THE STABLE ALBANESE HOMOLOGY OF THE IA-AUTOMORPHISM GROUPS OF FREE GROUPS

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Abstract. The IA-automorphism group IA_n of the free group F_n of rank n is a normal subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of F_n. We study the Albanese homology of IA_n, which is the quotient of the rational homology of IA_n defined as the image of the map induced by the abelianization map of IA_n on homology. The Albanese homology of IA_n is an algebraic $\mathrm{GL}(n,\mathbb{Q})$-representation. We determine the representation structure of the Albanese homology of IA_n for n greater than or equal to three times the homological degree. We also determine the structure of the stable Albanese homology of the analogue of IA_n to the outer automorphism group of F_n. Moreover, we identify the relation between the stable Albanese (co)homology of IA_n and the stable cohomology of $\mathrm{Aut}(F_n)$ with certain twisted coefficients.

1. Introduction

The IA-automorphism group IA_n of the free group F_n of rank n is the normal subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of F_n that is trivial under the canonical group homomorphism from $\mathrm{Aut}(F_n)$ to the general linear group $\mathrm{GL}(n,\mathbb{Z})$ induced by the abelianization map of F_n. Then we have a short exact sequence of groups

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1.$$

By this short exact sequence, the (co)homology of IA_n admits an action of $\mathrm{GL}(n,\mathbb{Z})$. The IA-automorphism group IA_n is analogous to the *Torelli groups* for surfaces, which are important objects in low-dimensional topology. Some strategies of studying the (co)homology of the Torelli groups can be used to study the (co)homology of IA_n and vice versa.

The structure of the first (co)homology was determined by Cohen–Pakianathan, Farb (both unpublished) and Kawazumi [8], independently. The *Johnson homomorphism* for $\mathrm{Aut}(F_n)$ induces an isomorphism

$$H_1(\mathrm{IA}_n,\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_{\mathbb{Z}}, \bigwedge^2 H_{\mathbb{Z}}), \quad H_{\mathbb{Z}} = H_1(F_n,\mathbb{Z}).$$

For $n=3$, it is known that IA_3 is not finitely presentable by Krstić–McCool [11] and $H_2(\mathrm{IA}_3,\mathbb{Z})$ has infinite rank by Bestvina–Bux–Margalit [1]. Pettet [15] determined the $\mathrm{GL}(n,\mathbb{Z})$-subrepresentation of $H^2(\mathrm{IA}_n,\mathbb{Q})$ that is detected by using the Johnson homomorphism, which is regarded as the second Albanese cohomology $H^2_A(\mathrm{IA}_n,\mathbb{Q})$ of IA_n explained below. Satoh [17] detected an irreducible subrepresentation of $H^2(\mathrm{IA}_3,\mathbb{Q})$ which is not included in $H^2_A(\mathrm{IA}_3,\mathbb{Q})$. For $n \geq 4$, it is still open whether IA_n is finitely presentable or not. However, it is known that $H_2(\mathrm{IA}_n,\mathbb{Z})$ is finitely generated as a $\mathrm{GL}(n,\mathbb{Z})$-representation by Day–Putman [4].

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In a stable range, that is, for sufficiently large n with respect to the (co)homological degree, the structure of the rational (co)homology of IA_n has been studied [7, 6, 13], and we have a conjectural structure of the stable rational (co)homology of IA_n (see Theorem 1.2).

The main interest of this paper is a subalgebra of the rational cohomology of IA_n which seems to play an essential role in the stable rational cohomology of IA_n. The subalgebra is defined to be the image of the map induced by the abelianization map of IA_n:

$$H_A^*(\mathrm{IA}_n, \mathbb{Q}) = \mathrm{im}(H^*(\mathrm{IA}_n^{\mathrm{ab}}, \mathbb{Q}) \to H^*(\mathrm{IA}_n, \mathbb{Q})).$$

Church–Ellenberg–Farb [2] called $H_A^*(\mathrm{IA}_n, \mathbb{Q})$ the *Albanese cohomology* of IA_n. The *Albanese homology* $H_*^A(\mathrm{IA}_n, \mathbb{Q})$ is predual to the Albanese cohomology defined by

$$H_*^A(\mathrm{IA}_n, \mathbb{Q}) = \mathrm{im}(H_*(\mathrm{IA}_n, \mathbb{Q}) \to H_*(\mathrm{IA}_n^{\mathrm{ab}}, \mathbb{Q})).$$

It follows from the definition of the Albanese (co)homology of IA_n and the computation of the first homology of IA_n that the Albanese (co)homology is an *algebraic* $\mathrm{GL}(n, \mathbb{Q})$*-representation*. The second and the third Albanese homology of IA_n was determined by Pettet [15] and the author [7], respectively. Moreover, in [7], the author detected a large subquotient $\mathrm{GL}(n, \mathbb{Q})$-representation W_i of $H_i^A(\mathrm{IA}_n, \mathbb{Q})$ for each $n \geq 3i$, and conjectured that $H_i^A(\mathrm{IA}_n, \mathbb{Q})$ is isomorphic to W_i for $n \geq 3i$.

The aim of this paper is to prove this conjecture on the representation structure of the Albanese homology of IA_n.

Theorem 1.1 (Theorem 2.5, cf. [7, Conjecture 6.2])**.** *We have an isomorphism of* $\mathrm{GL}(n, \mathbb{Q})$*-representations*

$$F_i : H_i^A(\mathrm{IA}_n, \mathbb{Q}) \xrightarrow{\cong} W_i$$

for $n \geq 3i$.

It follows from Theorem 1.1 that the Albanese homology of IA_n is representation stable in $n \geq 3i$ in the sense of Church–Farb [3].

The author received a draft version of [13] by Erik Lindell and noticed that some reinterpretation of [13, Proposition 6.3] can be used to determine the structure of the stable Albanese homology of IA_n. In the appendix of [13], she proved the statement of Theorem 1.1 only for $n \gg i$.

