Intelligent Machines and Incomplete Information

Sujata Goala,[*] Mridu Prabal Goswami[†] and Surajit Borkotokey,[‡]

Abstract

The distribution of efficient individuals in the economy and the efforts that they will put in if they are hired are two important concerns for a technologically advanced firm when the firm wants to open a new branch. The firm does not have the information about the exact level of efficiency of an individual when she is hired. We call this situation 'incomplete information'. The standard principal-agent models assume that employees know their efficiency levels. Hence these models design incentive compatible mechanisms. An incentive compatible mechanism ensures that a participant does not have the incentive to misreport her efficiency level. This paper does not assume that employees know how efficient they are. This paper assumes that the production technology of the firm is intelligent, i.e., the output of the machine reveals the efficiency levels of employees. Employees' marginal contributions to the total output of the intelligent machine, the probability distribution of the levels of efficiency, and employees' costs of efforts together define a game of incomplete information. A characterization of ex-ante Nash Equilibrium is established. The characterization results formalize the relationship between the distribution of efficiency levels and the distribution of output.

Keywords: intelligent machines, incomplete information, semi-value, firm location, talent distributions

1 Introduction

Machine operator efficiency is defined to be the performance of an employee who operates industrial machinery. The performance of an employee is decided by her effort and efficiency/talent.

[*]Dibrugarh University, Dibrugarh, India.
[†]Indian Statistical Institute, Tezpur, India.
[‡]Dibrugarh University, Dibrugarh, India

The importance of machine operator efficiency, or more broadly the importance of the interface between man and machine for machine productivity, can hardly be overemphasized, see (Wilson and Daugherty, 2015) for an elaborate discussion. In this paper by a machine we refer to a technologically sophisticated plant/factory/firm /a machine in the standard engineering sense such that efficiency of employees are of critical importance for productivity. Suppose a software firm requires its employees to know category theory. In this example by a machine we refer to the firm itself. Distinct levels of understanding of category theory refer to distinct levels of efficiency of an employee. Further, distinct numbers of hours put in by an employee for the firm are distinct levels of efforts. We call such an abstract machine **Task Aggregator Machine (TAM)**, i.e., a machine that takes efforts and efficiency levels as inputs to produce some output. Softwares designed by the software firm are its outputs. Further, instead of the 'number of softwares' the relevant measure of outputs maybe 'quality of the softwares'. The quality of a software is not a number. As a consequence we may consider the market value, i.e., the price at which a software is sold in the market, to be the measure of its quality. To formalize our ideas we assume that output of a firm can be measured by real numbers.[1] In order to avoid any confusion regarding units of measurements, output maybe interpreted as the total revenue.

The software firm mentioned above is an example of a TAM. Some real life examples of TAMs are Microsoft, Tesla and Spacex. To produce Tesla's self driven cars, considered as advanced AI models, require smart engineers.[2] The skills required by an engineer who works for Tesla are advanced and technical in nature, the skills required by a farmer for tilling a plot of land are not. When we imagine TAMs we do not imagine agricultural farms or 'relatively low tech' firms as examples. Thus, in our model the objective of the firm is not to maximize surplus. To elaborate this remark we recall that the objective of the principal in the standard principal-agent models in economic theory is to maximize surplus, i.e., the principal/employer maximizes profit by paying wages to the agents/employees that keep agents' payoffs at the lowest possible levels. In particular, the first best solution to the maximization problem in fact minimizes wage bill, see Chapter 14 in (Mas-Colell, Whinston and Green, 1995). Instead of maximizing surplus, modern technological firms, share revenues with their employees. Thus what matters for these technology firms is the marginal contribution of an employee to the firm's output, and not just efforts. The objective of the firm in our model is to find employees who can use the technology of the firm in such a way that produces the maximum revenue. The salary/remuneration to an employee in our model is based on the marginal contribution of the employee to the total revenue. A more efficient employee's marginal contribution is higher. Thus the firm in our model is not looking to minimize its wage bill. Minimizing the wage bill is not the appropriate objective for technologically advanced firms since such firms require innovations from the employees, see (Anderson, 2013) for a detailed discussion.

We assume that the firm in our model hires two employees. An employee's efficiency level is either high or low. An employee's effort level is either high or low. For every effort and efficiency vector TAM induces a cooperative game, i.e., a game in the characteristic form, see Definition 2. Although the total revenue is generated by the employees together, the employees put efforts strategically. The strategic behavior of the employees entail a non-cooperative game in which players are the employees. Now, the firm management is not informed about the efficiency levels of the prospective employees, we call this event a situation of **incomplete information**. Thus the probability distribution of the efficiency levels entails a game of incomplete information. We assume that the management knows this distribution. The pay-off function of an employee has two parts, benefit and cost. The benefits of the employees come in the form of shares of the total revenue. The cost that they incur is due to the efforts that they put in. A pure strategy of an employee is a function from the set of possible efficiency levels to the set of possible efforts. We provide a characterization of symmetric Ex-ante Nash equilibria in pure strategies. The three Nash equilibria outcomes that we obtain are (a) all employees put low efforts irrespective of their levels of efficiency (b) all employees put high efforts irrespective of their levels of efficiency (c) the relatively more efficient employee puts high effort, and the less efficient employee puts low effort. Our characterization results provide insights into how probability distributions of levels of efficiency are related to equilibria. In particular, given a probability distribution of levels efficiency, our results can tell us which strategies form equilibria and which do not. We may interpret a probability distribution of the levels of efficiency as a distribution of talent in the economy. Each Nash equilibrium entails a distribution on the set of outcomes. This distribution is obtained because employees come with different levels of efficiency. Thus our characterization results formally express the relationship between firm outputs and distribution of talents. The effect of the distribution of talent on productivity is an important variable that influences the decision of a firm when choosing a location for a new branch. Hence, our characterization results address one of the important aspects of the decision problem of choosing locations that many firms need to address. Example 1 finds that for the welfare of the economy it can be better that in equilibrium the less efficient employee puts low effort. In Example 1 one common distribution of talent gives rise to two Nash equilibria. In one equilibrium the less efficient employee puts low effort, and in the other equilibrium she puts high effort. It is difficult to predict which equilibrium will be played in such situations. To address this issue of multiple equilibria we consider **rationalizable** equilibrium strategies. This notion pins down the distribution of talents to unique equilibria.

An important aspect of a game in characteristic form is its set of all singleton coalitions. A singleton coalition is a situation in which an employee works alone. We do not assume that the productivity of an employee when she operates on TAM alone is zero, in fact it is a special case of our model. We interpret the singleton coalition that corresponds to an employee as the training or the probation period of that employee. Two employees that we consider in our model are the ones who survive their probation periods. Naturally the employees who are fired after the probation period do not have any marginal contribution to the firm and thus are not relevant for our study. Although the employees in our model survive the probation period, their efforts when they work alone may differ from their team efforts. We analyze Nash equilibria pertaining to this situation as well.

The firm in our model can form objective estimates of the contribution of the employees to the firm. In particular, TAM's output reveal effort and the level of efficiency of each employee. This revelation does not depend on what employees believe their levels of efficiency are. Therefore, we call such a firm an **intelligent TAM**, see Definition 3 for a formal exposition. On the contrary standard principal-agent models in economic theory analyze surplus maximizing wage-efforts contracts. These models assume that employees know how efficient they are. The efficiency levels of the employees appear in the pay-off functions of the employees. The pay-off functions of the employees appear as constraints in the optimization problem of the principal. Incentive constraints are very important. An incentive constraint requires that the pay-off obtained by an employee by pretending is not larger than the pay-off obtained by behaving according to her true level of efficiency, see (Mas-Colell, Whinston and Green, 1995) or (Laffont and Martimort, 2002). However, the assumption that employees know how efficient they are may not be considered realistic. For example, it is not obvious that an individual who is trained in category theory knows exactly how good she is in it.[3] Thus, an incentive constraint is not a well-defined notion if employees do not know how efficient they are.[4] In our model we do not assume that employees know about their efficiency levels. In our model the revelation of the efficiency levels is done by the intelligent TAM. Thus our model provides a novel way to approach the problem of hiring in the presence of incomplete information by incorporating intelligent technology instead of taking the approach that depends on the reported information of the employees about their levels of efficiency.

The organization of the paper is as follows. In Section 2 we explain how we model TAMs as games in characteristic forms. We provide a review of the related literature in Section 3. In Section 4 we discuss the technical conditions on TAM and the cost functions of the employees. In this section we also explain how we incorporate the probability distribution of efficiency levels in to our model so that we can analyze the resulting economic environment by using the Mathematics of games of incomplete information. In Section 4.3 we provide our main characterization result. In Section 4.3.1 we extend our model that incorporates strategic behavior that depends on the coalition that she is part of. We make concluding remarks in Section 5.

We discuss some preliminaries next about our model, this makes it convenient to put the related literature in the context of our research.

