eigenvalues $q_1,\dots,q_r$*, multiplicities and formal monodromies* γ *be given.* This gives rise to a family of formal differential operators $z\frac{d}{dz}+F$. The construction of the space of connections $\mathcal{M}$ amounts to computing a family of differential operators $z\frac{d}{dz}+A$ over $\mathbb{C}(z)$ with singular points $z=0$ and $z=\infty$. The condition is that this operator is at $z=\infty$ formally equivalent to $z\frac{d}{dz}+F$. Moreover, at $z=0$ the operator should be regular, or regular singular, or formally equivalent to another given formal operator.

Some details on the definition and the construction of the monodromy space $\mathcal{R}$. For each difference $q_k-q_l,\ k\neq l$ one considers a solution $y\neq 0$ of $z\frac{dy}{dz}=q_k-q_l$. The singular directions $d\in\mathbb{R}$ for q_k-q_l are defined by $y(e^{2\pi i d}r)$ tends to zero for $r \to +\infty$ with maximal speed (maximal descent).

The Stokes matrix $St_d \in \mathrm{GL}(V)$ for direction d reads $\mathbf{1}_V + \sum_{k,l} m_{k,l}$, where the sum is taken over the pairs such that d is singular for $q_k - q_l$ and $m_{k,l}$ denotes a linear map $V \stackrel{projection}{\to} V_k \to V_l \subseteq V$.

We note that $St_{d+1} = \gamma^{-1} St_d \gamma$ holds for $d \in \mathbb{R}$. Therefore the Stokes data can be identified with the space of all Stokes matrices St_d with $d \in [0,1)$ and can be identified with a vector space of dimension $N := \sum_{k \neq l} \deg(q_k - q_l) \cdot \dim V_k \cdot \dim V_l$.

The formal data combined with the data of the Stokes matrices classify the analytic singularity at $z = \infty$. In particular, mon_∞, the topological monodromy at $z = \infty$, is equivalent to the product $\gamma \circ St_{d_s} \circ \cdots \circ St_{d_1}$, where $d_s > \cdots > d_1$ are the singular directions in $[0,1)$. This property will be called *the monodromy identity.* For the construction of $\mathcal{R}$ we have to consider the various cases.

(i) $z = 0$ *is regular singular.* Given are $V = V_{q_1} \oplus \cdots \oplus V_{q_r}$, an action of σ on $\{q_1, \ldots, q_r\}$, the singular directions $1 > d_s > \cdots > d_1 \geq 0$ with the corresponding differences $q_k - q_l$. We note, in passing, that the highest coefficient of a difference $q_k - q_l$ may depend on t. In such a case the singular directions also depend on t. In the cases that we computed $\mathcal{R}$ itself is independent of t.

In general, $\mathcal{R}$ is defined as the set of equivalence classes of all possibilities for the Stokes data and the topological monodromies. In the present case, $\mathcal{R}$ consists of the equivalence classes of all possible tuples $(\gamma, St_{d_s}, \ldots, St_{d_1}) \in \mathrm{SL}(V)^{s+1}$, where by assumption γ is supposed to have distinct eigenvalues. We now make $\mathcal{R}$ explicit.

Let $(V, \{V_q\}, \gamma, \{St_d\})$ *be given.* The action of σ on the eigenvalues has orbits (i.e., Galois orbits) $\overline{Q}_1, \ldots, \overline{Q}_r$ and $\overline{Q}_i = \{q_{i,0}, \ldots, q_{i,\ell_i - 1}\}$ for all i. Let $d_i = \dim V_{q_{i,0}}$. Put $d = \sum_{i=1}^r d_i$ and as before we write $N = \sum \deg(q_{i,j} - q_{k,l}) \cdot \dim(V_{q_{i,j}}) \cdot \dim(V_{q_{k,l}})$.

Lemma 0.1. $\mathcal{R}$ *is isomorphic to the quotient of the space* $(\mathbb{C}^*)^{d-1} \times \mathbb{C}^N$ *by the action of a group isomorphic to* $(\mathbb{C}^*)^{d-1}$. *This quotient has an open, affine, dense subspace isomorphic to* $(\mathbb{C}^*)^{d-1} \times \mathbb{C}^{N-d+1}$. *In particular* $\dim \mathcal{R} = N$.

Proof. There is no restriction on the possibilities for the St_{d_j}. This produces the vector space $\mathbb{C}^N$. In order to make the restrictions on γ explicit, we consider a Galois orbit, here written as, $\overline{Q} = \{q_0, \ldots, q_{\ell-1}\}$ with $\dim V_{q_0} = f$. Choose a basis $e_1, \ldots e_f$ of eigenvectors for the action of γ^ℓ on V_{q_0}. Let $\alpha_1,\dots,\alpha_f$ denote the distinct eigenvalues. This basis is unique up to permuting and scaling of the basis vectors. Consider, for $i=1,\dots,\ell-1$, the basis of V_{q_i} to be $\gamma^i(e_1),\dots,\gamma^i(e_f)$. One concludes that the data for the matrix of γ on the space $\oplus V_{q_i}$ is equivalent to the tuple $(\alpha_1,\dots,\alpha_f)$. Moreover, the set of all ℓth roots of all α_i is the set of eigenvalues of γ.

Thus the total data for γ on V is given by the eigenvalues of γ^{l_i} on the space $V_{q_{i,0}}$ for $i=1,\dots,r$. Since, by assumption, γ has determinant 1, the space of possibilities for γ is $(\mathbb{C}^*)^{-1+\sum d_i}$.

The automorphisms of $(V,\{V_q\},\gamma)$ are the $\tau\in \mathrm{PGL}(V)$ such that $\tau(V_q)=V_q$ for all q and $\tau\gamma=\gamma\tau$. One concludes that τ is determined by its action on all $V_{q_{i,0}}$. Furthermore, τ has on this space the same eigenvectors as γ^{ℓ_i}. This implies that the group of automorphism is isomorphic to $(\mathbb{C}^*)^{d-1}$. It is seen that the group acts faithfully on the Stokes data. Finally, by scaling suitable Stokes data to 1, one obtains this affine, open, dense subspace of $\mathcal{R}$. □

(ii) $z=0$ *is regular.* As above in (i), but now with the additional restriction $\gamma\circ St_{d_s}\circ\cdots\circ St_{d_1}=\mathbf{1}_V$. For every candidate a computation is needed to find out whether $\mathcal{R}$ is not empty and to find its dimension.

(iii) $z=0$ *is irregular singular.* Let W denote the solution space at $z=0$. It has similar additional data as V, namely q's, γ, St_d, mon_0. The *link* (in [J-M-U] called 'connection') is a linear bijection $L:W\to V$ commuting with the mon_*. The space $\mathcal{R}$ is the space of equivalence classes of the data at V, W and the link L.

The parameter space $\mathcal{P}$ is defined by data of the topological and the formal monodromies, more precisely by their characteristic polynomials. By "fibre" we will mean a fibre of $\mathcal{R}\to\mathcal{P}$, which has the interpretation as space of initial conditions. Each fibre determines a Painlevé vector field (or scalar differential equation) of rank (or order) equal to the dimension of the fibre.

The rules used for composing our list of families of connections. The requirements concern the formal data at $z=\infty$ (and also at $z=0$ if this point irregular singular).

R1. The (distinct) eigenvalues $q_1,\dots,q_r$ with multiplicity $m_1,\dots,m_r$ satisfy: all $q_j\neq 0$, $\sum m_jq_j=0$ and (in case $e>1$) invariance under the Galois group of $\overline{\mathbb{C}((1/z))}$ over $\mathbb{C}((1/z))$. Further, one requires that the formal monodromy γ is "generic", meaning that it has n distinct eigenvalues.

R2. If $z=0$ is regular, then the formal data are normalized by the action of the group $\{z \mapsto az+b\}$. If $z=0$ is singular, the formal data are normalized using the group $\{z \mapsto az\}$. For the description of all formal data at $z=\infty$ and at $z=0$ (if this point is also irregular singular) only one variable t is needed. This is the translation of the requirement that the fibres of $\mathcal{M} \to \mathcal{R}$ are locally parametrized by a single t, called *the time variable*.

R3. The data should not define a subfamily of a family with more "time variables". However, in Section **??**, we will consider a "companion of P_1", which is a subfamily of an interesting "two time variables family".

R4. The emphasis is on hierarchies and not on individual families. By hierarchy we mean a sequence of families defined by certain properties of the eigenvalues. For example, $z=0$ regular singular; $e=1, \kappa=1$ defines the hierarchy given by the eigenvalues and multiplicities $(z)_{m_1}, (tz)_{m_2}, (-\frac{m_1+tm_2}{m_3}z)_{m_3}$ for $m_1, m_2, m_3 \geq 1$.

In § 1 a complete list for the cases with multiplicities 1 is presented. This includes of course the classical Painlevé equations with at most two singular points. This list extends in an obvious way to a complete list of hierarchies by allowing multiplicities.

In the next sections, the cases of the list which are not classical, are studied in more detail.

Details on definition and construction of the space of connections $\mathcal{M}$.

(i) Case $z=0$ is regular singular. We start by assuming that the irregular singularity $z=\infty$ is unramified and is given by data $(V, \{V_q\}, \gamma)$.

We choose a basis of V, consisting of eigenvectors of γ, and matrices w.r.t. this basis. This leads to a choice standard differential operator $z\frac{d}{dz}+S$ where S is a diagonal matrix with diagonal entries $(Q_1+a_1, \ldots, Q_n+a_n)$. The $Q_1, \ldots, Q_n$ are the eigenvalues $q_1, \ldots, q_r$, repeated according to their multiplicities. Thus $\sum Q_j = 0$. The $a_1, \ldots, a_n \in \mathbb{C}$ satisfy $\sum a_j = 0$ and are chosen such that the monodromy of the operator $z\frac{d}{dz} + diag(a_1, \ldots, a_n)$ equals γ.

Now we follow [vdP-Si, § 12] and consider the fine moduli space defined by the connections on $\mathbb{P}^1$ with the properties:

(a). the vector bundle of rank n is trivial;
(b). $z = 0$ is regular singular;
(c). A formal isomorphism at $z = \infty$ with $z\frac{d}{dz} + S$ is given.

According to [vdP-Si, Corollary 12.15 and its proof], the universal family of this fine moduli space is represented by the family of differential operators $Pr(g(z\frac{d}{dz} + S)g^{-1})$ where g runs in the N-dimensional affine space

$$\{\mathbf{1}_V + \sum_{k \neq \ell} \mathrm{Hom}(V_k, V_\ell) \otimes_{\mathbb{C}} (\mathbb{C}z^{-1} + \cdots + \mathbb{C}z^{-\deg(q_k - q_\ell)})\}.$$

The notation Pr denotes "principal part" and is defined here as

$$Pr(z\frac{d}{dz} + \sum_{k\ <<\infty} A_k z^k) = z\frac{d}{dz} + \sum_{0\leq\ k\ <<\infty} A_k z^k.$$

This completes the construction of $\mathcal{M}$ for the case $z = 0$ regular singular and $z = \infty$ is unramified. We note in passing that the above describes a (co-adjoint) orbit of a linear algebraic group over $\mathbb{C}$. Therefore $\mathcal{M}$ has a natural symplectic structure, see also [Bo].

For the ramified case one considers the cyclic covering of $\mathbb{P}^1$ of degree e, ramified over 0 and ∞. With respect to the variable $z^{1/e}$, one computes the universal family $z\frac{d}{dz} + A$ as above. The final step is a computation of the operator on a σ-invariant basis (compare [vdP-Si, § 12.5])).

(ii). The case $z = 0$ regular. From the data $(V, \{V_q\}, \gamma)$ one first computes, as above in (i), a universal family of matrix differential operators $z\frac{d}{dz} + \sum_{0\leq\ k\ <<\infty} A_k z^k$. We propose for $\mathcal{M}$ the subfamily defined by the condition that all the entries of A_0 are zero. An explicit computation is needed to verify whether $\mathcal{M}$ is not empty and to compute its dimension.

(iii). The case $z = 0$ and $z = \infty$ irregular singular. One expects a universal family of differential operators $z\frac{d}{dz} + \sum_{-\infty<<\ k\ <<\infty} A_k z^k$. For the righthand part $\sum_{0\leq\ k\ <<\infty} A_k z^k$ the method of (i) produces a proposal. The same holds for the lefthand part $\sum_{-\infty<<\ k\ \leq 0} A_k z^k$. Gluing of the two proposals may result in a suitable family. A priori, it is not clear whether the formal data at $z = 0$ and at $z = \infty$ can be combined to a family $\mathcal{M}$ and a corresponding monodromy space $\mathcal{R}$.

The explicit computation of $\mathcal{R}$ works quite well. The computation of $\mathcal{M}$ in cases (i) or (ii) may fail or may lead to a result unsuitable for further analysis, due to complexity. For case (iii) one needs a good guess to start the computation.

The number of cases where a complete computation of the Lax pairs can be given is, again due to complexity, rather small. In case (i), the differential equations involve $N = \sum_{k,\ell} \deg(q_k - q_\ell) \cdot \dim V_{q_k} \cdot \dim V_{q_\ell}$ functions of t. This system is mostly too large.

A first step is normalization by scaling the basis vectors of V. A next step is to reduce this system by the use of invariants, which are independent of t. The coefficients of the characteristic polynomials of the formal and the topological monodromy, generate the algebra of invariants. Their computation as expressions in the above functions and elimination some of the N variables is a source of complexity. For the cases (ii) and (iii) there are similar complexity problems.

A list of the most detailed and interesting explicit cases.
Here n denotes the rank of the connections and the formula gives representatives for the Galois orbits of the eigenvalues.