Habiro and the author [6] studied the structure of the stable rational cohomology of IA_n. By using Theorem 1.1, we can remove from one of the main results of [6] (cf. [6, Theorem 1.10 and Remark 7.9]) the assumption about the structure of the Albanese homology of IA_n, and obtain the following theorem.

Theorem 1.2 (Cf. [6, Theorem 1.10 and Remark 7.9])**.** *Suppose that* $H^i(\mathrm{IA}_n, \mathbb{Q})$ *is an algebraic* $\mathrm{GL}(n, \mathbb{Q})$*-representation for* $n \gg i$*. Then for* $n \gg i$*, we have*

$$H^i(\mathrm{IA}_n, \mathbb{Q}) \cong \bigoplus_{k+l=i} W_k^* \otimes \mathbb{Q}[z_1, z_2, \cdots]_l,$$

where $\mathbb{Q}[z_1, z_2, \cdots]_l$ *denotes the degree* l *part of* $\mathbb{Q}[z_1, z_2, \cdots]$ *and* $\deg z_j = 4j$.

Lindell [13] has recently weakened the assumption of Theorem 1.2 that the family $\{H^*(\mathrm{IA}_n, \mathbb{Q})\}_n$ is algebraic for $n \gg *$ to the assumption that $\{H^*(\mathrm{IA}_n, \mathbb{Q})\}_n$ satisfies *Borel vanishing* for $n \gg *$ (see [13, Definition 1.5]).

In this paper, we will prove Theorem 1.1 by taking care of the stable range. It follows from the proof of Theorem 1.1 that the Albanese cohomology algebra is quadratic for $n \geq 3*$. We also prove several related conjectures which are proposed in [7]. In particular, we will determine the structure of the Albanese homology of the analogue IO_n of IA_n to the outer automorphism group of F_n for $n \geq 3*$.

We will also prove the conjecture on the relation between the stable Albanese cohomology of IA_n and the stable cohomology of $\mathrm{Aut}(F_n)$ with coefficients in the tensor product $H^{p,q}$ of p copies of the standard representation $H = H_1(F_n, \mathbb{Q})$ of $\mathrm{GL}(n, \mathbb{Q})$ and q copies of the dual representation H^*.

Theorem 1.3 (Theorem 3.8, cf. [6, Conjecture 7.2])**.** *The inclusion map* $i : \mathrm{IA}_n \hookrightarrow \mathrm{Aut}(F_n)$ *induces an isomorphism of* $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$*-modules*

$$i^* : H^*(\mathrm{Aut}(F_n), H^{p,q}) \xrightarrow{\cong} [H_A^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$$

for $n \geq \min(\max(3 * +4, p + q), 2 * +p + q + 3)$.

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2. The stable Albanese homology of IA_n

In this section, we will prove [7, Conjecture 6.2] on the $\mathrm{GL}(n, \mathbb{Q})$-representation structure of the stable Albanese homology of IA_n.

2.1. **Algebraic $\mathrm{GL}(n, \mathbb{Q})$-representations.**

Here we briefly recall some facts from representation theory of $\mathrm{GL}(n, \mathbb{Q})$. See Fulton–Harris [5] for details.

A finite-dimensional $\mathrm{GL}(n, \mathbb{Q})$-representation (ρ, V) is called *algebraic* if after choosing a basis for V, the $(\dim V)^2$ coordinate functions of the group homomorphism $\rho : \mathrm{GL}(n, \mathbb{Q}) \to \mathrm{GL}(V)$ are rational functions on n^2 variables.

It is well known that algebraic $\mathrm{GL}(n, \mathbb{Q})$-representations are completely reducible and that irreducible representations are classified by *bipartitions*, i.e., pairs of partitions. Here, a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a non-increasing sequence of non-negative integers. Let $|\lambda| = \sum_{i=1}^{n} \lambda_i$ denote the *size* of λ and $l(\lambda) = \max(\{0\} \cup \{i \mid \lambda_i > 0\})$ the *length* of λ. For a bipartition (λ, μ), the irreducible algebraic $\mathrm{GL}(n, \mathbb{Q})$-representation $V_{\lambda,\mu}$ is constructed as follows. Consider $H = H_1(F_n, \mathbb{Q})$ as the standard representation of $\mathrm{GL}(n, \mathbb{Q})$ and set $H^{p,q} = H^{\otimes p} \otimes (H^*)^{\otimes q}$ for $p, q \geq 0$. The *traceless part* $T_{p,q}$ of $H^{p,q}$ is defined by

$$T_{p,q} = \bigcap_{1 \leq k \leq p,\ 1 \leq l \leq q} \ker(c_{k,l} : H^{p,q} \to H^{p-1,q-1}),$$

where $c_{k,l}$ is the *contraction map* that takes the dual pairing $\langle , \rangle : H \otimes H^* \to \mathbb{Q}$ at the k-th tensorand of $H^{\otimes p}$ and the l-th tensorand of $(H^*)^{\otimes q}$. For $|\lambda| = p$, $|\mu| = q$, let

$$V_{\lambda,\mu} = T_{p,q} \otimes_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} (S^\lambda \otimes S^\mu),$$

where S^λ and S^μ denote the Specht modules corresponding to λ and μ, respectively. If $n \geq l(\lambda) + l(\mu)$, then $V_{\lambda,\mu}$ is an irreducible algebraic $\mathrm{GL}(n, \mathbb{Q})$-representation and otherwise $V_{\lambda,\mu} = 0$.