2 Preliminaries: Relating TAMs with Cooperative Games

We denote TAM by M. We assume the number of employees to be two, the set of employees is denoted by $N = \{1, 2\}$. Let t_l and t_h be two possible efficiency levels. Let 0 denote the efficiency level of an individual who is not hired.[5] Thus, $T = \{t_l, t_h, 0\}$ is the set of efficiency levels, where t_l denotes the lower level of efficiency, i.e., low type; and t_h denotes the higher level of efficiency, i.e., high type. Formally, the notion of low and high type is expressed by an order $<_t$ on $\{t_l, t_h\}$ which is $t_l <_t t_h$. Analogously, $E = \{e_l, e_h, 0\}$ denotes the set of possible levels of efforts with $e_l <_e e_h$; e_l denotes the lower and e_h the higher level of efforts. $e_i = 0$ denotes the absence of effort of individual i as an employee. Let t_i denote a generic type of an individual, where $t_i = 0$ denotes the absence of individual i as an employee. A type profile is denoted by (t_1, t_2), and an efforts profile is denoted by (e_1, e_2). That is, while writing a profile we write the corresponding entry for individual 1 first and then for individual 2. A **machine state** refers to an ordered list $((e_1, e_2), (t_1, t_2))$ where e_1 is the efforts of individual 1 whose type is t_1. Analogously, e_2 denotes the effort of individual 2 whose type is t_2. The output of the machine M at the machine state $((e_1, e_2), (t_1, t_2))$ is $M((e_1, e_2), (t_1, t_2))$ and $M((e_1, e_2), (t_1, t_2)) \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We notice that $M((e_1, 0), (t_1, t_2)), t_2 \neq 0$, $M((e_1, e_2), (t_1, 0)), e_2 \neq 0$, $M((0, e_2), (t_1, t_2)), t_1 \neq 0$, and $M((e_1, e_2), (t_1, 0)), e_1 \neq 0$ are not defined. That is, if an individual is hired, then she cannot put zero effort; and if an individual is not hired, then speaking about her contributions to the firm is meaningless. Consider the machine state $((0, e_2), (0, t_2))$. In this state individual 1 is not hired and thus does not put effort. In this machine state only individual 2 is hired and she puts efforts e_2, and e_2 must be either e_h or e_l. Further, t_2 must be either t_h or t_l. This machine state denotes the singleton coalition in which only individual 2 is present whose type is t_2 and effort is e_2. An analogous interpretation holds for the machine state $((e_1, 0), (t_1, 0))$.[6] Further, $((0, 0), (0, 0))$ refers to the empty coalition, we set $M((0, 0), (0, 0)) = 0$, i.e., TAM cannot produce anything by itself. That is, we assume technology that requires humans to produce an output. The machine state $((e_1, e_2), (t_1, t_2))$ denotes the grand coalition in which both individuals are present and individual is efforts is e_i, her type is t_i and $e_i \neq 0, t_i \neq 0$. We assume that both individuals are hired so that the grand coalition is formed. [7] We call the machine state **grand machine state** in which both individuals are present. In the standard cooperative game theory a coalition is identified with a list of individuals, we identify a coalition with a list of individual specific characteristics, namely efficiency and effort. In particular, corresponding to every grand machine state there is a characteristic form game. For example, if $((e_l, e_h), (t_l, t_h))$ is the grand machine state, then $((e_l, e_h), (t_l, t_h)) \mapsto M((e_l, e_h), (t_l, t_h))$, $((e_l, 0), (t_l, 0)) \mapsto M((e_l, 0), (t_l, 0))$, $((0, e_h), (0, t_h)) \mapsto M((0, e_h), (0, t_h))$, and $((0, 0), (0, 0)) \mapsto M((0, 0), (0, 0))$ define a characteristic form game. We formalize the notion of TAM in Definition 1. Let $\mathbb{A} = \{((e_1, 0), (t_1, 0)), ((0, e_2), (0, t_2)), ((0, 0), (0, 0)), ((e_1, e_2), (t_1, t_2)) \mid e_i \in \{e_l, e_h\}, t_i \in \{t_l, t_h\}, i = 1, 2\}$, be the set of admissible coalitions. Let $\mathbb{R}_+$ denote the set of non-negative reals.

Definition **1** A **Task Aggregator Machine, TAM** is a function $M : \mathbb{A} \to \mathbb{R}_+$.

We define a game in characteristic form game corresponding to a grand machine state as follows.

Definition **2** Given M and the grand machine state $((e_1, e_2), (t_1, t_2))$, the restriction of M to $\{((e_1, 0), (t_1, 0)), ((0, e_2), (0, t_2)), ((0, 0), (0, 0)), ((e_1, e_2), (t_1, t_2))\}$ is called a **Game in Characteristic Form**.

A key assumption of our paper is that M is intelligent. A machine is intelligent if by observing the output of the machine the management can infer the effort-type combination that yields that output. We now proceed to define an intelligent machine formally.

Definition **3** We call a TAM M **intelligent** if

1. $M((e_1', e_2'), (t_1', t_2')) = M((e_2', e_1'), (t_2', t_1'))$. This condition can be interpreted as **symmetry**. That is, output depends only on effort and efficiency and not on the identities of the employees.
2. Except the **symmetry** condition given above, $M((e_1', e_2'), (t_1', t_2')) \neq M((e_1'', e_2''), (t_1'', t_2''))$ whenever $(e_1', e_2') \neq (e_1'', e_2'')$ or $(t_1', t_2') \neq (t_1'', t_2'')$.

The second property in the definition is the embodiment of the the notion of an intelligent TAM. Suppose the grand machine state is $((e_l, e_h), (t_l, t_h))$ and thus the output from the machine is $M((e_l, e_h), (t_l, t_h))$. By the second property outputs for different grand machine states are different. Thus the management can deduce the grand machine state that occurs. By applying the second property again we can deduce the effort and the efficiency combinations of all employees. To see this consider the first employee and $M((e_l, 0), (t_l, 0))$. By the second property $M((e_l, e_h), (t_l, t_h)) - M((e_l, 0), (t_l, 0))$ =marginal contribution of the second employee, with efficiency t_h who puts e_h, to the output of the grand machine state when the employee 1 with efficiency level t_l puts efforts $e_l \neq 0$. Further marginal contribution of employee $1 = M((e_l, e_h), (t_l, t_h)) - M((0, e_h), (0, t_h)) \neq 0$. Since by the second property $M((0, e_h), (0, t_h)) \neq M((e_l, 0), (t_l, 0))$, we obtain

$$M((e_l, e_h), (t_l, t_h)) - M((e_l, 0), (t_l, 0)) \neq M((e_l, e_h), (t_l, t_h)) - M((0, e_h), (0, t_h)).$$

Thus the management can deduce the efforts and efficiency combinations of all employees. Further, by the second property $M((e_1', 0), (t_1', 0)) \neq M((e_1'', 0), (t_1'', 0))$ if $(e_1', t_1') \neq (e_1'', t_1'')$. Thus the management can deduce the efforts and efficiency levels of all employees.[8] The notion of an intelligent machine as discussed in (Simkoff, 2019) puts our ideas into perspective. (Simkoff, 2019) describes machine intelligence as follows: "machine intelligence by necessity involves deductive logic. For example, systems exhibiting true machine intelligence come to understand when they've made mistakes, watch out for similar data that could lead to a similar mistake the next time, and avoid doing so." By following the line of thinking as described in (Simkoff, 2019) we consider TAMs that are capable of unambiguously deducing the efforts and the efficiency levels that are associated with the levels of outputs, hence TAMs are intelligent. In a low tech firm/agricultural farm it is harder to make such deductions about efficiency since technical skills do not matter a lot for the productivity in these firms. Therefore we do not consider them as examples of TAMs. Since output of an intelligent M reveals the effort and efficiency of the employees, we do not need to assume that employees know their efficiency levels. In standard economic theory it is assumed that employees know their efficiency levels, however the management does not. Such a situation is an example of a situation of asymmetric information. There can be situations where neither the employees know nor the outputs reveal the contributions of the employees. Consider a situation where two individuals have written a paper jointly. Let us assume that the paper receives an award and thus the two coauthors jointly receive a prize money. The question now is how the two individuals should share the prize money. It is difficult for an individual to know her contribution in the paper. Further from the outcome of the joint work, let the outcome be the prize money, it is difficult to deduce the contribution each author of the paper. In other words, this is an example of a situation where neither the marginal contribution of an individual be computed from the joint output, nor individuals know about their contribution. Thus such situations are not examples of intelligent M, and we do not consider them in our analysis.

Effort of an employee by itself is not enough for machine productivity. Even twelve hours of work every working day by an employee whose understanding of category theory is 'not good' may not be of significant importance to a software firm. Thus we consider situations where it matters for a firm whether efforts come from a high type or a low type employee. Formally, situations that describe whether efforts come from high or low type are modeled by functions from $\{t_l, t_h\}$ into $\{e_l, e_h\}$. We call such a function a **strategy**. Let s_i denote the strategy of employee i. Then (s_1, s_2) is called a **strategy profile**, and the profile is called **symmetric** if $s_1 = s_2$. For any type profile $(t_1, t_2) \in \{t_l, t_h\} \times \{t_l, t_h\}$, a strategy profile defines the grand machine state $((s_1(t_1), s_2(t_2)), (t_1, t_2))$, and thus defines a characteristic form game. Given a strategy-profile (s_1, s_2) we have the following collection of characteristic form games:

1. $M((s_1(t_l), s_2(t_h)), (t_l, t_h))$, $M((s_1(t_l), 0), (t_l, 0))$, $M((0, s_2(t_h)), (0, t_h))$, $M((0, 0), (0, 0))$

2. $M((s_1(t_h), s_2(t_l)), (t_h, t_l))$, $M((s_1(t_h), 0), (t_h, 0))$, $M((0, s_2(t_l)), (0, t_l))$, $M((0, 0), (0, 0))$

3. $M((s_1(t_h), s_2(t_h)), (t_h, t_h))$, $M((s_1(t_h), 0), (t_h, 0))$, $M((0, s_2(t_h)), (0, t_h))$, $M((0, 0), (0, 0))$

4. $M((s_1(t_l), s_2(t_l)), (t_l, t_l))$, $M((s_1(t_l), 0), (t_l, 0))$, $M((0, s_2(t_l)), (0, t_l))$, $M((0, 0), (0, 0))$.

We observe that for a fixed strategy profile the characteristic form games depend only on the types. We assume that the management is uninformed about the individual efficiency level of the job seekers. Thus given a strategy profile, uncertainty over the set of characteristics form games is the same as the uncertainty over the space $\{t_l, t_h\}$. We assume that the management knows the probability distribution on the type space $\{t_l, t_h\}$. We assume that the management knows the probability distribution on the type space $\{t_l, t_h\}$. A well known interpretation of a strategy s_i is that the nature reveals the type t_i to employee i, and then i decides the level of efforts as suggested by s_i which is $s_i(t_i)$. According to this interpretation s_i is a conscious contingent plan of actions of i. However we do not assume employee i to know her type. We may relax the assumption that i knows her type and instead assume that i holds a belief about her own type, and she knows what she truly believes. Such weakening from knowing to having a belief about being t_l or t_h is also not required. Since M is intelligent, eventually types will be reveled. Tesla's head of global recruiting once said that Tesla is a big believer in showing what an employee can do, versus telling them what an employee can do, see (Hess, 2018) for the details. Since we do not assume an employee to know her type, it follows that the employee does not know the strategy she plays. Thus we do not interpret a strategy as a conscious contingent plan of actions of an employee. We interpret s_i **to be a strategy that nature plays through individual** i. It is not important for our analysis whether employee i is aware of nature's move. An employee may believe that her type is t_h thus puts effort e_h. However it may turn out that her type actually is t_l. Thus the strategy that she believes she is following may not be what she is actually following. In Section 4.3.1 we consider the situation where employees' efforts depend on whether she works alone or in team. Therefore in Section 4.3.1 strategy of an employee is a 2-tuple.