§ 2 $z^{2/n} + tz^{1/n}$, $n \geq 3$. The hierarchy of M. Noumi and Y. Yamada.
§ 5.1 $z^{1/2}, tz^{1/2}$, $n = 4$. Conjectured to be related to P_4.
§ 7 $z, tz, (-1-t)z$, $n = 3$. M. Mazzocco's equation, related to P_6.
§ 9 $z^2, -z^2 - tz, tz$, $n = 3$. Complete results, including an explicit Hamiltonian. Conjectured to be related to P_1.
§ 12 $z^{-1/n}$ and $tz^{1/n}$, $n \geq 2$. A hierarchy, probably new, related to $P_3(D_8)$, with explicit description of the Lax pairs.
§ 13 $z^{5/2} + tz^{1/2}$, $n = 2$. "Companion of P_1", a variation on P_1.

1. List of cases with one time variable

With $z = 0$ regular singular or regular.
The condition "one time variable t" implies that there are at most three Galois orbits of eigenvalues. A regular singular case can restrict to a regular case, e.g., $z, tz, (-1-t)z$ and $z = 0$ regular exists and produces trivial Stokes data. In the table representatives for the Galois orbits of the eigenvalues are given; the rightmost column indicates in which section this example is discussed and which classical Painlevé equation P_j it relates to (the Flaschka-Newell equation which is equivalent to P_2, is denoted P_{2FN}); see, e.g., [vdP-Sa].

- One Galois orbit; $z=0$ regular singular;
  $z^{3/2}+tz^{1/2}$; P_{2FN}
  $z^{2/e}+tz^{1/e}$ for $e\geq 3$. § 2
- One Galois orbit; $z=0$ regular;
  $z^{5/2}+tz^{1/2}$; P_1
  $z^{4/3}+tz^{2/3}$. §3
- Two Galois orbits; $z=0$ regular singular;
  z^2+tz, $-(z^2+tz)$; P_4
  $z+tz^{1/2}$, $-2z$. §4
  z^{1/e_1}, tz^{1/e_2} for $e_1\geq e_2\geq 2$. §5
- Two Galois orbits; $z=0$ regular;
  z^3+tz, $-(z^3+tz)$. §6, P_2
- Three Galois orbits; $z=0$ regular singular;
  z,tz, $(-1-t)z$; §7
  $z^{1/e}$, tz, $-tz$ with $e>1$. §8
- Three Galois orbits; $z=0$ regular;
  z^2, $-z^2-tz$, tz. §9

With both $z=0$ and $z=\infty$ irregular singular.
- $1/z,-1/z$ at $z=0$ and $tz,-tz$ at $z=\infty$. §10, $P_3(D_6)$
- $z^{-1/2}$ at $z=0$ and $tz,-tz$ at $z=\infty$. §11, $P_3(D_7)$
- $z^{-1/n}$ at $z=0$ and $tz^{1/n}$ at $z=\infty$ with $n\geq 2$. §12, $P_3(D_8)$

More general:
$n_1,n_2\geq 2$ and z^{-1/n_1} at $z=0$ and tz^{1/n_2} at $z=\infty$
(with suitable multiplicities).

We discuss these cases in the indicated sections.

2. $z^{2/n}+tz^{1/n}, n\geq 3$, Hierarchy of M. Noumi and Y. Yamada

We study the structure of the moduli spaces $\mathcal{M}_n,\mathcal{R}_n$ and the Lax pair computations, separately for n odd and n even.

2.1. The moduli spaces $\mathcal{M}_n$ and $\mathcal{R}_n$ for odd n.

Computations for $\mathcal{R}_n$.
A module $M\in\mathcal{M}_n$ has the eigenvalues $q_j=\sigma^j(q_0)=\omega^{2j}z^{2/n}+t\omega^jz^{1/n}$ for $j=0,\dots,n-1$ at $z=\infty$, where $\omega:=e^{2\pi i/n}$.

The tuple $(V,\{V_q\},\gamma,\{St_d\})$ that classifies M at $z=\infty$ has the form $V=\mathbb{C}e_0\oplus\cdots\oplus\mathbb{C}e_{n-1}$ where $\mathbb{C}e_j=V_{q_j}$ for $j=0,\dots,n-1$. This basis is chosen such that γ satisfies $e_0\mapsto e_1\mapsto\cdots\mapsto e_{n-1}\mapsto e_0$.

The space of the Stokes matrices at $z = \infty$ is isomorphic to $\mathbb{C}^N$, where $N = n(n-1)\cdot\frac{2}{n} = 2(n-1)$. Since the basis $e_0, \ldots, e_{n-1}$ of V is unique up to multiplication of all basis vectors by the same constant, one finds $\mathcal{R}_n = \mathbb{C}^{2(n-1)}$ and $\dim \mathcal{M}_n = 1 + 2(n-1)$.

The parameter space $\mathcal{P}_n$ is the space of the conjugacy classes of SL_n. This is identified with a space of characteristic polynomials and it has dimension $n-1$. The fibres of $\mathcal{R}_n \to \mathcal{P}_n$ have dimension $n-1$. This can be verified by an explicit computation, as in [CM-vdP].

It has been verified for $n = 3$ and $n = 5$ that the combination with $z = 0$ regular is not possible.

The fibre for $n = 3$ is computed in [vdP-T] to be the affine surface $xyz + x^2 + p_1 x + p_2 y + p_3 z = 0$ for certain invariants p_1, p_2, p_3. This hints at a possible relation with P_4. A computation of the Lax pair equations produces indeed this relation.

For $n = 5$, we present details of the computation. Write $\omega = e^{2\pi i/5}$, $q_0 = z^{2/5} + tz^{1/5}$, $q_1 = \omega^2 z^{2/5} + \omega t z^{1/5}, \ldots, q_4 = \omega^3 z^{2/5} + \omega^4 t z^{1/5}$. The singular directions in $[0, 1)$ are $7/8$ for $q_0 - q_2,\ q_3 - q_4$, $5/8$ for $q_1 - q_4,\ q_3 - q_2$, $3/8$ for $q_1 - q_2,\ q_3 - q_0$ and $1/8$ for $q_1 - q_0,\ q_4 - q_2$. The fibres of $\mathcal{R}_5 \to \mathcal{P}_5$ are rational 4-folds. After eliminating 3 of the 8 variables for $\mathcal{R}_5$, the affine fibre is given by a degree 5 polynomial equation in 5 variables and with 4 parameters.

Construction of $\mathcal{M}_n$ and the Lax pairs.
We make the method explained in the Introduction and Summary, explicit. A differential module $M \in \mathcal{M}_n$ over $\mathbb{C}(z)$ is replaced by $N := \mathbb{C}(z^{1/n}) \otimes M$. Let D denote the differential operator $\nabla_{z\frac{d}{dz}}$ on M. Now D extends uniquely to a differential operator, also called D, on N. This D commutes with the semi-linear automorphism $\sigma : N \to N$, induced by the automorphism σ of $\mathbb{C}(z^{1/n})$, given by $\sigma z^{1/n} = \omega z^{1/n}$. Thus the $M \in \mathcal{M}$ are replaced by pairs (N, σ), as above.

Let $e_0, \ldots, e_{n-1}$ be a basis of N over $\mathbb{C}(z^{1/n})$ such that the map σ satisfies $\sigma : e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$. The operator D is determined by $D(e_0)$. The formula $D(e_0) = (z^{2/n} + tz^{1/n})e_0 + \sum_{i=1}^{n-1}(a_i + b_i z^{1/n})e_i$ is supported by [vdP-Si, §12.3-12.5] (compare the Introduction and Summary). For the operator $E := \frac{d}{dt} + B$ such that $\{D, E\}$ forms a Lax pair, one makes the guess that E is the σ-invariant operator with $E(e_0) = z^{1/n} e_0 + \sum_{j=1}^{n-1} c_j e_j$.

One deduces from this the matrix of D with respect to the basis $B_0,\dots,B_{n-1}$ of $M := N^{<\sigma>}$, where $B_j := \sum_{k=0}^{n-1}\sigma^k(z^{j/n}e_0)$ for $0\leq j\leq n-1$ and $B_n := zB_0, B_{n+1} := zB_1$. The formula is

$$D(B_j)=\frac{j}{n}B_j+\sum_{i=1}^{n-1}a_i\omega^{-ij}B_j+tB_{j+1}+\sum_{i=1}^{n-1}b_i\omega^{-i(j+1)}B_{j+1}+B_{j+2}.$$

The formula for E on this basis is $E(B_j)=B_{j+1}+(\sum_{k=1}^{n-1}\omega^{-kj}c_k)B_j$. Put $\epsilon_j=\frac{j}{n}+\sum_{i=1}^{n-1}a_i\omega^{-ij}$; $f_j=t+\sum_{i=1}^{n-1}b_i\omega^{-ij}$. The operator D is

$$z\frac{d}{dz}+\begin{pmatrix}\epsilon_0 & 0 & 0 & * & * & z & zf_0\\ f_1 & \epsilon_1 & 0 & 0 & * & 0 & z\\ 1 & f_2 & \epsilon_2 & 0 & * & * & 0\\ 0 & 1 & f_3 & \epsilon_3 & 0 & * & *\\ * & * & * & * & * & * & *\\ * & * & * & 1 & f_{n-2} & \epsilon_{n-2} & 0\\ * & * & * & 0 & 1 & f_{n-1} & \epsilon_{n-1}\end{pmatrix},$$

note that $\sum\epsilon_j=\frac{n-1}{2}$ and $\sum f_j=nt$. The $\epsilon_0,\dots,\epsilon_{n-1}$ are the parameters of the family. The $\{e^{2\pi i\epsilon_j}\}$ are the eigenvalues of the topological monodromy at $z=0$. These can be seen as parameters for $\mathcal{R}_n$. For an isomonodromic family the ϵ_j are constant and the $f_0,\dots,f_{n-1}$ are analytic functions of the parameter t.
The operator E reads on the above basis

$$\frac{d}{dt}+\begin{pmatrix}g_0 & 0 & 0 & 0 & * & * & z\\ 1 & g_1 & 0 & 0 & * & * & 0\\ 0 & 1 & g_2 & 0 & * & * & 0\\ * & * & * & * & * & * & *\\ * & * & * & * & * & * & *\\ 0 & 0 & 0 & 0 & 1 & g_{n-2} & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & g_{n-1}\end{pmatrix}$$

with $g_j=(\sum_{k=1}^{n-1}\omega^{-kj}c_k)$ and $\sum g_j=0$. For an isomonodromic family, the $\{g_j\}$ are functions of t and are in fact eliminated by the Lax pair condition $DE=ED$.
For $n=5$, the Painlevé type differential system for this Lax pair is

$$\begin{aligned}f_1'&=f_1(-f_1-2f_2-2f_4+t)+2\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4,\\ f_2'&=f_2(-2f_1+f_2-2f_4-t)-\epsilon_1+\epsilon_2,\\ f_3'&=f_3(-2f_1-f_3-2f_4+t)-\epsilon_2+\epsilon_3,\\ f_4'&=f_4(2f_1+2f_3+f_4-t)-\epsilon_3+\epsilon_4.\end{aligned}$$

For $n=7$, one has $\sum f_j = 7t,\ \sum \epsilon_j = 3$ and

$$\begin{aligned}
f_0' &= f_0(-f_1+f_2-f_3+f_4-f_5+f_6)+\epsilon_0-\epsilon_6+1,\\
f_1' &= f_1(f_0-f_2+f_3-f_4+f_5-f_6)-\epsilon_0+\epsilon_1,\\
f_2' &= f_2(-f_0+f_1-f_3+f_4-f_5+f_6)-\epsilon_1+\epsilon_2,\\
f_3' &= f_3(f_0-f_1+f_2-f_4+f_5-f_6)-\epsilon_2+\epsilon_3,\\
f_4' &= f_4(-f_0+f_1-f_2+f_3-f_5+f_6)-\epsilon_3+\epsilon_4,\\
f_5' &= f_5(f_0-f_1+f_2-f_4+f_4-f_6)-\epsilon_4+\epsilon_5,\\
f_6' &= f_6(-f_0+f_1-f_2+f_3-f_4+f_5)-\epsilon_5+\epsilon_6.
\end{aligned}$$

The general case for odd n *is similar.*

Observation. Apart from small changes the above is the symmetric Lax pair introduced by M. Noumi, Y. Yamada et al.(see [N-Y, S-H-C]). The changes are:
(a). ϵ_j is changed into $\epsilon_j - \frac{n-1}{2n}$ in order to obtain a matrix with trace zero. This corresponds to a small change in the definition of D, namely

$$D(e_0) = (z^{2/n}+tz^{1/n}-\frac{n-1}{2n})\,e_0 + \sum_{i=1}^{n-1}(a_i+b_iz^{1/n})\,e_i.$$

(b). A notational change of t into $\frac{t}{n}$.
(c). Transposing the matrix. This is due to the relation between a covariant solution space and a contravariant solution space.

An alternative method for $n=3$*, i.e., the Noumi-Yamada form for* P_4*.* The Lax pair equations are equivalent to $ED(e_0) = DE(e_0)$. The normalized operator D given as

$$De_0 = (z^{2/3}+tz^{1/3}+\frac{2}{3})e_0 + (a_1+b_1z^{1/3})e_1 + (a_2+b_2z^{1/3})e_2,$$

where a_1,a_2 are constants and b_1,b_2 are functions of t, commutes with the operator E such that $E(e_0) = z^{1/3}e_0 + c_1e_1 + c_2e_2$ (for suitable functions c_1,c_2 of t) if and only b_1,b_2 satisfy the differential equations

$$\begin{aligned}
b_1' &= a_1(1-\omega)+b_1t(2\omega+1)+b_2^2(-2\omega-1),\\
b_2' &= a_2(\omega+2)+b_1^2(2\omega+1)+b_2t(-2\omega-1).
\end{aligned}$$

This is in fact a Hamiltonian system $b_1' = \frac{\partial H}{\partial b_2}, b_2' = -\frac{\partial H}{\partial b_1}$ with $\omega = e^{2\pi i/3}$ and $H = -(\frac{b_1^3}{3}+\frac{b_2^3}{3})(2\omega+1)+b_1b_2t(2\omega+1)-b_1a_2(\omega+2)-b_2a_1(\omega-1)$.

