Differential equations on a k-dimensional torus: Poincaré type results

Lev Sakhnovich *

Abstract

Ordinary differential equations of the first order on the torus have been investigated in detail by H. Poincaré, P. Bohl and A. Denjoy. P. Bohl, back in 1916, emphasised the importance of the transfer of the results for the order $k = 1$ to the case $k > 1$, adding at the same time: "However, any attempt to do so would be hopeless". The following more than hundred years have only confirmed Bohl's forecast. It became clear that a new approach to this problem is needed.

In this paper, we propose a new (non-Hamiltonian) and promising approach. We use Hamiltonians, that is, ordinary differential systems of equations of the first order, only for heuristics. In the main scheme and corresponding proofs we do not use these systems. Instead of differential systems, we study sets of continuous vector functions $\phi(t, \eta)$ satisfying certain important conditions. Limit sets and left and right rotation vectors appear in the case $k > 1$. Some of our results are new even in the case $k = 1$. Under simple and natural conditions, the left and right rotation vectors coincide and a precise analog of the well-known H. Poincaré's result is derived for $k > 1$.

MSC(2020): 37J40, 53A45, 53A55.

Keywords: *Rotation number, k-dimensional torus, rotation vector, partially ordered space, Poincaré theorems, non-Hamiltonian approach.*

*Retired from Courant Institute, N.Y., USA, E-mail: lsakhnovich@gmail.com

1 Introduction

Let us formulate some classical Poincaré results [4, Ch. XVII]. For this purpose, consider the ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x), \tag{1.1}$$

where the function $f(t, x)$ is real, continuous and defined for all real numbers t and real numbers x. It is assumed in this theory that two conditions below are satisfied.

Condition I.
$$f(t + 1, x) = f(t, x + 1) = f(t, x). \tag{1.2}$$

Condition II. *Through every point of the* (t, s) *plane passes a unique solution of equation* (1.1)*.*

Let us study solutions of the differential system (1.1) with initial value (parameter) η. It is assumed also that the function $\phi(t, \eta)$ is known at the two fixed time points $t = 0$ and $t = 1$:

$$x = \phi(t, \eta), \quad \phi(0, \eta) = \eta, \quad \phi(1, \eta) = \psi(\eta). \tag{1.3}$$

It follows from (1.1)–(1.3) that

$$\phi(t, \eta + 1) = \phi(t, \eta) + 1, \quad \psi(\eta + 1) = \psi(\eta) + 1. \tag{1.4}$$

One may easily see that

$$\phi(t + 1, \eta) = \phi(t, \psi(\eta)). \tag{1.5}$$

Indeed, in view of (1.2) the function $\phi(t + 1, \eta)$ is a solution of (1.1) because $\phi(t, \eta)$ is its solution. Moreover, it follows from (1.3) that $\phi(1, \eta) = \phi(0, \psi(\eta)) = \psi(\eta)$ and so the solutions $\phi(t + 1, \eta)$ and $\phi(t, \psi(\eta))$ both have the same value $\psi(\eta)$ at $t = 0$. Hence, (1.5) is valid for all t. Here, we use the uniqueness theorem (see [4, Ch.1, Section 1]), instead of the Condition II, because the uniqueness theorem is also valid in the vector case (see [8, Ch. II, Section 1]), which we treat in the next section.

Remark 1.1. *Two points* $P_1 = (t_1, x_1)$ *and* $P_2 = (t_2, x_2)$ *are regarded as identical if* $t_1 - t_2$ *and* $x_1 - x_2$ *are integers.*

One may represent the solution paths (t,x) (on the torus $\mathbf{T^1}$) in the following way:

$$u = \big(a + b\cos(2\pi x)\big)\cos(2\pi t), \tag{1.6}$$

$$v = \big(a + b\cos(2\pi x)\big)\sin(2\pi t), \tag{1.7}$$

$$w = b\sin(2\pi x), \tag{1.8}$$

where a and b are constants such that $0 < b < a$, and (u,v,w) are rectilinear coordinates in a three-dimensional space. Now, let $x = \phi(t,\eta)$ be the solution of (1.1) such that $\phi(0,\eta) = \eta$ (see (1.3)). Let us consider the function

$$\psi(\eta) = \phi(1,\eta). \tag{1.9}$$

According to [4, Ch. XVII, Sect. 1], the function $\psi(\eta)$ is a continuous, monotonic and increasing homeomorphism of the real line into itself and (see (1.4))

$$\psi(\eta + 1) = \psi(\eta) + 1. \tag{1.10}$$

The function $\psi(\eta)$ *represents* a transformation T of the form $TP = P_1$ or, introducing it in a slightly different way,

$$T\eta = \psi(\eta). \tag{1.11}$$

Theorem 1.2. *[4, Ch. XVII, Sect. 2, Theorem 1.2] The limit*

$$\rho = \lim_{t\to\infty} \frac{\phi(t,\eta)}{t} \tag{1.12}$$

exists and does not depend on the initial value η; it is rational if and only if some integer and positive power $m \in \mathbb{N}$ of T has a fixed point.

