Massive Higher-Spin Fields in the Fractional Quantum Hall Effect

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Incompressibility plays a key role in the geometric description of fractional quantum Hall fluids. It is naturally related to quantum area-preserving diffeomorphisms and the underlying Girvin-MacDonald-Plazman algebra, which gives rise to an emergent non-relativistic massive spin-2 mode propagating in the bulk. The corresponding metric tensor can be identified with a nematic order parameter for the bulk states. In the linearised regime with a flat background, it has been shown that this mode can be described by a spin-2 Schroedinger action. However, quantum area-preserving diffeomorphisms also suggest the existence of higher-spin modes that cannot be described through nematic fractional quantum Hall states. Here, we consider p-atic Hall phases, in which the corresponding p-atic order parameters are related to higher-rank symmetric tensors. We then show that in this framework, non-relativistic massive chiral higher-spin fields naturally emerge and that their dynamics is described by higher-spin Schroedinger actions. We finally show that these effective actions can be derived from relativistic massive higher-spin theories in $2+1$ dimensions after taking a non-relativistic limit.

Introduction: The notion of order parameter is useful as a unifying concept for describing states of matter with emergent macroscopic properties. For instance, superconductors [1] and liquid crystals [2] are characterised by local-order parameters, which reflect the presence of spontaneously broken symmetries. On the other hand, topological phases such as fractional quantum Hall (FQH) states are instead characterised by topological invariants. Consequently, they cannot be described within Landau's theory of phase transitions [3–5]. However, there exist several states of matter where local order and topological phases can co-exist, such as topological superconductors [6–11] and nematic FQH phases [12–20]. In the latter, nematicity can be induced, for instance, through a tilted magnetic field, which naturally stretches the Landau orbits while preserving the areas the orbits encircle. The corresponding nematic order, which is associated to a rank-2 symmetric tensor order parameter, plays a central role in the identification of an emergent "spin-2" collective massive mode that appears in the bulk [21–26] and has been recently observed experimentally [27]. The importance of nematic order in this context can be qualitatively understood by noting that Hall nematic phases, which break rotational symmetry while preserving a C_2 discrete rotational symmetry, preserve the topological states while the corresponding deformations can be seen as special kinds of area-preserving deformations of the Hall fluids. The corresponding emergent quantum (non-commutative) geometry is, in fact, invariant under quantum area-preserving diffeomorphisms [28–31], that encode the incompressibility of quantum Hall fluids, and consequently are responsible for the existence of an energy bulk gap. These features are encoded in the so called Girvin-MacDonald-Plazman (GMP) algebra [32], also known as W_∞ algebra in the high-energy-physics literature [28][33]. Thus, the emergent geometry in Hall fluids is associated to a non-relativistic massive graviton-like mode, which at linearised level around a flat background can be described by a spin-2 Schroedinger action [25]. A characteristic feature of the GMP/W_∞ algebra is that, besides a generator of conformal spin two, it also contains an infinite number of generators of increasing conformal spin, with each spin occurring once [34]. This suggests the existence of additional bosonic bulk modes of spin greater than two (aka higher-spin modes), and this expectation is supported by several theoretical and numerical evidences [35–38]. However, the quantum-field-theory characterization of these massive modes remains unclear.

In this work, we partially address this important open question by employing a complementary approach in which we show that these higher-spin modes can naturally appear in the fractional quantum Hall effect (FQHE) via p-atic phases, with p even such that inversion symmetry is preserved [39]. In p-atic liquid crystals, the corresponding p-atic order parameters are higher-rank traceless symmetric tensors, which in two spatial dimensions are related to C_p symmetries [40, 41]. They are well known in soft matter and naturally generalize nematic phases [2, 42–46]. Notice that although several

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p-atic Hall phases have been already proposed and studied [47–51], their relation with higher-spin fields has been overlooked. Here, we will show that within this framework, massive higher-spin fields naturally emerge and their dynamics is described by higher-spin Schroedinger actions near the isotropic phase transition. In particular, we will focus on the chiral and massive spin-4 mode, which has been numerically discussed in Ref. [37] and that we will show to be related to a tetratic order parameter. We will finally show, through symmetry and field-theory arguments, that these effective actions can be derived from relativistic massive higher-spin theories after taking a suitable non-relativistic limit.

P-atic phases in the FQHE: In this section, firstly we summarize the effective-field-theory description of nematic order in the FQHE by mainly following Ref. [14]. Secondly we generalize those results to the case of tetratic Hall states. De Gennes' nematic order in liquid crystals is characterised by a rank-2 symmetric and traceless order parameter, which in two spatial dimensions takes the form $Q_{ab}(\mathbf{x},t) = Q_N(2d_a d_b - \delta_{ab})$, where $\mathbf{d} = (\cos\vartheta/2, \sin\vartheta/2)$ is the planar nematic director [52]. Since it has only two independent components, we can define from it the complex order parameter

$$\mathcal{Q}_N = Q_{11} + iQ_{12} = Q_N e^{i\vartheta}. \tag{1}$$

In the long-wavelength limit [5], the standard topological Chern-Simons (CS) sector of the Laughlin states reads

$$\mathcal{L}_{CS} = \frac{q}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda, \tag{2}$$

with A_μ and a_μ the electromagnetic and U(1) emergent gauge fields, respectively, and where $\mu,\nu \in \{0,1,2\}$. Here, q is an odd-integer number associated to the filling factor such that the Hall conductivity is quantised to $(1/q)(e^2/h)$. The corresponding effective action for the nematic Laughlin states is given by [14]

$$\begin{aligned}\mathcal{L}_{NFQHE} &= \mathcal{L}_{CS} + \mathcal{L}_N - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a^N_\mu\partial_\nu a_\lambda,\\ a^N_c &= -\frac{1}{8}\epsilon_{bd}Q_{ab}\partial_c Q_{ad} + C\epsilon_{bd}\partial_b Q_{dc},\\ a^N_0 &= -\frac{1}{8}\epsilon_{bd}Q_{ab}\dot{Q}_{ad},\end{aligned} \tag{3}$$

where $\dot{Q}_{ab} \equiv \partial Q_{ab}/\partial t$, a^N_0 can be associated to a Berry phase, while C is a constant parameter (it is related to the equal-time structure factor as shown in Ref. [14]). Close to the phase transition from an isotropic to a nematic FQH phase, $\mathcal{L}_N$ in the above equation identifies the effective Landau-de Gennes (LdG) Lagrangian

$$\mathcal{L}_N = \frac{k_1}{8}\epsilon_{bc}Q_{ab}\dot{Q}_{ac} - \frac{k_2}{8}(\partial_c Q_{ab})^2 - \frac{k_3}{2}(Q_{ab})^2 + \cdots, \tag{4}$$

where k_i are some given physical parameters and we omitted higher powers of the nematic order parameter. The first term proportional to the Berry phase a^N_0 originates from the broken time-reversal invariance of the FQHE and dictates the quantum dynamics of the chiral non-relativistic "graviton" [26]. Importantly, we omitted higher-powers in Q_{ab} because we consider the isotropic phase in presence of small nematic deformations. The corresponding Hamiltonian

$$H_N = \frac{k_2}{8}|\partial_i\mathcal{Q}_N|^2 + \frac{k_3}{2}|\mathcal{Q}_N|^2, \tag{5}$$

describes a quadratic energy spectrum that identifies a gapped spin-2 mode, namely the GMP mode.