After a linear change of variables this Hamiltonian coincides with Okamoto's standard Hamiltonian for P_4 (see [O1, p. 265]).

2.2. **The moduli spaces $\mathcal{M}_n$ and $\mathcal{R}_n$ for even n.**
We proceed as in §2.1. Write $n = 2m$ and $\omega = e^{2\pi i/n}$. The eigenvalues at $z = \infty$ are $q_j = \omega^{j/m} z^{1/m} + \omega^{j/2m} t z^{1/2m}$ for $j = 0, \dots, 2m-1$.

Now $N := \sum_{i \neq j} \deg(q_i - q_j) = 4m - 3$, $\mathcal{R}_{2m} \cong \mathbb{C}^N$ and $\mathcal{M}_{2m}$ has dimension $1 + 4m - 3$. The parameter space $\mathcal{P}_{2m}$ for the monodromy space consists of the characteristic polynomials of the monodromy at $z = 0$. Since $\Lambda^{2m} M$ is trivial, this monodromy has determinant 1. Thus $\dim \mathcal{P}_{2m} = 2m - 1$. The fibres of $\mathcal{R}_{2m} \to \mathcal{P}_{2m}$ have dimension $2m - 2$. We make the method of construction a differential operator, a Lax pair and Painlevé type equations explicit for $n = 4$. The general case is discussed after that.

2.2.1. The case $n = 4$. We expect, based on [vdP-Si, §12.3-12.5], that the differential operator D has on the basis e_0, e_1, e_2, e_3 the formula

$$D(e_0) = (z^{1/2} + \frac{t}{4} z^{1/4} - 3/8)\, e_0 + (a_1 + b_1 z^{1/4})\, e_1 + a_2\, e_2 + (a_3 + b_3 z^{1/4})\, e_3,$$

and D commutes with σ defined by $\sigma e_j = e_{j+1}$ for $j = 0, 1, 2$ and $\sigma e_3 = e_0$ and $\sigma z^{1/4} = i z^{1/4}$. Consider the following basis of invariants:

$$\begin{aligned} B_0 &= e_0 + e_1 + e_2 + e_3, & B_1 &= z^{1/4}(e_0 + i e_1 + i^2 e_2 + i^3 e_3), \\ B_2 &= z^{1/2}(e_0 - e_1 + e_2 - e_3), & B_3 &= z^{3/4}(e_0 - i e_1 - e_2 + i e_3). \end{aligned}$$

The matrix of D with respect to this basis is

$$\begin{pmatrix} -\frac{3}{8} + a_1 + a_2 + a_3 & 0 & z & z(\frac{t}{4} + b_1 + b_3) \\ \frac{t}{4} - i b_1 + i b_3 & -\frac{1}{8} - i a_1 - a_2 + i a_3 & 0 & z \\ 1 & \frac{t}{4} - b_1 - b_3 & \frac{1}{8} - a_1 + a_2 - a_3 & 0 \\ 0 & 1 & \frac{t}{4} + i b_1 - i b_3 & \frac{3}{8} + i a_1 - a_2 - i a_3 \end{pmatrix}$$

and D is equal to the differential operator $z\frac{d}{dz} + \begin{pmatrix} \epsilon_0 & 0 & z & z f_0 \\ f_1 & \epsilon_1 & 0 & z \\ 1 & f_2 & \epsilon_2 & 0 \\ 0 & 1 & f_3 & \epsilon_3 \end{pmatrix}$ with $\sum \epsilon_j = 0$, $f_0 + f_2 = f_1 + f_3 = \frac{t}{2}$. The $\epsilon_0, \dots, \epsilon_3$ are parameters.

The operator D is completed to a Lax pair by the differential operator E w.r.t. $\frac{d}{dt}$. This operator, written on the basis e_0, e_1, e_2 is σ-invariant and has the form $E(e_0) = z^{1/4} e_0 + \sum_{j=1}^{3} h_j e_j$ for suitable functions h_1, h_2, h_3 of t. On the basis B_0, B_1, B_2 one obtains $E := \frac{d}{dt} + \begin{pmatrix} g_0 & 0 & 0 & z \\ 1 & g_1 & 0 & 0 \\ 0 & 1 & g_2 & 0 \\ 0 & 0 & 1 & g_3 \end{pmatrix}$ with $\sum g_j = 0$. The assumption that E commutes with D produces equations for the derivatives of f_0, f_1, f_2, f_3, seen as functions of t. These formulas are similar to those derived by Noumi–Yamada. Moreover, combining the differential equations for f_0 and f_1 leads to the standard P_5 equation, see [N-Y, S-H-C] for details.

An alternative computation is a consequence of the observation that isomonodromy is given by $DE(e_0) = ED(e_0)$ and with $a_1, a_2, a_3 \in \mathbb{C}$. This produces equations with parameters a_1, a_2, a_3:

$$4t \cdot \frac{db_1}{dt} = -16ib_1^2b_3 + t^2b_1i + 16ib_3^3 - 4ia_1t + 4a_1t - 32a_2b_3$$

$$4t \cdot \frac{db_3}{dt} = -16ib_1^3 + 16ib_1b_3^2 - t^2b_3i + 4ia_3t - 32a_2b_1 + 4a_3t,$$

$$h_1 = -ib_1/2 + b_1/2,\ t \cdot h_2 = 2b_1^2i - 2b_3^2i + 4a_2,\ h_3 = b_3i/2 + b_3/2.$$

One observes that the equations for b_1, b_3 form a Hamiltonian system with

$$t\frac{db_1}{dt} = \frac{\partial H}{\partial b_3},\ t\frac{db_3}{dt} = -\frac{\partial H}{\partial b_1},$$

$$H = \frac{i(b_1^2 - b_3^2)^2}{t} + \frac{itb_1b_3}{4} + \frac{4a_2(b_1^2 - b_3^2)}{t} - (1+i)a_3b_1 + (1-i)a_1b_3.$$

Comments. The above Hamiltonian H and the differential equations for b_1, b_3 coincide, after a linear change of variables, with Okamoto's standard polynomial Hamiltonian for P_5, see [O1, p. 265].

The fibers of $\mathcal{R}_4 \to \mathcal{P}_4$. The eigenvalues at $z = \infty$ are: $q_0 = z^{1/2} + tz^{1/4}, q_1 = -z^{1/2} + itz^{1/4}, q_2 = z^{1/2} - tz^{1/4}, q_3 = -z^{1/2} - itz^{1/4}$. The differences $q_0 - q_1,\ q_0 - q_3,\ q_2 - q_1,\ q_2 - q_3$ have the form $2z^{1/2} + \dots$ and further $q_0 - q_2 = 2tz^{1/4}$ and $q_1 - q_3 = 2itz^{1/4}$. There is one singular directions in $[0, 1)$ for the terms $\pm 2z^{1/2}$. For the terms $\pm 2tz^{1/4}, \pm 2iz^{1/4}$ there is only one singular direction in $[0, 1)$. This leads to the monodromy identity

$$mon_\infty = \begin{pmatrix} & & & -1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ x_1 & 1 & x_2 & \\ & & 1 & \\ x_3 & & x_4 & 1 \end{pmatrix}.$$

One observes that mon_∞ cannot be the identity. Thus for the data considered here, $z = \infty$ singular with eigenvalues $z^{1/2} + tz^{1/4}$ and its conjugates and $z = 0$ *regular* cannot be combined.

The space $\mathcal{P}_4$ is parametrized by p_1, p_2, p_3, where the characteristic polynomial of mon_∞ is written as $T^4 + p_3T^3 + p_2T^2 + p_1T + 1$. For a suitable choice of elimination of two variables (e.g., x_1, x_2), the fibres are described by a cubic equation in three variables y, x_3, x_4 and parameters p_1, p_2, p_3. The equation reads $v_1v_2v_3 + *v_1^2 + *v_2^2 + *v_1 + *v_2 + *v_3 + * = 0$ for suitable affine expressions $*$'s in the parameters p_1, p_2, p_3. The cubic equation for the monodromy of P_5 has the same features.

2.2.2. The general case with $n = 2m$.
Consider the $\mathbb{C}(t)[z^{1/2m}]$-lattice with basis $e_0, \ldots, e_{2m-1}$, provided with the action of σ given by the formulas: $\sigma z^\lambda = e^{2\pi i\lambda} z^\lambda$ and $\sigma e_j = e_{j+1}$ for $j = 0, \ldots, 2m-2$ and $\sigma e_{2m-1} = e_0$. The operator D (with respect to the derivation $z\frac{d}{dz}$) representing $\mathcal{M}_{2m}$, is σ-invariant and is given by

$$D(e_0) = (z^{1/m} + \frac{t}{2m} z^{1/2m} - \frac{2m-1}{4m})e_0 + \sum_{j=1}^{2m-1} (a_j + b_j z^{1/2m})e_j,$$

with varying $a_j, b_j \in \mathbb{C}$ and where $b_m = 0$.
On the basis $B_0, \ldots B_{2m-1}$ of invariants, D has the form $z\frac{d}{dz} + A_0 + zA_1$. We make this explicit for $n = 6, m = 3$. The general $n = 2m$ case is similar.

$$D = z\frac{d}{dz} + \begin{pmatrix} \epsilon_0 & 0 & 0 & 0 & z & z(\frac{t}{6} + f_0) \\ \frac{t}{6} + f_1 & \epsilon_1 & 0 & 0 & 0 & z \\ 1 & \frac{t}{6} + f_2 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 1 & \frac{t}{6} + f_3 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 1 & \frac{t}{6} + f_4 & \epsilon_4 & 0 \\ 0 & 0 & 0 & 1 & \frac{t}{6} + f_5 & \epsilon_5 \end{pmatrix}.$$

The ϵ_j are linear combinations of $a_1, \ldots, a_5$ satisfying $\sum \epsilon_j = 0$, and the $f_0, \ldots, f_5$ are linear combinations of b_1, b_2, b_4, b_5 such that the relations $f_0 + f_2 + f_4 = f_1 + f_3 + f_5 = 0$ hold.

One observes that the data of the eigenvalues of A_0 are equivalent to $a_1, \ldots, a_{2m-1}$. Thus for an isomonodromic family the a_j are constant and the $b_1, \ldots, b_{2m-1}$ (with the condition $b_m = 0$) are functions of t. The differential equations for the b_j are derived from a Lax pair $z\frac{d}{dz} + A_0 + zA_1$, $\frac{d}{dt} + B$ with an, a priori, unknown matrix B depending on t and z. The action of the operator $\frac{d}{dt} + B$ on the $\mathbb{C}(t)[z^{1/2m}]$-lattice with basis $e_0, \ldots, e_{2m-1}$ is called E. It is σ-invariant and we make the "correct" guess

$$E(e_0) = z^{1/2m} e_0 + \sum_{j=1}^{2m-1} h_j e_j, \text{ for certain functions } h_1, \ldots, h_{2m-1} \text{ of } t.$$

We make this explicit for $n = 6, m = 3$ (again, the general case $n = 2m$ is similar).

$$E = \frac{d}{dt} + \begin{pmatrix} g_0 & 0 & 0 & 0 & 0 & z \\ 1 & g_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & g_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & g_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & g_5 \end{pmatrix} \text{ and } \sum g_j = 0.$$

The Painlevé type equations are similar to those in §2.2.1.

An alternative computation. From the equation $DE(e_0) = ED(e_0)$ and all a_j are constants, the differential equations for $b_1, \dots, b_{2m-1}$ follow. For the case $n = 6, m = 3$ one obtains the following Painlevé type equations for b_1, b_2, b_4, b_5 ($b_3 = 0, w = e^{2\pi i/6}$):

$$\begin{aligned}
-18t\tfrac{db_1}{dt} = {} & 288b_1b_2b_4w - 2b_1t^2w - 24b_2b_5tw - 288b_4^2b_5w + 18a_1tw \\
& -144b_1b_2b_4 + b_1t^2 + 12b_2b_5t + 144b_4^2b_5 - 18a_1t + 216a_3b_4, \\
-6t\tfrac{db_2}{dt} = {} & -4b_1^2tw + 96b_1b_2b_5w - 2b_2t^2w + 12b_4^2tw - 96b_4b_5^2w + 6a_2tw \\
& +2b_1^2t - 48b_1b_2b_5 + b_2t^2 - 6b_4^2t + 48b_4b_5^2 - 12a_2t + 72a_3b_5, \\
-6t\tfrac{db_4}{dt} = {} & 96b_1^2b_2w - 96b_1b_4b_5w - 12b_2^2tw + 2b_4t^2w + 4b_5^2tw - 6a_4tw \\
& -48b_1^2b_2 + 48b_1b_4b_5 + 6b_2^2t - b_4t^2 - 2b_5^2t + 72a_3b_1 - 6a_4t, \\
-18t\tfrac{db_5}{dt} = {} & 88b_1b_2^2w + 24b_1b_4tw - 288b_2b_4b_5w + 2b_5t^2w - 18a_5tw \\
& -144b_1b_2^2 - 12b_1b_4t + 144b_2b_4b_5 - b_5t^2 + 216a_3b_2.
\end{aligned}$$

There is a Hamiltonian function H such that

$$\frac{db_5}{dt} = -\frac{\partial H}{\partial b_1},\ \frac{db_1}{dt} = \frac{\partial H}{\partial b_5},\ \frac{db_4}{dt} = -\frac{\partial H}{\partial b_2},\ \frac{db_2}{dt} = \frac{\partial H}{\partial b_4}$$

with H defined by $3tH =$

$$\begin{aligned}
&(w-1/2)(b_2b_4 + b_1b_5/3)t^2 + ((-2w+1)b_2^3 + ((2b_5^2 - 3a_4)w - b_5^2 - 3a_4)b_2 \\
&\quad +(1-2w)b_4^3 + ((2b_1^2 - 3a_2)w - b_1^2 + 6a_2)b_4 - (3a_1b_5 + 3a_5b_1)w + 3a_1b_5)t \\
&\qquad +12(-b_1b_2 + b_4b_5)((-b_1(2w-1)b_2 + b_5(2w-1)b_4 - 3a_3),
\end{aligned}$$

where $w = e^{2\pi i/6}$.

