

Assessing frustration in real-world signed networks: traditional or relaxed balance?

Anna Gallo,^{1,2,*} Diego Garlaschelli,^{1,2,3} and Tiziano Squartini^{1,4,5}

¹IMT School for Advanced Studies, Piazza San Francesco 19, 55100 Lucca (Italy)

²INdAM-GNAMPA Istituto Nazionale di Alta Matematica ‘Francesco Severi’, P.le Aldo Moro 5, 00185 Rome (Italy)

³Lorentz Institute for Theoretical Physics, University of Leiden,
Niels Bohrweg 2, 2333 CA Leiden (The Netherlands)

⁴Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa (Italy)

⁵INdAM-GNAMPA Istituto Nazionale di Alta Matematica ‘Francesco Severi’, P.le Aldo Moro 5, 00185 Rome (Italy)

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According to traditional balance theory, individual social actors avoid establishing triads with an odd number of negative links. Generalising, mesoscopic balance is realised when the nodes of a signed graph can be grouped into positively connected subsets, mutually connected by negative links. If this prescription is interpreted rigidly without allowing for statistical noise, it quickly dismisses most real graphs as frustrated. As an alternative, a relaxed, yet qualitative, definition of balance has been advanced. After rephrasing both variants in statistically testable terms, we propose an inference scheme to unambiguously assess if a signed graph is traditionally or relaxedly balanced.

Introduction. The interest towards signed networks dates back to the *balance theory* (BT), firstly proposed by Heider as a theory of behaviour [1]. The choice of adopting signed graphs to model it has, then, led Cartwright and Harary [2] to introduce its structural version [2–6]. As the name suggests, BT deals with the concept of balance: a complete, signed graph is said to be balanced if *all triads* have an even number of negative edges, i.e. either zero (in this case, the three edges are all positive) or two. The so-called *structure theorem* states that a complete, signed graph is balanced if and only if its set of nodes can be partitioned into $k = 2$, disjoint subsets whose intra-modular links are all positive and whose inter-modular links are all negative. Cartwright and Harary extended the definition of balance to incomplete graphs [2] by including cycles of length larger than three: a network is, now, said to be balanced if *all cycles* have an even number of negative edges (although the points of each subset are no longer required to be connected). Taken together, the criteria above define the so-called *strong balance theory* (SBT). Such a framework has been further extended by Davis [7], who introduced the concept of *k-balanced* networks, according to which signed graphs are balanced if their set of nodes can be partitioned into $k \geq 2$, disjoint subsets with positive, intra-modular links and negative, inter-modular links. This generalised definition of balance has led to the formulation of the *weak balance theory* (WBT), according to which triads whose edges are all negative are balanced as well, since each node can be thought of as a group on its own. From a mesoscopic perspective, however, both versions of the BT require the presence of positive blocks along the main diagonal of the adjacency matrix ($k = 2$, according to the strong variant; $k > 2$, according to the weak variant) and of negative, off-diagonal blocks. Taken together, the SBT

and the WBT define what will be called *traditional balance theory* (TBT): hence, *k*-balanced networks are traditionally balanced.

Setting up the formalism. Each edge of a signed graph can be *positive*, *negative* or *missing*: as we will focus on binary, undirected, signed networks, a generic entry of the signed adjacency matrix $\mathbf{A}$ will be assumed to read $a_{ij} = -1, 0, +1$, with $a_{ij} = a_{ji}$, $\forall i < j$. To ease mathematical manipulations, let us employ Iverson’s brackets (a notation ensuring all quantities of interest to be non-negative - see Appendix A) and define the quantities $a_{ij}^- = [a_{ij} = -1]$, $a_{ij}^0 = [a_{ij} = 0]$, $a_{ij}^+ = [a_{ij} = +1]$: the new variables are mutually exclusive, sum to 1 and induce the two matrices $\mathbf{A}^+$ and $\mathbf{A}^-$, satisfying $\mathbf{A} = \mathbf{A}^+ - \mathbf{A}^-$ and $|\mathbf{A}| = \mathbf{A}^+ + \mathbf{A}^-$. The number of positive and negative links is defined as $L^+ = \sum_{i=1}^N \sum_{j(>i)} a_{ij}^+$ and $L^- = \sum_{i=1}^N \sum_{j(>i)} a_{ij}^-$, respectively.

Traditional Balance Theory. The top-down formulation of the TBT leads quite naturally to the definition of a score function for quantifying the ‘degree of compatibility’ of a given partition with the TBT itself. It is named *frustration*¹ and reads

$$\begin{aligned} F(\boldsymbol{\sigma}) &= \sum_{i=1}^N \sum_{j(>i)} a_{ij}^- \delta_{\sigma_i, \sigma_j} + \sum_{i=1}^N \sum_{j(>i)} a_{ij}^+ (1 - \delta_{\sigma_i, \sigma_j}) \\ &= L_{\bullet}^- + L^+ - L_{\bullet}^+ \\ &= L_{\bullet}^- + L_{\circ}^+ \end{aligned} \quad (1)$$

where $\boldsymbol{\sigma} \equiv \{\sigma_i\}$ stands for a vector of labels characterising a generic partition and $\delta_{\sigma_i, \sigma_j}$ is the Kronecker delta

* anna.gallo@imtlucca.it

¹More formally, *line index of imbalance* [8].

(i.e. $\delta_{\sigma_i, \sigma_j} = 1$ if $\sigma_i = \sigma_j$ and 0 otherwise). In words, $F(\sigma)$ counts the amount of misplaced links according to the TBT, i.e. the number of negative links within modules plus the number of positive links between modules. The simplest, operative criterion for singling out a k -balanced partition is based upon the following theorem.

Theorem I. $F(\sigma) = 0 \iff$ the partition σ is k -balanced.

Proof. Sufficiency condition: $F(\sigma) = 0 \implies$ the partition σ is k -balanced. Since $L_{\circ}^+ \geq 0$ and $L_{\bullet}^- \geq 0$, $F(\sigma) = 0$ implies that $L_{\circ}^+ = 0$ and $L_{\bullet}^- = 0$; hence, the definition of k -balanced partition is satisfied. Necessity condition: the partition σ is k -balanced $\implies F(\sigma) = 0$. Since a k -balanced partition is defined by the presence of a clustering with k subsets, no negative links within modules and no positive links between modules, $L_{\bullet}^- = 0$ and $L_{\circ}^+ = 0$; hence, $F(\sigma) = 0$. $\square$

In words, the bare, numerical value $F(\sigma)$ can be thought of as acting in a threshold-like fashion, classifying the configurations characterised by $F(\sigma) = 0$ as balanced and the configurations characterised by $F(\sigma) > 0$ as frustrated, according to the TBT. The criterion embodied by the F -test is, however, too strict for real-world networks, which are hardly (if ever) found to obey it: as table I shows, in fact, none of the listed configurations passes it.

Softening frustration. As noticed by Doreian and Mrvar [9], the block-structure defining the TBT is overly restrictive, dooming the vast majority of real-world, signed networks to be quickly dismissed as frustrated: in order to overcome what was perceived as a major limitation of the TBT, they proposed to replace $F(\sigma)$ with its softened variant

$$G(\sigma|\alpha) = \alpha L_{\bullet}^- + (1 - \alpha) L_{\circ}^+, \quad (2)$$

allowing *i)* the misplaced, positive links to be weighted more upon choosing $0 \leq \alpha < 1/2$; *ii)* the misplaced, negative links to be weighted more upon choosing $1/2 < \alpha \leq 1$. Even ignoring the ambiguity due to the lack of a principled way for selecting α (the so-called ‘ α problem’ in [8]), the criterion embodied by the G -test is still too strict: as table I shows, in fact, none of the listed configurations passes it either. This is rigorously stated by the following theorem, whose proof is immediate.

Theorem II. If $0 < \alpha < 1$, $F(\sigma) = 0 \iff G(\sigma|\alpha) = 0$, i.e. the partition σ is k -balanced.

Notice that the values $\alpha = 0$ and $\alpha = 1$ would, respectively, lead to the trivially balanced partition characterised by a single community gathering all nodes together and N single-node communities (or singletons).

	$F(\sigma)$	$G(\sigma \alpha)$	
		$\alpha = 0.2$	$\alpha = 0.8$
Fraternity [10]	1	0.2	0.4
N.G.H. Tribes [10]	2	1.4	0.4
Slovenian Parliament [11]	2	0.4	0.8
Monastery [10]	5	2.4	1.8
US Senate [10]	247	166.8	56
CoW, 1946-49 [12]	12	3.8	5.8
CoW, 1950-53 [12]	11	5.6	5.4
CoW, 1954-57 [12]	27	7	12.2
CoW, 1958-61 [12]	25	6.4	14.6
EGFR [10]	189	51.2	46.8
Macrophage [10]	316	91.4	77.2
Bitcoin Alpha [13]	1399	337.9	585.6
Bitcoin OTC [13]	3259	540.4	800.4

TABLE I: Empirical amount of frustration, detected by searching for the partition minimising $F(\sigma)$, that characterises the listed, real-world networks: according to the F -test, none of them turns out to satisfy the TBT. The same result holds true even when employing the generalised definition of the frustration index (here, implemented by posing $\alpha = 0.2$ and $\alpha = 0.8$).

Relaxed Balance Theory. In the light of the previous results, the second attempt pursued by Doreian and Mrvar to overcome the perceived limitations of the TBT was more radical, as they proposed to relax it by allowing for the presence of positive, off-diagonal blocks and negative, diagonal blocks - a generalisation that has gained the name of *relaxed balance theory* (RBT) [9]. Such a formulation of the RBT, however, lacks a proper mathematisation, as a score function such as $F(\sigma)$, or $G(\sigma|\alpha)$, cannot be easily individuated. Besides, it is affected by the problem highlighted in [14]: ‘[...] if the number of clusters is left unspecified a priori, the best partition is the singletons partition (i.e. each node in its own cluster) [...]’.

Statistical Balance Theory. Re-casting the theory of balance within a statistical framework solves all the aforementioned problems at once, allowing us to define an inference scheme to unambiguously assess if a signed graph is traditionally or relaxedly balanced - hence, overcoming the limitations of the F -based and G -based tests while providing a proper mathematisation of the RBT.

