THE GROTHENDIECK GROUP OF A TRIANGULATED CATEGORY

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ABSTRACT. We give a direct proof of the following known result: the Grothendieck group of a triangulated category with a silting subcategory is isomorphic to the split Grothendieck group of the silting subcategory.

1. INTRODUCTION

Let $\mathcal{T}$ be a skeletally small triangulated category. Denote by Σ its suspension functor. Recall from [2] that a full additive subcategory $\mathcal{M}$ of $\mathcal{T}$ is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \Sigma^i\mathcal{M}) = 0$ for any $i \geq 1$, or equivalently, $\mathrm{Hom}_{\mathcal{T}}(M, \Sigma^i(M')) = 0$ for any $M, M' \in \mathcal{M}$ and $i \geq 1$. It is called *silting*, if in addition $\mathcal{T} = \mathrm{tri}\langle\mathcal{M}\rangle$, that is, the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{M}$ coincides with $\mathcal{T}$ itself. The definition here is slightly different from [2, Definition 2.1], since we do not require that $\mathcal{M}$ is closed under direct summands. The study of silting objects is traced back to [7].

The Grothendieck group of $\mathcal{T}$ is denoted by $K_0(\mathcal{T})$. For a skeletally small additive category $\mathcal{A}$, we denote by $K_0^{\mathrm{sp}}(\mathcal{A})$ its split Grothendieck group.

The goal of this work is to give a direct proof of the following result.

Theorem A. *Let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$. Then the inclusion $\mathcal{M} \hookrightarrow \mathcal{T}$ induces an isomorphism $K_0^{\mathrm{sp}}(\mathcal{M}) \simeq K_0(\mathcal{T})$ of abelian groups.*

Theorem A is essentially due to [4, Theorem 5.3.1], whose indirect proof relies on the weight complex functor associated to a weight structure. Under the additional Krull-Schmidt assumption on $\mathcal{T}$, Theorem A is proved in [2, Theorem 2.27]. We mention that [2, Theorem 2.27] plays a fundamental role in the study of K-theoretical aspects of silting theory.

The surjectivity of the induced homomorphism $K_0^{\mathrm{sp}}(\mathcal{M}) \to K_0(\mathcal{T})$ above is immediate, but the injectivity is somehow nontrivial. For this, we establish the inverse homomorphism, whose argument resembles the one in [2] and relies on the octahedral axiom (TR4).

Denote by $\mathbf{K}^b(\mathcal{A})$ the bounded homotopy category of an additive category $\mathcal{A}$. We view $\mathcal{A}$ as a full subcategory of $\mathbf{K}^b(\mathcal{A})$ by identifying each object in $\mathcal{A}$ with the corresponding stalk complex concentrated in degree zero. It is clear that $\mathcal{A}$ is a silting subcategory of $\mathbf{K}^b(\mathcal{A})$. Therefore, Theorem A has the following immediate consequence, which seems to be well known to experts in K-theory; compare [10, Introuction, the fourth paragraph].

Corollary B. *The inclusion $\mathcal{A} \hookrightarrow \mathbf{K}^b(\mathcal{A})$ induces an isomorphism $K_0^{\mathrm{sp}}(\mathcal{A}) \simeq K_0(\mathbf{K}^b(\mathcal{A}))$ of abelian groups.*

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We mention that Corollary B is due to [5, Subsection 3.2.1, Lemma 3] and [9, Theorem 1.1].

We refer to [6, 3] for triangulated categories and to [12] for Grothendieck groups. All subcategories are required to be full and additive.

2. Filtrations

We will study filtrations on objects, which will be the key ingredient of the proof in the next section. Throughout this section, we fix a presilting subcategory $\mathcal{M}$ of a triangulated category $\mathcal{T}$.