2.2. **Invariant theory of** $\mathrm{GL}(n,\mathbb{Z})$**.** Let $\{e_i \mid 1 \leq i \leq n\}$ be a basis for H and $\{e_i^* \mid 1 \leq i \leq n\}$ the dual basis for H^*. Define a linear map $\omega : \mathbb{Q} \to H \otimes H^*$ by

$$\omega(1) = \sum_{i=1}^{n} e_i \otimes e_i^*, \tag{2.2.1}$$

which is dual to the pairing $\langle , \rangle$. By using $(p+q)$ copies of the element $\omega(1)$, we have a surjective linear map

$$\Omega : \mathbb{Q}[\mathfrak{S}_{p+q}] \twoheadrightarrow [H^{p,q} \otimes H^{q,p}]^{\mathrm{GL}(n,\mathbb{Z})},$$

which is defined by

$$\Omega(\sigma) = \sum_{1 \leq i_1, \cdots, i_{p+q} \leq n} \left(\bigotimes_{j=1}^{p} e_{i_j} \otimes \bigotimes_{j=1}^{q} e^*_{i_{\sigma^{-1}(j)}} \right) \otimes \left(\bigotimes_{j=p+1}^{p+q} e_{i_j} \otimes \bigotimes_{j=q+1}^{p+q} e^*_{i_{\sigma^{-1}(j)}} \right)$$

for $\sigma \in \mathfrak{S}_{p+q}$. See [13, Section 2.1] for details.

Let $\mathrm{pr} : [H^{p,q} \otimes H^{q,p}]^{\mathrm{GL}(n,\mathbb{Z})} \twoheadrightarrow [T_{p,q} \otimes T_{q,p}]^{\mathrm{GL}(n,\mathbb{Z})}$ denote the projection. Let

$$\Omega' = \mathrm{pr} \circ \Omega|_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]} : \mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q] \to [T_{p,q} \otimes T_{q,p}]^{\mathrm{GL}(n,\mathbb{Z})},$$

where $\Omega|_{\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]}$ is the restriction of Ω to $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$. By the surjectivity of Ω, the map Ω' is also surjective since we have $\mathrm{pr} \circ \Omega(\sigma) = 0$ for $\sigma \in \mathfrak{S}_{p+q} \setminus (\mathfrak{S}_p \times \mathfrak{S}_q)$. Let $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$ act on $[T_{p,q} \otimes T_{q,p}]^{\mathrm{GL}(n,\mathbb{Z})}$ by the place permutations of p copies of H^* and q copies of H in $T_{q,p}$. Then Ω' is a $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-module map.

We will generalize [13, Proposition 2.10] to obtain the following lemma.

Lemma 2.1. *For $n \geq \max(p+q, r+s)$, we have a linear isomorphism*

$$[T_{p,q} \otimes T_{r,s}]^{\mathrm{GL}(n,\mathbb{Z})} \cong \begin{cases} \mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q] & (p = s, q = r) \\ 0 & (\textit{otherwise}). \end{cases}$$

Therefore, the map Ω' is an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules for $n \geq p+q$.

Proof. We will compute the dimension of $[T_{p,q} \otimes T_{r,s}]^{\mathrm{GL}(n,\mathbb{Z})}$. By Koike [10], for $n \geq \max(p+q, r+s)$, we have irreducible decompositions

$$T_{p,q} \cong \bigoplus_{\lambda \vdash p, \mu \vdash q} (V_{\lambda,\mu})^{\oplus (\dim S^\lambda \dim S^\mu)}, \quad T_{r,s} \cong \bigoplus_{\xi \vdash r, \eta \vdash s} (V_{\xi,\eta})^{\oplus (\dim S^\xi \dim S^\eta)}.$$

Therefore, we have

$$T_{p,q} \otimes T_{r,s} \cong \bigoplus_{\substack{\lambda \vdash p, \mu \vdash q \\ \xi \vdash r, \eta \vdash s}} (V_{\lambda,\mu} \otimes V_{\xi,\eta})^{\oplus (\dim S^\lambda \dim S^\mu \dim S^\xi \dim S^\eta)}.$$

Since for each $\lambda \vdash p$, $\mu \vdash q$, $\xi \vdash r$, $\eta \vdash s$,

$$[V_{\lambda,\mu} \otimes V_{\xi,\eta}]^{\mathrm{GL}(n,\mathbb{Z})} \cong \mathrm{Hom}_{\mathrm{GL}(n,\mathbb{Z})}(V_{\eta,\xi}, V_{\lambda,\mu}) \cong \mathbb{Q}^{\oplus (\delta_{\lambda,\eta} \delta_{\mu,\xi})}$$

by Schur's lemma, we have

$$\dim([T_{p,q} \otimes T_{r,s}]^{\mathrm{GL}(n,\mathbb{Z})}) = \delta_{p,s} \delta_{q,r} \sum_{\lambda \vdash p, \mu \vdash q} (\dim S^\lambda)^2 (\dim S^\mu)^2.$$

It follows from

$$\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q] \cong \mathbb{Q}[\mathfrak{S}_p] \otimes \mathbb{Q}[\mathfrak{S}_q] \cong \bigoplus_{\lambda \vdash p, \mu \vdash q} (S^\lambda)^{\oplus \dim S^\lambda} \otimes (S^\mu)^{\oplus \dim S^\mu}$$

that we have

$$\dim(\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]) = \sum_{\lambda \vdash p, \mu \vdash q} (\dim S^{\lambda})^2 (\dim S^{\mu})^2,$$

which completes the proof. □

2.3. **The conjectural structure of the stable Albanese homology of IA_n.** We will briefly review our conjectural structure of the stable Albanese homology of IA_n. For $i \geq 1$, let

$$U_i = \mathrm{Hom}(H, \bigwedge^{i+1} H).$$

Let $U_* = \bigoplus_{i \geq 1} U_i$ be the graded $\mathrm{GL}(n, \mathbb{Q})$-representation. Define $W_* = \widetilde{S}^*(U_*)$ as the *traceless part* of the graded-symmetric algebra $S^*(U_*)$ of U_*. Here, the *traceless tensor product* $V_{\lambda,\mu} \widetilde{\otimes} V_{\xi,\eta}$ of two irreducible algebraic $\mathrm{GL}(n, \mathbb{Q})$-representations $V_{\lambda,\mu}$ and $V_{\xi,\eta}$ is defined by

$$V_{\lambda,\mu} \widetilde{\otimes} V_{\xi,\eta} = (V_{\lambda,\mu} \otimes V_{\xi,\eta}) \cap T_{|\lambda|+|\xi|,|\mu|+|\eta|} \subset H^{|\lambda|+|\xi|,|\mu|+|\eta|},$$

and the traceless part of the tensor algebra is defined by using the traceless tensor product instead of the usual tensor product. The traceless part of the graded-symmetric algebra is defined as the image of the traceless part of the tensor algebra under the canonical projection. See [7, Sections 2.5 and 2.6] for details of the notion of the traceless part.