In order to turn the TBT into a statistical theory of balance, let us suppose the presence of a probabilistic model behind the appearance of any, signed configuration: the TBT can be, then, rephrased by posing $p_{rr}^- = 0$, $r = 1 \dots k$ and $p_{rs}^+ = 0$, $\forall r < s$. The starting point of our approach is that of softening these positions, replacing them with the milder relationships $\text{sgn}[p_{rr}^+ - p_{rr}^-] = +1$, $r = 1 \dots k$ and $\text{sgn}[p_{rs}^+ - p_{rs}^-] = -1$, $\forall r < s$ which amount at requiring that $p_{rr}^+ > p_{rr}^-$, $r = 1 \dots k$ and $p_{rs}^+ < p_{rs}^-$, $\forall r < s$. A configuration satisfying these relationships will be claimed to obey the statistical variant of the TBT:

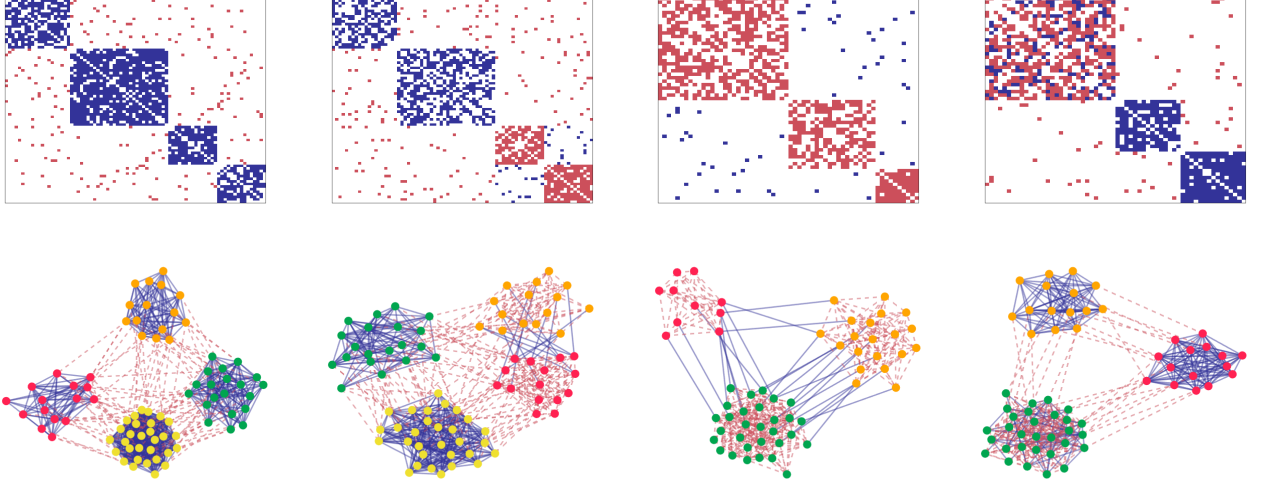


FIG. 1: Consistency checks on four, synthetic configurations (positive links are coloured in blue; negative links are coloured in red): minimising BIC always leads to recover the planted partition, be it homogeneous, balanced according to the TBT (first panel); homogeneous, balanced according to the RBT (second and third panel); heterogeneous, balanced according to the statistical variant of the RBT (fourth panel).

specifically, its strong variant if $k = 2$ and its weak variant if $k > 2$; otherwise (because $p_{rr}^+ \leq p_{rr}^-$ for some, diagonal blocks or $p_{rs}^+ \geq p_{rs}^-$ for some, off-diagonal blocks), it will be claimed to obey the statistical variant of the RBT. Additionally, we define a partition *homogeneous* if either $p_{rs}^+ = 0$ or $p_{rs}^- = 0$, $\forall r \leq s$; otherwise, it will be defined *heterogeneous* (see Appendix B).

Tuning the aforementioned parameters on a given, signed network requires a generative model to be specified: here, we will adopt the Signed Stochastic Block Model (SSBM), defined by the likelihood function

$$\mathcal{L}_{\text{SSBM}} = \prod_{r=1}^k (p_{rr}^+)^{L_{rr}^+} (p_{rr}^-)^{L_{rr}^-} (1 - p_{rr}^+ - p_{rr}^-)^{\binom{N_r}{2} - L_{rr}^+ - L_{rr}^-} \prod_{r=1}^k \prod_{s(>r)} (p_{rs}^+)^{L_{rs}^+} (p_{rs}^-)^{L_{rs}^-} (1 - p_{rs}^+ - p_{rs}^-)^{N_r N_s - L_{rs}^+ - L_{rs}^-} \quad (3)$$

where N_r is the number of nodes constituting block r , L_{rr}^+ (L_{rr}^-) is the number of positive (negative) links within block r , L_{rs}^+ (L_{rs}^-) is the number of positive (negative) links between blocks r and s , $\forall r < s$ and the probability coefficients read $p_{rr}^+ = 2L_{rr}^+/N_r(N_r - 1)$, $p_{rr}^- = 2L_{rr}^-/N_r(N_r - 1)$, $r = 1 \dots k$ and $p_{rs}^+ = L_{rs}^+/N_r N_s$, $p_{rs}^- = L_{rs}^-/N_r N_s$, $\forall r < s$. As maximising the bare likelihood is a recipe known to be affected by a number of limitations [8], we have opted for the minimisation of

$$\text{BIC} = \kappa_{\text{SSBM}} \ln n - 2 \ln \mathcal{L}_{\text{SSBM}}, \quad (4)$$

named *Bayesian Information Criterion* and embodying a trade-off between parsimony (accounted for by the first

addendum, with κ_{SSBM} being the number of parameters of the model² and $n = N(N - 1)/2$ proxying the system dimensions) and accuracy (accounted for by the second addendum, i.e. the log-likelihood term).

Results. First, let us test our prescription on a number of synthetic configurations. As fig. 1 shows, BIC minimisation always leads to recover the planted partition, irrespectively from the values of the sets of coefficients $\{p_{rr}^+\}$, $\{p_{rr}^-\}$, $\{p_{rs}^+\}_{r<s}$, $\{p_{rs}^-\}_{r<s}$, i.e. be it a homogeneous partition balanced according to the weak variant of the TBT (more precisely, a 4-balanced partition); two, homogeneous partitions, balanced according to the RBT (e.g. the third adjacency matrix is defined by $p_{11}^+ = p_{22}^+ = p_{33}^+ = 0$ and $p_{12}^- = p_{13}^- = p_{23}^- = 0$); an heterogeneous partition, balanced according to the statistical variant of the RBT (i.e. the fourth adjacency matrix, defined by $p_{11}^+ < p_{11}^-$, $p_{22}^+ > p_{22}^-$, $p_{33}^+ > p_{33}^-$ and $p_{12}^+ < p_{12}^-$, $p_{13}^+ < p_{13}^-$, $p_{23}^+ < p_{23}^-$).

Second, let us compare our recipe with that prescribing to minimise $F(\sigma)$. As fig. 2 shows, implementing the latter does not lead to recover the planted partition (in this case, a homogeneous one, compatible with the RBT): instead, it leads to a traditionally balanced configuration where the planted, negative cliques have been fragmented into singletons. More in general, minimising $F(\sigma)$ can lead to a number of ambiguous situations, such

²To avoid confusion with the number of modules, k , characterising k -balanced networks, we have indicated the number of a model parameters as κ : naturally, $\kappa_{\text{SSBM}} = k(k + 1)$ since we need to estimate two parameters per module.

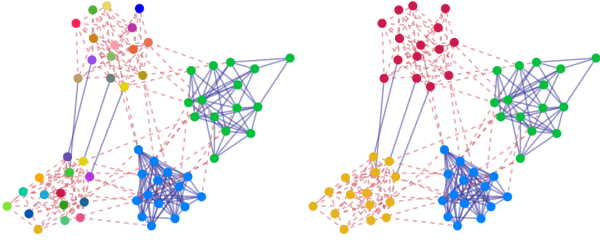


FIG. 2: Partitions recovered upon minimising $F(\sigma)$ (left panel) and upon minimising BIC (right panel): while minimising $F(\sigma)$ leads to recover a partition that is compatible with the TBT even if there is none ‘by design’, minimising BIC leads to recover the homogeneous planted partition, compatible with the RBT. Positive links are coloured in blue; negative links are coloured in red.

as *i*) returning configurations that are neither traditionally nor relaxedly balanced; *ii*) returning more than one frustrated configuration (see Appendix B).

Let us, now, apply our recipe to a number of real-world, signed configurations, i.e. four snapshots of the ‘Correlates of Wars’ (CoW) dataset [12], providing a picture of the international, political relationships over the years 1946-1997 and consisting of 13 snapshots of 4 years each: a positive edge between any two countries indicates an alliance, a political agreement or the membership to the same governmental organisation; conversely, a negative edge indicates that the two countries are enemies, have a political disagreement or are part of different, governmental organisations. As fig. 3 shows, minimising BIC leads to recover partitions that obey the statistical variant of the RBT (a blue block is characterised by a majority of positive links; a red block is characterised by a majority of negative links); other real-world, signed configurations, instead, are found to obey the statistical variant of the TBT (see Appendix B). All such partitions are heterogeneous.

Discussion. The present paper proposes a statistical approach to the theory of balance, assuming that any real-world, signed configuration is the result of a generative process, probabilistic in nature. As some degree of statistical noise is expected to affect a network structure, the criterion adopted to deem if it obeys the TBT or the RBT can be re-cast in terms of the signs of the differences of the block-wise probabilities to observe a positive and a negative link, i.e. $p_{rs}^+ - p_{rs}^-$, $\forall r \leq s$. Estimating these coefficients by minimising BIC allows one to unambiguously assess which variant of the theory is obeyed, from a statistical perspective, by any, signed configuration.

On the contrary, minimising $F(\sigma)$ is practically equiv-

alent at carrying out a sort of one-sided test of hypothesis, allowing one to state if a given partition *does not obey* the TBT (as a matter of fact, practically always) but incapable of providing a univocal classification for a generic, signed configuration. Moreover, it ‘works’ even with configurations generated by the Signed Random Graph Model (SRGM), i.e. a model carrying no information about a network modular structure, hence *overfitting* (i.e. misinterpreting statistical noise as a genuine signal - see Appendix B).

Under this respect, maximising the *signed modularity* $Q(\sigma)$ is of no help, being defined as

$$\begin{aligned} Q(\sigma) &= \sum_{i=1}^N \sum_{j(>i)} [(a_{ij}^+ - p_{ij}^+) - (a_{ij}^- - p_{ij}^-)] \delta_{\sigma_i, \sigma_j} \\ &= -F(\sigma) + \langle F(\sigma) \rangle + L^+ - \langle L^+ \rangle \end{aligned} \quad (5)$$

with obvious meaning of the symbols (the addendum $L^+ - \langle L^+ \rangle$ is just an offset not depending on the specific partition and amounting at zero for any model reproducing the total number of positive links) [15]. In words, the signed modularity compares the empirical amount of frustration of a given, signed configuration with the one predicted by a properly-defined reference model: one may, thus, define a partition as *statistically balanced* if satisfying the relationship $F(\sigma) < \langle F(\sigma) \rangle$, i.e. $Q(\sigma) > 0$. Although reasonable, such a criterion does not differ (much) from the one embodied by the F -test: more formally, it can be proven that the relationship $L^+ \gg L^-$ (often, if not always, found to hold true for real-world, signed networks) favours the fragmentation of the negative cliques into singletons, hence leading to recover traditionally balanced configurations even when there is none ‘by design’ (see fig. 2 and Appendix C).

