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Semistable reduction of covers of degree p

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Introduction

Let K be a field that is complete with respect to a non-trivial non-archimedean valuation v_K. We denote the valuation ring of K by $\mathcal{O}_K$, the residue field by κ, and we assume that κ has positive characteristic $p := \mathrm{char}(\kappa)$. The goal of this thesis is to develop a method for computing semistable models of smooth projective algebraic curves Y over K equipped with a degree-p morphism $\varphi\colon Y \to \mathbb{P}^1_K$.

Recall that an $\mathcal{O}_K$-model of a K-curve Y is a flat and proper $\mathcal{O}_K$-scheme $\mathcal{Y}$ equipped with an isomorphism $\mathcal{Y} \otimes_{\mathcal{O}_K} K \cong Y$. A model $\mathcal{Y}$ is called *semistable* if its special fiber $\mathcal{Y}_s := \mathcal{Y} \otimes_{\mathcal{O}_K} \kappa$ is reduced and only has ordinary double points as singularities. The Semistable Reduction Theorem of Deligne and Mumford ([DM69]) asserts that a semistable model $\mathcal{Y}$ always exists, possibly after replacing the curve Y with the base change Y_L to a finite separable field extension L/K. The proof of Deligne-Mumford suggests the following strategy for finding L and $\mathcal{Y}$: First, find a field extension L over which the Jacobian $J(Y)$ has semistable reduction by making sure that certain torsion points on $J(Y)$ are L-rational. Then find a minimal *regular* model of Y_L by resolving singularities (which is automatically semistable).

However, following this strategy currently seems unfeasible for curves of genus $g > 2$. We will use a different strategy, based on the covers $\varphi\colon Y \to \mathbb{P}^1_K$ that the curves Y we consider are always assumed to be equipped with. For the case that K is a finite extension of $\mathbb{Q}_3$ and Y is a plane quartic curve, our strategy is fully implemented, and many examples throughout are drawn from this class of curves.

Previous work. Much work has been done on computing semistable reductions of curves using covers. This approach may be summarized as follows. First one chooses a finite cover $Y \to X$; often the curve X is the projective line $\mathbb{P}^1_K$. Then one determines a suitable semistable model $\mathcal{X}$ of X such that the normalization $\mathcal{Y}$ of $\mathcal{X}$ in the function field F_Y of Y is a semistable model of Y (again, possibly after replacing K with a finite separable extension L/K and X and Y with their base changes to L). While the semistable models of $\mathbb{P}^1_K$ are well understood, choosing $\mathcal{X}$ so that $\mathcal{Y}$ is also semistable is a difficult problem.

The choice of $\mathcal{X}$ is comparatively simple if one can find $\varphi\colon Y \to \mathbb{P}^1_K$ such that the characteristic p is greater than $\deg(\varphi)$. In this case, a model $\mathcal{X}$ of $\mathbb{P}^1_K$ *separating the branch locus of* φ will always work. For example, this approach is leveraged in [BW17] for studying superelliptic curves. It also underlies approaches to semistable reduction phrased in terms of *cluster pictures*, as for example in [DDMM]. We briefly review the strategy of separating branch points in Section 1.8. Attention to the more difficult case $p \leqslant \deg(\varphi)$ has mostly been limited to degree-p Galois covers of $\mathbb{P}^1_K$, that is, to the study of *superelliptic curves*. A crucial part of the present work is that it applies to *all* degree-p covers of $\mathbb{P}^1_K$, be they Galois or not.

Much of the work on p-cyclic covers $Y \to \mathbb{P}^1_K$ goes back to Coleman's sketch of an algorithm for computing the semistable of Y presented in [Col87, Section 6]. It is written in rigid-analytic language. A complete proof of the Semistable Reduction Theorem in a similarly algorithmic and rigid-analytic spirit was eventually given in [AW12]. Coleman also introduces the notion of *p-approximations*, which in one way or another play a role in all the works mentioned in the following paragraph.

The papers of Lehr and Matignon [Leh01], [Mat03], [LM06] describe how to find a field extension L/K and a semistable model $\mathcal{Y}$ of Y in the case of p-cyclic $\varphi\colon Y \to \mathbb{P}^1_K$ conditional on the assumption that the branch locus of φ be *equidistant*. This assumption was no longer necessary in the local approach of [Arz12] and [AW12]. A modified form of this approach is used in the Sage package MCLF ([MCLF]), in which the computation of semistable reductions of superelliptic curves of degree p is implemented. Background and information on this can be found in [BW15, Section 4]. Similar is the recent work [FY23], which is mainly concerned with the semistable reduction of hyperelliptic curves at $p = 2$.

As mentioned above, we particularly consider the case of reduction of plane quartic curves at $p = 3$. These admit a degree-3 morphism $Y \to \mathbb{P}^1_K$, but this morphism is not usually Galois — if it is, Y is a *Picard curve*, which case was studied in [BBW17]. Mention should also be made of two recent works concerning the semistable reduction of plane quartics: In [BDLL], the stable reduction of plane quartics is described in terms of their *Dixmier-Ohno invariants*, but conditionally on the assumption $\mathrm{char}(\kappa) > 7$. In [BDDLL], the stable reduction of plane quartics is characterized in terms of combinatorial objects called *Cayley Octads*. This approach is implemented in Magma, but as of now conjectural.

The analytic viewpoint. When investigating the semistable reduction of an algebraic K-curve X, it is very convenient to employ *analytic geometry* over the non-archimedean field K. Above, mention was made of rigid-analytic geometry; we use analytic geometry in the sense of Berkovich ([Ber90]). This allows local work with objects like disks and anuli that is simply impossible in the algebraic setting.

Each algebraic curve X possesses an analytification X^{an}. While this is a complicated topological object (an example is drawn on the cover of this thesis[1]), it admits retracts to much simpler spaces, finite subgraphs of X^{an}

[1] Not included in the electronic version. You can take a look at my GitHub page.

called *skeletons*. In fact, the skeletons of X are the same as the intersection graphs of special fibers of semistable models of X. Thus the original problem of finding a semistable model of a given curve may be rephrased as the problem of finding a skeleton.

Since semistable models of $\mathbb{P}^1_K$ are well-understood, so is the structure of the analytic curve $(\mathbb{P}^1_K)^{\mathrm{an}}$. Its skeletons are simply finite trees. Given a finite morphism $\varphi\colon Y \to \mathbb{P}^1_K$, we stated above that our strategy consisted of finding a suitable model $\mathcal{X}$ of $\mathbb{P}^1_K$ such that the normalization of $\mathcal{X}$ in the function field F_Y is a semistable model of Y. In Berkovich's language, this translates to finding a suitable skeleton $\Gamma \subset (\mathbb{P}^1_K)^{\mathrm{an}}$ such that $\varphi^{-1}(\Gamma)$ is a skeleton of Y.

The Sage package MCLF is well-adapted to this strategy. It contains an implementation of trees $\Gamma \subset (\mathbb{P}^1_K)^{\mathrm{an}}$, and can check if the inverse image $\varphi^{-1}(\Gamma)$ of a tree is a skeleton of Y. The implementation of our algorithms is based on this functionality.

Outline and results. Chapter 1 begins by recalling (semistable) models of curves. Particularly important is the characterization of models of a curve X in terms of so-called *Type II* valuations on the function field F_X. This provides the connection to analytic geometry, which we develop to the point that we need. Special attention is given the projective line $(\mathbb{P}^1_K)^{\mathrm{an}}$. We also introduce the notion of *topological ramification*, following [Ber93] and [CTT]. The chapter ends with a discussion of reduction of analytic curves. As mentioned above, Section 1.8 contains an account of the strategy of "separating branch points"; there we also give an example illustrating that this strategy does not usually work for covers of degree p.

Chapter 2 introduces *admissible functions*. The most important class of these is *valuative functions*, which we discuss in Section 2.2. As a set, the analytification X^{an} of a curve X is obtained by adding to X a point ξ for each valuation v_ξ on the function field F_X extending the valuation v_K of K. The valuative function associated to an element $f \in F_X^\times$ is then the function that sends a point $\xi \in X^{\mathrm{an}}$ to the value $v_\xi(f)$.

Next, let $\varphi\colon Y \to \mathbb{P}^1_K$ be a morphism of degree p. We discuss in detail a function δ on Y^{an}, which is essentially the same as the *different function* studied in [CTT]. In Section 2.5, we then rephrase the main results of [CTT] in terms of an admissible function λ on X^{an}. The main result is Theorem 2.34, which states that if Γ is a skeleton of $\mathbb{P}^1_K$ separating the branch points of φ and on whose complement λ is locally constant, then $\varphi^{-1}(\Gamma)$ is a skeleton of Y.

Chapter 3 is concerned with finding explicit formulas for the value of the function λ at closed points $x_0 \in \mathbb{P}^1_K$. The main results are Theorems 3.30 and 3.33. They describe $\lambda(x_0)$ in terms of certain valuative functions on Y^{an} evaluated at a closed point $p_0 \in Y$ above x_0. These functions are derived from the coefficients of a "good" equation for the curve Y. In Section 3.5 we explain how one can always find such a "good" equation. Finally, in Section 3.6 we introduce the *tame locus* Y^{tame} associated to a curve Y equipped with a degree-

p morphism $Y \to \mathbb{P}^1_K$. Generalizing the *étale locus* of [BW15, Section 4.2], it is an affinoid subdomain of Y^{an} containing much of the same information as the function λ.

Chapter 4 deals with the reduction of plane quartic curves at $p = 3$. We begin by fixing a normal form for quartics in which a degree-3 morphism to $\mathbb{P}^1_K$ is given by a simple projection. Then we apply the general theory of Chapter 3 to describe the tame locus associated to a plane quartic. We descend part of the tame locus to an affinoid subdomain of $(\mathbb{P}^1_K)^{\mathrm{an}}$ using the "norm trick" of Section 4.5 — a crucial step, since the structure of $(\mathbb{P}^1_K)^{\mathrm{an}}$, including the description of affinoid subdomains and admissible functions, is much simpler than the structure of Y^{an}. In Section 4.6, we give a method to directly compute the different function δ above any given interval in $(\mathbb{P}^1_K)^{\mathrm{an}}$. Together with the norm trick, this allows us to concretely describe the image of the tame locus in $(\mathbb{P}^1_K)^{\mathrm{an}}$.

In Chapter 5, we work towards implementing our algorithm for computing the tame locus. An important point is that throughout Chapters 2, 3, and 4 we mostly worked over an algebraically closed field, ignoring the problem of finding a suitable field extension over which a semistable reduction exists. We describe how the computation of the tame locus may be carried out over a discretely valued subfield K_0 of the algebraically closed field K. This involves describing the structure of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ using so-called *discoids*, which generalize closed disks. In Section 5.5, we apply the resulting Algorithm 5.9 to compute the semistable reduction in many concrete examples of curves over $\mathbb{Q}_3$. In Section 5.6, we also consider the example of a plane quartic curve defined over the ramified extension $\mathbb{Q}_3(\zeta_3)$ of $\mathbb{Q}_3$ which arose in the study of the rational points on the modular curve $X^+_{\mathrm{ns}}(27)$ ([RSZ22]).

An appendix provides a short description of our implementation for computing the semistable reduction of plane quartics in Sage. It also contains various short code fragments that were used for computations in examples throughout the thesis.

Notations and conventions

For reference, we collect here various notations and conventions that are used throughout.

Valuations. All valuations we consider have rank 1. Thus a valuation on a field F is a map $v\colon F \to \mathbb{R} \cup \{\infty\}$ satisfying

$$v(x + y) \geqslant \min\{v(x), v(y)\}, \quad v(xy) = v(x) + v(y), \quad v(x) = \infty \iff x = 0$$

for all $x, y \in F$. The image $v(F^\times) \subseteq \mathbb{R}$ is called the *value group* of v and denoted Γ_v.

We will sometimes use the absolute value $|\cdot|_v$ associated to a valuation v, defined by $|x|_v = e^{-v(x)}$ for $x \in K$. Equivalently, we have $v(x) = -\log |x|_v$ for all $x \in F$.

The non-archimedean ground field. Throughout this entire thesis, K will denote a field that is complete with respect to a non-trivial non-archimedean valuation v_K. We denote the valuation ring of K by $\mathcal{O}_K$, the maximal ideal of $\mathcal{O}_K$ by $\mathfrak{m}_K$, and the residue field of K by κ. We assume that κ is a perfect field of positive characteristic, which we denote by $\mathrm{char}(\kappa) = p$. The field K may have characteristic 0 or characteristic p.

We always assume that we are in one of the following two cases:

(a) The field K is discretely valued

(b) The field K is the completion of an algebraic closure of a discretely valued field K_0

We will always indicate which of these holds. As a rule, K is usually assumed to be algebraically closed throughout Chapters 2, 3, and 4.

Our most important example of a valued field satisfying (a) is given by the p-adic numbers $\mathbb{Q}_p$, and by finite extensions of $\mathbb{Q}_p$. If K is a finite extension of $\mathbb{Q}_p$, we will always normalize v_K so that $v_K(p) = 1$. The most important example of a valued field satisfying (b) is the completion of an algebraic closure of $\mathbb{Q}_p$, denoted $\mathbb{C}_p$.

We will denote the value group of v_K by $\Gamma_K := \Gamma_{v_K}$ and the absolute value associated to v_K by $|\cdot|_K := |\cdot|_{v_K}$.

For convenience, we fix a multiplicative section

$$\Gamma_K \to K^\times, \qquad s \mapsto \pi^s,$$

of v_K. Thus $v_K(\pi^s) = s$ and $\pi^s\pi^t = \pi^{s+t}$ for $s, t \in \Gamma_K$.

Residue fields. As mentioned above, the residue field of K is denoted κ. We denote the residue field of any valuation v extending v_K by $\kappa(v)$, reflecting the fact that this residue field is in a natural way a field extension of κ.

The residue field of a point P on a K-curve X, that is the residue field of the local ring $\mathcal{O}_{X,P}$, is denoted by $k(P)$.

The completed residue field in the sense of Berkovich (see Definition 1.13) at a point ξ is denoted $\mathcal{H}(\xi)$. Its residue field is denoted $\kappa(\xi)$. Thus if ξ is of Type II, III, or IV, then we have $\kappa(\xi) = \kappa(v_\xi)$, where v_ξ denotes the valuation associated to ξ.

Algebraic curves. By a *curve* over a field F we mean a scheme of dimension 1 of finite type over F.

If X is a curve over the non-archimedean ground field K, then we always assume that X is projective, smooth, and geometrically connected. We denote the function field of such a curve by F_X and its genus by g_X.

The symbols $\pm\infty$. We will make use of the extended real line $\mathbb{R} \cup \{\pm\infty\}$, endowed with the usual topology. When it comes to arithmetic involving $\pm\infty$, we will only use the obvious rules $\infty + a = \infty$ for $a \in \mathbb{R} \cup \{\infty\}$, $-\infty + a = -\infty$ for $a \in \mathbb{R} \cup \{-\infty\}$, $a \cdot \infty = \infty$ and $a \cdot (-\infty) = -\infty$ for $a \in \mathbb{R}_{>0}$, as well as $a \cdot \infty = -\infty$ and $a \cdot (-\infty) = \infty$ for $a \in \mathbb{R}_{<0}$. We will never add or multiply ∞ and $-\infty$.

Fonts. We use normal italic capital letters to denote objects over the non-archimedean ground field K. In particular, X and Y always denote curves over K.

Objects over the residue field κ are often written with a bar; thus $\overline{X}$ is the common notation for a κ-curve associated to the K-curve X.

Finally, analytic curves in the sense of Berkovich are denoted by serifless letters such as $\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{U}, \mathsf{V}, \mathsf{X}$.

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Chapter 1

Models, valuations, and analytification

In this chapter, we collect various results that we will need later. We begin by recalling the theory of *models* of algebraic curves. In Section 1.2, we will see how to describe models using *Type II valuations*. This leads naturally to the notion of *analytification* in the sense of Berkovich ([Ber90]). We discuss the projective line $(\mathbb{P}^1_K)^{\mathrm{an}}$ in detail. It is both a useful illustration of the various notions we introduce and an indispensable tool for our cover-based approach to semistable reduction.

We briefly introduce *topological ramification* in Section 1.5. Then we turn to reduction in the context of analytic curves. Semistable models of a curve X correspond to *skeletons* of the analytification X^{an}, and it is this skeletal viewpoint that we will take in the next two chapters. In Section 1.8, we review *admissible reduction*. This approach is not sufficient to compute the semistable reduction of covers of degree p, but it is still a necessary ingredient and a good showcase of the concepts introduced in this chapter.

1.1 Models of curves

As mentioned in the section on notations and conventions, we work over a ground field K that is complete with respect to a non-archimedean valuation v_K. It is always assumed to be either discretely valued or to be the completion of an algebraic closure of a discretely valued field. In this section, we allow both possibilities.

Let us fix a smooth and geometrically irreducible projective curve X over K.

Definition 1.1. An $\mathcal{O}_K$*-model* (or simply a *model*) of X is a normal, flat, and proper $\mathcal{O}_K$-scheme $\mathcal{X}$ together with an isomorphism

$$X \cong \mathcal{X}_K = \mathcal{X} \otimes_{\mathcal{O}_K} K.$$

Chapter 10 of [Liu02] provides a comprehensive introduction to models of algebraic curves. A fairly self-contained account of most facts we need may be found in Chapter 3 of [Rüt14].

Remark 1.2. (a) The assumption that models be normal is non-standard, but essential for our approach. The point is that we want to describe models using certain sets of valuations (see Proposition 1.6). Normality guarantees that the local rings of a model at certain codimension-1 points are valuation rings.

(b) By [Liu02, Proposition 4.3.8], the flatness in the definition of an $\mathcal{O}_K$-model $\mathcal{X}$ implies that $\mathcal{X}$ is integral. Conversely, a surjective morphism from an integral scheme to $\operatorname{Spec} \mathcal{O}_K$ is automatically flat, because every torsion-free module over a valuation ring is flat ([Bou98, Chapter 1, §2, n° 4, Proposition 3]).

A *modification* of a model $\mathcal{X}$ of X is a second model $\mathcal{X}'$ of X together with a morphism $\varphi\colon \mathcal{X}' \to \mathcal{X}$ such that the diagram

$$\begin{tikzcd} \mathcal{X}'_K \arrow[rr, "\varphi_K"] \arrow[dr, "\cong"'] & & \mathcal{X}_K \arrow[dl, "\cong"] \\ & X & \end{tikzcd}$$

commutes.

Remark 1.3. We may regard the set of isomorphism classes of models of X as a partially ordered set, where $[\mathcal{X}_1] \geqslant [\mathcal{X}_2]$ means that there exists a modification $\mathcal{X}_1 \to \mathcal{X}_2$.

The base change of a model $\mathcal{X}$ of X to the residue field κ of K,

$$\mathcal{X}_s := \mathcal{X}_\kappa = \mathcal{X} \otimes_{\mathcal{O}_K} \kappa,$$

is called the *special fiber* of $\mathcal{X}$. The special fiber need not be smooth over κ, nor even irreducible. But it is a projective κ-curve, that is, a one-dimensional projective κ-scheme (cf. [Liu02, Lemma 8.3.3]).

Definition 1.4. The model $\mathcal{X}$ is called *semistable* if the special fiber $\mathcal{X}_s$ is a *semistable curve*. The latter means that $\mathcal{X}_s$ is reduced and has only ordinary double points as singularities.

The curve X is said to have *semistable reduction* if there exists a semistable $\mathcal{O}_K$-model of X. It was proved by Deligne and Mumford [DM69] that this is always the case after taking a finite separable field extension L/K — of course, if K is algebraically closed, this implies that every K-curve already has semistable reduction.

Semistable Reduction Theorem. *There exists a finite separable field extension L/K such that X_L has semistable reduction.*

Determining for certain classes of curves X a field extension L/K such that X_L has semistable reduction and a semistable model $\mathcal{X}$ of X_L is the ultimate goal of this thesis.

1.2 Type II valuations

We continue the notation from the last section. Thus the ground field K may still be algebraically closed or discretely valued, and X denotes a smooth and geometrically irreducible projective K-curve.

Definition 1.5. A valuation

$$v\colon F_X \to \mathbb{R} \cup \{\infty\}$$

is called a *Type II valuation* if v extends the valuation v_K on K and the transcendence degree of the residue field extension $\kappa(v)/\kappa$ is 1.

Suppose that $\mathcal{X}$ is a model of X and that Z is an irreducible component of the special fiber $\mathcal{X}_s$. Denote the generic point of Z by ξ_Z, so that $\overline{\{\xi_Z\}} = Z$. Then since $\mathcal{X}$ is normal, the local ring $\mathcal{O}_{\mathcal{X},\xi_Z}$ is a valuation ring. In the case that K is discretely valued and hence noetherian, this is clear, because every noetherian normal local domain of dimension 1 is a discrete valuation ring. For the case that K is algebraically closed, see [Gre96, Lemma 3.1].

We denote the valuation associated to the valuation ring $\mathcal{O}_{\mathcal{X},\xi_Z}$ by v_Z; it is a Type II valuation on F_X (for details see [Rüt14, Proposition 3.4] in the discretely valued case and again [Gre96, Lemma 3.1] in general).

Proposition 1.6. *The map*

$$\mathcal{X} \mapsto \{v_Z \mid Z \subseteq \mathcal{X}_s \text{ irreducible component}\}$$

induces an isomorphism of partially ordered sets

$$\begin{aligned} &\textit{isomorphism classes of models of } X \\ \longrightarrow\quad &\textit{finite non-empty sets of Type II valuations on } F_X. \end{aligned}$$

Proof. For the case that K is discretely valued, this is the main result of [Rüt14, Chapter 3], see [Rüt14, Corollary 3.18].

In general, this follows from [Gre96, Theorem 2]. Note that the "Local Skolem Property" on which this theorem depends is satisfied for the complete field K. The "proper, normal, integral $\mathcal{O}_K$-curves" considered in loc. cit. with function field F_X are the same as our $\mathcal{O}_K$-models of X, see Remark 1.2(b). ∎

The map described in Proposition 1.6 is an isomorphism of partially ordered sets, meaning that it is a bijection preserving the partial orders on domain and codomain. The partial order on isomorphism classes of models of X is explained in Remark 1.3. Sets of Type II valuations are simply ordered by inclusion.

Proposition 1.7. *Let $Y \to X$ be a cover of smooth irreducible projective curves over K. Let $\mathcal{X}$ be an $\mathcal{O}_K$-model of X and let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in the function field F_Y. Then $\mathcal{Y}$ is an $\mathcal{O}_K$-model of Y, and the valuations corresponding to $\mathcal{Y}$ via the bijection of Proposition 1.6 are the extensions of the valuations corresponding to $\mathcal{X}$.*

Proof. This is explained in [Rüt14, Section 5.1.2] for the case that K is discretely valued. The exact same proof is valid for algebraically closed K as well; all that is needed is replacing "discrete valuation ring" with "valuation ring". ■

Propositions 1.6 and 1.7 are our chief tools for describing models of curves. Given a finite cover of curves $\varphi\colon Y \to X$, we may describe a model of Y simply using a non-empty set of Type II valuations on X. The corresponding model of Y is obtained by normalization, as explained in Proposition 1.7.

The next lemma is an example of how valuation theory may be leveraged to gain geometric insight into the special fibers of models.

Lemma 1.8. *Let $\varphi\colon Y \to X = \mathbb{P}^1_K$ be a degree-p morphism of K-curves. Suppose that w is a Type II valuation on F_Y with restriction $v := w|_{F_X}$. If the residue field extension $\kappa(w)/\kappa(v)$ is inseparable, then the function field $\kappa(w)$ has genus 0.*

Proof. Since the special fiber of any model of $\mathbb{P}^1_K$ has arithmetic genus 0, the function field $\kappa(v)$ must have genus 0.

Consider the subfield $\kappa(w)^p$ of $\kappa(w)$ consisting of p-th powers. The extension $\kappa(w)/\kappa(w)^p$ is inseparable of degree p by [Sil09, Proposition II.2.11] — here we use the assumption that κ is perfect. Thus $\kappa(w)^p = \kappa(v)$. Since the Frobenius maps $\kappa(w)$ isomorphically to $\kappa(w)^p$, both these function fields must have the same genus 0. ■

For the rest of this section, we assume that the base field K is discretely valued.

For the following definition, suppose that L/K is the completion of an algebraic extension of K. Thus there is a unique extension v_L of v_K to L. We denote the ring of integers of L by $\mathcal{O}_L$. Of course, if L/K is finite, then L already is complete with respect to the unique extension of v_K to L.

Definition 1.9. Let X be a smooth and geometrically irreducible curve over K and let $\mathcal{X}$ be an $\mathcal{O}_K$-model of X. The *normalized base change* $\mathcal{X}_L$ of $\mathcal{X}$ to L is the normalization of $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$.

The normalized base change $\mathcal{X}_L$ is an $\mathcal{O}_L$-model of $X_L = X \otimes_K L$; in fact, it is the normalization of $\mathcal{X}$ in the function field of X_L.

Normalized base change is quite important to us. For certain curves X over K, we will see in subsequent chapters how to compute a *potentially semistable* $\mathcal{O}_K$-model $\mathcal{X}$ of X, meaning that the normalized base change $\mathcal{X}_L$ to a finite extension L/K is semistable. The problem of computing a semistable reduction of X is then reduced to determining this field extension L/K. We will return to this topic in Chapter 5.

