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CRITICAL BETA-SPLITTING, VIA CONTRACTION

BRETT KOLESNIK

ABSTRACT. The critical beta-splitting tree, introduced by Aldous, is a Markov branching phylogenetic tree of poly-logarithmic height. Recently, by a technical analysis, Aldous and Pittel proved, amongst other results, a central limit theorem for the height H_n of a random leaf.

We give an alternative proof, via contraction methods for random recursive structures. These techniques were developed by Neininger and Rüschendorf, motivated by Pittel's article "Normal convergence problem? Two moments and a recurrence may be the clues." Aldous and Pittel estimated the first two moments of H_n, with great precision. We show that a limit theorem follows, and bound the distance from normality.

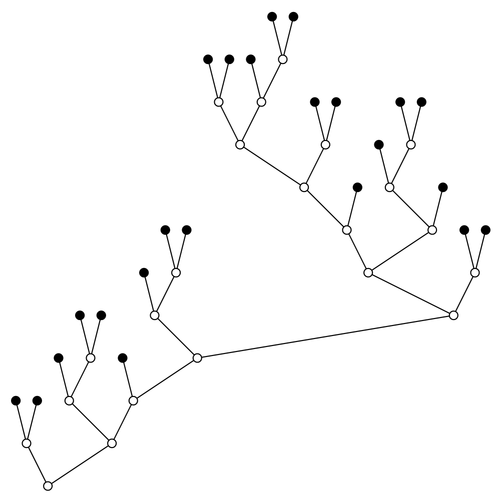

FIGURE 1. A critical beta-splitting tree with leaves labelled by $\{1,\dots,23\}$. Internal nodes indicate when and where the set of leaves above is split. Initially, $\{1,\dots,23\}$ splits into $\{1,2\}$ and $\{3,\dots,23\}$. Then, $\{1,2\}$ splits into $\{1\}$ and $\{2\}$, $\{3,\dots,23\}$ splits into $\{3,4,5\}$ and $\{6,\dots,23\}$, etc., until only the singleton sets $\{1\},\dots,\{23\}$ remain.

UNIVERSITY OF OXFORD, DEPARTMENT OF STATISTICS
E-mail address: brett.kolesnik@stats.ox.ac.uk.

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1. INTRODUCTION

The *critical beta-splitting tree* $\mathscr{T}_n$, introduced by Aldous [2], is a random recursive combinatorial structure, constructed in the following way. Assume that $n \geqslant 2$. We begin with the set $\{1,\dots,n\}$. Let

$$\vartheta(n) = \sum_{i=1}^{n} \frac{1}{i} \sim \log n \tag{1.1}$$

denote the harmonic sum. The first *split* occurs between some i and $i+1$ with probability

$$p(n,i) = \frac{1}{2\vartheta(n-1)} \frac{n}{i(n-i)}, \qquad 1 \leqslant i \leqslant n-1, \tag{1.2}$$

in which case $\{1,\dots,n\}$ separates into $\{1,\dots,i\}$ and $\{i+1,\dots,n\}$. We call $p(n,i)$ the *critical beta-splitting distribution*. See Figure 2.

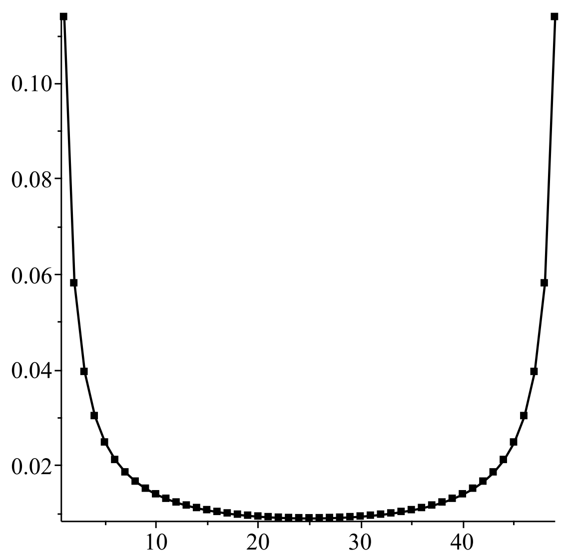

FIGURE 2. $p(50,i)$, for $1 \leqslant i \leqslant 49$.