Existing works have completely overlooked the issue of harmonising the request of having balanced configurations with that of having modular configurations, in most of the cases verifying either the ‘degree of balance’ of modular structures or the ‘degree of modularity’ of balanced structures *a posteriori*. Ignoring the interplay between the signs and the density of connections, solely accounting for the information carried by the first ones, may, in fact, lead to ‘resolution errors’, i.e. *i*) splitting modules (even fully-connected ones) into finer regions; *ii*) misinterpreting adjacent modules, characterised by the same, dominant sign, as single, coarser regions.

BIC, instead, is sensitive to the ‘signed density’ of the modules ‘by design’, hence capable of spotting the presence of groups of nodes as well as attributing to each of them the sign of the majority of its constituting links: beside lying the basis of a more comprehensive theory of balance, grounded on probability theory, such a desirable feature reconciles the generally contradictory results one gets when combining a purely structure-based community detection with a purely sign-based one.

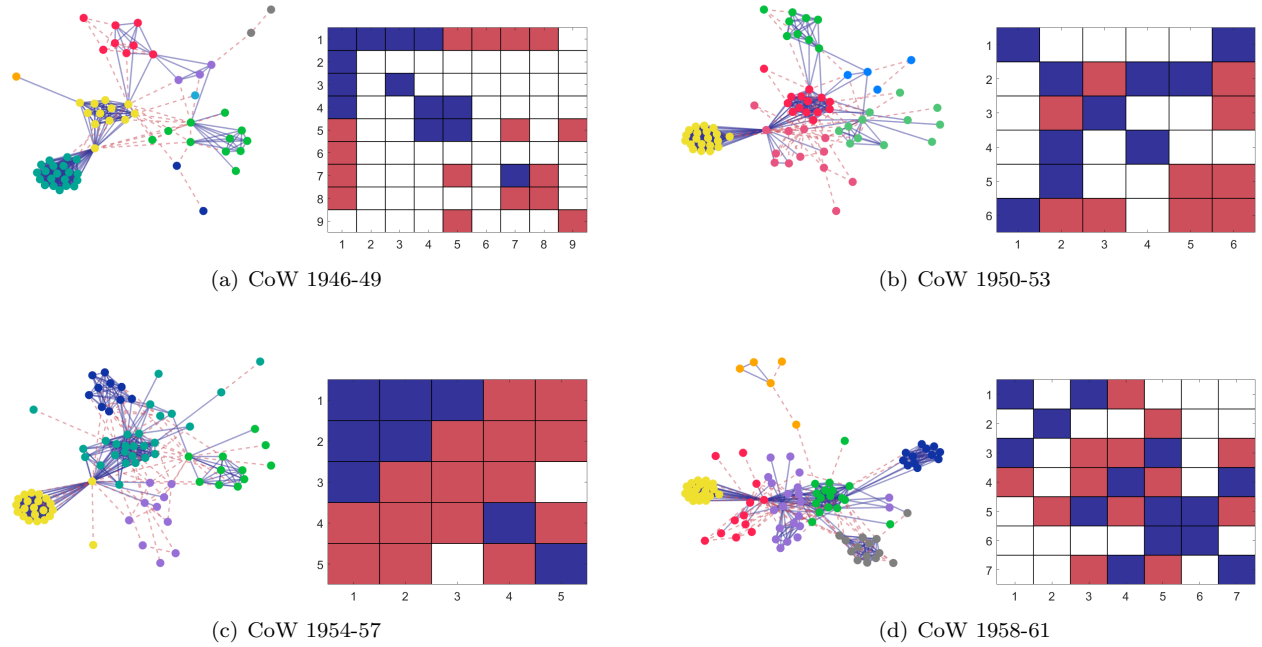


FIG. 3: Partitions recovered upon minimising BIC on four snapshots of the CoW dataset [12], providing a picture of the international, political relationships over the years 1946-1997. A generic block, indexed as rs , is coloured in blue if $L_{rs}^+ > L_{rs}^-$, in red if $L_{rs}^+ < L_{rs}^-$ and in white if $L_{rs}^+ = L_{rs}^-$: as blue blocks do not appear only on the diagonal and red blocks do not appear only off-diagonal, the considered configurations obey the statistical variant of the RBT.

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APPENDIX A.
REPRESENTING BINARY, UNDIRECTED, SIGNED NETWORKS

The three functions $a_{ij}^- = [a_{ij} = -1]$, $a_{ij}^0 = [a_{ij} = 0]$ and $a_{ij}^+ = [a_{ij} = +1]$ have been defined via the Iverson's brackets notation. Iverson's brackets work in a way that is reminiscent of the Heaviside step function, i.e. $\Theta[x] = [x > 0]$; in fact,

$$a_{ij}^- = [a_{ij} = -1] = \begin{cases} 1, & \text{if } a_{ij} = -1 \\ 0, & \text{if } a_{ij} = 0, +1 \end{cases} \quad (6)$$

(i.e. $a_{ij}^- = 1$ if $a_{ij} < 0$ and zero otherwise),

$$a_{ij}^0 = [a_{ij} = 0] = \begin{cases} 1, & \text{if } a_{ij} = 0 \\ 0, & \text{if } a_{ij} = -1, +1 \end{cases} \quad (7)$$

(i.e. $a_{ij}^0 = 1$ if $a_{ij} = 0$ and zero otherwise),

$$a_{ij}^+ = [a_{ij} = +1] = \begin{cases} 1, & \text{if } a_{ij} = +1 \\ 0, & \text{if } a_{ij} = -1, 0 \end{cases} \quad (8)$$

(i.e. $a_{ij}^+ = 1$ if $a_{ij} > 0$ and zero otherwise). These new variables are mutually exclusive, i.e. $\{a_{ij}^-, a_{ij}^0, a_{ij}^+\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and sum to 1, i.e. $a_{ij}^- + a_{ij}^0 + a_{ij}^+ = 1$. The matrices $\mathbf{A}^+ \equiv \{a_{ij}^+\}_{i,j=1}^N$ and $\mathbf{A}^- \equiv \{a_{ij}^-\}_{i,j=1}^N$ remain naturally defined, inducing the relationships $\mathbf{A} = \mathbf{A}^+ - \mathbf{A}^-$, i.e. $a_{ij} = a_{ij}^+ - a_{ij}^-$, $\forall i \neq j$ and $|\mathbf{A}| = \mathbf{A}^+ + \mathbf{A}^-$, i.e. $|a_{ij}| = a_{ij}^+ + a_{ij}^-$, $\forall i \neq j$.

APPENDIX B.
MINIMISATION OF THE BAYESIAN INFORMATION CRITERION

Signed Stochastic Block Model (SSBM) and Bayesian Information Criterion (BIC)

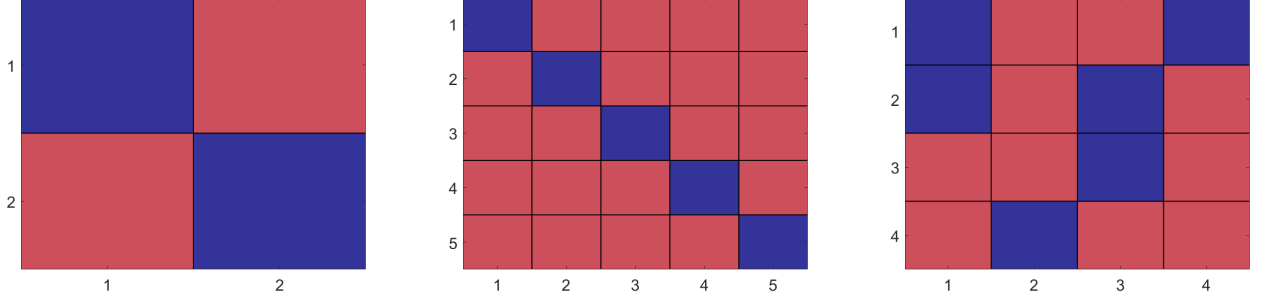


FIG. 4: Three, ideal partitions recoverable upon minimising BIC: one compatible with the strong balance theory ($k = 2$ with $p_{11}^+ > p_{11}^-$, $p_{22}^+ > p_{22}^-$ and $p_{12}^+ < p_{12}^-$ - left panel), one compatible with the weak balance theory ($k > 2$ with $p_{rr}^+ > p_{rr}^-$, $r = 1 \dots 5$, $p_{rs}^+ < p_{rs}^-$, $r, s = 1 \dots 5$, $\forall r < s$ - middle panel), one compatible with the relaxed balance theory ($p_{rr}^+ \leq p_{rr}^-$ for some, diagonal blocks and $p_{rs}^+ \geq p_{rs}^-$ for some, off-diagonal blocks - right panel).

