Complex Stochastic Optimal Control Foundation of Quantum Mechanics

Vasil Yordanov
Sofia University St. Kliment Ohridski
v.yordanov@phys.uni-sofia.bg

April 25, 2024

Abstract

Recent studies have expanded the use of the stochastic Hamilton–Jacobi–Bellman (HJB) equation to include complex variables for deriving quantum mechanical equations. However, these studies typically assume that it's valid to apply the HJB equation directly to complex numbers, an approach that overlooks the fundamental problem of comparing complex numbers to find optimal controls. This paper addresses how to properly apply the HJB equation in the context of complex variables. Our findings significantly reevaluate the stochastic movement of quantum particles, directly influenced by the Cauchy-Riemann theorem. These insights not only deepen our understanding of quantum dynamics but also enhance the mathematical rigor of the framework for applying stochastic optimal control in quantum mechanics.

1 Introduction

Recently, a number of studies [1, 2, 3] have derived non-relativistic quantum mechanical equations from the stochastic Hamilton–Jacobi–Bellman (HJB) equation. Although not explicitly stated in these works, the use of complex diffusion coefficients necessitates that the stochastic equations of motion incorporate complex coordinates and velocities of the particle, while the intermediate action is a complex function of the complex coordinates.

In their work, Yang et al. [4] introduce the concept of complex quantum mechanics, which also relies on the HJB equation. This study explicitly defines complex stochastic equations of motion and incorporates complex functions and variables within the stochastic HJB framework.

In my previous research [5], I extended these ideas by deriving the Dirac equation, assuming a complex equation of motion and applying the HJB equation to complex functions and variables.

Despite the innovative approaches in these papers, all assume that the four-coordinates and the intermediate actions can be straightforwardly replaced with their complex counterparts in the stochastic HJB equation.

This paper mathematically clarifies the correctness of using complex numbers in the HJB equation, as suggested in previous studies. It demonstrates that the formal replacement of real numbers with complex ones in the HJB equation is indeed valid. However, this formal replacement needs to be considered as a system of two HJB equations: one for the real part and one for the imaginary part of the intermediate action J.

We provide mathematical proof that the equations for finding the optimal control policy are consistent with those in the real-valued framework. However, for the complex-valued case, these equations require taking a complex derivative on the intermediate action J. This proof depends on the crucial assumption that both the Lagrangian $\mathcal{L}$ and the intermediate action J can be analytically continued into the complex plane. Additionally, we establish that, due to the Cauchy-Riemann equations, the diffusion coefficient is purely imaginary, indicating that stochastic movements are confined to the imaginary components of particle coordinates.

The derivation of the complex HJB equation is based on methodologies outlined in previous research [6, 7, 8, 9]. These works, which derive the HJB equation without considering complex numbers, establish a starting point for extending the equation to incorporate complex variables, as demonstrated in this paper.

2 Derivation of the complex stochastic HJB equation

According to the first postulate of the Stochastic Optimal Control Theory of Quantum Mechanics [5], we assume that the particle moves as a Brownian particle in four-dimensional spacetime, influenced by an external random spacetime force. However, this motion occurs within the complex plane of each four-coordinate component. The complex stochastic equation of motion governing this behavior is given by:

$$dz_\mu = w_\mu ds + \sigma_\mu dW_\mu, \quad \mu = 0..3, \tag{2.1}$$

where w_μ represents the complex velocity and σ_μ denotes the complex-valued diffusion coefficients, with dW_μ being the increments of a Wiener process that encapsulates the stochastic nature of the particle's trajectory.

As assumed either implicitly or explicitly in referenced above works, it is consider a particle moving within a complex plane. The four-coordinates of this particle are defined as complex numbers, enabling the inclusion of both real and imaginary components to fully describe its position in spacetime. We express these coordinates as follows:

$$z_\mu = x_\mu + iy_\mu, \quad \mu = 0..3, \tag{2.2}$$

where x_μ and y_μ represent the real and imaginary parts, respectively, and the index μ spans the four dimensions of spacetime, aligning with the notation used in relativistic mechanics.

The velocity of the particle is similarly complexified to account for motions in both the real and imaginary dimensions of the spacetime:

$$w_\mu = u_\mu + iv_\mu, \quad \mu = 0..3, \tag{2.3}$$

where u_μ and v_μ denote the real and imaginary components of the four-velocity, respectively.