Recall that $\mathbb{N}$ stands for the set of integer and positive numbers.

Definition 1.3. *The value ρ introduced in* (1.12) *is called rotation number.*

Let $\psi_n(\eta)$ be the function defined by the relations

$$\psi_0(\eta) = \eta, \quad \psi_n(\eta) = \psi[\psi_{n-1}(\eta)] \quad (n \in \mathbb{N}). \tag{1.13}$$

The function ψ_n is of the same type as ψ, that is, $\psi_n(\eta)$ is continuous, monotonic and satisfies (1.10). Next, ρ is expressed in terms of ψ_n.

Theorem 1.4. *[4, Ch. XVII, Sect. 5, Theorem 1.3] The rotation number satisfies the equality*

$$\rho = \lim_{n\to\infty} \psi_n(\eta)/n, \tag{1.14}$$

where the limit exists and does not depend on the initial value η.

P. Bohl, back in 1916 [3], emphasized the importance of the transfer of results for the order $k = 1$ to the case $k > 1$, adding at the same time: "However, any attempt to do so would be hopeless". In spite of many important developments of the considered here Poincaré theory (see, e.g., interesting recent works [2, 5, 7, 9–11, 13, 15, 16] and references therein), the following more than hundred years have confirmed Bohl's forecast. It became clear that a new approach to that problem is needed. In this paper, we propose a new (non-Hamiltonian) and promising approach.

Further we consider the k-dimensional vector space E_k of vectors $X = [x_1, x_2, ..., x_k]$ with real-valued elements, where the norm is defined by the relation

$$\|X\| = \max |x_p|, \quad 1 \le p \le k. \tag{1.15}$$

Definition 1.5. *(see [17]) Let vectors X and Y belong to E_k. We say that $X > Y$ if all the elements of the vector $X - Y$ are non-negative and at least one of them is positive.*

Thus, E_k is a partially ordered space.

Our approach has the following characteristic features:

1. We use Hamiltonians, that is, ordinary differential systems of equations of the first order of the form (1.1) only as a source of heuristics. The corresponding differential systems are not used in our statements and proofs.
2. Instead of differential systems, we study a set of continuous vector functions $\phi(t, \eta)$, which satisfy vector versions of conditions (1.3)–(1.5) and which we call solutions of *generalised vector systems* (see section 2). We do not assume that the functions $\phi(t, \eta)$ are differentiable.
3. We consider the corresponding problems in k-spaces E_k $(k \ge 1)$.
4. We use the theory of the partially ordered spaces E_k and Zorn's lemma.
5. Under natural conditions, we construct solutions $\phi(t, \eta)$ in an explicit form (see Theorem 2.3). This result is new even for the case $k = 1$.
6. We obtain vector versions of Theorems 1.2 and 1.4 (see section 3).
7. It is very difficult to check the critical points of the solution $\phi(t, \eta)$ within the framework of the classical theory. Our approach allows to effectively solve this problem (see Example 5.4).

Remark 1.6. *The avoidance of differential equations is not new in physics. In particular, the overcomplexity of the quantum electrodynamics equations is well known. R. Feynman [6] built an effective and fairly simple procedure for solving a number of problems of quantum electrodynamics without using the corresponding differential equations. Sure, classical equations served as a background and a hint for heuristics. Interesting interconnections between classical theory and R. Feynman theory are studied in our paper [14].*

In Section 2, generalised vector systems on torus are introduced and the problem of their recovery is studied. In Section 3, limit sets and left and right rotation vectors appear in the case $k > 1$. Under simple and natural conditions, the left and right rotation vectors coincide and a precise analog of the well-known H. Poincaré's result is derived for $k > 1$. This result is presented in Theorem 4.4 in Section 4. Finally, some interesting examples are considered in Section 5.

The notation $\mathbb{Z}$ stands for integer and the notation $\mathbb{R}$ for real numbers.

2 Generalised vector systems on torus

2.1 Differential systems

We start with a system of differential equations on a k-dimensional torus $\mathbf{T^k}$:

$$\frac{d}{dt}x(t) = f(t, x), \tag{2.1}$$

where the elements of the k-dimensional vector function $f(t, x)$ are real, continuous and well-defined for all the numbers $t \geq 0$ and real vectors $x(t) \in E_k$.

Further we always assume that $t \geq 0$. The case $t \leq 0$ is easily reduced to the case $t \geq 0$.

We also assume that

$$f(t + 1, x) = f(t, x + q) = f(t, x), \tag{2.2}$$

for each k-dimensional vector q with integer elements (i.e., for $q \in \mathbb{Z}^k$).

