Kinetic Model for Dark Energy - Dark Matter Interaction: Scenario for the Hubble Tension

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Abstract We analyze a model for Dark Energy - Dark Matter interaction, based on a decaying process of the former constituents into the latter ones. The dynamical equations are constructed following a kinetic formulation, which separates the interacting fluctuations from equilibrium distribution of the both the species. The emerging dynamical picture consists of coupled equations, which are specialized in the case of a Dark Energy equation of state parameter: we deal with a modified Lambda Cold Dark Matter model, which is investigated versus a possible interpretation of the Hubble tension. We compare our model with data corresponding to 6 points of the expansion rate from Type Ia Supernovae. We show that, the proposed model suitably fits data according to a value of the Hubble constant compatible with the SHOES Collaboration measurement. The tension is solved because, essentially for redshift greater than one, the correction to the Lambda Cold Dark Matter model vanishes and its presence does not affect the Planck measurements.

Keywords Hubble tension · Cosmological models · Dark Energy - Dark Matter

1 Introduction

Since the original Hubble measurement, the value of the Hubble constant, H_0, has played a central role in experimental and theoretical cosmology. Only in the epoch of the so-called "precision cosmology" it has been detected with a good degree of accuracy [1,2,3]. However, in recent years, the reliability of the Hubble constant has been subjected to a surprising fate: clear observational evidences [3,4,5,6,7] suggested a possible dependence of H_0 on the average redshift of the sources adopted for its determination. This observational discrepancy, known as the "Hubble tension", calls attention to an explanation based on a possible unrecognized redshift evolution of astrophysical sources (see, e.g. the analysis in [8,9,10], or on possible "new physics" as discussed in [11,12,13].

In [14] (see also [15,16,17,18]), it was argued that the decaying tendency of H_0 with the increasing different source redshift can also be recovered within the Type Ia Supernova (SNIa) Pantheon sample and it is well-reproduced by the functional form $H_0 \propto (1+z)^{-\alpha}$, where α is a parameter of the order 10^{-2}. Such behavior has been properly reproduced via modified $f(R)$-gravity in the Jordan frame [19,20,21], as investigated in [22]. For further studies facing the Hubble tension via a modified gravity theory, see [23,24], while for approaches involving also evolutionary phantom energy see [25,26]. A recent model using a slow-rolling scalar dynamics and dealing with the binned SNIa data, as processed in [14], see [27]. Finally, for a study of the creation of scalar matter via the cosmological gravitational field, see [28].

However, the difficulty to reconcile the Baryon Acoustic Oscillation (BAO) [29] data to those ones due to the SH0ES Collaboration on SNIa [7] (which live in overlapping redshift regions) has motivated early Universe modification of the dynamics, for instance affecting the sound horizon value, in addition to late Universe models. In [30,31,32], it has been suggested that the most reliable solution to the Hubble tension could require a combination of late and early Universe modified physics.

Here, to address the Hubble tension, we consider a Dark Energy (DE) - Dark Matter (DM) interaction model (for a review on this topic, see [33]), based on

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a kinetic approach to describe the decaying process of the DE constituents into DM particles. Thus, we assume that the equilibrium configurations for the species are perturbed by small fluctuations associated with a Boltzmann equation typical of a decay process from one species into another one. Actually, while the DM decay process is described via the kinetic theory, by reconstructing the macroscopic equation for the DE fluctuation, the transfer of energy to DM via particle creation is recovered by requiring the conservation of the total energy-momentum tensor for the two species. As a result of this underlying physical scenario, we get a modified Friedmann equation containing an additional evolutionary term with respect to the Lambda Cold DM (ΛCDM) model, which is the net effect of the DE decaying process. Clearly, when this additional term vanishes, the model is exactly reduced to a ΛCDM picture, in the correspondence to the instant when the DE-DM interaction starts.

We show how the proposed model is able to reconcile the measurements of the Hubble constants due to the Planck Satellite and the SHOES Collaboration. Actually, the former is unable to identify the interaction process and the corresponding modified dynamics. This is because we find that the Cosmic Microwave Background (CMB) luminosity distance is not affected by the obtained late Universe modification within the intrinsic error. The SNIa measurement is instead sensitive to the evolutionary term due to the interaction, and then the achieved value of H_0 is correspondingly enhanced. To reach this result, we use the data provided in [34] for the expansion rate, which give specific constraints on the range of the model free parameters.