The real and imaginary parts of the stochastic equation of motion of the particle are represented as follows:

$$dx_\mu = u_\mu ds + \sigma^x_\mu dW_\mu, \quad \mu = 0..3, \tag{2.4}$$

$$dy_\mu = v_\mu ds + \sigma^y_\mu dW_\mu, \quad \mu = 0..3, \tag{2.5}$$

where σ^x_μ and σ^y_μ denote the diffusion coefficients associated respectively with the stochastic dynamics of the real and imaginary components of the coordinates.

According to the third postulate of the Stochastic Optimal Control Theory of Quantum Mechanics [5], Nature tries to minimize the expected value for the action, in which the particle's velocity is consider to be a control parameter of the optimization. Formally the complex action is defined as the minimum of the expected value of stochastic action:

$$S(\mathbf{z}_i, \mathbf{w}(\tau_i \to \tau_f)) = \min_{\mathbf{w}(\tau_i \to \tau_f)} \left\langle \int_{\tau_i}^{\tau_f} ds\, \mathcal{L}(\mathbf{z}(s), \mathbf{w}(s), s) \right\rangle_{\mathbf{z}_i}, \tag{2.6}$$

However, this formal definition encounters difficulties as it is not possible to directly find a minimum of a complex-valued function. To address this, we redefine the complex action to separate its real and imaginary components, each optimized independently:

$$S(\mathbf{z}_i, \mathbf{w}) = S_R(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}, \mathbf{v}) + iS_I(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}, \mathbf{v}) \tag{2.7}$$

where the real and imaginary parts of the action are defined respectively as follows:

$$S_{R,I}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}(\tau_i \to \tau_f), \mathbf{v}(\tau_i \to \tau_f)) = \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \left\langle \int_{\tau_i}^{\tau_f} ds\, \mathcal{L}_{R,I}(\mathbf{x}(s), \mathbf{y}(s), \mathbf{u}(s), \mathbf{v}(s), s) \right\rangle_{\mathbf{x}_i, \mathbf{y}_i} \tag{2.8}$$

where $\mathcal{L}_R(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, s)$ and $\mathcal{L}_I(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, s)$ are the real and imaginary parts of the Lagrangian of the test particle. These functions depend on the real and imaginary components of the control policies $\mathbf{u}$ and $\mathbf{v}$, the and four-coordinates $\mathbf{x}$ and $\mathbf{y}$ at proper time s. The subscripts $\mathbf{x}_i$ and $\mathbf{y}_i$ on the expectation value indicate that the expectation is calculated over all stochastic trajectories that originate at the complex coordinate $\mathbf{z}_i = \mathbf{x}_i + i\mathbf{y}_i$.

We define the complex Lagrangian of the particle as:

$$\mathcal{L}(\mathbf{z}, \mathbf{w}, s) = \mathcal{L}_R(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, s) + i\mathcal{L}_I(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, s) \tag{2.9}$$

such that $\mathcal{L}(\mathbf{z}, \mathbf{w}, s)$ is the analytic continuation of the real Lagrangian of the particle.

The task of optimal control theory [10] is to find the controls $\mathbf{u}(s)$ and $\mathbf{v}(s)$, $\tau_i < s < \tau_f$, denoted as $\mathbf{u}(\tau_i \to \tau_f)$ and $\mathbf{v}(\tau_i \to \tau_f)$, that minimizes the expected value of the action $S_{R,I}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}(\tau_i \to \tau_f), \mathbf{v}(\tau_i \to \tau_f))$.

We introduce the optimal cost-to-go function for any intermediate proper time τ, where $\tau_i < \tau < \tau_f$:

$$J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) = \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \left\langle \int_{\tau}^{\tau_f} ds\, \mathcal{L}_{R,I}(\mathbf{x}(s), \mathbf{y}(s), \mathbf{u}(s), s) \right\rangle_{\mathbf{x}_i, \mathbf{y}_i} \tag{2.10}$$

We define the complex cost-to-go function as:

$$J(\tau, \mathbf{z}) = J_R(\tau, \mathbf{x}, \mathbf{y}) + iJ_I(\tau, \mathbf{x}, \mathbf{y}), \tag{2.11}$$

assuming that it is also an analytic function in the complex plane.