The vector solution of the differential system (2.1) is denoted by $x = \phi(t, \eta)$, and we (similar to section 1) assume that the function $\phi(t, \eta)$ is known at two fixed time points $t = 0$ and $t = 1$:

$$\phi(0, \eta) = \eta, \quad \phi(1, \eta) = \psi(\eta) \quad (\eta \in E_k). \tag{2.3}$$

Proposition 2.1. *Let $x = \phi(t, \eta)$ satisfy* (2.1) *and* (2.2).

Then, we have

$$\phi(t, \eta + q) = \phi(t, \eta) + q, \quad \psi(\eta + q) = \psi(\eta) + q \quad \text{for} \quad q \in \mathbb{Z}^k; \tag{2.4}$$

$$\phi(t + 1, \eta) = \phi(t, \psi(\eta)). \tag{2.5}$$

Remark 2.2. *We note that relations* (2.4) *and* (2.5) *are vector versions of* (1.4) *and* (1.5) *and are proved in the same way as* (1.4) *and* (1.5).

It follows from (2.4) and (2.5) that

$$\phi(t, \eta) = \eta + a(t, \eta) \tag{2.6}$$

$$\psi(\eta) = \eta + a(\eta), \tag{2.7}$$

where

$$a(t, \eta) = a(t, \eta + q) \quad \text{for} \quad q \in \mathbb{Z}^k, \quad a(\eta) = a(1, \eta). \tag{2.8}$$

2.2 Generalised systems

Neither equation (2.1) nor condition (2.2) are used in our further considerations.

We consider a set of real, continuous k-dimensional vector functions $\phi(t, \eta)$, where $t \geq 0$ and $\eta \in E_k$. We assume that relations (2.3)–(2.5) *are fulfilled.* The requirements above we call a *generalised system* and a set of the vector functions $\phi(t, \eta)$, which satisfy them, is called a *solution* of the generalised system.

Here and further, the continuity of the functions depending on t and η means that they continuously (in $\mathbb{R} \bigoplus E_k$ metrics) act on the elements (t, η) of the set $\{(t, \eta)\}$. The requirement for the function $\phi(t, \eta)$ to be differentiable is omitted. The restriction of ϕ on the interval $[0, 1]$ is denoted by Φ:

$$\Phi(t, \eta) = \phi(t, \eta) \quad (0 \leq t \leq 1, \quad \eta \in E_k). \tag{2.9}$$

Theorem 2.3. I. *Let a continuous vector function $\phi(t, \eta)$ satisfy relations* (2.3)–(2.5). *Then, the continuous vector function $\Phi(t, \eta)$ satisfies the following conditions.*

a) The relation

$$\Phi(t, \eta + q) = \Phi(t, \eta) + q, \tag{2.10}$$

holds for the vectors q with integer elements (*i.e., for $q \in \mathbb{Z}^k$*).

b) The following equalities are valid:

$$\Phi(0,\eta) = \eta, \quad \Phi(1,\eta) = \psi(\eta). \tag{2.11}$$

II. *Let a continuous vector function* $\Phi(t,\eta)$ $(0 \le t \le 1,\ \eta \in E_k)$ *satisfy conditions a) and b). Then, a solution* $\phi(t,\eta)$ *of the generalised system is recovered in the explicit form via the formula*

$$\phi(t+n,\eta) = \Phi(t,\psi_n(\eta)) \quad (0 \le t \le 1), \tag{2.12}$$

where $n \in \mathbb{N}$ *and* ψ_n *is expressed via* ψ *in* (1.13)*. This* $\phi(t,\eta)$ *is continuous and satisfies relations* (2.3)*–*(2.5)*.*

Proof. Part I of the theorem immediately follows from the theorem's conditions. Note that formula (2.12) follows (under Part I conditions) from (2.5) and (2.9).

Let us prove Part II. First, given (2.12), let us prove that the constructed vector function $\phi(t,\eta)$ is continuous. Indeed, discontinuities may only occur at the points $t = n$ for $n \in \mathbb{N}$. (Recall that the case $t \ge 0$ is considered.) In view of (2.11) and (2.12), we have the equalities:

$$\phi(n+0,\eta) = \Phi(+0,\psi_n(\eta)) = \Phi(0,\psi_n(\eta)) = \psi_n(\eta), \tag{2.13}$$
$$\phi(n-0,\eta) = \Phi(1-0,\psi_{n-1}(\eta)) = \Phi(1,\psi_{n-1}(\eta)) = \psi_n(\eta). \tag{2.14}$$

Thus, the constructed vector function $\phi(t,\eta)$ is continuous. Now, formula (2.5) follows from (2.9) and (2.12). Relations (2.10) and (2.11) yield the second equality in (2.4). The second equality in (2.4) and relations (2.5), (2.9) and (2.10) imply the first equality in (2.4). Formulas (2.3) follow from (2.9) and (2.10). In this way, we proved that the constructed solution $\phi(t,\eta)$ satisfies (2.3)–(2.5). □

Theorem 2.3 is new even in the scalar case.