2 Theoretical framework

We consider here a model of DE-DM interaction, based on the idea that a fraction of the DE constituents decay versus the creation of DM particles (this process is viewed in a kinetic formulation).

In order to describe the decaying process of DE particles (labelled by the suffix de), we first consider (according to the Planck data analysis in [35]) a flat isotropic Universe, described by the line element

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right) \,, \tag{1}$$

where t denotes the synchronous time (we used $c = 1$ units), (x, y, z) are Cartesian coordinates and $a(t)$ is the cosmic scale factor, governing the Universe expansion. Then, we set the DE distribution function as $f = f_{eq}(E_{de}) + \delta f(E_{de})$, where E_{de} denotes the particle energy, f_{eq} the dominant equilibrium contribution and δf describes the decaying process. Thus, while f_{eq} is responsible for the DE density of the Universe, i.e. $\rho_{de} = \rho^0_{de}(1+z)^{3w+3}$ (ρ^0_{de} being its present-day value and $w < -1/3$ the equation of state parameter), δf is subjected to the following Boltzmann equation [36]:

$$\partial_z \delta f + \frac{P_{de}}{1+z}\partial_{P_{de}}\delta f = \frac{\Gamma}{(1+z)H}\delta f \,, \tag{2}$$

where P_{de} is DE particle momentum, $H \equiv a^{-1}da/dt$ is the Hubble parameter and we are using the redshift time variable $z(t) = a_0/a - 1$ (a_0 is the today scale factor). We thus get $d(...)/dt = -(1+z)Hd(...)/dz$. In the equation above, Γ is a constant such that $1/\Gamma$ corresponds to the decaying time of the DE species.

We now recall the definition of the energy density fluctuation as:

$$\delta\rho_{de} \equiv \frac{g_{de}}{2\pi^2}\int P^2_{de}E_{de}\delta f\; dP_{de} \,, \tag{3}$$

where g_{de} is the particle degree of freedom and we used the isotropy of the momentum space. Hence, we get the following equation:

$$\frac{d\delta\rho_{de}}{dz} - \frac{3(1+w)}{1+z}\delta\rho_{de} = \frac{\Gamma}{(1+z)H}\delta\rho_{de} \,, \tag{4}$$

where we used the following phenomenological relation for the DE fluctuation pressure δp_{de}

$$\delta p_{de} \equiv \frac{g_{de}}{2\pi^2}\int \frac{P^4_{de}}{3E_{de}}\delta f\; dP_{de} = w\,\delta\rho_{de} \,. \tag{5}$$

This expression, compared with Eq.(3), for negative values of w (like DE), leads to particles with $E^2_{de} < 0$. This does not mean that we deal with tachyons but, more realistically, kinetic theory would suggest that the DE constituents are not elementary free particles, while corresponding to relativistic bounded states (self-interacting micro-clusters).

The solution of Eq.(4) reads

$$\delta\rho_{de} = \delta\rho^0_{de}(1+z)^{3w+3}\times \\ \times\exp\left[\Gamma\int_0^z \frac{dz'}{(1+z')H(z')}\right] \,, \tag{6}$$

where $\delta\rho^0_{de} = \delta\rho_{de}(z=0)$. As soon as, we require that the sum of the DE and DM energy-momentum tensors be conserved, we find the following equation for the DM energy density fluctuation $\delta\rho_{dm}$:

$$\frac{d\delta\rho_{dm}}{dz} - \frac{3}{1+z}\delta\rho_{dm} = -\frac{\Gamma}{(1+z)H}\delta\rho_{de} \,. \tag{7}$$

Now, defining $\delta \equiv \delta\rho_{de} + \delta\rho_{dm}$, from Eqs.(4) and (7), we get the following dynamical equation:

$$\frac{d\delta}{dz} - \frac{3}{1+z}\delta = -\frac{3|w|}{1+z}\delta\rho_{de} = \\ = -3 \mid w \mid \delta\rho_{de}^0(1+z)^{3w+2}\times \\ \times \exp\left[\Gamma \int_0^z \frac{dz'}{(1+z')H(z')}\right], \tag{8}$$

where we emphasized the negative values of w for the DE contribution and we made use of the solution (6).