By definition, the action $S_{R,I}(\mathbf{x}_i, \mathbf{u}(\tau_i \to \tau_f), \mathbf{v}(\tau_i \to \tau_f))$ is equal to the cost-to-go function $J_{R,I}(\tau_i, \mathbf{x}_{\tau_i}, \mathbf{y}_{\tau_i})$ at the initial proper time and spacetime coordinate:

$$S_{R,I}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{u}(\tau_i \to \tau_f), \mathbf{v}(\tau_i \to \tau_f)) = J_{R,I}(\tau_i, \mathbf{x}_{\tau_i}, \mathbf{y}_{\tau_i}) \tag{2.12}$$

We can rewrite recursive formula for $J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau)$ for any intermediate time τ', where $\tau < \tau' < \tau_f$:

$$\begin{aligned}
&J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) = \\
&= \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \left\langle \int_\tau^{\tau'} ds\, \mathcal{L}_{R,I}(s, \mathbf{x}_s, \mathbf{y}_s, \mathbf{u}_s, \mathbf{v}_s) + \int_{\tau'}^{\tau_f} ds\, \mathcal{L}_{R,I}(s, \mathbf{x}_s, \mathbf{y}_s, \mathbf{u}_s, \mathbf{v}_s) \right\rangle_{\mathbf{x}_\tau, \mathbf{y}_\tau} \\
&= \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \left\langle \int_\tau^{\tau'} ds\, \mathcal{L}_{R,I}(s, \mathbf{x}_s, \mathbf{y}_s, \mathbf{u}_s, \mathbf{v}_s) + \min_{\mathbf{u}(\tau' \to \tau_f)} \left\langle \int_{\tau'}^{\tau_f} ds\, \mathcal{L}_{R,I}(s, \mathbf{x}_s, \mathbf{y}_s, \mathbf{u}_s, \mathbf{v}_s) \right\rangle_{\mathbf{x}_{\tau'}, \mathbf{y}_{\tau'}} \right\rangle_{\mathbf{x}_\tau, \mathbf{y}_\tau} \\
&= \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \left\langle \int_\tau^{\tau'} ds\, \mathcal{L}_{R,I}(s, \mathbf{x}_s, \mathbf{y}_s, \mathbf{u}_s, \mathbf{v}_s) + J(\tau', \mathbf{x}_{\tau'}, \mathbf{y}_{\tau'}) \right\rangle_{\mathbf{x}_\tau, \mathbf{y}_\tau} .
\end{aligned} \tag{2.13}$$

In above equation we split the minimization over two intervals. These are not independent, because the second minimization is conditioned on the starting value $x_{\tau'}$, $y_{\tau'}$, which depends on the outcome of the first minimization.

If τ' is a small increment of τ, $\tau' = \tau + d\tau$ then:

$$J_{R,I}(\tau, x_\tau, y_\tau) = \min_{\substack{\mathbf{u}(\tau_i \to \tau_f) \\ \mathbf{v}(\tau_i \to \tau_f)}} \langle \mathcal{L}_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau, \mathbf{u}_\tau, \mathbf{v}_\tau) d\tau + J_{R,I}(\tau + d\tau, \mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) \rangle_{\mathbf{x}_\tau, \mathbf{y}_\tau} \tag{2.14}$$

We must take a Taylor expansion of J_R and J_I in $d\mathbf{x}$, $d\mathbf{y}$ and $d\tau$. However, since $\langle d\mathbf{x}^2 \rangle = \sigma_x^2 d\tau$ and $\langle d\mathbf{y}^2 \rangle = \sigma_y^2 d\tau$ is of order $d\tau$, we must expand up to order $d\mathbf{x}^2$ and $d\mathbf{y}^2$:

$$\begin{aligned}
&\langle J_{R,I}(\tau + d\tau, \mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) \rangle_{\mathbf{x}_\tau} = \\
&= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{x}_\tau), \sigma d\tau) J_{R,I}(\tau + d\tau, \mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) = \\
&= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma d\tau) \times \\
&\times (J_{R,I}(\tau, \mathbf{x}, \mathbf{y}) + d\tau \partial_\tau J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + dx^\mu \partial_{\mathbf{x}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + dy^\mu \partial_{\mathbf{y}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \\
&+ dx^\mu dx^\nu \frac{1}{2} \partial_{\mathbf{x}^\mu \mathbf{x}^\nu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + dy^\mu dy^\nu \frac{1}{2} \partial_{\mathbf{y}^\mu \mathbf{y}^\nu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + dx^\mu dy^\nu \partial_{\mathbf{x}^\mu \mathbf{y}^\nu} J(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau)) = \\
&= J_{R,I}(\tau, \mathbf{x}, \mathbf{y}) + d\tau \partial_\tau J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \langle dx^\mu \rangle \partial_{\mathbf{x}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \langle dy^\mu \rangle \partial_{\mathbf{y}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \\
&+ \frac{1}{2} \langle dx^\nu dx^\mu \rangle \partial_{\mathbf{x}^\nu \mathbf{x}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \frac{1}{2} \langle dy^\nu dy^\mu \rangle \partial_{\mathbf{y}^\nu \mathbf{y}^\mu} J_{R,I}(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau)
\end{aligned} \tag{2.15}$$