3 A partially ordered space, Zorn's lemma and rotation vectors

Recall the definition of the space E_k with the norm (1.15) and a partial order on E_k given by Definition (1.5) in introduction.

Remark 3.1. *Similar to the case $k = 1$ in Remark 1.1, the points $P_1 = (t_1, X)$ and $P_2 = (t_2, Y)$ $(X, Y \in E_k)$ are regarded as identical if $t_1 - t_2$ and all the elements of the vector $X - Y$ are integers. This condition shows that differential equations on the torus $\mathbf{T}^k$ are equivalent to corresponding differential equations in the space E_k.*

The vector function $\psi(\eta)$ plays a central role in this section. It follows from the requirements of the generalised system that $\psi(\eta)$ *is continuous and satisfies relations* (2.4), (2.7) *and* (2.8).

Note that formula (2.7) coincides with Poincare mapping in the case $k = 1$. Similar to (1.11), we introduce a transformation T defined by

$$T\eta = \psi(\eta). \tag{3.1}$$

The vector function ψ_n is again defined by (1.12). Our next lemma follows from (2.7).

Lemma 3.2. *The vector function $\psi_n(\eta)$ is continuous, satisfies the relation*

$$\psi_n(\eta + q) = \psi_n(\eta) + q \quad \text{for} \quad q \in \mathbb{Z}^k, \tag{3.2}$$

and has the form

$$\psi_n(\eta) = \eta + a(\eta) + a[\psi(\eta)] + ... + a[\psi_{n-1}(\eta)], \tag{3.3}$$

where $a(\eta) = \phi(1, \eta) - \eta$.

Formula (3.3) is well known for the case $k = 1$ (see [1, p. 104]).

Proposition 3.3. *If the operator T^m (for some $m \in \mathbb{N}$) has a fixed point, then the limit vector*

$$\rho = \lim_{n \to \infty} \psi_n(\eta)/n \tag{3.4}$$

exists at this point and all the elements of ρ are rational.

Proof. At the fixed point η, we have

$$T^m \eta = \psi_m(\eta) = \eta + q \quad \text{for some} \quad q \in \mathbb{Z}^k, \tag{3.5}$$

because the points η and $\eta + q$ are regarded as identical for the torus (see Remark 3.1). Taking into account (3.2) and (3.5), we obtain

$$\psi_{2m}(\eta) = \psi_m(\eta + q) = \psi_m(\eta) + q = \eta + 2q,$$
$$\psi_3 m(\eta + q) = \psi_m(\psi_{2m}(\eta)) = \eta + 3q, \quad \ldots,$$

and it follows by induction that

$$\psi_{mn}(\eta) = \eta + nq \quad (n = 0, 1, 2, ...). \tag{3.6}$$

Every $K \in \mathbb{N}$ admits representation $K = m\ell + s$, where the integer s satisfies $0 \le s < m$. Thus, (3.6) implies that

$$\psi_K(\eta) = \psi_s(\eta) + q\ell. \tag{3.7}$$

Using (3.7) we derive

$$\rho = \lim_{K\to\infty} \psi_K(\eta)/K = q/m, \tag{3.8}$$

which proves the lemma. □

Further we consider the general case, where fixed points (3.5) *are not necessary.* Let us introduce the following notation

$$a_n(\eta) = a(\eta) + a[\psi(\eta)] + ... + a[\psi_{n-1}(\eta)]. \tag{3.9}$$

It follows from (3.3) and (3.9) that $\psi_n(\eta) - a_n(\eta) = \eta$ so that

$$\lim_{n\to\infty} \big(\psi_n(\eta) - a_n(\eta)\big)/n] = 0. \tag{3.10}$$

Let us introduce the notion of a limit set for a sequence of vectors B_n $(n \in \mathbb{N})$ or a family of vectors B_t $\big(t \in (0, \infty)\big)$.

Definition 3.4. *The vector γ belongs to the limit set for a sequence of vectors B_n (for a family of vectors B_t) if there is a sequence $\{n_p\}$ $(\{t_p\})$, where $n_p \in \mathbb{N}$ $\big(t_p \in (0, \infty)\big)$ such that*

$$\gamma = \lim_{n_p\to\infty} B_{n_p} \quad \big(\gamma = \lim_{t_p\to\infty} B_{t_p}\big). \tag{3.11}$$

Since the vector function $a(\eta)$ is continuous (together with $\psi(\eta)$) and (2.8) holds, $a(\eta)$ is bounded:

$$\sup_{\eta\in E_k} \|a(\eta)\| \le M < \infty. \tag{3.12}$$

Taking into account (3.9) and (3.12), we see that $a_n(\eta)/n$ is also bounded:

$$\sup_{\eta\in E_k} \|a_n(\eta)/n\| \le M, \tag{3.13}$$

where M is the same as in (3.12). We denote the limit set of $a_n(\eta)/n$ by $Q(\eta)$. Relation (3.13) implies the following proposition.

