ON A GENERAL NOTION OF A POLYNOMIAL IDENTITY AND CODIMENSIONS

A. S. GORDIENKO

Abstract. Using the braided version of Lawvere's algebraic theories and Mac Lane's PROPs, we introduce polynomial identities for arbitrary algebraic structures in a braided monoidal category $\mathcal{C}$ as well as their codimensions in the case when $\mathcal{C}$ is linear over some field. The new cases include coalgebras, bialgebras, Hopf algebras, braided vector spaces, Yetter–Drinfel'd modules, etc. We find bases for polynomial identities and calculate codimensions in some important particular cases.

Contents

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1. Introduction

Vector spaces A with linear operations $A^{\otimes m} \to A^{\otimes n}$ where $m,n \in \mathbb{Z}_+$, e.g. coalgebras, bialgebras, Hopf algebras, braided vector spaces, Yetter–Drinfel'd modules, etc., find their applications in many areas of mathematics and physics. Such structures admit a universal approach via the notion of an Ω-*algebra* over a field where Ω is an arbitrary set of symbols ω of operations $\omega_A \colon A^{\otimes s(t)} \to A^{\otimes t(\omega)}$ of some arity $s(\omega)$ and coarity $t(\omega)$. (An example of studying (co)actions on Ω-algebras over fields see e.g. in [1].) On the other hand, analogous objects, which will be called below Ω-*magmas*, can be introduced in any monoidal category.

Studying polynomial identities in an algebraic structure is an important aspect of studying the algebraic structure itself that leads to a discovery of new classes of such structures defined in terms of polynomial identities, which in turn may help in solving known problems on them. Furthermore, when a basis for polynomial identities in a concrete algebra A over a field is being calculated, certain numerical characteristics of polynomial identities, called codimensions, come into play naturally. The nth codimension $c_n(A)$ is just the dimension of the vector space of all linear maps $A^{\otimes n} \to A$ that can be realized by multilinear polynomials. It is known [9, Section 6.2] that there exists an tight relationship between the structure of A and the asymptotic behavior of $c_n(A)$.

A polynomial identity is traditionally introduced as a functional equality $f \equiv g$ of polynomials (or just monomials) f, g in the corresponding signature where the polynomials f, g are usually elements of the corresponding free algebraic structure. (We recall this in Section 2.2 below.) Unfortunately, this approach may not work for Ω-algebras with $t(\omega) \geqslant 2$ for some $\omega \in \Omega$, since, say, free coalgebras do not exist. (There exist only cofree coalgebras, see [22].) In order to overcome this difficulty, Mikhail Kochetov introduced [13] in 2000 the notion of a polynomial identity for a coalgebra C as a polynomial identity for the algebra C^* dual to C. What would be a polynomial identity simultaneously involving all possible operations in, say, a Hopf algebra, which is both an algebra and a coalgebra, was still unclear.

Analyzing the definition of a polynomial identity in a traditional algebraic structure A (in the category **Sets** of small sets) of an arbitrary signature Ω, we see in Section 2.2 below that every Ω-monomial can be understood as a composition of arrows corresponding to symbols of Ω, swaps and operations Δ and ε defining the unique comonoid structure on the set A. It turns out that Δ and ε are needed to consider "non-multilinear" polynomial identities, i.e. where a letter occurs in one of the sides more than once or in only one side of the identity. All of this inspires to carry out Lawvere's approach to algebraic structures [14], which establishes a 1-1 correspondence between algebraic structures of a fixed signature Ω and product preserving functors from a certain category $\mathbb{A}$, called an *algebraic theory*, to **Sets**. Here Ω-monomials can be viewed just as morphisms in $\mathbb{A}$. Moreover, identifying certain morphisms in $\mathbb{A}$, one can describe in this way varieties of algebraic structures, i.e. classes of algebraic structures satisfying a fixed set of polynomial identities. In 1965, Mac Lane considered [16] symmetric monoidal analogs of algebraic theories, which he called PROPs (= "product and permutation categories"). In 1996 Martin Markl used linear PROPs (which he shortly referred to as just "theories") to study deformations of arbitrary Ω-algebras [17]. In 2002 Teimuraz Pirashvili described explicitly the PROP corresponding to the notion of a bialgebra over a field [19]. Using the terminology of the "traditional" PI-theory, Pirashvili's result can be compared with a description of a relatively free algebra for some variety of algebras.

In the current article we use braided strict monoidal categories, which we call *braided monoidal algebraic theories* (BMATs for short). BMATs are just braided versions of PROPs. Given a set Ω together with maps $s, t \colon \Omega \to \mathbb{Z}_+$, we construct a BMAT $\mathcal{M}(\Omega)$ such that for every braided monoidal category $\mathcal{C}$ there exists a 1-1 correspondence between Ω-magmas in

$\mathcal{C}$ and braided strong monoidal functors $\mathcal{M}(\Omega) \to \mathcal{C}$. The morphisms in $\mathcal{M}(\Omega)$ are called Ω-*monomials*. When hom-sets in $\mathcal{C}$ are vector spaces over a field $\Bbbk$, we define the category $\mathcal{P}(\Bbbk, \Omega)$ of Ω-polynomials with coefficients in $\Bbbk$ and codimensions of polynomial identities. Varieties of Ω-magmas are introduced in a natural way.

It turns out that if one considers Ω corresponding to a coalgebra, then Ω-polynomial identities with coefficients in $\Bbbk$ in a coalgebra C over $\Bbbk$ coincide with multilinear polynomial identities in C introduced by Mikhail Kochetov in [13]. There are two reasons why non-multilinear polynomial identities in coalgebras cannot fit into the framework of BMATs at all. First, as we have already mentioned, when we consider non-multilinear polynomial identities in algebras over $\Bbbk$, we must actually use maps Δ and ε from the comonoid structure in **Sets**, which are non-linear, so we drop out of the category $\mathbf{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces. Second, the procedure of making the algebra C^* out of a coalgebra C is essentially linear, so it is hard to expect that it would establish a correspondence between anything non-linear. On the other hand, over a field of characteristic 0 all polynomial identities in an algebra are consequences of its multilinear polynomial identities [9, Theorem 1.3.8].

If an Ω-algebra A over a field $\Bbbk$ is endowed with an additional structure, e.g. a group grading or a structure of a (co)module algebra over a Hopf algebra, it is natural to include this additional structure in the signature of polynomial identities [2, 4, 5, 9, 11].

On the one hand, if A is an H-module for a unital associative algebra H, e.g. A is endowed with a generalized H-action (we recall the definition at the end of Section 4.2), one can just add the symbols of operators from H to the signature Ω and consider polynomial $\Omega \sqcup H$-identities. On the other hand, if H is a Hopf algebra, the categories ${}_H\mathsf{Mod}$ and Comod^H of left H-modules and right H-comodules, respectively, are monoidal. If G is a group, then the category of G-graded vector spaces is monoidal too. Finally, a graded or a (co)module Ω-algebra A is just an Ω-magma in the category $\mathcal{C}$ of graded vector spaces or (co)modules, respectively, which prompts to take $\mathcal{C}$ as the base category and try to define polynomial identities in A in some intrinsic way. However if we consider the category $\mathcal{P}(\Bbbk, \Omega)$ of Ω-polynomals with coefficients in $\Bbbk$, the images of morphisms in $\mathcal{P}(\Bbbk, \Omega)$ under $\mathcal{E}_A$ will be the linear maps $A^{\otimes m} \to A^{\otimes n}$ resulting only from the operations ω_A, $\omega \in \Omega$, but not from the additional structure. In order to indeed include the additional structure in the signature of polynomial identities, in Section 4 we use the reconstruction technique (see e.g. [8, Chapter 5]). In this way for $\mathcal{C} = {}_H\mathsf{Mod}$ we get all multilinear polynomial H-identities. For $\mathcal{C} = \mathsf{Comod}^H$ we in fact recover multilinear polynomial H^*-identities where the H^*-action is induced by the H-comodule structure. However, if the Hopf algebra H is finite dimensional, the polynomial $\Omega \sqcup H^*$-identities obtained can be identified with multilinear polynomial H-identities in H-comodule algebras.

In Section 5 we study polynomial identities and their codimensions in vector spaces, (co)commutative Hopf algebras and Yetter — Drinfeld modules.

2. Ω-monomials and polynomial identities in Ω-magmas

2.1. **Monoids, comonoids and Hopf monoids.** We refer the reader to e.g. [8] for an account of monoidal categories.

Let $\mathcal{C}$ be a monoidal category with a monoidal product $\otimes$ and a monoidal unit $\mathbb{1}$. Recall that a *monoid* (M, μ, u) in $\mathcal{C}$ is an object M together with morphisms $\mu\colon M \otimes M \to M$ and

$u\colon \mathbb{1} \to M$ making the diagrams below commutative:

$$\begin{tikzcd}
(M\otimes M)\otimes M \arrow[rr, "\sim"] \arrow[d, "\mu\otimes\mathrm{id}_M"] & & M\otimes(M\otimes M) \arrow[d, "\mathrm{id}_M\otimes\mu"] \\
M\otimes M \arrow[dr, "\mu"] & & M\otimes M \arrow[dl, "\mu"'] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[d, "\sim"'] \arrow[ddr, equal] & M\otimes\mathbb{1} \arrow[d, "\mathrm{id}_M\otimes u"] \\
\mathbb{1}\otimes M \arrow[d, "u\otimes\mathrm{id}_M"'] & M\otimes M \arrow[d, "\mu"] \\
M\otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

Example 2.1. Recall that the category **Sets** of small sets is a monoidal category with the Cartesian monoidal product $\times$ and the neutral object $\{*\}$, a single element set. Monoids in **Sets** are just ordinary set-theoretical monoids M where $\mu(m,n) = mn$ for every $m,n \in M$ and $u(*) = 1_M$.

Example 2.2. The category $\mathbf{Vect}_\Bbbk$ of vector spaces over a field $\Bbbk$ is monoidal too where the monoidal product is the tensor product $\otimes$ and the monoidal unit is the base field $\Bbbk$. Monoids in $\mathbf{Vect}_\Bbbk$ are just unital associative algebras over the field $\Bbbk$.

A comonoid in a monoidal category $\mathcal{C}$ is a monoid (C,Δ,ε) in the dual category $\mathcal{C}^{\mathrm{op}}$. The corresponding morphisms $\Delta\colon C \to C\otimes C$ and $\varepsilon\colon C \to \mathbb{1}$ are called the *comultiplication* and the *counit*, respectively.

Comonoids in $\mathbf{Vect}_\Bbbk$ are called *coalgebras* over the field $\Bbbk$.

All comonoids in **Sets** are trivial. Namely, every set X admits exactly one structure of a comonoid where the comultiplication $\Delta\colon X \to X\times X$ is the diagonal map, i.e. $\Delta x = (x,x)$ for all $x\in X$, and the counit $\varepsilon\colon X\to\{*\}$ is trivial, i.e. $\varepsilon(x) = *$ for all $x \in X$.

Recall that if the category $\mathcal{C}$ is braided, then $\mathsf{Mon}(\mathcal{C})$ is a monoidal category too. Objects of the category $\mathsf{Comon}(\mathsf{Mon}(\mathcal{C}))$ (which is isomorphic to $\mathsf{Mon}(\mathsf{Comon}(\mathcal{C}))$) are called *bimonoids* in $\mathcal{C}$.

If (A,μ_A,u_A) is a monoid and $(C,\Delta_C,\varepsilon_C)$ is a comonoid in a monoidal category $\mathcal{C}$, then the set $\mathcal{C}(C,A)$ of all morphisms $C\to A$ in $\mathcal{C}$ admits a structure of an ordinary monoid: the multiplication is defined by

$$\varphi * \psi := \mu_A(\varphi\otimes\psi)\Delta_C \text{ for all } \varphi,\psi\in\mathcal{C}(C,A)$$

and $u_A\varepsilon_C$ is the identity element. The monoid $\mathcal{C}(C,A)$ is called the *convolution monoid*.

A bimonoid H in a braided monoidal category $\mathcal{C}$ is called a *Hopf monoid* if $\mathrm{id}_H \in \mathcal{C}(H,H)$ admits an inverse $S\colon H\to H$, which is called the *antipode*. We denote the category of Hopf monoids in $\mathcal{C}$ by $\mathsf{Hopf}(\mathcal{C})$.

Example 2.3. Hopf monoids in $\mathbf{Vect}_\Bbbk$, where $\Bbbk$ is a field, are exactly Hopf algebras over $\Bbbk$.

Example 2.4. If G is a group, then the group algebra $\Bbbk G$ is a Hopf algebra where $\Delta g := g\otimes g$, $\varepsilon(g) = 1$, $Sg = g^{-1}$ for all $g\in G$.

Example 2.5. Hopf monoids in **Sets** are exactly groups.

Denote by $\tau_{X,Y}\colon X\otimes Y \xrightarrow{\sim} Y\otimes X$ the braiding in $\mathcal{C}$.

For a monoid (A,μ_A,u_A) denote by A^{op} the monoid $(A,\mu_A\tau_{A,A},u_A)$. Analogously, for a comonoid $(C,\Delta_C,\varepsilon_C)$ denote by C^{cop} the comonoid $(C,\tau_{C,C}\Delta_C,\varepsilon_C)$.

Standard convolution techniques (see e.g. [6, Section 4.2] or [20, Lemma 35, Proposition 36]) combined with a diagram chasing shows that $S\colon H\to H^{\mathrm{op}}$ is a monoid homomorphism and $S\colon H^{\mathrm{cop}}\to H$ is a comonoid homomorphism. Moreover, every bimonoid homomorphism of Hopf monoids commutes with (or preserves) the antipode.

2.2. **Algebraic structures of a given signature.** Before considering different generalizations, below we recall a classical definition of an algebraic structure A of a given signature Ω and polynomial identities in A and notice that the trivial comonoid structure on A is present in the constructions implicitly.

Let Ω be a set with a map $s\colon \Omega \to \mathbb{Z}_+$. Recall that an *algebraic structure of signature* Ω is a set A endowed with maps $\omega_A\colon A^{s(\omega)} \to A$ for every $\omega \in \Omega$, which are called $s(\omega)$-*ary operations*. If A and B are algebraic structures of the same signature Ω, then a map $f\colon A \to B$ is called a *homomorphism* if

$$f\big(\omega_A(a_1,\dots,a_{s(\omega)})\big) = \omega_B\big(f(a_1),\dots,f(a_{s(\omega)})\big) \text{ for all } \omega\in\Omega \text{ and } a_1,\dots,a_{s(\omega)}\in A.$$

Example 2.6. A group can be treated as an algebraic structure of signature $\Omega = \{\cdot, 1, (-)^{-1}\}$ where $s(\cdot) = 2$, $s(1) = 0$ and $s\left((-)^{-1}\right) = 1$.

Example 2.7. A vector space over a field $\Bbbk$ can be treated as an algebraic structure of signature $\Omega = \Bbbk \sqcup \{+, 0, -\}$ where $s(\lambda) = 1$ for all $\lambda \in \Bbbk$, $s(+) = 2$, $s(0) = 0$ and $s\,(-) = 1$.

(Absolutely) free algebraic structures $\mathcal{F}_\Omega(X)$ are constructed inductively: given a set X, let

(1) $X \subseteq \mathcal{F}_\Omega(X)$;
(2) $\omega(w_1,\dots,w_{s(\omega)}) \in \mathcal{F}_\Omega(X)$ for all $\omega \in \Omega$ and all $w_1,\dots,w_{s(\omega)} \in \mathcal{F}_\Omega(X)$.