In order to prove our main theorem, we will review and give the stable range of [7, Proposition 12.3 and Lemma 12.4]. To state these proposition and lemma, we will review the wheeled PROP $\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$ that is introduced in [9], and the non-unital wheeled PROP $\mathcal{C}_{\mathcal{O}^{\circlearrowright}}$ that is introduced in [7], corresponding to the operad $\mathcal{C}om$ of non-unital commutative algebras.

Let $\mathcal{P}_0 = \bigoplus_{k \geq 1} \mathcal{P}_0(k)$ denote the operadic suspension of the operad $\mathcal{C}om$, i.e., we have $\mathcal{P}_0(0) = 0$ and $\mathcal{P}_0(k)$ is the sign representation of $\mathfrak{S}_k$ placed in cohomological dimension $k-1$ for $k \geq 1$. Let $\mathcal{P}_0^{\circlearrowright}$ denote the wheeled completion of $\mathcal{P}_0$ and $\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$ the wheeled PROP freely generated by $\mathcal{P}_0^{\circlearrowright}$.

Let $\mathcal{O} = \bigoplus_{k \geq 2} \mathcal{P}_0(k)$ denote the non-unital suboperad of $\mathcal{P}_0$. Let $\mathcal{O}^{\circlearrowright}$ denote the non-unital wheeled sub-operad of $\mathcal{P}_0^{\circlearrowright}$ and $\mathcal{C}_{\mathcal{O}^{\circlearrowright}}$ the non-unital wheeled sub-PROP of $\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$.

Remark 2.2. In [13], Lindell defined a wBr_n-module $\mathcal{P} \otimes \det$ (resp. a dwBr-module $\mathcal{P}' \otimes \det$), which is a functor from the walled Brauer category wBr_n (resp. the downward walled Brauer category dwBr) to the category of $\mathbb{Q}$-vector spaces, in terms of labelled partitions of sets. A wheeled PROP (resp. a non-unital wheeled PROP) admits a natural structure of a wBr_n-module (resp. a dwBr-module), and the wheeled PROP $\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$ corresponds to $\mathcal{P} \otimes \det$ and the non-unital wheeled PROP $\mathcal{C}_{\mathcal{O}^{\circlearrowright}}$ corresponds to $\mathcal{P}' \otimes \det$.

Lemma 2.3 (Cf. [7, Proposition 12.3])**.** *For $n \geq 3i$, we have an isomorphism of $\mathrm{GL}(n, \mathbb{Q})$-representations*

$$W_i \cong \bigoplus_{a-b=i} T_{a,b} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(a, b).$$

Here, we have $T_{a,b} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(a, b) = 0$ unless $i \leq a \leq 2i$, $0 \leq b \leq i$.

Proof. The proof of [7, Proposition 12.3] does not take care of the stable range, but the argument holds for $n \geq 3i$ since for any $k, l \geq 1$ such that $kl \leq i$, we have

$$H^{k(l+1),k} \otimes_{\mathbb{Q}[\mathfrak{S}_{l+1} \wr \mathfrak{S}_k]} \mathcal{O}(l)^{\otimes k} \cong S^k(V_{1^l,1}), \quad H^{kl,0} \otimes_{\mathbb{Q}[\mathfrak{S}_l \wr \mathfrak{S}_k]} \mathcal{O}^{\circlearrowright}_w(l)^{\otimes k} \cong S^k(V_{1^l,0}),$$

where $S^k(V_{1^l,i})$ denotes the graded-symmetric power of $V_{1^l,i}$ ($i = 0, 1$), and where $\mathcal{O}^{\circlearrowright}_w$ denotes the wheel part of the non-unital wheeled operad $\mathcal{O}^{\circlearrowright}$. □

Lemma 2.4 (Cf. [7, Lemma 12.4]). *For $n \geq \max(3i, p+q)$, we have an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules*

$$[(W_i)^* \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})} \cong \mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}(p,q)_i.$$

Proof. Since the proof of [7, Lemma 12.4] does not deal with the stable range, we will recall the proof of it to take care of the stable range. By Lemma 2.3, we have for $n \geq 3i$,

$$(W_i)^* \cong \bigoplus_{a-b=i} T_{b,a} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(a,b),$$

where the direct summand is trivial unless $i \leq a \leq 2i, 0 \leq b \leq i$. For $n \geq p+q$, we have $H^{p,q} \cong \bigoplus_{c=0}^{\min(p,q)} T_{p-c,q-c}^{\oplus \binom{p}{c}\binom{q}{c}c!}$. Therefore, we have

$$\begin{aligned}
&[(W_i)^* \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})} \\
&\cong \left[\bigoplus_{a-b=i} \left(T_{b,a} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(a,b) \right) \otimes \bigoplus_{c=0}^{\min(p,q)} \left(T_{p-c,q-c}^{\oplus \binom{p}{c}\binom{q}{c}c!} \right) \right]^{\mathrm{GL}(n,\mathbb{Z})} \\
&\cong \bigoplus_{a-b=i} \bigoplus_{c=0}^{\min(p,q)} [T_{p-c,q-c} \otimes T_{b,a}]^{\mathrm{GL}(n,\mathbb{Z})} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(a,b)^{\oplus \binom{p}{c}\binom{q}{c}c!} \\
&\cong \bigoplus_{c=0}^{\min(p,q)} \mathbb{Q}[\mathfrak{S}_{p-c} \times \mathfrak{S}_{q-c}] \otimes_{\mathbb{Q}[\mathfrak{S}_{p-c} \times \mathfrak{S}_{q-c}]} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(p-c, q-c)_i^{\oplus \binom{p}{c}\binom{q}{c}c!} \\
&\cong \bigoplus_{c=0}^{\min(p,q)} \mathcal{C}_{\mathcal{O}^{\circlearrowright}}(p-c, q-c)_i^{\oplus \binom{p}{c}\binom{q}{c}c!} \\
&\cong \mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}(p,q)_i
\end{aligned}$$

by Lemma 2.1. □

2.4. **The stable Albanese homology of IA_n.** Now we will state the main theorem of this paper.