The construction continues recursively, splitting $\{1,\dots,i\}$ and $\{i+1,\dots,n\}$ independently, etc., until only the singleton sets $\{1\},\dots,\{n\}$ remain.

Finally, the tree $\mathscr{T}_n$ is obtained as follows. Let $\mathscr{S}$ denote the set of subsets of $\{1,\dots,n\}$, determined by the splits in the above procedure. For each $1 \leqslant j \leqslant n$, a leaf $v(j) = v(\{j\})$ is placed at *height*

$$h(j) = \#\{S \in \mathscr{S} : j \in S\} - 1.$$

The "-1" above accounts for the singleton set $\{j\} \in \mathscr{S}$, which does not contribute to the height of $v(j)$.

An *internal node* $v(S)$ is added to the tree, for each $S \in \mathscr{S}$ with $\#S > 1$. The two children of $v(S)$ are $v(S_1)$ and $v(S_2)$, where $S_1, S_2 \in \mathscr{S}$ are the unique pair for which $S = S_1 \cup S_2$. There are $n-1$ internal nodes in total, one between each i and $i+1$. The first internal node $\rho = v(\{1,\dots,n\})$ is called the *root* of $\mathscr{T}_n$. The height $h(j)$ is simply the graph distance (number of edges) between ρ and $v(j)$. See Figure 1 for an example.

As discussed in [2], the tree $\mathscr{T}_n$ is "critical" in the following sense. A tree could be constructed in a similar way, but with $p(n,i)$ instead proportional to $i^\beta(n-i)^\beta$. The value $\beta = -1$ in the above construction is of particular interest, since at this point typical heights $h(j)$ switch from polynomial order $1/n^{\beta+1}$ to poly-logarithmic order $\log^2 n$.

1.1. **Results.** Amongst other results, Aldous and Pittel [3] recently proved a central limit theorem for $H_n = h(J)$, where J is uniformly random in $\{1,2,\dots,n\}$. In other words, H_n is the height of a random leaf in $\mathscr{T}_n$.

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

denote the *Riemann zeta function*.

Theorem 1 (Aldous and Pittel [3]). *As $n \to \infty$, we have that*

$$\frac{H_n - \frac{1}{2\zeta(2)}\log^2 n}{\sqrt{\frac{2\zeta(3)}{3\zeta^3(2)}\log^3 n}}$$

is asymptotically normal.

1.2. **Purpose.** Our purpose is to give another proof of Theorem 1, along with a bound on the rate of convergence, see Theorem 3 below. We will use the contraction methods of Neininger and Rüschendorf [9,10] (cf. Rösler [15,16], Rachev and Rüschendorf [14] and Rösler and Rüschendorf [17]), together with the estimates for the mean and variance of H_n obtained in [3], see (3.1) below.

We will also discuss, in Section 4.2 below, connections with a result of Iksanov, Marynych and Möhle [8], on collisions in the beta-coalescent.

1.3. **Discussion.** The limit theory in [9] was, in part, developed in response to work of Pittel [13], in which limit theorems are proved for various combinatorial quantities of interest (e.g., the independence number of a uniformly random labelled tree) with mean and variance that are close to linear.

The following line of reasoning is referred to as "Pittel's principle" in [9, p. 379]. Indeed, in [13, p. 1260], the author states that:

> For various global characteristics of large size combinatorial structures [...] one can usually estimate the mean and the variance, and also obtain a recurrence for the generating function [...]. As a heuristic principle based on our experience, we claim that such a characteristic is asymptotically normal if the mean and the variance are "nearly linear" [...]. The technical reason is that in such a case the moment generating function [...] of the normal distribution with the same two moments "almost" satisfies the recurrence.