For two subcategories $\mathcal{X}$ and $\mathcal{Y}$, we have the following subcategory

$$\mathcal{X} * \mathcal{Y} = \{E \in \mathcal{T} \mid \exists \text{ an exact triangle } X \to E \to Y \to \Sigma(X) \text{ with } X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

The operation $*$ on subcategories is associative; see [3, Lemme 1.3.10].

Lemma 2.1. *The following statements hold.*

(1) $\Sigma^i\mathcal{M} * \Sigma^j\mathcal{M} \subseteq \Sigma^j\mathcal{M} * \Sigma^i\mathcal{M}$ *for* $j < i$, *and* $\Sigma^i\mathcal{M} * \Sigma^i\mathcal{M} = \Sigma^i\mathcal{M}$.

(2) $\mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1}\mathcal{M} * \Sigma^{-2}\mathcal{M} * \cdots * \Sigma^{-n}\mathcal{M}, \Sigma^m\mathcal{M}) = 0$ *for any* $n \geq 1$ *and* $m \geq 0$.

Proof. For (1), we consider an exact triangle

$$\Sigma^i(M_1) \longrightarrow E \longrightarrow \Sigma^j(M_2) \stackrel{a}{\longrightarrow} \Sigma^{i+1}(M_1)$$

with $M_i \in \mathcal{M}$. Since $\mathcal{M}$ is presilting and $j < i+1$, the morphism $a = 0$. It follows that $E \simeq \Sigma^i(M_1) \oplus \Sigma^j(M_2)$, which belongs to $\Sigma^j\mathcal{M} * \Sigma^i\mathcal{M}$. If $i = j$, the object E belongs to $\Sigma^i\mathcal{M}$.

For (2), we observe that the subcategory

$$\mathcal{S} = \{E \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(E, \Sigma^m\mathcal{M}) = 0 \text{ for all } m \geq 0\}$$

is closed under extensions and Σ^{-1}. Moreover, for any $n \geq 1$, the subcategory $\Sigma^{-n}\mathcal{M}$ belongs to $\mathcal{S}$. Then we deduce (2). □

Definition 2.2. Let X be an object. A $\Sigma^{\leq 0}(\mathcal{M})$-*filtration* of *length* n for X means a sequence of morphisms

$$0 = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

such that each morphism fits into an exact triangle

$$X_{i+1} \longrightarrow X_i \longrightarrow \Sigma^{-i}(M_i^X) \longrightarrow \Sigma(X_{i+1})$$

with the i-th *factors* $M_i^X \in \mathcal{M}$ for each $0 \leq i \leq n-1$.

Remark 2.3. In the filtration above, each X_i belongs to

$$\Sigma^{-(n-1)}\mathcal{M} * \cdots * \Sigma^{-(i+1)}\mathcal{M} * \Sigma^{-i}\mathcal{M}.$$

In particular, by Lemma 2.1(2) we have

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma(M)) = 0 = \mathrm{Hom}_{\mathcal{T}}(X_1, M)$$

for any $M \in \mathcal{M}$.

Let $\mathcal{A}$ be a skeletally small additive category. For each object A, the corresponding element in the split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{A})$ is denoted by $\langle A \rangle$. Therefore, we have $\langle A \oplus B \rangle = \langle A \rangle + \langle B \rangle$.

Assume that there are two $\Sigma^{\leq 0}(\mathcal{M})$-filtrations of X:

$$0 = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X \tag{2.1}$$

and

$$0 = Y_m \longrightarrow Y_{m-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X \tag{2.2}$$

with factors M_i^X and M_j^Y. The two filtrations are said to be *equivalent* if

$$\sum_{i=0}^{n-1}(-1)^i\langle M_i^X\rangle = \sum_{j=0}^{m-1}(-1)^j\langle M_j^Y\rangle$$

holds in $K_0^{\mathrm{sp}}(\mathcal{M})$.

The argument in the following proof resembles the one in proving the Jordan-Hölder theorem for modules of finite length.