Theorem 2.5 ([7, Conjecture 6.2]). *For $n \geq 3i$, we have an isomorphism of $\mathrm{GL}(n,\mathbb{Q})$-representations*

$$F_i : H_i^A(\mathrm{IA}_n, \mathbb{Q}) \to W_i.$$

Remark 2.6. The range $n \geq 3i$ in the statement of Theorem 2.5 might be improved. However, the multiplicity of an irreducible algebraic $\mathrm{GL}(n,\mathbb{Q})$-representation which appears as a component of W_i is stable for $n \geq 3i$ (see [7, Section 6.1] for details). In the sense of Church–Farb [3], the Albanese homology of IA_n is representation stable in $n \geq 3i$.

Since the first Albanese homology of IA_n is isomorphic to the first rational homology of IA_n, the degree 1 case follows from Cohen–Pakianathan, Farb and Kawazumi [8]. The degree 2 case is proven by Pettet [15] and the degree 3 case is by the author [7]. Moreover, in [7, Theorem 6.1], the author proved the following.

Theorem 2.7 ([7, Theorem 6.1])**.** *We have a morphism of graded* $\mathrm{GL}(n,\mathbb{Q})$*-representations*

$$F_*: H_*(U_1,\mathbb{Q}) \to S^*(U_*)$$

such that $F_*(H_*^A(\mathrm{IA}_n,\mathbb{Q})) \supset W_*$ *for* $n \geq 3*$.

Remark 2.8. In order to detect W_* in the image of F_*, the author used abelian cycles in $H_*(\mathrm{IA}_n,\mathbb{Q})$. Therefore, it follows from Theorem 2.5 that the stable Albanese homology of IA_n is generated by abelian cycles.

In the appendix of [13], the author proved the statement of Theorem 2.5 for sufficiently large n with respect to the homological degree.

Proof of Theorem 2.5. Let $\mathcal{G}(p,q)$ denote the graded quotient vector space of $(2,1)$-valent marked directed oriented graphs with p incoming legs and q outgoing legs modulo the directed IH-relation, which is introduced in [13, Section 6]. In the proof of [13, Proposition 6.3], Lindell proved that $\mathcal{G}(p,q)$ is isomorphic to $(\mathcal{P}\otimes\det)(p,q)$, and constructed a surjective map

$$\alpha: \mathcal{G}(p,q) \twoheadrightarrow [R_{\mathrm{pres}} \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$$

by using the element $\omega(1)$ in (2.2.1), where R_{pres} is the graded-commutative ring that is constructed in [13, Definition 6.1]. By [13, Remark 6.2], we have an isomorphism of $\mathrm{GL}(n,\mathbb{Q})$-representations

$$R_{\mathrm{pres}} \cong H^*(U_1,\mathbb{Q})/\langle R_2\rangle,$$

where $\langle R_2\rangle$ denotes the two-sided ideal of $H^*(U_1,\mathbb{Q})$ generated by

$$R_2 = \ker\left(H^2(U_1,\mathbb{Q}) \cong \bigwedge^2 H^1(\mathrm{IA}_n,\mathbb{Q}) \xrightarrow{\cup} H^2(\mathrm{IA}_n,\mathbb{Q})\right).$$

Since the Albanese cohomology of IA_n is equal to the image of the cup product map $\bigwedge^* H^1(\mathrm{IA}_n,\mathbb{Q}) \xrightarrow{\cup} H^*(\mathrm{IA}_n,\mathbb{Q})$, we have a surjective morphism of graded $\mathrm{GL}(n,\mathbb{Q})$-representations

$$H^*(U_1,\mathbb{Q})/\langle R_2\rangle \twoheadrightarrow H_A^*(\mathrm{IA}_n,\mathbb{Q}). \tag{2.4.1}$$

Therefore, we have a surjective map

$$(\mathcal{P}\otimes\det)(p,q) \twoheadrightarrow [H_A^*(\mathrm{IA}_n,\mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}.$$

For $n \geq 3i$, by Lemma 2.4, we have

$$(\mathcal{P}\otimes\det)(2i,i)_i \cong \mathcal{C}_{\mathcal{P}_0^{\circlearrowleft}}(2i,i)_i \cong [(W_i)^* \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})}$$

and thus we have a surjective map

$$[(W_i)^* \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})} \twoheadrightarrow [H_A^i(\mathrm{IA}_n,\mathbb{Q}) \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})}.$$

By combining this with Theorem 2.7, for $n \geq 3i$ we have

$$[(W_i)^* \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})} \cong [H_A^i(\mathrm{IA}_n,\mathbb{Q}) \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})}.$$

Therefore, since $H_i^A(\mathrm{IA}_n,\mathbb{Q}) \subset H_i(U_1,\mathbb{Q}) \cong \bigwedge^i U_1 \subset H^{2i,i}$, we have $H_A^i(\mathrm{IA}_n,\mathbb{Q}) \cong (W_i)^*$, which completes the proof. $\square$

3. Several related conjectures

We will prove several conjectures related to the $\mathrm{GL}(n,\mathbb{Q})$-representation structure of the stable Albanese homology of IA_n, which are proposed in [7].