Here $\mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma d\tau)$ is the conditional probability starting from state $(\mathbf{x}_\tau, \mathbf{y}_\tau)$ to end up in state $(\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau})$. The integration is over the entire spacetime for $\mathbf{x}$ and $\mathbf{y}$.

We can calculate the expected values of dx^μ using Equation (2.4):

$$
\begin{aligned}
\langle dx^\mu \rangle &= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau) dx^\mu \\
&= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau)(\mathbf{u}^\mu d\tau + \sigma^\mu dW^\mu) \\
&= \mathbf{u}^\mu d\tau \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau) + \\
&+ \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau) \sigma^\mu dW^\mu
\end{aligned}
\tag{2.16}
$$

From this, we derive that:

$$\langle dx^\mu \rangle = u^\mu d\tau \tag{2.17}$$

Similarly, we can find the expected value for dy^μ as follows:

$$\langle dy^\mu \rangle = v^\mu d\tau \tag{2.18}$$

Similarly, the calculation of $\langle dx^\nu dx^\mu \rangle$ is performed using:

$$
\begin{aligned}
\langle dx^\nu dx^\mu \rangle &= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau) dx^\mu dx^\nu \\
&= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau)(u^\mu d\tau + \sigma^\mu dW^\mu)(u^\nu d\tau + \sigma^\nu dW^\nu) \\
&= \int d\mathbf{x}_{\tau+d\tau} d\mathbf{y}_{\tau+d\tau} \mathcal{N}((\mathbf{x}_{\tau+d\tau}, \mathbf{y}_{\tau+d\tau}) | (\mathbf{x}_\tau, \mathbf{y}_\tau), \sigma^\mu d\tau) \times \\
&\times (u^\mu u^\nu d^2\tau + u^\mu \sigma^\nu d\tau dW^\nu + \sigma^\mu dW^\mu u^\nu d\tau + \sigma^\mu dW^\mu \sigma^\nu dW^\nu)
\end{aligned}
\tag{2.19}
$$

From which we derive:

$$\langle dx^\nu dx^\mu \rangle = 0, \mu \neq \nu, \quad \left\langle (dx^\mu)^2 \right\rangle = \sigma_x^\mu \sigma_x^\mu d\tau \tag{2.20}$$

Respectively, for the imaginary components:

$$\langle dy^\nu dy^\mu \rangle = 0, \mu \neq \nu, \quad \left\langle (dy^\mu)^2 \right\rangle = \sigma_y^\mu \sigma_y^\mu d\tau \tag{2.21}$$

After substituting the above equations into equation (2.15) , we derive the stochastic HJB equation for the real part of our system:

$$
\begin{aligned}
&- \partial_\tau J_R(\tau, \mathbf{x}) = \min_{\mathbf{u},\mathbf{v}} \left(\mathcal{L}_R(\tau, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) + u^\mu \partial_{\mathbf{x}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + v^\mu \partial_{\mathbf{y}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) \right) + \\
&+ \frac{1}{2} \sum_{\mu=0}^{3} \sigma_x^\mu \sigma_x^\mu \partial_{\mathbf{x}^\mu \mathbf{x}^\mu} J_R(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \frac{1}{2} \sum_{\mu=0}^{3} \sigma_y^\mu \sigma_y^\mu \partial_{\mathbf{y}^\mu \mathbf{y}^\mu} J_R(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau)
\end{aligned}
\tag{2.22}
$$