Definition 1.10. An $\mathcal{O}_K$-model $\mathcal{X}$ of X is called *permanent* if $\mathcal{X}_s$ is reduced.

Remark 1.11. The motivation for the terminology "permanent" is the following. Let $\mathcal{X}$ be a permanent $\mathcal{O}_K$-model and consider a field extension L/K that is the completion of an algebraic extension of K. Then the special fiber of the normalized base change $\mathcal{X}_L$ is simply given by $\mathcal{X}_s \otimes_\kappa \lambda$, where λ denotes the residue field of L (cf. [AW12, Section 2.2]). To see this, note that κ being perfect, [Liu02, Corollary 3.2.14(a)] implies that the special fiber of $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$,

$$(\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L) \otimes_{\mathcal{O}_L} \lambda \cong \mathcal{X} \otimes_{\mathcal{O}_K} \lambda \cong (\mathcal{X} \otimes_{\mathcal{O}_K} \kappa) \otimes_\kappa \lambda = \mathcal{X}_s \otimes_\kappa \lambda,$$

is still reduced. From [Liu02, Lemma 4.1.18] it follows that $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is normal, so that $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is already equal to its normalized base change $\mathcal{X}_L$.

We end this section by explaining some further merits of the valuation-theoretic approach to models. As mentioned above, the key point is that much information on the special fiber $\mathcal{X}_s$ of a model $\mathcal{X}$ is encoded in the associated set of Type II valuations. We never describe models by their global geometry; there is no need, say, to worry about how a model might be embedded in some projective space $\mathbb{P}^N_{\mathcal{O}_K}$.

To elaborate further, suppose that $\mathcal{X}$ is an $\mathcal{O}_K$-model of X. Then if $Z \subseteq \mathcal{X}_s$ is an irreducible component with generic point ξ_Z, the residue field of the valuation v_Z is the same as the function field of the component $Z = \overline{\{\xi_Z\}}$. Thus knowledge of the valuations corresponding to $\mathcal{X}$ means knowing the components of $\mathcal{X}_s$ "birationally". Using the fact that the (arithmetic) genera of X and $\mathcal{X}_s$ coincide, it is easy to derive a criterion for $\mathcal{X}$ being semistable. This is explained in detail in [Rüt14, Section 5.3].

A simple example of this is the following: If the sum of the geometric genera g_Z over all components $Z \subseteq \mathcal{X}_s$ equals g_X, then $\mathcal{X}$ must be semistable. The curve X is said to have *abelian reduction* in this case; the arithmetic genus of $\mathcal{X}_s$ comes only from the geometric genera of the components $Z \subseteq \mathcal{X}_s$, not from any loops in the intersection graph of $\mathcal{X}_s$ (cf. Definition 1.46).

We can also detect non-reduced structure of $\mathcal{X}_s$ using valuations. Given a component $Z \subseteq \mathcal{X}_s$ with generic point ξ_Z, the *multiplicity* of Z is defined to be the length

$$m(Z) := \operatorname{len}(\mathcal{O}_{\mathcal{X}_s,\xi_Z}) = \operatorname{len}(\mathcal{O}_{\mathcal{X},\xi_Z}/\mathfrak{m}_K\mathcal{O}_{\mathcal{X},\xi_Z}).$$

(Recall that $\mathfrak{m}_K$ denotes the maximal ideal of the valuation ring $\mathcal{O}_K$.) The multiplicities of the components $Z \subseteq \mathcal{X}_s$ measure the degree to which $\mathcal{X}_s$ is non-reduced. Indeed, $\mathcal{O}_{\mathcal{X}_s,\xi_Z}$ is a local Artinian ring, and it is reduced if and only if its length equals 1. But if all the local rings $\mathcal{O}_{\mathcal{X}_s,\xi_Z}$ are reduced, then by normality of $\mathcal{X}$, the special fiber $\mathcal{X}_s$ must already be reduced ([AW12, Corollary 2.2(ii)]).

Lemma 1.12. *Let $\varphi\colon Y \to X = \mathbb{P}^1_K$ be a finite cover of K-curves, let $\mathcal{X}$ be a permanent $\mathcal{O}_K$-model of X and let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in F_Y. Let $Z \subseteq \mathcal{X}_s$ and $W \subseteq \mathcal{Y}_s$ be components with $\varphi(W) = Z$. Then the ramification index $e(v_W|v_Z)$ equals the multiplicity of the component W.*

Proof. Let η and ξ be the generic points of W and Z respectively. Note that we have $\mathfrak{m}_K\mathcal{O}_{\mathcal{X},\xi} = (\pi_Z^{m(Z)})$, where π_Z is a uniformizer for the discrete valuation ring $\mathcal{O}_{\mathcal{X},\xi}$. Thus $m(Z)$ equals the ramification index of the extension of valuations $v_Z|v_K$. Similarly, $m(W)$ equals the ramification index of $v_W|v_K$. Thus the result follows from the multiplicativity of ramification indices in the tower

$$\begin{tikzcd} v_W \arrow[d, dash] \\ v_Z \arrow[d, dash] \\ v_K. \end{tikzcd}$$

■

1.3 Analytic curves

In this section, we introduce analytic curves in the sense of Berkovich ([Ber90]). Berkovich defines a category of K-*analytic spaces*, which are certain locally ringed spaces. *Analytic curves* are treated in detail in Chapter 4 of [Ber90]. All the analytic spaces we consider will be curves. We will denote them by serifless letters such as $\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{U}, \mathsf{V}, \mathsf{X}$.

To every finite-type K-scheme X is attached an *analytification*, denoted X^{an}, see [Ber90, Section 3.4]. It is a K-analytic space representing the functor that sends an analytic space U to the set of morphisms of K-ringed spaces $\mathrm{Hom}(\mathsf{U}, X)$. Thus it has attached to it a universal morphism

$$\pi\colon X^{\mathrm{an}} \to X.$$

The association $X \mapsto X^{\mathrm{an}}$ is, of course, functorial: for every morphism $\varphi\colon X \to Y$ of finite-type K-schemes there is an associated morphism $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. We will usually omit the superscript and simply write φ for φ^{an}.

Suppose for example that $\iota\colon U \to X$ is the inclusion of an open subscheme. Then $\iota^{\mathrm{an}}\colon U^{\mathrm{an}} \to X^{\mathrm{an}}$ is also an open immersion ([Ber90, Proposition 3.4.6]). We will often use the universal morphism $\pi\colon X^{\mathrm{an}} \to X$ to consider functions on X as functions on X^{an}, by pulling back along the map of sheaves $\mathcal{O}_X \to \pi_*\mathcal{O}_{X^{\mathrm{an}}}$. Similarly, we will consider functions on open subschemes $U \subseteq X$ as functions on $U^{\mathrm{an}} \subseteq X^{\mathrm{an}}$.

Analytic spaces are made from local building blocks, so-called *affinoid spaces* ([Ber90, Section 2.2]). A subset of X^{an} is called an *affinoid subdomain* if it is the image of a morphism from an affinoid space to X^{an} satisfying a certain universal property (see [Ber90, Section 3.1]). In the next section, we will get to know some examples of these (*closed disks* and *anuli*), and later we will study affinoid subdomains of X^{an} determined by the values of certain algebraic functions on X^{an} (see Lemma 2.6). But for now, we take a different approach to defining X^{an}, based entirely on valuation theory.

As a set, X^{an} may be described as follows (compare [Ber90, Remark 3.4.2]). It consists of all pairs (P, v), where $P \in X$ is any point and where $v\colon k(P) \to \mathbb{R} \cup \{\infty\}$ is a valuation extending v_K. Usually, we will denote points in X^{an} by the Greek letters ξ and η (with various subscripts).

Definition 1.13. For a point $\xi = (P, v)$, the completion of $k(P)$ with respect to v is called the *completed residue field* at ξ and is denoted $\mathcal{H}(\xi)$.

We denote by $\kappa(\xi)$ the residue field of the valued field $\mathcal{H}(\xi)$.

The map $\pi\colon X^{\mathrm{an}} \to X$ is, on the level of sets, simply the "forgetful map" $(P, v) \mapsto P$. Since there exists a unique extension of v_K to every finite extension L/K, there is a unique point $\xi = (P, v)$ above every closed point $P \in X$.

In the case that X is an irreducible curve, the only non-closed point on X is the generic point. The points $\xi \in X^{\mathrm{an}}$ above the generic point correspond to the valuations on the function field F_X extending v_K. We denote the valuation belonging to such a point by v_ξ.

We cannot associate valuations on F_X to the points $\xi = (P, v)$ in X^{an} associated to closed points of X, but we can get close: precompose v with the natural map $\mathcal{O}_{X,P} \to k(P)$ to obtain a map $\tilde{v}_\xi\colon \mathcal{O}_{X,P} \to \mathbb{R} \cup \{\infty\}$. This map satisfies the usual rules for valuations

$$\tilde{v}_\xi(0) = \infty, \qquad \tilde{v}_\xi(1) = 0, \tag{1.1}$$

$$\tilde{v}_\xi(fg) = \tilde{v}_\xi(f) + \tilde{v}_\xi(g), \quad \tilde{v}_\xi(f+g) \geqslant \min(\tilde{v}_\xi(f), \tilde{v}_\xi(g)) \tag{1.2}$$

for all $f, g \in \mathcal{O}_{X,P}$, but it also sends every element of the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{X,P}$ to ∞.

Definition 1.14. A *pseudovaluation* on a ring R is a function $v\colon R \to \mathbb{R} \cup \{\infty\}$ satisfying the rules (1.1) and (1.2).

We extend the pseudovaluation $\tilde{v}_\xi$ to the function field F_X by defining

$$v_\xi\colon F_X \to \mathbb{R} \cup \{\pm\infty\}, \qquad f \mapsto \begin{cases} \tilde{v}_\xi(f) & \text{if } f \in \mathcal{O}_{X,P}, \\ -\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

Thus we have associated to each $\xi \in X^{\mathrm{an}}$ a function $v_\xi\colon F_X \to \mathbb{R} \cup \{\pm\infty\}$. Collectively, we call all functions v_ξ, where $\xi \in X^{\mathrm{an}}$, *pseudovaluations* on F_X, be they valuations or functions of the form (1.3).

Abhyankar's Inequality ([EP05, Theorem 3.4.3]) states that

$$\operatorname{trdeg}(\kappa(\xi)/\kappa) + \operatorname{rk}_{\mathbb{Q}}(\Gamma_{v_\xi}/\Gamma_K) \leqslant \operatorname{trdeg}(F_X/K) = 1.$$

In words: The transcendence degree of F_X/K is bounded by the sum of the transcendence degree of the extension of residue fields $\kappa(\xi)/\kappa$ and the rational rank of the quotient of value groups Γ_{v_ξ}/Γ_K. (The rational rank of an abelian group A is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes A$.)

The points of X^{an} are categorized into four *types* (the definitions below follow [Ber90, Section 1.4.4]; see also [Duc, Section 3.3.2] for the case of K not being algebraically closed):

- ξ is called *Type I* if $\mathcal{H}(\xi)$ may be embedded in the completion of an algebraic closure of K. In particular, the points of X^{an} associated to closed points of X are Type I.

- ξ is called *Type II* if v_ξ is a Type II valuation (Definition 1.5), that is, if the extension of residue fields $\kappa(\xi)/\kappa$ has transcendence degree 1. By Abhyankar's Inequality, in this case we have $\mathrm{rk}_{\mathbb{Q}}(\Gamma_{v_\xi}/\Gamma_K) = 0$. Note that [EP05, Theorem 3.4.3] also includes the statement that Γ_{v_ξ}/Γ_K is finitely generated if $\kappa(\xi)/\kappa$ has transcendence degree 1. Thus the index $[\Gamma_{v_\xi} : \Gamma_K]$ is finite; it the ramification index of the extension $\mathcal{H}(\xi)/K$.

- ξ is called *Type III* if $\mathrm{rk}_{\mathbb{Q}}(\Gamma_{v_\xi}/\Gamma_K) = 1$. In this case, we necessarily have $\mathrm{trdeg}(\kappa(\xi)/\kappa) = 0$.

- ξ is called *Type IV* if ξ is not Type I, but $\mathrm{rk}_{\mathbb{Q}}(\Gamma_{v_\xi}/\Gamma_K) = 0$ and $\mathrm{trdeg}(\kappa(\xi)/\kappa) = 0$.

Since for a Type II point $\xi \in X^{\mathrm{an}}$ the residue field $\kappa(\xi)$ has transcendence degree 1 over κ, that is, since $\kappa(\xi)$ is an algebraic function field over κ, the following definition makes sense.

Definition 1.15. (a) The smooth projective κ-curve with function field $\kappa(\xi)$ is called the *reduction curve* at ξ.

(b) The *genus* of a Type II point $\xi \in X^{\mathrm{an}}$, denoted $g(\xi)$, is the genus of the reduction curve at ξ.

By convention, we also define $g(\xi) = 0$ for all points $\xi \in X^{\mathrm{an}}$ not of Type II.

Definition 1.16. Let $\xi \in X^{\mathrm{an}}$ be a Type II point and let $f \in F_X^\times$ be a function with $v_\xi(f) \in \Gamma_K$. Then we define

$$[f]_\xi := \overline{f\pi^{-v_\xi(f)}} \in \kappa(\xi)^\times.$$

(Recall from the section on notations and conventions that for $s \in \Gamma_K$, we always denote by π^s an element of $K^\times$ with $v_K(\pi^s) = s$. The definition of $[f]_\xi$ depends on the choice of π^s, but a different choice of π^s will only change $[f]_\xi$ by multiplication with a non-zero constant in κ.)

Remark 1.17. Note that if K is algebraically closed, we have $v_\xi(f) \in \Gamma_K$ for all $f \in F_X^\times$. Indeed, the divisible group $\Gamma_K = \mathbb{Q}$ is a subgroup of Γ_{v_ξ}, which is itself a subgroup of $\mathbb{R}$. Since ξ is Type II, Γ_{v_ξ}/Γ_K is finite, so for every $\alpha \in \Gamma_{v_\xi}$, there exists some $n \geqslant 1$ with $n\alpha \in \mathbb{Q}$. But then $\alpha \in \mathbb{Q}$ as well, so $\Gamma_{v_\xi} = \Gamma_K$.

Lemma 1.18. *Let $f, g \in F_X^\times$ be non-zero functions with $v_\xi(f), v_\xi(g), v_\xi(f+g) \in \Gamma_K$. Then we have:*

(a) $[fg]_\xi = [f]_\xi \cdot [g]_\xi$

(b) $[f+g]_\xi = [f]_\xi + [g]_\xi$ *if and only if* $v_\xi(f) = v_\xi(g)$ *and* $[f]_\xi + [g]_\xi \neq 0$

Proof. We always assume that the section $s \mapsto \pi^s$ is multiplicative. Thus for (a), we can simply calculate

$$[fg]_\xi = \overline{fg\pi^{-v_\xi(fg)}} = \overline{f\pi^{-v_\xi(f)}g\pi^{-v_\xi(g)}} = \overline{f\pi^{-v_\xi(f)}} \cdot \overline{g\pi^{-v_\xi(g)}} = [f]_\xi[g]_\xi.$$

For (b), let us first remark that under the condition $v_\xi(f) = v_\xi(g)$ we have

$$[f]_\xi + [g]_\xi = 0 \qquad \text{if and only if} \qquad v_\xi(f+g) > v_\xi(f).$$

Indeed, we have

$$[f]_\xi + [g]_\xi = \overline{f\pi^{-v_\xi(f)}} + \overline{g\pi^{-v_\xi(g)}} = \overline{(f+g)\pi^{-v_\xi(f)}},$$

and this is 0 if and only if $v_\xi(f+g) > v_\xi(f)$.

Now the equivalence in (b) follows easily. Indeed, if $v_\xi(f) = v_\xi(g)$ and $[f]_\xi + [g]_\xi \neq 0$, then $v_\xi(f+g) = v_\xi(f) = v_\xi(g)$, and

$$[f+g]_\xi = \overline{(f+g)\pi^{-v_\xi(f+g)}} = \overline{f\pi^{-v_\xi(f+g)}} + \overline{g\pi^{-v_\xi(f+g)}} = [f]_\xi + [g]_\xi.$$

On the other hand, if $v_\xi(f) \neq v_\xi(g)$, say $v_\xi(f) > v_\xi(g)$, then $v_\xi(f+g) = v_\xi(g)$ and a similar calculation easily shows $[f+g]_\xi = [g]_\xi \neq [f]_\xi + [g]_\xi$. Likewise, if $[f]_\xi + [g]_\xi = 0$, then clearly $[f+g]_\xi \neq [f]_\xi + [g]_\xi$. ∎

So far, we have not discussed the topology of X^{an}. We now give a definition that is well-adapted to our valuation-theoretic definition of the analytification.

Suppose that $U \subseteq X$ is an affine open. For each function $f \in \mathcal{O}_X(U)$ and each point $\xi = (P, v)$ in U^{an} we may consider the value $v_\xi(f(P))$, where $f(P)$ denotes the image of f in $k(P)$. Then the topology on U^{an} is the coarsest one such that the evaluation map

$$U^{\mathrm{an}} \to \mathbb{R} \cup \{\infty\}, \qquad \xi \mapsto v_\xi(f(P)),$$

is continuous for every $f \in \mathcal{O}_X(U)$ (cf. [HR22, Section 2.1]).

Definition 1.19. A subset $I \subseteq X^{\mathrm{an}}$ is called an *interval* if it is homeomorphic to an interval $J \subseteq \mathbb{R} \cup \{\pm\infty\}$.

Let $\xi_1, \xi_2 \in X^{\mathrm{an}}$ be two points. If $I \subseteq X^{\mathrm{an}}$ is an interval containing ξ_1, ξ_2 homeomorphic to a compact interval $J \subset \mathbb{R} \cup \{\pm\infty\}$, with ξ_1, ξ_2 mapping to the boundary points of J, we will use the notation

$$I = [\xi_1, \xi_2]$$

and say that I is a *compact interval*. Similarly, we will write

$$[\xi_1, \xi_2) := [\xi_1, \xi_2]\backslash\{\xi_2\}, \quad (\xi_1, \xi_2] := [\xi_1, \xi_2]\backslash\{\xi_1\}, \quad (\xi_1, \xi_2) := [\xi_1, \xi_2]\backslash\{\xi_1, \xi_2\}.$$

Definition 1.20. A *branch* at a point $\xi \in X^{\mathrm{an}}$ is an equivalence class of intervals $[\xi, \xi']$ starting at ξ, where $[\xi, \xi_1]$ and $[\xi, \xi_2]$ are considered equivalent if there exists an interval $[\xi, \xi_3]$ contained in both $[\xi, \xi_1]$ and $[\xi, \xi_2]$.

The curve X^{an} is a *special quasipolyhedron* (see [Ber90, Definition 4.1.1] for what this means and [Ber90, Theorem 4.3.2] for the result). We will not discuss all technicalities tied up in this definition here, but note that this implies in particular that the topology of X^{an} has a basis consisting of open sets $U \subseteq X^{\mathrm{an}}$ with the following properties:

(a) The boundary of U is finite

(b) For any distinct $\xi_1, \xi_2 \in U$, there exists a unique compact interval $I \subseteq U$ with endpoints ξ_1 and ξ_2

If these two properties are even true for the open subset $U = X^{\mathrm{an}}$, then X^{an} is called a *simply connected special quasipolyhedron*. The most important example of a simply connected special quasipolyhedron is the analytification $(\mathbb{P}^1_K)^{\mathrm{an}}$ of the projective line $\mathbb{P}^1_K$, which we discuss in the next section.

1.4 The projective line

In this section, we assume that K is algebraically closed. However, we will revisit the projective line in the setting of discretely valued K in Section 5.2.

Our object of study is the analytification $(\mathbb{P}^1_K)^{\mathrm{an}}$ of the projective line $\mathbb{P}^1_K$. As explained in the previous section, the points of $(\mathbb{P}^1_K)^{\mathrm{an}}$ correspond to the pseudovaluations on the rational function field $K(x)$. By definition, these are either valuations on $K(x)$ extending v_K, or else are obtained from evaluation at closed points in $\mathbb{P}^1_K$. With the exception of the closed point $\infty \in \mathbb{P}^1_K$, all points of $(\mathbb{P}^1_K)^{\mathrm{an}}$ are induced by pseudovaluations on $K[x]$. Type I points different from ∞ may be identified with closed points in $\operatorname{Spec} K[x]$, or simply with the elements of K.

Definition 1.21. Let $a \in K$ be any element and let $r \in \mathbb{R}$ be a real number. The *closed disk* with center a and radius r is the subset

$$\mathsf{D}[a, r] := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x - a) \geqslant r\}.$$

The closed disk $\mathsf{D}[a,r]$ is an example of an affinoid subdomain of $(\mathbb{P}^1_K)^{\mathrm{an}}$. Affinoid spaces are defined to be the *spectra* of *affinoid K-algebras* ([Ber90, Section 2.1]), which are certain commutative Banach algebras. As a set, the spectrum $\mathcal{M}(\mathcal{A})$ of an affinoid K-algebra $\mathcal{A}$ consists of all pseudovaluations v on $\mathcal{A}$ such that the absolute value $|\cdot|_v$ associated to v is bounded by the given norm on $\mathcal{A}$ ([Ber90, Section 1.2]). The affinoid algebra giving rise to the closed disk $\mathsf{D}[a,r]$ is the *Tate algebra* of *restricted power series*

$$K\{\pi^{-r}t\} := \Big\{ \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!] : \lim_{i\to\infty} v_K(a_i) + ir = \infty \Big\}$$

given as a norm the absolute value associated to the *Gauss valuation* of radius r

$$v_r \colon K\{\pi^{-r}t\} \to \mathbb{R} \cup \{\infty\}, \qquad \sum_{i=0}^{\infty} a_i t^i \mapsto \min_{i\geqslant 0} \big\{ v_K(a_i) + ir \big\}.$$

(The Gauss valuations are so named because in proving that v_r, where $r \in \mathbb{Q}$, is really a valuation, one can use the same strategy as in the proof of Gauss's Lemma. See for example [Bos14, p. 13]. The notation $K\{\pi^{-r}t\}$ is so chosen because this Tate algebra is a rescaled version of the Tate algebra $K\{t\}$; one can think of this rescaling as replacing the parameter t with $\pi^{-r}t$, where π^r is an element of valuation r.)

The *spectrum* $\mathcal{M}(K\{\pi^{-r}t\})$ of $K\{\pi^{-r}t\}$ is by definition the set of all pseudovaluations v on $K\{\pi^{-r}t\}$ satisfying $v_r(f) \leqslant v(f)$ for all $f \in K\{\pi^{-r}t\}$. We can identify the affinoid space $\mathcal{M}(K\{\pi^{-r}t\})$ with the disk $\mathsf{D}[a,r]$ as follows. Write an arbitrary element $f \in K[x]$ as a polynomial in $t = x - a$, say $f = \sum_i a_i t^i$. Then we may regard f as an element of $K\{\pi^{-r}t\}$ and apply any $v \in \mathcal{M}(K\{\pi^{-r}t\})$ to it. In this way, v defines a pseudovaluation on $K[t]$, which extends uniquely to the rational function field $K(x)$. It is easy to see that the pseudovaluations obtained in this way are exactly the elements of $\mathsf{D}[a,r]$.

We denote the valuation on $K(x)$ corresponding to v_r by $v_{a,r}$. It is called the *Gauss valuation of radius r centered at a*. The point corresponding to $v_{a,r}$ is the unique boundary point of $\mathsf{D}[a,r]$. Given a closed disk D, we will also denote the valuation corresponding to the boundary point of D by v_{D} and call it the *Gauss valuation associated to* D.

Next, consider the *open disk* with center a and radius r, the subset

$$\mathsf{D}(a,r) := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x-a) > r\}.$$

It is the union of all closed disks of radius $r' > r$ centered at a. Correspondingly, we have

$$\mathcal{O}_{(\mathbb{P}^1_K)^{\mathrm{an}}}\big(\mathsf{D}(a,r)\big) = \varprojlim_{r'>r} K\{\pi^{-r'}\} \tag{1.4}$$

([Ber90, Section 2.3]). The *radius of convergence* of a power series $f = \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!]$ is defined to be

$$\rho(f) := -\liminf_{i\to\infty} \frac{v_K(a_i)}{i}. \tag{1.5}$$

Lemma 1.22. *The radius $\rho(f)$ is the smallest radius for which f is contained in $\mathcal{O}_{(\mathbb{P}^1_K)^{\mathrm{an}}}(\mathsf{D}(a,r))$.*

A similar statement can be found in [Gou20, Proposition 5.4.1], but not phrased in the language of analytic spaces in the sense of Berkovich, and using absolute values rather than valuations.