Proposition 2.4. *Any two $\Sigma^{\leq 0}(\mathcal{M})$-filtrations of an object X are equivalent.*

Proof. We assume that (2.1) and (2.2) are two given filtrations of X. By extending one of the filtrations by zeros, we may assume that they have the same length, that is, $n = m$. We use induction on the common length n. If $n = 1$, the statement is trivial, since both M_0^X and M_0^Y are isomorphic to X.

We assume that $n \geq 2$. We apply (TR4) to the exact triangles $Y_1 \to X \to M_0^Y \to \Sigma(Y_1)$ and $X \to M_0^X \to \Sigma(X_1) \to \Sigma(X)$, and obtain the following commutative diagram.

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M_0^Y \arrow[r] \arrow[d] & \Sigma(Y_1) \arrow[d, equal] \\
Y_1 \arrow[r, "a"] & M_0^X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma(Y_1) \arrow[d] \\
 & \Sigma(X_1) \arrow[r, equal] \arrow[d] & \Sigma(X_1) \arrow[r] \arrow[d, "\Sigma(b)"] & \Sigma(X) \\
 & \Sigma(X) \arrow[r] & \Sigma(M_0^Y) &
\end{tikzcd}$$

By Remark 2.3, we have $a = 0 = b$. Therefore, we have isomorphisms

$$\Sigma(X_1) \oplus M_0^Y \simeq Z \simeq \Sigma(Y_1) \oplus M_0^X.$$

The exact triangle $X_2 \to X_1 \to \Sigma^{-1}(M_1^X) \to \Sigma(X_2)$ gives rise to the following one

$$\Sigma(X_2) \longrightarrow Z \longrightarrow M_1^X \oplus M_0^Y \longrightarrow \Sigma^2(X_2).$$

Consequently, we have a $\Sigma^{\leq 0}(\mathcal{M})$-filtration of length $n-1$ for Z.

$$0 = \Sigma(X_n) \longrightarrow \Sigma(X_{n-1}) \longrightarrow \cdots \longrightarrow \Sigma(X_2) \longrightarrow Z$$

Its factors are given by $\{M_1^X \oplus M_0^Y, M_2^X, \cdots, M_{n-1}^X\}$. Similarly, we have another filtration of length $n-1$

$$0 = \Sigma(Y_n) \longrightarrow \Sigma(Y_{n-1}) \longrightarrow \cdots \longrightarrow \Sigma(Y_2) \longrightarrow Z$$

with factors $\{M_1^Y \oplus M_0^X, M_2^Y, \cdots, M_{n-1}^Y\}$. Now by induction, these two filtrations for Z are equivalent, that is, we have

$$\langle M_1^X \oplus M_0^Y\rangle + \sum_{i=2}^{n-1}(-1)^{i-1}\langle M_i^X\rangle = \langle M_1^Y \oplus M_0^X\rangle + \sum_{j=2}^{n-1}(-1)^{j-1}\langle M_j^Y\rangle.$$

This implies that $\sum_{i=0}^{n-1}(-1)^i\langle M_i^X\rangle = \sum_{j=0}^{m-1}(-1)^j\langle M_j^Y\rangle$, as required. □

The following result is analogous to the horseshoe lemma.

Lemma 2.5. *Let $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma(X)$ be an exact triangle. If*

$$0 = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

and

$$0 = Z_n \longrightarrow Z_{n-1} \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 = Z$$

are $\Sigma^{\leq 0}(\mathcal{M})$-filtrations of X and Z, respectively, then Y has a $\Sigma^{\leq 0}(\mathcal{M})$-filtration

$$0 = Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y$$

with its factors $M_i^Y \simeq M_i^X \oplus M_i^Z$ for $0 \leq i \leq n-1$.