Proposition 3.5. *The limit sets $Q(\eta)$ of the vector functions $a_n(\eta)/n$ are nonempty and uniformly bounded:*

$$\sup_{\gamma(\eta)\in Q(\eta)} \|\gamma(\eta)\| \leq M. \tag{3.14}$$

In view of (3.10) we obtain the corollary below.

Corollary 3.6. *The limit set of $\psi_n(\eta)/n$ coincides with he limit set $Q(\eta)$ of $a_n(\eta)/n$.*

Theorem 3.7. *The limit set of $\phi(t,\eta)/t$, for $t \to +\infty$, coincides with $Q(\eta)$.*

Proof. For $t \in (0,\infty)$ we choose $n_t \in \mathbb{N}$ so that $n_t \leq t < n_t + 1$. Hence t admits representation

$$t = n_t + s_t \quad (0 \leq s_t < 1). \tag{3.15}$$

In view of (2.5) and (3.15), we have

$$\phi(t,\eta) = \phi(s_t, \psi_{n_t}(\eta)). \tag{3.16}$$

We represent $\psi_{n_t}(\eta)$ in the form

$$\psi_{n_t}(\eta) = U_{n_t}(\eta) + V_{n_t}(\eta), \tag{3.17}$$

where all the elements of the vector $U_{n_t}(\eta)$ are integer and all the elements of the vector $V_{n_t}(\eta)$ are nonnegative and less then 1. Taking into account relations (2.4), (3.16) and (3.17), we obtain

$$\phi(t,\eta) = U_{n_t}(\eta) + \phi(s_t, V_{n_t}(\eta)). \tag{3.18}$$

Clearly, the norms of $V_{n_t}(\eta)$ and of $\phi(s_t, V_{n_t}(\eta))$ for our continuous vector function ϕ are bounded. Thus, the theorem follows from Corollary 3.6 and the equalities (3.17) and (3.18). □

Corollary 3.8. *The vector function $\phi(t,\eta)$ may be represented in the form*

$$\phi(t,\eta) = \psi_{n_t}(\eta) + b(t,\eta), \tag{3.19}$$

where $b(t,\eta)$ is a bounded vector function depending on $t \geq 0$ and $\eta \in E_k$.

According to (3.19), $\psi(\eta)$ defines $\phi(t,\eta)$ up to a bounded term. Using (3.3) and (3.9), we rewrite (3.19) in the form

$$\phi(t,\eta)=\eta+a_{n_t}(\eta)+b(t,\eta), \tag{3.20}$$

Taking into account (3.13) and (3.20), we obtain the next theorem.

Theorem 3.9. *The vector function*

$$F(t,\eta)=[\phi(t,\eta)-\eta]/t \quad (t>1)$$

is bounded and continuous. The limit set of $F(t,\eta)$, for $t\to+\infty$, coincides with $Q(\eta)$.

Theorem 3.9 implies (see [4, Ch.XYI, Theorem 1.1] or [8, Ch.X, Assertion 7.1]) the following proposition.

Proposition 3.10. *The limit set $Q(\eta)$ is closed, connected and nonempty.*

Note that the fact that $Q(\eta)$ is closed and nonempty easily follows from the considerations above Theorem 3.9.

Let q be a vector with integer elements. Then, taking into account (2.8) (3.2) and (3.9) we have

$$a_n(\eta+q)=a_n(\eta), \quad n=1,2,... \tag{3.21}$$

It follows from (3.21) that

$$Q(\eta+q)=Q(\eta). \tag{3.22}$$

Next, we will need Zorn's lemma [18]:

Lemma 3.11. *A partially ordered set containing upper bounds for every totally ordered subset contains at least one maximal element.*

The set $Q(\eta)$ is partially ordered, bounded and closed (see Definition 1.5, inequality (3.13) and Proposition 3.10). Hence, $Q(\eta)$ satisfies all the conditions of Zorn's lemma. Thus, we derive a vector analog of scalar Theorems 1.2 and 1.4.

Theorem 3.12. *There are vectors $\rho_1(\eta),\rho_2(\eta)\in Q(\eta)$, which satisfy the following relations for all the vectors $X\in Q(\eta)$.*

$$\rho_1(\eta)\leq X\leq\rho_2(\eta). \tag{3.23}$$

According to (3.22), we have

$$\rho_1(\eta+q)=\rho_1(\eta), \quad \rho_2(\eta+q)=\rho_2(\eta) \quad \text{for} \quad q\in\mathbb{Z}^k. \tag{3.24}$$

Definition 3.13. *By analogy with the scalar case, we call the vector functions $\rho_1(\eta)$ and $\rho_2(\eta)$ rotation vector functions (left and right, respectively).*