Proof of Lemma 1.22. That $-\rho(f)$ equals the limes inferior in (1.5) means that for every $\varepsilon > 0$ we have

$$\frac{v_K(a_i)}{i} \geqslant -\rho(f) - \varepsilon \tag{1.6}$$

for almost all $i \geqslant 0$ and that we have

$$\frac{v_K(a_i)}{i} \leqslant -\rho(f) + \varepsilon \tag{1.7}$$

for infinitely many $i \geqslant 0$. Now if $r > \rho(f)$, we choose $\varepsilon := \frac{r-\rho(f)}{2}$. Then by (1.6) we have

$$v_r(a_i t^i) \geqslant -i\rho(f) - i\frac{r-\rho(f)}{2} + ir = i\frac{r-\rho(f)}{2} = i\varepsilon$$

for almost all $i \geqslant 0$. Since $r > \rho(f)$ was arbitrary, f defines an element of (1.4). Now suppose that $r < \rho(f)$. Choose $\varepsilon := \rho(f) - r$. Then by (1.7) we have

$$v_r(a_i t^i) \leqslant -i\rho(f) + i\big(\rho(f) - r\big) + ir = 0$$

for infinitely many $i \geqslant 0$. Since $r < \rho(f)$ was arbitrary, f is not contained in (1.4) for any $r < \rho(f)$. ∎

A different type of affinoid space is the *closed anulus* centered at $a \in K$ with inner radius $R \in \mathbb{R}$ and outer radius $r \in \mathbb{R}$ satisfying $R > r$. It is the set

$$\mathsf{A}[a,R,r] := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid R \geqslant v_\xi(x-a) \geqslant r\}.$$

The affinoid K-algebra giving rise to $\mathsf{A}[a,R,r]$ is

$$K\{\pi^{-r}t, \pi^R t^{-1}\} := \Big\{ \sum_{i=-\infty}^{\infty} a_i t^i : \lim_{i\to\infty} v_K(a_i) + ir = \lim_{i\to-\infty} v_K(a_i) + iR = \infty \Big\}.$$

(See [BPR13, Section 2.1] for the definition of the norm on this Banach algebra.) Similarly, the *open anulus* centered at $a \in K$ with radii $R > r$ is the subset

$$\mathsf{A}(a,R,r) := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid R > v_\xi(x-a) > r\}.$$

Anuli have two boundary points, corresponding to their two radii. In fact, the two boundary points of $\mathsf{A}[a,R,r]$ are the Gauss valuations of radii r and R centered at a; the anulus $\mathsf{A}[a,R,r]$ equals $\mathsf{D}[a,r]\backslash\mathsf{D}(a,R)$, and similarly for $\mathsf{A}(a,R,r)$.

Remark 1.23. So far, we have defined open and closed disks and open and closed anuli as subdomains of $(\mathbb{P}^1_K)^{\mathrm{an}}$. In the sequel, we will call any affinoid space isomorphic to $\mathsf{D}[a,r]$ a closed disk. Similar remarks apply to open disks and closed and open anuli.

We come now to the elegant description of the points of $(\mathbb{P}^1_K)^{\mathrm{an}}$ using closed disks, following [Ber90, Section 1.4].

Definition 1.24. Let $\mathcal{D} = \{\mathsf{D}_i\}_{i\in I}$ be a family of closed disks in $(\mathbb{A}^1_K)^{\mathrm{an}}$. We call $\mathcal{D}$ a *nested* family of disks if $\mathsf{D}_i \subseteq \mathsf{D}_j$ whenever the radius of D_i is greater than the radius of D_j.

If $\mathcal{D}$ is a nested family of disks, we may define a pseudovaluation on $K[x]$ associated to $\mathcal{D}$ by

$$v_{\mathcal{D}}(f) = \sup_{\mathsf{D}\in\mathcal{D}} v_{\mathsf{D}}(f), \qquad f \in K[x].$$

It is shown in [Ber90, Section 1.4] that each pseudovaluation on $K[x]$ is of the form $v_{\mathcal{D}}$ for some family of nested disks $\mathcal{D}$ and that for each family $\mathcal{D}$, one of the following is true:

- There exist disks in $\mathcal{D}$ of arbitrarily large radius. In this case, $\bigcap_i \mathsf{D}_i$ consists of a single closed point, and $v_{\mathcal{D}}$ is the pseudovaluation associated to that point.
- $\bigcap_i \mathsf{D}_i$ is equal to one of the $\mathsf{D}_i \in \mathcal{D}$. In this case, $v_{\mathcal{D}}$ is the Gauss valuation associated to this disk. If the radius of D_i is a rational number, then $v_{\mathcal{D}}$ is a Type II valuation. Otherwise, $v_{\mathcal{D}}$ corresponds to a Type III point.
- $\bigcap_i \mathsf{D}_i = \varnothing$. In this case, $v_{\mathcal{D}}$ corresponds to a Type IV point.

In the case that K is discretely valued, not all points in $(\mathbb{A}^1_K)^{\mathrm{an}}$ correspond to families of nested disks. However, there is a similar classification involving so-called *discoids*. We will explain this in Section 5.2.

Disks are also helpful for describing paths in $(\mathbb{P}^1_K)^{\mathrm{an}}$. Recall that we have asserted above that $(\mathbb{P}^1_K)^{\mathrm{an}}$ is a simply connected special quasipolyhedron. In particular, for any pair of distinct points $\xi_1, \xi_2 \in (\mathbb{P}^1_K)^{\mathrm{an}}$, there exists a unique compact interval $[\xi_1, \xi_2]$. Let us illustrate this for the case that ξ_1 and ξ_2 are a pair of closed points, say corresponding to $a, b \in K$. Similar considerations apply for other types of points.

We write $r := v_K(a-b)$. Then the path connecting ξ_1 and ξ_2 is given by

$$\{\xi \mid v_\xi = v_{a,r'} \text{ for some } r' \geqslant r\} \cup \{\xi \mid v_\xi = v_{b,r'} \text{ for some } r' \geqslant r\}. \tag{1.8}$$

Intuitively, we consider increasingly larger disks centered at a up to the point where these disks contain b, and similarly we consider disks centered at b up to the point where they contain a. Then the path connecting ξ_1 and ξ_2 consists of the boundary points of all these disks.

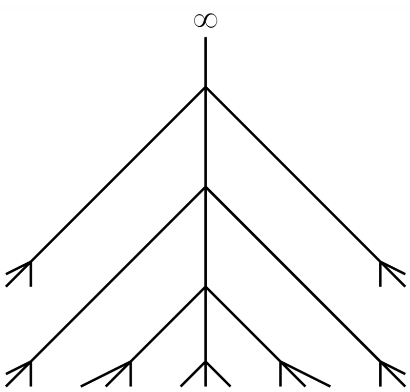


Figure 1.1: A sketch of $(\mathbb{P}^1_K)^{\mathrm{an}}$

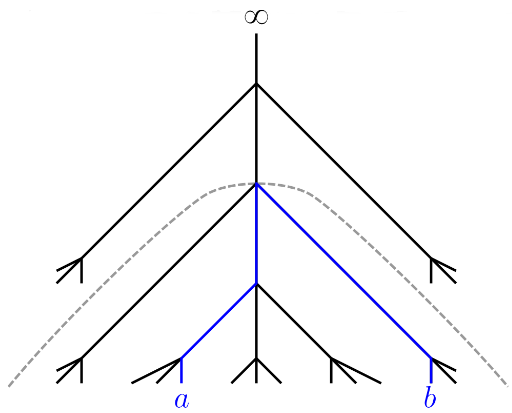


Figure 1.2: The same image with the path connecting two closed points highlighted

The "correct" way to visualize $(\mathbb{P}^1_K)^{\mathrm{an}}$ is shown in the famous drawing [Bak08, Figure 1] by Baker (based in turn on an illustration of Silverman). A similar drawing is in Figure 1.1. The curve $(\mathbb{P}^1_K)^{\mathrm{an}}$ is an infinite tree, with branching occurring at all Type II points. In fact, let D be a closed disk of radius r with boundary point ξ. Then the branches at ξ correspond to the open disks contained in D of radius r, plus one branch pointing towards the point ∞. Of course, only a small part of the branching behavior is indicated in Figure 1.1.

The Type I points are the endpoints at the bottom of Figure 1.1, except for the point ∞, which is at the very top. To simplify the picture, we have left out the Type IV points. The intervals between two given Type I or Type II points consist of further points of Type II or of Type III. The Type II points are boundary points of disks of rational radius, while the Type III points are boundary points of disks of irrational radius.

In Figure 1.2, the path (1.8) connecting two Type I points is shown. The smallest closed disk containing both points is indicated by the dashed gray line.

1.5 Topological ramification

In this section, K is algebraically closed. Let $\varphi\colon Y \to X$ be a finite separable morphism of K-curves and let $\xi \in X^{\mathrm{an}}$ and $\eta \in Y^{\mathrm{an}}$ be points with $\varphi(\eta) = \xi$.

Definition 1.25. The *topological ramification index* (called *geometric ramification index* in [Ber93]; see [CTT, Remark 3.2.2] for a brief discussion of the terminology) of φ at η is the degree

$$n_\eta := [\mathcal{H}(\eta) : \mathcal{H}(\xi)].$$

Remark 1.26. If ξ and η are of Type II, then we have $n_\eta = [\kappa(\eta) : \kappa(\xi)]$. Indeed, by [Tem10, Theorem 6.3.1(iii)], $\mathcal{H}(\xi)$ is *stable*, meaning that

$$n_\eta = [\kappa(\eta) : \kappa(\xi)] \cdot [\Gamma_{v_\eta} : \Gamma_{v_\xi}].$$

But by Remark 1.17, both Γ_{v_η} and Γ_{v_ξ} equal the subgroup Γ_K. Thus $[\Gamma_{v_\eta} : \Gamma_{v_\xi}] = 1$.

Remark 1.27. The topological ramification indices are closely related to the notion of *degree*. Given an open subspace $\mathsf{U} \subseteq X^{\mathrm{an}}$ and a connected component V of $\varphi^{-1}(\mathsf{U})$, the induced morphism $\pi\colon \mathsf{V} \to \mathsf{U}$ is also finite. As in the case of schemes, the degree $\deg(\pi)$ of π is defined to be the rank of the locally free $\mathcal{O}_{\mathsf{U}}$-module $\pi_*\mathcal{O}_{\mathsf{V}}$ ([Ber93, Remarks 6.3.1(iii)]). For any point $\xi \in \mathsf{U}$ we have the equality

$$\deg(\pi) = \sum_{\eta \in \pi^{-1}(\xi)} n_\eta.$$

In particular, we have

$$\deg(\varphi) = \sum_{\eta \in \varphi^{-1}(\xi)} n_\eta.$$

Definition 1.28. If $n_\eta > 1$, then η is called a *topological ramification point* of φ and $\xi = \varphi(\eta)$ is called a *topological branch point.*

If furthermore n_η is divisible by p, then η is called a *wild topological ramification point* of φ and ξ is called a *wild topological branch point.* Otherwise η is called a *tame topological ramification point* and ξ a *tame topological branch point.*

Remark 1.29.
(a) If φ has degree p, then it follows immediately from Remark 1.27 that a wild topological branch point of φ has a unique preimage.

(b) A Type II point $\eta \in Y^{\mathrm{an}}$ may be a wild topological ramification point even if the extension $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ (where $\xi = \varphi(\eta)$) is unramified. This choice of definition is briefly justified in [CTT, Remark 3.2.4]. We will say more on this in Lemma 2.17.

(c) The function $\eta \mapsto n_\eta$ is upper semicontinuous ([CTT, Lemma 3.6.10]). In particular, if φ is a morphism of degree p, then the topological ramification locus and the wild topological ramification locus associated to φ are closed. Likewise, the (wild) topological branch locus, which is the image of the (wild) topological ramification locus under the finite morphism φ, is closed.

It is convenient to extend the definition of the topological ramification index to the set of branches at a given point. As explained in [CTT, Section 3.6.11], if v

is a branch at a point η represented by an interval $[\eta, \eta']$, then the topological ramification index is constant on each subinterval $(\eta, \eta''] \subset [\eta, \eta']$ with η'' chosen close enough to η. We define the *topological ramification index of* v to be this constant value and denote it by n_v.

1.6 Skeletons

In this section, K is algebraically closed. Let X be a K-curve with analytification X^{an}. We begin by reviewing the central definitions, following [CTT, Section 3.5.1].

Definition 1.30. Let $G = (V, E, s, t)$ be a finite multigraph (i. e. V and E are finite sets, and $s, t \colon E \to V$ are any pair of functions).

The *topological realization* of G is the topological space obtained as follows: First introduce a point p_v for each vertex $v \in V$ and a compact interval I_e for each edge $e \in E$. Then, for each edge e, identify the two endpoints of I_e with $p_{s(e)}$ and $p_{t(e)}$ respectively.

We identify each vertex $v \in V$ with the point p_v in the topological realization of G and each edge $e \in E$ with the image of the interval I_e.

Definition 1.31. Consider a pair (Γ, S), where $\Gamma \subseteq X^{\mathrm{an}}$ is a closed subset and where $S \subseteq \Gamma$ is a finite subset consisting of Type I and Type II points, with at least one Type II point contained in S.

We say that (Γ, S) is a *graph* if there exist a finite multigraph $G = (V, E, s, t)$ and a homeomorphism φ between Γ and the topological realization of G with $\varphi(S) = V$.

We then also call the elements of S the *vertices* of (Γ, S) and call $\varphi^{-1}(e)$, $e \in E$, the *edges* of (Γ, S).

Remark 1.32. The condition that S contain at least one Type II point is omitted in [CTT]. Including it has the advantage that we may associate to every graph (Γ, S) a model of X, namely the model corresponding by Proposition 1.6 to the set of valuations

$$\{v_\xi \mid \xi \in S \text{ is of Type II}\}.$$

If S contained no Type II point, the graph would not correspond to any model, this set being empty.

Definition 1.33. A graph (Γ, S) is called a *skeleton* of X^{an} if the complement $X^{\mathrm{an}} \backslash \Gamma$ is a disjoint union of open disks and the set S contains all points in X^{an} of positive genus.

In the sequel, we will simply denote a graph or skeleton (Γ, S) by Γ, and write $S(\Gamma)$ for the vertex set S.

Remark 1.34. A skeleton $\Gamma \subset X^{\mathrm{an}}$ always exists. This fact is closely related to the Semistable Reduction Theorem. We will see below in Proposition 1.43 that the model determined by a skeleton (as explained in Remark 1.32) is semistable. Conversely, it can be shown that for a semistable model $\mathcal{X}$ of X, the incidence graph of the special fiber $\mathcal{X}_s$ is contained in X^{an} as a skeleton ([Ber90, Section 4.3], [BPR13, Corollary 4.7]).

Definition 1.35. Let Γ be a skeleton of X^{an}. A point $\xi \in S(\Gamma)$ is called a *leaf* of Γ if ξ maps to a leaf of $G = (V, E, s, t)$, where G is a finite multigraph to whose topological realization Γ is homeomorphic.

Remark 1.36. Let $\Gamma \subset X^{\mathrm{an}}$ be a skeleton and let $\xi \in X^{\mathrm{an}}$ be any point. Since $X^{\mathrm{an}} \backslash \Gamma$ is a disjoint union of open disks, it follows that there is a unique interval $[\xi, \xi']$ in X^{an} with $[\xi, \xi'] \cap \Gamma = \{\xi'\}$. (If $\xi \in \Gamma$, then $\xi' = \xi$ and $[\xi, \xi'] = \{\xi\}$.) We will frequently make use of this property of skeletons.

Definition 1.37. Let $\Gamma \subset X^{\mathrm{an}}$ be a skeleton. The *canonical retraction map* associated to Γ is the map

$$\mathrm{ret}_\Gamma \colon X^{\mathrm{an}} \to \Gamma$$

that sends each point $\xi \in \Gamma$ to itself and sends each point $\xi \in X^{\mathrm{an}} \backslash \Gamma$ to the unique boundary point of the connected component of $X^{\mathrm{an}} \backslash \Gamma$ containing ξ. Alternatively, $\mathrm{ret}_\Gamma(\xi)$ equals ξ', where $[\xi, \xi']$ is the unique interval in X^{an} with $[\xi, \xi'] \cap \Gamma = \{\xi'\}$.

Proposition 1.38. *Let $S \subset (\mathbb{P}^1_K)^{\mathrm{an}}$ be a finite set of Type I and Type II points, satisfying $|S| \geqslant 3$ or containing at least one Type II point. Then there exists a skeleton of $(\mathbb{P}^1_K)^{\mathrm{an}}$ whose underlying set is the union of all intervals $[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in S$.*

Proof. Let us first consider the case that S contains at least one Type II point. If $\xi \in (\mathbb{P}^1_K)^{\mathrm{an}}$ is a Type II point, then the subset $\{\xi\}$ is a skeleton of $\mathbb{P}^1_K$. By induction, it thus suffices to show the following claim: If ξ is a Type I or Type II point and Γ' is a skeleton of $(\mathbb{P}^1_K)^{\mathrm{an}}$ whose underlying set is the union of all intervals

$$[\xi_1, \xi_2], \qquad \xi_1, \xi_2 \in S',$$

for a finite set S' containing points of Type I and Type II, then one obtains again a skeleton Γ from Γ' by adding all intervals of the form $[\xi, \xi']$ for $\xi' \in S'$.

This follows easily from the fact that $(\mathbb{P}^1_K)^{\mathrm{an}}$ is uniquely path-connected. Indeed, there is a unique path $[\xi, \xi']$ such that $\xi' \in \Gamma'$ and $\xi'' \notin \Gamma'$ for any $\xi'' \in (\xi, \xi')$. We may construct Γ by adding $[\xi, \xi']$ to the underlying set of Γ' and adding ξ and ξ' to the vertex set of Γ' (if the latter is not already contained in it). On the level of the underlying finite multigraph of Γ', we simply attach a new leaf to Γ', after potentially subdividing one of its edges.

The case where S does not necessarily contain a Type II point, but contains at least three Type I points, works essentially the same. The point is that three distinct points ξ_1, ξ_2, ξ_3 in $(\mathbb{P}^1_K)^{\text{an}}$ determine a unique Type II point $\xi \in (\mathbb{P}^1_K)^{\text{an}}$ such that ξ_1, ξ_2, ξ_3 lie in distinct connected components of $(\mathbb{P}^1_K)^{\text{an}} \backslash \{\xi\}$. The skeleton Γ is obtained from $\{\xi\}$ by the same process as the one already discussed above. ■

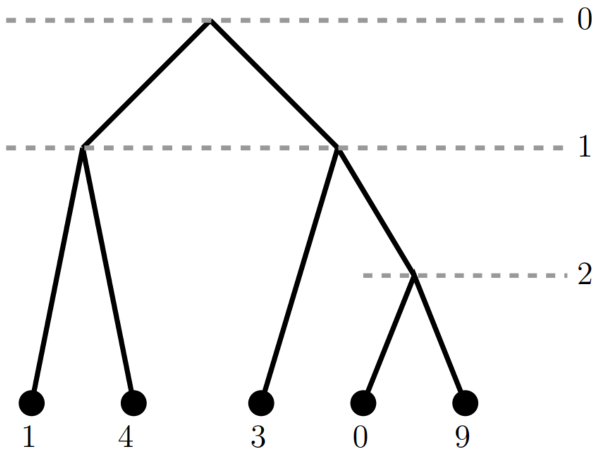


Figure 1.3: The tree spanned by the set of Type I points discussed in Example 1.39

The skeleton Γ described in Proposition 1.38 is called the *tree spanned by S*. We may visualize it by sketching the topological realization of the multigraph to which it is homeomorphic. We will do this in a similar style to the images in Section 1.4. Thus we will put the Type I leaves different from $\infty \in (\mathbb{P}^1_K)^{\text{an}}$ at the bottom. Then the edges beginning at such a leaf x_0 consist of the boundary points of disks of decreasing radii centered at x_0. The point ∞ will be drawn at the top of the graph, if it is contained in S.

Example 1.39. We consider $K = \mathbb{C}_3$ and take for S the set of Type I points corresponding to $0, 1, 3, 4, 9 \in \mathbb{C}_3$.

The tree Γ spanned by S is sketched in Figure 1.3. On top of the points in S, the vertex set of Γ contains three Type II points, which are the boundary points of the disk of radius 1 centered at 1 (or equivalently 4), the disk of radius 1 centered at 0 (or 3, or 9), and the disk of radius 2 centered at 0 (or 9). This is indicated by the dashed gray lines.

The tree spanned by S can also be visualized using a *cluster picture*. The connection between both viewpoints was already described and put to use by Bosch for computing semistable reductions in [Bos80]. Recently, cluster pictures have been used extensively in [DDMM] and related works.

A *cluster* is a subset of S that is obtained by intersecting S with a closed disk. In Figure 1.4, the five red balls correspond to the elements of S. The proper clusters (containing at least two points each) are drawn as rectangles, with the index indicating the radius of the smallest disk cutting out the cluster.

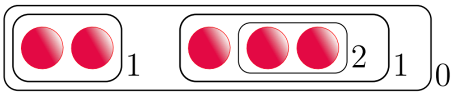


Figure 1.4: The cluster picture visualization of the set of Type I points discussed in Example 1.39

To end this section, we define a metric structure on X^{an}, following [BPR13, Section 5]. Consider first the standard open anulus of radius $r \in \mathbb{Q}_{>0}$,

$$\mathsf{A}(r) := \mathsf{A}(0, r, 0) = \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid r > v_\xi(x) > 0\}. \tag{1.9}$$

Denote by $\xi_0, \xi_r \in (\mathbb{P}^1_K)^{\mathrm{an}}$ the two boundary points of $\mathsf{A}(r)$. Then $\mathsf{A}(r)$ contains the interval (ξ_0, ξ_r) — in the language of [BPR13], this interval is called the *skeleton* of $\mathsf{A}(r)$. We define the length of this interval to be

$$\ell\big((\xi_0, \xi_r)\big) := r.$$

Next, suppose that $[\xi, \xi'] \subset X^{\mathrm{an}}$ is any interval consisting of Type II and Type III points. By Remark 1.34, it is possible to choose a skeleton Γ containing $[\xi, \xi']$. Removing the vertices of Γ yields a decomposition of (ξ, ξ') into disjoint open intervals

$$(\xi, \xi') \backslash S(\Gamma) = \coprod_{i=1}^{k} (\xi_i, \xi_{i+1}), \qquad \xi_1 = \xi, \quad \xi_{k+1} = \xi'.$$

By [CTT, Remark 3.5.2(ii)], there exist open anuli $\mathsf{A}_i \subset X^{\mathrm{an}}$ such that under an identification of A_i with a standard open anulus $\mathsf{A}(r_i)$ as in (1.9), the interval (ξ_i, ξ_{i+1}) is identified with the skeleton of $\mathsf{A}(r_i)$, an interval of length $r_i \in \mathbb{Q}_{>0}$. Thus we define $\ell(\xi_i, \xi_{i+1})$ to be r_i, and

$$\ell\big((\xi, \xi')\big) := \sum_{i=1}^{k} \ell\big((\xi_i, \xi_{i+1})\big) = \sum_{i=1}^{k} r_i.$$

It can be shown (see [BPR13, Section 5]) that this definition is independent of the choice of Γ. We extend the definition of the length to closed and half-open intervals consisting of Type II and Type III points by

$$\ell\big([\xi, \xi']\big) := \ell\big((\xi, \xi']\big) := \ell\big([\xi, \xi')\big) := \ell\big((\xi, \xi')\big).$$

Finally, we extend the length function to intervals $[\xi, \xi']$, $\xi \neq \xi'$, where one of the two endpoints is a Type I point, by setting $\ell([\xi, \xi']) := \infty$.

The following definition coincides with the one given in [CTT, Section 3.6.2].

Definition 1.40. A homeomorphism $\varphi\colon I \to J$ between an interval $I \subset X^{\mathrm{an}}$ and an interval $J \subseteq \mathbb{R} \cup \{\pm\infty\}$ is called a *radius parametrization* of I if

(a) the length $\ell(I')$ is the same as the length of $\varphi(I') \subseteq \mathbb{R} \cup \{\pm\infty\}$ (with respect to the usual metric on $\mathbb{R}$) for every subinterval $I' \subseteq I$,

(b) for some Type II point $\xi \in I$, we have $\varphi(\xi) \in \mathbb{Q}$.

Note that this implies that $\varphi(\xi) \in \mathbb{Q}$ for *every* Type II point $\xi \in I$. When I contains a point of Type I and at least one other point, then J must have ∞ or $-\infty$ as boundary point. The simplest example of a radius parametrization is the identification of the skeleton of the standard open anulus $\mathsf{A}(r)$ from (1.9) with the interval $(0, r)$, which identifies $s \in (0, r)$ with the point corresponding to the Gauss valuation of radius s.

Definition 1.41. The *radius function* associated to a skeleton Γ of X is the map

$$r_\Gamma \colon X^{\mathrm{an}} \to [0, \infty]$$

defined by

$$r_\Gamma(\xi) := \ell([\xi, \xi']),$$

where $[\xi, \xi']$ is the unique interval with $[\xi, \xi'] \cap \Gamma = \{\xi'\}$ (Remark 1.36).

In particular, if $\xi \in \Gamma$, then $\xi' = \xi$ and so $r_\Gamma(\xi) = 0$. In general, r_Γ measures the distance of points to the skeleton Γ.

1.7 Reduction maps

The field K is still assumed to be algebraically closed and X denotes a K-curve. In this section, we discuss reduction maps associated to $\mathcal{O}_K$-models of X.

Let $\mathcal{X}$ be an $\mathcal{O}_K$-model of X. Suppose that $x\colon \operatorname{Spec} K \to X$ is a closed point. By the valuative criterion for properness, x extends to a unique morphism $\operatorname{Spec} \mathcal{O}_K \to \mathcal{X}$:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_K \arrow[r, equal] \arrow[ur, dashed, "\exists!"] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

The image of the closed point of $\operatorname{Spec} \mathcal{O}_K$ under this map is called the *reduction* of P with respect to $\mathcal{X}$; it is denoted $\mathrm{red}_{\mathcal{X}}(x)$. The resulting map

$$\mathrm{red}_{\mathcal{X}} \colon \text{closed points of } X \longrightarrow \text{closed points of } \mathcal{X}_s$$

is surjective ([Liu02, Corollary 10.1.38]).