Proof. By Remark 2.3, the following square trivially commutes.

$$\begin{tikzcd}
Z \arrow[r, "c"] \arrow[d] & \Sigma(X) \arrow[d] \\
M_0^Z \arrow[r, "0"] & \Sigma(M_0^X)
\end{tikzcd}$$

Applying the 3×3 Lemma in [3, Proposition 1.1.11] and rotations, we have the following commutative diagram with exact columns and rows.

$$\begin{tikzcd}
\Sigma^{-1}(M_0^X) \arrow[r] \arrow[d] & \Sigma^{-1}(M_0^X \oplus M_0^Z) \arrow[r] \arrow[d] & \Sigma^{-1}(M_0^Z) \arrow[r] \arrow[d] & M_0^X \arrow[d] \\
X_1 \arrow[r, "a_1"] \arrow[d] & Y_1 \arrow[r, "b_1"] \arrow[d] & Z_1 \arrow[r, "c_1"] \arrow[d] & \Sigma(X_1) \arrow[d] \\
X \arrow[r, "a"] \arrow[d] & Y \arrow[r, "b"] \arrow[d] & Z \arrow[r, "c"] \arrow[d] & \Sigma(X) \arrow[d] \\
M_0^X \arrow[r] & M_0^X \oplus M_0^Z \arrow[r] & M_0^Z \arrow[r, "0"] & \Sigma(M_0^X)
\end{tikzcd}$$

The middle vertical triangle $\Sigma^{-1}(M_0^X \oplus M_Z^0) \to Y_1 \to Y \to M_0^X \oplus M_0^Z$ implies that $M_0^Y \simeq M_0^X \oplus M_0^Z$. We now repeat the argument to the exact triangle $X_1 \xrightarrow{a_1} Y_1 \xrightarrow{b_1} Z_1 \xrightarrow{c_1} \Sigma(X_1)$. Then we obtain the required filtration for Y. □

3. The proof of Theorem A

In this section, we give the proof of Theorem A and describe the original version [4] of Theorem A in terms of bounded weight structures. We fix a skeletally small triangulated category $\mathcal{T}$.

Let $\mathcal{C}$ be a full additive subcategory of $\mathcal{T}$. We define its Grothendieck group $K_0(\mathcal{C})$ to be the abelian group generated by $\{[C] \mid C \in \mathcal{C}\}$ subject to the relations $[C] - ([C_1] + [C_2])$ whenever there is an exact triangle $C_1 \to C \to C_2 \to \Sigma(C_1)$ in $\mathcal{T}$ with $C_i, C \in \mathcal{C}$. We emphasize that $K_0(\mathcal{C})$ depends on the embedding of $\mathcal{C} \hookrightarrow \mathcal{T}$.

The following result indicates that the Grothendieck group $K_0(\mathcal{C})$ of a certain subcategory $\mathcal{C}$ might be of useful.

Lemma 3.1. *Assume that the full subcategory $\mathcal{C}$ is closed under Σ^{-1} such that for any object $X \in \mathcal{T}$ there exists a natural number n satisfying $\Sigma^{-n}(X) \in \mathcal{C}$. Then the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ induces an isomorphism $K_0(\mathcal{C}) \simeq K_0(\mathcal{T})$.*

Proof. We make an easy observation: for each object C in $\mathcal{C}$, the trivial triangle $\Sigma^{-1}(C) \to 0 \to C \to C$ implies that $[\Sigma^{-1}(C)] = -[C]$ in $K_0(\mathcal{C})$. For each object X in $\mathcal{T}$, we choose a natural number n with $\Sigma^{-n}(X) \in \mathcal{C}$, and define an element $\phi(X) = [\Sigma^{-n}(X)]$ in $K_0(\mathcal{C})$. The observation above implies that $\phi(X)$ does not depend on the choice of n. Since any Σ^{-n} is a triangle functor, these $\phi(X)$ give rise to a well-defined homomorphism $\Phi\colon K_0(\mathcal{T}) \to K_0(\mathcal{C})$ such that $\Phi([X]) = \phi(X)$. It is routine to verify that Φ is inverse to the induced homomorphism $K_0(\mathcal{C}) \to K_0(\mathcal{T})$. □

Let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$. Denote by $\mathcal{F}$ the full subcategory of $\mathcal{T}$ formed by those objects having $\Sigma^{\leq 0}(\mathcal{M})$-filtrations.