A general theory is developed in [9], which, in particular, yields limit theorems in such situations (see [9, Corollary 5.2]). In fact, their results apply to a large family of random structures X_n, which satisfy a distributional recurrence of the form (see [9, (1)])

$$X_n \overset{d}{=} \sum_{i=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n. \tag{1.3}$$

As discussed in [9], such situations arise, e.g., in divide-and-conquer type algorithms. In this context, b_n is called the *toll function,* associated with the "cost" of splitting into smaller, but similar subproblems.

Under certain conditions, a limit theorem can be proved for X_n satisfying (1.3), via the so-called *contraction method*. Roughly speaking, this strategy aims to identify the limiting distribution of X_n, by means of the fixed point equation

$$X \overset{d}{=} \sum_{i=1}^{K} A_r^* X^{(r)} + b^*, \tag{1.4}$$

obtained by taking $n \to \infty$ in (1.3). The normal distribution is associated with the situation that $\sum_{i=1}^{K} (A_r^*)^2 = 1$ and $b_* = 0$.

See [9, Theorem 5.1 and Corollary 5.2] for their univariate results. See also [9, §5.4] for discussion on the multivariate case, and when $K = K_n$ is random, and potentially also $K_n \to \infty$.

The height H_n of a random leaf in the critical beta-splitting tree $\mathscr{T}_n$ satisfies a simple recurrence. Specifically, by (1.2), we have that

$$H_n \overset{d}{=} H_{I_n} + 1, \tag{1.5}$$

where

$$\mathbb{P}(I_n = i) = \frac{1}{(n-i)\vartheta(n-1)}, \qquad 1 \leqslant i \leqslant n-1. \tag{1.6}$$

That being said, the results in [9] do not apply. The problem is that $b_* = 0$, $K = 1$, and that, as it turns out (see (3.1) below), the mean and variance of H_n are of poly-logarithmic order. This leads to a trivial fixed point equation $X \overset{d}{=} X$, which yields no information about X.

However, the follow-up article [10], by the same authors, deals with this very situation, and it is these results that we will apply in the current article. Specifically, we will use Theorem 2.1 in [10]. In fact, this result does not apply as stated, but we will show that its proof can be suitably adapted.

We note that several applications of contraction methods are discussed in, e.g., [9, §5.2–5.3] and [10, §4–5]. In many cases, limit theorems follow quite easily using these techniques. We were introduced to them, while studying randomized importance sampling algorithms for perfect matchings [4] (cf. Neininger and Straub [11]).

1.4. **Time-heights.** In closing, let us mention that it is also natural to consider an alternative formation of $\mathscr{T}_n$, in which splitting events occur continuously in time. In [3], the authors analyze the case of exponential holding times on subsets, with rates $\vartheta(k-1)$ on subsets of size k. In this setting, a central limit theorem is proved for the *time-height* D_n of a random leaf. Aldous [1] has given an alternative, probabilistic proof, via martingales.

A limit theorem for D_n seems to be out of reach, however, by the methods in the current article, mainly due to the fact that D_n has smaller variance, see (3.1) and (4.1) below. Therefore, it would appear that the critical beta-splitting tree provides an example of a model, at the borderline of what can be analyzed using current contraction techniques.

1.5. **Acknowledgments.** We thank David Aldous and Boris Pittel for inspiring conversations. We are also grateful to Oleksandr Iksanov, who recently informed us that Theorem 1 can be derived from Theorem 12 in Gnedin, Pitman and Yor [7] (also proved using [10]) or by Theorem 3.1(a) in Gnedin and Iksanov [5] (proved using renewal theory).

2. CONTRACTION, WITH TRIVIAL FIXED POINT

To begin, let us state the main result in [10, Theorem 2.1].