Remark 3.14. *The case of $\rho=\rho_1=\rho_2$, which do not depend on η, is dealt with in greater detail in our next paper.*

4 Differentiable solutions of the generalised system

Suppose that a k-vector function $f(x)$ of k variables is differentiable. Then, the corresponding Jacobian matrix has the form

$$J_x(f)=\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_k} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_k} \end{pmatrix}. \tag{4.1}$$

Differentiable k-vector functions $f(x)$ and $g(f(x))$ satisfy the chain rule:

$$J_x(g(f(x))=J_f(g)J_x(f). \tag{4.2}$$

In this section we assume that all the first order derivatives of the k-vector function $\psi(\eta)$ ($\eta\in E_k$) exist and are continuous. Then, all the first order derivatives of the k-vector functions $\psi_n(\eta)$ ($n\in\mathbb{N}$) exist and are continuous as well. Clearly, we preserve our standard requirement

$$\psi(\eta+q)=\psi(\eta)+q \quad \text{for} \quad q\in\mathbb{Z}^k. \tag{4.3}$$

It follows from (4.2) that the Jacobian for $\psi_n(\eta)$ has the form:

$$J_\eta(\psi_n(\eta))=J_{\psi_{n-1}}(\psi_n)J_{\psi_{n-2}}(\psi_{n-1})...J_\eta(\psi), \tag{4.4}$$

where $\psi_0(\eta)=\eta$.

Corollary 4.1. *If* $\det[J_\eta(\psi)]$ *has no roots, then* $\det[J_\eta(\psi_n)]$ *has no roots too.*

Corollary 4.2. *If* $\det[J_\eta(\psi)$ *has no roots then the functions $\psi_n(\eta)$ ($n\in\mathbb{N}$) have not extremal points.*

By D_a, we denote the k-dimensional cube of the form:

$$-a \leq x_s \leq a \quad (a > 0,\ 1 \leq s \leq k). \tag{4.5}$$

The vectors from zero to the vertices of D_a (or the coordinates of the vertices) have the form ah, where each element of h equals either 1 or -1. A special role is played by the vectors ae and $-ae$, where all the elements of e equal one: $e = [1, 1, \ldots, 1]$.

Theorem 4.3. *Let all the first order derivatives of the k-vector function $\psi(\eta)$ $(\eta \in E_k)$ exist and be continuous and assume that the inequality*

$$\det[J_\eta(\psi)] \neq 0 \tag{4.6}$$

holds for $\eta \in E_k$.

Then, the following inequalities are valid for all the vectors $\eta \in D_a$:

$$\psi_n(-ae) \leq \psi_n(\eta) \leq \psi_n(ae). \tag{4.7}$$

Proof. According to Corollary 4.2, the function $\psi_n(\eta)$ has no extremum points inside of the cube D_a. If the extremum point τ_0 of the vector function $\psi_n(\eta)$ is on the boundary of the cube, but does not coincide with the vertex of the cube, then the elements of at least one of the columns of $J_\eta(\psi_n(\eta))$ are equal to zero, which contradicts condition (4.6) of our theorem. Thus, we have proved that ψ_n may attain its greatest and smallest values at the points belonging to the set V_a of the vertices of D_a only. Note that Zorn's lemma (Lemma 3.11) implies that the function $\psi_n(\eta)$ attains its greatest and smallest values in the domain D_a. Thus, these values are attained at the vertices.

Now suppose that ψ_n attains its greatest (or smallest) values at two vertices P and Q simultaneously. Considering the segment $[P, Q]$, we see again that the greatest and smallest values of ψ_n on it are attained at the ends of $[P, Q]$, that is, the greatest and the smallest values of ψ_n on $[P, Q]$ coincide. Therefore, ψ_n is constant on $[P, Q]$ and attains its greatest (or smallest) in D_a values on the whole segment $[P, Q]$. This contradicts the fact that the greatest and smallest values of ψ_n may be attained at the points of V_a only. Therefore, the function $\psi_n(\eta)$ takes its greatest (smallest) value in D_a at one and only one point, which belongs to V_a.

For sufficiently large a (in view of (4.3)), $\psi_n(\eta)$ may attain its greatest (smallest) value at the point ae $(-ae)$ only. Thus, it attains it there. Since the vector functions $\psi_n(ah)$ (for h with the elements taking values ± 1) are continuous with respect to a and the greatest (smallest) values are attained at one vertex only, the h at which $\psi_n(ah)$ takes the greatest (smallest) value cannot change when a decreases. Hence, (4.7) holds for all $a \geq 0$. $\square$

Relation (4.7) may be rewritten in the form

$$\psi_n(-ae)/n \leq \psi_n(\eta)/n \leq \psi_n(ae)/n \quad (\eta \in D_a). \tag{4.8}$$

We reduced our problem to the case $k = 1$. Using relation (4.8) and the proof of the classical Poincaré theorem ([4, Ch. XVII]) about rotation numbers, we obtain the main result of this section.