3. $z^{4/3} + tz^{2/3}$ AND REGULAR $z = 0$.

The $q_0 = z^{4/3} + tz^{2/3}, q_1 = \omega z^{4/3} + \omega^2 tz^{2/3}, q_2 = \omega^2 z^{4/3} + \omega tz^{2/3}$ with $\omega = e^{2\pi i/3}$ are the eigenvalues. First we assume that $z = 0$ is a regular singular point. Then the monodromy space $\mathcal{R}$ is isomorphic to $\mathbb{C}^8$. The monodromy *mon* at $z = \infty$ (or equivalently at $z = 0$) is a product of the formal monodromy and 8 Stokes matrices. The singular directions in $[0, 1)$ are $\frac{15}{16}, \frac{13}{16}, \frac{11}{16}, \frac{9}{16}, \frac{7}{16}, \frac{5}{16}, \frac{3}{16}, \frac{1}{16}$ for $q_1 - q_2, q_1 - q_0, q_2 - q_0, q_2 - q_1, q_0 - q_1, q_0 - q_2, q_1 - q_2, q_1 - q_0$. Each Stokes matrix has one nontrivial entry and these are in the same order $x_{12}, x_{10}, x_{20}, x_{21}, x_{01}, x_{02}, y_{12}, y_{10}$. A computation shows that $\mathcal{R} \to \mathrm{Sl}_3(\mathbb{C})$ is birational. Moreover, the preimage of $\mathbf{1} \in \mathrm{Sl}_3(\mathbb{C})$ is one point, namely

$$x_{01} = -1, x_{02} = 1, x_{10} = 1, x_{12} = -1, x_{20} = -1, x_{21} = 1, y_{10} = 1, y_{12} = -1.$$

The rather curious conclusion is that the monodromy space, for the case that $z = 0$ is regular, consists of one point.

The formal matrix differential operator is $z\frac{d}{dz} + \begin{pmatrix} -\frac{1}{3} & tz & z^2 \\ z & 0 & tz \\ t & z & \frac{1}{3} \end{pmatrix}$. The guess that $\mathcal{M}$ is represented by the family of operators of the form $\frac{d}{dz}+\begin{pmatrix} 0 & \frac{3t}{2} & z \\ 1 & 0 & \frac{3t}{2} \\ 0 & 1 & 0 \end{pmatrix}$ is verified by a Lax pair computation.

One concludes that the Stokes matrices, which are nontrivial, in this family do not depend on t.

4. $z+t^{1/2}z^{1/2}, z-t^{1/2}z^{1/2}, -2z$ AND REGULAR SINGULAR $z=0$.

The above formulas are the eigenvalues q_0, q_1, q_2 at $z=\infty$. For $t\in\mathbb{R}, t>0$, the singular directions in $[0,1)$ are $1/2$ for q_0-q_2, q_1-q_2 and 0 for $q_1-q_0, q_2-q_0, q_2-q_1$. The monodromy identity is

$$mon=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_{02} & x_{12} & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & x_{10} & x_{20} \\ 0 & 1 & x_{21} \\ 0 & 0 & 1 \end{pmatrix}.$$

This presentation is unique up to scaling the third basis vector. One has $\dim\mathcal{R}=4$, $\dim\mathcal{P}=2$ and the fibers are affine cubic surfaces with an equation of the form $x_1x_2x_3+x_1^2+*x_1+*x_2+*x_3+*=0$. This suggest that the corresponding Painlevé type equation is equivalent to P_4. The data do not match with $z=0$ regular, since $mon=\mathbf{1}$ has no solution.

In order to simplify the computation, the eigenvalues at $z=\infty$ are replaced by $z^{1/2}, -z^{1/2}, tz$. This is similar to the case $z\pm t^{1/2}z^{1/2}, -2z$. The method of [vdP-Si, §12.5] (especially, the proof of Corollary 12.15 and Lemma 12.16) produces an explicit formula for the universal family with these data. It depends on 5 variables. By scaling of the basis vectors, one of the variables is normalized and the resulting operator has the form $z\frac{d}{dz}+A$ with

$$A=\begin{pmatrix} -2a_1 & z & a_4t-a_3 \\ 1 & 2a_1+a_3t & a_3t \\ -(a_1+2a_8)t-1 & tz & tz-a_3t \end{pmatrix}.$$

The Lax pair formalism produces a set of differential equations $\frac{da_i}{dt}=R_i$ for $i=1,3,4,8$ where the R_i are rational functions in a_1,a_3,a_4,a_8,t. The eigenvalues of the residue matrix of A at $z=0$ are independent of t. One of the eigenvalues is $2a_1+a_3t$. Adding the equation $\frac{d(2a_1+a_3t)}{dt}=0$ to the above system of equations eliminates a_4 and produces: $a_1=c_1, a_3=c_2/t$ with constants c_1,c_2 and a Riccati equation for a_8. *In particular, the Painlevé type equation for this family is solvable by classical functions.*

We have no explanation of this computational result. In contrast, a computation of the monodromy space for the case $\pm z^{1/2}, tz$ produces again a 2-dimensional family of affine cubic surfaces of the type $x_1x_2x_3+x_1^2+*x_1+*x_2+*x_3+*=0$. This corresponds to the P_4 case.

5. $z^{1/e_1}, tz^{1/e_2},\ e_1 \geq e_2 \geq 2$ and regular singular $z=0$.

For $e_1 > e_2 \geq 2$, one has $\dim \mathcal{R} = 3e_1+e_2-3$ and $\dim \mathcal{P} = e_1+e_2-1$. For the smallest case $e_1 = 3, e_2 = 2$ one has $\dim \mathcal{R} = 8$, $\dim \mathcal{P} = 4$, $\mathcal{R} \to \mathcal{P}$ is (generically)surjective and the fibres have dimension 4. The formulas for the fibres are complicated. This makes the computation of $\mathcal{M}$ and the Lax pair nearly impossible.

For $e_1 = e_2 = m \geq 2,\ n = 2m$, one has $\dim \mathcal{R} = \frac{n(n-1)}{m} - 1 = 2n-3$, $\dim \mathcal{P} = n-1$. For the smallest case $m=2$ one has $\dim \mathcal{R} - \dim \mathcal{P} = 2$. The computation in § 5.1 below produces a second order Painlevé equation which is probably a pull back of the classical P_4 equation.

5.1. The case $z^{1/2}, tz^{1/2}$. The monodromy space $\mathcal{R}$.

The eigenvectors are $q_1 = z^{1/2}, q_2 = -z^{1/2}, q_3 = tz^{1/2}, q_4 = -tz^{1/2}$ with a basis f_1, f_2, f_3, f_4 of eigenvectors is such that γ permutes the two pairs $\{f_1, f_2\}$ and $\{f_3, f_4\}$. There are 6 variables present in the Stokes matrices. Now f_1 and f_3 can be scaled independently and so $\dim \mathcal{R} = 6-1 = 5$. Further $\mathcal{P} \cong \mathbb{C}^3$ is the space of the characteristic polynomials $T^4 - p_3T^3 + p_2T^2 - p_1T + 1$ of the elements of SL_4.

For the case $t = i$, the singular directions in $[0, 1)$ are 0 for $q_2 - q_1$, $\frac{1}{4}$ for $q_2 - q_4$ and for $q_3 - q_1$, $\frac{1}{2}$ for $q_3 - q_4$, $\frac{3}{4}$ for $q_1 - q_4$ and for $q_3 - q_2$.

The topological monodromy *mon*, which has the above characteristic polynomial, is equal to the product

$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \cdot \begin{pmatrix} 1 & & & x_{14} \\ & 1 & x_{23} & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & x_{43} & 1 \end{pmatrix} \cdot$$
$$\begin{pmatrix} 1 & & x_{13} & \\ & 1 & & \\ & & 1 & \\ & x_{42} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_{12} & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The fibres of $\mathcal{R} \to \mathcal{P}$ have the following data. Assume that $x_{42} \neq 0$ (note that $x_{42} = 0$ implies reducibility). After eliminating x_{12}, x_{23} there remains one cubic equation in the variables $x_{14}, x_{42}, x_{43}, x_{13}$, namely

$$x_{14}x_{42}x_{43} - p_3x_{43} + x_{13}x_{42} + x_{43}^2 + p_2 + 2 = 0.$$

After normalizing x_{13} to 1 (note that $x_{13} = 0$ produces a similar cubic equation), the cubic equation is almost identical to the one for P_4

([vdP-Sa]) which is $x_1x_2x_3+x_1^2-(s_2^2+s_1s_3)x_1-s_2^2x_2-s_2^2x_3+s_2^2+s_1s_2^3=0$. Therefore one expects a relation with the classical P_4.

Observation. The monodromy identity depends strongly on t. For instance, if $t\in\mathbb{R}_{>0}, t\neq 1$ then there is only one singular direction in $[0,1)$. The fibres of $\mathcal{R}\to\mathcal{P}$ are again rational surfaces. The dependence of these surfaces on t is somewhat mysterious.

The space of connections $\mathcal{M}$.
A differential module M over $\mathbb{C}(z)$ in this moduli space can be considered as a differential module N over $\mathbb{C}(z^{1/2})$ with an automorphism σ satisfying $\sigma\circ z^{1/2}=-z^{1/2}\circ\sigma$ and $\sigma^2=1$. Now N can be given a basis e_1,e_2,e_3,e_4 such that σ permutes the two pairs $\{e_1,e_2\}$ and $\{e_3,e_4\}$. The corresponding module M over $\mathbb{C}(z)$ has the basis

$$B_1=e_1+e_2, B_2:=z^{1/2}(e_1-e_2), B_3=e_3+e_4, B_4=z^{1/2}(e_3-e_4).$$

Let D denote the operator of the form $z\frac{d}{dz}+(a\ matrix)$ acting upon N. This operator commutes with σ and is determined by De_1, De_3. The formal part $\hat{D}$ of D is given by $\hat{D}e_1=z^{1/2}e_1, \hat{D}e_3=tz^{1/2}e_3$. Using [vdP-Si, §12] one concludes that D is given by the formulas

$$De_1=z^{1/2}e_1+a_1e_2+a_2e_3+a_3e_4,\quad De_3=tz^{1/2}e_3+a_4e_1+a_5e_2+a_6e_4$$

with constants $a_1,\dots,a_6$. For the generic case one can normalize to $a_5=1$. A computation of D on the basis $B_1,\dots,B_4$ produces the operator (normalized to trace equal to zero) $z\frac{d}{dz}+A$ with

$$A=\begin{pmatrix} a_1-1/4 & z & a_4+a_5 & 0\\ 1 & -a_1+1/4 & 0 & a_4-a_5\\ a_2+a_3 & 0 & a_6-1/4 & tz\\ 0 & a_2-a_3 & t & -a_6+1/4\end{pmatrix}.$$

As mentioned above one may normalize to $a_5=1$. Further the coefficients of the characteristic polynomial of residue matrix at $z=0$ are the parameters. We conclude that for fixed parameters, the above family of operators has dimension 2 (not counting the variable t). This is in agreement with the computation of the fibers of $\mathcal{R}\to\mathcal{P}$.

The operator $z\frac{d}{dz}+A$ is extended to a Lax pair by an operator of the form $\frac{d}{dt}+B$ with $B=B_0+B_1z$ for matrices B_0,B_1 depending on t only. The property $\frac{d}{dt}(A)=z\frac{d}{dz}(B)+[A,B]$ yields a vector field, represented by differential equations $\frac{d}{dt}a_j=R_j,\ j=1,2,3,4,6$ with R_j rational expressions in $a_1,\dots a_4,a_6,t$. The characteristic polynomial of the residue matrix at $z=0$ is written as $T^4+P_2T^2+P_1T+P_0$. We have used the formulas for P_1, P_2 and another invariant $P_0 = a_1 + a_6$ to eliminate (stepwise) the functions a_3, a_6, a_2. For the remaining a_1, a_4 one obtains the equations

$$\frac{da_1}{dt} = \frac{(2P_1a_4^2+(-4P_0^2+(8a_1+2)P_0-8a_1^2-4P_2)a_4+2P_1)}{(a_4^2-1)(t^2-1))},$$
$$\frac{da_4}{dt} = \frac{((-2a_4^2+2)a_1t^2+2(a_4^2+1)(P_0-2a_1)t+2(a_4-1)(a_4+1)(-a_1+P_0))}{(t^3-t)}.$$

The second equation can be used to write a_1 as an expression in a_4 and $\frac{da_4}{dt}$. Substitution in the first equation yields an explicit (and rather long) second order differential equation for a_4. The poles w.r.t. t are $0, 1, -1, \infty$. We did not find a relation to a classical Painlevé equation. A possible approach could be to search for a Hamiltonian and apply the ideas used in [D-F-L-S].

6. $(z^3 + tz)_{m_1}, (\frac{-m_1}{m_2}(z^3 + tz))_{m_2}$ AND REGULAR $z = 0$.

The case $m_1 = m_2 = 1$. Then $q_1 = z^3 + tz, q_2 = -(z^3 + tz)$; the singular directions for $q_1 - q_2$ are $\frac{1,3,5}{6}$; the singular directions for $q_2 - q_1$ are $\frac{0,2,4}{6}$. The monodromy identity reads

$$mon = \begin{pmatrix} g & 0 \\ 0 & 1/g \end{pmatrix}\begin{pmatrix} 1 & x_5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_4 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}.$$

The equation $mon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and division by $\mathbb{G}_m$ (made explicit by normalizing $x_5 = 1$) produce an explicit $\mathcal{R}$ of dimension 3. Furthermore $\dim \mathcal{P} = 1$ (parameter g). This is in fact the standard family for P_2.