Recall from Section 1.3 that the underlying set of X^{an} consists of pairs $\xi = (P, v)$, where $P \in X$ is any point and where v is a valuation on the residue field $k(P)$ extending v_K. Denote by $\mathcal{O}_v$ the valuation ring of v. Then as before, the valuative criterion for properness gives an extension of the natural morphism $\operatorname{Spec} k(P) \to \mathcal{X}$ to a morphism $\operatorname{Spec} \mathcal{O}_v \to \mathcal{X}$:

$$\begin{tikzcd} \operatorname{Spec} k(P) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_v \arrow[r] \arrow[ur, dashed, "\exists!"] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

We denote the image of the closed point of $\mathcal{O}_v$ under this morphism by $\operatorname{red}_{\mathcal{X}}(\xi)$. Thus we have extended $\operatorname{red}_{\mathcal{X}}$ to a map

$$\operatorname{red}_{\mathcal{X}} \colon X^{\mathrm{an}} \to \mathcal{X}_s.$$

The image of a Type II point $\xi \in X^{\mathrm{an}}$ under $\operatorname{red}_{\mathcal{X}}$ is the generic point of the component $Z \subseteq \mathcal{X}_s$ giving rise to v_ξ as explained in Section 1.2.

The above reduction maps are associated to models of X; there is no canonical reduction map associated to X alone. There is, however, a canonical reduction map associated to any affinoid subdomain of X^{an}.

Recall from Section 1.4 that affinoid subdomains are the building blocks of analytic curves in the sense of Berkovich. They are defined to be the *spectra* of certain commutative Banach algebras, the *affinoid K-algebras*, of which we saw two examples in Section 1.4. In fact, the affinoid subdomains whose reductions we consider below will be *strictly affinoid spaces*, which are the spectra of *strictly affinoid K-algebras* ([Ber90, Section 2.1]). Essentially, these are the affinoid spaces arising from affinoid spaces in the sense of rigid-analytic geometry ([Bos14, Chapter 3]).

If $\mathcal{A}$ is a strictly K-affinoid algebra, we define

$$\tilde{\mathcal{A}} := \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}, \qquad \text{where}$$

$$\begin{aligned} \mathcal{A}^\circ &= \{ f \in \mathcal{A} \mid \forall \xi \in \mathcal{M}(\mathcal{A}) : v_\xi(f) \geqslant 0 \}, \\ \mathcal{A}^{\circ\circ} &= \{ f \in \mathcal{A} \mid \forall \xi \in \mathcal{M}(\mathcal{A}) : v_\xi(f) > 0 \} \end{aligned}$$

(cf. [Ber90, Section 2.4, Theorem 1.3.1]). The association $\mathcal{A} \mapsto \tilde{\mathcal{A}}$ is functorial. The *canonical reduction* of the affinoid space $\mathcal{M}(\mathcal{A})$ is defined to be the scheme $\operatorname{Spec}(\tilde{\mathcal{A}})$. It is a reduced scheme of finite type over the residue field κ. There is a reduction map

$$\operatorname{red}_{\mathcal{M}(\mathcal{A})} \colon \mathcal{M}(\mathcal{A}) \to \operatorname{Spec}(\tilde{\mathcal{A}})$$

defined as follows. A given point $\xi \in \mathcal{M}(\mathcal{A})$ has an associated character $\mathcal{A} \to \mathcal{H}(\xi)$; applying the functor $(\cdot)^\sim$ yields a character $\tilde{\mathcal{A}} \to \kappa(\xi)$, which in turn defines a point in $\operatorname{Spec}(\tilde{\mathcal{A}})$.

Example 1.42. (a) Consider the closed unit disk

$$\mathsf{D} := \mathcal{M}(\mathcal{A}), \qquad \mathcal{A} = K\{T\}.$$

Its canonical reduction is the affine line $\mathbb{A}^1_\kappa$, and the reduction map sends a closed point $x \in \mathcal{O}_K$ to its reduction $\overline{x} \in \kappa$.

(b) Consider the closed anulus of radius $r \in \mathbb{Q}_{>0}$,

$$\mathsf{A}[r] := \mathsf{A}[0, r, 0] = \mathcal{M}(\mathcal{A}), \qquad \mathcal{A} = K\{t, \pi^r t^{-1}\}.$$

The canonical reduction of $\mathsf{A}[r]$ is the "cross" $\overline{X} := \operatorname{Spec} \kappa[T, U]/(TU)$. We may identify the set of rational points

$$\{x \in K \mid 0 \leqslant v_K(x) \leqslant r\}$$

with a subset of $\mathsf{A}[r]$. The reduction of such a point $x \in K$ is given by

- the reduction $\overline{x} \in \kappa$ of x, considered as a point in the affine line $\operatorname{Spec} \kappa[T] \subset \overline{X}$, if $v_K(x) = 0$,
- the nodal point of $\overline{X}$, if $0 < v_K(x) < r$,
- the reduction $\overline{x\pi^{-r}} \in \kappa$ (where π^{-r} denotes an element of K with valuation $-r$), considered as a point in the affine line $\operatorname{Spec} \kappa[U] \subset \overline{X}$, if $v_K(x) = r$.

We now explain how to piece together canonical reduction maps associated to strictly affinoid subdomains $\mathcal{M}(\mathcal{A}) \subset X^{\mathrm{an}}$ to obtain reduction maps on all of X^{an}, following [Ber90, Section 4.3].

Suppose that $\{\mathsf{U}_i\}_{i \in I}$ is a finite covering of X^{an} by strictly affinoid subdomains. Let us write $\mathsf{U}_{ij} := \mathsf{U}_i \cap \mathsf{U}_j$ and denote by $\overline{X}_i$ and $\overline{X}_{ij}$ the canonical reductions of U_i and U_{ij} respectively. The covering $\{\mathsf{U}_i\}_{i \in I}$ is called a *formal covering* of X^{an} if the induced morphism $\overline{X}_{ij} \to \overline{X}_i$ is an open embedding for all $i, j \in I$. In this case it is possible to glue the $\overline{X}_i$ along the $\overline{X}_{ij}$ to a κ-curve $\overline{X}$. Moreover, the canonical reductions $\operatorname{red}_{\mathsf{U}_i}$ combine to a reduction map $X^{\mathrm{an}} \to \overline{X}$. In the following proposition, we will see how to define a formal covering associated to a skeleton of X^{an}.

Proposition 1.43. *If $\Gamma \subset X^{\mathrm{an}}$ is a skeleton, then the model of X determined by Γ (see Remark 1.32) is semistable.*

Proof. A version of this folklore theorem was written down in [BPR13, Theorem 4.11]. Below, we use a similar construction, but our version involves models in the sense of Section 1.1, not formal models.

We assume that Γ has at least 2 edges (if this is not the case, the argument only needs to be slightly adjusted, as is done in [BPR13, Section 4.15.2–3]). We then define a formal covering of X^{an} as follows. For each edge $e = [\xi_1, \xi_2]$ of Γ, put

$$\mathsf{U}_e := \operatorname{ret}_\Gamma^{-1}(e).$$

The U_e satisfy the following ([BPR13, Section 4.15.1]):

- U_e is a strictly affinoid subdomain of X^{an}.
- The canonical reduction of U_e is a κ-curve $\overline{X}_e$ with one singularity, a node. If the start and end point of e are not the same, then $\overline{X}_e$ has two irreducible components intersecting in this node. Otherwise, $\overline{X}_e$ is irreducible.

- The complement of $\{\xi_1, \xi_2\}$ in U_e has as connected components an open anulus $\mathsf{A} := \mathrm{ret}_\Gamma^{-1}((\xi_1, \xi_2))$ and infinitely many open disks. The anulus A is the inverse image of the nodal point on $\overline{X}_e$ while each open disk is the inverse image of a smooth point on $\overline{X}_e$.

Suppose now that e and f are two distinct adjacent edges, say with $e \cap f = \{\xi\}$. Then $\mathsf{U}_e \cap \mathsf{U}_f$ is the retraction $\mathrm{ret}_\Gamma^{-1}(\xi)$. It follows from the above bullet points that $\mathrm{red}_{\mathsf{U}_e}(\mathsf{U}_e \cap \mathsf{U}_f)$ equals one of the components of $\overline{X}_e$ minus the nodal point. Thus the U_e form a formal covering of X^{an}. Moreover, it is clear that by gluing the canonical reductions $\overline{X}_e$, we obtain a semistable κ-curve $\overline{X}$.

By [Ber90, Proposition 2.4.4], the function fields of the irreducible components of the reduction $\overline{X}$ are the same as the residue fields $\kappa(\xi)$, where $\xi \in \Gamma$ is a Type II point. On the last two pages of [BL85] it is explained how to construct an honest $\mathcal{O}_K$-model $\mathcal{X}$ from the formal covering $\{\mathsf{U}_e\}_e$, whose special fiber $\mathcal{X}_s$ coincides with the semistable κ-curve $\overline{X}$. By Proposition 1.6, this $\mathcal{X}$ must be the model determined by Γ as explained in Remark 1.32, so we are done. ■

Remark 1.44. Let $\mathcal{X}$ be the model associated to a skeleton Γ of X. In Proposition 1.43, we have reconstructed the reduction map $\mathrm{red}_{\mathcal{X}}\colon X^{\mathrm{an}} \to \mathcal{X}_s$ working locally on X^{an}. From the proof of Proposition 1.43, it is clear how we may describe the reduction $\mathrm{red}_{\mathcal{X}}(x_0)$ of a closed point $x_0 \in X$ in terms of the retraction $\mathrm{ret}_\Gamma(x_0)$. In summary, we have the following:

- For every Type II vertex ξ of Γ, the special fiber $\mathcal{X}_s$ has an irreducible component $\overline{X}_\xi$

- Each edge e connecting two Type II vertices ξ_1, ξ_2 of Γ corresponds to a nodal point of intersection of the components $\overline{X}_{\xi_1}, \overline{X}_{\xi_2}$ (here we also allow $\xi_1 = \xi_2$, in which case the component $\overline{X}_{\xi_1} = \overline{X}_{\xi_2}$ has a singularity)

- If $\mathrm{ret}_\Gamma(x_0)$ lies in the interior of this edge e, then $\mathrm{red}_{\mathcal{X}}(x_0)$ is the point of intersection of $\overline{X}_{\xi_1}$ and $\overline{X}_{\xi_2}$ (or, if $\xi_1 = \xi_2$, then $\mathrm{red}_{\mathcal{X}}(x_0)$ is the singular point on $\overline{X}_{\xi_1} = \overline{X}_{\xi_2}$)

- If $\mathrm{ret}_\Gamma(x_0) = \xi$ for a Type II vertex ξ of Γ, then $\mathrm{red}_{\mathcal{X}}(x_0)$ is a smooth point on $\overline{X}_\xi$

- If x_0 lies on Γ, then it is a leaf of Γ, and $\mathrm{red}_{\mathcal{X}}(x_0)$ is a smooth point on $\overline{X}_\xi$, where ξ is the Type II vertex of Γ adjacent to x_0

- Suppose that $x_0, x_1 \in X$ are two closed points with the same retraction ξ to Γ. Then $\mathrm{red}_{\mathcal{X}}(x_0) = \mathrm{red}_{\mathcal{X}}(x_1)$ if and only if $[x_0, \xi]$ and $[x_1, \xi]$ represent the same branch at ξ in the sense of Definition 1.20

In light of the last point, it makes sense to talk of the *reduction of a branch* at a vertex ξ of Γ.

Corollary 1.45. *Let $\Gamma \subset X^{\mathrm{an}}$ be the tree spanned by a finite set of Type I points $S \subset X = \mathbb{P}^1_K$. Let $\overline{X}$ denote the reduction of X associated to Γ as constructed in Proposition 1.43. Then the points $x_0 \in S$ reduce to pairwise distinct smooth points on $\overline{X}$.*

Proof. This is immediate from Remark 1.44 and the construction of Γ in Proposition 1.38 — all x_0 are leaves of Γ. ∎

Definition 1.46. The *intersection graph* of a semistable curve C is the multigraph which has one vertex for each irreducible component of C and has one edge for each nodal point x on C, the endpoints of this edge corresponding to the components on which x lies.

It follows from Remark 1.44 that a skeleton Γ of X may be recovered as the topological realization of the intersection graph of the semistable curve $\overline{X}$, where $\overline{X}$ is the reduction associated to the formal covering induced by Γ.

1.8 Admissible reduction

We continue to assume that K is algebraically closed, but this assumption is not essential for this section. In fact, the main goal of works like [DDMM] and [BW17] cited below is computing arithmetic invariants of curves, and therefore a lot of emphasis is put on computing the minimal field extension over which a semistable reduction exists. We will consider the question of computing this field extension in Chapter 5; for now we stick to algebraically closed K for expository simplicity.

The goal of this section is to review a technique that is useful for computing the semistable reduction of many curves. The idea is to choose a finite morphism φ from the curve Y under consideration to $\mathbb{P}^1_K$ and then take as a model of Y the normalization $\mathcal{Y}$ in F_Y of a model $\mathcal{X}$ of $\mathbb{P}^1_K$ *separating the branch points* of φ. Our reason for discussing this technique here is twofold: First, it serves as an illustration and example for many notions introduced earlier in this chapter. And second, we will see in Example 1.49 that the method of separating branch points does not in general work without Assumption 1.47 below, thus motivating our developing a more general method in subsequent chapters. If Assumption 1.47 is satisfied, the induced map on special fibers $\mathcal{Y}_s \to \mathcal{X}_s$ is an *admissible cover* between semistable curves. (See [HM82, Section 4] — we will not use this terminology past this section.)

This approach has been used in many works treating the semistable reduction of curves. For example, [BW17] deals with the case of superelliptic curves. Separating branch points is also the basis of [DDMM] and related works, which treat the case of hyperelliptic curves in great detail.

Let $\varphi\colon Y \to X = \mathbb{P}^1_K$ be a cover of smooth and irreducible curves of degree $n \geqslant 2$. In this section only we make the following assumption. In subsequent chapters, we will usually consider the situation that φ is of degree exactly p.

Assumption 1.47. *The residue characteristic $p = \mathrm{char}(\kappa)$ is strictly greater than n.*

Let $\Gamma \subset X^{\mathrm{an}}$ be the tree spanned by the branch points of φ, as constructed in Proposition 1.38. Let $\mathcal{X}$ be the $\mathcal{O}_K$-model associated to Γ, as explained in Remark 1.32. Then by Corollary 1.45 the branch points of φ specialize to pairwise distinct points on $\mathcal{X}_s$. Finally, let $\mathcal{Y}$ denote the normalization of $\mathcal{X}$ in the function field F_Y; it is an $\mathcal{O}_K$-model of Y.

Proposition 1.48. *The model $\mathcal{Y}$ is semistable.*

Proof. In the case that φ is a Galois cover, this follows from [LL99, Theorem 2.3]. The general case is proved in detail in [Hel21, Section 3.3]. In either case, the needed tameness assumptions follow from Assumption 1.47. ∎

Since the most important concrete class of algebraic curves that we treat is that of plane quartic curves, many of our examples are drawn from this class, including the following Example 1.49. Though this is not necessary for the continuity of the argument, the reader might wish to consult Section 4.1 for a brief reminder on plane quartics and for an explanation of the normal form in which all our plane quartics are given.

Example 1.49. Consider the smooth plane quartic curve Y over $K := \mathbb{C}_3$ given by

$$Y\colon \quad F(x,y) = y^3 + x^3y + x^4 + 1 = 0. \tag{1.10}$$

The discriminant of F (considered as a polynomial in y whose coefficients are polynomials in x) is

$$\Delta_F = -4x^9 - 27x^8 - 54x^4 - 27.$$

The branch locus of the degree-3 cover $\varphi\colon Y \to \mathbb{P}^1_K$ given by $(x,y) \mapsto x$ consists of the zeros of Δ_F and the point ∞ (cf. Lemma 4.1 below)

In fact, the branch locus of φ is *equidistant* (a terminology found for example in [LM06, Section 2] or [BW15, Section 4.3]). This means that there exists a smooth $\mathcal{O}_K$-model $\mathcal{X}$ of X such that the branch points of φ specialize to distinct points on the special fiber $\mathcal{X}_s$. The set of Type II valuations associated to $\mathcal{X}$ via Proposition 1.6 contains only one element, which we denote ξ. The tree spanned by the branch points of φ is sketched in Figure 1.5; the branch point ∞ is at the top, the nine finite branch points at the bottom.
In fact, the point ξ is the boundary point of the disk of radius $1/2$ centered at one of the finite branch points, that is, at one of the zeros of the discriminant of the defining polynomial in (1.10).

However, the normalization $\mathcal{Y}$ of the model $\mathcal{X}$ determined by the point ξ does not have semistable reduction — thus, the conclusion of Proposition 1.48 is false for this example. To show this, we claim that the valuation v_ξ has a unique extension to F_Y and that the associated extension of residue fields is

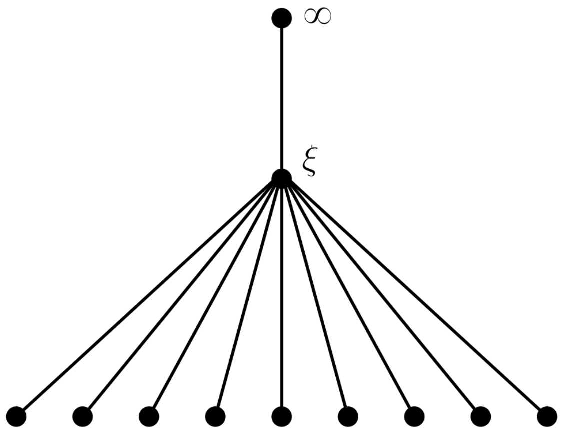


Figure 1.5: The tree spanned by the branch locus of the cover φ discussed in Example 1.49

inseparable. Then it follows from Lemma 1.8 that $\mathcal{Y}_s$ is an irreducible curve of geometric genus 0. In fact, the morphism $\mathcal{Y}_s \to \mathcal{X}_s$ is a homeomorphism ([Liu02, Exercise 5.3.9]), so $\mathcal{Y}_s$ cannot have nodal singularities. Thus $\mathcal{Y}_s$ is not a semistable curve.

It remains to prove the claim that v_ξ has a unique extension to F_Y with inseparable residue field extension. Let (x_0, y_0) be one of the finite ramification points of φ. Since $F(x_0, y_0) = 0$, the minimal polynomial of the element $z := (y - y_0)\pi^{1/2} \in F_Y$ over the subfield $K(t)$, where $t = x - x_0$, has the form

$$T^3 + a_0T^2 + (b_3t^3 + b_2t^2 + b_1t + b_0)T + c_4t^4 + c_3t^3 + c_2t^2 + c_1t. \tag{1.11}$$

Here the coefficients $a_0, b_0, \dots, b_3, c_1, \dots, c_4$ lie in a finite extension K_0 of $\mathbb{Q}$ — all that is needed is that the point (x_0, y_0) be K_0-rational and that K_0 contain the element $\pi^{1/2}$ of valuation $1/2$. It is therefore not hard to determine the valuations of these coefficients, for example using the functionality for extending valuations along extensions of number fields in Sage — see Code Listing A.1 in the appendix for details. The resulting valuations are

$$v_K(a_0) = \frac{1}{2}, \quad v_K(b_0) = 1,\ v_K(b_1) = \frac{2}{3},\ v_K(b_2) = \frac{1}{3},\ v_K(b_3) = -1,$$

$$v_K(c_1) = -\frac{1}{2},\ v_K(c_2) = -\frac{1}{6},\ v_K(c_3) = -\frac{3}{2},\ v_K(c_4) = -\frac{3}{2}.$$

(Note that these valuations are independent of the choice of ramification point (x_0, y_0), since the nine finite branch points of φ are Galois conjugate over $\mathbb{Q}_3$.) Recall that the valuation v_ξ is determined by

$$v_\xi\big(\sum_i a_it^i\big) = \min_i \big\{v_K(a_i) + \frac{i}{2}\big\}.$$

We write $\bar{t} := [t]_{v_\xi} \in \kappa(\xi)$ and let $\eta \in Y^{\mathrm{an}}$ be a point with $\varphi(\eta) = \xi$. Reducing the coefficients of (1.11) to $\kappa(v_\xi)$ shows that $\bar{z} := [z]_{v_\eta}$ satisfies an equation of the form

$$\bar{z}^3 + \bar{c}_3\bar{t}^3 + \bar{c}_1\bar{t} = 0,$$

where $\bar{c}_3, \bar{c}_1 \in \kappa^\times$. Clearly $\bar{c}_3\bar{t}^3 + \bar{c}_1\bar{t}$ is not a third power, so $\kappa(\eta)$ must be an inseparable residue field extension of $\kappa(v_\xi)$ of degree 3. The claim is proved.

We will see in Example 5.18 that Y has good reduction.

Chapter 2

Admissible functions

Throughout this entire chapter, the ground field K is algebraically closed.

We will introduce *admissible functions* on an analytic curve X^{an}. These are locally constant outside a skeleton of X^{an}. Thus despite the complicated structure of X^{an} we can work with and understand the structure of admissible functions. In Section 2.2, we discuss an important class of admissible functions: *valuative functions* attached to elements of the function field F_X. For the case that X is the projective line, an implementation of admissible functions exists in the Sage package MCLF. This is a crucial ingredient for the implementation of our algorithms in Chapter 5.

Suppose that we are given a degree-p morphism $\varphi\colon Y \to X$. In Sections 2.3, 2.4, and 2.5, we introduce three important functions on X^{an}, denoted by Greek letters. The first is the admissible function μ, which measures the distance to a given skeleton of X^{an}. The second is the function δ. It is essentially equivalent to the *different function* of [CTT] and measures the topological ramification (Section 1.5) of $\varphi\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$. While δ is not admissible, the function λ we derive from it is. We will see in Theorem 2.34 that λ controls the semistable reduction of Y^{an} to a large degree and exploit this fact in Chapter 3.

2.1 Definition

Let $I \subseteq \mathbb{R} \cup \{\pm\infty\}$ be an interval. A function $f\colon I \to \mathbb{R}$ is called an *affine function* if it is of the form

$$t \mapsto \alpha t + \beta, \qquad \alpha, \beta \in \mathbb{Q}.$$

It is called *piecewise affine* if it is continuous and for all but finitely many points $t_0 \in I$ there exists an open interval $t_0 \in J \subseteq I$ such that $f|_J$ is affine. The finitely many points t_0 where no such interval can be found are called the *kinks* of f.

Let X be a K-curve with analytification X^{an}. By choosing a radius parametrization of a compact interval $I \subset X^{\mathrm{an}}$, say

$$\varphi\colon I \xrightarrow{\sim} [a, b] \subseteq \mathbb{R} \cup \{\pm\infty\},$$

it makes sense to talk of *(piecewise) affine functions on I*; they are simply functions of the form $h = f \circ \varphi \colon I \to \mathbb{R}$, where f is (piecewise) affine in the previous sense. Similarly, the *kinks* of a piecewise affine function $h = f \circ \varphi \colon I \to \mathbb{R} \cup \{\pm\infty\}$ are given by $\{\varphi^{-1}(t_0) \mid t_0 \text{ is a kink of } f\}$. Clearly all these notions are independent of the radius parametrization chosen.

Let $U \subseteq X^{\mathrm{an}}$ be a subset. A function $h \colon U \to \mathbb{R} \cup \{\pm\infty\}$ is called *(piecewise) affine* if its restriction to every compact interval $I \subset U$ is (piecewise) affine.

Definition 2.1. An *admissible function* on X^{an} is a continuous function $h \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ such that there exists a skeleton $\Gamma \subset X^{\mathrm{an}}$ with the following properties:

(a) The restriction of h to every edge of Γ is an affine function

(b) The restriction of h to the complement $X^{\mathrm{an}} \backslash \Gamma$ is locally constant

Such a skeleton Γ is said to *trivialize* the admissible function h. A point $\xi \in X^{\mathrm{an}}$ with $h(\xi) = -\infty$ is called a *pole* of h.

We note that admissible functions are clearly piecewise affine, but not conversely.

Lemma 2.2. *Let $h_1, h_2 \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ be admissible functions. Then there exist unique admissible functions*

$$h_+, h_{\max}, h_{\min} \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$$

such that for all points $\xi \in X^{\mathrm{an}}$ with $h_1(\xi) \neq \pm\infty$, $h_2(\xi) \neq \pm\infty$ we have

$$h_+(\xi) = h_1(\xi) + h_2(\xi), \qquad h_{\max}(\xi) = \max(h_1(\xi), h_2(\xi)),$$
$$h_{\min}(\xi) = \min(h_1(\xi), h_2(\xi)).$$

Proof. Let $\Gamma \subset X^{\mathrm{an}}$ be a skeleton trivializing both h_1 and h_2. It follows from part (a) of Definition 2.1 that $h_1(\xi) = \pm\infty$ or $h_2(\xi) = \pm\infty$ is only possible if ξ is a Type I point on Γ. At all other points, we may simply define h_+ by pointwise addition, $h_{\max}$ by pointwise maximum, and $h_{\min}$ by pointwise minimum.