Finally, we observe that the baryonic component of the Universe is not involved in the particle creation process and its energy density ρ_b satisfies the standard equation

$$\frac{d\rho_b}{dz} - \frac{3}{1+z}\rho_b = 0\,. \tag{9}$$

In what follows, we denote by $\rho_m = \rho_{dm} + \rho_b$ the total equilibrium value of the matter component of the Universe.

3 Dynamics for the Hubble parameter

The Friedmann equation corresponding to the scenario depicted above takes the following form:

$$H^2(z) = \frac{\chi}{3}\left(\rho_m + \rho_{de} + \delta\right), \tag{10}$$

χ being the Einstein constant. We introduce the quantities

$$\Omega_m^0 \equiv \frac{\rho_m^0}{3H_*^2}\,, \qquad \Omega_{de}^0 \equiv \frac{\chi\rho_{de}^0}{3H_*^2}\,, \qquad \Delta \equiv \frac{\chi\delta}{3H_*^2}\,, \tag{11}$$

where ρ_m^0 and ρ_{de}^0 are the present-day value of the matter and DE energy density, respectively, while H_* is a fiducial Hubble constant.

In this scheme, Eq.(10) can be recast as

$$E_*^2(z) \equiv \frac{H^2(z)}{H_*^2} = \\ = \Omega_m^0(1+z)^3 + \Omega_{de}^0(1+z)^{3w+3} + \Delta(z)\,, \tag{12}$$

to be coupled with Eq.(8), here restated in the form

$$\frac{d\Delta}{dz} = \frac{3}{1+z}\Delta - 3|w|\bar{\Delta}(1+z)^{3w+2}\times \\ \times \exp\left[\bar{E}_* \int_0^z \frac{dz'}{(1+z')E_*(z')}\right], \tag{13}$$

where we set $\bar{\Delta} \equiv \chi\delta\rho_{de}^0/3H_*^2$ and $\bar{E}_* \equiv \Gamma/H_*$. Eqs.(12) and (13) formally describe the dynamics of the Hubble parameter.

3.1 Modified ΛCDM model

In order to develop a dynamical scheme close to the ΛCDM model, we study in detail the case $w = -1$, i.e. we set $\rho_{de} = const$. It is immediate to specialize Eqs.(12) and (13) to this scenario. By introducing the definition $\Delta \equiv (1+z)^3\mathcal{D}$ and the auxiliary function $F(z)$, the system rewrites

$$E_*^2(z) = \Omega_m^0(1+z)^3\Big(1 + \frac{\mathcal{D}}{\Omega_m^0}\Big) + 1 - \Omega_m^0\,, \tag{14}$$

$$\frac{d\mathcal{D}}{dz} = -\frac{3\bar{\Delta}}{(1+z)^4}e^{F(z)}\,, \tag{15}$$

$$\frac{dF}{dz} = \frac{\bar{E}_*}{(1+z)E_*(z)}\,. \tag{16}$$

where, according to the standard ΛCDM model, we assume $\Omega_{de}^0 = 1 - \Omega_m^0$ and, for completeness, we remark that we always coherently assume $F(z=0) = 0$.

From Eq.(15), $\mathcal{D}$ is found to be a decreasing function of z. Thus, if we indicate by z_i the instant when $\mathcal{D}$ vanishes (i.e. also $\Delta(z_i) = 0$), then it will increase up to the today value $\mathcal{D}(z=0) \equiv \mathcal{D}_0$. We regard z_i as the instant at which, in the past, the decaying process of DE versus DM starts. At $z \geq z_i$, the process is not present and the model exactly coincides with a ΛCDM dynamics of the Universe.

If we introduce $E(z) \equiv H(z)/H_0$, following our formalism, it results

$$E(z) = E_*(z)H_*/H_0\,, \tag{17}$$

where now

$$H_0 = H_*\sqrt{1+\mathcal{D}_0}\,, \tag{18}$$

and the Hubble parameter finally reads

$$H^2(z) = H_0^2\,\frac{\Omega_m^0(1+z)^3\big(1+\mathcal{D}/\Omega_m^0\big) + 1 - \Omega_m^0}{1+\mathcal{D}_0}\,. \tag{19}$$

In order to reconcile the late Universe predictions with the CMB data, we set (here and in the following H is in units: km s^{-1} Mpc^{-1})

$$H_* = H_0^{Pl} = 67.4\,, \tag{20}$$

$$\Omega_m^0 = \Omega_m^{0Pl} = 0.315\,, \tag{21}$$

thus implementing the corresponding values measured by Planck experiment. In this respect, in order the satellite be unable to recognize if the addressed late Universe modification is present (i.e. the term $\mathcal{D}$), the impact of such a contribution on the CMB luminosity distance must be smaller than the error in the measurement itself.