Case $m_1 = 2, m_2 = 1$, $z^3 + tz, z^3 + tz, -2z^3 - 2tz$ *and* $z = 0$ *is regular.*

Description of $\mathcal{R}$. The Stokes matrices are described by 12 variables, there are 2 variables describing the formal monodromy. The topological monodromy is supposed to be the identity. This produces 8 equations.

Actual computation, using the monodromy identity produces a space of dimension 6. Dividing by the action of $\mathbb{G}_m^2$, due to scaling the basis vectors, produces $\dim \mathcal{R} = 4$. Moreover $\dim \mathcal{P} = 2$ and the fibers of $\mathcal{R} \to \mathcal{P}$ have dimension 2. One expects a relation with a classical Painlevé equation with at most two singularities.

We follow the discussion of the cases (i) and (ii) on pages 6-7 for the construction of the matrix differential operator $z\frac{d}{dz} + A$. The formal operator $ST := z\frac{d}{dz} + diag(z^3 + tz + a_1, z^3 + tz + a_2, -2z^3 - 2tz - a_1 - a_2)$ is conjugated with the matrix $\mathbf{1} + M$, where $M =$

$$\begin{pmatrix} 0 & 0 & b_1/z + b_2/z^2 + b_3/z^3 \\ 0 & 0 & b_4/z + b_5/z^2 + b_6/z^3 \\ b_7/z + b_8/z^2 + b_9/z^3 & b_{10}/z + b_{11}/z^2 + b_{12}/z^3 & 0 \end{pmatrix}.$$

This produces an operator $z\frac{d}{dz}+\widetilde{A}=Prin(z\frac{d}{dz}+(\mathbf{1}+M)ST(\mathbf{1}+M)^{-1})$. Then A is obtained from $\widetilde{A}$ by adding the eight equations given by $\widetilde{A}(0)=0$. After scaling the basis vectors one has $b_2=1, b_{11}=1$ and the matrix A depends only on the variables b_1, b_5, b_7, b_8. The substitution $b_5=B_5b_8$ removes some denominators. The characteristic polynomial of the formal monodromy at $z=\infty$ is written as T^3+pT+q. The Lax pair equation together with the equations $\frac{dp}{dt}=0, \frac{dq}{dt}=0$ produce a differential system

$$\frac{dB_5}{dt}=0, \frac{db_1}{dt}=-6B_5b_1^3b_7-6b_1^3b_7-3b_1^2t-3,$$

$$\frac{db_7}{dt}=(27b_7^2(B_5+1)^2b_1^2+18tb_7(B_5+1)b_1+3t^2-p)/(3B_5+3).$$

This is a Hamiltonian system $\frac{db_1}{dt}=\frac{\partial H}{\partial b_7}, \frac{db_7}{dt}=-\frac{\partial H}{\partial b_1}$, depending on parameters B_5, p, with $H=-3(B_5+1)b_7^2b_1^3-3tb_7b_1^2+\frac{-3t^2+p}{3B_5+3}b_1-3b_7$. There is no evident relation between this Hamiltonian and Okamoto's list of polynomial Hamiltonians in [O1]. Possibly the method described in [D-F-L-S] can be applied here.

7. $z, tz, (-1-t)z$ and M. Mazzocco

7.1. **Computation of Stokes data and monodromy space $\mathcal{R}$.**

The formal solution space V at $z=\infty$ has a basis e_0, e_1, e_2 such that the formal differential operator has the form

$$z\frac{d}{dz}+\begin{pmatrix} z+a_0 & & \\ & tz+a_1 & \\ & & (-1-t)z-a_0-a_1\end{pmatrix}.$$

The formal monodromy and the Stokes matrices are given with respect to this natural basis. The basis is unique up to multiplying each e_j by a scalar. Since $z=0$ is regular singular, one has $\dim\mathcal{R}=6$. The formal monodromy at $z=\infty$ is the diagonal matrix $diag(g_1,g_2,g_3)$ with $g_1g_2g_3=1$ and $g_1=e^{2\pi i a_0}, g_2=e^{2\pi i a_1}$.

The singular directions depend on t. For t close to i, the singular direction $d_{kl}\in[0,1)$ for q_k-q_l are approximated by

$$0.93,\ 0.83,\ 0.62,\ 0.43,\ 0.33,\ 0.12 \text{ for } d_{20},\ d_{21},\ d_{01},\ d_{02},\ d_{12},\ d_{10}.$$

This determines the order of the six Stokes matrices in the monodromy identity, which states that the topological monodromy mon at $z=0$

is, up to conjugation, equal to the product

$$\begin{pmatrix} g_1 & & \\ & g_2 & \\ & & \frac{1}{g_1 g_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ x_{20} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & x_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_{01} & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot$$
$$\begin{pmatrix} 1 & & x_{02} \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & 1 & x_{12} \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ x_{10} & 1 & \\ & & 1 \end{pmatrix}.$$

If $z = 0$ is regular (i.e., $mon = \mathbf{1}$), then the Stokes matrices and the formal monodromy are equal to the identity. This case is uninteresting.

Suppose now that $z = 0$ is regular singular. Then $\dim \mathcal{P} = 4$ and the basic parameters are g_1, g_2 and the coefficients of the characteristic polynomial of mon. The fibers of $\mathcal{R} \to \mathcal{P}$ are affine cubic surfaces with equation $xyz + x^2 + y^2 + z^2 + p_1 x + p_2 y + p_3 z + p_4 = 0$ with $p_1, \ldots, p_4$ expressions in the parameters. This is an indication that the Painlevé type equation is related to P_6. We will compute $\mathcal{M}$ and the Lax pair. Due to the complexity, we did not find an explicit relation with P_6.

M. Mazzocco ([Maz]) studied the equivalent situation in which the eigenvalues at $z = \infty$ are $0, z, tz$. The matrix differential equation has the form $z\frac{d}{dz} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & tz & 0 \\ 0 & 0 & z \end{pmatrix} + W$, where the "constant" matrix W has a rather special form involving 4 parameters. This form is chosen in order to produce a relation with P_6 and its parameters by means of explicit formulas connecting the solutions of the two equations. The proof of this equivalence uses Fourier-Laplace transforms and Stokes matrices.

7.2. **Constructing the connection and the Lax pair.**

A Zariski open subset of the moduli space of connections $\mathcal{M}$ is obtained by considering the family of differential operators of the form

$$z\frac{d}{dz} + \begin{pmatrix} a_0 & m_1 & m_2 \\ m_3 & a_1 & m_4 \\ m_5 & m_6 & -a_0 - a_1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -1-t \end{pmatrix}$$

(compare [vdP-Si, Theorem 12.4]). Each of the basis vectors for this presentation can be multiplied by nonzero elements and the family has to be divided by this action of $\mathbb{G}_m^2$. A Zariski open part of the quotient space is obtained by assuming $m_3 m_4 \neq 0$ and normalizing $m_3 = m_4 = 1$. In this way one obtains the following explicit description of an open part of $\mathcal{M}$ in terms of

$$z\frac{d}{dz}+\begin{pmatrix} z+a_0 & v_1 & v_2 \\ 1 & tz+a_1 & 1 \\ v_3 & v_4 & (-1-t)z-a_0-a_1 \end{pmatrix}.$$

This operator (with $v_1,\dots,v_4$ as functions of t) is completed to a Lax pair with the operator $\frac{d}{dt}+B_0(t)+zB_1(t)$. The assumption that the two operators commute leads to a set of differential equations for $v_1,\dots,v_4$, namely

$$v_1'=\frac{-3v_3v_1+3v_2v_4}{2t^2+5t+2},\ v_2'=\frac{(6t+3)v_2^2-3v_3(t-1)v_2-3v_1(t+2)+(-9a_0t+9a_1)v_2}{2t^3+3t^2-3t-2},$$

$$v_3'=\frac{(3t-3)v_3^2+(-6t-3)v_2v_3+(9a_0t-9a_1)v_3+3v_4(t+2)}{2t^3+3t^2-3t-2},\ v_4'=\frac{3v_3v_1-3v_2v_4}{t^2+t-2}.$$

In a monodromic family the topological monodromy is constant and then also the characteristic polynomial of the residue matrix is constant. This means that there are constants δ_0,δ_1, explicitly

$$-a_0^2-a_0a_1-a_1^2-v_2v_3-v_1-v_4=\delta_1,$$

$$a_0^2a_1+a_0a_1^2+a_1v_2v_3-a_0v_1+a_0v_4-a_1v_1-v_1v_3-v_2v_4=\delta_0.$$

The algebra of functions on the parameter space for the connection is generated by $a_0,a_1,\delta_0,\delta_1$. They correspond to the 4 parameters for the moduli space $\mathcal{R}$ of the analytic data. Using these equations one eliminates v_1 and v_4 and one is left with the two differential equations

$$\begin{gathered}(2a_0+a_1-v_2+v_3)(2t^3+3t^2-3t-2)v_2'=\\ =(-6t-3)v_2^3+((21a_0+6a_1+6v_3)t+6a_0-6a_1-6v_3)v_2^2\\ +\{(-21a_0^2+(-12\,a_1-12\,v_3)a_0-3\,a_1^2-6a_1v_3-3v_3^2-3\delta_1)t-6a_0^2\\ +(12a_1+12v_3)a_0+3a_1^2+6a_1v_3+3v_3^2-6\delta_1\}v_2+3(a_0^3+a_0\delta_1+\delta_0)(t+2),\end{gathered}$$

$$\begin{gathered}(2a_0+a_1-v_2+v_3)(2t^3+3t^2-3t-2)v_3'=\\ (3t-3)v_3^3+((15a_0+3a_1-12v_2)t-6a_0-12a_1-6v_2)v_3^2\\ +\{(15a_0^2+(6a_1-24v_2)a_0-3a_1^2-12a_1v_2+6v_2^2-3\delta_1)t-6a_0^2\\ -(24a_1+12v_2)a_0-15a_1^2-6a_1v_2+3v_2^2-6\delta_1\}v_3\\ -3(t+2)(a_0^3+3a_0^2a_1+(3a_1^2+\delta_1)a_0+a_1^3+a_1\delta_1-\delta_0).\end{gathered}$$

This Painlevé vector field of dimension 2 can be transformed, as follows, into a second order differential equation. The equation for v_3' is used to eliminate v_2. This introduces a quadratic extension of $\mathbb{C}(t,v_3)$. Substitution of this formula for v_2 and its derivative in the equation involving v_2' produces an implicit order two equation $P(v_3'',v_3',v_3,t)=0$, of degree 2 in v_3''. Interchanging the roles of v_2 and v_3 produces a similar result. The formulas are too complicated to present here.

Remarks. There are explicit formulas for reducible loci and the corresponding Riccati equations. These turn out to be hypergeometric differential equations, which is again an indication of a relation with P_6. Example: $v_3 = v_4 = 0$ and $v_1' = 0$, $v_2' = \frac{3(2t+1)v_2^2 - 3(t+2)v_1 + 9(-a_0t+a_1)v_2}{(t-1)(2t+1)(t+2)}$.

7.3. **The hierarchy** $(z)_{m_1}, (tz)_{m_2}(\frac{-m_1-tm_2}{m_3}z)_{m_3}$.
The $m_1, m_2, m_3 \geq 1$ stand for multiplicities. A computation shows that "$z = 0$ is regular" leads to trivial Stokes matrices and formal monodromy. Consider the case $m_1 = 2, m_2 = m_3 = 1$. Then $\dim \mathcal{R} = 10$ and $\dim \mathcal{P} = 6$. According to [vdP-Si, Exercise 12.5, p. 300], the universal family is

$$z\frac{d}{dz} + \begin{pmatrix} z + a_1 & 0 & x_1 & x_2 \\ 0 & z + a_2 & x_3 & x_4 \\ x_5 & x_6 & tz + a_3 & x_7 \\ x_8 & x_9 & x_{10} & (-2-t)z - a_1 - a_2 - a_3 \end{pmatrix}.$$

Consider a normalization, say, $x_8 = x_9 = x_{10} = 1$, obtained by restricting to the open subspace defined by $x_8x_9x_{10} \neq 0$ and multiplying the base vectors by scalars. The Lax pair consists of the above operator and $\frac{d}{dt} + B_0 + zB_1$. An easy computation produces a system of differential equations (or vector field) $\frac{dx_j}{dt} \in \mathbb{C}(a_1, \ldots, a_4, x_1, \ldots, x_7)$, $j = 1, \ldots, 7$. Using three coefficients of the characteristic polynomial of the residue matrix at $z = 0$, one can eliminate x_2, x_4, x_7 in a rational way. The resulting vector field of rank 4 is a Painlevé type system. It is unfortunately too complicated. The system contains, of course, many closed subsystems corresponding to $z, tz, (-1-t)z$, that produce equations related to P_6.

8. $z^{1/e}, tz, -tz,\ e > 1$ AND REGULAR SINGULAR $z = 0$.

Write $q_1, \ldots, q_e$ for the conjugates of $z^{1/e}$; write r and s for tz and $-tz$. The value of the integer N (i.e., the dimension of the Stokes data) is equal to $e(e-1)\frac{1}{e}$ (for the $q_k - q_l$) plus $e + e + e + e$ (for $q_k - r, r - q_k, q_k - s, s - q_k$) plus 2 (for $r - s, s - r$) and sums up to $5e + 1$. The formal monodromy γ depends on 1 parameter. Normalization by the action of $\mathbb{G}_m^2$ results in $\dim \mathcal{R} = 5e$. The parameters are the $e + 1$ coefficients of the characteristic polynomial of the monodromy at $z = 0$ and the eigenvalues of γ. This leads to $\dim \mathcal{P} = e + 2$ and $\dim \mathcal{R} - \dim \mathcal{P} = 4e - 2$.