3.1. **The algebra structure of the Albanese cohomology of IA_n.** Here we study the algebra structure of the Albanese cohomology of IA_n. The cup product gives the algebra structure on the rational cohomology of IA_n and $\mathrm{IA}_n^{\mathrm{ab}}$, and the Albanese cohomology of IA_n is a subalgebra of $H^*(\mathrm{IA}_n,\mathbb{Q})$ generated by $H^1(\mathrm{IA}_n,\mathbb{Q})$. We refer the reader to [7, Section 8] for details.

The following theorem directly follows from the proof of Theorem 2.5.

Theorem 3.1 (Cf. [7, Conjecture 8.2])**.** *The Albanese cohomology algebra is stably quadratic in $n \geq 3*$. That is, the surjective $\mathrm{GL}(n,\mathbb{Q})$-equivariant morphism* (2.4.1) *of graded algebras*

$$H^*(U_1,\mathbb{Q})/\langle R_2\rangle \twoheadrightarrow H_A^*(\mathrm{IA}_n,\mathbb{Q})$$

is an isomorphism of $\mathrm{GL}(n,\mathbb{Q})$-representations for $n \geq 3*$.

3.2. **The coalgebra structure of the Albanese homology of IA_n.** Here we study the coalgebra structure of the Albanese homology of IA_n. We refer the reader to [7, Section 7] for details.

The rational homology of IA_n (resp. $\mathrm{IA}_n^{\mathrm{ab}}$) has a natural coalgebra structure, which is predual to the algebra structure of $H^*(\mathrm{IA}_n,\mathbb{Q})$ (resp. $H^*(\mathrm{IA}_n^{\mathrm{ab}},\mathbb{Q})$). The coalgebra structures of $H^*(\mathrm{IA}_n,\mathbb{Q})$ and $H^*(\mathrm{IA}_n^{\mathrm{ab}},\mathbb{Q})$ induce a coalgebra structure on the Albanese homology of IA_n. We also have a coalgebra structure on the graded-symmetric algebra $S^*(U_*)$ (see [7, Section 2.6] for details).

Let $F_* : H_*(U_1,\mathbb{Q}) \to S^*(U_*)$ be the morphism that appeared in Theorem 2.7. By [7, Proposition 7.1], the map F_* is a coalgebra morphism. Since we have

$$\mathrm{Prim}(S^*(U_*)) = U_* = \mathrm{Prim}(W_*),$$

where $\mathrm{Prim}(C)$ denotes the primitive part of a co-augmented coalgebra C, Theorem 2.5 implies the following.

Corollary 3.2 ([7, Conjecture 7.2])**.** *The $\mathrm{GL}(n,\mathbb{Q})$-equivariant coalgebra morphism F_* restricts to a $\mathrm{GL}(n,\mathbb{Q})$-equivariant morphism*

$$F_* : \mathrm{Prim}(H_*^A(\mathrm{IA}_n,\mathbb{Q})) \to U_*,$$

which is an isomorphism for $n \geq 3*$.

3.3. **The stable Albanese homology of IO_n.** Let $\mathrm{Out}(F_n)$ denote the outer automorphism group of F_n. Define IO_n as the kernel of the surjective group homomorphism from $\mathrm{Out}(F_n)$ to $\mathrm{GL}(n,\mathbb{Z})$, that is, we have a short exact sequence of groups

$$1 \to \mathrm{IO}_n \to \mathrm{Out}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1.$$

The Albanese homology of IO_n is defined in a way similar to IA_n by

$$H_*^A(\mathrm{IO}_n,\mathbb{Q}) = \mathrm{im}(H_*(\mathrm{IO}_n,\mathbb{Q}) \to H_*(\mathrm{IO}_n^{\mathrm{ab}},\mathbb{Q})).$$

The conjectural structure W_*^O of the stable Albanese homology of IO_n is constructed as the traceless part of the graded-symmetric algebra $S^*(U_*^O)$ of the graded

$\mathrm{GL}(n,\mathbb{Q})$-representation

$$U^O_* = \bigoplus_{i\geq 1} U^O_i, \quad \begin{cases} U^O_1 = \mathrm{Hom}(H, \bigwedge^2 H)/H \cong V_{1^2,1} & i=1 \\ U^O_i = U_i & i \geq 2. \end{cases}$$

We refer the reader to [7, Section 9] for details.

By using Theorem 2.5 and several results in [7], we will determine the stable Albanese homology of IO_n. The cases of degree $1, 2$ and 3 are proven by [8], [15] and [7], respectively.

Theorem 3.3 (Cf. [7, Conjecture 9.7]). *We have an isomorphism of* $\mathrm{GL}(n,\mathbb{Q})$-*representations*

$$H^A_i(\mathrm{IO}_n, \mathbb{Q}) \cong W^O_i$$

for $n \geq 3i$.

Proof. If we disregard the stable range, then the statement follows directly from Theorem 2.5 and [7, Proposition 9.11]. However, we will recall the argument and take care of the stable range.

We have isomorphisms of $\mathrm{GL}(n,\mathbb{Q})$-representations

$$H^A_i(\mathrm{IA}_n, \mathbb{Q}) \cong H^A_i(\mathrm{IO}_n, \mathbb{Q}) \oplus (H^A_{i-1}(\mathrm{IO}_n, \mathbb{Q}) \otimes H)$$

for $n \geq 2$ by [7, Proposition 9.8], and

$$W_i \cong W^O_i \oplus (W^O_{i-1} \otimes H)$$

for $n \geq 3i$ by [7, Lemma 9.9]. Since we have $H^A_1(\mathrm{IO}_n, \mathbb{Q}) \cong W^O_1$, the statement follows by induction on i. □

In the case of IA_n, the polynomiality of the dimension of the stable Albanese homology was known in [2]. By Theorem 2.5 and an irreducible decomposition of W_*, it is possible to compute the exact polynomial. It seems natural to expect the same thing holds for IO_n, but the author has not found any literature about the polynomiality of the stable Albanese homology of IO_n. By Theorem 3.3, we obtain the polynomiality in the case of IO_n as well.