The operations $\omega_{\mathcal{F}_\Omega(X)}\colon \mathcal{F}_\Omega(X)^{s(\omega)} \to \mathcal{F}_\Omega(X)$ are defined as follows:

$$\omega_{\mathcal{F}_\Omega(X)}(w_1,\dots,w_{s(\omega)}) := \omega(w_1,\dots,w_{s(\omega)}) \text{ for all } w_1,\dots,w_{s(\omega)} \in \mathcal{F}_\Omega(X).$$

Then for every algebraic structure A of signature Ω and every map $f_0\colon X \to A$ there exists a unique homomorphism $f\colon \mathcal{F}_\Omega(X) \to A$ such that $f\iota = f_0$ where ι is the embedding $X \hookrightarrow \mathcal{F}_\Omega(X)$. By this reason the elements of $\mathcal{F}_\Omega(X)$ are called *derived operations* and $f(w)$ is the *value* of $w \in \mathcal{F}_\Omega(X)$ under the substitution f_0. By w_A we denote the map $A^m \to A$ induced by each $w \in \mathcal{F}_\Omega(X)$ where m is the number of letters $x_1,\dots,x_m \in X$ actually appearing in w and $w_A(a_1,\dots,a_m) := f(w)$ for every $a_1,\dots,a_m \in A$ where $f_0\colon X \to A$ is some map such that $f_0(x_i) = a_i$ for every $1 \leqslant i \leqslant m$.

An expression $u \equiv v$ where $u, v \in \mathcal{F}_\Omega(X)$ is called a *polynomial identity* in an algebraic structure A of signature Ω if $f(u) = f(v)$ for all maps $f_0\colon X \to A$. In other words, $u \equiv v$ is a polynomial identity in A if $u_A = v_A$ as maps. (If some letter from X appears only in one of the expressions u and v, then we add a fictitious variable to the map corresponding to the other expression.)

Note that maps w_A, where $w \in \mathcal{F}_\Omega(X)$, are compositions of maps

(1) $(\mathrm{id}_A)^k \times \omega_A \times (\mathrm{id}_A)^\ell$ where $\omega \in \Omega$;
(2) $(\mathrm{id}_A)^k \times \tau \times (\mathrm{id}_A)^\ell$ where $\tau\colon A^2 \to A^2$ is the swap: $\tau(a,b) := (b,a)$ for all $a, b \in A$;
(3) $(\mathrm{id}_A)^k \times \Delta \times (\mathrm{id}_A)^\ell$ where $\Delta\colon A \to A^2$ is the diagonal map: $\Delta(a) := (a,a)$ for all $a \in A$;
(4) $(\mathrm{id}_A)^k \times \varepsilon \times (\mathrm{id}_A)^\ell$ where ε is the unique map from A to the one element set $\{*\}$;
(5) $(\mathrm{id}_A)^k \times \alpha \times (\mathrm{id}_A)^\ell$ where α is one of the identifications $\{*\} \times A \xrightarrow{\sim} A$ and $A \times \{*\} \xrightarrow{\sim} A$.

(The numbers $k, \ell \in \mathbb{Z}_+$ are arbitrary.)

The last two types of maps are needed to be able to include fictitious variables.

Recall that the category **Sets** is symmetric with the swap τ and the maps Δ and ε defined above coincide with those from the unique structure of a comonoid in **Sets** on the set A. All of this makes it natural to transfer the definition of a polynomial identity to similar structures in an arbitrary braided monoidal category.

Before making the formal definition, we notice that in this sense it is reasonable to treat the comonoid maps Δ and ε as the part of an algebraic structure A that allows "non-multilinear" polynomial identities.

2.3. **Ω-magmas.** Let Ω be a set together with maps $s, t\colon \Omega \to \mathbb{Z}_+$. We will refer to Ω as the *signature* too.

Definition 2.8. An Ω-*magma* in a monoidal category $\mathcal{C}$ is an object A endowed with morphisms $\omega_A\colon A^{\otimes s(\omega)} \to A^{\otimes t(\omega)}$ for every $\omega \in \Omega$. Here use the convention that $A^{\otimes 0} := \mathbb{1}$, the neutral object in $\mathcal{C}$.

Example 2.9. Every magma (i.e. a set with a binary operation) is just an Ω-magma in **Sets** for $\Omega = \{\mu\}$, $s(\mu) = 2$, $t(\mu) = 1$.

Example 2.10. Every (neither necessarily associative, nor necessarily unital) algebra over a field $\Bbbk$ is just an Ω-magma in $\mathbf{Vect}_\Bbbk$ for $\Omega = \{\mu\}$, $s(\mu) = 2$, $t(\mu) = 1$.

Example 2.11. Every unital algebra A over a field $\Bbbk$ is an example of an Ω-magma in $\mathbf{Vect}_\Bbbk$ for $\Omega = \{\mu, u\}$, $s(\mu) = 2$, $t(\mu) = 1$, $s(u) = 0$, $t(u) = 1$, where $u_A\colon \Bbbk \to A$ is defined by $u_A(\alpha) = \alpha 1_A$ for $\alpha \in \Bbbk$. An ordinary monoid is is an example of an Ω-magma in **Sets** for the same Ω.

Example 2.12. Every coalgebra C over a field $\Bbbk$ is an example of an Ω-magma in $\mathbf{Vect}_\Bbbk$ for $\Omega = \{\Delta, \varepsilon\}$, $s(\Delta) = 1$, $t(\Delta) = 2$, $s(\varepsilon) = 1$, $t(\varepsilon) = 0$.

In general, Ω-magmas in $\mathbf{Vect}_\Bbbk$ are called Ω-*algebras* over $\Bbbk$ [1].

Example 2.13. An object A endowed with a braiding $\sigma_A\colon A \otimes A \to A \otimes A$ is an example of an Ω-magma for $\Omega = \{\sigma\}$, $s(\sigma) = 2$, $t(\sigma) = 2$.

Let A and B be Ω-magmas in a monoidal category $\mathcal{C}$. A morphism $f\colon A \to B$ is called an Ω-*magma homomorphism* if for every $\omega \in \Omega$ the diagram below is commutative:

$$\begin{tikzcd}
A^{\otimes s(\omega)} \arrow[r, "f^{\otimes s(\omega)}"] \arrow[d, "\omega_A"'] & B^{\otimes s(\omega)} \arrow[d, "\omega_B"] \\
A^{\otimes t(\omega)} \arrow[r, "f^{\otimes t(\omega)}"] & B^{\otimes t(\omega)}
\end{tikzcd}$$

Denote by $\Omega\text{-}\mathsf{Magma}(\mathcal{C})$ the category of Ω-magmas in $\mathcal{C}$.

2.4. **Braided monoidal algebraic theories.** A *braided monoidal algebraic theory* (BMAT for short) is a braided strict monoidal category $\mathbb{A}$ where the objects are non-negative integers $n \in \mathbb{Z}_+$ and $m \otimes n = m + n$ for all $m, n \in \mathbb{Z}_+$. When $\mathbb{A}$ is symmetric, $\mathbb{A}$ is just a PROP in the sense of Mac Lane [16]. When the monoidal product in $\mathbb{A}$ is the categorical product, i.e. $\mathbb{A}$ is Cartesian, then $\mathbb{A}$ is just an algebraic theory in the sense of Lawvere [14].

Fix a set Ω and maps $s, t\colon \Omega \to \mathbb{Z}_+$. We need to introduce expressions in the signature Ω. In fact, they will be just morphisms in the BMAT $\mathcal{M}(\Omega)$ defined below.

Consider the free monoid $(M, \bullet)$ with the set of free generators

$$\Omega \sqcup \{\mathrm{id}_m, \tau_{m,n}, \tau^{-1}_{m,n} \mid m, n \in \mathbb{Z}_+\}$$

and define monoid homomorphisms $s\colon M \to \mathbb{Z}_+$ and $t\colon M \to \mathbb{Z}_+$ extending maps $s, t\colon \Omega \to \mathbb{Z}_+$, such that

(1) $s(\mathrm{id}_m) = t(\mathrm{id}_m) = m$;
(2) $s(\tau_{m,n}) = t(\tau_{m,n}) = s(\tau^{-1}_{m,n}) = t(\tau^{-1}_{m,n}) = m + n$.

Now consider the directed graph Γ with the set of vertices $\mathbb{Z}_+$ and the set of edges M where an edge $w \in M$ goes from the vertex $s(w)$ to the vertex $t(w)$.

Recall that an equivalence relation $\sim$ on hom-sets $X(a, b)$ of a category X is a *congruence* if $g_1 f_1 \sim g_2 f_2$ for all $f_1, f_2 \in X(a, b)$, $g_1, g_2 \in X(b, c)$ and objects a, b, c such that $f_1 \sim f_2$ and $g_1 \sim g_2$. The *factor category* $X/{\sim}$ is the category with the same objects as in X and the hom-sets $X(a, b)/{\sim}$.

Let X be the category where the objects are non-negative integers and the morphisms are finite paths in Γ. If $m, n \in \mathbb{Z}_+$, define $m \otimes n := m + n$. If $w_1 \dots w_s \in X(m,n)$ and $w'_1 \dots w'_t \in X(m', n')$ for some $m, n, m', n' \in \mathbb{Z}_+$ and $w_1, \dots, w_s, w'_1, \dots, w'_t \in M$, then define $w_1 \dots w_s \otimes w'_1 \dots w'_t \in X(m + m', n + n')$ as

$$(w_1 \bullet \mathrm{id}_{n'}) \dots (w_s \bullet \mathrm{id}_{n'})(\mathrm{id}_m \bullet w'_1) \dots (\mathrm{id}_m \bullet w'_t).$$

Denote by $\mathcal{M}(\Omega)$ the factor category $X/\sim$ where $\sim$ is the minimal congruence making

(1) id_m the identity morphisms of objects m;
(2) $\tau_{m,n}^{-1}$ the inverses of $\tau_{m,n}$;
(3) $X/\sim$ a braided strict monoidal category with the braidings $\tau_{m,n}$, the neutral object 0 and the monoidal product $\otimes$.

Then $\mathcal{M}(\Omega)$ is a BMAT, which we call the *category of* Ω*-monomials*. The morphisms in $\mathcal{M}(\Omega)$ are called Ω*-monomials*. Define $s(f) := m$ and $t(f) := n$ for $f \in \mathcal{M}(\Omega)(m,n)$. Note that 1 is an Ω-magma in $\mathcal{M}(\Omega)$.

Given braided monoidal categories $\mathcal{C}$ and $\mathcal{D}$, denote by $\mathbf{BSMF}(\mathcal{C}, \mathcal{D})$ the category of braided strong monoidal functors $\mathcal{C} \to \mathcal{D}$ where morphisms between $F, G \colon \mathcal{C} \to \mathcal{D}$ are all monoidal transformations $\alpha \colon F \Rightarrow G$.

Proposition 2.14. *Let Ω be a set together with maps $s, t \colon \Omega \to \mathbb{Z}_+$ and let $\mathcal{C}$ be a braided monoidal category. Then the categories Ω-$\mathsf{Magma}(\mathcal{C})$ and $\mathbf{BSMF}(\mathcal{M}(\Omega), \mathcal{C})$ are equivalent.*

Proof. Let A be an arbitrary Ω-magma in $\mathcal{C}$. Denote by $\mathcal{E}_A$ a braided strong monoidal functor $\mathcal{M}(\Omega) \to \mathcal{C}$ mapping

(1) 1 to A;
(2) ω to ω_A for every $\omega \in \Omega$.

Note that the functor $\mathcal{E}_A$ is completely determined by the choice of the natural transformation $\mathcal{E}_A(m) \otimes \mathcal{E}_A(n) \xrightarrow{\sim} \mathcal{E}_A(m+n)$ and an isomorphism $\mathbb{1} \xrightarrow{\sim} \mathcal{E}_A(0)$ where $\mathbb{1}$ is the monoidal unit in $\mathcal{C}$. Therefore, $\mathcal{E}_A$ is unique up to a monoidal isomorphism and we get a functor Ω-$\mathsf{Magma}(\mathcal{C}) \to \mathbf{BSMF}(\mathcal{M}(\Omega), \mathcal{C})$, $A \mapsto \mathcal{E}_A$.

Conversely, every braided strong monoidal functor $F \colon \mathcal{M}(\Omega) \to \mathcal{C}$ defines the Ω-magma $F(1)$ with the operations $\omega_{F(1)}$ that are equal to $F(\omega)$ composed with the corresponding isomorphisms $F(m) \xrightarrow{\sim} F(1)^{\otimes m}$, $\omega \in \Omega$. Given braided strong monoidal functors $F, G \colon \mathcal{M}(\Omega) \to \mathcal{C}$, the map $\alpha \mapsto \alpha_1$ defines a one-to-one correspondence between monoidal natural transformations $\alpha \colon F \Rightarrow G$ and Ω-magma homomorphisms $f \colon F(1) \to G(1)$. $\square$

2.5. **Polynomial identities in Ω-magmas.** Again fix a set Ω and maps $s, t \colon \Omega \to \mathbb{Z}_+$. Let f and g be Ω-monomials such that $s(f) = s(g)$ and $t(f) = t(g)$. Then the expression $f \equiv g$ is called a Ω*-polynomial identity*. (Here we use the word "polynomial" since $f \equiv g$ contains two monomials f and g.)

Let A be an Ω-magma in a braided monoidal category $\mathcal{C}$. We say that $f \equiv g$ is a *polynomial identity in A* or that *$f \equiv g$ holds in A* if $\mathcal{E}_A(f) = \mathcal{E}_A(g)$. Polynomial identities $f \equiv g$ in A can be identified with morphisms (f, g) in the category $\mathrm{Id}(A)$ where the set of objects is $\mathbb{Z}_+$ and $\mathrm{Id}(A)(m,n)$ for $m, n \in \mathbb{Z}_+$ is defined by the pullback

$$\begin{tikzcd}
\mathrm{Id}(A)(m,n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}(\Omega)(m,n) \arrow[d, "\mathcal{E}_A"] \\
\mathcal{M}(\Omega)(m,n) \arrow[r, "\mathcal{E}_A"] & \mathcal{C}(m,n)
\end{tikzcd}$$

Let V be some set of Ω-polynomial identities. The full subcategory $\mathrm{Var}(V)$ of Ω-$\mathsf{Magma}(\mathcal{C})$, consisting of all Ω-magmas satisfying V, is called a *variety* of Ω-magmas.

Example 2.15. Coalgebras and Hopf algebras form varieties for the corresponding sets of polynomial identities.

When one studies polynomial identities in an Ω-magma that already belongs to some variety $\mathrm{Var}(V)$, then it is natural identify such polynomial identities that can be derived one from the other using V. For example, in the variety of associative algebras it is natural to identify $x(yz) \equiv x(zy)$ and $(xy)z \equiv (xz)y$. Below we describe the corresponding construction for Ω-magmas in an arbitrary braided monoidal category $\mathcal{C}$.

If $\sim$ is a congruence on a monoidal category X and $f_1 \otimes g_1 \sim f_2 \otimes g_2$ for all $f_1, f_2 \in X(a,b)$, $g_1, g_2 \in X(c,d)$ and objects a,b,c,d such that $f_1 \sim f_2$ and $g_1 \sim g_2$, then we say that the congruence $\sim$ is *monoidal*. It is easy to see that in this case $X/{\sim}$ induces from X the structure of a monoidal category such that the canonical functor $X \to X/{\sim}$ is strict monoidal. If X is braided (symmetric), then the category $X/{\sim}$ and the functor $X \to X/{\sim}$ are braided (resp., symmetric) too.

Example 2.16. If A is an Ω-magma, then $\mathrm{Id}(A)$ is a monoidal congruence on $\mathcal{M}(\Omega)$.

Given a set V of Ω-polynomial identities, consider the minimal monoidal congruence $\sim$ on $\mathcal{M}(\Omega)$ such that $f \sim g$ for every $f = g$ from V. We call $\mathcal{M}_V(\Omega) := \mathcal{M}(\Omega)/{\sim}$ the category of *V-relative Ω-monomials*. Then $\mathcal{M}_V(\Omega)$ is a BMAT too and an Ω-magma A belongs to $\mathrm{Var}(V)$ if and only if there exists a braided strong monoidal functor $F\colon \mathcal{M}_V(\Omega) \to \mathcal{C}$ such that $F(1) = A$ and $F(\omega) = \omega_A$ for all $\omega \in \Omega$.