We conclude this section by discussing the physical meaning of the parameter Γ regarded as the inverse decaying time of the DE constituents. Since H_* is assumed the value of the Hubble constant measured by Planck, it is clear that, in order the decaying process to take place at the equilibrium (*de facto* being efficient), we have to require $\bar{E}_* \gtrsim 1$. In this respect, we observe that, since DE starts to dominate at the Universe half age [37], it is reasonable to require that the decaying time of DE be a fraction of the Universe age, greater than 1/2, in order to avoid a complete decaying of the DE contribution before $z = 0$. As a consequence, it is natural to require that $\bar{E}_* \lesssim 2$. We also observe that the quantity $\bar{\Delta}$ is the critical parameter for the DE fluctuations today. According to the perturbation scheme adopted above, we have to require that this contribution is small in comparison to the DE equilibrium component of the present Universe. Moreover, since the DM contribution is smaller than the DE one, we assume that the corresponding critical parameter for the DM fluctuations be even smaller.

4 Model testing

In this section, we constrain the model above using the 6 points of the expansion rate for $0.07 \leq z \leq 1.5$ provided in [34] for SNIa. This measurements alone actually constitutes an almost identical characterization of the complete SNIa sample and they thus represent a faithfully data compression, combining the Pantheon sample with 15 distant additional sources.

According the discussion above, we first set the physical constants of the model at the following nominal representative values: $\bar{E}_* = 1.5$ and $\bar{\Delta} = 0.15$. The model parameter $\mathcal{D}_0$ is than fixed by the best fit of the 6 points of $E(z)$ in the late Universe. The fit result for Eq.(17) is:

$$\mathcal{D}_0 = 0.1724 \pm 0.0458\ , \tag{22}$$

with a $\chi^2 = 0.12$ (p-value= 0.998). The obtained profile of $E(z)$ is depicted in Fig.1, together with the addressed measurements. According to Eq.(18), this result provides

$$H_0 = 72.98\ \pm 1.42, \tag{23}$$

which is compatible with the SHOES measurement of the Hubble constant [7]: 73.6 ± 1.1.

Moreover, by analyzing the evolution of the function $\mathcal{D}(z)$, we find for this case $z_i \simeq 0.88$. We recall that this corresponds to the instant at which $\mathcal{D} = 0$ and for $z \geq z_i$, the model coincides with a flat ΛCDM dynamics. The obtained profile of the Hubble parameter in Eq.(19) is finally reported in Fig.2, together with the expression specified for the Planck measured values of the parameters, i.e. $H_{Pl}(z) = H_0^{Pl}(\Omega_m^{0Pl}(1+z^3) + 1 - \Omega_m^{0Pl})^{1/2}$. We finally remark that, as discussed above, the viability of this scenario must be guaranteed the by the undetectable character of the additional late Universe contribution on the CMB luminosity distance. In this respect, the profile of $H(z)$ for the obtained parameters provides $d_L^{CMB} = 50.67$ (center value) which results compatible with the Planck value $d_{LPl}^{CMB} = 51.17 \pm 0.59$.

Although the limited number of data prevents a statistically meaningful test of all the free parameter of the model, nonetheless it is remarkable that a range of the physical constants exists for which the SHOES value of the Hubble constant is reconciled to the Planck measurements. Actually the latter result insensitive to the dynamical modification of the very late Universe here considered.

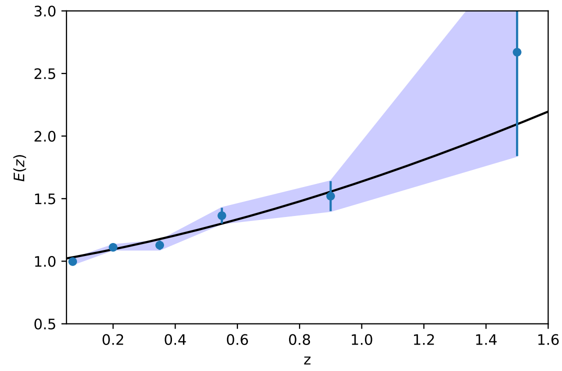

Fig. 1 Plot of the best fit for $E(z)$ constructed with Eq.(19) (black). Data points from [34] are in blue with the corresponding errors.