The case $e = 2$. For $t = 1$ the singular directions in $[0, 1)$ are $d = 1/2$ for $r - q_1, r - q_2, q_1 - s, q_2 - s, r - s$ and $q_2 - q_1, q_1 - r, q_2 - r, s - q_1, s - q_2, s - r$

for $d = 0$. This leads to a matrix formula for the topological monodromy

$$mon_0 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & g & \\ & & & 1/g \end{pmatrix} \cdot \begin{pmatrix} 1 & & x_1 & \\ & 1 & x_2 & \\ & & 1 & \\ x_3 & x_4 & x_5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_6 & & x_7 \\ & 1 & & x_8 \\ x_9 & x_{10} & 1 & x_{11} \\ & & & 1 \end{pmatrix}.$$

One may normalize to $x_{10} = x_{11} = 1$. The parameters are g and three coefficients of the characteristic polynomial of mon_0. A Maple computation verifies that the fibres of $\mathcal{R} \to \mathcal{P}$ are birational to $\mathbb{A}^6$.

For the differential operator $z\frac{d}{dz} + A$ we propose the following guess (based on [vdP-Si, §12])

$$A = \begin{pmatrix} a & z & * & * \\ 1 & -a & * & * \\ * & * & tz+b & * \\ * & * & * & -tz-b \end{pmatrix},$$

where a, b and $*$ are variables. Since two of the constants can be normalized to 0, this is a family of dimension 10 (over the field $\mathbb{C}(t)$). The invariants are three coefficients of the characteristic polynomial of the residue matrix at $z = 0$ and b at $z = \infty$.

The Lax pair $\{z\frac{d}{dz} + A, \frac{d}{dt} + B\}$ has the form $B = B_0 + B_1 z$, where B_0, B_1 are traceless matrices depending on t only. The Lax pair calculations for the case $e = 2$ produces a rational Painlevé vector field of dimension 6 which is too large to be presented here.

9. $z^2, -z^2 - tz, tz$ and $z = 0$ regular.

The assumptions: $z = 0$ is regular and $z = \infty$ is unramified and has Katz invariant 2 produces the eigenvalues $(z^2)_{m_1}, (a_2z^2 + a_1z)_{m_2}, (b_1z)_{m_3}$ such that the m_1, m_2, m_3 satisfy $m_1z^2 + m_2(a_2z^2 + a_1z) + m_3b_1z = 0$. Then $\dim \mathcal{R} = (n-1)^2$, where $n = m_1 + m_2 + m_3 \geq 3$.

We present computations for the case $m_1 = m_2 = m_3 = 1$. Then $\dim \mathcal{R} = 4$ and $\dim \mathcal{P} = 2$. The fibers of $\mathcal{R} \to \mathcal{P}$ are affine cubic surfaces, which have, after an affine linear change of the variables, the equation $xyz + x + y + 1 = 0$. We recall from [vdP-Sa], that in the P_1 case, the parameter space $\mathcal{P}$ is one point and that $\mathcal{R}$ is the affine cubic surface with the equation $xyz + x + y + 1 = 0$. This suggests a strong relation between P_1 and the case under consideration.

We *propose* a normalized differential operator

$$\frac{d}{dz}+\begin{pmatrix} z & a_1 & 1 \\ 1 & -z-t & a_2 \\ a_3 & a_4 & t \end{pmatrix}.$$

The reasoning for this proposal is the following. Observe that

$$z\frac{d}{dz}+\begin{pmatrix} z^2+c_1 & 0 & 0 \\ 0 & -z^2-tz+c_2 & 0 \\ 0 & 0 & tz-c_1-c_2 \end{pmatrix}$$

has at $z=\infty$ the universal deformation

$$z\frac{d}{dz}+\begin{pmatrix} z^2+c_1 & * & * \\ * & -z^2-tz+c_2 & * \\ * & * & tz-c_1-c_2 \end{pmatrix}.$$

Here the $*$'s are arbitrary polynomials in z of degree ≤ 1. The assumption that $z=0$ is regular implies that $c_1=c_2=0$ and the $*$ are elements of $\mathbb{C}z$. Finally, in the general case one can, by a change of the basis, arrive at two entries being z. Dividing by z produces the above proposal.

For the Lax pair situation a_1,a_2,a_3,a_4 are functions of t and the above operator is supposed to commute with $\frac{d}{dt}+B(z,t)$, where $B(z,t)$ has degree 1 in the variable z. The resulting differential equations are

$$\begin{array}{ll} a_1'=-3a_1a_2a_3+3a_4, & a_2'=-\frac{3}{2}a_1a_2^2+3a_2^2a_3-\frac{9}{2}a_2t-3/2, \\ a_3'=\frac{3}{2}a_3a_1a_2-\frac{3}{2}a_4, & a_4'=\frac{3}{2}a_4a_1a_2-3a_4a_2a_3+\frac{3}{2}a_1a_3+\frac{9}{2}a_4t. \end{array}$$

The two parameters describing the parameter space $\mathcal{P}$ are $p_1=a_1+2a_3$ and $p_2:=a_1+a_3+a_2a_4$ (thus $p_1'=0, p_2'=0$). Elimination of a_1,a_3 leads to the system of differential equations

$$\begin{array}{l} a_2'=-\frac{3}{2}+6a_2^3a_4+\frac{9p_1-12p_2}{2}a_2^2-\frac{9a_2t}{2}, \\ a_4'=\frac{3(-p_1+2p_2)(-p_2+p_1)}{2}+\frac{9t}{2}a_4+(-9p_1+12p_2)a_2a_4-9a_2^2a_4^2. \end{array}$$

The first equation can be used to write a_4 as rational expression in a_2,a_2'. Substitution in the second equation yields an explicit second order equation. The Hamiltonian H is equal to

$$-3a_4^2a_2^3-\frac{9p_1-12p_2}{2}a_4a_2^2-\frac{3p_1^2-9p_1p_2+6p_2^2}{2}a_2+\frac{3}{2}a_4+\frac{9a_2a_4}{2}t.$$

where $a_4'=\frac{\partial H}{\partial a_2}$, $a_2'=-\frac{\partial H}{\partial a_4}$ and p_1,p_2 are parameters (constants). We did not find a relation with some P_*.

10. $1/z, -1/z$ AT $z = 0$ AND $tz, -tz$ AT $z = \infty$.

The data above is the $m = 1$ case of the hierarchy $\mathcal{M}_m$ defined for $m \geq 1$ by the eigenvalues $(z^{-1})_m, -mz^{-1}$ at $z = 0$ and $(tz)_m, -mtz$ at $z = \infty$. One shows that $\dim \mathcal{R}_m = 4m$ and $\dim \mathcal{P} = 2m$. The case $m = 1$ is the family for $P_3(D_6)$.

For $m = 2$ a computation reveals that $\mathcal{R} \to \mathcal{P}$ is surjective with fibres of dimension 4. A computation of $\mathcal{M}$, indicated in the "Introduction and Summary", and a choice of normalizations give rise to an operator $z\frac{d}{dz} + A$ representing an open affine subset of $\mathcal{M}$, with $A =$

$$z^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} c_1 & 0 & -3m_1 \\ 0 & c_2 & -3 \\ 3 & 3m_3 & -c_1 - c_2 \end{pmatrix} + tz \cdot P \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} P^{-1},$$

where $P = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_3 & x_4 & 1 \end{pmatrix}$. A straightforward Lax pair computation produces formulas for $\frac{dx_i}{dt}, i = 1, ..., 4$, as rational functions in x_1, x_2, x_3, x_4, t. These are however too large to be displayed here. The computation does show that indeed in an isomonodromic family c_1, c_2, m_3, m_1 are constant.

11. $z^{-1/2}$ AT $z = 0$ AND $tz, -tz$ AT $z = \infty$.

The above defines the usual family for $P_3(D_7)$. By attaching multiplicities, e.g., $(z^{-1/2})_m, (tz)_m, (-tz)_m$ and $m \geq 1$, one obtains a hierarchy. For the case $m = 2$, the space $\mathcal{R}$ is given by the equation

$$L \circ mon_0 = mon_\infty \circ L \text{ with } L : V(0) \to V(\infty) \text{ a linear bijection}$$

where $mon_0 : V(0) \to V(0),\ mon_\infty : V(\infty) \to V(\infty)$ are the topological monodromies. The link L is considered up to multiplication by $\mathbb{C}^*$ and the matrices of mon_0 and mon_∞ have the form

$$mon_0 = \begin{pmatrix} & & * & \\ & & & * \\ 1 & & & \\ & 1 & & \end{pmatrix} \cdot \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$mon_\infty = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \cdot \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{pmatrix}.$$

After normalization one obtains $\dim \mathcal{R} = 12$ and $\dim \mathcal{P} = 4$. Therefore the relative dimension of $\mathcal{M}$ over $\mathbb{C}(t)$ is also 12. The construction of $\mathcal{M}$ and the Lax pair computations seem to be out of reach.

12. $z^{-1/n}$ AND $tz^{1/n}$, A HIERARCHY RELATED TO $P_3(D_8)$.

The assumption that $z=0$ and $z=\infty$ are both irregular singular and totally ramified leads, after normalization, to the Galois orbit of $z^{-1/n}$ at $z=0$ and the Galois orbit $tz^{1/n}$ at $z=\infty$. In the sequel we replace t by $t^{1/n}$. The moduli spaces will be denoted by $\mathcal{M}_n$ and $\mathcal{R}_n$. The standard isomonodromic family for $P_3(D_8)$ is derived from $\mathcal{M}_2$. First we study the structure of $\mathcal{R}_n$.

12.1. **The structure of the monodromy space $\mathcal{R}_n$.**

We refer to [vdP-Si, § 8,9] for notation and results. For a connection $M \in \mathcal{M}_n$, the solution space $V(\infty)$ at $z=\infty$ has the structure: $V(\infty) = \oplus_{j=0}^{n-1}\mathbb{C}e_j$ with $\mathbb{C}e_j = V(\infty)_{q_j}$, $q_j = \sigma^j(t^{1/n}z^{1/n}) = \zeta_n^j t^{1/n}z^{1/n}$, where $\zeta_n = e^{2\pi i/n}$. The basis $\{e_j\}$ is chosen such that the formal monodromy $\gamma_{V(\infty)}$ acts by $e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto (-1)^{n-1}e_0$.

By Lemma 0.1, the space of the Stokes data at $z=\infty$ can be identified with $\mathbb{C}^{n-1}$. The monodromy identity for the topological monodromy mon_∞ at $z=\infty$ has been studied in detail in [CM-vdP, p. 146-147]. The surprising property is:

Let the Stokes data be $(x_1,\dots,x_{n-1}) \in \mathbb{C}^{n-1}$. Then the characteristic polynomial of the topological monodromy mon_∞ is

$$T^n + x_{n-1}T^{n-1} + \cdots + x_1T + (-1)^n.$$

Thus the map from the Stokes data to the characteristic polynomial of mon_∞ is bijective.

The local solution space $V(0)$ at $z=0$ has a similar description. The map from the space of the Stokes matrices to the (non trivial) coefficients of the characteristic polynomial of the topological monodromy mon_0 at $z=0$, is bijective.

The monodromy space $\mathcal{R}_n$ consists of the local analytic data at $z=\infty$ and $z=0$ together with a link which glues the solution space above $\mathbb{P}^1 \setminus \{\infty\}$ to the solution space above $\mathbb{P}^1 \setminus \{0\}$. More precisely, the link $L : V(0) \to V(\infty)$ is a linear bijection such that $L \circ (mon_0)^{-1} = mon_\infty \circ L$. The "inverse sign" reflects the difference in directions of the paths for mon_0 and mon_∞.

It follows that all the structure of $V(0)$ is determined by mon_∞ and the link. In particular, mon_0^{-1} has the same characteristic polynomial as mon_∞. Furthermore, the Stokes matrices at $z=0$ are the same as those at $z=\infty$, however taken in the opposite order.

Let L_0 be a fixed choice for the link. Any other link has the form $M \circ L_0$ where $M = (m_{i,j}) \in \mathrm{GL}(V(\infty))$ commutes with mon_∞. Then $\mathcal{R}_n$ can be identified with the tuples $(M, x_1, \ldots, x_{n-1})$ as above and M taken modulo multiplication by a scalar (since the basis of $V(0)$ and $V(\infty)$ can be scaled).

A computation, using the formulas in [CM-vdP, § 3.3, § 3.4] for mon_∞, shows that M is determined by its last row $(m_{n,1}, \ldots, m_{n,n})$ and that this space can be identified with the open subspace of $\mathbb{P}^{n-1} \times \mathbb{A}^{n-1}$ consisting of the tuples

$$((m_{n,1} : \ldots : m_{n,n}), (x_1, \ldots, x_{n-1})) \in \mathbb{P}^{n-1} \times \mathbb{A}^{n-1}$$

such that the determinant F of the matrix M is not zero. One easily sees that F is homogeneous of degree n in the n variables $m_{n,1}, \ldots, m_{n,n}$ and its coefficients are polynomials in $x_1, \ldots, x_{n-1}$. In particular, $\mathcal{R}_n$ *is smooth, connected, quasi projective of dimension* $2(n-1)$.

Example $\mathcal{R}_2$. The local analytic data at $z = \infty$ are:
$V(\infty) = V(\infty)_{\sqrt{tz}} \oplus V(\infty)_{-\sqrt{tz}} = \mathbb{C}e_0 + \mathbb{C}e_1$, $\gamma : e_0 \mapsto e_1 \mapsto -e_0$. The singular directions depend on $t^{1/2}$. For $t^{1/2}$ in a neighbourhood of 1, the monodromy identity is: $mon_\infty = \binom{0\ -1}{1\ \ 0}\binom{1\ 0}{x\ 1} = \binom{-x\ -1}{1\ \ \ 0}$.