Suppose that a sequence (X_n) of random variables satisfies

$$X_n \stackrel{d}{=} X_{I_n} + b_n, \tag{2.1}$$

where (I_n, b_n) and (X_n) are independent, and I_n takes values in $\{1, \dots, n-1\}$. (In [10], I_n can take values in $\{0, 1, \dots, n\}$, but we have no use for this.)

Let $\mu_n = \mathbb{E}(X_n)$ and $\sigma_n^2 = \mathrm{Var}(X_n)$.

As usual, $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ denotes the L_p-norm.

Theorem 2 (Neininger and Rüschendorf [10]). *Suppose that* (X_n) *satisfies* (2.1)*, with* $\|X_n\|_3 < \infty$*,*

$$\limsup_{n\to\infty} \mathbb{E}[\log(I_n/n)] < 0, \qquad \sup_{n\geqslant 1} \|\log(I_n/n)\|_3 < \infty.$$

Assume also that, for some $\alpha, \lambda, \kappa \in \mathbb{R}$ *with* $0 \leqslant \lambda < 2\alpha$*, and some* $C > 0$*, we have that*

$$\|b_n - \mu_n + \mu_{I_n}\|_3 = O(\log^\kappa n), \qquad \sigma_n^2 = C\log^{2\alpha} n + O(\log^\lambda n),$$

and

$$\beta = \min\{3/2, 3(\alpha-\kappa), 3(\alpha-\lambda/2), \alpha-\kappa+1\} > 1.$$

Then, as $n \to \infty$*,*

$$\frac{X_n - \mu_n}{\sqrt{C}\log^\alpha n}$$

is asymptotically normal.

Furthermore, it is shown that the distance from normality is $O(1/\log^{\beta-1} n)$, with respect to the *Zolotarev metric* ζ_3 [19,20]. As discussed in [10],

$$\zeta_3(U,V) = \sup|\mathbb{E}f(U) - \mathbb{E}f(V)|,$$

where the supremum is over all twice differentiable f, with 1-Lipschitz f''. Convergence with respect to ζ_3 implies weak convergence.

3. Proof of Theorem 1

In proving Theorem 1, we will not apply Theorem 2 directly. Instead, we will adapt its proof.

We will use the following, remarkably precise, estimates in [3, Theorem 1.2]. Throughout this section, we let $\mu_n = \mathbb{E}(H_n)$ and $\sigma_n^2 = \mathrm{Var}(H_n)$. We have that

$$\begin{aligned}\mu_n &= \frac{1}{2\zeta(2)}\log^2 n + \frac{\gamma\zeta(2)+\zeta(3)}{\zeta^2(2)}\log n + O(1),\\ \sigma_n^2 &= \frac{2\zeta(3)}{3\zeta^3(2)}\log^3 n + O(1),\end{aligned} \tag{3.1}$$

where

$$\gamma = \lim_{n\to\infty}(\vartheta(n) - \log n)$$

is the *Euler–Mascheroni constant*.

The reason Theorem 2 does not apply directly is that, by (1.1) and (1.6), we have that

$$\begin{aligned}\mathbb{E}|\log(I_n/n)|^k &= \frac{1}{\vartheta(n-1)}\sum_{i=1}^{n-1}\frac{|\log(i/n)|^k}{n-i}\\ &\sim \frac{1}{\log n}\int_0^1 \frac{|\log(1-x)|^k}{x}dx = \frac{k!\zeta(k+1)}{\log n}.\end{aligned} \tag{3.2}$$

We do, however, have the distributional recurrence (1.5). Hence, by (3.1) and (3.2), we have, in the notation of Theorem 2, that $b_n = 1$, $\kappa = 2/3$, $\alpha = 3/2$

and $\lambda = 0$. In particular, to see that $\kappa = 2/3$, let us note that, by elementary arguments it can be shown, using (3.1) and (3.2), that

$$\|1-\mu_n+\mu_{I_n}\|_3 = O(\log n)\|\log(I_n/n)\|_3 = O(\log^{2/3} n). \qquad (3.3)$$

We will prove the following result, which, as we will see, follows by the proof of Theorem 2 (Theorem 2.1 in [10]), after a few adjustments.