Let us, first, recall the derivation of the SSBM. It is defined by the Hamiltonian

$$\begin{aligned}
 H(\mathbf{A}) &= \sum_{r \leq s} [\alpha_{rs} L_{rs}^+(\mathbf{A}) + \beta_{rs} L_{rs}^-(\mathbf{A})] \\
 &= \sum_{r \leq s} \left\{ \alpha_{rs} \left[\sum_{i=1}^N \sum_{j(>i)} \delta_{g_i r} \delta_{g_j s} a_{ij}^+ \right] + \beta_{rs} \left[\sum_{i=1}^N \sum_{j(>i)} \delta_{g_i r} \delta_{g_j s} a_{ij}^- \right] \right\} \\
 &= \sum_{i=1}^N \sum_{j(>i)} [\alpha_{g_i g_j} a_{ij}^+ + \beta_{g_i g_j} a_{ij}^-]
 \end{aligned} \tag{9}$$

leading to

$$\begin{aligned}
 Z &= \sum_{\mathbf{A} \in \mathbb{A}} e^{-H(\mathbf{A})} = \sum_{\mathbf{A} \in \mathbb{A}} \prod_{i=1}^N \prod_{j(>i)} e^{-[\alpha_{g_i g_j} a_{ij}^+ + \beta_{g_i g_j} a_{ij}^-]} = \prod_{i=1}^N \prod_{j(>i)} \sum_{a_{ij} = -1, 0, +1} e^{-[\alpha_{g_i g_j} a_{ij}^+ + \beta_{g_i g_j} a_{ij}^-]} \\
 &= \prod_{i=1}^N \prod_{j(>i)} [1 + e^{-\alpha_{g_i g_j}} + e^{-\beta_{g_i g_j}}].
 \end{aligned} \tag{10}$$

As a consequence,

$$\begin{aligned}
 P_{\text{SSBM}}(\mathbf{A}) &= \frac{e^{-H(\mathbf{A})}}{Z} = \frac{\prod_{i=1}^N \prod_{j(>i)} e^{-[\alpha_{g_i g_j} a_{ij}^+ + \beta_{g_i g_j} a_{ij}^-]}}{\prod_{i=1}^N \prod_{j(>i)} [1 + e^{-\alpha_{g_i g_j}} + e^{-\beta_{g_i g_j}}]} \equiv \prod_{i=1}^N \prod_{j(>i)} \frac{x_{g_i g_j}^{a_{ij}^+} y_{g_i g_j}^{a_{ij}^-}}{1 + x_{g_i g_j} + y_{g_i g_j}} \\
 &\equiv \prod_{i=1}^N \prod_{j(>i)} (p_{g_i g_j}^+)^{a_{ij}^+} (p_{g_i g_j}^0)^{a_{ij}^0} (p_{g_i g_j}^-)^{a_{ij}^-}
 \end{aligned} \tag{11}$$

having posed $e^{-\alpha_{g_i g_j}} \equiv x_{g_i g_j}$, $e^{-\beta_{g_i g_j}} \equiv y_{g_i g_j}$, $p_{ij}^+ \equiv x_{g_i g_j} / (1 + x_{g_i g_j} + y_{g_i g_j})$, $p_{ij}^- \equiv y_{g_i g_j} / (1 + x_{g_i g_j} + y_{g_i g_j})$, $p_{ij}^0 \equiv 1 / (1 + x_{g_i g_j} + y_{g_i g_j})$; let us notice that

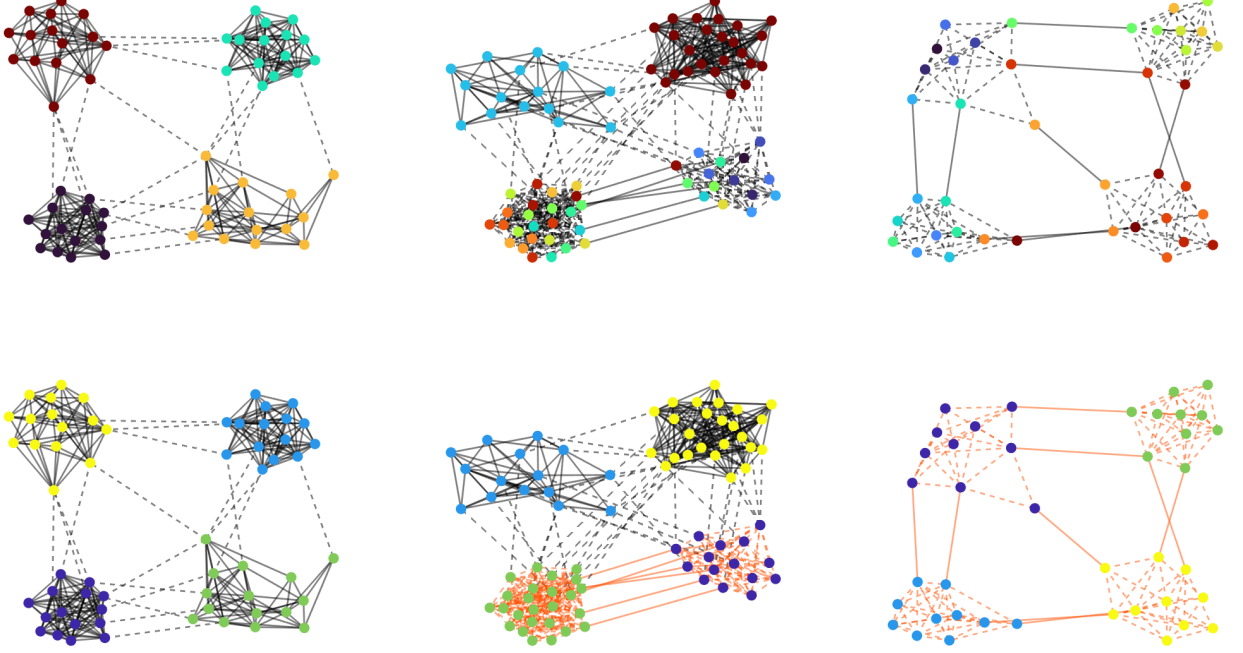


FIG. 5: Top panels: partitions recovered upon minimising $F(\sigma)$. Bottom panels: partitions recovered upon minimising BIC (orange links would be classified as misplaced according to the F -test). Minimising BIC leads us to find a community structure whose definition depends on the ‘signed density’ of connections: such a structure coincides with the one recovered upon minimising $F(\sigma)$ only if the former is k -balanced, i.e. satisfies the relationships $p_{rr}^- = 0$, $r = 1 \dots k$ and $p_{rs}^+ = 0$, $\forall r < s$.

$$\begin{aligned}
P_{\text{SSBM}}(\mathbf{A}) &= \prod_{i=1}^N \prod_{j(>i)} \prod_{r=1}^k \prod_{s(\geq r)} [(p_{rs}^+)^{a_{ij}^+} (p_{rs}^0)^{a_{ij}^0} (p_{rs}^-)^{a_{ij}^-}]^{\delta_{g_i r} \delta_{g_j s}} \\
&= \prod_{r=1}^k \prod_{s(\geq r)} \prod_{i=1}^N \prod_{j(>i)} [(p_{rs}^+)^{\delta_{g_i r} \delta_{g_j s} a_{ij}^+} (p_{rs}^0)^{\delta_{g_i r} \delta_{g_j s} a_{ij}^0} (p_{rs}^-)^{\delta_{g_i r} \delta_{g_j s} a_{ij}^-}] \\
&= \prod_{r=1}^k \prod_{s(\geq r)} [(p_{rs}^+)^{\sum_{i=1}^N \sum_{j(>i)} \delta_{g_i r} \delta_{g_j s} a_{ij}^+} (p_{rs}^0)^{\sum_{i=1}^N \sum_{j(>i)} \delta_{g_i r} \delta_{g_j s} a_{ij}^0} (p_{rs}^-)^{\sum_{i=1}^N \sum_{j(>i)} \delta_{g_i r} \delta_{g_j s} a_{ij}^-}] \\
&= \prod_{r=1}^k (p_{rr}^+)^{L_{rr}^+} (p_{rr}^0)^{L_{rr}^0} (p_{rr}^-)^{L_{rr}^-} \cdot \prod_{r=1}^k \prod_{s(>r)} (p_{rs}^+)^{L_{rs}^+} (p_{rs}^0)^{L_{rs}^0} (p_{rs}^-)^{L_{rs}^-} \\
&= \prod_{r=1}^k (p_{rr}^+)^{L_{rr}^+} (p_{rr}^-)^{L_{rr}^-} (1 - p_{rr}^+ - p_{rr}^-)^{\binom{N_r}{2} - L_{rr}} \cdot \prod_{r=1}^k \prod_{s(>r)} (p_{rs}^+)^{L_{rs}^+} (p_{rs}^-)^{L_{rs}^-} (1 - p_{rs}^+ - p_{rs}^-)^{N_r N_s - L_{rs}} \\
&= \prod_{r=1}^k \frac{x_{rr}^{L_{rr}^+} y_{rr}^{L_{rr}^-}}{[1 + x_{rr} + y_{rr}]^{\binom{N_r}{2}}} \cdot \prod_{r=1}^k \prod_{s(>r)} \frac{x_{rs}^{L_{rs}^+} y_{rs}^{L_{rs}^-}}{[1 + x_{rs} + y_{rs}]^{N_r N_s}} \tag{12}
\end{aligned}$$

where $p_{rr}^+ = x_{rr}/(1 + x_{rr} + y_{rr})$, $p_{rr}^- = y_{rr}/(1 + x_{rr} + y_{rr})$, $r = 1 \dots k$ and $p_{rs}^+ = x_{rs}/(1 + x_{rs} + y_{rs})$, $p_{rs}^- = y_{rs}/(1 + x_{rs} + y_{rs})$, $\forall r < s$. Its log-likelihood reads

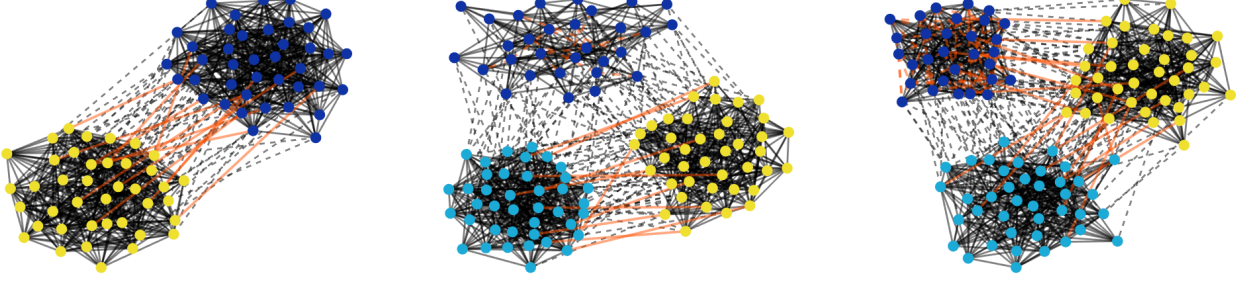


FIG. 6: Minimising $F(\sigma)$ can lead to a number of ambiguous situations, such as returning configurations that are neither traditionally nor relaxedly balanced (orange links are classified as misplaced according to the F -test; however, they can neither be arranged into homogeneous blocks). Within our, novel, statistical framework these configurations are unambiguously classified as balanced according to the statistical variant of the TBT.

$$\begin{aligned} \ln \mathcal{L}_{\text{SSBM}} = & \sum_{r=1}^k \left[L_{rr}^+(\mathbf{A}) \ln x_{rr} + L_{rr}^-(\mathbf{A}) \ln y_{rr} - \binom{N_r}{2} \ln[1 + x_{rr} + y_{rr}] \right] \\ & + \sum_{r=1}^k \sum_{s(>r)} [L_{rs}^+(\mathbf{A}) \ln x_{rs} + L_{rs}^-(\mathbf{A}) \ln y_{rs} - N_r N_s \ln[1 + x_{rs} + y_{rs}]] \end{aligned} \quad (13)$$

and its maximisation leads to recover the conditions $p_{rr}^+ = 2L_{rr}^+(\mathbf{A})/N_r(N_r - 1)$, $p_{rr}^- = 2L_{rr}^-(\mathbf{A})/N_r(N_r - 1)$, $r = 1 \dots k$ and $p_{rs}^+ = L_{rs}^+(\mathbf{A})/N_r N_s$, $p_{rs}^- = L_{rs}^-(\mathbf{A})/N_r N_s$, $\forall r < s$.