If ξ is a Type I point contained in Γ, then using part (a) again of Definition 2.1 choose an interval $[\xi', \xi] \subseteq \Gamma$ on which h_1 and h_2 are affine, say of the form

$$h_1(t) = \alpha_1 t + \beta_1, \qquad h_2(t) = \alpha_2 t + \beta_2$$

with respect to a radius parametrization. Then h_+ will be continuous if and only if we set

$$h_+(\xi) = \begin{cases} \infty, & \alpha_1 + \alpha_2 > 0, \\ -\infty, & \alpha_1 + \alpha_2 < 0, \\ \beta_1 + \beta_2, & \alpha_1 + \alpha_2 = 0. \end{cases}$$

Similar definitions work for $h_{\max}$ and $h_{\min}$. ∎

In the sequel, we will denote the functions h_+, $h_{\max}$, and $h_{\min}$ by $h_1 + h_2$, $\max(h_1, h_2)$, and $\min(h_1, h_2)$ respectively.

Remark 2.3. Note that given an admissible function $h\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ and a rational number $\alpha \in \mathbb{Q}$, the function

$$\alpha h\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}, \qquad \xi \mapsto \alpha h(\xi),$$

is obviously admissible as well. Applying Lemma 2.2 inductively, we may thus define, given admissible functions $h_1, \ldots, h_r\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ and rational numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{Q}$, the admissible functions

$$\alpha_1 h_1 + \ldots + \alpha_r h_r, \qquad \max(\alpha_1 h_1, \ldots, \alpha_r h_r), \qquad \min(\alpha_1 h_1, \ldots, \alpha_r h_r).$$

Let $h\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ be a piecewise affine function, let $\xi \in X^{\mathrm{an}}$ be a point, and let v be a branch at ξ. It follows from the definition that there exists an interval $[\xi, \xi']$ representing the branch v on which h is affine. Choose a radius parametrization

$$[\xi, \xi'] \xrightarrow{\sim} [a, b]$$

and suppose that h corresponds to the function $t \mapsto \alpha t + \beta$ on $[a, b]$, where $\alpha, \beta \in \mathbb{Q}$.

Definition 2.4. In the above situation, the rational number α is called the *slope* of h in the direction of v, denoted

$$\mathrm{slope}_v(h) := \alpha.$$

Remark 2.5. In [CTT, Section 3.6.3], *piecewise monomial functions* are considered. These are to our piecewise affine functions as the absolute value $|\cdot|_K$ is to the valuation v_K. To be precise, a monomial function on an interval $I = [\xi_1, \xi_2] \subset X^{\mathrm{an}}$ is defined to be a function of the form

$$\xi \mapsto \beta \psi(\xi)^n, \tag{2.1}$$

where $\psi\colon I \to [a, b]$ is a radius parametrization in the sense of [CTT, Section 3.6.2], where $n \in \mathbb{Z}$ and where β is the absolute value of some element in K. A radius parametrization in the sense of [CTT, Section 3.6.2] is of the form

$$\xi \mapsto e^{\varphi(\xi)},$$

where φ is a radius parametrization in our sense. Composing the monomial function (2.1) with $-\log$, we obtain the function

$$\xi \mapsto -\log(\beta\psi(\xi)^n) = -\log(\beta) - n\log(\psi(\xi)) = -n\varphi(\xi) - \log(\beta). \tag{2.2}$$

Note that $\log(\beta) \in \mathbb{Q}$. Thus this function is an affine function. In the same way, the piecewise monomial functions of [CTT, Section 3.6.3] correspond to our piecewise affine functions.

Let v be the branch at ξ_1 represented by the interval $[\xi_1, \xi_2]$. In [CTT, Section 3.6.5], the slope of the monomial function (2.1) in the direction of v is defined to be n. This is in contrast to the slope of the associated affine function (2.2), which according to our Definition 2.4 is $-n$. We will occasionally have to account for this discrepancy when appealing to results of [CTT].

Lemma 2.6. *Let $h_1, \ldots, h_r : X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ be admissible functions. Then the subset*

$$\mathsf{U} = \bigcup_{i=1}^{r} \{\xi \in X^{\mathrm{an}} \mid h_i(\xi) \geqslant 0\}$$

equals X^{an} or is an affinoid subdomain of X^{an}.

Proof. Since the maximum of the admissible functions $h_1, \ldots, h_r$ is again admissible, it suffices to consider the case of only one function $h := h_1$. Let Γ be a skeleton trivializing h. Then U may be expressed using the canonical retraction $X^{\mathrm{an}} \to \Gamma$ as

$$\mathsf{U} = r_\Gamma^{-1}(\{\xi \in \Gamma \mid h(\xi) \geqslant 0\}.$$

Thus U is obtained from X^{an} by deleting finitely many open disks and open anuli. By [BPR13, Lemma 4.12], the result is an affinoid subdomain of X^{an} (or all of X^{an} if the number of deleted disks and anuli is zero). ∎

2.2 Valuative functions

Definition 2.7. A *valuative function* is a function of the form

$$\widehat{f}_\alpha : X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}, \qquad \xi \mapsto \alpha v_\xi(f),$$

where $f \in F_X^\times$ is a non-zero rational function, and where $\alpha \in \mathbb{Q}$ is a rational number.

If $\alpha = 1$, we will denote this function simply by $\widehat{f} := \widehat{f}_1$.

Lemma 2.8. *Every valuative function $\widehat{f}_\alpha : X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ is admissible. It is trivialized by every skeleton $\Gamma \subset X^{\mathrm{an}}$ containing all zeros and poles of f.*

Proof. This follows from [BPR13, Theorem 5.15]. ∎

Lemma 2.9. *Let $f_1, \ldots, f_r \in F_X^\times$ be non-zero rational functions, and let $\alpha_1, \ldots, \alpha_r \in \mathbb{Q}$ be rational numbers. Then the admissible function $h := \alpha_1 \widehat{f}_1 + \ldots + \alpha_r \widehat{f}_r$ (see Remark 2.3) is a valuative function.*

For all points $\xi \in X^{\mathrm{an}}$ that are not poles or zeros of any of the $f_1, \ldots, f_r$, we have

$$h(\xi) = \alpha_1 \widehat{f}_1(\xi) + \ldots + \alpha_r \widehat{f}_r(\xi).$$

Proof. The equality $h(\xi) = \sum_i \alpha_i \widehat{f}_i(\xi)$ follows from Lemma 2.2, since the poles and zeros of f_i are exactly the points where $\widehat{f}_i(\xi) = \pm\infty$.

To show that h is a valuative function, choose a common denominator d for the $\alpha_1, \ldots, \alpha_r$ and write $\alpha_i = \nu_i/d$ for $i = 1, \ldots, r$. Define the function

$$f := \prod_{i=1}^{r} f_i^{\nu_i}.$$

Then the valuative function $\widehat{f}_{1/d}$ satisfies

$$\widehat{f}_{1/d}(\xi) = \frac{v_\xi(f)}{d} = \frac{\big(\nu_1 v_\xi(f_1) + \ldots + \nu_r v_\xi(f_r)\big)}{d} = \alpha_1 \widehat{f}_1(\xi) + \ldots + \alpha_r \widehat{f}_r(\xi) = h(\xi)$$

for all ξ that are not poles or zeros of any f_i. By the uniqueness part of Lemma 2.2, we must have $h = \widehat{f}_{1/d}$. ∎

We will often define a valuative function by a linear combination such as

$$\alpha_1 \widehat{f}_1 + \ldots + \alpha_r \widehat{f}_r,$$

which is justified by Lemma 2.9.

The following example illustrates the subtlety of defining a sum of valuative functions, and how it can be overcome using the trick in the proof of Lemma 2.9.

Example 2.10. Consider the curve $X = \mathbb{P}^1_K$ and the rational functions $f = x/(x-1)$ and $g = (x-1)^2$. The valuative function

$$h := \widehat{f} + \widehat{g}$$

may then be defined in terms of the product $fg = x(x-1)$; it is the function

$$(\mathbb{P}^1_K)^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}, \qquad \xi \mapsto v_\xi\big(x(x-1)\big).$$

It is well-defined at all points $\xi \in (\mathbb{P}^1_K)^{\mathrm{an}}$. For example, $h(0) = h(1) = \infty$. However, we have

$$\widehat{f}(1) = -\infty, \qquad \widehat{g}(1) = \infty,$$

and it is not possible to derive the value $h(1)$ from this.

Let $\varphi\colon Y \to X$ be a finite separable morphism of curves. We consider the norm map of the field extension of function fields F_Y/F_X,

$$\mathrm{Nm}\colon F_Y \to F_X.$$

To end this section, we will study how to relate valuative functions on Y^{an} and valuative functions on X^{an} using the norm map.

Definition 2.11. Let $h := \widehat{f}_\alpha \colon Y^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ be a valuative function. The *norm* of h, denoted $\mathrm{Nm}\, h$, is defined to be the valuative function

$$(\widehat{\mathrm{Nm}\, f})_\alpha \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}.$$

Remark 2.12. Let $f_1, \ldots, f_r \in F_Y^\times$ be rational functions and let $\alpha_1, \ldots, \alpha_r$ be rational numbers. It is an easy consequence of the multiplicativity of the map Nm that the norm $\mathrm{Nm}\, h$ of the valuative function $h := \alpha_1 \widehat{f}_1 + \ldots + \alpha_r \widehat{f}_r$ is the valuative function

$$\alpha_1(\widehat{\mathrm{Nm}\, f_1}) + \ldots + \alpha_r(\widehat{\mathrm{Nm}\, f_r}).$$

Now assume that $\deg(f) = p$ (the case that we will be concerned with in subsequent chapters) and that $\xi \in X^{\mathrm{an}}$ is a point with p distinct preimages, say $\varphi^{-1}(\xi) = \{\eta_1, \ldots, \eta_p\}$. Said differently, the valuation v_ξ has p distinct extensions $v_{\eta_1}, \ldots, v_{\eta_p}$ to F_Y. Let furthermore $h \colon Y^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ be a valuative function.

Lemma 2.13. *We have*

$$(\mathrm{Nm}\, h)(\xi) = \sum_{i=1}^{p} h(\eta_i).$$

Proof. This is obvious in case $h = \widehat{f}_\alpha$, where $f \in F_X^\times$. Indeed, in this case $\mathrm{Nm}\, f = f^p$ and $v_\xi(f) = v_{\eta_i}(f)$ for $i = 1, \ldots, p$.

Otherwise we have $h = \widehat{f}_\alpha$ for a function $f \in F_Y^\times \backslash F_X^\times$. Then f is necessarily a generator of the field extension of function fields F_Y/F_X. By [Neu99, Theorem II.8.2], the extensions of v_ξ to F_Y are obtained as follows. There are p embeddings of F_Y into an algebraic closure $\overline{(F_X)_{v_\xi}}$ of the completion $(F_X)_{v_\xi}$. The p extensions $v_{\eta_1}, \ldots, v_{\eta_p}$ are obtained by restricting the unique extension of v_ξ to $\overline{(F_X)_{v_\xi}}$ to F_Y along these p embeddings.

If we denote the images of f under the p embeddings $F_Y \to \overline{(F_X)_{v_\xi}}$ by $f_1, \ldots, f_p$, then because F_Y/F_X is separable of degree p we have $\mathrm{Nm}\, f = f_1 \cdots f_p$. Now everything follows by taking valuations in this equality. ∎

2.3 The function μ

Let Γ_0 be a skeleton of $\mathbb{P}^1_K$ containing the point ∞. For every closed point $x_0 \in (\mathbb{P}^1_K)^{\mathrm{an}} \backslash \Gamma_0$, the connected component of $(\mathbb{P}^1_K)^{\mathrm{an}} \backslash \Gamma_0$ containing x_0 is an open disk; let us denote it by $\mathsf{D}(x_0)$. We define the quantity

$$\mu_{\Gamma_0}(x_0) := \text{the radius of the disk } \mathsf{D}(x_0).$$

We will now derive a simple formula for $\mu_{\Gamma_0}(x_0)$ in the case that Γ_0 is the skeleton spanned by the zero set of a polynomial f and the point ∞. That is, Γ_0 is spanned by the set of Type I points

$$\{\infty\} \cup \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(f) = \infty\}$$

for a polynomial

$$f = \sum_{i=0}^{d} a_i x^i \in K[x].$$

Suppose first that x_0 is the closed point $x = 0$, corresponding to the pseudovaluation "evaluation at 0",

$$g \mapsto v_{x_0}(g) = v_K(g(0)).$$

The open disk of radius r centered at x_0 is contained in the complement $(\mathbb{P}^1_K)^{\mathrm{an}} \backslash \Gamma_0$ if and only if no zero of f has valuation greater than r. It follows from the theory of the Newton polygon that

$$\mu_{\Gamma_0}(x_0) = \max_{i \geqslant 1} \frac{v_K(a_0) - v_K(a_i)}{i}. \tag{2.3}$$

(A brief reminder on Newton polygons may be found in Section 3.2. See in particular Proposition 3.17 for the fact about the valuations of zeros of a polynomial we used here.) Notice that (2.3) makes sense because x_0 being contained in the complement of Γ_0 we have $v_K(a_0) < \infty$.

In the following lemma we generalize the above argument, which allows us to make sense of the formula (2.3) for all points $\xi \in X^{\mathrm{an}}$, including points contained in the skeleton Γ_0.

Lemma 2.14. *For every closed point $x_0 \in X^{\mathrm{an}} \backslash \Gamma_0$ we have*

$$\mu_{\Gamma_0}(x_0) = \max_{i \geqslant 1} \frac{\widehat{f}_0(x_0) - \widehat{f}_i(x_0)}{i}, \tag{2.4}$$

where the f_i are the polynomials

$$f_i = \frac{f^{(i)}}{i!} \in K[x], \qquad i \geqslant 0.$$

In particular, μ_{Γ_0} may be extended to an admissible function on X^{an}.

Proof. We may regard x_0 as an element of K (which is not a zero of the polynomial f). The Taylor expansion of f around x_0 reads

$$f(x_0 + t) = \sum_{i=0}^{d} f_i(x_0) t^i.$$

Now the open disk of radius r centered at x_0 is contained in the complement $(\mathbb{P}^1_K) \backslash \Gamma_0$ if and only if the polynomial $f(x_0 + t)$ has no root of valuation larger than r. By the theory of the Newton polygon, this is the case if and only if

$$r \geqslant \max_{i \geqslant 1} \frac{v_K(f_0(x_0)) - v_K(f_i(x_0))}{i}.$$

Thus (2.4) holds. The function

$$\max_{i \geqslant 1} \frac{\widehat{f}_0 - \widehat{f}_i}{i} \colon \quad X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$$

is admissible by Lemma 2.2, so the last statement of the lemma immediately follows. ■

Remark 2.15. (a) By Lemma 2.14 we can and will regard μ_{Γ_0} as a function

$$\mu_{\Gamma_0} \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}.$$

If the skeleton Γ_0 is clear from context, we will simply write $\mu := \mu_{\Gamma_0}$.

(b) The only pole of the function μ_{Γ_0} is ∞. This is because the function $\widehat{f}_d$ (where $d = \deg(f)$) is constant of finite value, so that at least one term appearing in the maximum in (2.4) is $> -\infty$ for any closed point $x_0 \neq \infty$.

2.4 The function δ

Let $\varphi \colon Y \to X$ be a separable degree-p morphism of K-curves. In [CTT], Cohen, Temkin, and Trushin define a *different function*

$$\delta_\varphi \colon Y^{\mathrm{an}} \to [0, 1].$$

Unlike the authors of [CTT] we prefer to work with additive valuations rather than with multiplicative absolute values. Just as valuations and absolute values are related by logarithms and exponentiation, so is the different function used by us to the one of [CTT]. Denoting the different function of [CTT] by $\delta_\varphi^{\mathrm{CTT}}$ and the different function used by us by $\delta_\varphi^{\mathrm{add}}$, we have the defining relations

$$\delta_\varphi^{\mathrm{CTT}} = \exp\Big(-\frac{p-1}{p}\delta_\varphi^{\mathrm{add}}\Big), \qquad \delta_\varphi^{\mathrm{add}} = -\frac{p}{p-1}\log(\delta_\varphi^{\mathrm{CTT}}).$$

Thus $\delta_\varphi^{\mathrm{add}}$ is a function

$$\delta_\varphi^{\mathrm{add}} \colon Y^{\mathrm{an}} \to [0, \infty].$$

The function $\delta_\varphi^{\mathrm{CTT}}$ is piecewise monomial, so by Remark 2.5 the function $\delta_\varphi^{\mathrm{add}}$ is piecewise affine. The reason for the factor $p/(p-1)$ in the definition of $\delta_\varphi^{\mathrm{add}}$ will become apparent in the next section; see in particular Proposition-Definition 2.27. If φ is a Galois cover, then this normalization makes $\delta_\varphi^{\mathrm{add}}$ equal to the *depth Swan conductor* in the sense of Kato ([Bre09, Definition 1.5.2]). See [OW16, Section 2.3] and in particular [OW16, Lemma 2.3] for a comparison of this Swan conductor and the different function of [CTT].

From now on, we will only write δ_φ instead of $\delta_\varphi^{\mathrm{add}}$. When φ is clear from context, we will simply denote this function by δ.

Remark 2.16. δ_φ never takes the value ∞ at points not of Type I. The behavior of δ_φ at Type I points, i. e. closed points in X, is explained in [CTT, Theorem 4.6.4]. At Type I points that are not totally ramified with respect to φ, the function δ_φ vanishes. If x_0 is totally ramified, then $\delta_\varphi(x_0) > 0$. In fact, $\delta_\varphi(x_0) = \infty$ if $\mathrm{char}(K) = p$ and δ_φ is constant in a neighborhood of x_0 otherwise.

Lemma 2.17. *Let $\eta \in Y^{\mathrm{an}}$ be a point of Type II with image $\xi := \varphi(\eta)$.*

(a) If $\delta_\varphi(\eta) > 0$, then η is a wild topological ramification point. In particular, we have $\varphi^{-1}(\xi) = \{\eta\}$.

(b) If $\delta_\varphi(\eta) = 0$ and η is a wild topological ramification point, then the extension of completed residue fields $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ is unramified of degree p.

Proof. If $\delta_\varphi(\eta) > 0$, then by [CTT, Lemma 4.2.2] the extension of completed residue fields $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ is wildly ramified. It follows from Remark 1.26 that the ramification index $[\Gamma_{v_\eta} : \Gamma_{v_\xi}]$ equals 1 and that $[\mathcal{H}(\eta) : \mathcal{H}(\xi)] = [\kappa(\eta) : \kappa(\xi)]$. Since $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ is wildly ramified, this degree must equal p. Thus η is a wild topological ramification point, proving (a).

Now suppose that $\delta_\varphi(\eta) = 0$ and that η is a wild topological ramification point. Again by [CTT, Lemma 4.2.2], it follows that $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ is not wildly ramified. But by assumption, this extension is of degree p. As above, the ramification index $[\Gamma_{v_\eta} : \Gamma_{v_\xi}]$ equals 1, so $\mathcal{H}(\eta)/\mathcal{H}(\xi)$ cannot be tamely ramified. Since it is not wildly ramified, it is unramified. This proves (b). ∎

Since φ has degree p, it is possible to regard δ_φ as a function not only on Y^{an}, but also on X^{an}. This is because $\delta_\varphi(\eta) = 0$ for every point $\eta \in Y^{\mathrm{an}}$ outside the wild topological ramification locus ([CTT, Section 4.1.1]). On the other hand, if $\xi \in X^{\mathrm{an}}$ is contained in the wild topological branch locus, then $\varphi^{-1}(\xi)$ contains only one point. Thus it makes sense to define

$$\delta_\varphi(\xi) := \delta_\varphi(\eta) \qquad \text{whenever } \varphi(\eta) = \xi.$$

However, some care is needed when considering δ_φ as a function on both Y^{an} and X^{an}, because the slopes of δ_φ at a point $\eta \in Y^{\mathrm{an}}$ and at $\xi := \varphi(\eta)$ will not agree. The following example illustrates this.

Example 2.18. Consider the Kummer cover

$$\varphi\colon \quad \mathbb{P}^1_K = Y \to X = \mathbb{P}^1_K, \qquad x \mapsto x^p.$$

of curves over $K = \mathbb{C}_p$. The function $\delta_\varphi\colon Y^{\mathrm{an}} \to [0, \infty)$ is constant with value $p/(p-1)$ on the interval I connecting the closed points $x = 0$ and ∞; and if for an interval $J = [\eta, \eta']$ of length $1/(p-1)$ with $J \cap I = \{\eta'\}$ we choose a radius parametrization $J \cong [0, 1/(p-1)]$, then $\delta|_J$ corresponds to the function

$$[0, 1/(p-1)] \to [0, \infty), \qquad t \mapsto pt,$$

(cf. [CTT, Remark 4.2.3(ii)]). Thus δ increases with slope p on J.

Let us fix such an interval $J = [\eta, \eta']$, say the one with η' corresponding to the Gauss valuation centered at $x = 0$ of radius 0 and with η corresponding to the Gauss valuation centered at $x = 1$ of radius $1/(p-1)$. Then $\xi' := \varphi(\eta')$ corresponds to the Gauss valuation centered at $x = 0$ of radius 0 as well, but $\xi := \varphi(\eta)$ corresponds to the Gauss valuation centered at $x = 1$ of radius $p/(p-1)$. Thus the interval $\varphi(J) = [\xi', \xi]$ has length $p/(p-1)$, and the slope of δ considered as a function on $\varphi(J)$ is 1.

In general, we have the following lemma. In its statement, we consider a pair of skeletons $\Gamma \subset X^{\mathrm{an}}$ and $\Sigma \subset Y^{\mathrm{an}}$, where Σ is defined as the inverse image of Γ under φ. By this we mean that we have an equality both of the underlying sets $\varphi^{-1}(\Gamma) = \Sigma$ and of the vertex sets $\varphi^{-1}(S(\Gamma)) = S(\Sigma)$.

Lemma 2.19. *Let $\Gamma \subset X^{\mathrm{an}}$ and $\Sigma := \varphi^{-1}(\Gamma) \subset Y^{\mathrm{an}}$ be skeletons. Suppose that $\eta_1, \eta_2 \in S(\Sigma)$ are adjacent vertices with images $\xi_1 = \varphi(\eta_1)$, $\xi_2 = \varphi(\eta_2)$. Then the topological ramification index is constant on (η_1, η_2), say equal to n, and*

$$\ell((\xi_1, \xi_2)) = n\ell((\eta_1, \eta_2)).$$

Proof. This follows from [CTT, Lemma 3.5.8], because the retractions $\mathrm{ret}_\Sigma^{-1}((\eta_1, \eta_2))$ and $\mathrm{ret}_\Gamma^{-1}((\xi_1, \xi_2))$ are open anuli (compare the proof of Proposition 1.43). ∎

An important tool for understanding the behavior of the function δ is the "genus formula for wide open domains" of [CTT, Section 6.2.6]. We will state this result in Proposition 2.20 below, adapted to our notation and applied to the morphism $\varphi\colon Y \to X$ of algebraic curves we fixed at the beginning of this section. A *wide open domain* on the analytification X^{an} of an algebraic curve X is the same as the inverse image under the reduction map $\mathrm{red}_{\mathcal{X}}$ associated to a model $\mathcal{X}$ of X of a closed point on the special fiber $\mathcal{X}_s$, say

$$\mathsf{D} = \mathrm{red}_{\mathcal{X}}^{-1}(\overline{x})$$

(cf. [CTT, Remark 6.2.5(i)]). Denote the normalization of the model $\mathcal{X}$ in the function field F_Y by $\mathcal{Y}$; it is a model of Y. Let us write $\{\overline{y}_1, \ldots, \overline{y}_r\}$ for the inverse image of $\overline{x}$ under the induced map on special fibers $\mathcal{Y}_s \to \mathcal{X}_s$. Then the inverse image of $\mathsf{D} = \mathrm{red}_{\mathcal{X}}^{-1}(\overline{x})$ under φ is the disjoint union of wide open domains

$$\varphi^{-1}(\mathsf{D}) = \mathsf{C}_1 \amalg \ldots \amalg \mathsf{C}_r, \qquad \mathsf{C}_i = \mathrm{red}_{\mathcal{Y}}^{-1}(\overline{y}_i).$$

Given a wide open domain $\mathsf{D} = \mathrm{red}_{\mathcal{X}}^{-1}(\overline{x})$, let us denote by $\xi_1, \ldots, \xi_m$ the generic points of the components of $\mathcal{X}_s$ on which $\overline{x}$ lies. For example, if $\mathcal{X}$ is semistable, then $m = 1$ (if $\overline{x}$ is a smooth point) or $m = 2$ (if $\overline{x}$ is an ordinary double point). Each of the ξ_i, $i = 1, \ldots, m$, corresponds to a Type II point on X^{an}, which we also denote ξ_i. Following [CTT, Section 6.2.6], we denote by D_∞ the set of all branches at any of the points $\xi_1, \ldots, \xi_m$ pointing inside of D.

The statement of the genus formula below also includes the *genera* of wide open domains. The genus of a wide open domain D is by definition ([CTT, Section 6.2.4]) the quantity

$$g(\mathsf{D}) = h^1(\mathsf{D}) + \sum_{\xi\in\mathsf{D}} g(\xi).$$

That is, $g(\mathsf{D})$ is the sum of the first Betti number of D (essentially the number of loops in the topological space D) and the sum of the genera of all points of positive genus contained in D.