Theorem 3. *Let H_n be the height of a uniformly random leaf in the critical beta-splitting tree $\mathscr{T}_n$. Then*

$$H_n^* = \frac{H_n - \mu_n}{\sigma_n}$$

is asymptotically normal, where $\mu_n = \mathbb{E}(H_n)$ and $\sigma_n^2 = \mathrm{Var}(H_n)$. Furthermore, for any $\varepsilon > 0$,

$$\zeta_3(H_n^*, Z) = O\left(\frac{1}{\log^{1/2-\varepsilon} n}\right),$$

where Z is a standard normal random variable.

In what follows, we will assume familiarity with the proof of Theorem 2.1 in [10], and the notation introduced therein. Since only a few changes are required, we will not explain the full proof here, but rather only discuss the places that need adjustment.

Proof. We put $C = 2\zeta(3)/3\zeta^3(2)$, so that, by (3.1),

$$\sigma_n^2 = C\log^3 n + O(1).$$

There are two main parts of the proof of [10, Theorem 2.1] that need attention. The first is the technical result [10, Lemma 3.1]. In fact, the proof of this result simplifies. Secondly, we will revisit the upper bound [10, (19)], as this estimate is used in the inductive proof of [10, Lemma 3.1].

Let us start with the second part. Recall that $\mathbb{P}(I_n \in \{0,n\}) = 0$. We set $\delta = 1$, and let $\ell_n = \log n + \mathbf{1}_{n=1}$ play the role of $L_\delta(n)$.

As noted above, $\alpha = 3/2$. In particular, we simply have

$$b^{(n)} = \frac{1-\mu_n+\mu_{I_n}}{\sqrt{C}\ell_n^{3/2}},$$
$$\tau_n = \frac{\sigma_n}{\sqrt{C}\ell_n^{3/2}},$$
$$G_n = \frac{\sigma_{I_n}}{\sqrt{C}\ell_n^{3/2}}.$$

We claim that the right hand side of [10, (23)] (and so also the left hand side of [10, (19)]) is $O(1/\log^{5/2} n)$. To see this, we first note, using (3.3), that $\|b^{(n)}\|_3^3 = O(1/\log^{5/2} n)$. Similarly, it can be shown that $\|b^{(n)}\|_2 =$

$O(1/\log n)$. Next, we observe that, clearly, $|\tau_n - 1| = O(1/\log^3 n)$. Finally, we note, by similar arguments as (3.3), that

$$|G_n - 1| = O\left(\frac{1 + (\log n)^2 |\log(I_n/n)|}{\log^3 n}\right).$$

It follows that $\|G_n - 1\|_2 = O(1/\log^{3/2} n)$, $\|G_n - 1\|_3^3 = O(1/\log^4 n)$ and

$$\|\Delta_n\|_3^3 = \mathbb{E}|\tau_n^2 - G_n^2|^{3/2} = O(1/\log^{5/2} n).$$

Altogether, the right hand side of [10, (23)] is $O(1/\log^{5/2} n)$, as claimed.

Therefore, to the complete the proof, it remains only the prove the following analogue of the technical result in [10, Lemma 3.1].

Claim 4. *Let I_n be as in* (1.6)*. Suppose that nonnegative sequences (d_n) and (r_n) satisfy*

$$d_n \leqslant \mathbb{E}\left[\left(\frac{\ell_{I_n}}{\ell_n}\right)^{9/2} d_{I_n}\right] + r_n, \qquad n \geqslant 2, \tag{3.4}$$

and

$$r_n = O\left(\frac{1}{\log^{5/2} n}\right). \tag{3.5}$$

Then, for all small $\varepsilon > 0$, it follows that

$$d_n = O\left(\frac{1}{\log^{1/2-\varepsilon} n}\right).$$

To see this, we will follow the proof of [10, Lemma 3.1]. We can, in fact, make some simplifications in this special case. Using (1.6), (3.2) and (3.5), let $M > 0$ and n_1 be such that $r_n \leqslant M/\log^{5/2} n$ and

$$\frac{\mathbb{E}[\log(I_n/n)]}{\log n} + \frac{\mathbb{P}(I_n = 1)}{\log n} + \frac{1}{\log^{2+\varepsilon} n} \leqslant 0$$

for all $n \geqslant n_1$.