Each of the four characteristic form games entails the Shapley shares for each employee. The Shapely share of employee i is the average of the marginal contributions, see (Roth, 1988) for the formula for a model with arbitrary number of agents. Each employee contributes to two coalitions, the empty coalition and the grand machine state. Let the grand machine state be $((e_l, e_h), (t_l, t_h))$.

Then the Shapely share of employee 1 is

$$Sh_1((e_l, e_h), (t_l, t_h)) = \frac{M((e_l, e_h), (t_l, t_h)) - M((0, e_h), (0, t_h)) + M((e_l, 0), (t_l, 0))}{2},$$

where $M((e_l, 0), (t_l, 0))$ is employee 1's marginal contribution to the empty coalition, and $M((e_l, e_h), (t_l, t_h)) - M((0, e_h), (0, t_h))$ is employee 1's marginal contribution when 1 joins employee 2 whose type is t_h and puts effort e_h. We interpret the singleton coalition as the probation period of an employee. An analogues formula hod for employee 2, see subsection 4.2. In our model an employee receives her Shapely share as the remuneration.

REMARK **1** The Shapley share is efficient, i.e., for all $(e_1, e_2) \in \{e_l, e_h\} \times \{e_l, e_h\}$ and $(t_1, t_2) \in \{t_l, t_h\} \times \{t_l, t_h\}$, $Sh_1((e_1, e_2), (t_1, t_2)) + Sh_2((e_1, e_2), (t_1, t_2)) = M((e_1, e_2), (t_1, t_2))$. We can generalize Shapley value to Semi-values. Semi-values consider a generalized notion of marginal contribution. There is a subclass of semi-values in which the sum of the shares of the individuals is less than $M((e_1, e_2), (t_1, t_2))$, i.e., the entire revenue is not shared with the employees. See Chapter 7 in (Roth, 1988) for a discussion on Semi-values. Considering such a class of values only complicates our computations and does not lead to any qualitative change in the results.

Although the firm shares the revenue with the employees, there is a non-cooperative side to our model. An employee can strategically decide what effort to put. For example, it is conceivable that if employee 2 puts high efforts, then the best response of employee 1 is to put low efforts. Alternatively, an employee may put high effort to increase her Shapley share. However high effort from a less efficient employee may not be desirable for the firms or the economy's net surplus. In our model the economy consists of two employees and the firm. Example 1 in Section 4.3 makes this point formally. Given a probability distribution over $\{t_l, t_h\}$ one can compute the expected Shapley share and expected cost from a strategy profile, see Section 4.2. An employee's pay-off is "expected Shapley share - expected cost". The next section reviews some literature.

3 Relation to the Literature

Companies are increasingly using artificial intelligence based algorithms when hiring employees, see (Morris, 2022). These algorithms analyze resumes of the job applicants to measure their personalities, thus these algorithms can be perceived as indirect mechanisms whose objective is to elicit private information about the applicants. These algorithms are prediction machines, the notion of a prediction machine is well discussed in (Agrawal, et al., 2018). (Rhea et al., 2022) find these algorithms to be not reliable, for instance "recruiters do not know why certain candidates are on page one of the ranking, or why certain people are on page ten of the ranking when they search for candidates", see (Morris, 2022). In other words, intelligent machines that are being used currently to screen job seekers are at their infancies. (Moloi and Marwala, 2020) imagine intelligent machines that can moderate agent-behaviour to be in line with the expected behaviour. Thus the TAM, as a machine in a physical form if can be built, falls into the category of intelligent machines that are used in hiring. Since intelligent TAMs are based on cooperative game theory solution concepts that have fairness properties, see (Myerson, 1997), intelligent TAMs have these fairness properties as well. Next we discuss some literature related to solutions concepts of cooperative game theory with uncertainty.

Fix a type profile (t_1, t_2), and consider all possible efforts profiles. This profile entails a game in non-transferable utility (NTU) described as follows. Let $V(S)$ denote a coalitional function, where S is a non-empty subset of $\{1, 2\}$. Then $V(\{1,2\}) = \Big\{\Big(Sh_1((e_1', e_2'), (t_1, t_2)) - C(e_1', t_1), Sh_2((e_1', e_2'), (t_1, t_2)) - C(e_2', t_2)\Big) \mid (e_1', e_2') \in \{e_l, e_h\} \times \{e_l, e_h\}\Big\} \subseteq \mathbb{R}^2$. Then $V(\{1\}) = \{M((e_1', 0), (t_1, 0)) - C(e_1', t_1) \mid e_1' \in \{e_l, e_h\}\}$. Further,

$$V(\{2\}) = \{M((0, e_2'), (0, t_2)) - C(e_2', t_2) \mid e_2' \in \{e_l, e_h\}\} \subseteq \mathbb{R},$$

for details on NTU games see (Hart, 2004). An example in Section 1 in https://drive.google.com/file/d/1S4QP demonstrates that NTU Shapley Value pay-off vector is different from the Nash equilibrium pay-off vector of the non-cooperative game where the set of actions of the players is $\{e_l, e_h\}$ and the pay-off vectors of the non-cooperative game is given by the elements from the set $V(\{1,2\})$.[9] Hence the solution concept discussed in this paper is an extension of Shapley Value for Transferable Utility (TU) Games to Nash equilibrium which is a solution concept applied to non-cooperative games. Usually TU solutions are extended to NTU solutions, see (Hart, 2004). Further, once we allow for the type profiles to be probabilistic, then we extend our solution concept from a TU game solution concept to a solution concept for games with incomplete information i.e., ex-ante Nash equilibrium. In (Myerson, 1984) strategies are functions from type spaces to type spaces and they are continent plans of actions of the players. Since we do not assume that employees know their types we do not consider direct mechanisms and hence we do not consider the kind of strategies that (Myerson,

1984) considers. (Masuya, 2016) studies a model in which the worth of the singleton and grand coalitions are known and provides an axiomatic characterization of complete and superadditive extension of such games. In our model the worth of all coalitions is unknown. (Pongou and Tondji, 2018) consider a model in which there are n inputs that produces some output. The employees in our model are the analog of the inputs in (Pongou and Tondji, 2018). The quality of the inputs in (Pongou and Tondji, 2018) is unknown. The level of efficiency can be thought of as the level of quality of an employee. (Pongou and Tondji, 2018) characterize ex-ante, which (Pongou and Tondji, 2018) call a priori, and Bayesian Shapley value.

4 Model and Results

In this section we discuss our model and results. First we discuss the technology i.e., assumptions on M and the cost function of the employees.

4.1 Definitions and Assumptions on M and C

To carry out our analysis we impose certain restrictions on M. The effort and type profiles are denoted by $e_S = (e_i)_{i \in S}$ and $t_S = (t_i)_{i \in S}$ respectively in which the i^{th} coordinate is 0, if $i \notin S$. A typical coalition is written as $((e_1, e_2), (t_1, t_2))$. We consider the following definitions.

4.1.1 Assumptions on M

We assume M to be intelligent. **Definition of ordering on type-coalition:** For all pairs of type profiles (t_1, t_2) and (t'_1, t'_2) we say $(t_1, t_2) <_{tt} (t'_1, t'_2)$ if and only if, $t_i <_t t'_i$ for all i or if $t_i = t'_i$ for some i then $t_j <_t t'_j$ for $i \neq j$. **Definition of ordering on effort-coalition:** For all pairs of effort profiles $(e_1, e_2) <_{ee} (e'_1, e'_2)$ if and only if, $e_i <_e e'_i$ for all i or if $e_i = e'_i$ for some i then $e_j <_e e'_j$ for $i \neq j$. We assume M to satisfy **Monotonicity within types i.e., higher effort profile generates more return**, which is defined as follows. Let (t_1, t_2) be a type profile and two effort profiles (e'_1, e'_2) and (e_1, e_2) with $(e'_1, e'_2) <_{ee} (e_1, e_2)$ then, $M((e'_1, e'_2), (t_1, t_2)) < M((e_1, e_2), (t_1, t_2))$. We assume M to satisfy **Monotonicity within efforts i.e., more efficient type profile generates more return** which is defined as follows. Let (e_1, e_2) be an effort profile and two type profiles (t'_1, t'_2) and (t_1, t_2) with $(t'_1, t'_2) <_{tt} (t_1, t_2)$ then, $M((e_1, e_2), (t'_1, t'_2)) < M((e_1, e_2), (t_1, t_2))$. We assume M to satisfy **Supermodularity i.e., increments in return for more efficient types are larger** which is defined as follows. For all two type profiles $t^{'}$, $t^{''}$ and two effort profiles, $e^{'}$, $e^{''}$ with $t^{'} <_{tt} t^{''}$ and $e^{'} <_{ee} e^{''}$, $M(e^{''}, t^{'}) - M(e^{'}, t^{'}) < M(e^{''}, t^{''}) - M(e^{'}, t^{''})$. As an example let $(t^{'}, 0) <_{tt} (t^{''}, 0)$, and $(e^{'}, 0) <_{ee} (e^{''}, 0)$. Supermodularity implies $M((e^{''}, 0), (t^{'}, 0)) - M((e^{'}, 0), (t^{'}, 0)) < M((e^{''}, 0), (t^{''}, 0)) - M((e^{'}, 0), (t^{''}, 0))$.