We are now ready to discuss the generalization of the above results to the case of tetratic order. The latter refers to a generalizion of nematic order, in which a fourfold rotational symmetry C_4 replaces the C_2 of nematic phases. This phase can be either considered as the result of a deformation of the magnetic field or of the confining potential preserving the C_4 symmetry, or as a result of local orientational ordering introduced by interactions, possibly supplemented by additional p-atic modes [47–51]. Being inversion invariant, tetratic phases are compatible with FQH states [39]. The tetratic order parameter is a symmetric traceless rank-4 tensor Q_{abcd}. Importantly, in two space dimensions, like for the nematic order parameter, the traceless symmetric tensor $Q_{abcd} = Q_T(8d_a d_b d_c d_d - 8\delta_{(ab}d_c d_{d)} + \delta_{(ab}\delta_{cd)})$, where $\mathbf{d} = (\cos\vartheta/4, \sin\vartheta/4)$ is the planar tetratic director, admits only two independent components that can be used to define the complex order parameter

$$\mathcal{Q}_T = Q_{1111} + iQ_{1112} = Q_T\, e^{i\vartheta}. \tag{6}$$

This peculiar feature that holds only in two dimensions allows us to generalize most of the results derived in Ref. [14] in the long-wavelength limit. In fact, also in the current case, we can define an action for the tetratic Laughlin states, which reads

$$\begin{aligned}\mathcal{L}_{TFQHE} &= \mathcal{L}_{CS} + \mathcal{L}_T - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a^T_\mu\partial_\nu a_\lambda,\\ a^T_c &= -\frac{1}{8}\epsilon_{bd}Q_{afgb}\partial_c Q_{afgd} + G\epsilon_{bd}\epsilon_{fg}\epsilon_{mn}\partial_b\partial_f\partial_m Q_{dgnc},\\ a^T_0 &= -\frac{1}{8}\epsilon_{bd}Q_{afgb}\dot{Q}_{afgd},\end{aligned} \tag{7}$$

where G is a constant parameter. Close to the transition from an isotropic to a tetradic FQH phase, the corresponding effective LdG Lagrangian $\mathcal{L}_T$ [40, 41] augmented by the Berry phase term a^T_0 is given by

$$\mathcal{L}_T = \frac{k'_1}{8}\epsilon_{bd}Q_{afgb}\dot{Q}_{afgd} - \frac{k'_2}{8}(\partial_d Q_{abcd})^2 - \frac{k'_3}{2}(Q_{abcd})^2, \tag{8}$$

where k'_i are some given physical parameters. Similarly to the nematic case, we have

$$H_T = \frac{k'_2}{8}|\partial_i\mathcal{Q}_T|^2 + \frac{k'_3}{2}|\mathcal{Q}_T|^2, \tag{9}$$

and the corresponding spectrum identifies a gapped spin-4 collective mode. All the results presented above can be naturally generalised to spin-p modes associated to $C_{\rm p}$ symmetries and p-atic Hall states. See also Ref. [36] for a complementary way to introduce effective bulk higher-spin fields in the FQHE, that however does not seem to be directly linked to p-atic phases.

From the Fierz-Pauli equations to higher-spin Schroedinger actions using symmetries: Before discussing the spin-4 and higher-spin massive modes, we first show how the LdG effective-field-theory description of the GMP mode can be obtained by concentrating on symmetries only [53]. Our starting point are the equations of motion for a free relativistic massive spin-2 particle of mass m

$$\left(\Box - m^2c^2\right) h_{\mu\nu} = 0, \qquad \partial_\mu h^{\mu\nu} = 0 = \eta_{\mu\nu}h^{\mu\nu}, \quad (10)$$

where $h_{\mu\nu} = h_{\nu\mu}$ is a symmetric tensor field and $\eta^{\mu\nu} = \mathrm{diag}(-1/c^2, 1, 1)$. The Fierz-Pauli (FP) equations (10) describe two modes with helicities ± 2 and are invariant under translations and Lorentz transformations. We decompose the traceless field $h_{\mu\nu}$ as follows

$$h_{\mu\nu} = \left(\begin{array}{c|c} \phi & v_b \\ \hline v_a & \frac{1}{2c^2}\phi\,\delta_{ab} + S_{ab} \end{array}\right), \qquad a = 1,2 \quad (11)$$

with S_{ab} symmetric and traceless. We then observe that under a Lorentz boost a time index transforms to a spatial index but that the inverse transformation from a spatial index to a time index is suppressed by a factor of $1/c^2$. This implies that, after taking the limit $c \to \infty$, the boost becomes a transformation that only acts in one direction on the different components of $h_{\mu\nu}$ as follows

$$\phi \longrightarrow v_a \longrightarrow S_{ab} \longrightarrow 0. \quad (12)$$

From this we deduce that, unlike the ϕ and v_a components, the symmetric traceless field S_{ab} forms by itself a representation of the contracted Poincaré group which is the Galilei group. In $2+1$ dimensions one can combine the two independent components S_{11} and S_{12} of S_{ab} into the complex field $\mathcal{S}_N = S_{11} + iS_{12}$, that transforms under infinitesimal spatial rotations generated by $\Lambda_{ab} = \epsilon_{ab}\lambda$ not as a scalar but as

$$\mathcal{S}'_N(x') = \mathcal{S}_N(x) - 2i\lambda\mathcal{S}_N(x). \quad (13)$$

The second term on the r.h.s. represents an emerging U(1) symmetry. Furthermore, we find that $\mathcal{S}(x)$ transforms under internal parity (P) $x^1 \to -x^1$, time-reversal (T) $t \to -t$ and inversion (I) $x^{1,2} \to -x^{1,2}$ as follows

$$\text{P: } \mathcal{S}'_N(x') = \mathcal{S}^\star_N(x)\,, \quad \text{T, I: } \mathcal{S}'_N(x') = \mathcal{S}_N(x). \quad (14)$$

It is well known that the Galilei group cannot be used to describe a massive mode. One argument is that at the non-relativistic level one expects two Noether symmetries describing the conservation of energy and mass. The second Noether symmetry extends the Galilei algebra to the Bargmann algebra [54]. To achieve this extension we redefine, before taking the limit $c \to \infty$, the complex field $\mathcal{S}_N$ in terms of another complex field $\mathcal{Q}_N$ as follows [53]

$$\mathcal{S}_N = e^{-i(mc^2 - E_0)t}\mathcal{Q}_N, \quad (15)$$

thereby introducing a new constant E_0. The phase factor has the effect that after taking the limit $c \to \infty$, the field $\mathcal{Q}_N(x)$ transforms under boosts with parameter Λ_a as

$$\delta\mathcal{Q}_N = -(\Lambda^a t)\partial_a\mathcal{Q}_N + im\Lambda_a x^a\mathcal{Q}_N. \quad (16)$$

Furthermore, in the limit $c \to \infty$ the symmetry under parity P and time-reversal T is broken. This is due to the fact that both the $t \to -t$ time-reversal transformation and the complex conjugation that occurs in the parity transformation (14) have the effect that when taking the limit of the transformation rule of the non-relativistic field $\mathcal{Q}_N$, the phase factor does not drop out which gives an invalid limit. Instead, we find that the inversion symmetry I is preserved. There exists a unique action invariant under translations, spatial rotations together with the modified boost of Eq. (16), that is of first order in the time derivatives, thus breaking the invariance under time-reversal symmetry. It is given by the spin-2 planar Schroedinger action

$$I_{\text{Schroedinger}} \propto \int d^3x \quad \Big[im\mathcal{Q}_N\dot{\mathcal{Q}}^*_N - im\mathcal{Q}^*_N\dot{\mathcal{Q}}_N + \partial^a\mathcal{Q}^*_N\partial_a\mathcal{Q}_N + 2mE_0\mathcal{Q}_N\mathcal{Q}^*_N\Big] \quad (17)$$

This action corresponds to the complex rewriting of Eq. (4), thus allowing one to interpret $\mathcal{Q}_N$ as the nematic order parameter (1). Whereas the FP equations (10) describes two helicity modes, that under parity are mapped to each other, the Schroedinger action (17) describes just a single mode and parity is broken. The reason that we lost a degree of freedom is that instead of complexifying all FP fields, we combined two real FP fields into a single complex field. In summary, we conclude that the LdG description of the fluctuations of the nematic phase order parameter around the isotropic phase in the long wavelength limit given in Eq. (4) can be recovered from a non-relativistic limit of the three-dimensional FP theory.