In a similar manner, the equation for the imaginary part is derived:

$$
\begin{aligned}
&- \partial_\tau J_I(\tau, \mathbf{x}) = \min_{\mathbf{u},\mathbf{v}} \left(\mathcal{L}_I(\tau, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) + u^\mu \partial_{\mathbf{x}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) + v^\mu \partial_{\mathbf{y}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) \right) + \\
&+ \frac{1}{2} \sum_{\mu=0}^{3} \sigma_x^\mu \sigma_x^\mu \partial_{\mathbf{x}^\mu \mathbf{x}^\mu} J_I(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \frac{1}{2} \sum_{\mu=0}^{3} \sigma_y^\mu \sigma_y^\mu \partial_{\mathbf{y}^\mu \mathbf{y}^\mu} J_I(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau)
\end{aligned}
\tag{2.23}
$$

The optimal control policies $\mathbf{u}$ and $\mathbf{v}$, which minimize the expected cost, can be determined from the following conditions:

$$\begin{aligned} \partial_{u_\mu}\mathrm{Re}(L_z(w_\mu, z)) + \partial_{x^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) = 0 \\ \partial_{v_\mu}\mathrm{Re}(L_z(w_\mu, z)) + \partial_{y^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) = 0 \end{aligned} \tag{2.24}$$

For the imaginary part of the action, the minimization conditions are obtained from:

$$\begin{aligned} \partial_{u_\mu}\mathrm{Im}(L_z(w_\mu, z)) + \partial_{x^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) = 0 \\ \partial_{v_\mu}\mathrm{Im}(L_z(w_\mu, z)) + \partial_{y^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) = 0 \end{aligned} \tag{2.25}$$

These conditions ensure that the stochastic Hamilton-Jacobi-Bellman (HJB) equations for both the real and imaginary actions are satisfied, leading to the minimum expected value of the action across all possible trajectories of the system. Later, we will prove that the equations derived for the optimal control policies, specifically equations (2.24) and (2.25), are equivalent.

By multiplying equation (2.23) by the imaginary unit i and adding it to equation (2.22), we obtain the combined formal form of the stochastic Hamilton-Jacobi-Bellman (HJB) equation for the complex action.

$$\begin{aligned} &-\partial_\tau J(\tau, \mathbf{x}) = \min_{\mathbf{u},\mathbf{v}}(\mathcal{L}(\tau, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) + u^\mu \partial_{\mathbf{x}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + v^\mu \partial_{\mathbf{y}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + \\ &+ i u^\mu \partial_{\mathbf{x}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) + i v^\mu \partial_{\mathbf{y}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y})) + \\ &+ \frac{1}{2}\sum_{\mu=0}^{3} \sigma_x^\mu \sigma_x^\mu \partial_{\mathbf{x}^\mu\mathbf{x}^\mu} J_R(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \frac{1}{2}\sum_{\mu=0}^{3} \sigma_y^\mu \sigma_y^\mu \partial_{\mathbf{y}^\mu\mathbf{y}^\mu} J_R(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + \\ &+ i\frac{1}{2}\sum_{\mu=0}^{3} \sigma_x^\mu \sigma_x^\mu \partial_{\mathbf{x}^\mu\mathbf{x}^\mu} J_I(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) + i\frac{1}{2}\sum_{\mu=0}^{3} \sigma_y^\mu \sigma_y^\mu \partial_{\mathbf{y}^\mu\mathbf{y}^\mu} J_I(\tau, \mathbf{x}_\tau, \mathbf{y}_\tau) \end{aligned} \tag{2.26}$$

Since the complex intermediate action $J(\tau, \mathbf{z})$ is analytic, the following equations are satisfied:

$$\partial_{\mathbf{z}^\mu} J(\tau, \mathbf{z}) = \partial_{\mathbf{x}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + i\partial_{\mathbf{x}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) = \partial_{\mathbf{y}^\mu} J_I(\tau, \mathbf{x}, \mathbf{x}) - i\partial_{\mathbf{y}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) \tag{2.27}$$

The Cauchy-Riemann equations are:

$$\partial_{\mathbf{x}^\mu} J_R(\tau, \mathbf{x}) = \partial_{\mathbf{y}^\mu} J_I(\tau, \mathbf{y}), \quad \partial_{\mathbf{x}^\mu} J_I(\tau, \mathbf{y}) = -\partial_{\mathbf{y}^\mu} J_R(\tau, \mathbf{x}) \tag{2.28}$$