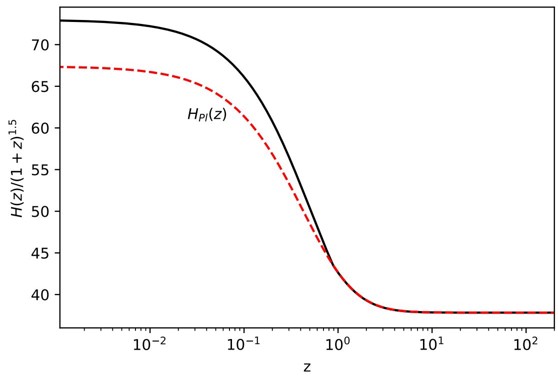

Fig. 2 Plot of the (normalized) Hubble parameters $H(z)$ from Eq.(19) (black) and $H_{Pl}(z)$ (cfr. the text) with the errors (red).

5 Concluding remarks

We derived a kinetic model for the DE-DM interaction, which separates the equilibrium distribution function of the two species into their equilibrium component and small corrections, accounting for the decaying process of DE constituents into DM particles. This way, we arrive to a set of coupled equations, which are reduced to their simplified form when the DE parameter is taken $w = -1$. This choice has been justified in order to construct a modified ΛCDM dynamics, appropriate for a comparison with the SNIa data.

The model we analyzed in detail contained three free parameters, $\bar{E}_*$ (corresponding to the ratio between the Universe age and the decaying time of DE constituents), $\bar{\Delta}$ (which fixes the value in $z = 0$ for the DE fluctuation critical parameter) and $\mathcal{D}_0$ (which is the most important quantity since it determines $H(z = 0)$). Then, we compared the model we set up with the 6 points of the expansion rate for the SNIa, which are provided in [34]. The data analysis has been performed by fixing the values of $\bar{E}_*$ and $\bar{\Delta}$, while $\mathcal{D}_0$ has been left free for the fitting procedure. The resulting best fit indicates that the value of H_0 is reconciled with the value provided by the SH0ES collaboration within the errors. This achievement is relevant for the Hubble tension since the modification of the ΛCDM model vanishes as $z \gtrsim 1$ and, hence the Planck Satellite can not detect such a modification, because the obtained fluctuation induced on the luminosity distance is smaller than the corresponding fluctuation due to the intrinsic error of the measurement. Moreover, since we fixed $\bar{E}_* = 1.5$, it is worth noting that the relatively late triggering of the DE decay process is the consequence of the significantly large value of the corresponding characteristic time.

The proposed scenario has the merit to demonstrate how a suitable late Universe phenomenon is unable to significantly affect the Planck Satellite data (and the corresponding ΛCDM model as the best fit), but it is strong enough to provide a higher value of H_0, which becomes compatible with the SH0ES data analysis.