Theorem 4.4. *If the conditions of Theorem 4.3 are fulfilled, then the vector function $\psi(ae)$ is monotonically increasing and the following equality holds:*

$$\rho_1(\eta) = \rho_2(\eta) = \rho. \tag{4.9}$$

.

5 Examples

Example 5.1. *Starting with the simplest case, where relations* (2.1) *and* (2.2) *hold, let*

$$\frac{dx}{dt} = G, \quad x(0) = \eta \qquad (t \in \mathbb{R};\ \eta,\, G,\, x(t) \in E_k). \tag{5.1}$$

The solution $\phi(t, \eta)$ of equation (5.1) and $\psi(\eta)$ have the form

$$\phi(t, \eta) = Gt + \eta, \quad \psi(\eta) = G + \eta. \tag{5.2}$$

It follows that

$$\psi_n(\eta) = nG + \eta, \quad Q(\eta) = \{G\} \quad (\text{i.e., } \rho_1 = \rho_2 = G). \tag{5.3}$$

The next assertion is easily checked directly.

Proposition 5.2. *In the case* (5.1)*, relations* (2.3)*–*(2.5) *are fulfilled. The vector functions $\psi(\eta)$ and $\phi(t, \eta)$ (for each t) are injective.*

Remark 5.3. *We note that injective property of the vector function $\psi(\eta)$ is an analog of the monotonic property of $\psi(\eta)$ in the scalar case. Injective property of the vector function $\phi(t,\eta)$ is an analog of Condition II on $\phi(t,\eta)$ in the scalar case. Thus, injective properties of $\psi(\eta)$ and $\phi(t,\eta)$ are of interest.*

Next, consider a more non-trivial example.

Example 5.4. *Let $k=2$, $\eta=[\eta_1,\,\eta_2]$ and $\psi(\eta)$ have the form*

$$\psi(\eta)=[\eta_1+r\sin(2\pi\eta_2),\,\eta_2-r\sin(2\pi\eta_1)] \quad (r\in\mathbb{R}). \tag{5.4}$$

Let the 2-vector function $\Phi(t,\eta)$ $(0\le t\le 1)$ have the form

$$\Phi(t,\eta)=[\Phi_1(t,\eta),\,\Phi_2(t,\eta)], \quad \Phi_1(t,\eta)=\eta_1+r\sin(\pi t/2)\sin(2\pi\eta_2), \tag{5.5}$$
$$\Phi_2(t,\eta)=\eta_2-r\sin(\pi t/2)\sin(2\pi\eta_1). \tag{5.6}$$

Clearly, conditions (2.10) and (2.11) are fulfilled and we may apply Theorem 2.3. (Clearly, the second equality in (2.4) holds for ψ as well.)

Proposition 5.5. *Given* (5.4)–(5.6) *we recover solution $\phi(t,\eta)$ of a generalised system using* (2.12).

Let us consider $\psi(\eta)$ in greater detail. The following inequality is well known:

$$|\sin x-\sin y|\le|x-y|. \tag{5.7}$$

Using (5.4) and (5.7), we derive

$$\psi(\eta)-\psi(p)=\eta-p+rb, \quad \|b\|<2\pi\,\|\eta-p\|\,, \tag{5.8}$$

where $b=[\sin(2\pi\eta_2)-\sin(2\pi p_2),\,\sin(2\pi p_1)-\sin(2\pi\eta_1)]$. Taking into account (5.8), we obtain

$$\|\psi(\eta)-\psi(p)\|\neq 0 \quad \text{for} \quad \eta\neq p \quad \text{and} \quad |r|<1/(2\pi). \tag{5.9}$$

Using relation (5.9) we have:

Proposition 5.6. *The vector function $\psi(\eta)$ given by* (5.4), *where $|r|<1/(2\pi)$, is an injective mapping of the space E_2 into the space E_2.*

Remark 5.7. *Proposition 5.6 may be generalised using representation* (2.7) *of ψ in the k-dimensional generalised system. Namely, if the inequality*

$$\|a(\eta) - a(p)\| < \|\eta - p\| \,. \tag{5.10}$$

is fulfilled for all the k-vectors η and p, then the vector function $\psi(\eta)$ is an injective mapping of the space E_k into the space E_k. It is easy to see that the vector function $\psi(\eta)$ considered in Proposition 5.6 satisfies the condition (5.10).

Now, let us calculate the Jacobian matrix $\mathbf{J}$ for this $\psi(\eta)$:

$$\mathbf{J} = \begin{pmatrix} 1 & 2r\pi\cos(2\pi\eta_2) \\ -2r\pi\cos(2\pi\eta_1) & 1 \end{pmatrix}. \tag{5.11}$$

Hence, we have

$$\det \mathbf{J} = 1 + (2\pi r)^2 \cos(2\pi\eta_2)\cos(2\pi\eta_1) > 0 \quad \text{for} \quad |r| < 1/(2\pi). \tag{5.12}$$

Formula (5.12) yields the next proposition.