We can now generalize the above discussion to the spin-4 case, which is the next higher-spin field that preserves inversion symmetry. We start from a rank-4 symmetric tensor field $\Phi_{\mu\nu\rho\sigma}(x)$ that satisfies

$$\left(\Box - m^2c^2\right)\Phi_{\mu\nu\rho\sigma} = 0, \quad \partial_\mu\Phi^{\mu\nu\rho\sigma} = 0 = \eta_{\mu\nu}\Phi^{\mu\nu\rho\sigma}. \quad (18)$$

This field can be decomposed into the different helicity modes as $\Phi_{\mu\nu\rho\sigma} = \{\beta\,, \alpha_a\,, \phi_{ab}\,, v_{abc}\,, S_{abcd}\}$, where each component is completely symmetric and traceless in its spatial indices. The independent components decompose as

$$\begin{aligned}\Phi_{abcd} &= S_{abcd} + \frac{1}{6c^2}\phi_{(ab}\delta_{cd)} + \frac{1}{8c^4}\delta_{(ab}\delta_{cd)}\beta\,,\\ \Phi_{abc0} &= v_{abc} + \frac{1}{4c^2}\alpha_{(a}\delta_{bc)}\,,\end{aligned} \tag{19}$$

with parentheses denoting a symmetrisation, where dividing by the number of terms in the symmetrisation is understood. The decomposition of the remaining components follows from the traceless condition. Except for β, each component describes two helicity modes. Under boost transformations these components transform, after taking the limit $c \to \infty$, as follows

$$\beta \longrightarrow \alpha_a \longrightarrow \phi_{ab} \longrightarrow v_{abc} \longrightarrow S_{abcd} \longrightarrow 0. \tag{20}$$

This shows that the highest component S_{abcd} forms a representation of the Galilei algebra by itself. Defining $\mathcal{S}_T = S_{1111} + iS_{1112}$, we find that under spatial rotations the complex field $\mathcal{S}$ transforms as

$$\mathcal{S}'_T(x') = \mathcal{S}_T(x) - 4i\lambda\mathcal{S}_T(x), \tag{21}$$

which shows that we are dealing with a spin-4 field. Making the same redefinition (15) as in the spin-2 case, we find in the $c \to \infty$ limit the same continuous and discrete symmetries as in the spin-2 case except for the spatial rotations. This leads to a planar spin-4 Schroedinger action, which has the same form as the planar spin-2 Schroedinger action given in Eq. (17) in terms of $\mathcal{Q}_T$, which in turn is equivalent to the tetratic LdG Lagrangian (8) when written in complex form.

From massive higher-spins to p-atic LdG actions: The LdG description for p-atic phases in the FQHE can also be recovered from the $c \to \infty$ limit of relativistic actions describing massive higher-spin fields. Let us stress that moving to an off-shell derivation of the Schroedinger actions might provide a rationale for introducing nonlinearities and background couplings as limits of their relativistic counterparts (see, e.g., [55, 56] for recent works on massive higher-spin couplings) although complementary on-shell techniques have been explored too [57].

We first show how to recover the spin-2 LdG action and then we generalise the analysis to the spin-4 case by rescaling the fields as in Eq. (15). Motivated by the description of the spin-2 GMP mode in terms of unimodular gravity [25], we choose to work with the action resulting from the coupling of linearised unimodular gravity with auxiliary Stueckelberg fields B_μ and S

$$\begin{aligned}\kappa^{-1}\mathcal{L} &= \frac{1}{2}h^{\mu\nu}\left(\Box h_{\mu\nu} - 2\partial_\mu\partial\cdot h_\nu - m^2c^2h_{\mu\nu}\right)\\ &+ \frac{1}{2}B^\mu\left(\Box B_\mu - \partial_\mu\partial\cdot B\right) + \frac{1}{2}S\left(\Box S + 3m^2c^2S\right)\\ &+ \sqrt{2}mc\,h^{\mu\nu}\partial_\mu B_\nu + 2mc\,B^\mu\partial_\mu S,\end{aligned} \tag{22}$$

where $h_{\mu\nu}$ is traceless and κ is the gravitational constant, as we are assuming that this action results from the linearisation of a massive gravity theory. This strategy to build quadratic actions for massive fields is the analogue of that described, e.g., in [58, 59] and we provide more details in the Supplemental Material (SM). The action (22) is invariant under the gauge transformations

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu - \frac{2}{3}\,\eta_{\mu\nu}\partial\cdot\epsilon,\\ \delta B_\mu &= \partial_\mu\lambda + \sqrt{2}mc\,\epsilon_\mu, \quad \delta S = 2mc\,\lambda,\end{aligned} \tag{23}$$

where the gauge parameters satisfy the constraint

$$\partial\cdot\epsilon - \frac{3mc}{\sqrt{2}}\,\lambda = 0. \tag{24}$$

As discussed in the SM, the equations of motion following from this action reduce to the FP one (10). We can also gauge away the scalar field S using its algebraic gauge transformation. This leads to a transverse condition $\partial\cdot\epsilon = 0$ for the vector parameter, implying that only residual volume-preserving diffeomorphisms in $2+1$ dimensions are allowed. We now decompose $h_{\mu\nu}$ as in (11) in terms of the spatial tensors S_{ab}, v_a and ϕ and we introduce $B_\mu = (v, S_a)$. We also introduce the complex fields $\mathcal{S}_{[2]} = \mathcal{S}_N = S_{11} + iS_{12}$, $\mathcal{S}_{[1]} = S_1 + iS_2$ and $\mathcal{V}_{[1]} = v_1 + iv_2$. We then define

$$\mathcal{S}_{[k]} = e^{-i(mc^2 - E_0)t}\mathcal{Q}_{[k]}, \quad \mathcal{V}_{[k]} = e^{-i(mc^2-E_0)t}\mathcal{U}_{[k]}, \tag{25}$$

where k denotes the spin of the field and $\mathcal{Q}_{[2]}$ has to be identified with the $\mathcal{Q}_N$ introduced in Eq. (1). Integrating out the field $\mathcal{U}_{[1]}$ and taking the limit $c \to \infty$ leads to

$$\kappa^{-1}\mathcal{L} = 2im\dot{\mathcal{Q}}_{[2]}\mathcal{Q}^*_{[2]} - 2mE_0\mathcal{Q}_{[2]}\mathcal{Q}^*_{[2]} - \partial_a\mathcal{Q}_{[2]}\partial_a\mathcal{Q}^*_{[2]}\,, \tag{26}$$

where we omitted all terms that do not depend on $\mathcal{Q}_{[2]}$ and thus decouple (see SM). In the non-relativistic limit we thus recover the nematic LdG Lagriangian (4) with parameters: $k_1 = -8\kappa m$, $k_2 = 4\kappa$, $k_3 = 2\kappa mE_0$.