4.1.2 Assumption on C

While putting effort each individual incurs a cost that depends on the type of the individual. Let $e \in E$ and $t \in T$ then the dependency of cost on effort and type is denoted by $C(e, t)$, and the cost function C admits the following properties. As Analogous to M we do not define $C(e, t)$ if either one of them is 0. Thus when we say C to be function we mean that C maps (e, t) to real a number when neither e nor t is 0. We assume C to satisfy **monotonicity of cost in efforts i.e., higher level of effort costs more** which is defined as follows. For all $t \in \{t_l, t_h\}$, $C(e_l, t) < C(e_h, t)$ where $e_l <_e e_h$. We assume C to satisfy **efficiency in type i.e., higher type incurs lower cost** which is defined as follows. For all $e \in \{e_l, e_h\}$, $C(e, t_h) < C(e, t_l)$ where $t_l <_t t_h$. We assume C to satisfy **Submodularity i.e., increment in cost for the efficient type is lower** which is defined as follows. For the pair of t_l, t_h with $t_l <_t t_h$ and e_l, e_h with $e_l <_e e_h$, $C(e_h, t_h) - C(e_l, t_h) < C(e_h, t_l) - C(e_l, t_l)$. In our study we can include the case, $C(e_h, t_h) - C(e_l, t_h) < C(e_h, t_l) - C(e_l, t_l)$, this condition incorporate the case : Cost function is zero and then we can go back to the Shapley value as a particular case.

The technology entails a game of incomplete information. This is discussed next.

4.2 The Game of Incomplete Information

In this section we describe the game of incomplete information that we utilize to analyze the set of possible outcomes. Let $\Gamma = \{N, T^{'2}, E^{'2}, p, G, (g_t)_{t \in T^{'2}}\}$ denote the game of incomplete information whose components are defined as follows: $N = \{1, 2\}$ denote the set of players, $T^{'2} = T' \times T'$, $E^{'2} = E' \times E'$, where $T' = \{t_l, t_h\}, E' = \{e_l, e_h\}$. We assume players' types are drawn interdependently and identically according to the probability distribution p, and $p(t_l) > 0, p(t_h) > 0$ with $p(t_l) + p(t_h) = 1$. Let $\mathbb{P} = \{(p(t_h), p(t_l)) \mid p(t_l) > 0, p(t_h) > 0, p(t_l) + p(t_h) = 1\}$ denote the set of all probability distributions with support $\{t_l, t_h\}$. The probability measure p represents the belief of the management about the distribution of types in the economy from which employees are drawn. For instance, p may be a relative frequency empirical distribution. The shape of this distribution maybe an important factor that determines the high concentration of high tech firms in some geographical region. For instance, (Audretsch,Lehmann, and Warning, 2003) and (Figueiredo, Guimaraes and Woodward, 2003) look at the effect of educational attainments of the labor force on the firm location decision.[10] Since we assume that both individuals are hired, the grand coalition is formed. Consider a grand machine state $((e_1, e_2), (t_1, t_2))$. By definition of a grand machine state $e_i \in \{e_l, e_h\}$ and $t_i \in \{t_l, t_h\}$ for $i = 1, 2$. Thus consider the game in characteristic form defined by $M((e_1, e_2), (t_1, t_2)), M((e_1, 0), (t_1, 0)), M((0, e_2), (0, t_2)), M((0, 0), (0, 0))$. Let the Shapley shares of this game be denoted by $Sh_i((e_1, e_2), (t_1, t_2))$, $i = 1, 2$. The Shapley share of employee 1 is $Sh_1((e_1, e_2), (t_1, t_2)) = \frac{M((e_1,e_2),(t_1,t_2)) - M((0,e_2),(0,t_2)) + M((e_1,0),(t_1,0))}{2}$, and that of employee 2 is $Sh_2((e_1, e_2), (t_1, t_2)) = \frac{M((e_1,e_2),(t_1,t_2)) - M((e_1,0),(t_1,0)) + M((0,e_2),(0,t_2))}{2}$. The pay-off of employee 1 at $(e_1, e_2) \in \{e_l, e_h\} \times \{e_l, e_h\}$ and $(t_1, t_2) \in \{t_l, t_h\} \times \{t_l, t_h\}$ is $Sh_1((e_1, e_2), (t_1, t_2)) - C(e_1, t_1)$, and that of employee 2 is $Sh_2((e_1, e_2), (t_1, t_2)) - C(e_2, t_2)$. Given a profile of efficiency levels, and employee i's effort level, the value of the grand coalition depends on the effort of the other individual, and thus marginal contribution of i is affected by effort of agent j. That is, for any fixed type profile $Sh_i((e_1, e_2), (t_1, t_2)) - C(e_1, t_1)$ depends on the effort level of j. Thus we have a well defined stage game at each type profile as defined in according to (Maschler Solan and Zamir, 2013). G is the set of stage games, g_t refers to the stage game at the type profile $(t_1, t_2) \in \{t_l, t_h\} \times \{t_l, t_h\}$. Since type profiles are probabilistic, stage games are probabilistic as well. This defines a game of incomplete information. Next we define a pure strategy. Since this is the only notion of strategy we use in this paper from now on we call a pure strategy to be a strategy. A pure strategy or simply a strategy is a function from the set of types $\{t_l, t_h\}$ to the set of actions $\{e_l, e_h\}$. There are four possible pure strategies defined below.

Definition 4 A **strategy (or a pure strategy)** $s_{e_1 e_2}$ is a function $s_{e_1 e_2} : \{t_l, t_h\} \to \{e_l, e_h\}$, defined as

$$s_{e_1 e_2}(t) = \begin{cases} e_1 & \text{if } t = t_l \\ e_2 & \text{if } t = t_h, \end{cases}$$

Given a strategy profile (s_1, s_2) at $((s_1(t_1), s_2(t_2)), (t_1, t_2))$ pay-off of player 1 is given by $Sh_i((s_1(t_1), s_2(t_2)), (t_1, t_2))-$ $C(s_1(t_1), t_1)$. Likewise $Sh_2((s_1(t_1), s_2(t_2)), (t_1, t_2)) - C(s_2(t_2), t_2)$ denotes the Shapley share of player 2. The ex-ante expected-payoff of employee 1 for the play of the strategy-profile (s_1, s_2) is

$$\Pi_1(s_1, s_2) = \sum_{(t_1,t_2)\in T'^2} [Sh_1((s_1(t_1), s_2(t_2)), (t_1, t_2)) - C(s_1(t_1), t_1)]p(t_1)p(t_2)$$

and that for employee 2 is

$$\Pi_2(s_1, s_2) = \sum_{(t_1,t_2)\in T'^2} [Sh_2((s_1(t_1), s_2(t_2)), (t_1, t_2)) - C(s_2(t_2), t_2)]p(t_1)p(t_2).$$

Since the firm management knows only the distribution of types in the population, and we do not assume individuals to know their types, ex-ante expected pay-off is the appropriate notion of expected pay-off. Let S_i be the set of all strategies of individual i. The notion of ex-ante Nash equilibrium is defined below.

DEFINITION **5** A strategy profile (s_1^*, s_2^*) is an **ex-ante Nash equilibrium** for the game Γ if and only if (i) $\Pi_1(s_1^*, s_2^*) \geq \Pi_1(s_1, s_2^*)$ for all $s_1^* \neq s_1,\ s_1^*, s_1 \in S_1$; and (ii) $\Pi_2(s_1^*, s_2^*) \geq \Pi_2(s_1^*, s_2)$ for all $s_2^* \neq s_2,\ s_2^*, s_2 \in S_2$ hold.

An ex-ante Nash equilibrium is a notion of stability for the profiles of strategies. If (s_1^*, s_2^*) is such an equilibrium, then no individual i is expected to be better off if i does not end up behaving, consciously or unconsciously, according to s_i^* and j behaves according to s_j^*. Symmetric ex ante Nash equilibrium is defined below.

DEFINITION **6** (Maschler Solan and Zamir, 2013) A strategy profile (s_1^*, s_2^*) is a **symmetric ex-ante Nash equilibrium (SNE)** for Γ if and only if: (i) (s_1^*, s_2^*) is an ex-ante Nash equilibrium and (ii) $s_1^* = s_2^*$.

We shall call the SNE (s_1^*, s_2^*) as the SNE in the strategy $s_{e_1e_2}$ if $s_{e_1e_2} = s_i^*$ for $i = 1, 2$. An SNE in the the strategy $s_{e_he_l}$, i.e., $s_{e_he_l}(t_l) = e_h$, $s_{e_he_l}(t_h) = e_l$, does not exist in our model. We may interpret the other three symmetric equilibria as follows: $s_{e_le_l}$ is the equilibrium in which TAM receives low efforts from all individuals irrespective of their efficiency levels; $s_{e_he_h}$ is the equilibrium in which TAM receives high efforts from all individuals irrespective of their efficiency levels; $s_{e_le_h}$ is the equilibrium in which TAM receives low efforts from an individual who is of low type and receives high efforts from a type who is of high type. A firm may consider the distribution of efficiency levels as a factor before deciding on a location because equilibrium outcomes depend on the distributions.

Therefore we provide a characterization of SNEs that provides a classification of SNEs by the probability distributions on $\{t_l, t_h\}$. In particular our computations provide information on what range of $p(t_h)$ gives what kind of equilibria. For example we show that symmetric equilibrium in $s_{e_l e_l}$ exist if and only if p takes values in an interval around 0. Further, we identify the upper bound of the interval which depends on the Shapley value and the cost of efforts. It may be easier for a firm to take a decision about locating itself if for a given distribution the corresponding SNE is unique. The uniqueness pins down the possible equilibrium behavior uniquely. Thus we consider a notion of rationalizable equilibrium.

Definition 7 A strategy $s_{e_1 e_2}$ is **rationalizable** if there is a probability distribution p that makes $s_{e_1 e_2}$ a unique SNE.

Since there is no SNE in $s_{e_h e_l}$, this strategy is not rationalizable. The other three equilibria are rationalizable. Since different strategies entail different kinds of grand coalitions, rationalizable strategies provide information about the nature of the grand coalition that may form. The notion of rationalizibility in (Pongou and Tondji, 2018) is different from ours. In (Pongou and Tondji, 2018) a player is an input to a production function, and a pure strategy of an input is quality. A mixed strategy of an input is defined to be a probability distribution on the set of pure strategies. (Pongou and Tondji, 2018) call a vector, i.e., a vector in which each entry refers to a mixed strategy of an input, of mixed strategies rationalizable if the vector constitutes a Nash equilibrium of the associated complete information game. In the associated game the pay-off from a vector of pure strategy is the Shapley value from the characteristic form game in which each player is identified with a quality of an input. We discuss our main results next.