(Co)restricting the functors from Proposition 2.14, we get

Proposition 2.17. *Let Ω be a set together with maps $s, t\colon \Omega \to \mathbb{Z}_+$, let V be a set of Ω-polynomial identities and let $\mathcal{C}$ be a braided monoidal category. Then the categories* $\mathrm{Var}(V)$ *and* $\mathbf{BSMF}(\mathcal{M}_V(\Omega), \mathcal{C})$ *are equivalent.*

In other words, to every variety $\mathrm{Var}(V)$ of Ω-magmas we assign its BMAT $\mathcal{M}_V(\Omega)$. As in the case of ordinary algebraic theories, in this way we can obtain every BMAT:

Proposition 2.18. *An arbitrary BMAT $\mathbb{A}$ is braided monoidally isomorphic to $\mathcal{M}_V(\Omega)$ for some V and Ω.*

Proof. Denote by Ω the set of all morphisms ω in $\mathbb{A}$ and define the maps $s, t\colon \Omega \to \mathbb{R}_+$ by $s(\omega) := m$, $s(\omega) := n$ for $\omega \in \mathbb{A}(m,n)$. Then 1 is an Ω-magma in $\mathbb{A}$ and there exists a unique strict braided monoidal functor $F\colon \mathcal{M}(\Omega) \to \mathbb{A}$ such that $F(1) = 1$ and $F(\omega) = \omega$ for all $\omega \in \Omega$. Note that F is surjective on hom-sets. Denote by V the set of polynomial identities resulting from the kernel congruence $\sim$ of F. Then $\mathbb{A} \cong \mathcal{M}(\Omega)/{\sim} = \mathcal{M}_V(\Omega)$. □

As we have already mentioned, if an Ω-magma A belongs to $\mathrm{Var}(V)$ for some set V of polynomial identities, it is natural to use in the definition of additional polynomial identities that A may satisfy, the category $\mathcal{M}_V(\Omega)$ instead of $\mathcal{M}(\Omega)$. For A belonging to $\mathrm{Var}(V)$ the functor $\mathcal{E}_A$ factors through $\mathcal{M}_V(\Omega)$. Denote the corresponding functor $\mathcal{M}_V(\Omega) \to \mathcal{C}$ again by $\mathcal{E}_A$. For V-relative Ω-monomials f, g we say that $f \equiv g$ is a *polynomial identity in* A or that $f \equiv g$ *holds in* A if $\mathcal{E}_A(f) = \mathcal{E}_A(g)$.

Definition 2.19. Let V be a set of Ω-polynomial identities. We say that a polynomial identity $f \equiv g$ *follows* from V for some Ω-monomials f and g if the images of f and g in $\mathcal{M}_V(\Omega)$ coincide. Let A be an Ω-magma. We say that V *generates* polynomial identities in A or that V is a *basis* for polynomial identities in A if the polynomial identities from V hold in A and the functor $\mathcal{E}_A\colon \mathcal{M}_V(\Omega) \to \mathcal{C}$ is faithful. In other words, V generates polynomial identities in A if any polynomial identity $f \equiv g$ in A follows from V.

Remark 2.20. Recall that if $\tau_{C,A}\tau_{A,C} = \mathrm{id}_{A\otimes C}$ and $\tau_{C,B}\tau_{B,C} = \mathrm{id}_{B\otimes C}$ for some objects A, B, C in a braided monoidal category $\mathcal{C}$ with a braiding τ, then $\tau_{C,A\otimes B}\tau_{A\otimes B,C} = \mathrm{id}_{(A\otimes B)\otimes C}$ too.

Now the induction argument implies that if the polynomial identity $\tau_{1,1}^2 \equiv \mathrm{id}_2$ belongs to V, then the category $\mathcal{M}_V(\Omega)$ is symmetric.

Remark 2.21. If A is an ordinary algebraic structure of signature Ω, i.e. $t(\omega) = 1$ for all $\omega \in \Omega$ and $\mathcal{C} = \mathbf{Sets}$, then Ω-polynomial identities defined above will all be “multilinear”, i.e. every variable will appear in each side exactly once. As we have mentioned at the end of Section 2.2, in order to get all polynomial identities in the sense of Section 2.2, one has to add to Ω two new symbols, ε and Δ, where $s(\Delta) = s(\varepsilon) = 1$, $t(\Delta) = 2$, $t(\varepsilon) = 0$, define $\varepsilon_A(a) = *$, $\Delta_A a := (a, a)$ for all $a \in A$ and consider $\Omega \sqcup \{\Delta, \varepsilon\}$-polynomial identities. Let V be the set consisting of the following polynomial identities:

(1) $\tau_{1,1}^2 \equiv \mathrm{id}_2$, forcing the braiding $\tau_{k,\ell} \colon (k+\ell) \to (k+\ell)$, $k, \ell \in \mathbb{Z}_+$, to be symmetric (see Remark 2.20);
(2) $(\Delta \otimes \mathrm{id}_1)\Delta \equiv (\mathrm{id}_1 \otimes \Delta)\Delta$, $(\varepsilon \otimes \mathrm{id}_1)\Delta = (\mathrm{id}_1 \otimes \varepsilon)\Delta \equiv \mathrm{id}_1$, forcing $(1, \Delta, \varepsilon)$ to be a comonoid;
(3) $\varepsilon\omega \equiv \varepsilon^{\otimes s(\omega)}$ and $\Delta\omega \equiv (\omega \otimes \omega)\tau\Delta^{\otimes s(\omega)}$ for all $\omega \in \Omega$ where τ is a composition of swaps $\tau_{k,\ell}$ that corresponds to the permutation $\begin{pmatrix} 1 & 2 & \ldots & s(\omega) & s(\omega)+1 & s(\omega)+3 & \ldots & 2s(\omega) \\ 1 & 3 & \ldots & 2s(\omega)-1 & 2 & 4 & \ldots & 2s(\omega) \end{pmatrix}$, making it possible to swap ε and Δ with ω in the same way as in ordinary algebraic structures of signature Ω.

Then every $f \in \mathcal{M}_V(\Omega \sqcup \{\Delta, \varepsilon\})(n, 1)$ equals $\alpha\beta\gamma$ where α is a composition of $\omega \in \Omega$, β is a composition of swaps $\tau_{k,\ell}$ and γ is a composition of Δ and ε where ω, $\tau_{k,\ell}$, Δ and ε can appear monoidally multiplied by id_r for some $r \in \mathbb{N}$. Substituting the generators $x_1, \ldots, x_n$ of the free algebraic structure $\mathcal{F}_\Omega(x_1, \ldots, x_n)$ for the corresponding arguments of f, we see that the set $\mathcal{M}_V(\Omega \sqcup \{\Delta, \varepsilon\})(n, 1)$ can be identified with $\mathcal{F}_\Omega(x_1, \ldots, x_n)$ for every $n \in \mathbb{N}$. Under this identification, polynomial identities of A correspond to polynomial identities of A.

3. Polynomial identities in Ω-magmas in linear categories

3.1. **Polynomial identities in algebras over a field.** Again, before making a generalization, we recall the classical definition of polynomial identities and their codimensions in associative algebras [7, 9].

Let $\Bbbk$ be a field. Denote by $\Bbbk\langle X \rangle$ the free non-unital associative algebra on the countable set $X = \{x_1, x_2, \ldots\}$, i.e. the algebra of polynomials without a constant term with coefficients from $\Bbbk$ in the non-commuting variables from X. Let A be an associative $\Bbbk$-algebra. We say that $f \in \Bbbk\langle X \rangle$ is a *polynomial identity* in A and write $f \equiv 0$ if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in A$ where $n \in \mathbb{N}$ is the number of variables that appear in f. In other words, $f \equiv 0$ if and only if $\varphi(f) = 0$ for every algebra homomorphism $\varphi \colon \Bbbk\langle X \rangle \to A$. The set $\mathrm{Id}(A)$ forms an ideal in $\Bbbk\langle X \rangle$ that is invariant under all endomorphisms of $\Bbbk\langle X \rangle$.

Consider the vector space

$$P_n = \left\{ \sum_{\sigma \in S_n} \alpha_\sigma \, x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} \,\middle|\, \alpha_\sigma \in \Bbbk \right\} \subset \Bbbk\langle X \rangle$$

of multilinear polynomials in $x_1, \ldots, x_n$. (Here S_n is the nth symmetric group.)

The number $c_n(A) := \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$, $n \in \mathbb{N}$, is called the nth *codimension* of polynomial identities of A. Let $\mathrm{Hom}_\Bbbk(A^{\otimes n}, A)$ be the space of all linear maps $A^{\otimes n} \to A$. Then $c_n(A)$ is just the dimension of the subspace of $\mathrm{Hom}_\Bbbk(A^{\otimes n}, A)$ consisting of all the maps that can be realized using the multiplication in A. A detailed study of the asymptotic behaviour of the codimension sequence can be found in [9].

Analogous definitions can be made for not necessarily associative algebras too. Instead of $\Bbbk\langle X \rangle$ one must use the absolutely free non-associative algebra $\Bbbk\{X\}$ and the corresponding non-associative multilinear polynomials. The alternative definition of $c_n(A)$ as the dimension of the corresponding subspace in $\mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$ implies that for an associative algebra A the numbers $c_n(A)$ do not depend on whether we use $\Bbbk\langle X\rangle$ or $\Bbbk\{X\}$ in their definition.

3.2. **Ω-polynomials with coefficients in a ring.** Let R be a unital commutative associative ring. (Below all commutative rings will be associative too and the word "associative" will usually be omitted.) Recall that a category $\mathcal{C}$ is *R-linear* if it is enriched over the category ${}_R\mathsf{Mod}$ of R-modules, i.e. if all hom-sets of $\mathcal{C}$ are R-modules and the composition of morphisms is R-bilinear. For R-linear monoidal categories we require that the monoidal product is R-bilinear too. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between R-linear categories $\mathcal{C}$ and $\mathcal{D}$ is called *R-linear* if F is an R-linear map on hom-sets.

For R-linear braided monoidal categories $\mathcal{C}$ and $\mathcal{D}$ denote by R-$\mathbf{LBSMF}(\mathcal{C},\mathcal{D})$ the category of R-linear braided strong monoidal functors $\mathcal{C} \to \mathcal{D}$ where morphisms between $F, G\colon \mathcal{C} \to \mathcal{D}$ are all monoidal transformations $\alpha\colon F \Rightarrow G$.

For Ω-magmas in R-linear braided monoidal categories one can introduce polynomial identities with coefficients in R.

Consider the braided monoidal category $\mathcal{P}(R,\Omega)$ where the objects, the monoidal product on them and the braiding are the same as in $\mathcal{M}(\Omega)$ and for every $m,n \in \mathbb{Z}_+$ the hom-set $\mathcal{P}(R,\Omega)(m,n)$ is just the free R-module with the basis $\mathcal{M}(\Omega)(m,n)$. The compositions and the monoidal product on morphisms are extended from $\mathcal{M}(\Omega)$ to $\mathcal{P}(R,\Omega)$ by the R-linearity.

Then $\mathcal{P}(R,\Omega)$ is an R-linear BMAT, which we call the *category of Ω-polynomals with coefficients in R*. Morphisms in $\mathcal{P}(R,\Omega)$ are called *Ω-polynomials with coefficients in R*.

For every Ω-magma A in an R-linear braided monoidal category $\mathcal{C}$ extend the functor $\mathcal{E}_A\colon \mathcal{M}(\Omega) \to \mathcal{C}$ defined above to the functor $\mathcal{P}(R,\Omega) \to \mathcal{C}$ by linearity. Namely, if $k,m,n \in \mathbb{Z}_+$ and $\alpha_i \in R$, $f_i \in \mathcal{M}(\Omega)(m,n)$ for $1 \leqslant i \leqslant k$, let

$$\mathcal{E}_A\left(\sum_{i=1}^{k} \alpha_i f_i\right) := \sum_{i=1}^{s} \alpha_i \mathcal{E}_A(f_i).$$

An Ω-polynomial f is a *polynomial identity in A with coefficients in R* if $\mathcal{E}_A(f) = 0$. In this case we say that A satisfies $f \equiv 0$. Polynomial identities in A with coefficients in R are morphisms in the category $\mathrm{Id}(A,R)$ where the set of objects is $\mathbb{Z}_+$ and $\mathrm{Id}(A,R)(m,n) := \mathrm{Ker}\left(\mathcal{E}_A\big|_{\mathcal{P}(R,\Omega)(m,n)}\right)$ for every $m,n \in \mathbb{Z}_+$.

Let V be a set of Ω-polynomials with coefficients in R. The full subcategory $\mathrm{Var}(V)$ of Ω-$\mathsf{Magma}(\mathcal{C})$, consisting of all Ω-magmas satisfying $f \equiv 0$ for all $f \in V$, is again called a *variety* of Ω-magmas.

Recall that a *monoidal ideal* I in an R-linear monoidal category X is a system of R-submodules $I(a,b) \subseteq X(a,b)$, where a,b are objects in X, such that

(1) $gf \in I(a,c)$ for all $f \in X(a,b)$, $g \in X(b,c)$ where either $f \in I(a,b)$ or $g \in I(b,c)$;
(2) $f \otimes g \in I(a\otimes c, b \otimes d)$ for all $f \in X(a,b)$, $g \in X(c,d)$ where either $f \in I(a,b)$ or $g \in I(c,d)$

for all objects a,b,c,d in X. Then the *factor category* X/I is the category with the same objects as in X and the hom-sets $X(a,b)/I(a,b)$. It is easy to see that X/I is again an R-linear monoidal category and the canonical functor $X \to X/I$ is an R-linear strict monoidal functor. If X is braided (symmetric), then the category X/I and the functor $X \to X/I$ are braided (resp., symmetric) too.

Example 3.1. For every Ω-magma A the hom-sets $\mathrm{Id}(A,R)(m,n)$ of $\mathrm{Id}(A,R)$ form a monoidal ideal in $\mathcal{P}(R,\Omega)$.

Again, given a set V of Ω-polynomials with coefficients in R, we consider the minimal monoidal ideal I in $\mathcal{P}(R,\Omega)$ such that $V \subseteq \bigcup\limits_{m,n\in\mathbb{Z}_+} I(m,n)$. We call the category

$\mathcal{P}_V(R,\Omega) := \mathcal{P}(R,\Omega)/I$ the category of *V-relative Ω-polynomials with coefficients in R*. Then an Ω-magma A belongs to $\mathrm{Var}(V)$ if and only if there exists a R-linear braided strong monoidal functor $F\colon \mathcal{P}_V(R,\Omega) \to \mathcal{C}$ such that $F(1) = A$ and $F(\omega) = \omega_A$ for all $\omega \in \Omega$. The category $\mathcal{P}_V(R,\Omega)$ is an R-linear BMAT and the analogs of Propositions 2.14, 2.17 and 2.18 hold too:

Proposition 3.2. *Let Ω be a set together with maps $s,t\colon \Omega \to \mathbb{Z}_+$ and let $\mathcal{C}$ be a R-linear braided monoidal category for a unital commutative ring R. Then the categories Ω-$\mathsf{Magma}(\mathcal{C})$ and R-$\mathbf{LBSMF}(\mathcal{P}(R,\Omega),\mathcal{C})$ are equivalent. Moreover, if V is a set of Ω-polynomials with coefficients in R, then the categories $\mathrm{Var}(V)$ and R-$\mathbf{LBSMF}(\mathcal{P}_V(R,\Omega),\mathcal{C})$ are equivalent.*

Proof. Again, consider the correspondence $A \mapsto \mathcal{E}_A$ and $F \mapsto F(1)$ where A is an Ω-magma and $F\colon \mathcal{P}(R,\Omega) \to \mathcal{C}$ is an R-linear braided strong monoidal functor. □

Proposition 3.3. *Let R be a unital commutative ring. An arbitrary R-linear BMAT $\mathbb{A}$ is linearly braided monoidally isomorphic to $\mathcal{P}_V(R,\Omega)$ for some V and Ω.*

Proof. Denote by Ω the union of generating sets for all hom-R-modules of $\mathbb{A}$ and define the maps $s,t\colon \Omega \to \mathbb{R}_+$ by $s(\omega) := m$, $s(\omega) := n$ for $\omega \in \mathbb{A}(m,n)$. Then 1 is an Ω-magma in $\mathbb{A}$ and there exists a unique R-linear strict braided monoidal functor $F\colon \mathcal{P}_V(R,\Omega) \to \mathbb{A}$ such that $F(1) = 1$ and $F(\omega) = \omega$ for all $\omega \in \Omega$. Again, F is surjective on hom-sets. Denote by V the set of polynomial identities resulting from the kernel ideal I of F. Then $\mathbb{A} \cong \mathcal{P}(R,\Omega)/I = \mathcal{P}_V(R,\Omega)$. □

Again, if an Ω-magma A belongs to $\mathrm{Var}(V)$ for some set V of Ω-polynomials with coefficients in R, it is natural to use in the definition of additional polynomial identities that A may satisfy, V-relative Ω-polynomials with coefficients in R. The functor $\mathcal{P}_V(R,\Omega) \to \mathcal{C}$ induced by $\mathcal{E}_A$ is denoted again by $\mathcal{E}_A$.