Put

$$R = M \vee \max\{d_k \ell_k^{1/2-\varepsilon} : 1 \leqslant k \leqslant n_1\}.$$

To prove the claim, we will show, by induction, that $d_n \leqslant R/\ell_n^{1/2-\varepsilon}$. By the choice of R, there is nothing to prove for $n \leqslant n_1$. On the other hand, for

$n > n_1$, by (3.4), the choice of n_1, and the inductive hypothesis,

$$
\begin{aligned}
d_n &\leqslant \mathbb{E}\left[\left(\frac{\ell_{I_n}}{\ell_n}\right)^{9/2}\frac{R}{\ell_{I_n}^{1/2-\varepsilon}}\right]+\frac{M}{\log^{5/2} n}\\
&\leqslant \frac{R}{\ell_n^{1/2-\varepsilon}}\left[\mathbb{E}\left(\frac{\ell_{I_n}}{\ell_n}\right)+\frac{1}{\log^{2+\varepsilon} n}\right]\\
&\leqslant \frac{R}{\ell_n^{1/2-\varepsilon}}\left(1+\frac{\mathbb{E}[\log(I_n/n)]}{\log n}+\frac{\mathbb{P}(I_n=1)}{\log n}+\frac{1}{\log^{2+\varepsilon} n}\right)\\
&\leqslant R/\ell_n^{1/2-\varepsilon},
\end{aligned}
$$

as required.

This finishes the proof, as the rest of the proof of Theorem 2.1 in [10] applies, without any further changes. ∎

4. Final remarks

4.1. **Time-height.** Recall, as discussed in Section 1.4 above, that D_n is the time-height of the critical beta-splitting tree with exponential holding times. In [3, Theorem 1.1], it is shown that

$$
\begin{aligned}
\mathbb{E}(D_n) &= \frac{1}{\zeta(2)}\log n + O(1),\\
\operatorname{var}(D_n) &= (1+o(1))\frac{2\zeta(3)}{\zeta^3(2)}\log n. \qquad (4.1)
\end{aligned}
$$

Finer estimates are available, assuming a certain "h-ansatz," see [3, §2.2].

The proof of Theorem 3 does not seem to work for D_n. First of all, the $o(1)$ in the above variance estimate effectively gives only $\lambda = 2\alpha$. Furthermore, since $\alpha = 1/2$ for D_n, rather than $\alpha = 3/2$ for H_n, the right hand side of [10, (23)] is $\gg 1/\log^2 n$. As a result, the contribution from $\mathbb{E}[\log(I_n/n)]/\log n$ is not enough to yield an analogue of Claim 4.

4.2. **Collisions.** Finally, as mentioned in Section 1.2 above, let us discuss the central limit theorem proved by in [8] for the number of collisions Ξ_n in the $\beta(2,b)$-*coalescent*. See, e.g., Pitman [12], Sagitov [18] and the survey by Gnedin, Iksanov and Marynych [6] for background.

In [8, (2)], there is a similar recurrence as (1.5) above. Also, Theorem 2 does not apply for similar reasons (compare (3.2) with [8, Remark 3.2]). To overcome this issue, an alternative, and more complicated, recurrence is derived [8, (14)], and then Theorem 2 is applied. However, the authors ask [8, Remark 1.6] if a more direct proof, using the simpler recursion [8, (2)], is possible.