4.3 Main Results: Characterizations of SNEs

We state our results in this section. For the sake of convenience of exposition instead of calling a game by Γ we call it by Γ_p since the only parameter that we vary while studying equilibria is p. Further we fix individual 2's strategy. For example while studying $(s_{e_h e_h}, s_{e_h e_h})$ as equilibrium we fix individual 2's strategy at $s_{e_h e_h}$, and then argue that player 1 cannot be made better off from deviation from the strategy that is assumed for player 2. To study individual 1 let $\Delta C_{t_1} = C(e_h, t_1) - C(e_l, t_1) \equiv$ increment in cost due to an increase in efforts at type t_1, and $\Delta_{e_1 e_2}^{e_1' e_2'} Sh_1(t_1 t_2) =$

$Sh_1((e_1', e_2'), (t_1, t_2)) - Sh_1((e_1, e_2), (t_1, t_2)) \equiv$ change in Shapley share due to change in efforts at the type profile (t_1, t_2).

Proposition **1** Let M be super-modular and C sub-modular: (i) $(s_{e_h e_h}, s_{e_h e_h})$ is an SNE of Γ_p for some $p \in \mathbb{P}$ and (ii) $(s_{e_h e_h}, s_{e_h e_h})$ is not an SNE of $\Gamma_{p'}$ for some $p \neq p' \in \mathbb{P}$; if and only if $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$.

Proof: See the Appendix at the end.

■

Proposition 1 establishes that for a pair of super-modular M and sub-modular C there is a probability distribution over $\{t_h, t_l\}$ for which there is SNEs in strategy $s_{e_h e_h}$. The next lemma gives a range of probabilities on $p(t_h)$ for which one can obtain a symmetric ex-ante equilibrium.

Corollary **1** Let M be super-modular and C sub-modular. Then $(s_{e_h e_h}, s_{e_h e_h})$ is an SNE of Γ_p if and only if $p(t_h) \in [\frac{\Delta C_{t_l} - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)}{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)}, 1)$.

Proof: See the Appendix at the end.

■

The next result is about $s_{e_l e_l}$.

Proposition **2** Let M be super-modular and C sub-modular: (i) $(s_{e_l e_l}, s_{e_l e_l})$ is an SNE of Γ_p, for some $p \in \mathbb{P}$; (ii) $(s_{e_l e_l}, s_{e_l e_l})$ is not an SNE of $\Gamma_{p'}$, for some $p' \in \mathbb{P}$ if and only if $\Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l) < \Delta C_{t_h} < \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_h)$.

A proof similar to the proof of Proposition 1 can be found in Section 2 at https://drive.google.com/file/d/1S4
The next corollary gives a range of probabilities on $p(t_h)$ for which one can obtain such symmetric ex-ante equilibria.

Corollary **2** Let M be super-modular and C be sub-modular. Then $(s_{e_l e_l}, s_{e_l e_l})$ is an symmetric ex-ante equilibrium of Γ_p if and only if $p(t_h) \in (0, \frac{\Delta C_{t_h} - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l)}{\Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_h) - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l)}]$.

Proof: A proof can be found in Section 2 of
https://drive.google.com/file/d/1S4QPtCl5wq4sSdM4j13IfCpiNOTdfjTY/view?usp=sharing.

$\blacksquare$

The next two results are pertaining to symmetric ex-ante equilibria in strategy $s_{e_l e_h}$.

PROPOSITION **3** Let M be super-modular and C sub-modular: (a) $(s_{e_l e_h}, s_{e_l e_h})$ is an SNE of Γ_p, for some $p \in \mathbb{P}$; (b) $(s_{e_l e_h}, s_{e_l e_h})$ is not an SNE of $\Gamma_{p'}$, for some $p' \in \mathbb{P}$;

if and only if exactly one of the following holds:

(i) at least one of the following holds

(a) $\Delta^{e_h e_l}_{e_l e_l} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$,

(b) $\Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l) < \Delta C_{t_h} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_h)$,

(ii) at least one of the following holds

(a) $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) < \Delta C_{t_l} < \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_l t_l)$,

(b) $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_h) < \Delta C_{t_h} < \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l)$,

Proof: A proof of the result can be found in Section 3 at https://drive.google.com/file/d/1S4QPtCl5wq4sSdM

$\blacksquare$

The next corollary gives a range of probabilities on $p(t_h)$ for which one can obtain such symmetric ex-ante equilibria.

COROLLARY **3** Let M be super-modular and C sub-modular. Then $(s_{e_l e_h}, s_{e_l e_h})$ is an SNE of Γ_p if and only if $p(t_h) \in [\frac{\Delta C_{t_h} - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l)}{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_h) - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_h t_l)}, \frac{\Delta C_{t_l} - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_l t_l)}{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta^{e_h e_l}_{e_l e_l} Sh_1(t_l t_l)}] \subseteq (0, 1)$.

Proof: A proof of the result can be found in Section 3 at https://drive.google.com/file/d/1S4QPtCl5wq4sSdM

$\blacksquare$

PROPOSITION **4** Let M be super-modular and C sub-modular. There exists no p for which $(s_{e_h e_l}, s_{e_h e_l})$ is an SNE.

Proof: A proof of the result can be found in Section 4 at https://drive.google.com/file/d/1S4QPtCl5wq4sSdM

■

Our characterization results lay down necessary and sufficient conditions for SNEs in terms of the parameters of our model. These conditions can be computed by using the parameters of the model. The three main propositions provide us with nontrivial intervals, i.e., an interval which is neither a singleton set nor an empty set, for which SNEs in $s_{e_h e_h}, s_{e_l e_l}$ and $s_{e_l e_h}$ exist. The three corollaries provide the ranges of these intervals. Our results also tell us when a particular equilibrium does not exist. If a probability distribution on $\{t_l, t_h\}$ represents a distribution of efficiency, and if efficiency is interpreted as talent, then our results provide us with information about what kind of stable or equilibrium behavior outcomes may be expected if a distribution is given. The bounds on the intervals in the three corollaries are given by quantities that are functions of M and C. Since the pair M, C define technology in our paper, our results provide an indirect mechanism to study observed behavior from the perspective of existing technology. As an example consider $\frac{\Delta C_{t_h} - \Delta_{e_l e_l}^{e_h e_l} Sh_1(t_h t_l)}{\Delta_{e_l e_l}^{e_h e_l} Sh_1(t_h t_h) - \Delta_{e_l e_l}^{e_h e_l} Sh_1(t_h t_l)}$. This ratio can be interpreted as

$$\frac{\text{the surplus in incremental cost of efforts over the change in Shapley share of emplpoyee 1 due an incraese in her efficiency}}{\text{the surplus in the change in the Shapely share of employee 1 due to change in efficiency of employee 2}}$$
$$\equiv \frac{\text{Net internal effect of change in efforts and efficiency in cost}}{\text{Net external effect of efficiency on benefits}}$$

This ratio lies between 0 and 1. Thus this ratio can also be interpreted as a price of being employed in a firm in which cooperation entails external benefits for the employee. Supermodularity of M and Submodularity of C, i.e., two important features of technology in our paper, play an important role in making the price lie between 0 and 1. The lower bound of the interval in Corollary 2 is smaller than the lower bound of the interval in Corollary 3, and the lower bound of the interval in Corollary 1 is bigger than 0. Thus the strategies $s_{e_l e_l}$ and $s_{e_h e_h}$ are rationalizable. However if we assume M to be concave then $s_{e_l e_h}$ is also rationalizable. See Section 6 at https://drive.google.com/file/d/1S4QPtCl5wq4sSdM4j13IfCpiNOTdfjTY/view?usp=sharing for an example that shows that if M is not concave then the strategy $s_{e_l e_h}$ is not rationalizable. The notion of a concave M is defined below.

DEFINITION **8** Consider a grand machine state. M is said to be **concave within the type profile** t' if for three effort profiles, e', e'', e''' with $e' <_{ee} e'' <_{ee} e'''$, $M(e''', t') - M(e'', t') < M(e'', t') - M(e', t')$.

Concavity of M says the increase in the output of M is smaller at higher efforts. The following corollary provides a characterization of rationalizable strategies.

Corollary **4** Let M be super-modular and C sub-modular, then $S_{e_le_l}$ and $s_{e_he_h}$ are rationalizable. If M is also concave within type profile, then $s_{e_le_h}$ is also rationalizable.

Proof: The first part follows from the discussion above. If M is concave, then the intervals in all the three corollaries above are non empty and they are pairwise disjoint. For the details see the Appendix at the end of the paper. ■

It is possible that if we take the union of the intervals obtained in the corollaries above, then we may not obtain $[0, 1]$. However this should not be surprising because for certain games Γ_p SNE may not exist since SNEs are pure strategies. If the intervals in corollaries 1 and 3 intersect, then we may wonder whether it is better for the firm that both employees put high effort according to the SNE in the strategy $s_{e_he_h}$, Example 1 demonstrates that it may not be so. First we define expected welfare for the economy. The expected welfare of the economy from SNE in the strategy $s_{e_le_h}$ is:
$EW(s_{e_le_h}, p) = 2[p(t_l)p(t_l)\{Sh_1(e_le_l, t_lt_l) - C(e_l, t_l)\} + p(t_l)p(t_h)\{M(e_le_h, t_lt_h) - (C(e_l, t_l) + C(e_h, t_h))\} + p(t_h)p(t_h)\{Sh_1(e_he_h, t_ht_h) - (C(e_h, t_h))\}]$
The expected welfare from SNE in the strategy $s_{e_he_h}$ is:
$EW(s_{e_he_h}, p) = [p(t_l)p(t_l)\{Sh_1(e_he_h, t_lt_l) - C(e_h, t_l)\} + p(t_l)p(t_h)\{M(e_he_h, t_lt_h) - (C(e_h, t_l) + C(e_h, t_h))\} + p(t_h)p(t_h)\{Sh_1(e_he_h, t_ht_h) - (C(e_h, t_h))\}]$
In Example 1 we construct a super-modular M and sub-modular C and show that expected welfare from $s_{e_le_h}$ is higher than that of $s_{e_he_h}$.