Corollary 3.4 (Cf. [7, Conjecture 9.6]). *There is a polynomial* $P^O_i(T)$ *of degree* $3i$ *such that we have* $\dim_{\mathbb{Q}}(H^A_i(\mathrm{IO}_n, \mathbb{Q})) = P^O_i(n)$ *for* $n \geq 3i$.

3.4. **Relation between the stable Albanese cohomology of IA_n and the stable twisted cohomology of $\mathrm{Aut}(F_n)$.** Here we will study the relation between the stable Albanese cohomology of IA_n and the stable cohomology of $\mathrm{Aut}(F_n)$ with coefficients in $H^{p,q}$.

The stability of the twisted homology of $\mathrm{Aut}(F_n)$ was shown by Randal-Williams–Wahl [16]. We will use the recent improvement of the stable range in [14].

Theorem 3.5 (Miller–Patzt–Petersen–Randal-Williams [14, Theorem 1.2]). *For any bipartition* (λ, μ), *the homology group* $H_i(\mathrm{Aut}(F_n), V_{\lambda,\mu})$ *stabilizes for* $n \geq 3i + 4$.

By using this improved stable range, we will prove the following.

Theorem 3.6 (Cf. [7, Conjectures 12.5 and 12.6])**.** *We have an isomorphism of* $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$*-modules*

$$H^i(\mathrm{Aut}(F_n), H^{p,q}) \cong [H_A^i(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$$

for $n \geq \min(\max(3i+4, p+q), 2i+p+q+3)$*. The statement also holds for coefficients in* $V_{\lambda,\mu}$ *with* $|\lambda| = p, |\mu| = q$ *instead of* $H^{p,q}$*.*

Proof. By [12, Theorem A], for $n \geq 2i+p+q+3$, we have an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$H^i(\mathrm{Aut}(F_n), H^{p,q}) \cong (\mathcal{P} \otimes \det)(p,q)_i. \tag{3.4.1}$$

Since $\mathcal{P} \otimes \det$ is independent of n, it follows from Theorem 3.5 that the isomorphism (3.4.1) holds for $n \geq \max(3i+4, p+q)$. On the other hand, by Theorem 2.5 and Lemma 2.4, we have isomorphisms of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$[H_A^i(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Q})} \cong [(W_i)^* \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Q})} \cong \mathcal{C}_{\mathcal{P}_0^\circlearrowright}(p,q)_i \cong (\mathcal{P} \otimes \det)(p,q)_i$$

for $n \geq \max(3i, p+q)$, which completes the proof. $\square$

Remark 3.7. For the non-vanishing case $p - q = i$, we have

$$\min(\max(3i+4, p+q), 2i+p+q+3) = \begin{cases} \max(3i+4, p+q) & q \geq 1, \\ 3i+3 & q = 0. \end{cases}$$

Moreover, in [6], Habiro and the author conjectured that the isomorphism in Theorem 3.6 is realized by the morphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$i^* : H^*(\mathrm{Aut}(F_n), H^{p,q}) \to H^*(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})} = [H^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$$

that is induced by the inclusion map $i : \mathrm{IA}_n \hookrightarrow \mathrm{Aut}(F_n)$. To prove this conjecture, we will construct a wheeled PROP corresponding to $[H_A^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$ in a way similar to the wheeled PROP $\mathcal{H}$ corresponding to $H^*(\mathrm{Aut}(F_n), H^{p,q})$ that was introduced in [9].

In what follows, let $H(n)$ denote $H = H_1(F_n, \mathbb{Q})$ to make the dependence on n explicit. The inclusion map $F_n \hookrightarrow F_{n+1}$ that maps the generator $x_i \in F_n$ to $x_i \in F_{n+1}$ for $1 \leq i \leq n$ induces an inclusion map $\mathrm{IA}_n \hookrightarrow \mathrm{IA}_{n+1}$. We have the projection $H(n+1) \twoheadrightarrow H(n)$ that sends the basis element $e_i \in H(n+1)$ to $e_i \in H(n)$ for $1 \leq i \leq n$ and $e_{n+1} \in H(n+1)$ to 0. We also have the projection $H(n+1)^* \twoheadrightarrow H(n)^*$ that sends the dual basis element $e_i^* \in H(n+1)^*$ to $e_i^* \in H(n)^*$ and $e_{n+1}^* \in H(n+1)^*$ to 0. These three maps induce the linear map

$$[H_A^*(\mathrm{IA}_{n+1}, \mathbb{Q}) \otimes H^{p,q}(n+1)]^{\mathrm{GL}(n+1,\mathbb{Z})} \to [H_A^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}(n)]^{\mathrm{GL}(n,\mathbb{Z})}$$

and the stable $\mathrm{GL}(n,\mathbb{Z})$-invariant part of the twisted Albanese cohomology is the limit $\varprojlim_n [H_A^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}(n)]^{\mathrm{GL}(n,\mathbb{Z})}$. By Theorem 2.5, we have

$$\varprojlim_n [H_A^i(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}(n)]^{\mathrm{GL}(n,\mathbb{Z})} \cong [H_A^i(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}(n)]^{\mathrm{GL}(n,\mathbb{Z})}$$

for $n \geq \max(3i, p+q)$.