It seems that we cannot quite answer this question. The reason is that, by [8, Theorem 1.1],

$$\mathbb{E}(\Xi_n) = A\log^2 n + B\log n + O(1),$$
$$\mathrm{var}(\Xi_n) = C\log^3 n + O(\log^2 n), \tag{4.2}$$

for explicit constants A, B, C. Therefore, $\lambda = 2$ for Ξ_n, whereas $\lambda = 0$ for H_n. As such, once again, the right hand side of [10, (23)] is $\gg 1/\log^2 n$.

REFERENCES

1. D. Aldous, *The critical beta-splitting random tree II: Overview and open problems*, preprint available at `https://arxiv.org/abs/2303.02529`.
2. ———, *Probability distributions on cladograms*, Random discrete structures (Minneapolis, MN, 1993), IMA Vol. Math. Appl., vol. 76, Springer, New York, 1996, pp. 1–18.
3. D. Aldous and B. Pittel, *The critical beta-splitting random tree: Heights and related results*, preprint available at `https://arxiv.org/abs/2303.02529`.
4. P. Diaconis and B. Kolesnik, *Randomized sequential importance sampling for estimating the number of perfect matchings in bipartite graphs*, Adv. in Appl. Math. **131** (2021), Paper No. 102247, 41.
5. A. Gnedin and A. Iksanov, *Regenerative compositions in the case of slow variation: a renewal theory approach*, Electron. J. Probab. **17** (2012), no. 77, 19.
6. A. Gnedin, A. Iksanov, and A. Marynych, *Λ-coalescents: a survey*, J. Appl. Probab. **51A** (2014), no. Celebrating 50 Years of The Applied Probability Trust, 23–40.
7. A. Gnedin, J. Pitman, and M. Yor, *Asymptotic laws for regenerative compositions: gamma subordinators and the like*, Probab. Theory Related Fields **135** (2006), no. 4, 576–602.
8. A. Iksanov, A. Marynych, and M. Möhle, *On the number of collisions in* $\mathrm{beta}(2,b)$-*coalescents*, Bernoulli **15** (2009), no. 3, 829–845.
9. R. Neininger and L. Rüschendorf, *A general limit theorem for recursive algorithms and combinatorial structures*, Ann. Appl. Probab. **14** (2004), no. 1, 378–418.
10. ———, *On the contraction method with degenerate limit equation*, Ann. Probab. **32** (2004), no. 3B, 2838–2856.
11. R. Neininger and J. Straub, *On the contraction method with reduced independence assumptions*, 33rd International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, LIPIcs. Leibniz Int. Proc. Inform., vol. 225, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022, pp. Art. No. 14, 13.
12. J. Pitman, *Coalescents with multiple collisions*, Ann. Probab. **27** (1999), no. 4, 1870–1902.
13. B. Pittel, *Normal convergence problem? Two moments and a recurrence may be the clues*, Ann. Appl. Probab. **9** (1999), no. 4, 1260–1302.
14. S. T. Rachev and L. Rüschendorf, *Probability metrics and recursive algorithms*, Adv. in Appl. Probab. **27** (1995), no. 3, 770–799.
15. U. Rösler, *A limit theorem for "Quicksort"*, RAIRO Inform. Théor. Appl. **25** (1991), no. 1, 85–100.
16. ———, *A fixed point theorem for distributions*, Stochastic Process. Appl. **42** (1992), no. 2, 195–214.

17. U. Rösler and L. Rüschendorf, *The contraction method for recursive algorithms*, vol. 29, 2001, Average-case analysis of algorithms (Princeton, NJ, 1998), pp. 3–33.
18. S. Sagitov, *The general coalescent with asynchronous mergers of ancestral lines*, J. Appl. Probab. **36** (1999), no. 4, 1116–1125.
19. V. M. Zolotarev, *Approximation of the distributions of sums of independent random variables with values in infinite-dimensional spaces*, Teor. Verojatnost. i Primenen. **21** (1976), no. 4, 741–758.
20. ———, *Ideal metrics in the problem of approximating the distributions of sums of independent random variables*, Teor. Verojatnost. i Primenen. **22** (1977), no. 3, 449–465.