Example **1** All tables related to this example are in the Appendix of the paper. Table 1 describes the TAM. For instance the the number 7 in the table is $M((e_l, 0), (t_l, 0))$. That is 7 is the value for the singleton coalition when only player 1 is present. Analogously 25 is the value of the grand coalition when both player are of type t_h and puts efforts e_h. That is $M((e_h, e_h), (t_h, t_h)) = 25$. Table 2 provides describes a cost function. From Table 3 we see that for $p_h \in [0.578948, 0.794872]$, which is the intersection of range of p_h in the second and the third row of Table 3, expected welfare for SNE in the strategy $s_{e_le_h}$ is higher than in the strategy $s_{e_he_h}$.

We end this section with a remark about extending our model to more than two types and two levels of effort.

Remark 2 We can extend our analysis to models with $n \geq 3$ employees. Instead of intervals we obtain subsets of $n-1$ dimensional simplices such that a distribution in the subset tells us about a strategy being an equilibrium and for distributions outside the subset that particular strategy is not an equilibrium. We have carried out an explicit computation of an equilibrium for three employees in Section 7 of https://drive.google.com/file/d/1S4QPtCl5wq4sSdM4j13IfCpiN0TdfjTY/view?usp=sharing.The computations are cumbersome. The employees put low efforts In this equilibrium. The three types are t_l, t_m, t_h and $t_l <_t< t_m <_t t_h$. Finding general solutions involve inequalities in $n-1$ degree, however the analytical framework for the three agents can be generalized in a straightforward manner.

The extension of our main model to incorporate the strategic behavior across coalitions is discussed next.

4.3.1 Strategic Behavior Across Coalitions

In this section we allude to how the results from the earlier section chnges if employees put different efforts at different coalitions. Let $C_i = \{\{i\}, \{i,j\}\}$ denote the set of coalitions that employer i can be part of. Next we define a strategy of employee i below.

Definition 9 A **strategy (or a pure strategy)** of i is a collection of two functions $s_i^\alpha : \{t_l, t_h\} \to \{e_l, e_h\}$, $\alpha \in C_i$. Here s_i^α denotes the strategy when α is the coalition. We denote a strategy of i by $(s_i^{\{i\}}, s_i^{\{i,j\}}) = (s_i^\alpha)_{\alpha \in C_i}$.

If we assume efforts not to vary across coalitions, then $s_i^{\{i,j\}}(t_l) = s_i^{\{i\}}(t_l)$ and $s_i^{\{i,j\}}(t_h) = s_i^{\{i\}}(t_h)$, $i = 1, 2$. We have analyzed this situation in the earlier sections. We explain the pay-off from a strategy profile below. Fix a strategy profile $((s_1^\alpha)_{\alpha \in C_1}, (s_2^\alpha)_{\alpha \in C_2})$ and a type profile (t_1, t_2). The worth of admissible coalitions corresponding to the corresponding game are: $M((s_1^{\{1,2\}}(t_1), s_2^{\{1,2\}}(t_2)), (t_1, t_2))$, $M((s_1^{\{1\}}(t_1), 0), (t_1, 0)), M((0, s_2^{\{2\}}(t_2)), (0, t_2,)), M((0, 0), (0, 0))$. The Shapley value of employee 1 corresponding to this game is denoted by,

$$Sh_1((s_1^\alpha)_{\alpha \in C_1}, (s_2^\alpha)_{\alpha \in C_2}, t_1, t_2))$$

The expected payoff of employee 1 for the play of the strategy-profile $((s_1^\alpha)_{\alpha \in C_1}, (s_2^\alpha)_{\alpha \in C_2})$

is:$\Pi_1((s_1^\alpha)_{\alpha\in C_1},(s_2^\alpha)_{\alpha\in C_2}) =$

$$\sum_{(t_1,t_2)\in T^2}[Sh_1((s_1^\alpha)_{\alpha\in C_1},(s_2^\alpha)_{\alpha\in C_2},t_1,t_2)) - C(s_1^{\{1\}}(t_1),t_1) - C(s_1^{\{1,2\}}(t_1),t_1)]p(t_1)p(t_2).$$

The expected pay-off of employee 2 is computed analogously. The notion of SNE is defined analogously. We note that a deviation by a player can occur in many ways. For instance $(s_i^{\{i\}'}, s_i^{\{i,j\}})$ is a deviation from $(s_i^{\{i\}}, s_i^{\{i,j\}})$, if $s_i^{\{i\}'}$ is a function that is distinct from $s_i^{\{i\}}$. Since the next proposition provides a characterization of symmetric equilibria we drop the suffix i from the strategies. For $\alpha \in C_i$, the function $s^\alpha_{e_1e_2}$ is defined as $s^\alpha_{e_1e_2}(t_l) = e_1, s^\alpha_{e_1e_2}(t_h) = e_2$. The next proposition says that it is possible that in equilibrium a player may put different efforts across coalitions.

Proposition **5** Let M be super-modular and C be sub-modular. Further let M be concave within type. Then for any probability distribution over $\{t_l, t_h\}$ exactly one of the following holds.

(i) If there are SNEs in the strategy $(s^{\{1\}}_{e_le_l}, s^{\{1,2\}}_{e_le_l})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_le_l})$, then there are no other SNEs.

(ii) If there are SNEs in the strategy $(s^{\{1\}}_{e_he_h}, s^{\{1,2\}}_{e_le_l})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_le_l})$, then there are no other SNEs.

(ii) If there are SNEs in the strategy $(s^{\{1\}}_{e_le_l}, s^{\{1,2\}}_{e_le_h})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_le_h})$, then there are no other SNEs.

(iv) If there are SNEs in the strategy $(s^{\{1\}}_{e_he_h}, s^{\{1,2\}}_{e_le_h})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_le_h})$, then there are no other SNEs.

(v) If there are SNEs in the strategy $(s^{\{1\}}_{e_he_h}, s^{\{1,2\}}_{e_he_h})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_he_h})$, then there are no other SNEs.

(vi) If there are SNEs in the strategy $(s^{\{1\}}_{e_le_l}, s^{\{1,2\}}_{e_he_h})$ or $(s^{\{1\}}_{e_le_h}, s^{\{1,2\}}_{e_he_h})$, then there are no other SNEs.

In the equilibria in (ii) we see an employee putting high efforts when working alone, and low efforts during joint work. It can be shown, under some mild conditions, that if one of the equilibria in (ii) exists, then the other also exists. In fact this holds in all six situations described in Proposition 5. This is shown Section 5 in `https://drive.google.com/file/d/1S4QPtCl5wq4sSdM4j13IfCpiNOTdfjTY/view?us` In Proposition 5 only one of the six possibilities can arise. Further, in both equilibria in all cases the strategies for the grand coalition are the same. Also, the output at a grand machine state is what matters for the firm management. Therefore without loss of generality we may consider the scenario in which employees put the same efforts in all coalitions.

5 Concluding Remarks

We consider a firm whose production technology is intelligent. The management of the firm hires two employees. The management of the firm does not know how efficient an employee is when hiring her. This entails a game of incomplete information, where corresponding to each type profile there is a stage game in which the available actions to the employees are their effort levels. The payoff function of the employees incorporates a cooperative and a non-cooperative aspect. Our characterization result explains the dependence of symmetric ex-ante equilibria on the distribution of type profiles. In turn, the characterization results formalize the relationship between the distribution of efficiency levels and the distribution of output.

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Notes

[1]A conversions of quality to a real number may involve measurement error. However, for the analysis carried out in this paper such measurement errors are not the matters of discussions of this paper because we assume that a measure of quality in terms of a real number is given.

[2] We have used the word advanced only in a suggestive sense and not in a sense in which it implies any continuity, the issues that arise while using it in the latter sense are pointed out in (Mitchell, 2021).

[3](Vazire and Carlsosn, 2010) find self-perception of personality to be far from accurate. Further (Parks and Wellman, 2015) observe that people seem to not understand incentive compatibility well.

[4]Incentive constraints are well defined in an auction environment. For example, buyers who participate in an auction know their valuations, i.e., the maximum they want to pay for the object to be sold.

[5]This notation is introduced for the convenience of exposition.

[6]Often software firms give online coding tests to prospective employees, and the individuals who are eventually hired are asked to work in teams. If we imagine such firms to be TAMs, then the individual who takes the test can be thought of as a singleton coalition.

[7]Since we wish to study cooperative outcomes we need to define cooperative outcomes. To define such outcomes we need at least two individuals. Thus we assume the firm to hire both individuals.

[8]The chess engine Stockfish can differentiate between a "bad" move and a "good" move, which makes it an intelligent machine because the classification of moves into bad and good is depends on the winning probability and probability of winning is an outcome. However Stockfish is not a TAM.

[9]We do not need assumptions such as closed or convex on the set of pay-off vectors of coalitions while applying the algorithm to compute NTU value described in (Hart, 2004). In our case since the number of pay-off vectors is finite for each coalition the set of pay-off vectors is closed. We could make the set of pay-off convex by allowing for randomized strategies. Since the TAM in this paper does not take probability distribution as inputs we do not consider randomization.

[10]Since efficiency is latent, educational attainment may not be a good proxy for efficiency.

Appendix

Proof of Proposition 1

An outline of the steps in the proof are follows. First we show that $(s_{e_h e_h}, s_{e_h e_h})$ is an SNE of Γ_p for some $p \in \mathbb{P}$ if and only if

$$\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h).$$

Then we show that there is a probability distribution $p \in \mathbb{P}$ for which

$$\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$$

holds if

$$\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$$

holds. This establishes existence of an SNE for some $p \in \mathbb{P}$. Then we also show that if $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ holds then there is $p \in \mathbb{P}$ for which $\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ does not hold; we use the Farkas' lemma to show this. This establishes that sufficiency of $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ in Proposition 1. Next we assume that (i) and (ii) hold and use Farkas' lemma to show that $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ holds.

LEMMA **1** Let M be super-modular C be sub-modular and $p \in \mathbb{P}$, $(s_{e_h e_h}, s_{e_h e_h})$ is an SNE of Γ_p if and only if $\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$.