Define a wheeled PROP $\mathcal{A}$ as follows. The morphisms are the graded $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$\mathcal{A}(p,q) = \varprojlim_n [H_A^*(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}(n)]^{\mathrm{GL}(n,\mathbb{Z})},$$

where the actions of $\mathfrak{S}_p$ and $\mathfrak{S}_q$ are given by place permutations of the copies of H and H^*. The horizontal composition

$$\mathcal{A}(p_1, q_1) \otimes \mathcal{A}(p_2, q_2) \to \mathcal{A}(p_1 + p_2, q_1 + q_2)$$

is induced by the cup product map for the cohomology of IA_n, and the vertical composition

$$\mathcal{A}(q, r) \otimes \mathcal{A}(p, q) \to \mathcal{A}(p, r)$$

is induced by the composition of the cup product map for the cohomology of IA_n and the map $H^{q,r} \otimes H^{p,q} \cong \mathrm{Hom}(H^{\otimes r}, H^{\otimes q}) \otimes \mathrm{Hom}(H^{\otimes q}, H^{\otimes p}) \to H^{p,r}$ defined by the composition of linear maps. The contraction map

$$\xi^i_j : \mathcal{A}(p, q) \to \mathcal{A}(p-1, q-1)$$

is induced by the contraction map $c_{i,j} : H^{p,q} \to H^{p-1,q-1}$. We refer the reader to [9, Definition 6.1, Proposition 6.2].

Theorem 3.8 (Cf. [6, Conjecture 7.2])**.** *The inclusion map $i : \mathrm{IA}_n \hookrightarrow \mathrm{Aut}(F_n)$ induces an isomorphism of wheeled PROPs*

$$i^* : \mathcal{H} \xrightarrow{\cong} \mathcal{A},$$

and thus induces an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$i^* : H^*(\mathrm{Aut}(F_n), H^{p,q}) \to [H^*_A(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$$

for $n \geq \min(\max(3 * +4, p+q), 2 * +p+q+3)$.

Proof. Since we have observed the stability of $H^*(\mathrm{Aut}(F_n), H^{p,q})$ and $[H^*_A(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$, by Theorem 3.6, it suffices to check that the inclusion map $i : \mathrm{IA}_n \hookrightarrow \mathrm{Aut}(F_n)$ induces a surjective morphism of wheeled PROPs $i^* : \mathcal{H} \twoheadrightarrow \mathcal{A}$.

The Johnson homomorphism for $\mathrm{Aut}(F_n)$ is the group homomorphism

$$\tau : \mathrm{IA}_n \to \mathrm{Hom}(H, \bigwedge^2 H)$$

defined by $\tau(f)([x]) = [f(x)x^{-1}]$ for $f \in \mathrm{IA}_n$ and $x \in F_n$, where $[x]$ denotes the image under the projection $F_n \twoheadrightarrow H$ and $[f(x)x^{-1}]$ the image under the projection $[F_n, F_n] \twoheadrightarrow \bigwedge^2 H$. By abuse of the notation, let $\tau \in H^1(\mathrm{IA}_n, H^{2,1})$ denote the cohomology class that is represented by the composition map of the Johnson homomorphism and the inclusion $\mathrm{Hom}(H, \bigwedge^2 H) \hookrightarrow H^{2,1}$. We can check that the cohomology class τ is non-trivial for $n \geq 3$ by using the dual pairing

$$\langle , \rangle : H^1(\mathrm{IA}_n, H^{2,1}) \otimes H_1(\mathrm{IA}_n, H^{1,2}) \to \mathbb{Q}$$

defined by $\langle [f], [g \otimes x] \rangle = f(g)(x)$ for $f : \mathrm{IA}_n \to H^{2,1} \cong \mathrm{Hom}_{\mathbb{Q}}(H^{1,2}, \mathbb{Q})$ and $g \in \mathrm{IA}_n$, $x \in H^{1,2}$, where $[f]$ (resp. $[g \otimes x]$) denotes the cohomology class (resp. homology class).

Kawazumi [8] extended the Johnson homomorphism to a cohomology class $h_1 \in H^1(\mathrm{Aut}(F_n), H^{2,1})$, which is non-trivial. Therefore, we have

$$i^*(h_1) = \tau \in H^1(\mathrm{IA}_n, H^{2,1})^{\mathrm{GL}(n,\mathbb{Z})}. \tag{3.4.2}$$

By [9] and [12], $\mathcal{H}$ is generated by h_1 as a wheeled PROP, which means that for any $p, q \geq 0$, the hom-space $\mathcal{H}(p, q)$ is a $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-module spanned by morphisms obtained from some copies of h_1 and $\mathrm{id}_H \in H^{1,1}$ by the horizontal composition, the vertical composition and the contraction map of $\mathcal{H}$. Since the map i^* sends the generator h_1 of $\mathcal{H}$ to $\tau \in \mathcal{A}(2,1)$, in order to prove that i^* is surjective, we have only to prove that $\mathcal{A}$ is generated by τ as a wheeled PROP.

The Albanese cohomology of IA_n is generated by the first cohomology as an algebra. Therefore, $\mathcal{A}$ is generated by the degree 1 part $[H^1_A(\mathrm{IA}_n, \mathbb{Q}) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})} = H^1(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})}$. For $p, q \geq 0$ with $p-q \neq 1$, we have $H^1(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})} = 0$ for $n \geq \max(3, p+q)$. For $p = 1, q = 0$, we have

$$H^1(\mathrm{IA}_n, H^{1,0})^{\mathrm{GL}(n,\mathbb{Z})} = \mathbb{Q}\xi^1_1(\tau)$$

for $n \geq 3$. For $p = 2, q = 1$, we have

$$H^1(\mathrm{IA}_n, H^{2,1})^{\mathrm{GL}(n,\mathbb{Z})} = \mathbb{Q}\{\tau, \xi^1_1(\tau) \cup \mathrm{id}_H, \mathrm{id}_H \cup \xi^1_1(\tau)\}$$

for $n \geq 3$. For $p \geq 3$, $q = p-1$, the horizontal composition map

$$H^1(\mathrm{IA}_n, H^{2,1})^{\mathrm{GL}(n,\mathbb{Z})} \otimes (H^{p-2,p-2})^{\mathrm{GL}(n,\mathbb{Z})} \to H^1(\mathrm{IA}_n, H^{p,p-1})^{\mathrm{GL}(n,\mathbb{Z})}$$

is surjective for $n \geq 2p-1$. Therefore, $\mathcal{A}$ is generated by τ as a wheeled PROP, which completes the proof. □

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