The second derivative of the intermediate action is:

$$\begin{aligned} \partial_{\mathbf{z}^\mu\mathbf{z}^\mu} J(\tau, \mathbf{z}) =& \partial_{\mathbf{x}^\mu\mathbf{x}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + i\partial_{\mathbf{x}^\mu\mathbf{x}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) = \\ =& -\partial_{\mathbf{y}^\mu\mathbf{y}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) - i\partial_{\mathbf{y}^\mu\mathbf{y}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y}) \end{aligned} \tag{2.29}$$

Using the above equations, we can simplify the HJB equation:

$$\begin{aligned} &-\partial_\tau J(\tau, \mathbf{z}) = \min_{\mathbf{w}}(\mathcal{L}(\tau, \mathbf{z}, \mathbf{w}) + w^\mu \partial_{\mathbf{x}^\mu} J_R(\tau, \mathbf{x}, \mathbf{y}) + i w^\mu \partial_{\mathbf{x}^\mu} J_I(\tau, \mathbf{x}, \mathbf{y})) + \\ &+ \frac{1}{2}\sum_{\mu=0}^{3} \sigma_x^\mu \sigma_x^\mu \partial_{\mathbf{z}^\mu\mathbf{z}^\mu} J(\tau, \mathbf{z}_\tau) - i\frac{1}{2}\sum_{\mu=0}^{3} \sigma_y^\mu \sigma_y^\mu \partial_{\mathbf{z}^\mu\mathbf{z}^\mu} J(\tau, \mathbf{z}_\tau) = \\ &= \min_{\mathbf{w}}(\mathcal{L}(\tau, \mathbf{z}, \mathbf{w}) + w^\mu \partial_{\mathbf{z}^\mu} J(\tau, \mathbf{z})) + \frac{1}{2}\sum_{\mu=0}^{3} \sigma^\mu \sigma^\mu \partial_{\mathbf{z}^\mu\mathbf{z}^\mu} J(\tau, \mathbf{z}_\tau) \end{aligned} \tag{2.30}$$

Finally, we can formally write the complex HJB equation:

$$-i\partial_\tau J_z(\tau,\mathbf{z}) = \min_{\mathbf{w}}\left(\mathcal{L}_z(\tau,\mathbf{z},\mathbf{w}) + w^\mu \partial_{z^\mu} J(\tau,\mathbf{x}) + \frac{1}{2}\sum_{\mu=0}^{3}\sigma^\mu\sigma^\mu \partial_{\mathbf{z}^\mu\mathbf{z}^\mu} J_z(\tau,\mathbf{z}_\tau)\right), \tag{2.31}$$

where the complex diffusion coefficient satisfies

$$\sigma^\mu\sigma^\mu = \sigma^\mu_x\sigma^\mu_x - i\sigma^\mu_y\sigma^\mu_y \tag{2.32}$$

It is crucial to emphasize again that this formal form of the HJB equation is conceptually meaningful only when considering the distinct equations for its real (2.23) and imaginary (2.22) parts.

It is clear from its definition that the boundary condition for $J_z(\tau,\mathbf{z}_\tau)$ is:

$$J_z(\tau_f,\mathbf{z}_{\tau_f}) = 0 \tag{2.33}$$

Sinces $L_z(\mathbf{w}_\mu,\mathbf{z})$ is an analytic function, the derivative operator and the operator for taking the real part commute. Consequently, equation (2.24) can be expressed as:

$$\begin{aligned} \mathrm{Re}(\partial_{u_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{x^\mu} J_R(\tau,\mathbf{x},\mathbf{y}) = 0 \\ \mathrm{Re}(\partial_{v_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{y^\mu} J_R(\tau,\mathbf{x},\mathbf{y}) = 0 \end{aligned} \tag{2.34}$$

Similarly, from equation (2.25), we can derive the equations for the optimal control that result from minimizing the imaginary part of the intermediate action:

$$\begin{aligned} \mathrm{Im}(\partial_{u_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{x^\mu} J_I(\tau,\mathbf{x},\mathbf{y}) = 0 \\ \mathrm{Im}(\partial_{v_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{\mathbf{y}^\mu} J_I(\tau,\mathbf{x},\mathbf{y}) = 0 \end{aligned} \tag{2.35}$$