The following result contains Theorem A.

Theorem 3.2. *Let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$. Then the inclusions $\mathcal{M} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{T}$ induce isomorphisms $K_0^{\rm sp}(\mathcal{M}) \simeq K_0(\mathcal{F}) \simeq K_0(\mathcal{T})$ of abelian groups.*

Proof. For each $X \in \mathcal{F}$, we choose a $\Sigma^{\leq 0}(\mathcal{M})$-filtration

$$0 = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

with factors M_i^X. We define an element

$$\gamma(X) = \sum_{i=0}^{n-1} (-1)^i \langle M_i^X \rangle$$

in $K_0^{\rm sp}(\mathcal{M})$. By Proposition 2.4, the element $\gamma(X)$ does not depend on the choice of the filtrations. By Lemma 2.5, the map $(X \mapsto \gamma(X))$ is compatible with exact triangles in $\mathcal{F}$. Therefore, such a map induces a well-defined homomorphism $\Gamma\colon K_0(\mathcal{F}) \to K_0^{\rm sp}(\mathcal{M})$ such that $\Gamma([X]) = \gamma(X)$. It is rountine to verify that Γ is inverse to the induced homomorphism $K_0^{\rm sp}(\mathcal{M}) \to K_0(\mathcal{F})$.

In view of Remark 2.3, we have

$$\mathcal{F} = \bigcup_{n \geq 0} \Sigma^{-n}\mathcal{M} * \cdots * \Sigma^{-1}\mathcal{M} * \mathcal{M}.$$

In particular, $\mathcal{F}$ is closed under Σ^{-1}. Since $\mathcal{T} = {\rm tri}\langle \mathcal{M} \rangle$, each object X of $\mathcal{T}$ belongs to

$$\Sigma^{i_1}\mathcal{M} * \cdots * \Sigma^{i_{n-1}}\mathcal{M} * \Sigma^{i_n}\mathcal{M}$$

for some $i_1, \cdots, i_{n-1}, i_n \in \mathbb{Z}$. By Lemma 2.1(1), we may assume that $i_1 < i_2 < \cdots < i_n$. Consequently, for any sufficiently large n, the object $\Sigma^{-n}(X)$ belongs to $\mathcal{F}$. So, the conditions in Lemma 3.1 are fulfilled. Then the required isomorphism $K_0(\mathcal{F}) \simeq K_0(\mathcal{T})$ follows immediately. □

Recall from [4, Definition 1.1.1] that a *weight structure* on $\mathcal{T}$ is a pair $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ of subcategories subject to the following conditions:

(1) Both $\mathcal{U}_{\geq 0}$ and $\mathcal{U}_{\leq 0}$ are closed under direct summands;
(2) $\mathcal{U}_{\geq 0}$ is closed under Σ^{-1}, and $\mathcal{U}_{\leq 0}$ is closed under Σ;
(3) ${\rm Hom}_{\mathcal{T}}(\mathcal{U}_{\geq 0}, \Sigma\mathcal{U}_{\leq 0}) = 0$;
(4) $\mathcal{U}_{\geq 0} * \Sigma\mathcal{U}_{\leq 0} = \mathcal{T}$.

The *core* of the weight structure is defined to be the subcategory $\mathcal{C} = \mathcal{U}_{\geq 0} \cap \mathcal{U}_{\leq 0}$. It is a presilting subcategory of $\mathcal{T}$. We mention that a weight structure is called a co-t-structure in [8, Definition 1.4].