As above, there is a surjective morphism $\mathcal{R}_2 \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$. The fibres consist of the $(b_3 : b_4) \in \mathbb{P}^1$ such that the determinant $F = -b_3b_4x + b_3^2 + b_4^2$ of the matrix $M := \binom{b_1\ b_2}{b_3\ b_4}$, commuting with mon_∞, is nonzero. Thus $\mathcal{R}_2 \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ is the complement of the quadratic curve $F = 0$ over $\mathbb{A}^1$.

We note that the description in [vdP-Sa] of monodromy space for the classical case $P_3(D_8)$ is slightly different. There the link L is normalized by the assumption $\det L = 1$.

Example $\mathcal{R}_3$. The local analytic data at $z = \infty$ are
$V(\infty) = \mathbb{C}e_0 + \mathbb{C}e_1 + \mathbb{C}e_2$ with $\mathbb{C}e_j = V(\infty)_{q_j}$ for $j = 0, 1, 2$ and $q_0 = t^{1/3}z^{1/3}$, $q_1 = \zeta_3 t^{1/3}z^{1/3}$, $q_2 = \zeta_3^2 t^{1/3}z^{1/3}$ and $\zeta_3 = e^{2\pi i/3}$. The basis vectors e_0, e_1, e_2 are chosen such that the formal monodromy γ satisfies $e_0 \mapsto e_1 \mapsto e_2 \mapsto e_0$. The basis e_0, e_1, e_2 is unique up to a simultaneous multiplication by a scalar.

For $t^{1/3}$ equal to 1, the topological monodromy mon_∞ at $z=\infty$ is $mon_\infty = \gamma \cdot St_{3/4} \cdot St_{1/4}$ which is explicitly

$$mon_\infty = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{01} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{21} & 1 \\ 1 & x_{01} & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of mon_∞ is $X^3 - x_{01}X^2 - x_{21}X - 1$.

The space $\mathbb{A}^2$ of the topological monodromies at $z=\infty$, consists of the pairs $(x_{01}, x_{21}) \in \mathbb{C}^2$. The fibres of the obvious map $\mathcal{R}_3 \to \mathbb{A}^2$ consist of the elements $(m_{3,1} : m_{3,2} : m_{3,3}) \in \mathbb{P}^2$ such that the determinant F of M is invertible. Explicitly,

$$\begin{gathered} F = a_7^3 x_1 x_2 + a_7^2 a_8 x_1^2 + a_7^2 a_9 x_2^2 + a_7 a_8 a_9 x_1 x_2 - a_7^2 a_8 x_2 + a_7^2 a_9 x_1 \\ -2a_7 a_8^2 x_1 + 2a_7 a_9^2 x_2 - a_8^2 a_9 x_2 + a_8 a_9^2 x_1 + a_7^3 - 3a_7 a_8 a_9 + a_8^3 + a_9^3, \end{gathered}$$

where $(a_7, a_8, a_9) = (m_{3,1}, m_{3,2}, m_{3,3})$ and $x_1 = x_{0,1},\ x_2 = x_{2,1}$. Thus $\mathcal{R}_3 \subseteq \mathbb{P}^2 \times \mathbb{A}^2$ is the complement of the cubic curve over $\mathbb{A}^2$ with equation $F = 0$.

12.2. **Construction of $\mathcal{M}_n$ by using a cyclic covering.**

The space $\mathcal{M}_n$ will be represented by the "universal" matrix differential operator $L = z\frac{d}{dz} + A$ of size $n \times n$ over $\mathbb{C}(z)$. This operator has only at $z=0$ and $z=\infty$ singularities and these are given by the Galois orbits of $z^{-1/n}$ at $z=0$ and $tz^{1/n}$ at $z=\infty$. As in § 2, we use the n-cyclic covering of $\mathbb{P}^1$ to produce explicit formulas.

Now L is seen as a map on a vector space V of dimension n over $\mathbb{C}(z)$. Let σ denote the automorphism of $\mathbb{C}(z^{1/n})$ over $\mathbb{C}(z)$, given by $\sigma(z^{1/n}) = \omega z^{1/n}$ with $\omega = e^{2\pi i/n}$. Then σ acts as semi-linear map on $W := \mathbb{C}(z^{1/n}) \otimes V$ and L extends uniquely to a derivation D on W. The operator D has no ramification. Define the trace $tr : W \to V$ by $tr(w) = \sum_{j=0}^{n-1} \sigma^j(w)$.

We apply the method of [vdP-Sa, Chapter 12], *to construct a universal family.* One considers a basis $e_0, e_1, \ldots, e_{n-1}$ of W over $\mathbb{C}(z^{1/n})$ such that σ acts by $e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$ and D has, w.r.t. this basis, poles of order 1 at $z^{1/n} = 0$ and at $z^{1/n} = \infty$ and no further singularities. By construction, D commutes with σ. In particular, $D(e_0)$ determines D and $D(e_0)$ has the form $\sum_{j=0}^{n-1} (a_j z^{-1/n} + b_j + c_j z^{1/n}) e_j$ where the a_j, b_j, c_j are variables, parametrizing the family.

Define the V-basis $\{B_0, B_1, \ldots, B_{n-1}\}$ by $B_j = tr(z^{j/n}e_0)$ for all j. In the computations, we change B_{n-1} into $z^{-1}B_{n-1}$. The given data for $D(e_0)$ induces a formula $z\frac{d}{dz} + A$ for D on the basis $B_0, \ldots, B_{n-1}$.

It is seen that the matrix A has at most singularities at $z = 0$ and $z = \infty$ (in fact poles of order at most 1). The characteristic polynomial of A is seen to have the form $T^n + p_{n-1}T^{n-1} + \cdots + p_1T + p_0 - (\alpha z^{-1} + \beta z)$ with all entries $p_0, \ldots, p_{n-1}, \alpha, \beta$ are in $\mathbb{C}[a_0, b_0, c_0, \ldots, b_{n-1}, c_{n-1}]$. In particular, there are explicit expressions $\neq 0$ for α and β. In the family given by A we *require* that α and β are invertible. Indeed, this implies that $z = 0$ and $z = \infty$ are totally ramified and have Katz invariant $\frac{1}{n}$ for the operator $z\frac{d}{dz} + A$.

The resulting affine family of operators $z\frac{d}{dz} + A$ is parametrized by the spectrum of $\mathbb{C}[a_0, b_0, \ldots, c_{n-1}, \frac{1}{\alpha}, \frac{1}{\beta}]$. Next, we make the following restrictions and normalizations. We *require* that A has trace zero. This is equivalent to giving b_0 the value $\frac{3-n}{2n}$. The variable z is scaled such that $\alpha = 1$ and we write t for β. The next step is to divide by the action, by conjugation, of the group of the (constant) diagonal matrices on the differential operator $z\frac{d}{dz} + A$. In examples this is done by replacing $(n-1)$ suitable entries of A by 1 (for example, resulting in $a_1 = 1, a_2 = \cdots = a_{n-1} = 0$). One sees that the dimension of the final family (not counting t) is $3n - 1 - 2 - (n-1) = 2n - 2$ (-1 for b_0 and -2 for α, β and $-(n-1)$ for conjugation). This is in accordance with $\dim \mathcal{R}_n = 2(n-1)$.

We do not attempt to describe the full moduli space $\mathcal{M}_n$, but claim that the constructed family $D = z\frac{d}{dz} + A$ describes an affine open subset. The operator $E := \frac{d}{dt} + B$ such that $\{D, E\}$ forms a Lax pair is also considered as σ-equivariant operator on W and is determined by $E(e_0)$. One computes $E(e_0) = z^{1/n}\sum_{j=0}^{n-1} c_je_j$, under the assumption that $a_0 = 1, a_1 = \cdots = a_{n-1} = 0$. Below, the above construction is made explicit for $n = 3$, extended to $n = 4$ and to general $n \geq 3$. For $n = 2$, it is compared to the classical formula.

12.2.1. *Case $n = 3$.* The matrix of D w.r.t. the basis $B_0, B_1, z^{-1}B_2$ is

$$\begin{pmatrix} b_0 + b_1 + b_2 & a_0 + a_1 + a_2 & c_0 + c_1 + c_2 \\ c_0 + c_1\omega^2 + c_2\omega & \frac{1}{3} + b_0 + b_1\omega^2 + b_2\omega & z^{-1}(a_0 + a_1\omega^2 + a_2\omega) \\ a_0 + a_1\omega + a_2\omega^2 & z(c_0 + c_1\omega + c_2\omega^2) & -1/3 + b_0 + b_1\omega + b_2\omega^2 \end{pmatrix}$$

with $\alpha = (a_0 + a_1 + a_2)(a_0 + a_1\omega + a_2\omega^2)(a_0 + a_1\omega^2 + a_2\omega)$ and
$\beta = (c_0 + c_1 + c_2)(c_0 + c_1\omega + c_2\omega^2)(c_0 + c_1\omega^2 + c_2\omega)$.

Normalization: $(a_0+a_1+a_2) = (a_0+a_1\omega+a_2\omega^2) = (a_0+a_1\omega^2+a_2\omega) = 1$ (thus $a_0 = 1, a_1 = a_2 = 0$), $b_0 = 0$ and $\beta = t$. This produces

$$z\frac{d}{dz} + \begin{pmatrix} d_0 & 1 & f_0 \\ f_1 & d_1 & \frac{1}{z} \\ 1 & f_2 z & d_2 \end{pmatrix} \text{ with } f_0 f_1 f_2 = t \text{ and } d_0 + d_1 + d_2 = 0.$$

It is completed to a Lax pair by $t\frac{d}{dt} + \begin{pmatrix} 0 & 0 & f_0 \\ f_1 & 0 & 0 \\ 0 & f_2 z & 0 \end{pmatrix}$. The Painlevé type equations are

$$t\frac{f_0'}{f_0} = d_0 - d_2,\ t\frac{f_1'}{f_1} = d_1 - d_0,\ t\frac{f_2'}{f_2} = d_2 - d_1 + 1,$$

$$td_0' = f_1 - f_0,\ td_1' = f_2 - f_1,\ td_2' = f_0 - f_2.$$

The system of equations has symmetries ρ and σ defined by

$\rho :\quad f_0, f_1, f_2 \mapsto f_1, f_2, f_0$ and $d_0, d_1, d_2 \mapsto d_1 - 1/3, d_2 + 2/3, d_0 - 1/3,$

$\sigma :\quad f_0, f_1, f_2 \mapsto f_1, f_0, f_2$ and $d_0, d_1, d_2 \mapsto -d_0, -d_2, -d_1,$

and generating $D_3 = S_3$.

The above formulas are very similar to the ones of Kawakami [K3, p. 35]. In the latter the trace of the operator is $+1$ instead of 0 and the entries of the matrix are written in terms of canonical variables p_1, p_2, q_1, q_2 for a certain Hamiltonian.

One substitutes $F = f_0, G = f_1$ and obtains the equivalent system

$$F'' = \frac{(F')^2}{F} - \frac{F'}{t} + \frac{FG - 2F^2}{t^2} + \frac{1}{tG},$$

$$G'' = \frac{(G')^2}{G} - \frac{G'}{t} + \frac{FG - 2G^2}{t^2} + \frac{1}{tF}.$$

For a solution $(f_0, f_1, f_2, d_0, d_1, d_2)$, invariant under the symmetry $f_0 \leftrightarrow f_1$, one has $F = G$ and the resulting equation is close to $P_3(D_8)$, namely

$$F'' = \frac{(F')^2}{F} - \frac{F'}{t} + \frac{-F^2}{t^2} + \frac{1}{tF}.$$

The invariant solutions under $D_3 = S_3$ are $F = G = \zeta t^{1/3}$ with $\zeta^3 = 1$.

A computation of the infinitesimal symmetries at the invariant solution $f_0 = f_1 = f_2 = t^{1/3}$ produces ugly formulas.

12.2.2. *The general case.* We make the following normalization: $D(e_0) = (z^{-1/n} + b_0 + c_0 z^{1/n})e_0 + \sum_{j=1}^{n-1}(b_j + c_j z^{1/n})e_{n-1}$, $b_0 = \frac{3-n}{2n}$, $\beta = t$ and $E(e_0) = z^{1/n}\sum_{j=0}^{n-1} c_j e_j$ and basis $B_0, \ldots, B_{n-2}, z^{-1}B_{n-1}$.

For general $n \geq 3$, the formulas for the Lax pair are:

$$z\frac{d}{dz} + \begin{pmatrix} d_0 & 1 & 0 & . & 0 & f_0 \\ f_1 & d_1 & 1 & . & 0 & 0 \\ 0 & f_2 & d_2 & . & 0 & 0 \\ . & . & . & . & 1 & . \\ 0 & . & . & f_{n-2} & d_{n-2} & \frac{1}{z} \\ 1 & 0 & . & 0 & f_{n-1}z & d_{n-1} \end{pmatrix}, \; t\frac{d}{dt} + \begin{pmatrix} 0 & 0 & 0 & . & 0 & f_0 \\ f_1 & 0 & 0 & . & 0 & 0 \\ 0 & f_2 & 0 & . & 0 & 0 \\ . & . & . & . & 0 & . \\ 0 & . & . & f_{n-2} & 0 & 0 \\ 0 & 0 & . & 0 & f_{n-1}z & 0 \end{pmatrix}$$

with $\sum d_j = 0$, $\prod f_j = t$. The Painlevé type equations are

$$t\frac{f_0'}{f_0} = d_0 - d_{n-1},\; t\frac{f_1'}{f_1} = d_1 - d_0, \cdots, t\frac{f_{n-1}'}{f_{n-1}} = d_{n-1} - d_{n-2} + 1,$$

$$td_0' = f_1 - f_0,\; td_1' = f_2 - f_1, \cdots\cdots, td_{n-1}' = f_0 - f_{n-1}.$$

The symmetries observed for $n = 3$ generalize to $n \geq 3$ as follows:

$$\rho\colon \begin{array}{l} (f_0, f_1, \ldots, f_{n-1}) \mapsto (f_1, f_2, \ldots, f_{n-1}, f_0), \\ (d_0, d_1, \ldots, d_{n-1}) \mapsto (d_1 - \frac{1}{n}, \ldots, d_{n-2} - \frac{1}{n}, d_{n-1} + \frac{n-1}{n}, d_0 - \frac{1}{n}) \end{array}$$

and

$$\sigma\colon\; (f_0, f_1, \ldots, f_{n-1}) \mapsto (f_{n-1}, f_{n-2}, \ldots, f_1, f_0),$$

combined with $\sigma(d_j) = -d_{\pi(j)} + c_j$ for a permutation π satisfying $\pi^2 = 1$ and constants $c_j \in \{-\frac{1}{n}, \frac{n-1}{n}\}$ such that $\sum c_j = 0$. These symmetries generate the dihedral group D_n of order $2n$.