Let us, now, recall that BIC is defined as

$$\text{BIC} = \kappa \ln n - 2 \ln \mathcal{L} \quad (14)$$

where $\mathcal{L}$ is a model likelihood and κ indicates the number of parameters entering into its definition. Here, we have posed $\mathcal{L} \equiv \mathcal{L}_{\text{SSBM}}$, $\kappa \equiv \kappa_{\text{SSBM}} = k(k+1)$ (i.e. $k + k(k-1)/2$ parameters to be tuned on the set of values L_{rs}^+ , $\forall r \leq s$ and $k + k(k-1)/2$ parameters to be tuned on the set of values L_{rs}^- , $\forall r < s$) and $n = N(N-1)/2$.

Figure 4 shows three, ideal partitions: one compatible with the strong balance theory, one compatible with the weak balance theory and one compatible with the relaxed balanced theory.

Comparing BIC minimisation with F minimisation

Let us, now, carry out another comparison between the recipe prescribing to minimise $F(\sigma)$ and the one prescribing to minimise BIC. As fig. 5 shows, the partitions that are recovered upon minimising BIC match the planted ones, a result confirming that BIC is sensitive to the ‘signed density’ of the modules. As a consequence, the partitions recovered upon minimising it coincide with the ones recovered upon minimising $F(\sigma)$ only if the former ones are k -balanced, i.e. satisfy the relationships $p_{rr}^- = 0$, $r = 1 \dots k$ and $p_{rs}^+ = 0$, $\forall r < s$.

Minimising $F(\sigma)$ can lead to a number of ambiguous situations, such as *i*) returning configurations that are neither traditionally nor relaxedly balanced; *ii*) returning more than one frustrated configuration.

Figure 6 depicts the first situation. Nodes of the same colour are those put together as a consequence of minimising $F(\sigma)$: although the presence of negative links between and within modules makes the recovered partitions ‘traditionally’ frustrated, the original formulation of the RBT would lead us to conclude that they are ‘relaxedly’ frustrated as well; only within our, novel, statistical framework these configurations can be unambiguously classified as balanced according to the statistical variant of the TBT.

Figure 7 depicts the second situation: the minimisation of $F(\sigma)$ can return more than one frustrated configuration; our BIC-based test, however, ‘prefers’ the one on the left, classifying it as balanced according to the statistical variant of the TBT.

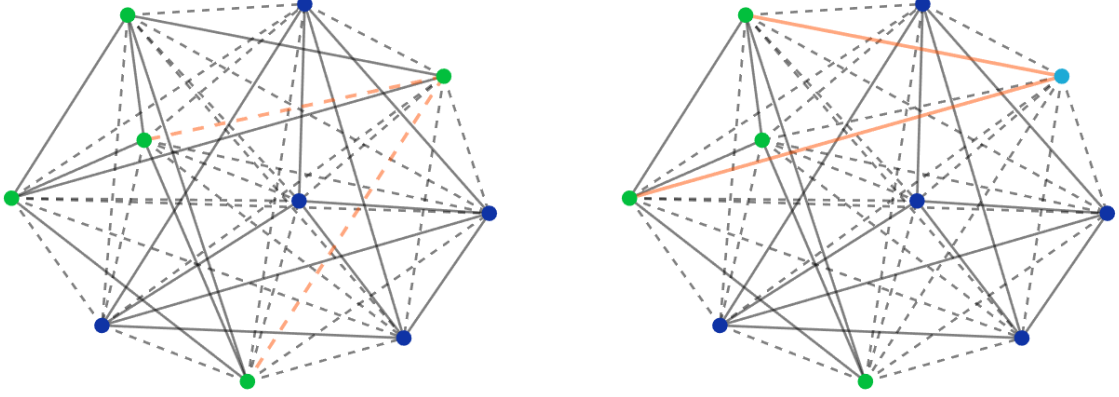


FIG. 7: Minimising $F(\sigma)$ can lead to a number of ambiguous situations, such as returning more than one configuration in correspondence of which $F(\sigma)$ attains its minimum. In this, particular case, the Slovenian Parliament admits two, different arrangements of nodes characterised by $F(\sigma) = 2$ (orange links are classified as misplaced according to the F -test). Our BIC-based test, however, ‘prefers’ the one on the left, classifying it as balanced according to the statistical variant of the TBT.

Minimising BIC on configurations generated by the Signed Random Graph Model (SRGM)

Let us, first, recall the derivation of the SRGM. It is defined by the Hamiltonian

$$H(\mathbf{A}) = \alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A}) = \sum_{i=1}^N \sum_{j(>i)} [\alpha a_{ij}^+ + \beta a_{ij}^-] \quad (15)$$

leading to

$$\begin{aligned} Z &= \sum_{\mathbf{A} \in \mathbb{A}} e^{-H(\mathbf{A})} = \sum_{\mathbf{A} \in \mathbb{A}} \prod_{i=1}^N \prod_{j(>i)} e^{-[\alpha a_{ij}^+ + \beta a_{ij}^-]} = \prod_{i=1}^N \prod_{j(>i)} \sum_{a_{ij} = -1, 0, +1} e^{-[\alpha a_{ij}^+ + \beta a_{ij}^-]} \\ &= \prod_{i=1}^N \prod_{j(>i)} [1 + e^{-\alpha - \beta}] = [1 + e^{-\alpha - \beta}]^{\binom{N}{2}}. \end{aligned} \quad (16)$$

As a consequence,

$$P_{\text{SRGM}}(\mathbf{A}) = \frac{e^{-H(\mathbf{A})}}{Z} = \frac{e^{-[\alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A})]}}{[1 + e^{-\alpha - \beta}]^{\binom{N}{2}}} \equiv \frac{x^{L^+(\mathbf{A})} y^{L^-(\mathbf{A})}}{[1 + x + y]^{\binom{N}{2}}} \equiv (p^-)^{L^-(\mathbf{A})} (p^0)^{L^0(\mathbf{A})} (p^+)^{L^+(\mathbf{A})} \quad (17)$$

where $p^+ = x/(1 + x + y)$, $p^- = y/(1 + x + y)$ and $p^0 = 1/(1 + x + y)$. Its log-likelihood reads

$$\ln \mathcal{L}_{\text{SRGM}} = L^+(\mathbf{A}) \ln x + L^-(\mathbf{A}) \ln y - \binom{N}{2} \ln[1 + x + y] \quad (18)$$

and its maximisation leads to recover the conditions $p^+ = 2L^+(\mathbf{A})/N(N-1)$, $p^- = 2L^-(\mathbf{A})/N(N-1)$.

Figure 8 shows three configurations generated by the SRGM. As such a model does not carry any information about a network modular structure, no groups of nodes should be recognised. This is precisely the output of our BIC-based

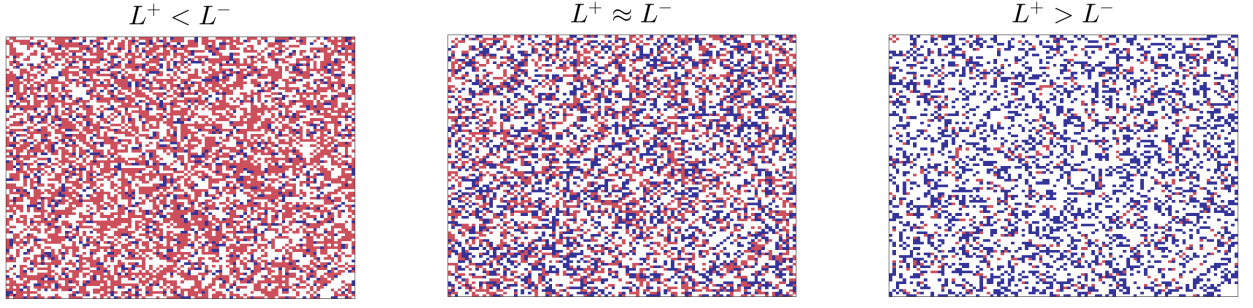


FIG. 8: Three configurations generated by the SRGM (positive links are coloured in blue; negative links are coloured in red). Since this model does not carry any information about a network modular structure, no groups of nodes should be detected: minimising BIC, in fact, always returns a single community gathering all nodes together, irrespectively from the choice of the parameters.

recipe, returning a single community gathering all nodes together (i.e. $k = 1$), irrespectively from the choice of the parameters (i.e. be $p^+ < p^-$, $p^+ \simeq p^-$ or $p^+ > p^-$). Minimising $F(\sigma)$ (or maximising $Q(\sigma)$ - see Appendix C), instead, leads to recover a number of modules $k \geq 1$ (in the case $L^+ < L^-$, $k_F = 25$ and $k_Q = 13$; in the case $L^+ \simeq L^-$, $k_F = 6$ and $k_Q = 5$; in the case $L^+ > L^-$, $k_F = k_Q = 1$).

Minimising BIC on negative and positive cliques

Let us, now, consider a negative clique composed by N nodes; evaluating BIC on the partition defined by k modules returns the value $k(k+1) \ln[N(N-1)/2]$; since $N \geq k \geq 1$, keeping all nodes together is the most convenient choice. The same result holds true if we consider a positive clique composed by N nodes, the reason lying in the completely symmetric role played by negative and positive links, both contributing to the density of the (potential) network modules.

Minimising BIC on complete, signed graphs

When dealing with complete graphs, the signs come into play in a quite peculiar fashion. Let us, in fact, consider a complete graph of size $N = N_1 + N_2$, constituted by two, negative cliques having, respectively, N_1 and N_2 nodes and such that each node of a clique is connected to each node of the other via a positive link. Let us, now, pose $k > 2$ and consider the following inequality

$$k(k+1) \ln \binom{N}{2} - 2 \ln \mathcal{L}_{\text{SSBM}} > 6 \ln \binom{N}{2} \quad (19)$$

stating that evaluating BIC on a generic partition defined by $k > 2$ modules returns a value that is strictly larger than the value of BIC calculated on the bi-partition whose modules coincide with the cliques themselves (in fact, $N \geq k > 2$ and $\ln \mathcal{L}_{\text{SSBM}} > 0$).