We will prove that Equations (2.39) and (2.35) are equivalent. To do this, we find the derivatives:

$$\begin{aligned} \partial_{u_\mu} L_z(w_\mu,z) = \partial_{w_\mu} L_z(\mathbf{w},\mathbf{z})\frac{\partial w_\mu}{\partial u_\mu} = \partial_{w_\mu} L_z(\mathbf{w},\mathbf{z}) \\ \partial_{v_\mu} L_z(w_\mu,z) = \partial_{w_\mu} L_z(\mathbf{w},\mathbf{z})\frac{\partial w_\mu}{\partial v_\mu} = i\partial_{w_\mu} L_z(\mathbf{w},\mathbf{z}) \end{aligned} \tag{2.36}$$

If we substitute the above derivatives into Equation (2.39), multiply the second equation by the imaginary unit, and add it to the first equation, we obtain:

$$\mathrm{Re}(\partial_{w_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + i\mathrm{Re}(i\partial_{w_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{x^\mu} J_R(\tau,\mathbf{x},\mathbf{y}) + i\partial_{y^\mu} J_R(\tau,\mathbf{x},\mathbf{y}) = 0 \tag{2.37}$$

Similarly, if we substitute the above derivatives into Equation (2.35), multiply the second equation by the imaginary unit, and subtract it from the first equation, we obtain

$$\mathrm{Im}(\partial_{w_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) - i\mathrm{Im}(i\partial_{w_\mu} L_z(\mathbf{w}_\mu,\mathbf{z})) + \partial_{x^\mu} J_I(\tau,\mathbf{x},\mathbf{y}) - i\partial_{y^\mu} J_I(\tau,\mathbf{x},\mathbf{y}) = 0 \tag{2.38}$$

From the Cauchy-Riemann equation (2.28), the definition of the complex derivative (2.27), and the identities $\mathrm{Re}(Z) + i\mathrm{Re}(iZ) = Z$ and $-\mathrm{Im}(iZ) + i\mathrm{Im}(Z) = Z$, where Z is any complex number, we prove that both equations can be written as:

$$\partial_{w_\mu} L_z(\mathbf{w}_\mu,\mathbf{z}) + \partial_{z^\mu} J_R(\tau,\mathbf{x},\mathbf{y}) = 0 \tag{2.39}$$

The equation for the optimal control policy, $\mathbf{w} = \mathbf{u} + i\mathbf{v}$, maintains the same form as that of the real-valued HJB equation. However, it requires taking a complex derivative of the intermediate action, as can be observed.

In the next section, we will illustrate this concept with a specific example, applying a concrete Lagrangian for a relativistic particle in an electromagnetic field.

3 Analytic continuation of the Covariant Relativistic Lagrangian

The relativistic Lagrangian for a particle in an electromagnetic field is given by:

$$\Lambda = \tilde{\sigma} mc\sqrt{\tilde{\sigma} u_\mu u^\mu} + qA_\mu u^\mu, \tag{3.1}$$

where q represents the charge of the particle and A_μ denotes the 4-vector potential. The symbol $\tilde{\sigma}$ indicates the sign convention for the metric tensor: it takes the value of $+1$ for the metric with diagonal elements $(1, -1, -1, -1)$ and -1 for the metric $(-1, 1, 1, 1)$, as elaborated in [11].

The components of the four-velocity of the particle are related to the speed of light by the equation:

$$u_\mu u^\mu = \tilde{\sigma} c^2. \tag{3.2}$$

This relation, referred to as the "weak equation" by Dirac, allows us to treat u^μ as unconstrained quantities until all differentiation operations have been carried out, at which point we impose the condition of equation (3.2) (see [12] Chapter 7.10). This will be the approach we employ as we seek to minimize the expected value of the stochastic action.

The Lagrangian in equation (3.1) is a real-valued function of real arguments – the coordinates and velocity of the particle. In complex stochastic optimal control, we assume that this Lagrangian is the analytic continuation of the real-valued Lagrangian referenced in equation (3.1).

$$L_z(\mathbf{z}, w_\mu) = \tilde{\sigma} mc\sqrt{\tilde{\sigma} w_\mu w^\mu} + qA_\mu(\tau, \mathbf{z}) w^\mu, \tag{3.3}$$

The "weak equation" should be also analytically continued:

$$w_\mu w^\mu = \tilde{\sigma} c^2. \tag{3.4}$$

The derivative of the complex Lagrangian can be calculated using the "weak equation" (3.4):

$$\partial_{w_\mu} L_z(w_\mu, z) = \partial_{w_\mu}(\tilde{\sigma} mc\sqrt{\tilde{\sigma} w_\mu w^\mu} + qA_\mu(\tau, \mathbf{z}) w^\mu) = \frac{1}{2\sqrt{\tilde{\sigma} w_\mu w^\mu}} 2w_\mu + qA_\mu(\tau, \mathbf{z}), \tag{3.5}$$