Taking D_n-invariants $f_0 = \ldots = f_{n-1} := t^{1/n}$ and corresponding d_j's produces algebraic solutions of the Painlevé type equations.

Case $n = 4$. For invariant solutions under $f_0 \leftrightarrow f_1, f_2 \leftrightarrow f_3$ one has $(f_0, f_1, f_2, f_4) = (f, f, \frac{\sqrt{t}}{f}, \frac{\sqrt{t}}{f})$ and $d_0 = -1/8, d_1 = t\frac{f'}{f} - 1/8, d_2 = 3/8$, $d_3 = -t\frac{f'}{f} - 1/8$ and the equation

$$f'' = \frac{(f')^2}{f} - \frac{f'}{t} - \frac{f^2}{t^2} + \frac{1}{t^{3/2}}.$$

This is again close to the equation for $P_3(D_8)$. The D_4-symmetric solutions are $f = f_0 = f_1 = f_2 = f_3 = \zeta t^{1/4}$ with $\zeta^4 = 1$.

Case $n = 5$. Consider solutions invariant under $f_0 \leftrightarrow f_4, f_1 \leftrightarrow f_3$. Then $(f_0, f_1, f_2, f_3, f_4) = (f, g, \frac{t}{f^2g^2}, g, f)$. Moreover,

$$(d_0, d_1, d_2, d_3, d_4) = (t\frac{f'}{f} - \frac{2}{5},\; t\frac{f'}{f} + t\frac{g'}{g} - \frac{2}{5},\; -t\frac{f'}{f} - t\frac{g'}{g} + \frac{3}{5},\; t\frac{f'}{f} + \frac{3}{5},\; \frac{-2}{5}).$$

The system of equations for f, g reads

$$f'' = \frac{(f')^2}{f} - \frac{f'}{t} + \frac{fg - f^2}{t^2}, \quad g'' = \frac{(g')^2}{g} - \frac{g'}{t} + \frac{fg - 2g^2}{t^2} + \frac{1}{f^2 g t}.$$

The D_5-invariant solutions are $f = g = \zeta t^{1/5}$ with $\zeta^5 = 1$.
The presented examples for small n suggest that subgroups of D_n produce interesting subsystems.

For completeness we consider also the case $n = 2$.
The Lax pair is $z\frac{d}{dz} + \begin{pmatrix} d_0 & \frac{1}{z}+f_0 \\ 1+f_1 z & d_1 \end{pmatrix}, \quad t\frac{d}{dt} + \begin{pmatrix} 0 & f_0 \\ f_1 z & 0 \end{pmatrix}$ with $f_0 f_1 = t$ and $d_0 + d_1 = 0$. The equations are

$$t\frac{f_0'}{f_0} = d_1 - d_0, \; t\frac{f_1'}{f_1} = d_0 - d_1 + 1, \; td_0' = f_1 - f_0, \; td_1' = f_0 - f_1.$$

One observes that $q = f_0$ satisfies the classical equation for $P_3(D_8)$, namely $q'' = \frac{(q')^2}{q} - \frac{q'}{t} + \frac{2q^2}{t^2} - \frac{2}{t}$. There is a symmetry ρ, given by $f_0, f_1 \mapsto f_1, f_0$ and $d_0, d_1 \mapsto d_1 - 1/2, d_0 + 1/2$.

The symmetry means that if q is a solution, then so is $\frac{t}{q}$. Further the invariant element $r = q + \frac{t}{q}$ satisfies the equation

$$r'' = \frac{r}{r^2 - 4t}(r')^2 - \frac{r^2}{t(r^2 - 4t)}r' + \frac{2r^4 - 16tr^2 + tr + 32t^2}{t^2(r^2 - 4t)}.$$

The symmetric solutions are $q = \pm\sqrt{t}$ and $r = \pm 2\sqrt{t}$.

13. A companion of P_1.

What we like to call the *companion of* P_1 is the family $\mathcal{M}$ of connections, given by the set of differential modules M over $\mathbb{C}(z)$ with dimension 2, $\Lambda^2 M$ is trivial, $z = 0$ is regular singular and the generalized eigenvalues at $z = \infty$ are $\pm w$ with $w = z^{5/2} + \frac{t}{2} z^{1/2}$. This is the P_1 case except for allowing a regular singularity at $z = 0$.

Description of $\mathcal{R}$. The singular directions at $z = \infty$, lying in $[0, 1)$ are $\frac{1}{5}, \frac{3}{5}$ for the difference of eigenvalues $w - (-w)$, and $0, \frac{2}{5}, \frac{4}{5}$ for $(-w) - w$. Thus $\mathcal{R} \cong \mathbb{A}^5$. Let *mon* : $\mathcal{R} \to \mathrm{SL}_2(\mathbb{C})$ denote the morphism which sends the Stokes matrices to the monodromy matrix at $z = 0$.

The fibre of *mon* above $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ is given by the monodromy identity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_5 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One eliminates x_1, x_2 by $x_1 = -c(x_3 + x_5 + x_3x_4x_5) - a(1 + x_3x_4)$ and $x_2 = d(1 + x_4x_5) + bx_4$. Since $ad - bc = 1$ we are left with the equation $d(x_3 + x_5 + x_3x_4x_5) + b(1 + x_3x_4) + 1 = 0$.

For $d \neq 0$, the fibre is as usual an affine cubic surface with three lines at infinity. In fact this is the cubic surface "associated to" P_1 (see [vdP-Sa]).
For $d = 0$ the equation of the fibre reads $x_3x_4 = -b^{-1} - 1$ and x_5 has no relations. For $b \neq -1$ it is the surface $\mathbb{C}^* \times \mathbb{C}$. In particular, $\mathcal{R} \to \mathrm{SL}_2$ is surjective. Furthermore, the parameter space $\mathcal{P}$ has dimension 1.

Description of $\mathcal{M}$. Since the fibres of $RH : \mathcal{M} \to \mathcal{R}$ are parametrized by t, one has $\dim \mathcal{M} = 6$. The isomorphy classes of the residue matrix at $z = 0$ form the parameter space. It can be shown that the family

$$z\frac{d}{dz} + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)z^3 + \left(\begin{smallmatrix} 0 & b_2 \\ 1 & 0 \end{smallmatrix}\right)z^2 + \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{smallmatrix}\right)z + \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{smallmatrix}\right),$$

with $t = b_1 - b_2^2 + c_0$, is the universal family of connections $\mathcal{M}$. We eliminate b_1 by $b_1 = t + b_2^2 - c_0$. Furthermore, $p_0 := a_0^2 + b_0c_0$ is the basic parameter.

For the Lax pair computation we suppose that the above operator commutes with $\frac{d}{dt} + \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{smallmatrix}\right) + z\left(\begin{smallmatrix} y_4 & y_5 \\ y_6 & -y_4 \end{smallmatrix}\right)$ and that $\frac{d(a_0^2+b_0c_0)}{dt} = 0$. This eliminates $y_1, \dots, y_6$ and produces the equations

$$a_0' = 2b_2c_0 - b_0,\ b_0' = -4a_0b_2,\ a_1' = -3b_2^2 + 2c_0 - t,\ b_2' = -2a_1,\ c_0' = 2a_0.$$

For $c_0 = 0$, and also for a fixed residue matrix (i.e., $a_0' = b_0' = c_0' = 0$), one obtains the P_1 equation. For $c_0 \neq 0$ one eliminates $b_0 = (p_0 - a_0^2)/c_0$ and $b_0' = (-2a_0a_0'c_0 - (p_0 - a_0^2)c_0')/c_0^2$. This results in the Painlevé type vector field

$$a_0' = 2b_2c_0 - \frac{p_0 - a_0^2}{c_0},\ c_0' = 2a_0,\ a_1' = -3b_2^2 + 2c_0 - t,\ b_2' = -2a_1.$$

One eliminates a_0, a_1, and c_0 in the above equations by:
$a_1 = -\frac{1}{2}b_2'$, $c_0 = \frac{t}{2} + \frac{3}{2}b_2^2 - \frac{1}{4}b_2''$, $a_0 = \frac{1}{4} + \frac{3}{2}b_2b_2' - \frac{1}{8}b_2^{(3)}$. The remaining equation produces the following fourth order explicit differential equation for $f := b_2$:

$$-2(6f^2 - f^{(2)} + 2t)f^{(4)} = 288f^5 - 240f^3f^{(2)} + 192tf^3 - 24ff^{(1)}f^{(3)} + 32f(f^{(2)})^2 - 80tff^{(2)}$$

$$+32ft^2 + 24(f^{(1)})^2f^{(2)} - 48(f^{(1)})^2 + 48ff^{(1)} + (f^{(3)})^2 - 4f^{(3)} + 64p_0 + 4$$

with $f = b_2,\ f^{(j)} := (\frac{d}{dt})^j(b_2)$ for $j = 1, 2, 3, 4$.

We note that the denominator of the formula for $f^{(4)}$ is the equation for P_1. It seems probable that the field $\mathbb{C}(t)(b_2, \frac{d}{dt}b_2, (\frac{d}{dt})^2b_2, (\frac{d}{dt})^3b_2)$ has, for generic p_0, transcendence degree 4 over $\mathbb{C}(t)$. This would fit with the observation that the fibres of $\mathcal{R} \to \mathcal{P}$ have dimension 4.

Comments. There are two reasons why this "companion of P_1" is not in the classical list $P_1 - P_6$. The Painlevé type equations describe in fact a vector field of rank 4 (written above as explicit differential equation of order 4).

Secondly, the monodromic family is a subfamily of the natural monodromic family with "two time variables" given by the data: $z = 0$ is regular singular and $z = \infty$ is irregular singular with generalized eigenvalues $\pm(z^{5/2} + \frac{t_1}{2}z^{3/2} + \frac{t_2}{2}z^{1/2})$ with time variables t_1, t_2. We extend our computations to this case.

Isomonodromy and Lax pairs for $q = (z^{5/2} + \frac{t_1}{2}z^{3/2} + \frac{t_2}{2}z^{1/2})$.
As in the case $q = z^{5/2} + \frac{t}{2}z^{1/2}$ the famlily of connections $z\frac{d}{dz} + A$ can be normalized to

$$z\frac{d}{dz} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot z^3 + \begin{pmatrix} 0 & b_2 \\ 1 & 0 \end{pmatrix} \cdot z^2 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \cdot z + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix},$$

where $c_0 = b_2^2 - b_2t_1 + \frac{1}{4}t_1^2 - b_1 + t_2$ and $c_1 = -b_2 + t_1$. The variables a_0, a_1, b_0, b_1, b_2 are seen as functions of t_1, t_2. The Lax pairs are expressed by $[z\frac{d}{dz} + A, \frac{d}{dt_i} + B_i] = 0$ for $i = 1, 2$ and B_i a matrix depending on t_1, t_2, z and polynomial in z of degree ≤ 2. One obtains the following system of closed one-forms for $d(a_0), \cdots, d(b_2)$

$$d(a_0) = \frac{1}{48}\{16b_2^4 - 16b_2^3t_1 + 4b_2t_1^3 - t_1^4 - 48b_1b_2^2 + 32b_1b_2t_1 - 4b_1t_1^2 + 32b_2^2t_2 - 16b_2t_1t_2 - 16b_0b_2 +$$

$$8b_0t_1 + 32b_1^2 - 48b_1t_2 + 16t_2^2\}dt_1 + \{-2b_2^3 + 3b_2^2t_1 - \frac{3b_2t_1^2}{2} + \frac{t_1^3}{4} + 2b_1b_2 - b_1t_1 - 2b_2t_2 + t_1t_2 + b_0\}dt_2$$

$$d(a_1) = \frac{1}{24}\{-16b_2^3 + 20b_2^2t_1 - 4b_2t_1^2 - t_1^3 + 16b_1b_2 - 16b_1t_1 - 16b_2t_2 + 12t_1t_2 + 8b_0\}dt_1$$

$$+\{b_2^2 - 2b_2t_1 + 3/4(t_1^2) + 2b_1 - t_2\}dt_2$$

$$d(b_0) = \frac{1}{6}\{-4a_0b_2^2 + a_0t_1^2 + 8a_0b_1 - 4a_0t_2 - 4a_1b_0\}dt_1 + \{(4b_2 - 2t_1)a_0\}dt_2$$

$$d(b_1) = \frac{1}{6}\{-4a_1b_2^2 + a_1t_1^2 + 4a_0b_2 - 2a_0t_1 + 4a_1b_1 - 4a_1t_2 + 2b_2 - t_1\}dt_1 + \{4a_1b_2 - 2a_1t_1 + 2a_0 + 1\}dt_2$$

$$d(b_2) = \frac{1}{3}\{-a_1t_1 + 2a_0 + 2\}dt_1 + 2a_1dt_2.$$

Note that $p_0 := a_0^2 + b_0c_0$ satisfies $d(p_0) = 0$ and p_0 is a generating parameter for this system.

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