Definition 3.4. Let V be a set of Ω-polynomials with coefficients in R. We say that a polynomial identity $f \equiv 0$ *follows* from V for some Ω-polynomial f if the image of f in $\mathcal{P}_V(\Omega,R)$ is zero. Let A be an Ω-magma. We say that V *generates* polynomial identities in A with coefficients in R or that V is a *basis* for polynomial identities in A with coefficients in R if A belongs to $\mathrm{Var}(V)$ and the functor $\mathcal{E}_A\colon \mathcal{P}_V(\Omega,R) \to \mathcal{C}$ is faithful. In other words, V generates polynomial identities in A if any polynomial identity $f \equiv 0$ in A follows from V.

Example 3.5. Let $\Omega = \{\mu\}$, $s(\mu) = 2$, $t(\mu) = 1$, $V_{\text{assoc}} := \{(\mu \otimes \mathrm{id}_1)\mu - (\mathrm{id}_1 \otimes \mu)\mu\} \cup V_{\text{symm}}$ where $V_{\text{symm}} := \{\tau_{1,1}^2 - \mathrm{id}_2\}$ and let $\Bbbk$ be a field. In other words, V_{assoc} consists of the associator $(xy)z - x(yz)$ and the identities that force the braiding $\tau_{k,\ell}\colon (k+\ell) \to (k+\ell)$ to be symmetric (see Remark 2.20). Then $\mathcal{P}_{V_{\text{assoc}}}(\Bbbk,\Omega)(n,1)$, where $n \in \mathbb{N}$, coincides with the vector space P_n of associative multilinear polynomials in the variables $x_1,\dots,x_n$. Under this identification, polynomial identities in an arbitrary associative algebra A correspond to polynomial identities in A.

3.3. **Codimensions of Ω-polynomial identities.** Consider an Ω-magma A in a $\Bbbk$-linear braided monoidal category $\mathcal{C}$ where $\Bbbk$ is a field. Let $m,n \in \mathbb{Z}_+$. Then

$$c_{m,n}(A) := \dim \mathcal{E}_A\big(\mathcal{P}(\Bbbk,\Omega)(m,n)\big)$$

is called the *(m,n)-codimension* of polynomial identities of A.

Proposition 3.6. *Let A be an Ω-algebra over a field $\Bbbk$, i.e. an Ω-magma in $\mathbf{Vect}_\Bbbk$. If $\dim A < +\infty$, then $c_{m,n}(A) \leqslant (\dim A)^{m+n}$ for every $m,n \in \mathbb{Z}_+$.*

Proof. The space $\mathcal{E}_A\big(\mathcal{P}(\Bbbk,\Omega)(m,n)\big)$ is the subspace of $\mathrm{Hom}_\Bbbk(A^{\otimes m}, A^{\otimes n})$ that consists of all linear maps $A^{\otimes m} \to A^{\otimes n}$ that can be constructed using the operations from Ω together with permutations of variables. Now the upper bound on $c_{m,n}(A)$ follows from the equality $\dim \mathrm{Hom}_{\Bbbk}(A^{\otimes m}, A^{\otimes n}) = (\dim A)^{m+n}$. □

Example 3.7. If A is an ordinary algebra over a field $\Bbbk$, then $c_{n,1}(A)$ is the nth ordinary codimension $c_n(A)$ of polynomial identities of A for $n \in \mathbb{N}$.

3.4. **Ω^*-magmas.** For a set Ω together with maps $s,t\colon \Omega \to \mathbb{Z}_+$ define the set $\Omega^* := \{\omega^* \mid \omega \in \Omega\}$ and the maps $s,t\colon \Omega^* \to \mathbb{Z}_+$ by $s(\omega^*) := t(\omega)$ and $t(\omega^*) := s(\omega)$ for all $\omega \in \Omega$. We call the signature Ω^* *dual* to Ω. By the definition, $\Omega^{**} := \Omega$.

Example 3.8. Unital algebras are Ω-magmas for Ω from Example 2.11 while coalgebras are Ω^*-magmas for the same Ω.

The map $(-)^*\colon \Omega \to \Omega^*$ induces the contravariant functor $(-)^*\colon \mathcal{M}(\Omega) \to \mathcal{M}(\Omega^*)$ where $m^* := m$ for objects, $(u \otimes v)^* := u^* \otimes v^*$ for morphisms, $\tau^*_{m,n} := \tau^{-1}_{m^*,n^*}$. For every unital commutative ring R the functor $(-)^*$ extends uniquely to an R-linear contravariant functor $(-)^*\colon \mathcal{P}(R,\Omega) \to \mathcal{P}(R,\Omega^*)$

Consider finite dimensional Ω-algebras A over a field $\Bbbk$, which can be treated as Ω-magmas in the category $\mathbf{Vect}^{\text{f.d.}}_{\Bbbk}$ of finite dimensional vector spaces over $\Bbbk$. After the natural identifications $(A^{\otimes n})^* \cong (A^*)^{\otimes n}$, the space A^* of linear functions $A \to \Bbbk$ becomes an Ω^*-algebra with $\omega_{A^*} := \omega_A^*$ for all $\omega \in \Omega$. The commutativity of the diagram below, where both horizontal arrows are bijections, implies that f^* is a polynomial identity in A^* if and only if f is a polynomial identity in A and $c_{m,n}(A) = c_{n,m}(A^*)$ for all $m,n \in \mathbb{Z}_+$:

$$\begin{tikzcd}
\mathcal{P}(\Bbbk,\Omega)\,(m,n) \arrow[r, "(-)^*"] \arrow[d, "\mathcal{E}_A"] & \mathcal{P}(\Bbbk,\Omega^*)\,(n,m) \arrow[d, "\mathcal{E}_{A^*}"] \\
\mathbf{Vect}^{\text{f.d.}}_{\Bbbk}\,(A^{\otimes m}, A^{\otimes n}) \arrow[r, "(-)^*"] & \mathbf{Vect}^{\text{f.d.}}_{\Bbbk}\,((A^*)^{\otimes n}, (A^*)^{\otimes m})
\end{tikzcd}$$

4. Polynomial identities in graded and (co)module algebras

4.1. **Polynomial U-identities.** Motivated by applications of the reconstruction technique to H-(co)module algebras (see e.g. [8, Chapter 5]), we give the following definitions.

Let $U\colon \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between a monoidal category $\mathcal{C}$ and a braided monoidal category $\mathcal{D}$. Consider the set-theoretical monoid $\mathrm{End}(U)$ of (not necessarily monoidal) natural transformations $U \Rightarrow U$. For every Ω-magma A in $\mathcal{C}$ the object UA admits the structure of an $\Omega \sqcup \mathrm{End}(U)$-magma in $\mathcal{D}$ where for $h \in \mathrm{End}(U)$ we have $s(h) := t(h) := 1$ and the operation $h_{UA}\colon UA \to UA$ is just the component h_A of the natural transformation h. Then the *polynomial U-identities* of the Ω-magma A in $\mathcal{C}$ are the polynomial identities of the $\Omega \sqcup \mathrm{End}(U)$-magma A in $\mathcal{D}$. If $\mathcal{D}$ is linear over a field $\Bbbk$, one can define the codimensions $c^U_{m,n}(A)$ of polynomial U-identities too. Let $m,n \in \mathbb{Z}_+$. Then

$$c^U_{m,n}(A) := \dim \mathcal{E}_{UA}\big(\mathcal{P}(\Bbbk, \Omega \sqcup \mathrm{End}(U))(m,n)\big)$$

is called the (m,n)-*codimension of polynomial U-identities* of A.

4.2. **Polynomial H-identities in H-module algebras.** In order to show why multilinear polynomial H-identities in H-module algebras and their codimensions are indeed a particular case of polynomial U-identities and their codimensions introduced above, we recall the corresponding definitions [4, 5, 11]. (We refer the reader to [6, 18, 22] for an account of Hopf algebras and algebras with Hopf algebra actions.)

The free associative non-unital algebra $\Bbbk\langle X\rangle$ over a field $\Bbbk$ admits a $\mathbb{Z}$-grading $\Bbbk\langle X\rangle = \bigoplus_{n=1}^{\infty} \Bbbk\langle X\rangle^{(n)}$ where $\Bbbk\langle X\rangle^{(n)}$ is the linear span of all monomials of total degree n.

Let H be a Hopf algebra over the field $\Bbbk$. Recall that the category ${}_H\mathsf{Mod}$ of left H-modules is monoidal where the monoidal product of H-modules M and N is their tensor product over the base field $\Bbbk$ and $h(m \otimes n) := h_{(1)}m \otimes h_{(2)}n$ for all $h \in H$, $m \in M$, $n \in N$. (Here we use Sweedler's notation $\Delta h = h_{(1)} \otimes h_{(2)}$ for the comultiplication $\Delta \colon H \to H \otimes H$ and $h \in H$, where the sign of sum is omitted.) An algebra A over a field $\Bbbk$ is an *H-module algebra* if A is a (left) H-module and the multiplication $A \otimes A \to A$ is an H-module homomorphism.

Consider the algebra

$$\Bbbk\langle X|H\rangle := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes \Bbbk\langle X\rangle^{(n)}$$

with the multiplication $(u \otimes v)(t \otimes w) := u \otimes t \otimes vw$ for $u \in H^{\otimes m}$, $t \in H^{\otimes n}$, $v \in \Bbbk\langle X\rangle^{(m)}$, $w \in \Bbbk\langle X\rangle^{(n)}$, $m, n \in \mathbb{N}$. Then $\Bbbk\langle X|H\rangle$ is a left H-module algebra where $h(u \otimes v) = hu \otimes v$ for all $u \in H^{\otimes m}$, $v \in \Bbbk\langle X\rangle^{(m)}$, $m \in \mathbb{N}$. The subset $\{1 \otimes x_i \mid i \in \mathbb{N}\}$ can be identified with X. The algebra $\Bbbk\langle X|H\rangle$ is called the *free H-module algebra*. Note that $\Bbbk\langle X|H\rangle$ is just the (non-unital) tensor algebra of the free H-module generated by the set X. Elements of $\Bbbk\langle X|H\rangle$ are called *H-polynomials*. Every map $X \to A$, where A is an associative H-module algebra, extends to a homomorphism $\Bbbk\langle X|H\rangle \to A$ of algebras and H-modules in a unique way. An H-polynomial $f \in \Bbbk\langle X|H\rangle$ is called a *polynomial H-identity* of A if $\varphi(f) = 0$ for all homomorphisms $\varphi \colon \Bbbk\langle X|H\rangle \to A$ of algebras and H-modules. The set $\mathrm{Id}^H(A)$ of polynomial H-identities of A is an H-invariant ideal of $k\langle X|H\rangle$.

Let $x_{i_1}^{h_1} \dots x_{i_n}^{h_n} := h_1 \otimes h_2 \otimes \dots \otimes h_n \otimes x_{i_1} \dots x_{i_n}$ for $h_1, \dots, h_n \in H$, $i_1, \dots, i_n, n \in \mathbb{N}$. The subspace

$$P_n^H = \left\langle x_{\sigma(1)}^{h_1} \dots x_{\sigma(n)}^{h_n} \,\middle|\, \sigma \in S_n,\ h_i \in H \right\rangle_{\Bbbk} \subset \Bbbk\langle X|H\rangle$$

is called the space of *multilinear H-polynomials* in variables $x_1, \dots, x_n$. The dimension $c_n^H(A) := \dim \frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)}$ is called the *nth codimension of polynomial H-identities* of A.

Let $\mathcal{C} := {}_H\mathsf{Mod}$, $\mathcal{D} := \mathbf{Vect}_{\Bbbk}$ and let $U \colon \mathcal{C} \to \mathcal{D}$ be the corresponding forgetful functor. For an arbitrary natural transformation $\theta \colon U \Rightarrow U$ define $h_\theta := \theta_H(1_H)$ where H is considered to be a left module over itself. Let $m \in M$ where M is a left H-module. Define the H-module homomorphism $f \colon H \to M$ by $f(h) := hm$, $h \in H$. The diagram below is commutative by the naturality of θ:

$$\begin{tikzcd} H \arrow[r, "\theta_H"] \arrow[d, "f"'] & H \arrow[d, "f"'] \\ M \arrow[r, "\theta_M"] & M \end{tikzcd}$$

Hence $\theta_M(m) = \theta_M f(1_H) = f\theta_M(1_H) = f(h_\theta) = h_\theta m$. Conversely, every element $h \in H$ defines a natural transformation $\theta \colon U \Rightarrow U$ by $\theta_M(m) := hm$ for $m \in M$ where M is a left H-module. Therefore, $\mathrm{End}(U)$ can be identified with H.

Now consider an H-module algebra A and let $\Omega = \{\mu\}$, $s(\mu) = 2$, $t(\mu) = 1$. Every element of $\mathcal{P}(\Bbbk, \{\mu\} \sqcup \mathrm{End}(U))(n, 1) = \mathcal{P}(\Bbbk, \{\mu\} \sqcup H)(n, 1)$, where $n \in \mathbb{N}$, corresponds to a linear map $A^{\otimes n} \to A$ that can be represented by an H-polynomial. Therefore, the codimension $\dim \mathcal{E}_{UA}\big(\mathcal{P}(\Bbbk, \{\mu\} \sqcup H)(n, 1)\big)$ of polynomial U-identities equals $c_n^H(A)$. In contrast with traditional polynomial H-identities, where operators from H are applied only to variables themselves, not their products, in the elements of $\mathcal{P}(\Bbbk, \{\mu\} \sqcup H)(m, n)$ operators from H can be applied in any place. However if we let $V := V_{\mathrm{assoc}} \cup \{h(xy) - (h_{(1)}x)(h_{(2)}y) \mid h \in H\}$, then $\mathcal{P}_V(\Bbbk, \{\mu\} \sqcup H)(n, 1)$ can be identified with P_n^H. Under this identification, polynomial identities correspond to polynomial identities.