The weight structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ is *bounded* if for each object X, there exist natural numbers $n \leq m$ such that $X \in \Sigma^{-n}\mathcal{U}_{\geq 0} \cap \Sigma^{-m}\mathcal{U}_{\leq 0}$. In this case, the core $\mathcal{C}$ is a silting subcategory; see [4, Corollary 1.5.7]. Moreover, by [2, Proposition 2.23(b)] any silting subcategory which is closed under direct summands arises as the core of a bounded weight structure.

The following result is due to [4, Theorem 5.3.1], which might be viewed as a version of Theorem 3.2.

Corollary 3.3. *Let* $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ *be a bounded weight structure on* $\mathcal{T}$ *with core* $\mathcal{C}$*. Then the inclusions* $\mathcal{C} \hookrightarrow \mathcal{U}_{\geq 0} \hookrightarrow \mathcal{T}$ *induce isomorphisms* $K_0^{\rm sp}(\mathcal{C}) \simeq K_0(\mathcal{U}_{\geq 0}) \simeq K_0(\mathcal{T})$ *of abelian groups.*

Proof. As mentioned above, the core $\mathcal{C}$ is a silting subcategory of $\mathcal{T}$. Moreover, by [2, Proposition 2.23(b)] an object has a $\Sigma^{\leq 0}(\mathcal{C})$-filtration if and only if it belongs to $\mathcal{U}_{\geq 0}$. Then we deduce these isomorphisms by Theorem 3.2. □

Remark 3.4. (1) By applying the corollary above to the opposite category of $\mathcal{T}$, one might deduce isomorphisms $K_0^{\rm sp}(\mathcal{C}) \simeq K_0(\mathcal{U}_{\leq 0}) \simeq K_0(\mathcal{T})$ of ablian groups.

(2) The corollary above is analogous to the following well-known result; see [1, Proposition A.9.5]. Let $\mathcal{T}$ have a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Then the inclusions $\mathcal{A} \hookrightarrow \mathcal{D}^{\leq 0} \hookrightarrow \mathcal{T}$ induce isomorphisms $K_0(\mathcal{A}) \simeq K_0(\mathcal{D}^{\leq 0}) \simeq K_0(\mathcal{T})$ of abelian groups.

(3) We mention that the isomorphism $K_0^{\rm sp}(\mathcal{C}) \simeq K_0(\mathcal{T})$ above is extended to isomorphisms between the corresponding higher K-groups in [11]. One expects that the higher K-groups of $\mathcal{U}_{\geq 0}$ are also isomorphic to them.

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References

[1] P.N. Achar, Perverse Sheaves and Applications to Representation Theory, Math. Survey Monograph **258**, Amer. Math. Soc., Province Rhode Island, 2021. 6

[2] T. Aihara, and O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. **85** (3) (2012), 633–668. 1, 5, 6

[3] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100**, Soc. Math. France, Paris, 1982. 2, 4

[4] M.V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (from motives and in general)*, J. K-Theory **6** (2010), no.3, 387–504. 1, 4, 5

[5] H. Gillet, and C. Soule, *Descent, motives and K-theory*, J. reine angew. Math. **478** (1996), 127–176. 2

[6] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc. Lect. Note Ser. **119**, Cambridge Univ. Press, Cambridge, 1988. 2

[7] B. Keller, and D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. **40** (1988), 239–253. 1

[8] D. Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. **6** (2008), 25–42. 5

[9] D.E.V. Rose, *A note on the Grothendieck group of an additive category*, arXiv:1109.2040v1, 2011. 2

[10] M. Schlichting, *Negatuve K-theory of derived categories*, Math. Z. **253** (2006), 97–134. 1, 6

[11] V. Sosnilo, *Theorem of the heart in negative K-theory for weight structures*, Doc. Math. **24** (2019), 2137–2158. 6

[12] R.G. Swan, Algebraic K-theory, Lect. Notes Math. **76**, Springer, Berlin, Heidelberg, 1968. 2

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