Now we turn our attention to a relativistic massive spin-4 field [60]. The analogue of the Stueckelberg Lagrangian (48) involves the Maxwell-like kinetic terms that have been introduced in [61, 62] to generalize linearised unimodular gravity and reads

$$\begin{aligned}\mathcal{L} = \sum_{k=0}^{4}\frac{\kappa}{2}\,\varphi^{\mu(k)}\Bigg[&\left(\Box - \frac{(k-3)(k+5)m^2c^2}{2k+1}\right)\varphi_{\mu(k)}\\ &- k\,\partial_\mu\partial\cdot\varphi_{\mu(k-1)} + 2mc\sqrt{\frac{k(5-k)(k+3)}{2k-1}}\,\partial_\mu\varphi_{\mu(k-1)}\Bigg],\end{aligned} \tag{27}$$

where $\mu(k)$ stands for a set of k symmetrised indices, while repeated indices denote a symmetrisation. This action is invariant under

$$\delta\varphi_{\mu(k)} = k\,\partial_\mu\xi_{\mu(k-1)} - \frac{2}{2k-1}\binom{k}{2}\eta_{\mu\mu}\partial\cdot\xi_{\mu(k-2)} + a_k\,\xi_{\mu(k)}, \tag{28}$$

where $\xi_{\mu\nu\rho}$ and $\xi_{\mu\nu}$ are traceless tensors and the gauge parameters satisfy the constraints

$$\partial \cdot \xi_{\mu(k-1)} + b_{k-1}\, \xi_{\mu(k-1)} = 0, \tag{29}$$

with fixed a_k and b_k (see SM). We then decompose the rank-4 field $\varphi_{\mu(4)} \equiv \Phi_{\mu\nu\rho\sigma}$ in traceless spatial components as in (19) and we introduce the complex variables $\mathcal{S}_{[4]} = \mathcal{S}_T = S_{1111} + iS_{1112}$ and $\mathcal{V}_{[3]} = v_{111} + iv_{112}$. We also introduce a similar decomposition for the auxiliary fields. After redefining the fields as in (25) and integrating out from the action the field $\mathcal{U}_{[3]}$, the field $\mathcal{Q}_{[4]}$ (which coincides with $\mathcal{Q}_T$ in Eq. (6)) decouples from the other fields in the limit $c \to \infty$ (see SM). Its dynamics is described by the Lagrangian in Eq. (8) with the identifications $k_1' = -8\kappa m$, $k_2' = 4\kappa$, $k_3' = 2\kappa m E_0$.

Conclusions and outlook: In this work, we have shown and discussed the relation between p-atic phases and non-relativistic higher-spin fields in the FQHE. We have derived the effective field theories for the higher-spin collective modes in the Laughlin states, given by planar spin-p Schroedinger actions, from (2+1)-dimensional relativistic massive higher-spin theories. There are several important open questions that we aim to investigate in future work: i) the effective-field-theory description of higher-spin collective modes in the presence of a curved background [25, 63–66]. For instance, it would be interesting to see whether one can construct a planar spin-p Schroedinger action in a Newton-Cartan background, generalising the spin-0 analysis of [67]; ii) a supergeometric extension to describe the spin-3/2 massive mode [68–70] and possible further semi-integer higher-spin modes; iii) higher-spin interactions allowing to describe bulk modes away from the isotropic point. For all these developments, one could also exploit the $c \to \infty$ limit of the plethora of relativistic higher-spin interactions that only exist in three dimensions. These have been mainly studied for massless higher-spin fields (see [71] for a review), but investigations of massive interactions appeared, e.g., in [56, 72]; iv) more general higher-spin field theories possibly describing incompressibility and collective modes in higher-dimensional FQH fluids [73]. Finally, we wish to mention that in three dimensions there exist alternative descriptions of the dynamics of massive helicities in terms of higher-derivative theories, for both the spin-2 and higher-spin cases; see, e.g., [57, 72, 74–77] and references therein. We defer to future work an analysis of our non-relativistic limit within this approach.

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SUPPLEMENTAL MATERIAL

Proca-like actions for massive particles of any spin

We presented quadratic actions describing relativistic massive spin-s fields that involve a number of auxiliary fields. The rationale for building these actions is that a massive spin-s field in D spacetime dimensions propagates the same helicities as a tower of massless fields of spin comprised between 0 and s, with each spin appearing once. As a result, one can build a quadratic action for a massive spin-s field by coupling the free actions for massless fields with spin comprised between 0 and s while preserving a deformation of their free gauge symmetry.

To fix the ideas, let us consider the case of Proca's theory, describing a massive spin-1 field. The Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{m^2c^2}{2}\,B_\mu B^\mu\,, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \tag{30}$$

does not have any gauge symmetry, but one can reinstate the gauge symmetry of the Maxwell Lagrangian by introducing a "Stueckelberg field" via the shift

$$B_\mu \to B_\mu - \frac{1}{mc}\,\partial_\mu S\,. \tag{31}$$

The resulting Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{m^2c^2}{2}\,B_\mu B^\mu - \frac{1}{2}\,\partial_\mu S\,\partial^\mu S + mc\,B^\mu\partial_\mu S, \tag{32}$$

and it is manifestly invariant under the gauge transformations

$$\delta B_\mu = \partial_\mu\lambda\,, \tag{33a}$$

$$\delta S = mc\,\lambda\,. \tag{33b}$$

The gauge transformation of the scalar field S is algebraic: this means that one can use it to set the field to zero and go back to the action (30). We denote all fields with an algebraic gauge transformation as Stueckelberg fields. We obtained the Lagrangian in Eq. (32) performing a shift in the Proca Lagrangian. Alternatively, one can also consider the most general coupling of a Maxwell field with a free scalar and fix the coefficients demanding that the action be invariant under the transformations (33). This strategy was used in Ref. [58] to describe a massive spin-s field via the coupling of s Fronsdal's Lagrangians [84], describing massless fields of spin $0 \leq k \leq s$. The same outcome can be obtained by recovering the massive theory from the Kaluza-Klein reduction of a spin-s Fronsdal's action in one more dimension, although in this case one might need to consider field redefinitions in order to diagonalize the resulting kinetic terms; see, e.g., [85]. Fixing completely the algebraic Stueckelberg symmetries then gives the Singh-Hagen Lagrangian [86], which describes a massive spin-s field in terms of a traceless rank-s tensor and a number of auxiliary fields growing with s.

We used a strategy similar to that employed in [58], but substituting Fronsdal's Lagrangians with Maxwell-like ones [61, 62]. The $s = 2$ instance of the latter corresponds to the linearization of the Lagrangian of unimodular gravity and, in general, they provide an alternative description of the free dynamics of massless particles in terms of traceless fields. Besides being motivated by the role played by unimodular gravity in the FQHE [25], our choice has also the advantage of allowing one to work with traceless fields while still preserving a gauge symmetry. Working with traceless fields is convenient for defining the non-relativistic limits along the lines of [53], while the gauge symmetry (that is absent in the Singh-Hagen approach) might be useful to introduce systematically non-linearities; see, e.g., [55, 56].