Proposition 5.8. *The vector function $\psi(\eta)$ given by* (5.4)*, where $|r| < 1/(2\pi)$, has no critical points.*

Recall that a *critical point* of the vector function is a point where the rank of the Jacobian matrix is not maximal.

Similar to Proposition 5.6 we derive a more general proposition.

Proposition 5.9. *The vector function $\Phi(t,\eta)$ given by* (5.5) *and* (5.6)*, where $|r| < 1/(2\pi)$, is an injective mapping of the space E_k into the space E_k for each $t \in [0,1]$.*

For the Jacobian matrix $\mathbf{J} = \{\mathbf{J}_{ik}\}$ $(1 \le i \le 2,\ 1 \le k \le 3)$ of $\Phi(t,\eta)$ we easily obtain: $\mathbf{J}_{11} = \mathbf{J}_{22} = 1$,

$$\mathbf{J}_{12} = 2r\pi\sin(\pi t/2)\cos(2\pi\eta_2), \quad \mathbf{J}_{21} = -2r\pi\sin(\pi t/2)\cos(2\pi\eta_1).$$

Thus, the rows of $\mathbf{J}$ are linearly independent for $|r| < 1/(2\pi)$ and the Jacobian matrix $\mathbf{J}$ has the maximal rank 2.

Proposition 5.10. *The vector function $\Phi(t,\eta)$ given by* (5.5) *and* (5.6)*, where $|r| < 1/(2\pi)$, has no critical points.*

Finally, we apply important Theorem 4.4 to Example 5.4. From (5.12) and Theorem 4.4 follows the proposition below.

Proposition 5.11. *If the vector function* $\psi(\eta)$ *is given by* (5.4), *where* $|r| < 1/(2\pi)$, *the equality* (4.9) *is valid.*

Acknowledgement. The author is very grateful to Alexander Sakhnovich for the careful reading of the work and many useful remarks.

COMPETING INTERESTS.. There is no competing interests related to this paper.

References

[1] Arnold, V.I.: Geometrical methods in the theory of ordinary differential equations. Springer, New York (1988)

[2] Berger, P., Biebler, S.: Emergence of wandering stable components. J. Amer. Math. Soc. **36**, 397–482 (2023)

[3] Bohl, P.: Über die Hinsichtlich der Unabhängigen und Abhängigen Variabeln Periodische Differentialgleichung Erster Ordnung. Acta Math. **40**, 321–336 (1916)

[4] Coddington, E.A., Levinson, N.: Theory of ordinary differential equations. McGraw-Hill Book Co., New York (1955)

[5] Damanik, D., Zhou, Zhe: The rotation number for almost periodic Schrödinger operators with δ-potentials. J. Dynam. Differential Equations **34**, 155–177 (2022)

[6] Feynman, R.: QED: The strange theory of light and matter. Prinston University Press, Prinston (1985)

[7] Hale, J.K., Raugel, G.: A modified Poincare method for the persistence of periodic orbits and applications. J. Dynam. Differential Equations **22**, 3–68 (2010)

[8] Lefschetz, S.: Differential equations: geometric theory. Interscience Publishers, New York (1957)

[9] Lellouch, G.: On rotation sets for surface homeomorphisms in genus ≥ 2. (French) Mem. Soc. Math. Fr. (N.S.) No. 178 (2023)

[10] Li, Ke, Li, Shimin: On the Poincare-Bendixson index theorem for a class of piecewise linear differential systems. Qual. Theory Dyn. Syst. **23**, Paper No. 9 (2024)

[11] Llibre, J., Valls, C.: Darboux theory of integrability in $\mathbb{T}^n$. Eur. J. Math. **10**, Paper No. 26 (2024)

[12] Poincaré, H.: Sur les courbes definies par une equation differentielle. Oeuvres, vol. 1, Paris (1892)

[13] Roeder, R.K.W.: On Poincare's fourth and fifth examples of limit cycles at infinity. Rocky Mountain. J. Math. **33**, 1057–1082 (2003)

[14] Sakhnovich, L.A.: The deviation factor and divergences in quantum electrodynamics, concrete examples. Phys. Lett. A **383**, 2846–2853 (2019)

[15] Silva, C.M.: Nonuniform μ-dichotomy spectrum and kinematic similarity. J. Differential Equations **375**, 618–652 (2023)

[16] Valls, C.: A generalization of the Poincare compactification and the real Jacobian conjecture. J. Dynam. Differential Equations **36**, 619–631 (2024).

[17] Vulikh, B.Z.: Introduction to the theory of partially ordered spaces. Wolters-Noordhoff Scientific Publications, Gröningen (1967)

[18] Zorn, M.: A remark on method in transfinite algebra. Bull. Amer. Math. Soc. **41**, 667–670 (1935)