In addition, one can consider a more general situation. Recall that an algebra A is an *algebra with a generalized H-action* if A is a left H-module for an associative unital algebra

H and for every $h \in H$ there exist $k \in \mathbb{N}$ and $h_i', h_i'', h_i''', h_i'''' \in H$ where $1 \leqslant i \leqslant k$ such that

$$h(ab) = \sum_{i=1}^{k} \big((h_i'a)(h_i''b) + (h_i'''b)(h_i''''a)\big) \text{ for all } a, b \in A.$$

Multilinear H-polynomials are introduced in the same way as for H-module algebras [5, 11]. Again, multilinear H-polynomials of degree n can be identified with elements of $\mathcal{P}_V(\Bbbk, \{\mu\} \sqcup H)(n, 1)$ for

$$V = V_{\text{assoc}} \cup \left\{ h(xy) - \sum_{i=1}^{k} \big((h_i'x)(h_i''y) + (h_i'''y)(h_i''''x)\big) \,\middle|\, h \in H \right\}.$$

As before, polynomial identities correspond to polynomial identities.

4.3. **Graded polynomial identities and polynomial H-identities in H-comodule algebras.** Here we first recall the definition of polynomial H-identities in H-comodule algebras introduced in [2] as a generalization of graded polynomial identities (see e.g. [9, Section 10.5]).

Let H be again a Hopf algebra H over a field $\Bbbk$. Then the category Comod^H of right H-comodules is monoidal where for H-comodules M and N the comodule map $\rho_{M\otimes N} \colon M \otimes N \to M \otimes N \otimes H$ is defined by $\rho_{M\otimes N}(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}$ for all $m \in M$ and $n \in N$. (Here we use Sweedler's notation $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ for $m \in M$ and the comodule map $\rho_M \colon M \to M \otimes H$.) Below will denote the structure maps for all right comodules by the same letter ρ.

Let A be an H-*comodule algebra*, i.e. a $\Bbbk$-algebra that is a (right) H-comodule where the multiplication $A \otimes A \to A$ is an H-comodule homomorphism. In order to define a polynomial H-identity, E. Aljadeff and C. Kassel use $\Bbbk\langle X|H\rangle$ too, however considering on $\Bbbk\langle X|H\rangle$ a structure of an H-comodule algebra. Namely, H is now a right H-comodule where the comodule map ρ coincides with the comultiplication Δ on H, the spaces $\Bbbk\langle X\rangle^{(n)}$ are trivial H-comodules where $\rho(w) := w \otimes 1$ and structure of an H-comodule on $\Bbbk\langle X|H\rangle$ is induced from H and $\Bbbk\langle X\rangle^{(n)}$ via the tensor product $\otimes$. An element $f \in \Bbbk\langle X|H\rangle$ is a polynomial H-identity in A if $\varphi(f) = 0$ for all homomorphisms $\varphi \colon \Bbbk\langle X|H\rangle \to A$ of algebras and H-comodules.

Example 4.1. Let G be a group and let $\Bbbk G$ be its group Hopf algebra. Recall that the comultiplication $\Delta \colon \Bbbk G \to \Bbbk G \otimes \Bbbk G$, the counit $\varepsilon \colon \Bbbk G \to \Bbbk$ and the antipode $S \colon \Bbbk G \to \Bbbk G$ are defined on $\Bbbk G$ as follows: $\Delta g := g \otimes g$, $\varepsilon(g) = 1$, $Sg := g^{-1}$ for all $g \in G$. Moreover, right $\Bbbk G$-modules are just G-graded vector spaces $M = \bigoplus_{g \in G} M^{(g)}$ where $\rho \colon M \to M \otimes \Bbbk G$ is defined by $\rho(m) = m \otimes g$ for $m \in M$, $g \in G$. Let $x_i^{(g)} := g \otimes x_i \in \Bbbk\langle X|\Bbbk G\rangle$ for $g \in G$ and $i \in \mathbb{N}$. Then $\Bbbk\langle X|\Bbbk G\rangle$ can be identified with the free associative non-unital algebra $\Bbbk\langle X^{G\text{-gr}}\rangle$ where $X^{G\text{-gr}} := \left\{ x_i^{(g)} \,\middle|\, g \in G,\ i \in \mathbb{N} \right\}$. Expanding the definition of polynomial $\Bbbk G$-identity for a G-graded algebra $A = \bigoplus_{g \in G} A^{(g)}$, we obtain that $f \in \Bbbk\langle X^{G\text{-gr}}\rangle$ is a polynomial $\Bbbk G$-identity if and only if $\varphi(f) = 0$ for all algebra homomorphisms $\varphi \colon \Bbbk\langle X^{G\text{-gr}}\rangle \to A$ such that $\varphi(X^{(g)}) \subseteq A^{(g)}$, which is exactly the definition of a G-*graded polynomial identity*. Denote by $\operatorname{Id}^{G\text{-gr}}(A)$ the set of G-graded polynomial identities of A, which is a graded ideal of $\Bbbk\langle X^{G\text{-gr}}\rangle$. The subspace

$$P_n^{G\text{-gr}} = \left\langle x_{\sigma(1)}^{(g_1)} \dots x_{\sigma(n)}^{(g_n)} \,\middle|\, \sigma \in S_n,\ g_i \in G \right\rangle_{\Bbbk} \subset \Bbbk\langle X^{G\text{-gr}}\rangle$$

is called the space of *multilinear G-graded polynomials* in variables $x_1, \ldots, x_n$. The dimension $c_n^{G\text{-gr}}(A) := \dim \frac{P_n^{G\text{-gr}}}{P_n^{G\text{-gr}} \cap \mathrm{Id}^{G\text{-gr}}(A)}$ is called the *nth codimension of G-graded polynomial identities* of A.

When H is different from a group algebra, the description of all H-comodule algebra homomorphisms $\Bbbk\langle X|H\rangle \to A$ may become not as explicit as in the case of H-module algebras. The main difference is the following. While $\Bbbk\langle X|H\rangle$ is the tensor algebra of the free H-module $H \otimes \langle x_1, x_2, x_3 \ldots\rangle_\Bbbk$ and all H-module algebra homomorphisms from $\Bbbk\langle X|H\rangle$ are defined by images of x_i, the same vector space $H \otimes \langle x_1, x_2, x_3 \ldots\rangle_\Bbbk$ is not free, but a cofree right H-comodule.

On the other hand, every right H-comodule is a left H^*-module where H^* is the algebra dual to the coalgebra H. Namely, $\gamma m := \gamma(m_{(1)})m_{(0)}$ for all $\gamma \in H^*$, $m \in M$ and an H-comodule M. Moreover, all H^*-module homomorphisms between two comodules are H-comodule homomorphisms and vice versa. The dual H^* of a finite dimensional Hopf algebra H is again a Hopf algebra. Moreover, by the Larson — Sweedler theorem, H is co-Frobenius, that is $H \cong H^*$ as left H^*-modules (see e.g. [15] and [18, Theorem 2.1.3]), and

$$\Bbbk\langle X|H\rangle \cong \Bbbk\langle X|H^*\rangle$$

as H^*-module algebras.

Let $\mathcal{C} := \mathsf{Comod}^H$ for an arbitrary Hopf algebra H over a field $\Bbbk$, $\mathcal{D} := \mathbf{Vect}_\Bbbk$ and let $U\colon \mathcal{C} \to \mathcal{D}$ be the corresponding forgetful functor. For an arbitrary natural transformation $\theta\colon U \Rightarrow U$ define the linear map $\gamma\colon H \to \Bbbk$ by $\gamma(h) := \varepsilon(\theta_H(h))$ for $h \in H$ where H is considered to be a right module over itself via the comultiplication $\Delta\colon H \to H \otimes H$. Let M be a right H-comodule and let $\alpha\colon M \to \Bbbk$ be an arbitrary linear function. Denote by $\rho\colon M \to M \otimes H$ the comodule structure map on M. Then $(\alpha \otimes \mathrm{id}_H)\rho\colon M \to H$ is a comodule homomorphism. Therefore the diagram below is commutative by the naturality of θ:

$$\begin{tikzcd}
M \arrow[r, "\theta_M"] \arrow[d, "(\alpha\otimes\mathrm{id}_H)\rho"'] & M \arrow[d, "(\alpha\otimes\mathrm{id}_H)\rho"'] \\
H \arrow[r, "\theta_H"] & H
\end{tikzcd}$$

Hence $(\alpha \otimes \mathrm{id}_H)\rho\theta_M = \theta_H(\alpha \otimes \mathrm{id}_H)\rho$. Applying the counit $\varepsilon\colon H \to \Bbbk$ to both sides and substituting an arbitrary element m for the argument, we get

$$(\alpha \otimes \varepsilon)\rho\theta_M(m) = \gamma(m_{(1)})\alpha(m_{(0)}),$$

$$\alpha(\theta_M(m)_{(0)})\varepsilon(\theta_M(m)_{(1)}) = \alpha\big(\gamma(m_{(1)})m_{(0)}\big),$$

$$\alpha(\theta_M(m)) = \alpha\big(\gamma(m_{(1)})m_{(0)}\big).$$

Since $\alpha \in M^*$ is arbitrary, we get $\theta_M(m) = \gamma(m_{(1)})m_{(0)}$. Conversely, every $\gamma \in H^*$ defines the natural transformation $\theta\colon U \Rightarrow U$ by $\theta_M(m) := \gamma(m_{(1)})m_{(0)}$ for $m \in M$ where M is a right H-comodule. Therefore, $\mathrm{End}(U)$ can be identified with H^*.

Let $\Omega = \{\mu\}$, $s(\mu) = 2$, $t(\mu) = 1$. If H is finite dimensional, then, as we have mentioned above, $H \cong H^*$ as left H^*-modules and $\mathcal{P}_V(\Bbbk, \{\mu\} \sqcup H^*)(n, 1)$ for $V = V_{\text{assoc}} \cup \{h(xy) - (h_{(1)}x)(h_{(2)}y) \mid h \in H^*\}$ can be identified with the subspace of $\Bbbk\langle X|H\rangle$ consisting of all multilinear H-polynomials in $x_1, \ldots, x_n$. Again, under this identification polynomial identities correspond to polynomial identities. In particular, for an algebra A graded by a finite group G we have the equality of codimensions

$$\dim \mathcal{E}_{UA}\big(\mathcal{P}(\Bbbk, \{\mu\} \sqcup (\Bbbk G)^*)(n, 1)\big) = c_n^{G\text{-gr}}(A).$$

By [10, Lemma 7.1], the equality above still holds if G is infinite but A is finite dimensional.

5. Examples and applications

5.1. **Vector spaces.** Every object A in a braided monoidal category $\mathcal{C}$ can be considered as an $\varnothing$-magma. However, the category $\mathcal{M}(\varnothing)$ still has non-trivial morphisms resulting from the braiding in $\mathcal{C}$. At the same time, $\mathcal{M}(\varnothing)(m,n) = \varnothing$ for $m \neq n$.

Let W be a vector space over a field $\Bbbk$. Recall that the category $\mathbf{Vect}_\Bbbk$ is symmetric, which implies that the polynomial identity $\tau_{1,1}^2 - \mathrm{id}_2 \equiv 0$ holds in W. By the Coherence Theorem for the symmetric categories applied to $\mathcal{M}_{V_{\mathrm{symm}}}(\varnothing)$, where $V_{\mathrm{symm}} := \{\tau_{1,1}^2 - \mathrm{id}_2\}$ (see Remark 2.20), there exists an isomorphism between the nth symmetric group S_n and the monoid $\mathcal{M}_{V_{\mathrm{symm}}}(\varnothing)(n,n)$ where the morphism $f\colon n \to n$ corresponds to the permutation $\sigma \in S_n$ such that

$$(\mathcal{E}_W f)(a_1 \otimes a_2 \otimes \dots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \dots \otimes a_{\sigma^{-1}(n)} \text{ for all } a_1, \dots, a_n \in W.$$

This bijection extends to the isomorphism $\theta\colon \Bbbk S_n \xrightarrow{\sim} \mathcal{P}_{V_{\mathrm{symm}}}(\varnothing, \Bbbk)(n,n)$ of algebras.

Theorem 5.1. *Let W be a vector space over a field $\Bbbk$, $\operatorname{char} \Bbbk = 0$, $\dim W = d \in \mathbb{Z}_+$. Then all polynomial $\varnothing$-identities in W with coefficients in $\Bbbk$ follow from the set*

$$V_{\mathrm{symm},d} := V_{\mathrm{symm}} \cup \left\{ \sum_{\sigma \in S_{d+1}} (\operatorname{sign}\sigma)\theta(\sigma) \right\}.$$

Moreover $c_{m,n}(W) = 0$ for $m \neq n$ and

$$c_{n,n}(W) \sim \alpha_d n^{-\frac{d^2-1}{2}} d^{2n} \text{ as } n \to \infty$$

where $\alpha_d := \left(\frac{1}{\sqrt{2\pi}}\right)^{d-1} \left(\frac{1}{2}\right)^{(d^2-1)/2} \cdot 1! \cdot 2! \cdot \ldots \cdot (d-1)! \cdot d^{d^2/2}$.

Proof. One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. Recall that irreducible representations of S_n are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ where $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_s \geqslant 0$, $\sum_{i=1}^{s} \lambda_i = n$, and their diagrams D_λ. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign}\sigma)\sigma$, be the Young symmetrizers corresponding to a Young tableau T_λ. Then $M(\lambda) = \Bbbk S_n e_{T_\lambda} \cong \Bbbk S_n e^*_{T_\lambda}$ is an irreducible $\Bbbk S_n$-module corresponding to a partition $\lambda \vdash n$. We refer the reader to [3, 7, 9] for an account of S_n-representations and their applications to polynomial identities.

The polynomial identity $\sum_{\sigma \in S_{d+1}} (\operatorname{sign}\sigma)\theta(\sigma) \equiv 0$ holds in W since $\dim W = d$. Therefore, $\mathcal{E}_W\theta$ factors through $\mathcal{P}_{V_{\mathrm{symm},d}}(\varnothing, \Bbbk)(n,n)$. Hence all Young symmetrizers corresponding to Young diagrams D_λ of height $\operatorname{ht}\lambda$ greater than d are mapped by $\mathcal{E}_W\theta$ to 0. On the other hand, given a Young tableau T_λ where $\operatorname{ht}\lambda \leqslant d$, apply $\mathcal{E}_W\theta\left(e^*_{T_\lambda}\right)$ to the tensor product $a_1 \otimes \dots \otimes a_n$ of basis elements $b_1, \dots, b_d$ of W where $a_i := b_j$ if the number i appears in the jth row of T_λ. Then the result is nonzero, whence $\mathcal{E}_W\theta\left(e^*_{T_\lambda}\right) \neq 0$.

Recall that the group algebra $\Bbbk S_n$ is the direct sum of minimal ideals, each of which is the direct sum of isomorphic irreducible representations of S_n. Every Young symmetrizer belongs to the corresponding minimal ideal. Thus $\operatorname{Ker}(\mathcal{E}_W\theta)$ is generated as an ideal by Young symmetrizers $e^*_{T_\lambda}$ with $\operatorname{ht}\lambda > d$ and the image of $\operatorname{Ker}(\mathcal{E}_W\theta)$ in $\mathcal{P}_{V_{\mathrm{symm},d}}(\varnothing, \Bbbk)(n,n)$ is zero. Therefore, all polynomial identities of W are indeed generated by the set $V_{\mathrm{symm},d}$. Moreover, the image of $\mathcal{E}_W\theta$ is isomorphic as a vector space to the direct sum of $M(\lambda)$ taken with the same multiplicity as in $\Bbbk S_n$, i.e. $\dim M(\lambda)$, where $\lambda \vdash n$, $\operatorname{ht}\lambda \leqslant d$. Hence

$$c_{n,n}(W) = \sum_{\substack{\lambda\vdash n,\\ \operatorname{ht}\lambda\leqslant d}} \big(\dim M(\lambda)\big)^2.$$

By [21, Section 4.5, Case 2], $c_{n,n}(W)$ has the required asymptotic behavior. □

Remark 5.2. If $d := \dim W = 0$, then $\sum\limits_{\sigma\in S_{d+1}} (\operatorname{sign}\sigma)\theta(\sigma) \equiv 0$ reduces to $\mathrm{id}_1 \equiv 0$ and $c_{m,n}(W) = 0$ for all $m, n \in \mathbb{Z}_+$.