References

1. P. de Bernardis, P.A.R. Ade, J.J. Bock, J.R. Bond, J. Borrill, A. Boscaleri, K. Coble, B.P. Crill, G. De Gasperis, P.C. Farese, P.G. Ferreira, K. Ganga, M. Giacometti, E. Hivon, V.V. Hristov, A. Iacoangeli, A.H. Jaffe, A.E. Lange, L. Martinis, S. Masi, P.V. Mason, P.D. Mauskopf, A. Melchiorri, L. Miglio, T. Montroy, C.B. Netterfield, E. Pascale, F. Piacentini, D. Pogosyan, S. Prunet, S. Rao, G. Romeo, J.E. Ruhl, F. Scaramuzzi, D. Sforna, N. Vittorio, Nature **404**(6781), 955 (2000). DOI 10.1038/35010035
2. C.L. Bennett, D. Larson, J.L. Weiland, N. Jarosik, G. Hinshaw, N. Odegard, K.M. Smith, R.S. Hill, B. Gold, M. Halpern, E. Komatsu, M.R. Nolta, L. Page, D.N. Spergel, E. Wollack, J. Dunkley, A. Kogut, M. Limon, S.S. Meyer, G.S. Tucker, E.L. Wright, ApJ S.R. **208**(2), 20 (2013). DOI 10.1088/0067-0049/208/2/20
3. N. Aghanim, et al., A& A **641**, A6 (2020). DOI 10.1051/0004-6361/201833910. [Erratum: Astron.Astrophys. 652, C4 (2021)]
4. R.I. Anderson, N.W. Koblischke, L. Eyer, ApJ Lett. **963**(2), L43 (2024). DOI 10.3847/2041-8213/ad284d
5. D. Scolnic, D. Brout, A. Carr, A.G. Riess, T.M. Davis, A. Dwomoh, D.O. Jones, N. Ali, P. Charvu, R. Chen, E.R. Peterson, B. Popovic, B.M. Rose, C.M. Wood, P.J. Brown, K. Chambers, D.A. Coulter, K.G. Dettman, G. Dimitriadis, A.V. Filippenko, R.J. Foley, S.W. Jha, C.D. Kilpatrick, R.P. Kirshner, Y.C. Pan, A. Rest, C. Rojas-Bravo, M.R. Siebert, B.E. Stahl, W. Zheng, ApJ **938**(2), 113 (2022). DOI 10.3847/1538-4357/ac8b7a. URL `https://dx.doi.org/10.3847/1538-4357/ac8b7a`
6. D.M. Scolnic, D.O. Jones, A. Rest, Y.C. Pan, R. Chornock, R.J. Foley, M.E. Huber, R. Kessler, G. Narayan, A.G. Riess, S. Rodney, E. Berger, D.J. Brout, P.J. Challis, M. Drout, D. Finkbeiner, R. Lunnan, R.P. Kirshner, N.E. Sanders, E. Schlafly, S. Smartt, C.W. Stubbs, J. Tonry, W.M. Wood-Vasey, M. Foley, J. Hand, E. Johnson, W.S. Burgett, K.C. Chambers, P.W. Draper, K.W. Hodapp, N. Kaiser, R.P. Kudritzki, E.A. Magnier, N. Metcalfe, F. Bresolin, E. Gall, R. Kotak, M. McCrum, K.W. Smith, ApJ **859**(2), 101 (2018). DOI 10.3847/1538-4357/aab9bb
7. D. Brout, et al., ApJ **938**(2), 110 (2022). DOI 10.3847/1538-4357/ac8e04
8. M.G. Dainotti, V. Petrosian, L. Bowden, ApJ Lett. **914**(2), L40 (2021). DOI 10.3847/2041-8213/abf5e4
9. M.G. Dainotti, A.L. Lenart, A. Chraya, G. Sarracino, S. Nagataki, N. Fraija, S. Capozziello, M. Bogdan, Mon.Not. RAS **518**(2), 2201 (2023). DOI 10.1093/mnras/stac2752
10. M.G. Dainotti, G. Sarracino, S. Capozziello, Pub. Astro. Soc. Japan **74**(5), 1095 (2022). DOI 10.1093/pasj/psac057
11. E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D.F. Mota, A.G. Riess, J. Silk, Class. Quant. Grav. **38**(15), 153001 (2021). DOI 10.1088/1361-6382/ac086d
12. S. Capozziello, G. Sarracino, G. De Somma, Universe **10**(3), 140 (2024). DOI 10.3390/universe10030140
13. S. Capozziello, G. Sarracino, A.D.A.M. Spallicci, Phys. Dark Univ. **40**, 101201 (2023). DOI 10.1016/j.dark.2023.101201
14. M.G. Dainotti, B. De Simone, T. Schiavone, G. Montani, E. Rinaldi, G. Lambiase, ApJ **912**(2), 150 (2021). DOI 10.3847/1538-4357/abeb73
15. M.G. Dainotti, B. De Simone, T. Schiavone, G. Montani, E. Rinaldi, G. Lambiase, M. Bogdan, S. Ugale, Galaxies **10**(1) (2022). DOI 10.3390/galaxies10010024. URL `https://www.mdpi.com/2075-4434/10/1/24`
16. C. Krishnan, E.O. Colgáin, Ruchika, A.A. Sen, M.M. Sheikh-Jabbari, T. Yang, Phys. Rev. D **102**(10), 103525 (2020). DOI 10.1103/PhysRevD.102.103525
17. C. Krishnan, E.O. Colgáin, M.M. Sheikh-Jabbari, T. Yang, Phys. Rev. D **103**(10), 103509 (2021). DOI 10.1103/PhysRevD.103.103509
18. L. Kazantzidis, L. Perivolaropoulos, Phys. Rev. D **102**(2), 023520 (2020). DOI 10.1103/PhysRevD.102.023520