Within this framework, the Lagrangian describing a massive spin-s field contains s traceless fields that we distinguish by the number of indices they carry: $\{\varphi_{\mu(k)}\}$ with $0 \leq k \leq s$. We recall that the shorthand $\mu(k)$ denotes a set of k symmetrised indices; for instance, $\varphi_{\mu(2)} := \varphi_{\mu_1\mu_2}$. The Lagrangian reads

$$\mathcal{L} = \kappa\sum_{k=0}^{s}\left\{\frac{1}{2}\,\varphi^{\mu(k)}\left(\Box\varphi_{\mu(k)} - k\,\partial_\mu\partial\cdot\varphi_{\mu(k-1)} - m_k\,(mc)^2\,\varphi_{\mu(k)}\right) + c_k\,mc\,\varphi^{\mu(k)}\partial_\mu\varphi_{\mu(k-1)}\right\}, \tag{34}$$

with

$$m_k = -\frac{(s-k-1)(D+s+k-2)}{D+2k-2}\,, \qquad c_k = \sqrt{\frac{k(s-k+1)(D+s+k-4)}{D+2k-4}}\,. \tag{35}$$

Here and in the following, repeated covariant or contravariant indices denote a symmetrization, where dividing by the number of terms in the symmetrization is understood. For instance, $\partial_\mu \xi_\mu := \frac{1}{2}(\partial_{\mu_1}\xi_{\mu_2} + \partial_{\mu_2}\xi_{\mu_1})$. The action corresponding to (34) is invariant under the gauge transformations

$$\delta\varphi_{\mu(k)} = k\,\partial_\mu \xi_{\mu(k-1)} - \frac{2}{D+2k-4}\binom{k}{2}\eta_{\mu\mu}\partial\cdot\xi_{\mu(k-2)} + c_{k+1}\,mc\,\xi_{\mu(k)}\,, \tag{36}$$

where the gauge parameters $\{\xi_{\mu(k)}\}$ with $0 \le k \le s-1$ are traceless and satisfy the constraints

$$\partial\cdot\xi_{\mu(k-1)} - \frac{(D+2k-2)c_{k+1}}{(k+1)(D+2k-4)}\,mc\,\xi_{\mu(k-1)} = 0\,, \qquad 1 \le k \le s-1\,. \tag{37}$$

The Lagrangian (34) depends on a single free mass parameter m and its coefficients can be fixed by demanding invariance under gauge transformations of the form (36). In the limit $m \to 0$ one obtains the sum of s Maxwell-like Lagrangians, which are invariant under gauge transformations generated by traceless and divergenceless gauge parameters, thus presenting themselves as the natural higher-spin generalization of unimodular gravity (linearised volume-preserving diffeomorphisms are indeed generated by transverse vectors).

Alternatively, the coefficients in the Lagrangian (34) can also be fixed by demanding that the equations of motion reduce to the Fierz-Pauli ones. Let us discuss this in the example of a massive spin-2 field. To this end, it is convenient to first eliminate the scalar field by using its algebraic Stueckelberg symmetry. After this step, the Euler-Lagrange equations of (34) read

$$\Box h_{\mu\nu} - 2\,\partial_{(\mu}\partial\cdot h_{\nu)} - \frac{2}{D}\,\eta_{\mu\nu}\partial\cdot\partial\cdot h - m^2c^2 h_{\mu\nu} + c_2\,mc\left(\partial_{(\mu}B_{\nu)} - \frac{1}{D}\,\eta_{\mu\nu}\partial\cdot B\right) = 0\,, \tag{38}$$

$$\Box B_\mu - \partial_\mu\partial\cdot B - c_2\,mc\,\partial\cdot h_\mu = 0\,, \tag{39}$$

where we renamed the fields to distinguish them more easily. Notice that the equations of motion must be traceless as the fields entering the Lagrangian. For instance, the double divergence of $h_{\mu\nu}$ appears in Eq. (38) as a result of the traceless projection. Taking a divergence of Eq. (39) one obtains $\partial\cdot\partial\cdot h = 0$ and substituting this result in the divergence of Eq. (38) one obtains

$$\frac{(D-1)}{D}\,c_2\,\partial_\mu\partial\cdot B = \left(1 - \frac{c_2^2}{2}\right) mc\,\partial\cdot h\,. \tag{40}$$

The value of c_2 fixed by Eq. (35) is precisely that setting to zero the right-hand side. This allows one to conclude $\partial_\mu\partial\cdot B = 0$ and, eventually, $\partial\cdot B = 0$ up to constant functions that do not affect the propagated local degrees of freedom. This on-shell constraint is crucial because, after the gauge fixing setting the scalar to zero, the spin-2 gauge parameter satisfies $\partial\cdot\xi = 0$, so that the divergence of the vector potential becomes gauge invariant. On the other hand, the latter is forced to vanish by the equations of motion, so that one can gauge away the remaining portion of the vector potential by using the algebraic gauge transformation generated by ξ_μ. After one reaches the condition $B_\mu = 0$ by combining the equations of motion and a gauge fixing, Eq. (38) reduces to the Fierz-Pauli equation (10).

Non-relativistic limit

In this section we look at a particular Galilean limit of the higher-spin actions presented in the previous section. We consider the examples of fields of spin two and four and restrict the analysis to $D = 2+1$ dimensions. The relativistic indices will be split in the form $\mu = \{0, a\}$ and we consider coordinates $x^\mu = \{x^0 \equiv t, x^a\}$ so that the Minkowski metric and its inverse are given by $\eta_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1)$ and $\eta^{\mu\nu} = \mathrm{diag}(-1/c^2, 1, 1)$, respectively. Let us consider now an explicit traceless parametrisation of the fields entering the action (34) up to spin-4, which can be written in terms of a set of six completely symmetric and traceless tensors $\{S_{abcd}, S_{abc}, S_{ab}, v_{abc}, v_{ab}, \phi_{ab}\}$, four vectors $\{S_a, v_a, \phi_a, \alpha_a\}$ and five scalar fields $\{\beta, \alpha, v, \phi, S\}$ as follows. In order to avoid confusion, we rename the fields as

$$\varphi_{\mu(0)} = S, \quad \varphi_{\mu(1)} = B_\mu, \quad \varphi_{\mu(2)} = h_{\mu\nu}, \quad \varphi_{\mu(3)} = \chi_{\mu\nu\rho}, \quad \varphi_{\mu(4)} = \Phi_{\mu\nu\rho\sigma}, \tag{41}$$

For the spin-1 and spin-2 auxiliary fields we define

$$B_\mu = \begin{pmatrix} B_0 \\ B_a \end{pmatrix} \equiv \begin{pmatrix} v \\ S_a \end{pmatrix}, \qquad h_{\mu\nu} = \left(\begin{array}{c|c} h_{00} & h_{0b} \\ \hline h_{a0} & h_{ab} \end{array}\right) \equiv \left(\begin{array}{c|c} \phi & v_b \\ \hline v_a & \frac{1}{2c^2}\phi\,\delta_{ab} + S_{ab} \end{array}\right), \tag{42}$$

and we similarly decompose the spin-3 auxiliary field in traceless spatial components. We thus define

$$\chi_{ab0} = \frac{1}{2c^2}\alpha\delta_{ab} + v_{ab}, \qquad \chi_{abc} = \frac{3}{4c^2}\phi_{(a}\delta_{bc)} + S_{abc}, \tag{43}$$

while the remaining components are fixed by the constraint $\eta^{\mu\nu}\chi_{\mu\nu\rho}$, for instance $\chi_{a00} = c^2\delta^{bc}\chi_{abc} = \phi_a$. For the spin-4 field we define

$$\Phi_{abc0} = \frac{3}{4c^2}\alpha_{(a}\delta_{bc)} + v_{abc}, \qquad \Phi_{abcd} = \frac{3}{8c^4}\beta\delta_{(cd}\,\delta_{ab)} + \frac{1}{c^2}\phi_{(cd}\,\delta_{ab)} + S_{abcd}, \tag{44}$$

where again the remaining components can be obtained using the traceless constraint for $\Phi_{\mu\nu\rho\sigma}$. In two spatial dimensions, any traceless symmetric tensor has only two independent components. Therefore, for every spatial vector or spatial traceless symmetric tensor of the form $Z_{a_1\cdots a_k}$ with $k = 1, 2, \ldots$, it is possible to construct a complex field $\mathcal{Z}_{[k]}$ out of its independent components as

$$Z_{a_1\cdots a_k} \quad\longrightarrow\quad \mathcal{Z}_{[k]} = Z_{\underbrace{1\,1\,\cdots\,1}_{k\text{ times}}} \;+\; i\,Z_{\underbrace{1\,1\,\cdots\,1}_{k-1\text{ times}}\,2}. \tag{45}$$