Proof of Lemma 1 Let $(s_{e_h e_h}, s_{e_h e_h})$ be an SNE for the game Γ_p, then

$$\Pi_1((s_{e_h e_h}, s_{e_h e_h})) \geq \Pi_1(s_{e_l e_h}, s_{e_h e_h}) \tag{1}$$

$$\Leftrightarrow p(t_l)p(t_l)\{Sh_1((e_h, e_h), (t_l, t_l)) - C(e_h, t_l)\} + p(t_l)p(t_h)\{Sh_1((e_h, e_h), (t_l, t_h)) - C(e_h, t_l)\}$$

$$\geq p(t_l)p(t_l)\{Sh_1((e_l, e_h), (t_l, t_l)) - C(e_l, t_l)\} + p(t_l)p(t_h)\{Sh_1((e_l, e_h), (t_l, t_h)) - C(e_l, t_l)\}$$

$$p(t_l)[p(t_l)\{Sh_1((e_h, e_h), (t_l, t_l)) - C(e_h, t_l) - Sh_1((e_l, e_h), (t_l, t_l)) + C(e_l, t_l)\}$$

$$+p(t_h)\{Sh_1((e_h, e_h), (t_l, t_h)) - Sh_1((e_l, e_h), (t_l, t_h)) - C(e_h, t_l) + C(e_l, t_l)\}] \geq \tag{2}$$

$$\Leftrightarrow p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) \geq \Delta C_{t_l}.$$

Therefore player 1 does not have an incentive to deviate to $s_{e_l e_h}$ if and only if $\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$, given this we show next that other deviations are also not profitable for player 1. Since C is sub-modular $\Delta C_{t_l} = C(e_h, t_l) - C(e_l, t_l) > \Delta C_{t_h} = C(e_h, t_h) - C(e_l, t_h)$. Since M is super modular $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_l) > \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)$ and $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_h) > \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$. Now $\Delta C_{t_h} < p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$.

$$\Rightarrow p(t_l)\{\Delta C_{t_h} - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_l)\} + p(t_h)\{\Delta C_{t_h} - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_h t_h)\} < 0$$

$$\Leftrightarrow p(t_h)[p(t_l)\{Sh_1((e_l, e_h), (t_h, t_l)) - C(e_l, t_h) - Sh_1((e_h, e_h), (t_h, t_l)) + C(e_h, t_h)\}$$

$$+p(t_h)\{Sh_1((e_l, e_h), (t_h, t_h)) - Sh_1((e_h, e_h), (t_h, t_h)) - C(e_l, t_h) + C(e_h, t_h)\}] < 0 \tag{3}$$

$$\Leftrightarrow \Pi_1(s_{e_he_l}, s_{e_he_h}) < \Pi_1(s_{e_he_h}, s_{e_he_h})$$

Therefore player 1 does not have a profitable deviation opportunity to $s_{e_le_h}$ if and only if $\Delta C_{t_l} \leq p(t_l)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) + p(t_h)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)$ implies player 1 does not have a profitable deviation opportunity to $s_{e_he_l}$. From $\Delta C_{t_l} \leq p(t_l)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) + p(t_h)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)$, sub-modular C and Super Modular M it follows that Inequality (2) + inequality (3)>0 and hence player 1 does not have any incentive to unilaterally deviate to the strategy $s_{e_le_l}$.

End of Proof of Lemma 1

Lemma **2** Let M be super-modular and C be sub-modular. If $\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) < \Delta C_{t_l} < \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)$ then there is $p \in \mathbb{P}$ such that $\Delta C_{t_l} \leq p(t_l)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) + p(t_h)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)$.

Proof of Lemma 2 It is enough to show that the following system has a solution.

$$p(t_l)\{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)\} + p(t_h)\{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)\} \leq 0$$
$$p(t_l) > 0,\ p(t_h) > 0,\ p(t_l) + p(t_h) = 1$$

This system of inequalities is rewritten as, call the rewritten system (Ph):

$$\begin{aligned} p(t_l)\{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)\} + p(t_h)\{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h)\} &\leq 0 \\ p(t_l) + p(t_h) &\leq 1 \\ -p(t_l) - p(t_h) &\leq -1 \\ p(t_l).0 - p(t_h) &< 0 \\ -p(t_l) + p(t_h).0 &< 0. \end{aligned} \tag{Ph}$$

This system then can be seen succinctly in the form $Ax \leq b$, $Bx < c$, with $x \equiv (p(t_l), p(t_h))$ with;

$$A = \begin{bmatrix} \Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) & \Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In order to establish that this system has solution we show that the appropriate Farkas' dual does not have a solution. We recall the following version of Farkas' lemma.

Theorem **1** [**Farkas' Lemma** (Motzkin, 2001)] Exactly one of the following statements is true.

(1) There exists x satisfying $Ax \leq b$ and $Bx < c$.

(2) There exist y, z such that,

$$y \geq 0, \quad z \geq 0, \quad A^T y + B^T z = 0$$

and

$$b^T y + c^T z < 0 \text{ or } b^T y + c^T z = 0, \quad z \neq 0$$

The corresponding Farkas' dual, i.e. the set of inequalities (2) in Theorem 1, for our system of inequalities is the following:

$y = (y_1, y_2, y_3), y_i \geq 0, i = 1, 2, 3, \ z = (z_1, z_2), z_j \geq 0, j = 1, 2;$

$$A^T y + B^T z = 0 \text{ and } b^T y + c^T z < 0 \text{ or } b^T y + c^T z = 0 \ ; \ z \neq 0.$$

We show this dual has no solution; and hence a proof of Lemma 2 follows by Theorem 1. We prove this claim by the way of contradiction.

We note $b^T y + c^T z < 0 \Leftrightarrow y_2 < y_3$.

$$A^T y + B^T z = 0 \Leftrightarrow \begin{bmatrix} \Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) & 1 & -1 \\ \Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l)\} y_1 + y_2 - y_3 - z_2 = 0$$

$$\{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h)\} y_1 + y_2 - y_3 - z_1 = 0$$

$$\Leftrightarrow \quad \{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h)\} y_1 - z_1 = y_3 - y_2 > 0 \ \text{ contradiction as } \ \Delta C_{t_l} < \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h).$$

Now consider the case,

$$A^T y + B^T z = 0 \ ; \ b^T y + c^T z = 0 \Leftrightarrow y_2 = y_3, \ z \neq 0 \Leftrightarrow z_1 > 0 \text{ or } z_2 > 0.$$

$$\text{Then, } \ \{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l)\} y_1 + y_2 - y_3 - z_2 = 0$$

$$\{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h)\} y_1 + y_2 - y_3 - z_1 = 0$$

$$\Leftrightarrow \quad \{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l)\} y_1 - z_2 = 0$$

$$\{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h)\} y_1 - z_1 = 0$$

$$\Leftrightarrow \quad \{\Delta C_{t_l} - \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h)\} y_1 - z_1 = 0 \ \text{ contradiction if } \ y_1 \neq 0 \text{ as } \Delta C_{t_l} < \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h);$$

$$\text{if } y_1 = 0 \text{ then } z_1 = z_2 = 0.$$

End of Proof of Lemma 2

Now we go back to the proof of Proposition 1.

Let $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$, and we show (i) and (ii) to hold.

We first show (i) in Proposition 1. Choose the the probability distribution for which Lemma 2 holds, and then by Lemma 1, (i) follows.

Now we show (ii) in Proposition 1 i.e., we show if $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ then (ii) holds. In particular by Lemma 1 it is sufficient to show that there is $p \in \mathbb{P}$ such that $\Delta C_{t_l} \leq p(t_l)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) + p(t_h)\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ does not hold i.e., $p(t_l)\{\Delta C_{t_l} - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)\} + p(t_h)\{\Delta C_{t_l} - \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)\} > 0$. In order to show this we show that the following system of inequalities has a solution.

$$
\begin{aligned}
p(t_l)\{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\} + p(t_h)\{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\} < 0 \\
-p(t_l) + 0.p(t_h) < 0 \\
0.p(t_l) - p(t_h) < 0 \\
p(t_l) + p(t_h) \leq 1 \\
-p(t_l) - p(t_h) \leq -1
\end{aligned}
\tag{Qh}
$$

Which is of the form $Ax \leq b$, $Bx < c$, with $x \equiv (p(t_l), p(t_h))$,
$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B = \begin{bmatrix} \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l} & \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The Dual of (Qh) is

$y \equiv (y_1, y_2) \geq 0,\ z \equiv (z_1, z_2, z_3) \geq 0;\ A^T y + B^T z = 0\ ;\ b^T y + c^T z < 0$ or $b^T y + c^T z = 0\ ;\ z \neq 0$.

We show that this dual has no solution. We prove this claim by the way of contradiction. First consider $b^T y + c^T z < 0$.

$$A^T y + B^T z = 0;\ b^T y + c^T z < 0 \Leftrightarrow y_1 < y_2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l} & -1 & 0 \\ \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l} & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0$$

$$
\begin{aligned}
\Leftrightarrow\quad & y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\}.z_1 - z_2 = 0 \\
& y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\}.z_1 - z_3 = 0 \text{ a contradiction as } \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l}.
\end{aligned}
$$

Now consider the situation in which $b^T y + c^T z = 0$. That is,

$$
\begin{aligned}
& b^T y + c^T z = 0 \Leftrightarrow y_1 = y_2,\ z \neq 0 \Leftrightarrow z_1 > 0 \text{ or } z_2 > 0 \text{ or } z_3 > 0 \\
\Leftrightarrow\ & y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\}.z_1 - z_2 = 0 \\
& y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\}.z_1 - z_3 = 0 \\
\Leftrightarrow\ & \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\}.z_1 - z_2 = 0 \\
& \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\}.z_1 - z_3 = 0 \ \text{ contradiction if } \ z_1 \neq 0 \text{ as } \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l}; \\
& \text{if } z_1 = 0 \text{ then } z_3 = 0 = z_2.
\end{aligned}
$$

Therefore the dual of (Qh) does not have a solution, hence (ii) in Proposition 1 is established by Theorem1.