5.2. **Hopf algebras.** Every Hopf algebra H over a field $\Bbbk$ is an Ω-magma in $\mathbf{Vect}_\Bbbk$ where $\Omega = \{\mu, u, \Delta, \varepsilon, S\}$ are symbols for the multiplication $\mu_H \colon H\otimes H \to H$, the unit $u_H \colon \Bbbk \to H$, the comultiplication $\Delta_H \colon H \to H\otimes H$ the counit $\varepsilon_H \colon H \to \Bbbk$ and the antipode $S_H \colon H \to H$. A vector space H endowed with the maps listed above is a Hopf algebra if the diagrams corresponding to the conditions from Section 2.1 are commutative. Since the commutativity of each diagram means that the corresponding polynomial identity holds in H, Hopf algebras form a variety $\operatorname{Var}(V_{\mathbf{Hopf}})$ where

$$\begin{aligned} V_{\mathbf{Hopf}} = V_{\mathrm{symm}} \cup \{&\mu(\mu\otimes \mathrm{id}_1) - \mu(\mathrm{id}_1 \otimes\mu),\quad \mu(u\otimes \mathrm{id}_1) - \mathrm{id}_1,\quad \mu(\mathrm{id}_1 \otimes u) - \mathrm{id}_1,\\ &(\Delta\otimes \mathrm{id}_1)\Delta - (\mathrm{id}_1 \otimes\Delta)\Delta,\quad (\varepsilon\otimes \mathrm{id}_1)\Delta - \mathrm{id}_1,\quad (\mathrm{id}_1 \otimes\varepsilon)\Delta - \mathrm{id}_1,\\ &\varepsilon u - \mathrm{id}_0,\quad \varepsilon\mu - \varepsilon\otimes\varepsilon,\\ &\Delta u - u\otimes u,\quad \Delta\mu - (\mu\otimes\mu)(\mathrm{id}_1 \otimes\tau_{1,1} \otimes \mathrm{id}_1)(\Delta\otimes\Delta),\\ &\mu(S\otimes \mathrm{id}_1)\Delta - u\varepsilon,\quad \mu(\mathrm{id}_1 \otimes S)\Delta - u\varepsilon\}. \end{aligned} \tag{5.1}$$

On the one hand, in order to depict maps from $\mathcal{E}_H\big(\mathcal{M}(\Bbbk,\Omega)(m,n)\big)$ one can use the graphical calculus (see e.g. [12]). On the other hand, one can represent compositions in the usual functional notation. For example, $\mu(\mu\otimes \mathrm{id}_1)$ can be written as $(xy)z$ or $(x_1x_2)x_3$ and $(\Delta\otimes \mathrm{id}_1)\Delta$ can be written as $x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)}$.

A Hopf monoid H is *cocommutative* if the polynomial identity $x_{(1)} \otimes x_{(2)} \equiv x_{(2)} \otimes x_{(1)}$ holds in H. Denote the category $\mathcal{P}_{V_{\mathbf{CocHopf}}}(\Bbbk,\Omega)$, where

$$V_{\mathbf{CocHopf}} := V_{\mathbf{Hopf}} \cup \{x_{(1)} \otimes x_{(2)} - x_{(2)} \otimes x_{(1)}\},$$

by $\mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)$. Then 1 is a cocommutative Hopf monoid in $\mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)$. The standard convolution techniques (see e.g. [6, Proposition 4.2.7]) show that $S^2 = \mathrm{id}_1$. Using the compatibility conditions and properties of the antipode S, in every monomial f in $\mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)$ we move the multiplication, the antipode and the unit to the codomain and the comultiplication and the counit to the domain. We get

$$f(x_1,\dots,x_m) = \varepsilon(x_{i_1})\dots\varepsilon(x_{i_s}) \bigotimes_{k=1}^{n} S^{\alpha_{k1}} x_{j_{k1}(\ell_{k1})} \dots S^{\alpha_{kt_k}} x_{j_{kt_k}(\ell_{kt_k})}$$

where $\alpha_{kr} \in \{0,1\}$, $i_1 < \ldots < i_s$, every symbol x_q or $x_{q(r)}$ for given $1 \leqslant q \leqslant m$ and $r \in \mathbb{Z}_+$ appears no more than once and there exist numbers $p_q \in \mathbb{Z}_+$ such that

$$\begin{aligned} &\{x_{i_1},\dots,x_{i_s}\} \sqcup \bigsqcup_{k=1}^{n} \{x_{j_{k1}(\ell_{k1})},\dots,x_{j_{kt_k}(\ell_{kt_k})}\}\\ &= \{x_q \mid 1 \leqslant q \leqslant m,\ p_q = 0\} \sqcup \{x_{q(r)} \mid 1 \leqslant q \leqslant m,\ p_q > 0,\ 1 \leqslant r \leqslant p_q\}. \end{aligned}$$

Above we use the Sweedler notation

$$x_{q(1)} \otimes \dots \otimes x_{q(p_q)} := \begin{cases} (\Delta\otimes \mathrm{id}_{p_q-2})\dots(\Delta\otimes \mathrm{id}_1)\Delta x_q & \text{if } p_q \geqslant 2,\\ x_q & \text{if } p_q = 1. \end{cases}$$

By the cocommutativity, we may assume that if $j_{k\alpha} = j_{q\beta}$ but either $k < q$ or $k = q$ and $\alpha < \beta$, then $\ell_{k\alpha} < \ell_{q\beta}$. In addition, using the identity $h_{(1)}Sh_{(2)} = (Sh_{(1)})h_{(2)} = \varepsilon(h)$ for all $h \in H$, we exclude the entries $\left(Sx_{q(r)}\right) x_{q(r+1)}$ and $x_{q(r)}Sx_{q(r+1)}$.

Now we define the linear map $\varphi_{m,n} \colon \Bbbk\mathcal{F}(x_1,\dots,x_m)^{\otimes n} \to \mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)(m,n)$ where $m,n \in \mathbb{Z}_+$ and $\mathcal{F}(X)$ is the free group on a set X. Consider the standard basis in $\Bbbk\mathcal{F}(x_1,\dots,x_m)^{\otimes n}$ consisting of the elements $w_1 \otimes w_2 \otimes \dots \otimes w_n$ where $w_i \in \mathcal{F}(x_1,\dots,x_m)$, $1 \leqslant i \leqslant n$. By the definition, $\varphi_{m,n}(w_1 \otimes w_2 \otimes \dots \otimes w_n)$ is obtained from $w_1 \otimes w_2 \otimes \dots \otimes w_n$ as follows:

(1) we replace all x_j^{-1} with Sx_j;
(2) if some x_j appears more than once, we replace each x_j with $x_{j(r)}$ where r is the number of the entry;
(3) if some x_j does not appear, we add $\varepsilon(x_j)$ in the beginning of the expression.

The map $\varphi_{m,n}$ is extended on $\Bbbk\mathcal{F}(x_1,\dots,x_m)^{\otimes n}$ by the linearity.

Example 5.3.
$$\varphi_{5,2}\left(x_1x_2x_1^{-1} \otimes x_5x_2^{-1}x_1^2\right) = \varepsilon(x_3)\varepsilon(x_4)\, x_{1(1)}x_{2(1)}Sx_{1(2)} \otimes x_5\left(Sx_{2(2)}\right)x_{1(3)}x_{1(4)}.$$

Theorem 5.4. *For every $m,n \in \mathbb{Z}_+$ the map $\varphi_{m,n}$ is a linear bijection. Moreover, if $\mathcal{F}(x_1,x_2,\dots)$ is the free group of the countable rank, then the (multilinear) polynomial identities with coefficients in $\Bbbk$ of $\Bbbk\mathcal{F}(x_1,x_2,\dots)$ as a Hopf algebra are generated by the identity $x_{(1)} \otimes x_{(2)} - x_{(2)} \otimes x_{(1)} \equiv 0$ of cocommutativity.*

Proof. The surjectivity of $\varphi_{m,n}$ follows from the remarks made above. Replacing every variable x_j in an element of $\mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)(m,n)$ with the generator x_j of $\mathcal{F}(x_1,x_2,\dots)$, we see that the images of $w_1 \otimes w_2 \otimes \dots \otimes w_n$ under the composition $\mathcal{E}_{\mathcal{F}(x_1,x_2,\dots)}\varphi_{m,n}$ are linearly independent, which implies that $\varphi_{m,n}$ is injective.

If there exists a polynomial identity of $\Bbbk\mathcal{F}(x_1,x_2,\dots)$ that does not follow from the cocommutativity, then $\mathcal{E}_{\mathcal{F}(x_1,x_2,\dots)}f$ is the zero linear map for some $f \in \mathcal{P}_{\mathbf{CocHopf}}(\Bbbk)(m,n)$ and $m,n \in \mathbb{Z}_+$. The substitution made above implies $f = 0$. $\square$

Remark 5.5. The linear bijections $\varphi_{m,n}$ provide a description of the category $\mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)$.

5.3. **Commutative cocommutative Hopf algebras.** Again, let $\Bbbk$ be a field. Denote the category $\mathcal{P}_{V_{\mathbf{CommCocHopf}}}(\Bbbk,\Omega)$, where
$$V_{\mathbf{CommCocHopf}} := V_{\mathbf{CocHopf}} \cup \{xy - yx\},$$
by $\mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)$. Then 1 is a commutative cocommutative Hopf monoid in $\mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)$. Using the commutativity, every monomial $f \in \mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)$ can be presented in the following form:
$$f(x_1,\dots,x_m) = \varepsilon(x_{i_1})\dots\varepsilon(x_{i_s}) \bigotimes_{k=1}^{n}\prod_{r=1}^{m}\prod_{j=1}^{p_{kr}} S^{\alpha_{kr}} x_{r(\beta_{krj})} \tag{5.2}$$
where $p_{kr} \in \mathbb{Z}_+$, $\alpha_{kr} \in \{0,1\}$, $i_1 < \ldots < i_s$ and
$$(\beta_{1r1},\beta_{1r2},\dots,\beta_{2r1},\dots,\beta_{n,r,p_{nr}}) = \left(1,\dots,\sum_{k=1}^{n} p_{kr}\right)$$
for every $1 \leqslant r \leqslant m$,
$$\{x_{i_1},\dots,x_{i_s}\} = \left\{ r \,\middle|\, 1 \leqslant r \leqslant m,\ \sum_{k=1}^{n} p_{kr} = 0 \right\},$$

$x_{r(1)} := x_r$ if $\sum\limits_{k=1}^{n} p_{kr} = 1$.

Now we define the linear map $\psi_{m,n} \colon \Bbbk\mathcal{F}_{\mathbf{Ab}}(x_1,\ldots,x_m)^{\otimes n} \to \mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)(m,n)$ where $m,n \in \mathbb{Z}_+$ and $\mathcal{F}_{\mathbf{Ab}}(X)$ is the free abelian group a set X written in a multiplicative form. The standard basis in $\Bbbk\mathcal{F}_{\mathbf{Ab}}(x_1,\ldots,x_m)^{\otimes n}$ consists of elements $g = \bigotimes\limits_{k=1}^{n} \prod\limits_{r=1}^{m} x_r^{s_{kr}}$ where $s_{kr} \in \mathbb{Z}$. By the definition, $\psi_{m,n}(g) := f$ where f is defined in (5.2) above for $p_{kr} := s_{kr}$ and $\alpha_{kr} := 0$ if $s_{kr} \geqslant 0$ and $p_{kr} := -s_{kr}$ and $\alpha_{kr} := 1$ if $s_{kr} < 0$. The map $\psi_{m,n}$ is extended on $\Bbbk\mathcal{F}_{\mathbf{Ab}}(x_1,\ldots,x_m)^{\otimes n}$ by the linearity.

Example 5.6.

$$\psi_{5,2}\left(x_1^{-2}x_2^3 \otimes x_1^2 x_2^{-1} x_5\right) = \varepsilon(x_3)\varepsilon(x_4)\left(Sx_{1(1)}\right)\left(Sx_{1(2)}\right) x_{2(1)}x_{2(2)}x_{2(3)} \otimes x_{1(3)}x_{1(4)}\left(Sx_{2(4)}\right) x_5.$$

Theorem 5.7. *For every $m,n \in \mathbb{Z}_+$ the map $\psi_{m,n}$ is a linear bijection. Moreover, if $\mathcal{F}_{\mathbf{Ab}}(x_1,x_2,\ldots)$ is the free abelian group of the countable rank, then the (multilinear) polynomial identities with coefficients in $\Bbbk$ of $\Bbbk\mathcal{F}_{\mathbf{Ab}}(x_1,x_2,\ldots)$ as a Hopf algebra are generated by the identities $xy-yx \equiv 0$ of commutativity and $x_{(1)}\otimes x_{(2)} - x_{(2)}\otimes x_{(1)} \equiv 0$ of cocommutativity.*

Proof. The surjectivity of $\psi_{m,n}$ follows from the remarks made above. Replacing every variable x_j in an element of $\mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)(m,n)$ with the generator x_j of $\mathcal{F}_{\mathbf{Ab}}(x_1,x_2,\ldots)$, we see that the images of basis elements g under the composition $\mathcal{E}_{\mathcal{F}_{\mathbf{Ab}}(x_1,x_2,\ldots)}\psi_{m,n}$ are linearly independent, which implies that $\psi_{m,n}$ is injective.

If there exists a polynomial identity of $\Bbbk\mathcal{F}(x_1,x_2,\ldots)$ that does not follow from the commutativity and cocommutativity, then $\mathcal{E}_{\mathcal{F}_{\mathbf{Ab}}(x_1,x_2,\ldots)}f$ is the zero linear map for some $f \in \mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)(m,n)$ and $m,n \in \mathbb{Z}_+$. The substitution made above implies $f = 0$. □

Remark 5.8. The linear bijections $\psi_{m,n}$ provide a description of the category $\mathcal{P}_{\mathbf{CommCocHopf}}(\Bbbk)$.

5.4. **Group Hopf algebra of the cyclic group of order** 2. Let $C_2 = \{1,c\}$, $c^2 = 1$, be the cyclic group of order 2 and let $\Bbbk$ be a field, $\operatorname{char}\Bbbk \neq 2$. Then the following polynomial identities hold in the group algebra $\Bbbk C_2$ of C_2:

$$\begin{aligned} x_{(1)} \otimes x_{(2)} &\equiv x_{(2)} \otimes x_{(1)},\\ xy &\equiv yx,\\ x_{(1)}x_{(2)} &\equiv \varepsilon(x)1,\\ \left(x_{(1)} - \varepsilon(x_{(1)})1\right) \otimes \left(x_{(2)} - \varepsilon(x_{(2)})1\right) &\otimes \left(y - \varepsilon(y)1\right)\\ \equiv \left(x - \varepsilon(x)1\right) \otimes \left(y_{(1)} - \varepsilon(y_{(1)})1\right) &\otimes \left(y_{(2)} - \varepsilon(y_{(2)})1\right). \end{aligned} \tag{5.3}$$

(It is sufficient to substitute basis elements.) Below we show that all polynomial identities in $\Bbbk C_2$ follow from (5.3).