When writing the terms in the action (34) in complex form the following identities will be useful

$$\begin{aligned}
Y_{a_1\cdots a_k}Z_{a_1\cdots a_k} &= 2^{k-2}\left(\mathcal{Y}^*_{[k]}\mathcal{Z}_{[k]} + \mathcal{Y}_{[k]}\mathcal{Z}^*_{[k]}\right),\\
Y_{b_1\cdots b_k}\partial_a Z_{ab_1\cdots b_k} &= 2^{k-2}\left(\mathcal{Y}^*_{[k]}\bar\partial\mathcal{Z}_{[k+1]} + \mathcal{Y}_{[k]}\partial\mathcal{Z}^*_{[k+1]}\right),\\
\partial_a Z_{ac_1\cdots c_{k-1}}\partial_b Z_{bc_1\cdots c_{k-1}} &= \frac{1}{2}\partial_a Z_{b_1\cdots b_k}\partial_a Z_{b_1\cdots b_k} = 2^{k-1}\bar\partial\mathcal{Z}_{[k]}\partial\mathcal{Z}^*_{[k]},
\end{aligned} \tag{46}$$

where in the last identity we have neglected total derivative terms and defined the complex derivatives $\partial = \partial_1 + i\partial_2$ and $\bar\partial = \partial_1 - i\partial_2$ and T.D. stands for total derivative terms that we neglect.

Using the prescription (45), we define complex fields $\mathcal{S}_{[i]}$ and $\mathcal{V}_{[j]}$ associated to the fields $S_{a_1\cdots b_i}$ and $v_{a_1\cdots a_j}$ entering in the parametrisation (42), (43) and (44) of the relativistic fields. The non-relativistic limit will be carried out after introducing the following field redefinition for the fields (45)

$$\mathcal{S}_{[i]} = e^{-i(mc^2-E_0)t}\mathcal{Q}_{[i]}, \qquad \mathcal{V}_{[j]} = e^{-i(mc^2-E_0)t}\mathcal{U}_{[j]}, \tag{47}$$

where m coincides with the mass parameter in the action (34) and E_0 is an arbitrary constant. This, in turn, introduces new vectors and symmetric traceless tensors $Q_{a_1\cdots a_i}$ and $u_{a_1\cdots a_j}$, whose components are given by the real and imaginary parts of $\mathcal{Q}_{[k]}$ and $\mathcal{U}_{[k]}$ as indicated in (45). In the following we will show that when considering the Lagrangian for a spin-2 (spin-4) field and taking the non-relativistic limit defined by implementing (47) and setting $c \to \infty$, the field Q_{ab} (Q_{abcd}) decouples from all the lower-spin fields and is described by the a Schroedinger action of the form given in Eq. (17).

Spin-2 case

We start considering the Lagrangian (34) for a massive traceless spin-2 field. Using the λ-symmetry to gauge the scalar field S away we find the action

$$\kappa^{-1}\mathcal{L} = \frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} - h_{\mu\lambda}\partial^\mu\partial_\nu h^{\nu\lambda} - \frac{m^2c^2}{2}h^{\mu\nu}h_{\mu\nu} + \frac{1}{2}B_\mu\Box B^\mu - \frac{1}{2}B^\mu\partial_\mu\partial_\nu B^\nu + \sqrt{2}mc\,h^{\mu\nu}\partial_\mu B_\nu. \tag{48}$$

We adopt the parametrisation (42) and use the complex notation (45). Thus, the identities (46) allow to write the action as (up to total derivatives)

$$\begin{aligned}
\kappa^{-1}\mathcal{L} = {} & \frac{3}{4c^6}\dot\phi^2 - \frac{3}{4c^4}\partial\phi\bar\partial\phi + \frac{1}{c^2}\dot{\mathcal{S}}_{[2]}\dot{\mathcal{S}}^*_{[2]} - \frac{1}{c^6}\dot\phi^2 + \frac{1}{4c^2}\partial\phi\bar\partial\phi + \frac{1}{2c^2}\left(\bar\partial\mathcal{S}_{[2]}\bar\partial\phi + \partial\mathcal{S}^*_{[2]}\partial\phi\right)\\
& - \frac{3m^2}{4c^2}\phi^2 - m^2c^2\mathcal{S}_{[2]}\mathcal{S}^*_{[2]} + \frac{1}{2c^2}\left(\dot{\mathcal{S}}_{[1]}\dot{\mathcal{S}}^*_{[1]} - \dot{\mathcal{S}}_{[1]}\bar\partial v - \dot{\mathcal{S}}^*_{[1]}\partial v + \partial v\bar\partial v\right)\\
& - \frac{1}{4}\left(\partial\mathcal{S}_{[1]}\bar\partial\mathcal{S}^*_{[1]} + \bar\partial\mathcal{S}_{[1]}\partial\mathcal{S}^*_{[1]}\right) + \frac{1}{8}\left(\partial\mathcal{S}^*_{[1]} + \bar\partial\mathcal{S}_{[1]}\right)^2 + \frac{\sqrt{2}m}{c^3}\phi\dot v\\
& + \frac{m}{2\sqrt{2}c}\phi\left(\partial\mathcal{S}^*_{[1]} + \bar\partial\mathcal{S}_{[1]}\right) - \frac{mc}{\sqrt{2}}\left(\bar\partial\mathcal{S}_{[2]}\mathcal{S}^*_{[1]} + \partial\mathcal{S}^*_{[2]}\mathcal{S}_{[1]}\right) + \kappa^{-1}\mathcal{L}_{\mathcal{V}_{[1]}},
\end{aligned} \tag{49}$$

where we have isolated the terms that depend on $\mathcal{V}_{[1]}$ in the Lagrangian

$$\begin{aligned}\kappa^{-1}\mathcal{L}_{\mathcal{V}_{[1]}} &= m^2\mathcal{V}_{[1]}\mathcal{V}^*_{[1]} + \frac{1}{2c^2}\left(\partial\mathcal{V}_{[1]}\bar{\partial}\mathcal{V}^*_{[1]} + \bar{\partial}\mathcal{V}_{[1]}\partial\mathcal{V}^*_{[1]}\right) - \frac{1}{c^2}\left(\bar{\partial}\mathcal{S}_{[2]}\dot{\mathcal{V}}^*_{[1]} + \partial\mathcal{S}^*_{[2]}\dot{\mathcal{V}}_{[1]}\right)\\ &+ \frac{1}{2c^4}\left(\dot{\mathcal{V}}_{[1]}\bar{\partial}\phi + \dot{\mathcal{V}}^*_{[1]}\partial\phi\right) - \frac{1}{4c^2}\left(\partial\mathcal{V}^*_{[1]} + \bar{\partial}\mathcal{V}_{[1]}\right)^2 - \frac{m}{\sqrt{2}c}\left(\mathcal{V}_{[1]}\bar{\partial}v + \mathcal{V}^*_{[1]}\partial v + \dot{\mathcal{S}}_{[1]}\mathcal{V}^*_{[1]} + \dot{\mathcal{S}}^*_{[1]}\mathcal{V}_{[1]}\right).\end{aligned} \tag{50}$$