Proposition 2.20. *Suppose that $\mathsf{D} \subseteq X^{\mathrm{an}}$ is a wide open domain and that $\mathsf{C} \subseteq Y^{\mathrm{an}}$ is a connected component of $\varphi^{-1}(\mathsf{D})$. Write n for the degree of the induced morphism $\mathsf{C} \to \mathsf{D}$ and n_v for the topological ramification index of φ at a branch v (as defined in Section 1.5). Then we have*

$$2g(\mathsf{C}) - 2 - n(2g(\mathsf{D}) - 2) = \sum_Q d_Q + \sum_{v\in\mathsf{C}_\infty} n_v - 1 - \frac{p-1}{p}\,\mathrm{slope}_v(\delta),$$

where the first sum is over all ramification points Q of φ contained in C, and where d_Q is the differential length $d_Q = \mathrm{len}(\Omega_{Y/X,Q})$.

Proof. This is [CTT, Theorem 6.2.7]. The quantity R_Q appearing in loc. cit. is replaced with the differential length d_Q, as per [CTT, Theorem 4.6.4]. If $\mathrm{char}(K) = 0$, it simply equals $e_Q - 1$, where e_Q is the ramification index of Q over $\varphi(Q)$.

The slopes $\mathrm{slope}_v(\delta)$ appear with the opposite sign in our version, as explained in Remark 2.5. ∎

From now on we assume that $X = \mathbb{P}^1_K$. The following example illustrates the kind of insight into the structure of Y^{an} the genus formula then gives us.

Example 2.21. Let $x_0 \in X^{\mathrm{an}}$ be a closed point different from ∞ and let D be an open disk of rational radius centered at x_0. It is a wide open domain. Let us assume that $\delta(\xi) > 0$, where $\xi \in X^{\mathrm{an}}$ is the boundary point of D. Then ξ is a wild topological branch point and the inverse image $\mathsf{C} := \varphi^{-1}(\mathsf{D})$ is connected. The degree of the induced morphism $\mathsf{C} \to \mathsf{D}$ equals p, and so does the topological ramification index of φ at the unique branch $v \in \mathsf{C}_\infty$. Thus Proposition 2.20 implies

$$\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 1 - p - 2g(\mathsf{C}) + \sum_Q d_Q,$$

where the sum is over the ramification points of φ in C. We see that the slope of δ is the steeper the more branch points of φ are contained in D. And the slope is the less steep the larger the genus of C is. Though we will not use this explicitly, we note that there is a simple formula for the genus of a wide open domain in terms of the special fiber $\mathcal{X}_s$, where $\mathcal{X}$ is the model of X such that the wide open domain is defined using the reduction map $\mathrm{red}_{\mathcal{X}}$ ([CTT, Remark 6.2.5(ii)]).

Gradually increasing the radius of D will decrease the number of branch points in D, which changes the slope of δ in a predictable way. Thus knowledge of δ would allow us to compute $g(\mathsf{C})$ and hence determine if any points of positive genus lie above D. We will return to this approach in Section 4.6.

Proposition 2.22. *Let Γ_0 be a skeleton of $\mathbb{P}^1_K$ containing the branch points of φ. Then for every point $\eta \in Y^{\mathrm{an}} \backslash \varphi^{-1}(\Gamma_0)$ of positive genus (necessarily of Type II), there exists a unique path $[\eta, \eta_1]$ with $[\eta, \eta_1] \cap \varphi^{-1}(\Gamma_0) = \{\eta_1\}$.*

Moreover, δ is strictly positive on $(\eta, \eta_1]$.

Proof. Let η be a point as in the statement of the proposition and let $\xi := \varphi(\eta)$ be its image in $(\mathbb{P}^1_K)^{\mathrm{an}}$. Because $(\mathbb{P}^1_K)^{\mathrm{an}}$ is uniquely path-connected, there is a unique path $[\xi, \xi_1]$ with $[\xi, \xi_1] \cap \Gamma_0 = \{\xi_1\}$. It suffices to show that δ is strictly positive on $(\xi, \xi_1]$, because then there will also be a unique path connecting η and η_1.

To this end, let us consider an arbitrary Type II point $\xi_2 \in (\xi, \xi_1)$. Let D be the open disk containing ξ with boundary point ξ_2 and let C be a connected component of $\varphi^{-1}(\mathsf{D})$. The genus formula for wide open domains of Proposition 2.20 applied to C reads

$$2g(\mathsf{C}) - 2 + 2n = \sum_{v \in \mathsf{C}_\infty} n_v - 1 - \frac{p-1}{p} \operatorname{slope}_v(\delta),$$

where n is the degree of the induced morphism $\mathsf{C} \to \mathsf{D}$. Since $\sum_{v \in C_\infty} n_v = n$ (Remark 1.27), it follows that

$$0 < 2g(\mathsf{C}) \leqslant 1 - n - \sum_{v \in \mathsf{C}_\infty} \frac{p-1}{p} \operatorname{slope}_v(\delta),$$

which is only possible if $\sum_{v \in \mathsf{C}_\infty} \operatorname{slope}_v(\delta) < 0$. This implies in particular that δ is strictly positive on the interval (ξ_2, ξ_1). Varying ξ_2 we obtain as desired that δ is strictly positive on $(\xi, \xi_1]$. ∎

Lemma 2.23. *If Γ_0 is a skeleton of $X = \mathbb{P}^1_K$ containing the branch points of φ, then every tame topological branch point $\xi \in X^{\mathrm{an}}$ of φ is contained in Γ_0. Moreover, if $\xi \in X^{\mathrm{an}} \backslash \Gamma_0$ is a wild topological branch point with $\delta(\xi) = 0$, then $\operatorname{slope}_v(\delta) > 0$, where v is the branch at ξ pointing towards Γ_0.*

Proof. This is another application of the genus formula for wide open domains. The argument is a variation of the one given in [CTT, Lemma 6.3.2].

Suppose that $\xi \in X^{\mathrm{an}} \backslash \Gamma_0$ is *not* a wild topological branch point such that $\operatorname{slope}_v(\delta) > 0$ for the branch v at ξ pointing towards Γ_0. We will then show that ξ is not a topological branch point. We have $\operatorname{slope}_v(\delta) = 0$, by assumption, if ξ is a wild topological branch point, and because the wild topological branch locus is closed (Remark 1.29(c)) otherwise. Thus we may choose an open disk containing ξ contained in the complement $X^{\mathrm{an}}\backslash\Gamma_0$ such that $\mathrm{slope}_u(\delta) = 0$ for the unique branch $u \in \mathsf{D}_\infty$. Let C be a component of $\varphi^{-1}(\mathsf{D})$. The genus formula for wide open domains applied to C reads

$$2g(\mathsf{C}) - 2 + 2n = \sum_{v \in \mathsf{C}_\infty} n_v - 1,$$

where n is the degree of the induced morphism $\mathsf{C} \to \mathsf{D}$. Since $\sum_{v \in \mathsf{C}_\infty} n_v = n$, it follows that

$$2g(\mathsf{C}) \leqslant 1 - n \leqslant 0.$$

This is only possible if $n = 1$ and $g(\mathsf{C}) = 0$. Thus $\mathsf{C} \to \mathsf{D}$ is an isomorphism. Since the component C was arbitrary, $\xi \in \mathsf{D}$ is not a topological branch point. ∎

2.5 The function λ

We continue the notation from the previous section. Thus $\varphi\colon Y \to X = \mathbb{P}^1_K$ is a separable degree-p morphism of K-curves. Let $\Gamma_0 \subset X^{\mathrm{an}}$ be the tree spanned by a finite set of Type I points which includes the branch locus of φ and the point ∞. In this section we will define an admissible function

$$\lambda\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$$

that will be very important in the subsequent discussion. To define λ we will need the following construction.

Let $\Sigma \subset Y^{\mathrm{an}}$ be the graph obtained from $\Sigma_0 := \varphi^{-1}(\Gamma_0)$ by

1) adding to the vertex set $S(\Sigma_0)$ all points of positive genus on Σ_0,
2) attaching to Σ_0 all paths $[\eta, \eta_1]$, where $\eta \in Y^{\mathrm{an}}\backslash\Sigma_0$ is a point of positive genus and $\eta_1 \in \Sigma_0$.

As in the previous section, by the inverse image of Γ_0 under φ we mean the graph with underlying set $\varphi^{-1}(\Gamma_0)$ and vertex set $\varphi^{-1}(S(\Gamma_0))$. We write $\Gamma := \varphi(\Sigma)$. Note that the construction of Σ makes sense because of Proposition 2.22: For each positive genus point $\eta \in Y^{\mathrm{an}}\backslash\Sigma_0$, there exists a unique path connecting η and Σ_0, which consists of wild topological ramification points.

Proposition 2.24. *The graph Σ is the unique minimal skeleton of Y containing $\Sigma_0 = \varphi^{-1}(\Gamma_0)$.*

Proof. We first prove that Σ is a skeleton. We use [CTT, Theorem 6.3.4], according to which it suffices to show that the ramification points of φ are among the vertices of Σ and that Σ "locally trivializes" δ in the sense of [CTT, Section 6.3.3] — but see Remark 2.26 below. The first condition is true, because Γ_0 contains the branch locus of φ. The latter condition means that for every point $\eta \in \Sigma$ and every branch v at η pointing outside of Σ we have

$$\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 1 - n_v \tag{2.5}$$

(the sign here is opposite from the one in [CTT, Section 6.3.3], recall Remark 2.5). We now check this second condition. Let us write $\xi := \varphi(\eta)$ and denote the image of the branch v by u. Denote by D the open disk with boundary point ξ such that u points into D.

We claim that δ is monotonously increasing on every interval $I = [\xi', \xi] \subset \mathsf{D}$. To show this, fix such an interval $I = [\xi', \xi]$, and consider any interval $[\xi_1, \xi_2] \subset I$ on which δ is strictly positive. Then $[\xi_1, \xi_2]$ consists of wild topological branch points, so above the branch at ξ_2 pointing in the direction of ξ_1 there is a unique branch w. Let D_{ξ_2} denote the open disk with boundary point ξ_2 containing ξ_1 and let C_{ξ_2} be a component of $\varphi^{-1}(\mathsf{D}_{\xi_2})$. The genus formula for wide open domains, Proposition 2.20, applied to C_{ξ_2} reads

$$2g(\mathsf{C}_{\xi_2}) + 2p - 2 = p - 1 - \frac{p-1}{p}\,\mathrm{slope}_w(\delta).$$

It follows immediately that $\mathrm{slope}_w(\delta)$ is negative, proving our claim that δ is monotonously increasing on the interval $[\xi', \xi] \subset \mathsf{D}$.

Now we show that Σ locally trivializes δ, that is, we show that (2.5) holds. Let C be a component of $\varphi^{-1}(\mathsf{D})$. Suppose first that $\delta(\xi) = 0$. Then Lemma 2.23 and our claim about monotonicity of δ show that there are no topological branch points contained in D. Thus $\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 0 = 1 - n_v$.

Next, suppose that $\delta(\xi) > 0$. Then Lemma 2.23 and our claim show that the locus of topological branch points in D is non-empty, connected, adjacent to ξ, and consisting of *wild* topological branch points. In particular, there are no loops in C. Moreover, C contains no points of positive genus by construction of Σ, so $g(\mathsf{C}) = 0$. Now Proposition 2.20 applied to C shows that

$$2p - 2 = p - 1 - \frac{p-1}{p}\,\mathrm{slope}_v(\delta),$$

so $\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 1-p = 1-n_v$. Thus we have shown that Σ locally trivializes δ, so is a skeleton.

Finally, it is obvious that Σ is the unique minimal skeleton containing Σ_0. Indeed, we have only added to Σ_0 points of positive genus and the unique paths connecting them to Σ_0. ■

Definition 2.25. The skeleton Σ is called the *φ-minimal skeleton* of Y with respect to Γ_0.

Remark 2.26. Adding to the vertex set of Σ the points of positive genus lying on $\Sigma_0 = \varphi^{-1}(\Gamma_0)$ is necessary to ensure that Σ is a skeleton. This condition is missing in [CTT, Theorem 6.3.4]. Using our notation, it is asserted there that a graph containing Σ_0 and locally trivializing δ is already a skeleton. For a concrete example where this is not the case, see Example 5.16.

We now define the function λ that is the subject of this section, in terms of the functions δ_φ (Section 2.4), r_{Γ_0} (Definition 1.41), and μ_{Γ_0} (Section 2.3).

Proposition-Definition 2.27. *There exists a unique piecewise affine function $\lambda_{\varphi,\Gamma_0}\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following:*

(a) If $\xi \in X^{\mathrm{an}}\backslash\{\infty\}$ lies in Γ_0 or ξ is a topological branch point, then

$$\lambda_{\Gamma_0,\varphi}(\xi) = \delta_\varphi(\xi) + r_{\Gamma_0}(\xi) + \mu_{\Gamma_0}(\xi). \tag{2.6}$$

(b) It is locally constant on the locus

$$\{\xi \in X^{\mathrm{an}} \mid \xi \notin \Gamma_0 \textit{ and } \xi \textit{ is not a topological branch point}\}.$$

Proof. Suppose that $\xi \in X^{\mathrm{an}}\backslash\{\infty\}$ lies in Γ_0 or is a topological branch point. Then $\lambda_{\Gamma_0,\varphi}(\xi)$ is unambiguously determined by (2.6), except perhaps if several of $\delta_\varphi(\xi), r_{\Gamma_0}(\xi), \mu_{\Gamma_0}(\xi)$ equal $\pm\infty$. However, the only pole of μ_{Γ_0} is by Remark 2.15 given by $\xi = \infty$, and we have $\delta_\varphi(\xi), r_{\Gamma_0}(\xi) > -\infty$ for all $\xi \in X^{\mathrm{an}}$. Thus we may use (2.6) to define $\lambda_{\Gamma_0,\varphi}(\xi)$ if $\xi \neq \infty$.

Moreover, there clearly exists a unique value $\lambda_{\Gamma_0,\varphi}(\infty)$ compatible with $\lambda_{\Gamma_0,\varphi}$ being a piecewise affine function; the argument is the same as in Lemma 2.2. Note that in the case $\mathrm{char}(K) = 0$, which is most important to us, we have $\delta_\varphi(\infty) < \infty$ by [CTT, Theorem 4.6.4(b)]. So in this case we can also compute $\lambda_{\Gamma_0,\varphi}(\infty)$ using (2.6).

In any case, imposing the condition (2.6) is no issue, and it remains to show that we may extend λ to all of X^{an} in a way compatible with condition (b). We begin by showing that the set

$$R := \Gamma_0 \cup \{\xi \in X^{\mathrm{an}} \mid \xi \text{ is a topological branch point}\}$$

is connected.

Choose a skeleton Γ of X containing Γ_0 and such that $\Sigma := \varphi^{-1}(\Gamma)$ is a skeleton of Y. It follows from [CTT, Theorem 7.1.4] that the topological ramification locus of φ is a certain "radial set". It is the set obtained by adding to

$$S := \{\eta \in \Sigma \mid \eta \text{ is a topological ramification point}\}$$

all closed intervals $[\eta, \eta']$ where $\eta \in S$, $\eta' \notin \Sigma$, and the length of the interval $[\eta, \eta']$ is $\delta(\eta)/p$. (In [CTT], the length of these intervals is instead $\delta(\eta)/(p-1)$. As usual, the discrepancy is because of our different normalization of δ. See Example 2.18 for an explicit example showing the intervals of length $\delta(\eta)/p$.)

According to Proposition 2.24, a skeleton Γ as above is obtained from Γ_0 by adding to Γ_0 finitely many intervals consisting of topological branch points of φ; then $\Sigma = \varphi^{-1}(\Gamma)$ is the φ-minimal skeleton of Y with respect to Γ_0. Thus

$$\varphi^{-1}(\Gamma_0) \cup \{\eta \in Y^{\mathrm{an}} \mid \eta \text{ is a topological ramification point}\}$$

equals the union of Σ and the topological ramification locus of φ, and is therefore connected. The image of this set under φ is the set R used in the definition of λ; we have now shown that this set R is connected.

Now it follows easily from the fact that X^{an} is uniquely path-connected that λ is well-defined. For every point $\xi \in X^{\mathrm{an}} \backslash R$ there exists a unique path $[\xi, \xi']$ with $\xi' \in R$, but $\xi'' \notin R$ for any $\xi'' \in (\xi, \xi')$, and we define $\lambda_{\varphi,\Gamma_0}(\xi) = \lambda_{\varphi,\Gamma_0}(\xi')$. ■

Remark 2.28. Since φ and Γ_0 will usually be fixed and clear from context, we will often simply write $\lambda := \lambda_{\varphi,\Gamma_0}$ in the sequel.

Remark 2.29. Lemma 2.23 allows us to replace "topological branch point" in the definition of λ with "*wild* topological branch point". That is, $\lambda(\xi)$ is defined by (2.6) if $\xi \in \Gamma_0$ or ξ is a *wild* topological branch point, and λ is locally constant on the locus consisting of points that are neither contained in Γ_0 nor *wild* topological branch points.

Lemma 2.30. *The function* $\lambda\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ *is admissible.*

Proof. Let Γ and Σ denote the skeletons from the proof of Proposition-Definition 2.27. That is, Σ is the φ-minimal skeleton of Y with respect to Γ_0 and $\Gamma = \varphi(\Sigma)$. We will show that λ is locally constant on the complement $X^{\mathrm{an}} \backslash \Gamma$. Let $\xi \in X^{\mathrm{an}} \backslash \Gamma$ be any point, and let v be the unique branch at ξ pointing in the direction of Γ. We want to show that $\mathrm{slope}_v(\lambda) = 0$.

This is obvious from the definition of λ if ξ is not a topological branch point. We consider the remaining case that ξ is a topological branch point not contained in Γ (necessarily wild, by Lemma 2.23). Since $\xi \notin \Gamma_0$, we have

$$\mathrm{slope}_v(\mu_{\Gamma_0} + r_{\Gamma_0}) = \mathrm{slope}_v(\mu_{\Gamma_0}) + \mathrm{slope}_v(r_{\Gamma_0}) = 0 - 1 = -1.$$

Denote by η the unique point in the preimage of ξ and denote by w the unique branch above v. It follows from [CTT, Theorem 7.1.4] that δ (when considered as a function on Y^{an}) satisfies $\mathrm{slope}_w(\delta) = p$. By Lemma 2.19 it follows (now considering δ as a function on X^{an}) that $\mathrm{slope}_v(\delta) = 1$. Adding up the slopes of δ, of r_{Γ_0}, and of μ_{Γ_0} immediately shows $\mathrm{slope}_v(\lambda) = 0$.

Since δ, μ_{Γ_0}, and r_{Γ_0} are all piecewise affine, we can refine Γ so that λ is affine on each edge of the resulting skeleton. This proves that λ is admissible. ■

Next, we will study the value λ takes on closed points not contained in the skeleton Γ_0. Let us fix such a point $x_0 \in (\mathbb{P}^1_K)^{\mathrm{an}} \backslash \Gamma_0$. We introduce some notation associated to the point x_0 that will be used throughout.

For $r \in \mathbb{R}$, we denote by

$$\mathsf{D}(r) := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x - x_0) > r\} \tag{2.7}$$

the *open disk of radius* r centered at x_0 and by

$$\mathsf{D}[r] := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x - x_0) \geqslant r\} \tag{2.8}$$

the *closed disk of radius* r centered at x_0. We will denote the boundary point of $\mathsf{D}[r]$ by ξ_r and the valuation v_{ξ_r} simply by v_r. By convention, we set $\xi_\infty := x_0$. The valuation v_r is the Gauss valuation centered at x_0 of radius r, defined by

$$v_r\Big(\sum_i a_i(x-x_0)^i\Big) = \min_i(v_K(a_i)+ir).$$

The dependence on the closed point x_0 of $\mathsf{D}(r)$, $\mathsf{D}[r]$, v_r, and ξ_r is not reflected in the notation. This should not cause confusion, since we will usually keep one choice of x_0 fixed.

We begin with a simple formula for the value of λ at the closed point x_0.

Proposition 2.31. *We have*

$$\lambda(x_0) = \min\big\{r \geqslant \mu(x_0) \mid \delta(\xi_r) = 0\big\}.$$

Proof. Consider the unique path connecting x_0 and Γ_0; it is the path

$$[x_0, \xi_{\mu(x_0)}] = \{\xi_r \mid \mu(x_0) \leqslant r \leqslant \infty\}.$$

By Remark 2.16, we have $\delta(x_0) = 0$. If $\delta(\xi) = 0$ for every point ξ in $[x_0, \xi_{\mu(x_0)}]$, then by Lemma 2.23 there are no topological branch points in the interval $[x_0, \xi_{\mu(x_0)})$. Thus λ is by its definition constant on the whole path, the value equal to $\lambda(\xi_{\mu(x_0)})$. Since $\xi_{\mu(x_0)} \in \Gamma_0$, this value is

$$\lambda(\xi_{\mu(x_0)}) = \delta(\xi_{\mu(x_0)}) + r_{\Gamma_0}(\xi_{\mu(x_0)}) + \mu(\xi_{\mu(x_0)}) = \mu(\xi_{\mu(x_0)}).$$

Suppose conversely that δ is not identically zero on this path, say that the radius r is chosen minimal among $r \geqslant \mu(x_0)$ with $\delta(\xi_r) = 0$. We have $r = r_{\Gamma_0}(\xi_r) + \mu(\xi_r)$. Thus

$$\lambda(x_0) = \lambda(\xi_r) = \delta(\xi_r) + r_{\Gamma_0}(\xi_r) + \mu(\xi_r) = r$$

as desired. ∎

In the following, we characterize the value of λ at $x_0 \in X^{\mathrm{an}}\backslash\Gamma_0$ as the smallest radius r for which we can guarantee that the open disk of radius r around x_0 splits into p disjoint open disks along φ.

Proposition 2.32. *Let $x_0 \in X^{\mathrm{an}}$ be a Type I point that is not contained in Γ_0. Let $\mathsf{D} := \mathsf{D}(\lambda(x_0))$ be the open disk of radius $\lambda(x_0)$ around x_0. Then $\varphi^{-1}(\mathsf{D})$ is a disjoint union of p open disks,*

$$\varphi^{-1}(\mathsf{D}) = \mathsf{C}_1 \amalg \ldots \amalg \mathsf{C}_p,$$

and for each $1 \leqslant i \leqslant p$, the induced morphism $\varphi|_{\mathsf{C}_i} : \mathsf{C}_i \to \mathsf{D}$ is an isomorphism.

Proof. It follows from Proposition 2.31 that $\delta = 0$ on D. Lemma 2.23 then implies that there are no topological branch points in D. By Remark 1.27, the restriction of φ to each component of $\varphi^{-1}(\mathsf{D})$ is a finite morphism of degree 1, that is, an isomorpism. Thus every component of $\varphi^{-1}(\mathsf{D})$ must be an open disk, mapped isomorphically to D by φ, and everything follows. ∎

Corollary 2.33. *Suppose that the skeleton Γ_0 is the tree spanned by ∞ and the branch points of φ. Then the radius $\lambda(x_0)$ is the smallest radius $r \in \mathbb{Q}$ for which $\varphi^{-1}(\mathsf{D}(r))$ is a disjoint union of p open disks.*

Proof. By Proposition 2.32, the inverse image of $\mathsf{D}(\lambda(x_0))$, and hence of $\mathsf{D}(r)$ for every $r \geqslant \lambda(x_0)$, is a disjoint union of p open disks. By Proposition 2.31, if $r < \lambda$, then we have $r < \mu(x_0)$ or $\delta(\xi_r) > 0$. In the first case, $\mathsf{D}(r)$ contains a branch point of φ, in the second case it contains a point ξ with $\delta(\xi) > 0$. In either case, $\varphi^{-1}(\mathsf{D})$ cannot be a disjoint union of p open disks. ∎

Theorem 2.34. *Suppose that Γ is a skeleton of X containing Γ_0 and such that λ is locally constant on $X^{\mathrm{an}}\backslash\Gamma$.*

If all points of positive genus on $\Sigma := \varphi^{-1}(\Gamma)$ are contained in the vertex set $S(\Sigma)$, then Σ is a skeleton of Y^{an}.

Proof. This is similar to the proof of Proposition 2.24. By [CTT, Theorem 6.3.4], it suffices to show that the ramification points of φ are among the vertices of Σ and that Σ "locally trivializes" δ in the sense of [CTT, Section 6.3.3] (and that the vertex set of Σ contains all points of positive genus, recall Remark 2.26).

By assumption, Γ_0 contains the branch locus of φ, so we need only show that for every point $\eta \in \Sigma$ and every branch v at η pointing outside of Σ

$$\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 1 - n_v.$$

We write $\xi := \varphi(\eta)$ and denote the image of the branch v by u. First suppose that $\delta(\eta) > 0$. Then $\varphi^{-1}(\xi) = \{\eta\}$. Since u points outside of Γ, the function $r_{\Gamma_0} + \mu_{\Gamma_0}$ increases with slope 1 in the direction of u. Since λ is locally constant along u, it follows that δ (considered as a function on X^{an}) decreases with slope 1 in the direction of u. By Lemma 2.19, δ considered as a function on Y^{an} decreases with slope p in the direction of v. Thus we have proved that $\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 1 - n_v$.