Let us, now, compare the bi-partition induced by the negative cliques with the partition induced by imposing the presence of just one module: in this, last case, evaluating BIC returns the value

$$\begin{aligned} \text{BIC}(k=1) &= 2 \ln \binom{N}{2} - 2 \ln \left[(p^+)^{L^+} (p^-)^{L^-} \right] \\ &= 2 \ln \binom{N}{2} - 2 \ln \left[\left(\frac{2N_1N_2}{N(N-1)} \right)^{N_1N_2} \left(\frac{N_1(N_1-1) + N_2(N_2-1)}{N(N-1)} \right)^{\binom{N_1}{2} + \binom{N_2}{2}} \right] \\ &= 2 \ln \binom{N}{2} - 2 \ln \left[\left(\frac{2N_1(N-N_1)}{N(N-1)} \right)^{N_1(N-N_1)} \left(\frac{N_1(N_1-1) + (N-N_1)(N-N_1-1)}{N(N-1)} \right)^{\binom{N_1}{2} + \binom{N-N_1}{2}} \right] \end{aligned} \quad (20)$$

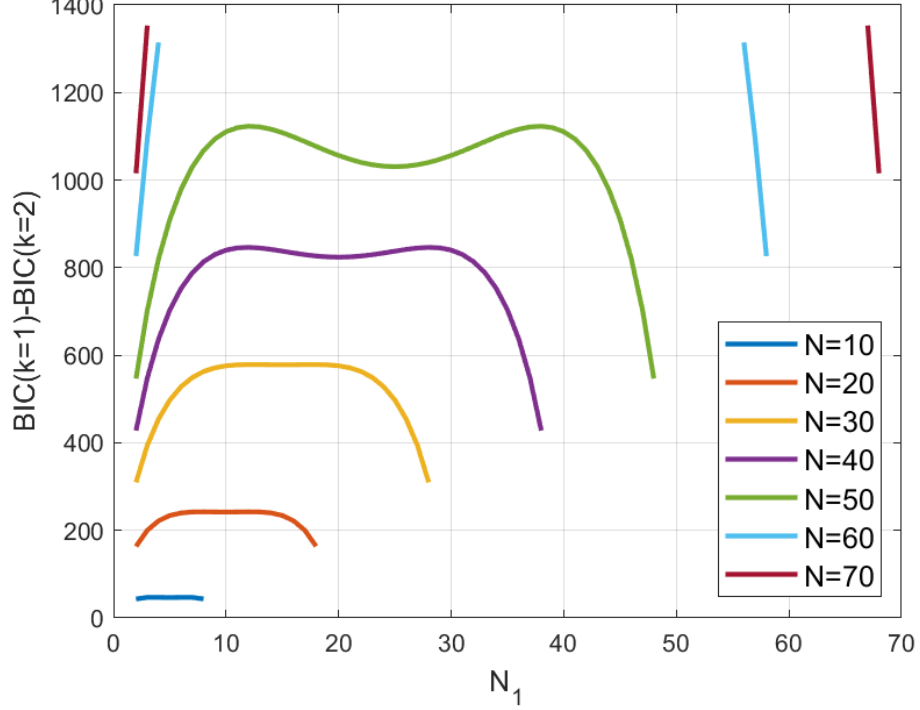


FIG. 9: Plotting the difference between $BIC(k=1)$ and the value of BIC characterised by the bi-partition whose modules coincide with the cliques themselves, as a function of $2 < N_1 < N-2$, reveals it to be always positive. This result confirms that such a bi-partition is the one in correspondence of which BIC attains its minimum value.

where we have used the relationship $N_2 = N - N_1$. As depicted in fig. 9, splitting nodes according to the partition induced by the signs is always ‘more convenient’ than partitioning them in a different way, a result suggesting that signs keep playing a role as long as the information embodied by the network density becomes irrelevant.

The same reasoning can be repeated for any number of cliques, the last relationship becoming

$$c(c+1) \ln \binom{N}{2} < 2 \ln \binom{N}{2} - 2 \ln \left[\left(\frac{2 \sum_{i < j} N_i N_j}{N(N-1)} \right)^{\sum_{i < j} N_i N_j} \left(\frac{\sum_i N_i (N_i - 1)}{N(N-1)} \right)^{\sum_i \binom{N_i}{2}} \right], \quad (21)$$

i.e. the partition defined by c modules coinciding with the cliques is the one characterised by the minimum value of BIC (see also fig. 10).

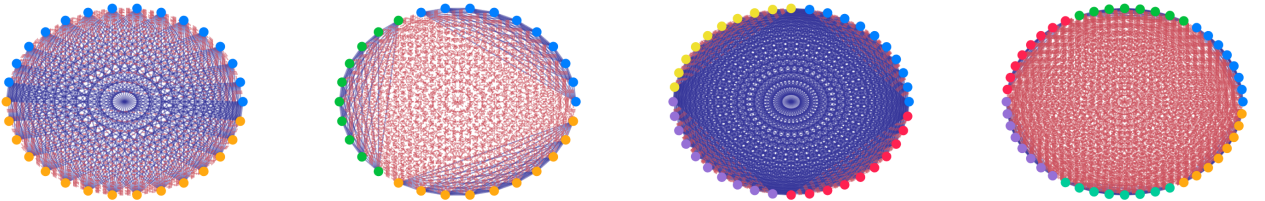


FIG. 10: Results of testing BIC minimisation on four, complete, signed graphs balanced according to the TBT (second panel: a 3-balanced configuration; fourth panel: a 6-balanced configuration) and maximally frustrated according to the TBT but perfectly balanced according to the RBT (first panel: $k=2$; third panel: $k=4$). Minimising BIC leads to the partition induced by signs, since always ‘more convenient’ than any, other partition. Positive links are coloured in blue; negative links are coloured in red.

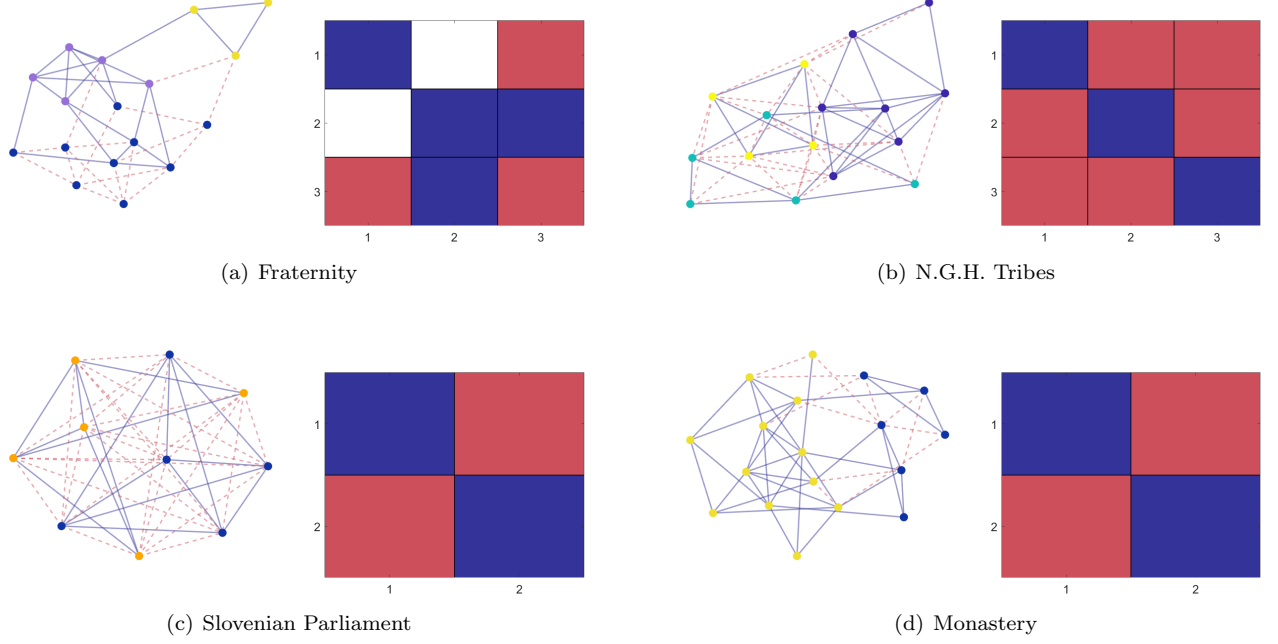


FIG. 11: Partitions recovered upon minimising BIC on the Fraternity, N.G.H. Tribes, Slovenian Parliament and Monastery datasets. A generic block, indexed as rs , is coloured in blue if $L_{rs}^+ > L_{rs}^-$, in red if $L_{rs}^+ < L_{rs}^-$ and in white if $L_{rs}^+ = L_{rs}^-$. Minimising BIC leads to recover partitions that obey the statistical variant of the TBT in most of the cases. Positive links are coloured in blue; negative links are coloured in red.

Minimising BIC on more, real-world configurations

Let us, now, apply our recipe to a number of real-world, signed configurations, i.e. Fraternity, N.G.H. Tribes, Monastery and [10] and Slovenian Parliament [11]: as fig. 11 shows, minimising BIC leads to recover partitions that obey the statistical variant of the TBT (either in its strong or weak form) in most of the cases (a blue block is characterised by a majority of positive links; a red block is characterised by a majority of negative links).

On the other hand, US Senate, EGFR, Macrophage, Bitcoin Alpha and Bitcoin OTC datasets [13] are characterised by $k = 1$ (either because $p^+ \simeq p^-$ or because $p^+ \gg p^-$).

APPENDIX C. MORE ON SIGNED MODULARITY

A deterministic theory of balance can be turned into a statistical theory of balance by answering the following question: is it possible to define a reference level of misplaced links by means of which discerning frustrated graphs from balanced ones? The answer is affirmative and calls for comparing the empirical amount of frustration of a given, signed configuration with the one predicted by a properly-defined reference model: in formulas, one may define a partition as *statistically balanced* if satisfying the relationship $F(\sigma) < \langle F(\sigma) \rangle$.