Finally the complex velocity is:

$$w_\mu = -\frac{1}{m}(\partial_{z^\mu} J(\tau, \mathbf{z}) + qA_\mu(\tau, \mathbf{z})) \tag{3.6}$$

4 Stochastic equation of motion

In my previous work [5], I demonstrated that to linearize the HJB equation, the diffusion coefficient must be purely imaginary:

$$g_{\mu\mu}\sigma^{\mu}\sigma^{\mu} = -\frac{2i\epsilon_r\hbar}{m} \tag{4.1}$$

From equation (2.32), it is evident that the real stochastic coefficient is equal to zero:

$$g_{\mu\mu}\sigma_x^{\mu}\sigma_x^{\mu} = 0 \tag{4.2}$$

Conversely, the imaginary stochastic coefficient is represented by:

$$g_{\mu\mu}\sigma_y^{\mu}\sigma_y^{\mu} = \frac{2\epsilon_r\hbar}{m} \tag{4.3}$$

This leads us to the stochastic equations of motion:

$$dx_{\mu} = u_{\mu}ds \tag{4.4}$$

$$dy_{\mu} = v_{\mu}ds + \epsilon_r g_{\mu\mu}\sqrt{\frac{2\hbar}{m}}dW_{\mu}, \quad \mu = 0..3 \tag{4.5}$$

This result is particularly interesting because it reveals that only the imaginary part of the particle's motion is governed by a random process, while the real part of the coordinate remains deterministic.

5 Conclusion

In this paper, we demonstrate that it is possible to formally substitute real variables such as the four-coordinates, control policy, and cost-to-go function in the HJB equation with complex-valued ones. However, such substitution can only be performed if we define the complex-valued Lagrangian as an analytical continuation of the real-valued Lagrangian of the system.

In consequence of Cochy-Rieman theorem we prove that the diffusion coefficient in the complex stochastic HJB equation is: $\sigma^{\mu}\sigma^{\mu} = \sigma_x^{\mu}\sigma_x^{\mu} - i\sigma_y^{\mu}\sigma_y^{\mu}$.

The last result is quite interesting because in all previous works [1, 2, 3, 4, 5] that derive quantum mechanical equations it is proved that in order to linerize the HJB equation it is required the square of the diffusion coefficient to be a pure imaginary number. This result tell us that the equation only for the imaginary coordinates in the quantum mechnical equations is stochastic.

Bibliography

[1] Lech Papiez. Stochastic optimal control and quantum mechanics. *Journal of Mathematical Physics*, 23(6):1017–1019, 06 1982.

[2] L Papiez. Stochastic optimal control quantization of a free relativistic particle, 1981.

[3] Jussi Lindgren and Jukka Liukkonen. Quantum mechanics can be understood through stochastic optimization on spacetimes. *Scientific Reports*, 9(1), December 2019.

[4] Ciann-Dong Yang and Shiang-Yi Han. Extending quantum probability from real axis to complex plane. *Entropy*, 23:210, 02 2021.

[5] Vasil Yordanov. Derivation of dirac equation from the stochastic optimal control principles of quantum mechanics. *Scientific Reports*, 14(1):6507, 2024.

[6] Hilbert J. Kappen. Path integrals and symmetry breaking for optimal control theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2005(11):P11011–P11011, nov 2005.

[7] Hilbert Kappen. Optimal control theory and the linear bellman equation. *Neural Networks*, 08 2011.

[8] Wendell H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Stochastic Modelling and Applied Probability. Springer, New York, NY, 2 edition, 2006. Number of Pages: XVII, 429.

[9] Vasil Yordanov. Derivation of the stochastic hamilton-jacobi-bellman equation, 2023.

[10] Richard Bellman. The theory of dynamic programming. *Operations Research*, 2(3):275–285, 1954.

[11] Alain J. Brizard. On the proper choice of a Lorentz-covariant relativistic Lagrangian. *arXiv e-prints*, page arXiv:0912.0655, December 2009.

[12] Herbert Goldstein and Charles Poole. *Classical mechanics*. Pearson education asia, Delhi, 3rd ed edition, 2002. Includes subject and author index, selected bibliography.