Next, suppose that $\delta(\eta) = 0$. Again, $r_{\Gamma_0} + \mu_{\Gamma_0}$ increases with slope 1 in the direction of u. Since $\delta \geqslant 0$, it cannot be the case that λ is defined using (2.6) along u. Rather, points close to ξ in the direction of u are not topological branch points, so we have

$$\frac{p-1}{p}\,\mathrm{slope}_v(\delta) = 0 = 1 - n_v$$

as desired. ∎

Chapter 3

An analytic expression for λ

Throughout this entire chapter, the ground field K is algebraically closed. We denote by X the projective line $\mathbb{P}^1_K$ over K. Moreover, we fix a K-curve Y (as usual assumed to be smooth, projective, and geometrically connected) and a finite separable morphism $\varphi\colon Y \to X$ of degree p. Our main goal is to find a characterization of the value at a closed point $x_0 \in X$ of the function λ associated to φ that was introduced in Section 2.5. By Theorem 2.34, knowledge of λ is intimately linked to the problem of finding a skeleton of Y.

We begin in Section 3.1 by introducing several notions and notations that we will need in the sequel. The main thrust of the chapter will then be concerned with studying suitable generators of the field extension F_Y/F_X and their minimal polynomials to obtain information on λ. In the final Section 3.6 of this chapter, we define the *tame locus* associated to φ, an affinoid subdomain of Y^{an} encoding the behavior of λ. It will be key for computing skeletons in concrete examples.

3.1 Preliminaries

For the following definition and for Lemma 3.4, it is not necessary to assume that $\varphi\colon Y \to X$ be of degree p.

Definition 3.1. An *étale affine chart* for φ (or simply a *chart*) is a pair (U, y), where $U \subseteq \mathbb{P}^1_K \backslash \{\infty\}$ is a non-empty open subscheme, and where y is a generator of the field extension of function fields F_Y/F_X, such that

(i) y is a regular function on $V := \varphi^{-1}(U)$, that is, $y \in \mathcal{O}_Y(V)$,

(ii) the morphism $V \to \mathbb{A}^1_U$ corresponding to the function y on the U-scheme V is a closed embedding, and

(iii) the induced morphism $\varphi|_V : V \to U$ is étale.

Example 3.2. Assume that $\mathrm{char}(K) \neq 2$ and let Y be the hyperelliptic curve with Weierstrass equation

$$Y\colon \quad y^2 = f(x), \qquad f(x) \in K[x].$$

Define $U := \mathbb{P}^1_K \backslash (\{\infty\} \cup B)$, where B is the set of zeros of f. Then (U, y) is an étale affine chart for the hyperelliptic map $\varphi\colon Y \to \mathbb{P}^1_K$, $(x, y) \mapsto x$.

Example 3.3. Let Y be a smooth plane curve and suppose that an affine chart on Y is given by $\operatorname{Spec} S \subset Y$, where

$$S \cong K[x, y]/(F), \qquad F \in K[x, y].$$

After possibly replacing x by $x + cy$ for a suitable $c \in K$, we may assume that S is finite over $K[x]$. (This is a special case of Noether normalization, see for example [Vak23, Sections 12.2.4–12.2.6].)

The corresponding finite morphism $\operatorname{Spec} S \to \mathbb{A}^1_K$ extends to a cover $\varphi\colon Y \to \mathbb{P}^1_K$. If we define $U := \mathbb{P}^1_K \backslash (\{\infty\} \cup B)$, where B is the set of branch points of φ, then (U, y) is an étale affine chart for φ.

Lemma 3.4. *Let $U \subseteq \mathbb{P}^1_K \backslash \{\infty\}$ be a non-empty open subscheme and let y be a generator of the field extension F_Y/F_X with minimal polynomial*

$$F(T) = T^p + f_{p-1}T^{p-1} + \ldots + f_1 T + f_0.$$

Then the pair (U, y) is an étale affine chart for φ if and only if the following hold:

(a) U is disjoint from the branch locus of φ

(b) The coefficients $f_{p-1}, \ldots, f_1, f_0$ are regular functions on U, that is, are contained in $\mathcal{O}_X(U)$

(c) The discriminant of the minimal polynomial F is an invertible function on U, that is, a unit in $\mathcal{O}_X(U)$

Proof. Assume first that the conditions (i), (ii), and (iii) from Definition 3.1 hold. Of course (iii) implies (a). Write $A := \mathcal{O}_X(U)$ and $B := \mathcal{O}_Y(V)$, so that B is the normalization of A in the function field F_Y. Since $y \in B$, its minimal polynomial F has coefficients in A, so (b) holds. Moreover, (i) and (ii) mean that we have the following commutative diagrams:

$$\begin{tikzcd} V \arrow[rr, hook] \arrow[dr, "\varphi|_V"'] & & \mathbb{A}^1_U \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} B & & A[T] \arrow[ll, two heads, "y \mapsfrom T"'] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It follows that $B \cong A[T]/(F)$. By [Neu99, Theorem III.2.6], the places in $\operatorname{Spec} A$ that ramify in $\operatorname{Spec} A[T]/(F)$ are precisely the places at which the discriminant of F vanishes. Because of (iii), this implies that the discriminant is a unit in A, so we have (c).

Conversely, assume that the conditions (a), (b), and (c) from Lemma 3.4 hold. As before, we write $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(B)$. Now B being the normalization of A in F_Y, it follows from (b) that y is a regular function on $V = \operatorname{Spec} B$. Of course, (iii) holds because of (a). Thus to finish the proof it suffices to show that the subring $A[y]$ of B is actually equal to B. Because the discriminant of the minimal polynomial F is a unit, it follows that $A[y] \cong A[T]/(F)$ is étale over A. But this is only possible if $A[y] = B$, since otherwise $A[y]$ is not integrally closed and cannot be étale over A ([Stacks, Tag 025P]). ∎

From now on, we assume as in the previous chapter that φ is a degree-p cover. Suppose that (U, y) is a chart for φ and let $x_0 \in U$ be any closed point. The open subscheme U and the point x_0 will remain fixed for most of this chapter, but in time we will modify the generator y.

The complement of the chart U consists of a finite number of closed points. Let $\Gamma_0 \subset (\mathbb{P}^1_K)^{\mathrm{an}}$ be the tree spanned by these points. By construction, Γ_0 contains the point ∞ and the branch locus of φ. In particular, Γ_0 satisfies the requirements made of the skeleton used to define the functions μ and λ from Chapter 2. Thus we have at our disposal the functions

$$\mu = \mu_{\Gamma_0}, \quad \lambda = \lambda_{\varphi,\Gamma_0} : \quad X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$$

with respect to φ and Γ_0. The main results of this chapter are Theorems 3.30 and 3.33, in which we find an analytic formula for the value $\lambda(x_0)$ of λ at the closed point x_0.

We begin by discussing power series expansions of rational functions on X. As in Section 2.5, we denote by $\mathsf{D}(r)$, $r \in \mathbb{Q}$, the open disk of radius r centered at x_0 and by $\mathsf{D}[r]$ the closed disk. Moreover, v_r denotes the Gauss valuation of radius r centered at x_0 and ξ_r the corresponding Type II point. The function

$$t := x - x_0$$

is a parameter for the family of disks $\mathsf{D}(r)$, $r \in \mathbb{Q}$, in the sense that

$$\mathsf{D}(r) = \{\xi \in X^{\mathrm{an}} \mid v_\xi(t) > r\}.$$

Let $f \in \mathcal{O}_X(U)$ be a function. If $r \geqslant \mu(x_0)$, then we have $\mathsf{D}(r) \subset U^{\mathrm{an}}$ by definition of $\mu(x_0)$. Applying the restriction map

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}}) \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$$

we can by Lemma 1.22 write f as a power series in the parameter t converging on the open disk $\mathsf{D}(r)$, say

$$f = \sum_{i=0}^{\infty} a_i t^i, \qquad \rho(f) = -\liminf_{i\to\infty} \frac{v_K(a_i)}{i} \leqslant r. \tag{3.1}$$

In the following, we use the notation introduced in Section 1.3, writing

$$[f]_r := [f]_{v_r} \in \kappa(\xi_r).$$

Note that $\kappa(\xi_r)$ is the rational function field $\kappa(\bar{t})$, where the transcendental $\bar{t}$ is the element $\bar{t} := [t]_r$. Since an element $f \in \mathcal{O}_X(U)$ has no poles in $\mathsf{D}(r)$, the reduction $[f]_r$ has no pole at $\bar{t} = 0$. Thus $[f]_r$ can be written as a power series in $\kappa[\![\bar{t}]\!]$. In the following lemma we investigate the relationship between reduction and passing to power series expansions.

Lemma 3.5. *Suppose that $r \geqslant \mu(x_0)$. Let f be an element of $\mathcal{O}_X(U)$ with power series expansion* (3.1)*. Then the following hold:*

(a) The minimum $\min_{i\geqslant 0}\{v_K(a_i) + ir\}$ exists and equals $v_r(f)$

(b) The power series expansion of $[f]_r$ is $\sum_{i\geqslant 0} \bar{a}_i\bar{t}^i \in \kappa[\![t]\!]$, where $\bar{a}_i$ is the reduction of $a_i\pi^{ir-v_r(f)}$ to κ

Proof. For convenience, in this proof we will use the notation

$$\tilde{v}_r\Big(\sum_{i=0}^{\infty} a_i t^i\Big) := \min_{i\geqslant 0}\{v_K(a_i) + ir\},$$

whenever the right-hand side is well-defined, that is, whenever the set $\{v_K(a_i)+ ir \mid i \geqslant 0\}$ has a smallest element.

To prove (a), we assume by way of contradiction that the set $\{v_K(a_i) + ir \mid i \geqslant 0\}$ does not have a smallest element. Then there exist arbitrarily large $N > 0$ such that for all $i < N$ we have $v_K(a_N) + Nr < v_K(a_i) + ir$, or equivalently

$$\frac{v_K(a_i) - v_K(a_N)}{N - i} > r.$$

Choose $s \in \mathbb{Q}$ with $\frac{v_K(a_i)-v_K(a_N)}{N-i} > s > r$. It follows from [Gou20, Corollary 7.4.11] that f has at least N zeros on the closed disk $\mathsf{D}[s] \subset U$. (Strictly speaking, Gouvêa only treats the case where $\mathrm{char}(K) = 0$, but the proof of [Gou20, Corollary 7.4.11] is also valid in general — the result is an easy consequence of the Weierstrass Preparation Theorem for restricted power series, [Bos14, Section 2.2, Corollary 9].) Since f is an algebraic function, this cannot be true for arbitrarily large N. Thus we have reached a contradiction, showing that the minimum $\min_{i\geqslant 0}\{v_K(a_i) + ir\}$ exists.

Next, let us consider a product decomposition

$$f = gh, \qquad g = \sum_{i=0}^{\infty} b_i t^i, \quad h = \sum_{i=0}^{\infty} c_i t^i.$$

Assuming that $\min_{i\geqslant 0}\{v_K(b_i) + ir\}$ and $\min_{i\geqslant 0}\{v_K(c_i) + ir\}$ both exist, we have the relation

$$\tilde{v}_r(f) = \tilde{v}_r(g) + \tilde{v}_r(h). \tag{3.2}$$

Indeed, the inequality "$\geqslant$" is obvious. To show "$\leqslant$", it suffices to show that if $\tilde{v}_r(g) = \tilde{v}_r(h) = 0$, then $\tilde{v}_r(f) = 0$ as well. This follows by considering the images of g and h under the following reduction homomorphism π, which is the composite of the maps "evaluate t at $\pi^r t$" and "reduce coefficients to κ":

$$\begin{array}{ccccc} \pi\colon\ K[\![t]\!] & \longrightarrow & K[\![t]\!] & \longrightarrow & \kappa[\![\overline{t}]\!] \\ \sum_{i=0}^{\infty} a_i t^i & \mapsto & \sum_{i=0}^{\infty} a_i \pi^{ir} t^i & \mapsto & \sum_{i=0}^{\infty} \overline{a_i \pi^{ir}}\,\overline{t}^i \end{array} \tag{3.3}$$

The fact that $\tilde{v}_r(g) = \tilde{v}_r(h) = 0$ implies that $\pi(g) \neq 0$ and $\pi(h) \neq 0$. Since $\kappa[\![\overline{t}]\!]$ is an integral domain, $\pi(f) \neq 0$ as well, which means $\tilde{v}_r(f) = 0$. Thus we have established (3.2).

To finish the proof of (a), we need to show that $v_r(f) = \tilde{v}_r(f)$ for all $f \in \mathcal{O}_X(U)$. This is clear for polynomials $f \in K[t]$. And since every $f \in \mathcal{O}_X(U)$ can be written as a quotient $f = g/h$ of two polynomials $g, h \in K[t]$, we deduce $v_r(f) = \tilde{v}_r(f)$ in general from (3.2) and the corresponding property of v_r. This finishes the proof of (a).

For (b) we may assume that $v_r(f) = \tilde{v}_r(f) = 0$. Then we need to show that $[f]_r = \pi(f)$ for all $f \in \mathcal{O}_X(U)$. Again, this is clearly correct for polynomials $f \in K[t]$. To prove it in general, we write $f = g/h$ for $g, h \in K[t]$ and without loss of generality assume $v_r(g) = v_r(h) = 0$. Then it follows from Lemma 1.18(a) and the fact that π is a homomorphism that

$$[f]_r = [g/h]_r = [g]_r/[h]_r = \pi(g)/\pi(h) = \pi(g/h) = \pi(f).$$

■

In accordance with Lemma 3.5, we will from now on use the notation

$$v_r(f) = \min_{i \geqslant 0}\{v_K(a_i) + ir\}$$

for $f \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ with power series expansion $f = \sum_i a_i t^i$. Of course, this only makes sense if the set $\{v_K(a_i) + ir \mid i \geqslant 0\}$ has a minimum. If this is the case we will say that $v_r(f)$ *is well-defined*. Note that v_r satisfies the axioms of a valuation on the set of elements in $\mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ where it is well-defined — the only hard part is the relation $v_r(fg) = v_r(f) + v_r(g)$, which we verified in the proof of Lemma 3.5.

For the following definition, let $f \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ be a function with power series expansion $\sum_i a_i t^i$ for which $v_r(f)$ is well-defined (for example, if $r \geqslant \mu(x_0)$, then this is the case for all $f \in \mathcal{O}_X(U)$).

Definition 3.6. (a) The *order* of the function f on the open disk $\mathsf{D}(r)$ is defined to be

$$\mathrm{ord}_r(f) := \min\big\{i \geqslant 0 \mid v_K(a_i) + ir = v_r(f)\big\}.$$

(b) We define

$$[f]_r := \sum_{i=0}^{\infty} \overline{a_i \pi^{ir - v_r(f)}} \cdot \overline{t}^i \in \kappa[\![\overline{t}]\!].$$

Note that by Lemma 3.5, Definition 3.6(b) agrees with the previous Definition 1.16 for $f \in \mathcal{O}_X(U)$. In the following lemma we record the fact that the statements of Lemma 1.18 also remain valid.

Lemma 3.7. *Let $f, g \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ be functions with power series expansions $f = \sum_i a_i t^i$ and $g = \sum_i b_i t^i$ respectively such that $v_r(f)$ and $v_r(g)$ are well-defined. Then we have:*

(a) $[fg]_r = [f]_r \cdot [g]_r$

(b) $[f+g]_r = [f]_r + [g]_r$ *if and only if* $v_r(f) = v_r(g)$ *and* $[f]_r + [g]_r \neq 0$

Proof. In the proof of Lemma 3.5 we have seen that $[f]_r$ is obtained from f by first multiplying f with $\pi^{-v_r(f)}$ and then applying the reduction homomorphism (3.3). With these facts in mind, the proof of the present lemma is identical to that of Lemma 1.18, with v_r taking the role of v_ξ and the homomorphism (3.3) taking the role of the reduction denoted $f \mapsto \overline{f}$ in Definition 1.16. ∎

Lemma 3.8. *For an element $f \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ with power series expansion $\sum_i a_i t^i$ such that $v_r(f)$ is well-defined, the following quantities are equal:*

(a) $\mathrm{ord}_r(f)$

(b) the order of vanishing of $[f]_r$ at $\bar{t} = 0$, i. e. the $\bar{t}$-adic valuation of $[f]_r$

(c) the number of zeros of f on the disk $\mathsf{D}(r)$, counted with multiplicity

Proof. The equivalence of (a) and (b) is obvious from Lemma 3.5(b). To show that these are equivalent to (c) we may assume without loss of generality that $r = 0$. Indeed, replacing t by $t\pi^r$ transforms f into a power series converging on the open disk $\mathsf{D}(0)$; the valuation of each monomial $a_i t^i$ with respect to v_r is the same as the valuation of $a_i(t\pi^r)^i$ with respect to v_0.

Now the desired equivalence follows from the Weierstrass Preparation Theorem [Bou98, Chapter 7, §3, n° 8, Proposition 6]. It asserts that there is a unique factorization

$$f = \pi^{v_0(f)} \cdot u \cdot g,$$

where $u \in \mathcal{O}_K[\![t]\!]^\times$ is a unit and where $g = \sum_{i=0}^s b_i t^i \in \mathcal{O}_K[t]$ is a polynomial of degree $s = \mathrm{ord}_0(f)$ with $v_K(b_s) = 0$ and $v_K(b_i) > 0$ for $0 \leqslant i < s$. (Note that this is a slightly different Weierstrass Preparation Theorem from the one mentioned in the proof of Lemma 3.5, which was about *restricted* power series.) The zeros of f on $\mathsf{D}(0)$ are the zeros of g on $\mathsf{D}(0)$, and there are s of these, counted with multiplicity — the latter is an easy consequence of the theory of the Newton polygon, specifically, of Proposition 3.17 below. ∎

Motivated by Lemma 3.8, we extend the function ord_r to $r = \infty$ by letting

$$\mathrm{ord}_\infty(f) := \text{the order of vanishing of } f \text{ at } x_0.$$

Suppose now that $f \in K[\![t]\!]$ is a function on the *closed* disk $\mathsf{D}[r]$, that is, an element of the Tate algebra $K\{\pi^{-r}t\}$. For example, this is the case if

$r > \mu(x_0)$, so that $\mathsf{D}[r] \subset U^{\mathrm{an}}$. As explained in Section 1.4, the elements of the Tate algebra are restricted power series

$$f = \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!], \qquad \lim_{i\to\infty} v_K(a_i) + ir = \infty.$$

Clearly, the minimum $\min_{i\geqslant 0}\{v_K(a_i) + ir\}$ exists and $v_r(f)$ is the same as the Gauss valuation on restricted power series as discussed in Section 1.4. In particular, Definition 3.6 and Lemmas 3.7 and 3.8 all apply to f.

In Lemma 3.8 we saw that the number of zeros of f on the open disk $\mathsf{D}(r)$ equals the order of vanishing of the reduction $[f]_r$ at $\bar{t} = 0$. The analogue for the closed disk $\mathsf{D}[r]$ is the following.

Lemma 3.9. *If $f \in K\{\pi^{-r}t\}$, we have the following:*

(a) $[f]_r$ is a polynomial

(b) The degree of $[f]_r$ equals the number of zeros of f on $\mathsf{D}[r]$

Proof. That $[f]_r$ is a polynomial follows immediately from the formula in Lemma 3.5 and the fact that f is represented by a *restricted* power series. That the degree of this polynomial equals the number of zeros of f on $\mathsf{D}[r]$ follows from the Weierstrass Preparation Theorem for restricted power series [Bos14, Section 2.2, Corollary 9]. ∎

Remark 3.10. Let F be the minimal polynomial over F_X of the generator y that is part of the affine étale chart (U, y) we fixed earlier in this section. By Lemma 3.4, the discriminant Δ_F is an invertible function on U. In particular, it has no zeros in the disk $\mathsf{D}(r)$ for any $r \geqslant \mu(x_0)$. Thus Lemma 3.8 shows that the reduction $[\Delta_F]_r$ does not vanish at $\bar{t} = 0$ for any $r \geqslant \mu(x_0)$. And Lemma 3.9 shows that $[\Delta_F]_r$ is a constant in κ for any $r > \mu(x_0)$.

For the following lemma, note that the function

$$\mathbb{Q} \to \mathbb{Q}, \qquad r \mapsto v_r(f)$$

is by Lemma 2.8 induced by a piecewise affine function $\mathbb{R} \to \mathbb{R}$. Therefore it makes sense to talk of its right derivative $\partial_+ v_r(f)$ with respect to r.

Lemma 3.11. *If $f \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ and $v_r(f)$ is well-defined, then we have*

$$\mathrm{ord}_r(f) = \big(\partial_+ v_r(f)\big)(r).$$

Proof. As in the proof of Lemma 3.8 we may assume without loss of generality that $r = 0$ and use the Weierstrass Preparation Theorem [Bou98, Chapter 7, §3, n° 8, Proposition 6] to write

$$f = \pi^{v_0(f)} \cdot u \cdot (t - \alpha_1) \cdots (t - \alpha_s),$$

where $u \in \mathcal{O}_K[\![t]\!]^\times$ and where the $\alpha_1, \ldots, \alpha_s$ are the $s := \mathrm{ord}_0(f)$ zeros of f on $\mathsf{D}(0)$ (this number of zeros is counted with multiplicity, so the α_i may not be pairwise distinct). We may choose $\varepsilon > 0$ with $v_K(\alpha_1), \ldots, v_K(\alpha_s) > \varepsilon$. Then for $0 \leqslant r < \varepsilon$,

$$v_r(f) = v_0(f) + \sum_{i=1}^{s} v_r(t - \alpha_i) = v_0(f) + sr.$$

Thus the right derivative of $v_r(f)$ at 0 equals $s = \mathrm{ord}_0(f)$. ■

Now let $r \geqslant \lambda(x_0)$ be a rational number. We consider the decomposition from Proposition 2.32,

$$\varphi^{-1}(\mathsf{D}(r)) = \mathsf{C}_1 \amalg \ldots \amalg \mathsf{C}_p.$$

The function y that is part of the affine étale chart (U, y) is a regular function on the affine open subset $V = \varphi^{-1}(U)$, that is, an element of $\mathcal{O}_Y(V)$. Thus we can apply to it the following composite maps

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_{Y^{\mathrm{an}}}(V^{\mathrm{an}}) \longrightarrow \mathcal{O}_{Y^{\mathrm{an}}}(\mathsf{C}_i) \xrightarrow{\sim} \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r)), \qquad i = 1, \ldots, p. \quad (3.4)$$

Here the first map is induced by the analytification morphism $Y^{\mathrm{an}} \to Y$, the second is a restriction map of the sheaf $\mathcal{O}_{Y^{\mathrm{an}}}$, and the last is induced by the isomorphism $\varphi|_{\mathsf{C}_i} : \mathsf{C}_i \to \mathsf{D}(r)$.

In this way, we obtain p images of the function y in $\mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$. We denote these by

$$y_1, y_2, \ldots, y_p,$$

and consider them as power series in the parameter t. Note that we have $y_i(0) = y(p_0)$, where p_0 is the point lying above x_0 contained in C_i, for $i = 1, \ldots, p$. In other words, the constant coefficients of the power series $y_1, \ldots, y_p$ correspond to the p preimages in $\varphi^{-1}(x_0)$. In particular, the $y_1, \ldots, y_p$ are pairwise distinct.

Remark 3.12. Let $f \in \mathcal{O}_Y(V)$ be any function and let $\sum_i a_i t^i$ be its image under one of the maps (3.4). Then $v_r(f)$ is well-defined (i. e. the minimum $\min_{i \geqslant 0}\{v_K(a_i) + ir\}$ exists) by the same argument as in the proof of Lemma 3.5; the key point is that the algebraic function f only has finitely many zeros. Thus Definition 3.6 and the lemmas following it are applicable to f.

Now let us furthermore suppose that $r = \lambda(x_0)$ and that $\xi_{\lambda(x_0)}$ is a wild topological branch point of φ, so that $\varphi^{-1}(\xi_{\lambda(x_0)}) = \{\eta\}$ for a Type II point $\eta \in Y^{\mathrm{an}}$. Then for $f = \sum_i a_i t^i$ as above we have

$$v_\eta(f) = v_{\lambda(x_0)}(f) = \min_{i \geqslant 0}\{v_K(a_i) + i\lambda(x_0)\}. \quad (3.5)$$

Indeed, if $r > \lambda(x_0)$ and $\eta_r \in \mathsf{C}_i$ is chosen so that $\varphi(\eta_r) = \xi_r$, we have $v_{\eta_r}(f) = \min_{i \geqslant 0}\{v_K(a_i) + ir\}$, because $\varphi|_{\mathsf{C}_i}$ is an isomorphism. Then (3.5) follows because of continuity of evaluation at f.

Since the power series $y_1, \dots, y_p$ are contained in $\mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(\lambda(x_0)))$, they have radius of convergence at most $\lambda(x_0)$. In fact, it follows from Corollary 2.33 that in the case that Γ_0 is the tree spanned by ∞ and the branch points of φ, the radius of convergence of the $y_1, \dots, y_p$ is exactly $\lambda(x_0)$. We explore this in the following example.