Note that (5.3) implies

$$\left(x_{(1)} - \varepsilon(x_{(1)})1\right)\left(x_{(2)} - \varepsilon(x_{(2)})1\right) \equiv -2\left(x - \varepsilon(x)1\right), \qquad Sx \equiv x. \tag{5.4}$$

Let V_1 be the union of $V_{\mathbf{Hopf}}$, defined in (5.1) above, and the polynomials corresponding to (5.3). Consider the presentation (5.2). Using $x_r = \varepsilon(x_r)1 + (x_r - \varepsilon(x_r)1)$, (5.3) and (5.4), every polynomial from $\mathcal{P}_{V_1}(\Bbbk,\Omega)(m,n)$, where $m,n \in \mathbb{Z}_+$, can be written as a linear combination of polynomials

$$\begin{aligned} f(x_1,\dots,x_m) = \varepsilon(x_{i_1})\dots\varepsilon(x_{i_s})\,1\otimes\dots\otimes 1\otimes(x_{j_1(1)}-\varepsilon(x_{j_1(1)})1)\left(\prod_{k=2}^{t}(x_{j_k}-\varepsilon(x_{j_k})1)\right)&\\ \otimes 1\otimes\ldots\otimes 1\otimes(x_{j_1(2)}-\varepsilon(x_{j_1(2)})1)&\\ \ldots&\\ \otimes 1\otimes\ldots\otimes 1\otimes(x_{j_1(q)}-\varepsilon(x_{j_1(q)})1)&\\ \otimes 1\otimes\dots\otimes 1& \end{aligned} \tag{5.5}$$

for $t \geqslant 1$, $q \geqslant 1$ and

$$f(x_1,\dots,x_m) = \varepsilon(x_1)\dots\varepsilon(x_n)\,1\otimes\dots\otimes 1 \tag{5.6}$$

for $t = 0$ where $i_1 < \ldots < i_s$, $j_1 < \ldots < j_t$,

$$\{i_1,\dots,i_s\}\sqcup\{j_1,\dots,j_t\} = \{1,\dots,n\}.$$

Theorem 5.9. *Let $\Bbbk$ be a field, $\operatorname{char}\Bbbk \neq 2$. Monomials (5.5) and (5.6) are linearly independent modulo polynomial identities in $\Bbbk C_2$ (as a Hopf algebra). As a consequence, (multilinear) polynomial identities with coefficients in $\Bbbk$ of $\Bbbk C_2$ are generated by (5.3). Moreover, $c_{m,n}(\Bbbk C_2) = 2^{m+n} - 2^m - 2^n + 2$ for all $m, n \in \mathbb{Z}_+$.*

Proof. Suppose that for some $m, n \in \mathbb{Z}_+$ a linear combination of monomials (5.5) and (5.6) is a polynomial identity. Among all the monomials that occur in this linear combination with non-zero coefficients, choose the monomial f_0 with the least $t =: t_0$ and substitute $x_{i_k} = 1$ for $1 \leqslant k \leqslant s$, $x_{j_\ell} = c$ for $1 \leqslant \ell \leqslant t_0$. Then the monomials with $t > t_0$ or different j_ℓ will be zero. The values of monomials with 1 on different places in the tensor product (5.5) will be linearly independent with the value of f_0. Thus f_0 must occur in the polynomial identity with the zero coefficient, and we get a contradiction. Hence monomials (5.5) and (5.6) are linearly independent modulo polynomial identities in $\Bbbk C_2$. Therefore, polynomial identities of $\Bbbk C_2$ are generated by (5.3).

The codimension $c_{m,n}(\Bbbk C_2)$ equals the number of monomials (5.5) and (5.6). Thus

$$c_{m,n}(\Bbbk C_2) = 1 + \sum_{t=1}^{m}\binom{m}{t}\sum_{k=1}^{n}\binom{n}{k} = 1 + (2^n-1)(2^m-1) = 2^{m+n} - 2^m - 2^n + 2.$$

□

5.5. **A Yetter — Drinfel'd module of dimension 2 as an $\{\sigma\}$-magma in $\mathbf{Vect}_\Bbbk$.** Let H be a Hopf algebra over a field $\Bbbk$ with an invertible antipode S. Denote by ${}^H_H\mathcal{YD}$ the category of *left Yetter — Drinfel'd modules* (or ${}^H_H\mathcal{YD}$*-modules* for short), i.e. left H-modules and H-comodules M such that the H-action and the H-coaction $\delta\colon M \to H \otimes M$ satisfy the following compatibility condition:

$$\delta(hm) = h_{(1)}m_{(-1)}Sh_{(3)} \otimes h_{(2)}m_{(0)} \text{ for every } m \in M \text{ and } h \in H.$$

The category ${}^H_H\mathcal{YD}$ is braided monoidal where the monoidal product of ${}^H_H\mathcal{YD}$-modules M and N is their usual tensor product $M \otimes N$ over $\Bbbk$ with the induced structures of a left H-module and a left H-comodule. The braiding $\sigma_{M,N}\colon M \otimes N \to N \otimes M$ is defined by the formula

$$\sigma_{M,N}(m \otimes n) := m_{(-1)}n \otimes m_{(0)} \text{ for } m \in M,\ n \in N.$$

Every Yetter — Drinfel'd module (or, more generally, a braided vector space) M can be considered an $\{\sigma\}$-algebra over $\Bbbk$ where $\sigma_M\colon M \otimes M \to M \otimes M$ is just the braiding $\sigma_{M,M}$.

Let $\operatorname{char}\Bbbk \neq 2$ and $M := \langle a, b\rangle_\Bbbk$. Define on M the structure of a Yetter — Drinfel'd module over the Hopf algebra $\Bbbk C_2$ by $\delta a := 1 \otimes a$, $\delta b := c \otimes b$, $ca := -a$, $cb = b$.

Below we calculate $c_{m,n}(M)$ where M is considered a $\{\sigma\}$-algebra over $\Bbbk$. In addition to the Yetter — Drinfel'd braiding σ_M, which is now an operation, in $\mathcal{P}(\{\sigma\}, \Bbbk)$ there are ordinary swaps $\tau_{m,n} \colon m+n \to n+m$ where $\mathcal{E}_M(\tau_{m,n})(u \otimes v) := v \otimes u$ for all $u \in M^{\otimes m}$ and $v \in M^{\otimes n}$, $m, n \in \mathbb{Z}_+$.

Again, define the embedding $\theta \colon \Bbbk S_n \hookrightarrow \mathcal{P}_{V_{\text{symm}}}(\{\sigma\}, \Bbbk)(n, n)$ in the way that

$(\mathcal{E}_M\theta(\rho))(a_1 \otimes a_2 \otimes \dots \otimes a_n) = a_{\rho^{-1}(1)} \otimes a_{\rho^{-1}(2)} \otimes \dots \otimes a_{\rho^{-1}(n)}$ for all $a_1, \dots, a_n \in M$ and $\rho \in S_n$.

Let $p^{(n)}_{21} := \frac{1}{2}(\mathrm{id}_2 - \tau_{1,1}\sigma) \otimes \mathrm{id}_{n-2}$ and $p^{(n)}_{ij} := \theta(\rho)p^{(n)}_{21}\theta(\rho)^{-1}$ where $\rho \in S_n$ such that $\rho(2) = i$, $\rho(1) = j$, $1 \leqslant i, j \leqslant n$, $i \neq j$, $n \in \mathbb{N}$, $n \geqslant 2$. Then

$$\left(\mathcal{E}_M p^{(n)}_{ij}\right)(u_1 \otimes u_2 \otimes \dots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \dots \otimes u_n & \text{if } u_i = a \text{ and } u_j = b, \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

for every $u_1, \dots, u_n \in \{a, b\}$.

For every $\varnothing \subsetneqq I \subsetneqq \{1, 2, \dots, n\}$ define $p^{(n)}_I := \prod\limits_{\substack{i \in I, \\ j \in \bar I}} p^{(n)}_{ij}$ where $\bar I := \{j \mid 1 \leqslant j \leqslant n,\ j \notin I\}$.

Let

$$p^{(n)}_\varnothing := \mathrm{id}_n - \sum_{\varnothing \subsetneqq I \subsetneqq \{1,2,\dots,n\}} p^{(n)}_I.$$

Theorem 5.10. *Let $\operatorname{char}\Bbbk \neq 2$ and let $M = \langle a, b\rangle_\Bbbk$ be the Yetter — Drinfel'd module over $\Bbbk C_2$ defined above. Then polynomial identities with coefficients in $\Bbbk$ of M as an $\{\sigma\}$-magma in $\mathbf{Vect}_\Bbbk$ are generated by the set*

$$\begin{aligned} V_2 := V_{\text{symm}} \cup \Big\{ & p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2} - p^{(n)}_{i_2 j_2}p^{(n)}_{i_1 j_1}, \quad p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2} - p^{(n)}_{i_1 j_2}p^{(n)}_{i_2 j_1}, \\ & p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2}p^{(n)}_{i_1 j_2} - p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2}, \quad p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 i_1}, \quad \left(p^{(n)}_{i_1 j_1}\right)^2 - p^{(n)}_{i_1 j_1}, \\ & p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2}\theta\big((i_1 i_2)\big) - p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2}, \quad p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2}\theta\big((j_1 j_2)\big) - p^{(n)}_{i_1 j_1}p^{(n)}_{i_2 j_2} \\ & \mid 1 \leqslant i_1, j_1, i_2, j_2 \leqslant n,\ n = 2, 3, 4;\ \textit{in all } p^{(n)}_{ij} \textit{ we have } i \neq j \Big\} \\ & \cup \left\{ p^{(n)}_\varnothing \theta(\rho) - p^{(n)}_\varnothing \,\middle|\, n \geqslant 2,\ \rho \in S_n \right\}. \end{aligned}$$

Moreover $c_{m,n}(M) = 0$ for $m \neq n$, $c_{0,0}(M) = c_{1,1}(M) = 1$ and for $n \geqslant 2$ we have

$$c_{n,n}(M) = \binom{2n}{n} - 1 \sim \frac{4^n}{\sqrt{\pi n}} \textit{ as } n \to \infty.$$

Proof. By (5.7), for every $\varnothing \subsetneqq I \subsetneqq \{1, 2, \dots, n\}$

$$\left(\mathcal{E}_M p^{(n)}_I\right)(u_1 \otimes u_2 \otimes \dots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \dots \otimes u_n & \text{if } u_i = a \text{ for all } i \in I \\ & \text{and } u_j = b \text{ for all } j \in \bar I, \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

for every $u_1, \dots, u_n \in \{a, b\}$. Hence

$$\left(\mathcal{E}_M p^{(n)}_\varnothing\right)(u_1 \otimes u_2 \otimes \dots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \dots \otimes u_n & \text{if } u_i = a \text{ for all } 1 \leqslant i \leqslant n \\ & \text{or } u_i = b \text{ for all } 1 \leqslant i \leqslant n, \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

for every $u_1, \dots, u_n \in \{a, b\}$. By (5.7) and (5.9), the elements of the set V_2 are indeed polynomial identities in M.

Tensoring $p^{(m)}_{ij}$ with id_{n-m} and conjugating them by $\theta(\rho)$ where $\rho \in S_n$, we obtain that the polynomial identities appearing in the set V_2 only for $n \leqslant 4$ imply the same polynomial identities for $n > 4$.

Denote the images of morphisms from $\mathcal{P}_{V_{\text{symm}}}(\{\sigma\}, \Bbbk)(n,n)$ in $\mathcal{P}_{V_2}(\{\sigma\}, \Bbbk)(n,n)$ by the same symbols.

The identities from V_2 imply that $p^{(n)}_I$ for $\varnothing \subsetneqq I \subsetneqq \{1, 2, \dots, n\}$ are orthogonal idempotents. Moreover,

$$p^{(n)}_I p^{(n)}_{ij} = p^{(n)}_{ij} p^{(n)}_I = \begin{cases} p^{(n)}_I & \text{if } i \in I \text{ and } j \in \bar{I}, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, $\theta(\rho) p^{(n)}_{\varnothing} \theta(\rho)^{-1} = p^{(n)}_{\varnothing}$ for all $\rho \in S_n$.

Therefore, $p^{(n)}_{\varnothing}$ is a central idempotent in the algebra $\mathcal{P}_{V_2}(\{\sigma\}, \Bbbk)(n,n)$ and is orthogonal to the other $p^{(n)}_I$.

Let $I, J \subsetneqq \{1, 2, \dots, n\}$. Recall that every permutation $\rho \in S_n$ is a composition of transpositions. By the identities from V_2, we have $p^{(n)}_I \theta(\rho) = p^{(n)}_I$ if $\rho(I) = I$ and

$$p^{(n)}_I \theta(\rho) p^{(n)}_J = p^{(n)}_I \theta(\rho) p^{(n)}_J \theta(\rho)^{-1} \theta(\rho) = p^{(n)}_I p^{(n)}_{\rho(J)} \theta(\rho)^{-1} = 0$$

if $\rho(J) \ne I$. Hence the vector space $\left\langle p^{(n)}_I \theta(\rho) p^{(n)}_J \,\middle|\, \rho \in S_n \right\rangle_{\Bbbk}$ is one-dimensional if $|I| = |J|$ and zero otherwise.

For every $I, J \subsetneqq \{1, 2, \dots, n\}$, $|I| = |J|$, fix some permutation $\rho_{I,J} \in S_n$ such that $\rho_{I,J}(J) = I$. Rewriting $\sigma = \theta\big((12)\big)\left(\mathrm{id}_2 - 2p^{(2)}_{21}\right)$ and inserting $\mathrm{id}_n = \sum\limits_{I \subsetneqq \{1,2,\dots,n\}} p^{(n)}_I$, we obtain that $\mathcal{P}_{V_2}(\{\sigma\}, \Bbbk)(n,n)$ is the linear span of the V_2-relative $\{\sigma\}$-polynomials

$$p^{(n)}_I \theta(\rho_{I,J}) p^{(n)}_J. \tag{5.10}$$

By (5.8) and (5.9), the V_2-relative $\{\sigma\}$-polynomials (5.10) form a basis in $\mathcal{P}_{V_2}(\{\sigma\}, \Bbbk)(n,n)$, the restriction of $\mathcal{E}_M$ on $\mathcal{P}_{V_2}(\{\sigma\}, \Bbbk)(n,n)$ is an injection, V_2 generates $\{\sigma\}$-polynomial identities in M with coefficients in $\Bbbk$ and

$$c_{n,n}(M) = \sum_{k=0}^{n-1} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 - 1 = \binom{2n}{n} - 1 \sim \frac{4^n}{\sqrt{\pi n}} \text{ as } n \to \infty.$$

□

5.6. **A Yetter — Drinfel'd module of dimension 2 as an $\varnothing$-magma in ${}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$.** In this section we consider the Yetter — Drinfel'd module M defined in Section 5.5 above, but without swaps from $\mathbf{Vect}_{\Bbbk}$. The functor $\mathcal{E}_M$ now maps $\tau_{k,\ell}$ to the corresponding components of the ${}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$-braiding.

Before treating our particular case, we first describe the category $M(\varnothing)$.

Let

$$B_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leqslant i \leqslant n-2 \rangle$$

be the nth braid group, $B_0 = B_1 := \{1\}$. The invertibility of the braiding implies that each $M(\varnothing)(n,n)$ is a group too. Below we show that $M(\varnothing)(n,n) \cong B_n$.

Denote by $\mathcal{B}$ the braid category, i.e. the braided monoidal category where the set of objects coincides with $\mathbb{Z}_+$, $\mathcal{B}(m,n) = \varnothing$ for $m \ne m$ and $\mathcal{B}(n,n) = B_n$ for every $n \in \mathbb{Z}_+$.

Define the group homomorphisms $\theta \colon B_n \to \mathcal{M}(\varnothing)(n,n)$ by $\theta(\sigma_i) := \mathrm{id}_{i-1} \otimes \tau_{1,1} \otimes \mathrm{id}_{n-i-1}$ for $1 \leqslant i \leqslant n-1$, $n \in \mathbb{Z}_+$.