A quantity embodying such a comparison already exists: it is the *signed modularity*, reading

$$\begin{aligned}
 Q(\sigma) &= \sum_{i=1}^N \sum_{j(>i)} [(a_{ij}^+ - p_{ij}^+) - (a_{ij}^- - p_{ij}^-)] \delta_{\sigma_i, \sigma_j} \\
 &= L_{\bullet}^+ - \langle L_{\bullet}^+ \rangle - (L_{\bullet}^- - \langle L_{\bullet}^- \rangle) \\
 &= (L^+ - L_{\circ}^+) - \langle L^+ - L_{\circ}^+ \rangle - (L_{\bullet}^- - \langle L_{\bullet}^- \rangle) \\
 &= -(L_{\circ}^+ + L_{\bullet}^-) + \langle L_{\circ}^+ + L_{\bullet}^- \rangle + L^+ - \langle L^+ \rangle \\
 &= -F(\sigma) + \langle F(\sigma) \rangle + L^+ - \langle L^+ \rangle
 \end{aligned} \tag{22}$$

with obvious meaning of the symbols (the addendum $L^+ - \langle L^+ \rangle$ is just an offset not depending on the specific partition and amounting at zero for any model reproducing the total number of positive links) [15]. Since the total number of positive links is preserved under any model considered in the present paper, we obtain

$$Q(\sigma) = -F(\sigma) + \langle F(\sigma) \rangle. \tag{23}$$

$Q(\sigma)$ has been widely employed to spot communities on signed networks, with the positions $p_{ij}^+ = k_i^+ k_j^+ / 2L^+$ and $p_{ij}^- = k_i^- k_j^- / 2L^-$, $\forall i < j$. Such a recipe, instantiating the Chung-Lu (CL) model, is applicable only in case $p_{ij}^+ \leq 1$ and $p_{ij}^- \leq 1$, $\forall i < j$: these conditions, however, do not hold in several cases of interest, an example of paramount importance being provided by sparse networks whose degree distribution is scale-free [16]. In order to overcome the aforementioned limitation, a different framework is needed.

One may follow the analytical approach introduced in [17] and further developed in [18], aimed at identifying the functional form of the maximum-entropy probability distribution that preserves a desired set of empirical constraints, on average. Specifically, this approach looks for the graph probability $P(\mathbf{A})$ that maximises Shannon entropy $S = -\sum_{\mathbf{A} \in \mathbb{A}} P(\mathbf{A}) \ln P(\mathbf{A})$, under constraints enforcing the expected value of a chosen set of properties. The formal solution to this problem is the exponential probability $P(\mathbf{A}) = e^{-H(\mathbf{A})}/Z$ where the *Hamiltonian* $H(\mathbf{A})$ is a linear combination of the constrained properties and the *partition function* $Z = \sum_{\mathbf{A} \in \mathbb{A}} e^{-H(\mathbf{A})}$ plays the role of normalising constant, the sum running over the set $\mathbb{A}$ of all binary, undirected, signed graphs whose cardinality amounts to $|\mathbb{A}| = 3^{\binom{N}{2}}$. Two examples of models of the kind are the Signed Random Graph Model (SRGM) and the Signed Configuration Model (SCM) [19].

According to the TBT, several ways exist in which a given configuration can be frustrated. Let us, now, analyse them in detail.

1. Evaluating frustration due to negative links

Positive subgraphs connected by negative links. In order to understand how a Q -based test would work, let us consider two subgraphs with, respectively, m and n nodes, positive intra-modular links and negative inter-modular links. Let us denote with $V_{\bullet} = m(m-1)/2 + n(n-1)/2$ the total number of intra-modular pairs of nodes and with Q_0 the value of modularity associated to the partition of the entire graph, except our, two subgraphs; let us also call $L_{\bullet}^+$ the total number of positive links within modules and $L_{\circ}^-$ the total number of negative links between modules. Then,

$$Q_A = Q_0 + [L_{\bullet}^+ - p^+ V_{\bullet}] - [0 - p^- V_{\bullet}], \tag{24}$$

$$Q_B = Q_0 + [L_{\bullet}^+ - p^+(V_{\bullet} + mn)] - [L_{\circ}^- - p^-(V_{\bullet} + mn)] \tag{25}$$

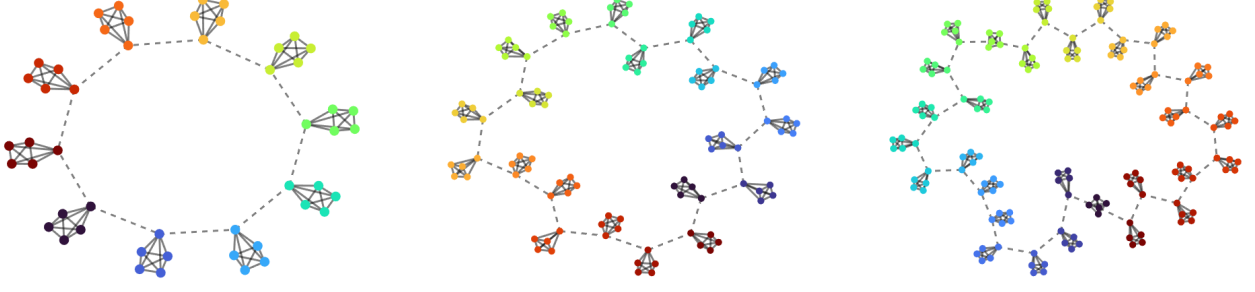


FIG. 12: Partitions recovered upon maximising the signed modularity on three rings of cliques, i.e. a set of 10 (left), 20 (middle), 30 (right) cliques, constituted by 5 nodes each, internally connected by positive links and inter-connected by negative links.

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to be fully consistent with the TBT, one should require

$$Q_A > Q_B = Q_A - L_o^- - (p^+ - p^-)mn, \quad (26)$$

a condition that it is satisfied whenever $L_o^- > (p^- - p^+)mn$, i.e. whenever the probability $p_o^- \equiv L_o^-/mn$ of establishing a negative link within modules is larger than $p^- - p^+ = 2(L^- - L^+)/N(N-1)$. This condition sheds light on the role played by the signed variant of the resolution limit, naturally re-interpretable as a threshold-based criterion for discerning if a given, signed configuration is balanced or not: in words, the ‘acceptable’ level of frustration, according to which our subgraphs can be safely interpreted as a single community, is represented by $(p^- - p^+)mn$.

2. Evaluating frustration due to positive links

Positive subgraphs connected by positive links. Let us, now, consider two subgraphs with, respectively, m and n nodes, positive intra- and inter-modular links; let us also call $L_\bullet^+$ the total number of positive links within modules and L_o^+ the total number of positive links between modules. Then,

$$Q_A = Q_0 + [L_\bullet^+ - p^+V_\bullet] - [0 - p^-V_\bullet], \quad (27)$$

$$Q_B = Q_0 + [L_\bullet^+ + L_o^+ - p^+(V_\bullet + mn)] - [0 - p^-(V_\bullet + mn)] \quad (28)$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to be fully consistent with the TBT, one should require

$$Q_B = Q_A + L_o^+ - (p^+ - p^-)mn > Q_A, \quad (29)$$

a condition that it is satisfied whenever $L_o^+ > (p^+ - p^-)mn$, i.e. whenever the probability $p_o^+ \equiv L_o^+/mn$ of establishing a positive link between modules is larger than $p^+ - p^- = 2(L^+ - L^-)/N(N-1)$. The threshold-based criterion for discerning balance represented by the signed variant of the resolution limit, now, sets the ‘acceptable’ level of frustration, according to which our subgraphs can be safely interpreted as two, separate communities, at $(p^+ - p^-)mn$.

3. Evaluating frustration in real-world networks

Interestingly enough, when studying real-world, signed networks, the relationship $L^+ \gg L^-$ is often (if not always) found to hold true: as a consequence, the condition

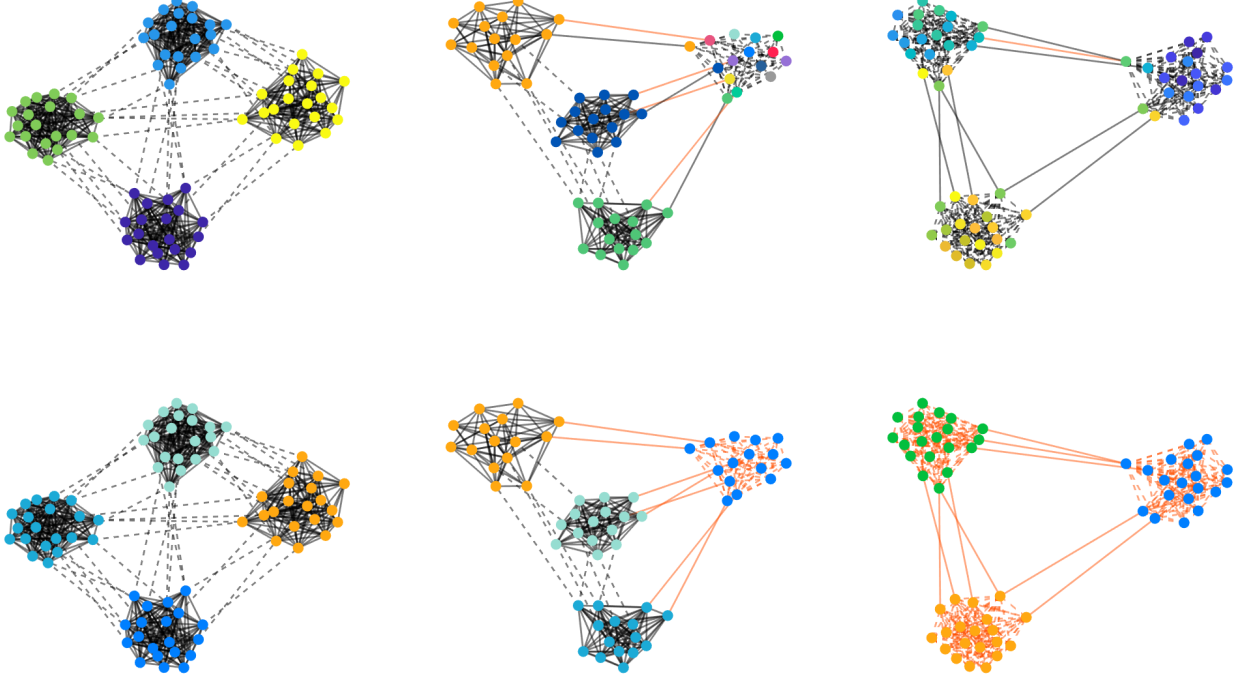


FIG. 13: Top panels: partitions recovered upon maximising $Q(\sigma)$. Bottom panels: partitions recovered upon minimising BIC. Left panels: minimising BIC returns partitions that coincide with those returned upon maximising the signed modularity only in case they are k -balanced. Middle and right panels: minimising BIC (bottom panels) leads to recover the planted partitions, balanced according to the RBT; maximising $Q(\sigma)$ (top panels), instead, leads to the fragmentation of the subgraphs constituted by negative links into singletons. In other words, the Q -based test seeks to recover a configuration obeying the TBT even when there is none ‘by design’ (orange links are classified as misplaced according to the F -test).

$$L_{\circ}^{-} > (p^{-} - p^{+})mn < 0 \quad (30)$$

is trivially satisfied. Such an evidence has several consequences. In order to discuss them, let us focus on *i*) the case of negative subgraphs connected by negative links; *ii*) the case of negative subgraphs connected by positive links.