Now we introduce complex fields in terms of $\mathcal{Q}_{[2]}$, $\mathcal{Q}_{[1]}$ and $\mathcal{U}_{[1]}$ by means of the field redefinition (47), in terms of which the Lagrangian takes the form

$$\begin{aligned}\kappa^{-1}\mathcal{L} &= -2mE_0\mathcal{Q}_{[2]}\mathcal{Q}^*_{[2]} - 2im\mathcal{Q}_{[2]}\dot{\mathcal{Q}}^*_{[2]} - mE_0\mathcal{Q}_{[1]}\mathcal{Q}^*_{[1]} - im\mathcal{Q}_{[1]}\dot{\mathcal{Q}}^*_{[1]}\\ &+ \frac{m^2c^2}{2}\mathcal{Q}_{[1]}\mathcal{Q}^*_{[1]} - \frac{mc}{\sqrt{2}}\left(\bar{\partial}\mathcal{Q}_{[2]}\mathcal{Q}^*_{[1]} + \partial\mathcal{Q}^*_{[2]}\mathcal{Q}_{[1]}\right) + \kappa^{-1}\mathcal{L}_{\mathcal{U}_{[1]}}\\ &- \frac{1}{4}\left(\partial\mathcal{Q}_{[1]}\bar{\partial}\mathcal{Q}^*_{[1]} + \bar{\partial}\mathcal{Q}_{[1]}\partial\mathcal{Q}^*_{[1]}\right) + \frac{1}{8}\left(e^{i(mc^2-E_0)t}\partial\mathcal{Q}^*_{[1]} + e^{-i(mc^2-E_0)t}\bar{\partial}\mathcal{Q}_{[1]}\right)^2 + O(1/c).\end{aligned} \tag{51}$$

Note that the field redefinition (47) introduced some factors $e^{-i(mc^2-E_0)t}$ that will oscillate in the $c\to\infty$ limit. On the other hand, these terms will decouple in non-relativistic the limit, so that we will ignore this issue as in [53]. After introducing the new variables, the Lagrangian for $\mathcal{U}_{[1]}$ now has the form

$$\kappa^{-1}\mathcal{L}_{\mathcal{U}_{[1]}} = m^2\mathcal{U}_{[1]}\mathcal{U}^*_{[1]} - \mathcal{U}_{[1]}\mathcal{J}^*_{[1]} - \mathcal{U}^*_{[1]}\mathcal{J}_{[1]} + O(1/c^2), \tag{52}$$

where we have defined

$$\mathcal{J}_{[1]} = -\frac{im^2c}{\sqrt{2}}\mathcal{Q}_{[1]} + im\bar{\partial}\mathcal{Q}_{[2]} + \frac{1}{c}\left(\frac{m}{\sqrt{2}}\partial v + \frac{imE_0}{\sqrt{2}}\mathcal{Q}_{[1]} + \frac{m}{\sqrt{2}}\dot{\mathcal{Q}}_{[1]}\right) + O(1/c^2), \tag{53}$$

Now we note that for large values of c the field equations for $\mathcal{U}^*_{[1]}$ lead to

$$\mathcal{U}_{[1]} = \frac{1}{m^2}\mathcal{J}_{[1]} + O(1/c^2), \tag{54}$$

and therefore integrating out $\mathcal{U}_{[1]}$ yields the following effective Lagrangian

$$\begin{aligned}\kappa^{-1}\mathcal{L}^{eff}_{\mathcal{U}_{[1]}} = -\frac{1}{m^2}\mathcal{J}_{[1]}\mathcal{J}^*_{[1]} &= -\frac{m^2c^2}{2}\mathcal{Q}_{[1]}\mathcal{Q}^*_{[1]} + \frac{mc}{\sqrt{2}}\left(\mathcal{Q}_{[1]}\partial\mathcal{Q}^*_{[2]} + \mathcal{Q}^*_{[1]}\bar{\partial}\mathcal{Q}_{[2]}\right) - \bar{\partial}\mathcal{Q}_{[2]}\partial\mathcal{Q}^*_{[2]}\\ &+ \frac{im}{2}\mathcal{Q}_{[1]}\left(\bar{\partial}v + \dot{\mathcal{Q}}^*_{[1]} - iE_0\mathcal{Q}^*_{[1]}\right) - \frac{im}{2}\mathcal{Q}^*_{[1]}\left(\partial v + \dot{\mathcal{Q}}_{[1]} + iE_0\mathcal{Q}_{[1]}\right) + O(1/c),\end{aligned} \tag{55}$$

When replacing this expression by $\mathcal{L}_{\mathcal{U}_{[1]}}$ in the Lagrangian (51), the divergent terms cancel and $\mathcal{Q}_{[2]}$ decouples from the other fields. Thus, in the limit $c\to\infty$, up to total derivative terms, we find the spin-2 Schroedinger Lagrangian Eq. (26)

$$\mathcal{L} = 2i\kappa m\dot{\mathcal{Q}}_{[2]}\mathcal{Q}^*_{[2]} - \kappa\bar{\partial}\mathcal{Q}_{[2]}\partial\mathcal{Q}^*_{[2]} - 2\kappa mE_0\mathcal{Q}_{[2]}\mathcal{Q}^*_{[2]} + \ldots \tag{56}$$

where the dots denote terms that do not depend on $\mathcal{Q}_{[2]}$. Using the fact that up to total derivatives $\epsilon_{bc}Q_{ab}\dot{Q}_{ac} = -2i\dot{\mathcal{Q}}_{[2]}\mathcal{Q}^*_{[2]}$ and the identities (46), the $\mathcal{Q}_{[2]}$-dependent part of this Lagrangian can be put in the form (4) with coefficients $k_1 = -8\kappa m$, $k_2 = 4\kappa$ and $k_3 = 2\kappa mE_0$.

Spin-4 case

The Lagrangian (34) evaluated for $D = 2+1$ and $s = 4$ can be written as

$$\mathcal{L} = \kappa\sum_{k=0}^{4}\left(K_{\varphi_{\mu(k)}} + M_{\varphi_{\mu(k)}\varphi_{\nu(k-1)}}\right), \tag{57}$$

where, using (41), the different kinetic and mixing terms different terms take the form

$$
\begin{aligned}
K_\Phi &= \frac{1}{2}\Phi^{\mu\nu\rho\sigma}\Box\Phi_{\mu\nu\rho\sigma} - 2\Phi^{\mu\nu\rho\sigma}\partial_\mu\partial^\lambda\Phi_{\lambda\nu\rho\sigma} - m^2c^2\,\Phi^{\mu\nu\rho\sigma}\Phi_{\mu\nu\rho\sigma},\\
K_\chi &= \frac{1}{2}\chi^{\mu\nu\rho}\Box\chi_{\mu\nu\rho} - \frac{3}{2}\chi^{\mu\nu\rho}\partial_\mu\partial^\lambda\chi_{\lambda\nu\rho},\\
K_h &= \frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} - h^{\mu\nu}\partial_\mu\partial^\lambda h_{\lambda\nu} + \frac{7}{10}\,m^2c^2\,h^{\mu\nu}h_{\mu\nu},\\
K_B &= \frac{1}{2}B^\mu\Box B_\mu - \frac{1}{2}B^\mu\partial_\mu\partial^\lambda B_\lambda + 2\,m^2c^2\,B^\mu B_\mu,\\
K_S &= \frac{1}{2}S\Box S + \frac{15}{2}\,m^2c^2\,S^2,
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
M_{\Phi\chi} &= 2\,mc\,\Phi^{\mu\nu\rho\sigma}\partial_\mu\chi_{\nu\rho\sigma}, \qquad & M_{\chi h} &= \frac{6}{\sqrt{5}}\,mc\,\chi^{\mu\nu\rho}\partial_\mu h_{\nu\rho},\\
M_{hB} &= \sqrt{10}\,mc\,h^{\mu\nu}\partial_\mu B_\nu, \qquad & M_{BS} &= 2\sqrt{2}\,mc\,B^\mu\partial_\mu S.
\end{aligned}
\tag{59}
$$