Proposition 5.11. *The maps $\theta\colon B_n \to M(\varnothing)(n,n)$ defined above are group isomorphisms for every $n \in \mathbb{Z}_+$. Their extensions $\Bbbk B_n \to \mathcal{P}(\varnothing, \Bbbk)(n,n)$ by the linearity, which we denote below by the same letter θ, are algebra isomorphisms. The category $M(\varnothing)$ is isomorphic as a braided monoidal category to the braid category $\mathcal{B}$.*

Proof. By the properties of the braiding, all $\tau_{k,\ell}$ are compositions of $\mathrm{id}_s \otimes \tau_{1,1} \otimes \mathrm{id}_t$, $s+t+2 = k+\ell$, $s,t \in \mathbb{Z}_+$, whence θ is surjective. Note that 1 is an $\varnothing$-magma in $\mathcal{B}$. Consider the braided strong monoidal functor $\mathcal{E}_1\colon \mathcal{M}(\Omega) \to \mathcal{B}$ from Proposition 2.14. We have $\mathcal{E}_1 = \theta^{-1}$ on each $\mathcal{M}(\varnothing)(n,n)$. Therefore, the homomorphisms θ are in fact isomorphisms and the categories $\mathcal{M}(\varnothing)$ and $\mathcal{B}$ are isomorphic. Now extend θ to algebra homomorphisms $\Bbbk B_n \to \mathcal{P}(\varnothing, \Bbbk)(n,n)$ by the linearity. Then θ are algebra isomorphisms. $\square$

Denote by PB_n the kernel of the surjective homomorphism $\bar{(\ )}\colon B_n \to S_n$ where $\sigma_i \mapsto (i, i+1)$. Elements of PB_n are called *pure braids*. The subgroup PB_n is the normal closure of the elements $\sigma_1^2, \ldots, \sigma_{n-1}^2$. Note that

$$\left(\mathcal{E}_M \theta(\rho)\right)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \pm\, u_{\bar\rho^{-1}(1)} \otimes u_{\bar\rho^{-1}(2)} \otimes \cdots \otimes u_{\bar\rho^{-1}(n)} \tag{5.11}$$

for all $u_1, \ldots, u_n \in \{a, b\}$ and $\rho \in B_n$.

For every $1 \leqslant i < j \leqslant n$, where $n \geqslant 2$, choose some braid $\rho_{ij} \in B_n$ such that $\bar\rho_{ij}(1) = i$ and $\bar\rho_{ij}(2) = j$ and define

$$q_{ij}^{(n)} := \frac{1}{2}\theta(\rho_{ij})(\mathrm{id}_n - \tau_{1,1}^2 \otimes \mathrm{id}_{n-2})\theta(\rho_{ij})^{-1}, \quad r_{ij}^{(n)} := \frac{1}{2}\theta(\rho_{ij})(\mathrm{id}_n + \tau_{1,1}^2 \otimes \mathrm{id}_{n-2})\theta(\rho_{ij})^{-1}.$$

For our convenience, let $q_{ji}^{(n)} := q_{ij}^{(n)}$, $r_{ji}^{(n)} := r_{ij}^{(n)}$.

By (5.11),

$$\left(\mathcal{E}_M q_{ij}^{(n)}\right)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \cdots \otimes u_n & \text{if } u_i \ne u_j, \\ 0 & \text{otherwise} \end{cases} \tag{5.12}$$

and

$$\left(\mathcal{E}_M r_{ij}^{(n)}\right)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \cdots \otimes u_n & \text{if } u_i = u_j, \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

for every $u_1, \ldots, u_n \in \{a, b\}$.

Denote by Ξ_n the set of all decompositions $\{I, \bar I\}$ where $I \sqcup \bar I = \{1, 2, \ldots, n\}$, $I, \bar I \ne \varnothing$. Here we stress that $\{I, \bar I\} = \{\bar I, I\}$.

For every $\{I, \bar I\} \in \Xi_n$ define

$$q_{\{I,\bar I\}}^{(n)} := \left(\prod_{\substack{i \in I,\\ j \in \bar I}} q_{ij}^{(n)}\right)\left(\prod_{\substack{i,j \in I,\\ i<j}} r_{ij}^{(n)}\right)\left(\prod_{\substack{i,j \in \bar I,\\ i<j}} r_{ij}^{(n)}\right).$$

Let

$$q_{\{\varnothing,\bar\varnothing\}}^{(n)} := \mathrm{id}_n - \sum_{\{I,\bar I\} \in \Xi_n} q_{\{I,\bar I\}}^{(n)}$$

where for brevity we use the notation $\bar\varnothing := \{1, 2, \ldots, n\}$.

Let $\tilde\Xi_n := \Xi_n \cup \big\{\{\varnothing, \bar\varnothing\}\big\}$. Then, by (5.12) and (5.13),

$$\left(\mathcal{E}_M q_{\{I,\bar I\}}^{(n)}\right)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \begin{cases} u_1 \otimes u_2 \otimes \cdots \otimes u_n & \begin{array}{l} \text{if } u_i = u_j \text{ for all } i,j \in I, \\ u_i = u_j \text{ for all } i,j \in \bar I, \\ u_i \ne u_j \text{ for all } i \in I,\ j \in \bar I, \end{array} \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

for every $\{I, \bar I\} \in \tilde\Xi_n$.

Theorem 5.12. *Let* $\operatorname{char} \Bbbk \neq 2$ *and let* $M = \langle a, b\rangle_\Bbbk$ *be the Yetter — Drinfel'd module over* $\Bbbk C_2$ *defined in Section 5.5. Then polynomial identities with coefficients in* $\Bbbk$ *of* M *as an* $\varnothing$*-magma in* ${}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$ *are generated by the set*

$$\begin{aligned} V_3 := \left\{\tau_{1,1}^4 - \mathrm{id}_2\right\} \cup \left\{\theta\left(\sigma_i^2\rho - \rho\sigma_i^2\right) \mid i = 1, \ldots, n-1,\ \rho \in PB_n,\ n \geqslant 2\right\} \\ \cup \left\{ q_{ij}^{(n)}\theta(\rho) - \theta(\rho)q_{ij}^{(n)} \,\middle|\, 1 \leqslant i < j \leqslant n,\ \rho \in B_n,\ n \geqslant 2,\ \bar\rho(i) = i,\ \bar\rho(j) = j\right\} \\ \cup \left\{q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}}\left(\mathrm{id}_j \otimes \tau_{1,1} \otimes \mathrm{id}_{n-j-2}\right) - q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}}\right. \\ \left. \mid 0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant n-2,\ j \neq i-1\right\}. \end{aligned}$$

Moreover $c_{m,n}(M) = 0$ *for* $m \neq n$, $c_{0,0}(M) = c_{1,1}(M) = 1$ *and for* $n \geqslant 2$ *we have*

$$c_{n,n}(M) = \frac{1}{2}\binom{2n}{n} \sim \frac{4^n}{2\sqrt{\pi n}} \text{ as } n \to \infty.$$

Proof. By (5.11), the image of every pure braid may change only the sign of a tensor product of several a and b. Therefore, such images commute and $\theta\left(\sigma_i^2\rho - \rho\sigma_i^2\right) \equiv 0$ holds in M. An explicit check shows that the other elements of the set V_3 are polynomial identities in M too.

Denote the images of morphisms from $\mathcal{P}(\varnothing, \Bbbk)(n, n)$ in $\mathcal{P}_{V_3}(\varnothing, \Bbbk)(n, n)$ by the same symbols. Then $\tau_{1,1}^4 = \mathrm{id}_2$ implies that both $q_{ij}^{(n)}$ and $r_{ij}^{(n)}$ are idemponents and

$$q_{ij}^{(n)}r_{ij}^{(n)} = r_{ij}^{(n)}q_{ij}^{(n)} = 0.$$

Now from $\theta\left(\sigma_i^2\right)\theta(\rho) = \theta(\rho)\theta\left(\sigma_i^2\right)$ for every $1 \leqslant i \leqslant n-1$ and $\rho \in PB_n$ it follows that the images of every two pure braids commute and $q^{(n)}_{\{I,\bar I\}}$ are orthogonal idempotents for $\{I, \bar I\} \in \tilde\Xi_n$.

The equality $q_{ij}^{(n)}\theta(\rho) = \theta(\rho)q_{ij}^{(n)}$ for $\rho \in B_n$, such that $\bar\rho(i) = i$ and $\bar\rho(j) = j$, implies $r_{ij}^{(n)}\theta(\rho) = \theta(\rho)r_{ij}^{(n)}$ since $r_{ij}^{(n)} = \mathrm{id}_n - q_{ij}^{(n)}$. In particular, $q_{ij}^{(n)}$ and $r_{ij}^{(n)}$ do not depend on the braid ρ_{ij} used in their definition. Moreover, $q^{(n)}_{\{I,\bar I\}}\theta(\rho) = \theta(\rho)q^{(n)}_{\{I,\bar I\}}$ for all $\{I, \bar I\} \in \tilde\Xi_n$ and $\rho \in B_n$ such that $\bar\rho(I) = I$. Finally, $\theta(\rho)q^{(n)}_{\{J,\bar J\}}\theta(\rho)^{-1} = q^{(n)}_{\{I,\bar I\}}$ for every $\{I, \bar I\}, \{J, \bar J\} \in \tilde\Xi_n$ and $\rho \in B_n$ such that $\bar\rho(J) = I$. Hence

$$q^{(n)}_{\{I,\bar I\}}\theta(\rho)q^{(n)}_{\{J,\bar J\}} = q^{(n)}_{\{I,\bar I\}}\theta(\rho)q^{(n)}_{\{J,\bar J\}}\theta(\rho)^{-1}\theta(\rho) = q^{(n)}_{\{I,\bar I\}}q^{(n)}_{\{J,\bar J\}}\theta(\rho)^{-1} = 0 \tag{5.15}$$

if $\bar\rho(J) \neq I$ and $\bar\rho(J) \neq \bar I$.

Let $\{I, \bar I\}, \{J, \bar J\} \in \tilde\Xi_n$. Consider the vector space $\left\langle q^{(n)}_{\{I,\bar I\}}\theta(\rho)q^{(n)}_{\{J,\bar J\}} \,\middle|\, \rho \in B_n\right\rangle_\Bbbk$. By (5.15), it this space is zero if $|I| \neq |J|$ and $|I| \neq n - |J|$. Suppose $|I| = |J| =: i$. (If $|I| = n - |J|$, we just swap J and $\bar J$.) Choose $\rho_1, \rho_2 \in B_n$ such that $\bar\rho_1(I) = \bar\rho_2(J) = \{1, \ldots, i\}$. Then

$$\begin{aligned} \theta(\rho_1)\left\langle q^{(n)}_{\{I,\bar I\}}\theta(\rho)q^{(n)}_{\{J,\bar J\}} \,\middle|\, \rho \in B_n\right\rangle_\Bbbk \theta(\rho_2)^{-1} \\ = \left\langle \theta(\rho_1)q^{(n)}_{\{I,\bar I\}}\theta(\rho_1)^{-1}\theta\left(\rho_1\rho\rho_2^{-1}\right)\theta(\rho_2)q^{(n)}_{\{J,\bar J\}}\theta(\rho_2)^{-1} \,\middle|\, \rho \in B_n\right\rangle_\Bbbk \\ = \left\langle q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}}\theta(\rho)q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}} \,\middle|\, \rho \in B_n\right\rangle_\Bbbk \end{aligned}$$

If $\bar\rho(\{1, \ldots, i\}) \neq \{1, \ldots, i\}$ and $\bar\rho(\{1, \ldots, i\}) \neq \{i+1, \ldots, n\}$, then, by (5.15), $q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}}\theta(\rho)q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}} = 0$. If $\bar\rho(\{1, \ldots, i\}) = \{1, \ldots, i\}$, then $\theta(\rho)$ is a product of an image of a pure braid, which can be expressed via $q_{k\ell}^{(n)}$ and $r_{k\ell}^{(n)}$, and elements $\mathrm{id}_j \otimes \tau_{1,1} \otimes \mathrm{id}_{n-j-2}$ for $j \neq i-1$, which by V_3 do not alter $q^{(n)}_{\{\{1,\ldots,i\},\{i+1,\ldots,n\}\}}$. If $\bar\rho(\{1, \ldots, i\}) = \{i+1, \ldots, n\}$, then $\rho = \rho_0\rho_1$ where ρ_0 is a fixed braid such that

$\bar{\rho}_0(\{1,\ldots,i\}) = \{i+1,\ldots,n\}$ and ρ_1 is a braid such that $\bar{\rho}_1(\{1,\ldots,i\}) = \{1,\ldots,i\}$. Now the argument above can be applied to ρ_1. Hence we get that

$$\dim \left\langle q^{(n)}_{\{I,\bar{I}\}}\theta(\rho)q^{(n)}_{\{J,\bar{J}\}} \,\middle|\, \rho \in B_n \right\rangle_{\Bbbk}$$

is zero if $|I| \ne |J|$ and $|I| \ne n - |J|$, does not exceed 1 if $|I| = |J| \ne \frac{n}{2}$ and does not exceed 2 if $|I| = |J| = \frac{n}{2}$.

For every $\{I,\bar{I}\},\{J,\bar{J}\} \in \tilde{\Xi}_n$, $|I| = |J|$, fix some braid $\rho_{\{I,\bar{I}\}} \in B_n$ such that $\bar{\rho}_{\{I,\bar{I}\}}(J) = I$. In addition, for every $\{I,\bar{I}\},\{J,\bar{J}\} \in \tilde{\Xi}_n$, $|I| = |J| = \frac{n}{2}$, fix some braid $\xi_{\{I,\bar{I}\}} \in B_n$ such that $\bar{\xi}_{\{I,\bar{I}\}}(J) = \bar{I}$.

Inserting $\mathrm{id}_n = \sum\limits_{\{I,\bar{I}\}\in\tilde{\Xi}_n} q^{(n)}_{\{I,\bar{I}\}}$ and using the argument above, we obtain that $\mathcal{P}_{V_3}(\varnothing,\Bbbk)(n,n)$ is the linear span of the V_3-relative $\varnothing$-polynomials

$$\begin{gathered} q^{(n)}_{\{I,\bar{I}\}}\theta(\rho_{\{I,\bar{I}\}})q^{(n)}_{\{J,\bar{J}\}} \text{ for } \{I,\bar{I}\},\{J,\bar{J}\} \in \tilde{\Xi}_n,\ |I| = |J|, \\ q^{(n)}_{\{I,\bar{I}\}}\theta(\xi_{\{I,\bar{I}\}})q^{(n)}_{\{J,\bar{J}\}} \text{ for } \{I,\bar{I}\},\{J,\bar{J}\} \in \tilde{\Xi}_n,\ |I| = |J| = \frac{n}{2}. \end{gathered} \tag{5.16}$$

By (5.11) and (5.14), the V_3-relative $\varnothing$-polynomials (5.16) form a basis in $\mathcal{P}_{V_3}(\varnothing,\Bbbk)(n,n)$, the restriction of $\mathcal{E}_M$ on $\mathcal{P}_{V_3}(\varnothing,\Bbbk)(n,n)$ is an injection and V_3 generates $\varnothing$-polynomial identities in M with coefficients in $\Bbbk$.

For $n = 2k+1$ we have

$$c_{n,n}(M) = \sum_{i=0}^{k} \binom{2k+1}{i}^2 = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}^2 = \frac{1}{2}\binom{2n}{n} \sim \frac{4^n}{2\sqrt{\pi n}} \text{ as } n \to \infty,$$

for $n = 2k$ we have

$$\begin{aligned} c_{n,n}(M) &= 2\left(\frac{1}{2}\binom{2k}{k}\right)^2 + \sum_{i=0}^{k-1}\binom{2k}{i}^2 = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}^2 \\ &= \frac{1}{2}\binom{2n}{n} \sim \frac{4^n}{2\sqrt{\pi n}} \text{ as } n \to \infty. \end{aligned}$$

$\square$

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Email address: alexey.gordienko@math.msu.ru

Department of Higher Algebra, Faculty of Mechanics and Mathematics, M. V. Lomonosov Moscow State University, Leninskiye Gory, d.1, Main Building, GSP-1, 119991 Moskva, Russia