19. T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010). DOI 10.1103/RevModPhys.82.451. URL `https://link.aps.org/doi/10.1103/RevModPhys.82.451`
20. S. Nojiri, S.D. Odintsov, Physics Reports **505**(2), 59 (2011). DOI https://doi.org/10.1016/j.physrep.2011.04.001
21. S. Capozziello, M. de Laurentis, Phys. Rept. **509**(4), 167 (2011). DOI 10.1016/j.physrep.2011.09.003
22. T. Schiavone, G. Montani, F. Bombacigno, Mon. Not. RAS **522**(1), L72 (2023). DOI 10.1093/mnrasl/slad041. URL `https://doi.org/10.1093/mnrasl/slad041`
23. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Nucl. Phys. B **980**, 115850 (2022). DOI 10.1016/j.nuclphysb.2022.115850
24. S.D. Odintsov, D. Sáez-Chillón Gómez, G.S. Sharov, Nucl. Phys. B **966**, 115377 (2021). DOI 10.1016/j.nuclphysb.2021.115377
25. G. Montani, M. De Angelis, F. Bombacigno, N. Carlevaro, Mon. Not. RAS **527**(1), L156–L161 (2024). DOI 10.1093/mnrasl/slad159
26. S. Adil, Ö. Akarsu, E. Di Valentino, R. Nunes, E. Özülker, A. Sen, E. Specogna, Phys. Rev. D **109**(2), 023527 (2024). DOI 10.1103/PhysRevD.109.023527
27. G. Montani, N. Carlevaro, M.G. Dainotti, Phys. Dark Univ. **44**, 101486 (2024). DOI 10.1016/j.dark.2024.101486
28. R. Erdem, arXiv e-prints arXiv:2402.16791 (2024). DOI 10.48550/arXiv.2402.16791
29. G. Efstathiou, Mon. Not. RAS **505**(3), 3866 (2021). DOI 10.1093/mnras/stab1588
30. S. Vagnozzi, S. Dhawan, M. Gerbino, K. Freese, A. Goobar, O. Mena, Phys. Rev. D **98**, 083501 (2018). DOI 10.1103/PhysRevD.98.083501. URL `https://link.aps.org/doi/10.1103/PhysRevD.98.083501`
31. S. Vagnozzi, Universe **9**(9), 393 (2023). DOI 10.3390/universe9090393
32. S. Vagnozzi, Phys. Rev. D **102**(2), 023518 (2020). DOI 10.1103/PhysRevD.102.023518
33. B. Wang, E. Abdalla, F. Atrio-Barandela, D. Pavón, Rep. Prog. Phys. **79**(9), 096901 (2016). DOI 10.1088/0034-4885/79/9/096901
34. A.G. Riess, S.A. Rodney, D.M. Scolnic, D.L. Shafer, L.G. Strolger, H.C. Ferguson, M. Postman, O. Graur, D. Maoz, S.W. Jha, B. Mobasher, S. Casertano, B. Hayden, A. Molino, J. Hjorth, P.M. Garnavich, D.O. Jones, R.P. Kirshner, A.M. Koekemoer, N.A. Grogin, G. Brammer, S. Hemmati, M. Dickinson, P.M. Challis, S. Wolff, K.I. Clubb, A.V. Filippenko, H. Nayyeri, V. U, D.C. Koo, S.M. Faber, D. Kocevski, L. Bradley, D. Coe, ApJ **853**(2), 126 (2018). DOI 10.3847/1538-4357/aaa5a9
35. G. Efstathiou, S. Gratton, Mon. Not. RAS **496**(1), L91 (2020). DOI 10.1093/mnrasl/slaa093
36. G. Montani, M.V. Battisti, R. Benini, G. Imponente, *Primordial Cosmology* (World Scientific, Singapore, 2011). DOI 10.1142/7235
37. A. Melchiorri, L. Pagano, S. Pandolfi, Phys. Rev. D **76**(4), 041301 (2007). DOI 10.1103/PhysRevD.76.041301