Using the parametrisation (42), (43), (44), together with the identities (46) we find that in complex notation the kinetic terms and the mixing terms can be written in a unified notation for $k \geq 2$ as

$$
\begin{aligned}
K_{\varphi_{\mu(k)}} &= \frac{2^{k-1}}{2c^2}\dot{\mathcal{S}}_{[k]}\dot{\mathcal{S}}^*_{[k]} - \frac{2^{k-3}k}{c^2}\left(\dot{\mathcal{V}}_{[k-1]}\partial\mathcal{S}^*_{[k]} + \dot{\mathcal{V}}^*_{[k-1]}\bar{\partial}\mathcal{S}_{[k]}\right) + 2^{k-2}\left(\frac{k}{2}-1\right)\partial_a\mathcal{S}_{[k]}\partial_a\mathcal{S}^*_{[k]}\\
&+ 2^{k-3}\,k\,m_k\,m^2\,\mathcal{V}_{[k-1]}\mathcal{V}^*_{[k-1]} - 2^{k-2}m_k\,m^2c^2\,\mathcal{S}_{[k]}\mathcal{S}^*_{[k]} + O(1/c^2),\\
M_{\varphi_{\mu(k)}\varphi_{\nu(k-1)}} &= -2^{k-3}c_k\left(\frac{m}{c}\mathcal{V}^*_{[k-1]}\dot{\mathcal{S}}_{[k-1]} + \frac{m}{c}\mathcal{V}_{[k-1]}\dot{\mathcal{S}}^*_{[k-1]} + mc\,\mathcal{S}_{[k-1]}\partial\mathcal{S}^*_{[k]} + mc\,\mathcal{S}^*_{[k-1]}\bar{\partial}\mathcal{S}_{[k]}\right) + O(1/c^2).
\end{aligned}
\tag{60}
$$

where we have reinstated the coefficients m_k and c_k, whose explicit form is given in Eq. (35). The non-relativistic limit can be taken after redefining the variables by means of (47), in terms of which (60) takes the form

$$
\begin{aligned}
K_{\varphi_{\mu(k)}} &= 2^{k-1}im\dot{\mathcal{Q}}_{[k]}\mathcal{Q}^*_{[k]} + 2^{k-1}\left(\frac{m^2c^2}{2} - mE_0\right)\mathcal{Q}_{[k]}\mathcal{Q}^*_{[k]} + 2^{k-3}im\,k\Big(\mathcal{U}_{[k-1]}\partial\mathcal{Q}^*_{[k]} - \mathcal{U}^*_{[k-1]}\bar{\partial}\mathcal{Q}_{[k]}\Big)\\
&+ 2^{k-2}\left(\frac{k}{2}-1\right)\partial_a\mathcal{Q}_{[k]}\partial_a\mathcal{Q}^*_{[k]} + 2^{k-3}\,k\,m_k\,m^2\,\mathcal{U}_{[k-1]}\mathcal{U}^*_{[k-1]} - 2^{k-2}m_k\,m^2c^2\,\mathcal{Q}_{[k]}\mathcal{Q}^*_{[k]} + O(1/c^2),\\
M_{\varphi_{\mu(k)}\varphi_{\nu(k-1)}} &= -2^{k-3}c_k\,mc\left(-im\left(\mathcal{U}^*_{[k-1]}\mathcal{Q}_{[k-1]} - \mathcal{U}_{[k-1]}\mathcal{Q}^*_{[k-1]}\right) + \mathcal{Q}_{[k-1]}\partial\mathcal{Q}^*_{[k]} + \mathcal{Q}^*_{[k-1]}\bar{\partial}\mathcal{Q}_{[k]}\right) + O(1/c^2).
\end{aligned}
\tag{61}
$$

From these expressions we see that in order to eliminate the divergent term contained in $K_{\varphi_{\mu(s)}}$ term, the constants m_s should take precisely the form (35). Moreover, the term $M_{\varphi_{\mu(s)}\varphi_{\mu(s-1)}}$ also contains divergent terms, which disappear after integrating out $\mathcal{U}_{[s-1]}$ from the action (57). An inspection of the different terms in (61) shows that the terms in the action involving the $\mathcal{U}_{[s-1]}$ have the following structure

$$
\mathcal{L}_{\mathcal{U}_{[s-1]}} = 2^{s-3}\kappa\Big[s\,m_s\,m^2\,\mathcal{U}_{[s-1]}\mathcal{U}^*_{[s-1]} - im\mathcal{U}_{[s-1]}\Big(c_s\,mc\,\mathcal{Q}^*_{[s-1]} - s\partial\mathcal{Q}^*_{[s]}\Big) + im\mathcal{U}^*_{[s-1]}\Big(c_s\,mc\,\mathcal{Q}_{[s-1]} - s\bar{\partial}\mathcal{Q}_{[s]}\Big)\Big] + O(1/c^2)
\tag{62}
$$

from which we find the following field equation of $\mathcal{U}_{[s-1]}$ for large values of c

$$
\mathcal{U}_{[s-1]} = -\frac{i}{s\,m_s}\Big(c_s\,c\mathcal{Q}_{[s-1]} - \frac{s}{m}\bar{\partial}\mathcal{Q}_{[s]}\Big) + O(1/c).
\tag{63}
$$

Integrating out $\mathcal{U}_{[s-1]}$ yields the effective Lagrangian

$$
\mathcal{L}^{eff}_{\mathcal{U}_{[s-1]}} = -\frac{2^{s-3}\kappa}{s\,m_s}\Big(c_s\,mc\,\mathcal{Q}_{[s-1]} - s\bar{\partial}\mathcal{Q}_{[s]}\Big)\Big(c_s\,mc\,\mathcal{Q}^*_{[s-1]} - s\partial\mathcal{Q}^*_{[s]}\Big) + \cdots\,,
\tag{64}
$$

where $\cdots$ represents either terms that do not involve $\mathcal{Q}_{[s]}$ or are subleading in $1/c$. Replacing (62) by (64), inserting it back in the action (57), and taking the limit $c \to \infty$ gives rise to the following action for $\mathcal{Q}_{[s]}$

$$\mathcal{L} = 2^{s-1}\kappa\, m\left[i\dot{\mathcal{Q}}_{[s]}\mathcal{Q}^*_{[s]} - \frac{1}{2m}\bar{\partial}\mathcal{Q}_{[s]}\partial\mathcal{Q}^*_{[s]} - E_0\mathcal{Q}_{[s]}\mathcal{Q}^*_{[s]}\right] + \cdots\,, \tag{65}$$

where $\mathcal{Q}_{[s]}$ is decoupled from all the other fields in the Lagrangian. Even though this result was considered for $s = 4$, it is written in a general form that allows to generalize the result to arbitrary values of s. Using the identities (46) plus the fact that up to total derivatives $\epsilon_{mn}Q_{abcm}\dot{Q}_{abcn} = -8i\dot{\mathcal{Q}}_{[4]}\mathcal{Q}^*_{[4]}$, the Lagrangian for $\mathcal{Q}_{[4]}$ takes the form (8) with parameters $k'_1 = -8\kappa m$, $k'_2 = 4\kappa$ and $k'_3 = 2\kappa m E_0$.