Now we prove the converse of Proposition 1. That is we assume (i) and (ii) to hold; and show $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$. Since (i) and (ii) hold the duals of Ph and Qh do not have a solution by Theorem 1. First we note that in both systems of inequalities Ph and Qh the scalar vector b has both positive and negative entries. This means that the reason for the duals of Ph and Qh not to have solutions is not because $b^T y + c^T z > 0$ for all $y \geq 0, z \geq 0$ and $z \neq 0$. Also it is not true that $b^T y + c^T z = 0$ implies $z = 0$. Hence it is enough to consider the situations described by the duals of Ph and Qh and look at the implications if they are violated in order to establish the converse of Proposition 1. First the following intermediary result is needed. Let $A = \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}$ and $B = \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}$.

LEMMA **3** For (M, C) the following holds,

(a) If $A \geq 0$ then $B > 0$.

(b) If $B \leq 0$ then $A < 0$.

Proof of Lemma 3 Since M is super-modular therefore, $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) > \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)$ **Proof of** (a)**:** Let,

$$
\begin{aligned}
& A \geq 0 \Rightarrow \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) \geq \Delta C_{t_l} \\
\Rightarrow\ & \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) > \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) \geq \Delta C_{t_l} \\
\Rightarrow\ & \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l} > 0 \Leftrightarrow B > 0
\end{aligned}
$$

End of the Proof of (a)**.**

Proof of (b)**:** If

$$
\begin{aligned}
& B \leq 0 \Rightarrow \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) \leq \Delta C_{t_l} \\
\Rightarrow \quad & \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) < \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) \leq \Delta C_{t_l} \\
\Rightarrow \quad & \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) - \Delta C_{t_l} < 0 \Leftrightarrow A < 0
\end{aligned}
$$

End of the Proof of (b)

End of Proof of Lemma 3 Now we go back to the proof of the converse in Proposition 1.

Consider the dual of (Ph)**.**

Step 1**:** We note $b^T y + c^T z < 0 \Leftrightarrow y_3 > y_2$; and the dual does not have a solution means for all $y \geq 0, z \geq 0$ the following system has no solution, which in turn implies at least one of them has no solution.

$$
\begin{aligned}
& \{\Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\} y_1 + y_2 - y_3 - z_2 = 0 \\
& \{\Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\} y_1 + y_2 - y_3 - z_1 = 0
\end{aligned}
$$

We argue that if $A = \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) - \Delta C_{t_l} > 0$ and $B = \Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) - \Delta C_{t_l} > 0$ then the dual has a solution. To see this let $A > 0, B > 0$ and consider $A = y_3 - y_2 + z_2$ and $B = y_3 - y_2 + z_1$. Since $y \geq 0, z \geq 0$ $y_3 - y_2 + z_2 \in (0, \infty)$ and $y_3 - y_2 + z_1 \in (0, \infty)$, and hence choose $A = y_3^* - y_2^* + z_2^*$ and $0 < y_3^* - y_2^* < B$. Then set $z_1^* = B - [y_3^* - y_2^*]$. This means $(1, y_2^*, y_3^*), (z_1^*, z_2^*)$ is a solution to the dual of (Ph).
Hence either $\Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) \leq \Delta C_{t_l}$ or $\Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_h) \leq \Delta C_{t_l}$.

Step 2**:** Analogous to the last step, for $b^T y + c^T z = 0 \Leftrightarrow y_3 = y_2$; and the dual does not have a solution means for all $y \geq 0, z \geq 0, z \neq 0$ at least one equation in the system

$$
\begin{aligned}
& \{A = [\Delta_{e_l e_h}^{e_h e_h} Sh_1(t_l t_l) - \Delta C_{t_l}]\} y_1 + y_2 - y_3 - z_2 = 0 \\
& \{B = [\Delta_{e_l e_l}^{e_h e_l} Sh_1(t_l t_h) - \Delta C_{t_l}]\} y_1 + y_2 - y_3 - z_1 = 0 \\
\Leftrightarrow \quad & A y_1 - z_2 = 0; B y_1 - z_1 = 0
\end{aligned}
$$

does not hold. By Lemma 3 if $A = 0$, then $B > 0$. Then setting $z_2 = 0$ we can find a solution to the dual. Hence $A \neq 0$. If $A > 0$, by Step 1 $B \leq 0$; which contradicts Lemma 3 since $B \leq 0$ implies $A < 0$ by Lemma 3. Also by Lemma 3 if $A > 0$ then $B > 0$, which contradicts Step 1.

Hence the only possibility is $A < 0$ and $B \geq 0$, i.e. $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l}$ and $\Delta C_{t_l} \leq \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$.

Now consider the dual of (Qh)**.**

Step 3**:** We note $b^T y + c^T z < 0 \Leftrightarrow y_2 > y_1$, and the dual of (Qh) does not have a solution means the system of inequalities for all $y \geq 0, z \geq 0$ at least one equation in the system

$$y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\}.z_1 - z_2 = 0$$
$$y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\}.z_1 - z_3 = 0$$

does not hold. By an argument analogous to Step 1 either $\Delta C_{t_l} \geq \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$ or $\Delta C_{t_l} \geq \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l)$.

Step 4**:** Also $b^T y + c^T z = 0 \Leftrightarrow y_2 = y_1$, and the dual of (Qh) does not have a solution means the system of inequalities for all $y \geq 0, z \geq 0, z \neq 0$, has no solution. Which in turn implies following equation has no solution.

$$\begin{aligned} & y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) - \Delta C_{t_l}\}.z_1 - z_2 = 0 \\ & y_1 - y_2 + \{\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h) - \Delta C_{t_l}\}.z_1 - z_3 = 0 \\ \Leftrightarrow \quad & -Az_1 - z_2 = 0 \text{ and } -Bz_1 - z_3 = 0 \\ \Leftrightarrow \quad & Az_1 + z_2 = 0 \text{ and } Bz_1 + z_3 = 0 \end{aligned}$$

By Lemma 3, if $B < 0$ then $A < 0$. In this situation we can find a solution to the dual of (Qh). In particular $y = (y_1, y_2), y_1 = y_2$, $z = (1, -A, -B)$ is a solution. Hence $B \geq 0$. If $B = 0$, then by Lemma 3, $A < 0$. Then $y = (y_1, y_2), y_1 = y_2$, $z = (1, -A, 0)$ is a solution to the dual of (Qh). If $B > 0$ and $A > 0$ then contradicts Step 1. Hence, $B > 0$ and $A \leq 0$.

Now from Step 2 and Step 4 its follows $A < 0, B > 0$. Hence we have established that if (i) and (ii) hold then $\Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_l) < \Delta C_{t_l} < \Delta^{e_h e_h}_{e_l e_h} Sh_1(t_l t_h)$.

End of the proof of Proposition 1.

Proof of Corollary 1: From Lemma 1 it follows that $(s_{e_he_h}, s_{e_he_h})$ is an SNE of Γ_p if and only if

$$
\begin{aligned}
& p(t_l)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) + p(t_h)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) \geq \Delta C_{t_l} \\
\Leftrightarrow \quad & (1-p(t_h))\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) + p(t_h)\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) \geq \Delta C_{t_l} \\
\Leftrightarrow \quad & p(t_h)\{\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)\} \geq \{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)\} \\
\Leftrightarrow \quad & p(t_h) \geq \frac{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)}
\end{aligned}
$$

Using Proposition 1, $\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l) < \Delta C_{t_l} < \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) \implies 0 < \frac{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)} < 1$ and hence $(s_{e_he_h}, s_{e_he_h})$ is an SNE of Γ_p if and only if $p(t_h) \in [\frac{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h}Sh_1(t_lt_l)}, 1)$.

End of the proof of Corollary 1

Tables related to Example 1

M	$(e_l, 0)$	$(e_h, 0)$	(e_l, e_l)	(e_l, e_h)	(e_h, e_l)	(e_h, e_h)
$(t_l, 0)$	7	12.9				
$(t_h, 0)$	10	16				
(t_l, t_l)			8	11	11	16
(t_l, t_h)			9	13.1	13	20
(t_h, t_l)			9	13	13.1	20
(t_h, t_h)			10	16	16	25

Table 1: Table for TAM

C	$(e_l, 0)$	$(e_h, 0)$
$(t_l, 0)$	2	8
$(t_h, 0)$	1	6.3

Table 2: Table for Cost Function

Equilibrium	Range of p_h	Total Expected Revenue
$(s_{e_le_l}, s_{e_le_l})$	$(0, 0.263158]$	$EW(s_{e_le_l}, p) = 0.000000p_h^2 + 2.000000p_h + 2.000000$
$(s_{e_le_h}, s_{e_le_h})$	$[0.102041, 0.794872]$	$EW(s_{e_le_h}, p) = 3.400000p_h^2 + 0.800000p_h + 2.000000$
$(s_{e_he_h}, s_{e_he_h})$	$[0.578948, 1)$	$EW(s_{e_he_h}, p) = 0.500000p_h^2 + 5.700000p_h + 0.000000$

Table 3: Table representing equilibrium range and expected revenue function.

Proof of corollary (4)

From Corollary (1), Corollary (3) and Corollary (2) we have, $[\dfrac{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l)}, 1) \subseteq (0,1), (0, \dfrac{\Delta C_{t_h} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}{\Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}] \subseteq (0,1)$, $[\dfrac{C_{t_h} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_ht_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}, \dfrac{C_{t_l} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l)}] \subseteq (0,1)$. Given M satisfies concavity within each type profile, the following holds.

$$
\begin{aligned}
& \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l) \leq \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l) \\
\Rightarrow \quad & C_{t_l} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l) \leq \Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l) \\
\Rightarrow \quad & \frac{C_{t_l} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_lt_l)} \leq \frac{\Delta C_{t_l} - \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_h) - \Delta^{e_he_h}_{e_le_h} Sh_1(t_lt_l)}
\end{aligned}
$$

Similarly it can be shown that, $\dfrac{\Delta C_{t_h} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}{\Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)} \leq \dfrac{C_{t_h} - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}{\Delta^{e_he_h}_{e_le_h} Sh_1(t_ht_h) - \Delta^{e_he_l}_{e_le_l} Sh_1(t_ht_l)}$. We conclude that, the three intervals mentioned above are pairwise disjoint consequently the strategies $(s_{e_le_l}, s_{e_le_l}), (s_{e_le_h}, s_{e_le_h}), (s_{e_he_h}, s_{e_he_h})$ are rationalizable.