Negative subgraphs connected by negative links. Let us consider two subgraphs with, respectively, m and n nodes, negative intra- and inter-modular links; let us also call $L_{\bullet}^{-}$ the total number of negative links within modules and $L_{\circ}^{-}$ the total number of negative links between modules. Then,

$$Q_A = Q_0 + [0 - p^{+}V_{\bullet}] - [L_{\bullet}^{-} - p^{-}V_{\bullet}], \quad (31)$$

$$Q_B = Q_0 + [0 - p^{+}(V_{\bullet} + mn)] - [L_{\bullet}^{-} + L_{\circ}^{-} - p^{-}(V_{\bullet} + mn)] \quad (32)$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to limit the number of links that would be deemed as misplaced according to the TBT, one should require

$$Q_A > Q_B = Q_A - L_{\circ}^{-} - (p^{+} - p^{-})mn, \quad (33)$$

a condition that it is satisfied whenever $L_o^- > (p^- - p^+)mn$, i.e. whenever the probability $p_o^- \equiv L_o^-/mn$ of establishing a negative link between modules is larger than $p^- - p^+ = 2(L^- - L^+)/N(N-1) < 0$. Hence, it is always convenient to separate negatively connected modules and, if such a line of reasoning is repeated in a hierarchical fashion, it is always convenient to separate negatively connected nodes. Otherwise stated, one should not expect the presence of negative links within blocks since negatively connected modules will always lead to singletons: in this sense, the signed modularity is resolution limit-free.

Negative subgraphs connected by positive links. Let us, now, focus on the case of negative subgraphs connected by positive links and consider two subgraphs with, respectively, m and n nodes, negative intra-modular links and positive inter-modular links; let us also call $L_\bullet^-$ the total number of negative links within modules and L_o^+ the total number of positive links between modules. Then,

$$Q_A = Q_0 + [0 - p^+ V_\bullet] - [L_\bullet^- - p^- V_\bullet], \quad (34)$$

$$Q_B = Q_0 + [L_o^+ - p^+(V_\bullet + mn)] - [L_\bullet^- - p^-(V_\bullet + mn)] \quad (35)$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to limit the number of links that would be deemed as misplaced according to the TBT, one should require

$$Q_B = Q_A + L_o^+ - (p^+ - p^-)mn > Q_A, \quad (36)$$

a condition that it is satisfied whenever $L_o^+ > (p^+ - p^-)mn$, i.e. whenever the probability $p_o^+ \equiv L_o^+/mn$ of establishing a positive link between modules is larger than $p^+ - p^- = 2(L^+ - L^-)/N(N-1)$. Now, as a consequence of eq. (30), it is convenient to fragment negatively connected modules into singletons; hence, according to the TBT, frustration can only occur because of misplaced, positive links appearing between blocks.

Figure 12 depicts the results of the signed modularity maximisation on three rings of cliques: since the relationship $L^+ \gg L^-$ holds true, one should not expect the presence of negative links within blocks (as we said, the signed modularity is resolution limit-free, in this sense). Notice that our exercise is defined in such a way that the numerical value of the generic addendum $(a_{ij}^+ - p_{ij}^+) - (a_{ij}^- - p_{ij}^-)$ is fixed, once and for all, by the choice of the benchmark to be solved: in other words, the definition of modularity does not change with the level of aggregation, being just recomputed (as any other score function) as the partition changes [19].

As fig. 13 shows, minimising BIC returns partitions that coincide with those returned by maximising modularity, or minimising $F(\sigma)$, solely in case they are k -balanced (i.e. obey the TBT). In case subgraphs constituted by negative links are, instead, present, minimising BIC leads to the planted partition induced by gathering such nodes together while maximising $Q(\sigma)$ leads to their fragmentation, hence recovering singletons. In other words, the Q -based test (exactly as the F -based test and the G -based test) seeks to recover traditionally balanced configurations even when there is none ‘by design’.

From a purely numerical perspective, partitioning nodes by minimising BIC is accomplished as described in Algorithms 1 - 3.

Algorithm 1: Pseudocode to partition nodes by minimising BIC - step I

```

1: function BICBasedCommunityDetectionStepI( $N, \mathbf{A}$ )
2:  $C \leftarrow$  array of labels of length  $N$ , initialised as  $(1, 2, \dots, N)$ ;
3:  $\text{BIC} \leftarrow \text{UpdateBIC}(\mathbf{A}, C)$ ;
4:  $E \leftarrow$  randomly sorted edges;
5: for  $(u, v) \in E$  do
6:    $C_0 \leftarrow C$ ;
7:    $\text{BIC}_0 \leftarrow \text{BIC}$ ;
8:   if  $C(u) \neq C(v)$  then
9:      $C_1 \leftarrow C$ ;
10:    for node  $w \in C(u)$  do
11:       $C_1(w) \leftarrow C(v)$ ;
12:    end for
13:     $\text{BIC}_1 \leftarrow \text{UpdateBIC}(\mathbf{A}, C_1)$ ;
14:  end if
15:  if  $\text{BIC}_1 < \text{BIC}_0$  then
16:     $C \leftarrow C_1$ ;
17:     $\text{BIC} \leftarrow \text{BIC}_1$ 
18:  else
19:     $C \leftarrow C_0$ ;
20:     $\text{BIC} \leftarrow \text{BIC}_0$ 
21:  end if
22: end for
23:  $\Rightarrow$  repeat the for-loop to improve the chance of finding the best partition

```

Algorithm 2: Pseudocode to partition nodes by minimising BIC - step II

```

1: function BICBasedCommunityDetectionStepII( $N, \mathbf{A}$ )
2:  $C \leftarrow \text{BICBasedCommunityDetectionStepI}(N, \mathbf{A})$ ;
3:  $\text{BIC} \leftarrow \text{UpdateBIC}(\mathbf{A}, C)$ ;
4:  $E \leftarrow$  randomly sorted edges;
5: for  $(u, v) \in E$  do
6:    $C_0 \leftarrow C$ ;
7:    $\text{BIC}_0 \leftarrow \text{BIC}$ ;
8:   if  $C(u) \neq C(v)$  then
9:      $C_1 \leftarrow C$ ;
10:     $C_1(u) \leftarrow C(v)$ ;
11:     $\text{BIC}_1 \leftarrow \text{UpdateBIC}(\mathbf{A}, C_1)$ ;
12:     $C_2 \leftarrow C$ ;
13:     $C_2(v) \leftarrow C(u)$ ;
14:     $\text{BIC}_2 \leftarrow \text{UpdateBIC}(\mathbf{A}, C_2)$ ;
15:  else if  $C(u) = C(v)$  then
16:     $C_1 \leftarrow C$ ;
17:     $C_1(u) \leftarrow$  randomly sorted community different from  $C(v)$ ;
18:     $\text{BIC}_1 \leftarrow \text{UpdateBIC}(\mathbf{A}, C_1)$ ;
19:     $C_2 \leftarrow C$ ;
20:     $C_2(v) \leftarrow$  randomly sorted community different from  $C(u)$ ;
21:     $\text{BIC}_2 \leftarrow \text{UpdateBIC}(\mathbf{A}, C_2)$ ;
22:  end if
23:   $i \leftarrow \text{argmin}\{\text{BIC}_0, \text{BIC}_1, \text{BIC}_2\}$ ;
24:   $C \leftarrow C_i$ ;
25:   $\text{BIC} \leftarrow \text{BIC}_i$ ;
26: end for
27:  $\Rightarrow$  repeat the for-loop to improve the chance of finding the best partition

```

Algorithm 3: Pseudocode to update BIC

```

1: function UpdateBIC(A, C)
2:  $k \leftarrow$  number of modules, i.e. number of distinct labels in C;
3:  $\mathbf{P}^- \leftarrow k \times k$  matrix whose entry  $(c_1, c_2)$  is the probability that a node  $u \in C(u) = c_1$  is linked
   via a  $-1$  to a node  $v \in C(v) = c_2$ ;
4:  $\mathbf{P}^+ \leftarrow k \times k$  matrix whose entry  $(c_1, c_2)$  is the probability that a node  $u \in C(u) = c_1$  is linked
   via a  $+1$  to a node  $v \in C(v) = c_2$ ;
5:  $\mathbf{L}^- \leftarrow k \times k$  matrix whose entry  $(c_1, c_2)$  is i) the number of  $-1$ s between  $c_1$  and  $c_2$ , if  $c_1 \neq c_2$ ;
   ii) the number of  $-1$ s within  $c_1$ , otherwise;
6:  $\mathbf{L}^+ \leftarrow k \times k$  matrix whose entry  $(c_1, c_2)$  is i) the number of  $+1$ s between  $c_1$  and  $c_2$ , if  $c_1 \neq c_2$ ;
   ii) the number of  $+1$ s within  $c_1$ , otherwise;
7:  $\mathbf{n} \leftarrow k \times 1$  array whose  $c$ -th entry is the number of nodes belonging to  $c$ ;
8:  $\mathcal{L} \leftarrow 1$ ;
9: for  $c = 1 \dots k$  do
10:    $\mathcal{L} = \mathcal{L} \cdot \mathbf{P}^-(c, c)^{\mathbf{L}^-(c, c)} \mathbf{P}^+(c, c)^{\mathbf{L}^+(c, c)} (1 - \mathbf{P}^-(c, c) - \mathbf{P}^+(c, c))^{\binom{\mathbf{n}(c)}{2} - \mathbf{L}^-(c, c) - \mathbf{L}^+(c, c)}$ ;
11:   for  $d = c + 1 \dots k$  do
12:      $\mathcal{L} = \mathcal{L} \cdot \mathbf{P}^-(c, d)^{\mathbf{L}^-(c, d)} \mathbf{P}^+(c, d)^{\mathbf{L}^+(c, d)} (1 - \mathbf{P}^-(c, d) - \mathbf{P}^+(c, d))^{\mathbf{n}(c)\mathbf{n}(d) - \mathbf{L}^-(c, d) - \mathbf{L}^+(c, d)}$ ;
13:   end for
14: end for
15:  $\text{BIC} = k(k+1) \ln \binom{N}{2} - 2 \ln \mathcal{L}$ 

```