Example 3.13. Consider the plane quartic

$$Y\colon \quad F(y) = y^3 + 3xy^2 - 3y - 2x^4 - x^2 - 1 = 0$$

over $K = \mathbb{C}_3$. The branch points of the degree-3 cover $\varphi\colon Y \to \mathbb{P}^1_K$ given by $(x, y) \mapsto x$ consist of the point ∞ and the zeros of the discriminant of F,

$$\Delta_F = -108x^8 + 216x^7 - 108x^6 + 432x^5 - 135x^4 + 270x^3 + 27x^2 + 162x + 81.$$

We use the étale affine chart as explained in Example 3.3, that is to say, U is just the complement of the branch locus of φ. Thus Γ_0 is the tree spanned by the zeros of Δ_F and ∞. Two of the zeros of Δ_F have valuation $1/2$ and the other six have valuation 0. In particular, we may take $x_0 = 0$ as a closed point in the complement of Γ_0.

Let us fix a point $p_0 = (x_0, y_0)$ with $\varphi(p_0) = x_0$. We denote by

$$y_1 = \sum_{i=0}^{\infty} b_i t^i \in K[\![t]\!]$$

the image of y corresponding to p_0. It is easy to calculate the coefficients $b_1, b_2, \dots$ and their valuations (see Code Listing A.2). We have the following:

i	1	2	3	4	5	6	7	8	9	10
$\frac{v_K(b_i)}{i}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{7}{8}$	$-\frac{1}{2}$	$-\frac{4}{5}$	$-\frac{8}{9}$	$-\frac{6}{7}$	$-\frac{7}{8}$	$-\frac{25}{27}$	$-\frac{9}{10}$

It looks as though the radius of convergence y_1,

$$\rho(y_1) = -\liminf_{i \to \infty} \frac{v_K(b_i)}{i},$$

equals 1. And indeed we will see in Example 5.17 that $\lambda(x_0) = 1$.

We discuss now another perspective on the power series $y_1, \dots, y_p$ that will become important later, when we will consider not only one fixed closed point x_0, but vary x_0. Let us write $A := \mathcal{O}_X(U)$ and $B := \mathcal{O}_Y(V)$, where $V = \varphi^{-1}(U)$, and denote by

$$F(T) = T^p + f_{p-1}T^{p-1} + \dots + f_1 T + f_0 \in A[T]$$

the minimal polynomial of y. We now introduce a "generic" version of the minimal polynomial F, the polynomial

$$\tilde{F}(T) = T^p + \tilde{f}_{p-1}T^{p-1} + \dots + \tilde{f}_1 T + \tilde{f}_0 \in A[\![t]\!][T], \tag{3.6}$$

where the $\tilde{f}_i \in A[\![t]\!]$ are given by the Taylor expansions

$$\tilde{f}_i(t) = \sum_{\ell=0}^{\infty} a_{i,\ell} t^{\ell}, \qquad a_{i,\ell} = \frac{f_i^{(\ell)}}{\ell!}, \qquad 0 \leqslant i \leqslant p-1. \tag{3.7}$$

The use of the tildes will always indicate that we are considering such a "generic" polynomial, whose coefficients are themselves not just elements of K, but functions. Evaluation of the $\tilde{f}_i$ at x_0 yields the minimal polynomial of y over the subfield $K(t)$ of F_Y. That is, the minimal polynomial of y over the subfield $K(t)$ is given by

$$G(T) = T^p + g_{p-1}T^{p-1} + \ldots + g_1 T + g_0, \tag{3.8}$$

$$\text{where} \qquad g_i = \sum_{\ell=0}^{\infty} a_{i,\ell}(x_0) t^{\ell}.$$

We need the following lemma, which may be viewed as an algebraic version of the Implicit Function Theorem:

Lemma 3.14. *Let R be a ring and let $F \in R[\![t]\!][T]$ be a monic polynomial whose coefficients are formal power series over R in the indeterminate t, say*

$$F = T^n + f_{n-1}T^{n-1} + \ldots + f_1 T + f_0, \qquad f_0, \ldots, f_{n-1} \in R[\![t]\!].$$

Suppose that there exists a zero $a_0 \in R$ of the polynomial

$$F_0 := T^n + f_{n-1}(0)T^{n-1} + \ldots + f_1(0)T + f_0(0)$$

and that $F_0'(a_0) \in R^{\times}$, where F_0' is the derivative

$$F_0' = nT^{n-1} + (n-1)T^{n-2} f_{n-1}(0) + \ldots + f_1(0).$$

Then there exists a unique lift of a_0 to a power series $P = a_0 + a_1 t + a_2 t^2 + \ldots \in R[\![t]\!]$ with $F(P) = 0$.

Proof. We construct P inductively. First let us make the ansatz $P_1 = a_0 + a_1 t + O(t^2)$. We have

$$F(a_0 + a_1 t) = F(a_0) + F_0'(a_0) a_1 t + O(t^2).$$

Since $F_0'(a_0) \in R^{\times}$, we may and must choose a_1 so that $F(a_0 + a_1 t) \in O(t^2)$. Similarly, the ansatz $P_2 = a_0 + a_1 t + a_2 t^2$ leads to

$$F(a_0 + a_1 t + a_2 t^2) = F(a_0 + a_1 t) + F_0'(a_0) a_2 t^2 + O(t^3),$$

which forces us to choose a_2 so that $F(P_2) \in O(t^3)$. Continuing in this manner uniquely defines the power series P with $F(P) = 0$ as desired. ∎

We apply Lemma 3.14 to the monic polynomial $\tilde{F} \in A[\![t]\!][T] \subseteq B[\![t]\!][T]$ of (3.6) and the zero $y \in B$. Because φ is étale over U, we have $\tilde{F}_0'(y) \in B^\times$. It follows that there exists a formal solution in $B[\![t]\!]$ to (3.6), which we denote

$$\tilde{y} = y + b_1 t + b_2 t^2 + b_3 t^3 + \dots. \tag{3.9}$$

The power series $y_1, \dots, y_p$ constructed above are obtained from $\tilde{y}$ by evaluating each of the coefficients of $\tilde{y}$ at the points $p_0 \in \varphi^{-1}(x_0)$. That is,

$$y_i = y(p_0) + b_1(p_0)t + b_2(p_0)t^2 + \dots,$$

where $p_0 \in \varphi^{-1}(x_0)$ is contained in C_i. This may be seen by applying Lemma 3.14 again, to the polynomial in (3.8), and to the root $y(p_0)$ of this polynomial.

3.2 Newton polygons

In this section, we recall some basic facts about Newton polygons and introduce some terminology that we will need. We will later impose further restrictions, but for now let (F, v_F) be any valued field, with ring of integers $\mathcal{O}_F$. We begin by recalling the central definition.

Definition 3.15. Let $G = \sum_{i=0}^n a_i T^i \in F[T]$ be a monic polynomial. The *Newton polygon* of G is the boundary of the lower convex hull of the set

$$\big\{(0, v_F(a_0)), (1, v_F(a_1)), \dots, (n, v_F(a_n))\big\} \subset \mathbb{R} \times (\mathbb{R} \cup \{\infty\}). \tag{3.10}$$

We denote it by N_G.

Geometrically, N_G is obtained as follows. Begin with the ray $\{(n, t) \mid t \leqslant 0\}$ pointing downward from the rightmost point $(n, 0)$ in (3.10). Rotate this ray clockwise until it hits another point $(i, v_F(a_i))$. Then remove the line segment connecting the two points $(n, 0)$ and $(i, v_F(a_i))$ from the ray and continue rotating its remainder, again until hitting another point. Continue this procedure until the point $(0, v_F(a_0))$ in (3.10) is hit. The Newton polygon N_G then consists of the line segments connecting the points that were hit.

The points $(i, v_F(a_i))$ that lie on N_G are called the *vertices* of N_G.

Note that our definition allows certain coefficients a_i, $i < n$, to vanish. If this is the case, the corresponding vertex $(i, v_F(a_i))$ is (i, ∞). If there exists an $i_0 > 0$ such that $a_{i_0} \neq 0$, but $a_i = 0$ for all $i < i_0$, then N_G contains a vertical line segment, connecting the vertices $(i_0, v_F(i_0))$ and $(0, \infty)$.

Definition 3.16. We denote by θ_G the negative of the steepest slope of the Newton polygon N_G. Thus

$$\theta_G = \max_{1 \leqslant i \leqslant p} \frac{v_F(a_0) - v_F(a_i)}{i},$$

with the convention that $\theta_G = \infty$ if $a_0 = 0$.

The following proposition is the most important fact about the Newton polygon of a polynomial.

Proposition 3.17. *If the vertices $(i, v_F(a_i))$ and $(j, v_F(a_j))$ of N_G, where $i < j$, are the endpoints of one of the line segments making up N_G, then in its splitting field, the polynomial G has exactly $j - i$ roots of valuation*

$$\frac{v_F(a_i) - v_F(a_j)}{j - i}.$$

Proof. This is essentially the same statement as [Neu99, Theorem II.6.3]; however, in loc. cit. it is assumed that $a_0 \neq 0$. Our version is easily reduced to that case. Indeed, write $G = T^r H$, where $H(0) \neq 0$. Then N_G is obtained from N_H by translating N_H to the right by r and adding a vertical line segment. The version of [Neu99, Theorem II.6.3] connects the non-vertical slopes of N_G with the non-zero roots of G, and the first and vertical line segment of N_G correctly predicts r roots of infinite valuation. ∎

From now on, we assume that F has residue field of characteristic $p > 0$ and that G is a monic irreducible polynomial of degree p.

Definition 3.18. The Newton polygon N_G is called *inseparable* if its only vertices are the points $(0, v_F(a_0))$ and $(p, 0)$. Otherwise, it is called *separable*.

Remark 3.19. If N_G is inseparable, then it consists of a single line segment. If N_G is separable, then it may or may not consist of a single line segment. In particular, if the valuations of the roots of G are not all equal, then it follows from Proposition 3.17 that N_G is separable.

Lemma 3.20. *Suppose that v_F has a unique extension v_L to $L := F(\alpha)$, where α is a root of G with $v_L(\alpha) = 0$. Then the following hold:*

- *(a) $G \in \mathcal{O}_F[T]$, and the polynomial $\overline{G}$ obtained from G by mapping each coefficient to the residue field of v_F is inseparable if and only if N_G is inseparable.*
- *(b) If $\overline{G}$ is reducible, then N_G (and hence $\overline{G}$) is inseparable.*

Proof. That v_F has a unique extension to L implies that G is irreducible over the completion of F with respect to v_F ([Neu99, Theorem II.8.2]). It follows that N_G consists of a single line segment ([Neu99, Theorem II.6.4]). Since $v_L(\alpha) = 0$, it is the line segment connecting $(0, 0)$ and $(p, 0)$. Thus we have

$$v_F(a_i) \geqslant v_F(a_0) = 0, \qquad 0 < i < p. \tag{3.11}$$

In particular, all coefficients $a_0, \ldots, a_p$ lie in $\mathcal{O}_F$, so it makes sense to define the reduction $\overline{G}$. Moreover, N_G is inseparable if and only all the inequalities in (3.11) are strict.

The reduction $\overline{G}$ cannot have more than one distinct irreducible factor, since any factorization into distinct factors would by Hensel's Lemma lift to a factorization of G over the completion of F.

There are two possible cases: either the irreducible factor is of degree 1 or it is of degree p. In the first case, write $T - \overline{\alpha}$ for this factor; then

$$\overline{G}(T) = (T - \overline{\alpha})^p = T^p - \overline{\alpha}^p$$

is inseparable. Clearly the inequalities (3.11) are all strict, so N_G is inseparable as well. This shows (b), and (a) in the case that $\overline{G}$ is reducible.

In the second case, $\overline{G}$ is itself irreducible, so it is inseparable if and only if its derivative $\overline{G}'$ vanishes. We have

$$\overline{G} = T^p + \overline{a}_{p-1}T^{p-1} + \ldots + \overline{a}_1 T + \overline{a}_0,$$

$$\overline{G}' = (p-1)\overline{a}_{p-1}T^{p-2} + \ldots + 2\overline{a}_2 T + \overline{a}_1,$$

where $\overline{a}_i$, $0 \leqslant i < p$, is the image of a_i in the residue field of v_F. Clearly $\overline{G}' = 0$ if and only if all inequalities in (3.11) are strict, that is, if and only if N_G is inseparable. This shows (a) in the case that $\overline{G}$ is irreducible. ■

Our most important application of Newton polygons is to Type II valuations on the function field of a curve over the complete ground field K. For the rest of this section, suppose in particular that $X = \mathbb{P}^1_K$. As before, denote by $\mathsf{D}(r)$ the open disk of radius $r \in \mathbb{Q}$ around the fixed closed point $x_0 \in X$. We may identify the function field of X with the rational function field $K(t)$ in the parameter $t = x - x_0$ associated to the closed point x_0. Denoting as usual the boundary point of $\mathsf{D}(r)$ by ξ_r, we obtain a family of valuations $v_r = v_{\xi_r}$ on $K(t)$. Thus given a polynomial $G = \sum_i a_i T^i \in K(t)[T]$, varying r we also obtain a family of Newton polygons of G with respect to the different v_r. We denote the Newton polygon of G with respect to v_r by $N_G(r)$. Similarly, varying r, we may regard θ_G as a function of r,

$$\theta_G \colon \mathbb{Q} \to \mathbb{Q}, \qquad r \mapsto \theta_G(r) = \max_{1 \leqslant i \leqslant r} \frac{v_r(a_0) - v_r(a_i)}{i}.$$

We end this section with a lower bound for the value of the function λ at the closed point x_0, before tackling the more delicate question of an upper bound in the next section. Recall that we have fixed in the previous section an affine étale chart (U, y). As before, we write

$$G(T) = T^p + g_{p-1}T^{p-1} + \ldots + g_1 T + g_0$$

for the minimal polynomial of y over $K(t)$.

Note that if $\lambda(x_0) = \mu(x_0)$, the following proposition is never applicable.

Proposition 3.21. *If $r \in \mathbb{Q}$ is a rational number with $\lambda(x_0) > r \geqslant \mu(x_0)$, then the Newton polygon $N_G(r)$ is inseparable.*

Proof. By Proposition 2.31 we have $\delta(\xi_r) > 0$. It follows from Lemma 2.17 that we have $\varphi^{-1}(\xi_r) = \{\eta\}$ for a point $\eta \in Y^{\mathrm{an}}$. In other words, v_η is the unique extension of the valuation v_r to F_Y. Moreover, the extension $\kappa(\eta)/\kappa(\xi_r)$ is purely inseparable of degree p by Remark 1.26.

To place ourselves in the position to apply Lemma 3.20, we replace y with $z = y\pi^{-s}$, where $s = v_\eta(y)$. Then $v_\eta(z) = 0$, and the minimal polynomial of z is

$$H(T) = T^p + g_{p-1}\pi^{-s}T^{p-1} + \ldots + g_1\pi^{(1-p)s}T + g_0\pi^{-ps}.$$

Denote by $\overline{H}$ the polynomial obtained from H by mapping each coefficient to the residue field $\kappa(\xi_r)$. If $\overline{H}$ is reducible, then $N_H(r)$ is inseparable by Lemma 3.20(b). Otherwise, $\overline{z} := [z]_r$, the reduction of z, is a generator of the inseparable field extension $\kappa(\eta)/\kappa(\xi_r)$. Thus $N_H(r)$ is inseparable by Lemma 3.20(a).

Now it suffices to notice that $N_G(r)$ is inseparable if and only if $N_H(r)$ is. This is an easy consequence of the definition. Indeed, let us write $h_i := g_i\pi^{(i-p)s}$, $i = 0, \ldots, p-1$, for the coefficients of H. Then $N_H(r)$ is inseparable if and only if

$$v_r(g_i) + (i-p)s = v_r(h_i) > v_r(h_0) = 0, \qquad 0 < i < p.$$

Rewriting this yields

$$\frac{v_r(g_i)}{p-i} > \frac{v_r(g_0)}{p} = s,$$

which is true for $i = 1, \ldots, p-1$ if and only if $N_G(r)$ is inseparable. ■

Remark 3.22. In the proof of Proposition 3.21, we had to distinguish two cases: Either the reduction $\overline{z}$ of the generator z was a generator of the extension of residue fields $\kappa(\eta)/\kappa(\xi_r)$, or it was not.

This distinction, and the fact that we cannot read off from the shape of the Newton polygon in which case we are, is significant. In the next section, we will formulate a condition (see Assumption 3.23) for ruling out that $\overline{z}$ is not a generator.

3.3 An upper bound for λ: tail case

Recall that in Section 3.1 we have fixed an étale affine chart (U, y) for φ. We also retain the closed point $x_0 \in U$ and the associated notations (introduced in Section 2.5) for the open and closed disks of radius $r \in \mathbb{Q}$ around x_0, $\mathsf{D}(r)$ and $\mathsf{D}[r]$. Also recall the family of functions ord_r of Definition 3.6, which associate an order on $\mathsf{D}(r)$ to all $f \in \mathcal{O}_{X^{\mathrm{an}}}(\mathsf{D}(r))$ for which $v_r(f)$ is well-defined. As in (3.8), we denote the minimal polynomial of y over $K(t)$ by

$$G(T) = T^p + g_{p-1}T^{p-1} + \ldots + g_1T + g_0,$$

where $t = x - x_0$ is the parameter associated to x_0.

The results we prove in this section and the next only hold conditional on an assumption on G:

Assumption 3.23. *For some integer $m \geqslant 1$, we have the inequalities*

$$\mathrm{ord}_\infty(g_0) \geqslant m, \qquad \mathrm{ord}_{\mu(x_0)}(g_0) < mp.$$

We note that this assumption is in terms of the parameter t, and hence dependent on the closed point x_0. This should cause no confusion, since the closed point x_0 is still fixed throughout this section and the next.

The condition in Assumption (3.23) will not usually hold *a priori*. For now, we will prove our results assuming that it holds; then in Section 3.5, we will see how to modify the generator y so that the condition is guaranteed to hold.

The "tail case" in the title of this section refers to the fact that the main result of this section, Theorem 3.30, depends on another additional assumption, namely that $\lambda(x_0) > \mu(x_0)$. In the next section, we will remove this condition at the cost of replacing the formula for $\lambda(x_0)$ of Theorem 3.30 with a more complicated formula.

The reason for dubbing the case $\lambda(x_0) > \mu(x_0)$ the "tail case" will become apparent once in Section 3.6 we introduce the *tame locus* associated to the tree Γ_0. As explained in Lemma 3.45, a closed point x_0 in the tame locus is contained in the *tail locus* if and only if $\lambda(x_0) > \mu(x_0)$.

Recall that in Section 3.1 we have introduced the p roots $y_1, \ldots, y_p \in \mathsf{D}(\lambda(x_0))$ of the minimal polynomial G of y. The following simple lemma contains the central combinatorial argument based on Assumption 3.23.

Lemma 3.24. *Suppose that Assumption 3.23 holds for some $m \geqslant 1$. Then there exist $i_0, i_1 \in \{1, \ldots, p\}$ such that*

$$\mathrm{ord}_r(y_{i_0}) < m \leqslant \mathrm{ord}_r(y_{i_1})$$

for all rational numbers $r \geqslant \lambda(x_0)$.

Proof. Viewing the $y_1, \ldots, y_p$ and the coefficients of G as elements of $K[\![t]\!]$, we have

$$y_1 \cdots y_p = g_0.$$

Thus by Assumption 3.23, at least one of the $y_1, \ldots, y_p$, say y_{i_1}, must have a zero in x_0. Since x_0 is not a branch point of φ, the discriminant

$$\Delta_G = \prod_{i<j} (y_i - y_j)^2$$

does not have a zero at x_0, so neither do any of the y_i, $i \neq i_1$. Thus the zero of y_{i_1} must be of order $\geqslant m$.

In particular, it follows from Lemma 3.8 that $\mathrm{ord}_r(y_{i_1}) \geqslant m$ for all $r \geqslant \lambda(x_0)$. But we cannot have $\mathrm{ord}_{\lambda(x_0)}(y_i) \geqslant m$ for all $i \in \{1, \ldots, p\}$, since this would imply

$$\mathrm{ord}_{\mu(x_0)}(g_0) \geqslant \mathrm{ord}_{\lambda(x_0)}(g_0) = \sum_{i=1}^{p} \mathrm{ord}_{\lambda(x_0)}(y_i) \geqslant mp,$$

contradicting the second part of Assumption 3.23. Thus there exists some $i_0 \neq i_1$ with

$$\operatorname{ord}_r(y_{i_0}) \leqslant \operatorname{ord}_{\lambda(x_0)}(y_{i_0}) < m$$

for all $r \geqslant \lambda(x_0)$ as desired. ■

Lemma 3.25. *Suppose that Assumption 3.23 is satisfied and let $r > \lambda(x_0)$ be a rational number. Then the Newton polygon $N_r(G)$ has a kink.*

Proof. We consider the reductions

$$\overline{y}_i := [y_i]_r \in \kappa[\![t]\!], \qquad i = 1, \ldots, p,$$

(Definition 3.6). By Lemma 3.9, the $\overline{y}_i$ are polynomials, of degree equal to the number of zeros of y_i in $\mathsf{D}[r]$. Because of Lemma 3.24, there exist $i_0, i_1 \in \{1, \ldots, p\}$ with $\deg(\overline{y}_{i_0}) < \deg(\overline{y}_{i_1})$.

By Lemma 3.7(a), the reduction of the discriminant Δ_G satisfies

$$[\Delta_G]_r = \prod_{i<j} [y_i - y_j]_r^2.$$

It is a constant by Remark 3.10. Because of Lemma 3.9, all the $[y_i - y_j]_r$ are polynomials, so they must be constant too. Since $\overline{y}_{i_0}$ and $\overline{y}_{i_1}$ are distinct polynomials, one of which is not constant, it follows from Lemma 3.7(b) that $v_r(y_{i_0}) \neq v_r(y_{i_1})$. Since the valuations $v_r(y_i)$, $i = 1, \ldots, p$, are the negatives of the slopes of $N_G(r)$, we have shown that this Newton polygon must have a kink. ■

Corollary 3.26. *Suppose that Assumption 3.23 holds. Then the following are true:*

(a) The Newton polygon $N_G(\lambda(x_0))$ is separable.

(b) If $\lambda(x_0) > \mu(x_0)$, then $N_G(\lambda(x_0))$ is a straight line.

Proof. By Lemma 3.25, the Newton polygon $N_r(G)$ has a kink for any $r > \lambda(x_0)$. By Proposition 3.21, $N_r(G)$ is inseparable for $\lambda(x_0) > r \geqslant \mu(x_0)$.

Thus both statements of the corollary are reduced to the following elementary facts about deformations of Newton polygons: If a Newton polygon $N_r(G)$ is inseparable, then so is $N_G(r + \varepsilon)$ for $|\varepsilon|$ small enough. And if $N_G(r)$ has a kink, then so does $N_G(r + \varepsilon)$ for $|\varepsilon|$ small enough. ■

Lemma 3.27. *Assume that $\lambda(x_0) > \mu(x_0)$ and that Assumption 3.23 is satisfied. Then $\xi_{\lambda(x_0)}$ is a wild topological branch point, and in particular $\varphi^{-1}(\xi_{\lambda(x_0)}) = \{\eta\}$ for a point $\eta \in Y^{\mathrm{an}}$.*

Moreover, the extension $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ of residue fields is separable of degree p with generator $\overline{y} := [y]_\eta \in \kappa(\eta)$.

Proof. Because $\lambda(x_0) > \mu(x_0)$, Proposition 2.31 shows that $\delta(\xi_r) > 0$ for $\lambda(x_0) > r > \mu(x_0)$, while $\delta(\xi_{\lambda(x_0)}) = 0$. The ξ_r for $\lambda(x_0) > r > \mu(x_0)$ are wild topological branch points by Lemma 2.17(a), and so is $\xi_{\lambda(x_0)}$ because the wild topological branch locus is closed (Remark 1.29(c)). As in the statement of the lemma, we denote the unique preimage of $\xi_{\lambda(x_0)}$ by η. Because $\delta(\xi_{\lambda(x_0)}) = 0$, Lemma 2.17(b) shows that $\mathcal{H}(\eta)/\mathcal{H}(\xi_{\lambda(x_0)})$ is an unramified extension of degree p, so that the extension $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ is indeed separable of degree p.

We are left to verify that $\overline{y}$ generates the field extension $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$. Write $V := \varphi^{-1}(U)$ for the inverse image of the chart U. Recall from Remark 3.12 that we have described maps $\mathcal{O}_Y(V) \to K[\![t]\!]$ that send y to the power series y_i, where $i = 1, \ldots, p$. By passing to quotient fields, these extend to embeddings of fields $F_Y \to K(\!(t)\!)$. Valuations are preserved in the sense that we have

$$v_\eta(f) = \min_{i \geqslant 0}\{v_K(a_i) + i\lambda(x_0)\}$$

for $f \in \mathcal{O}_Y(V)$ with image $\sum_i a_i t^i$ in $K[\![t]\!]$. Thus the map $F_Y \to K(\!(t)\!)$ descends to a map of residue fields $\kappa(\eta) \to \kappa(\!(\overline{t})\!)$ with $\overline{y} \mapsto \overline{y}_i := [y_i]_{\lambda(x_0)}$.

Each such embedding defines an extension of the $\overline{t}$-adic valuation on $\kappa(\overline{t})$ to $\kappa(\eta)$. But Assumption 3.23 guarantees that not all $\overline{y}_i$, $i = 1, \ldots, p$, have the same $\overline{t}$-adic valuation (combine Lemmas 3.24 and 3.8). In particular, $\overline{y}$ cannot lie in $\kappa(\xi_{\lambda(x_0)})$. Since $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ is of degree p, we conclude that $\overline{y}$ generates this field extension. ∎

Remark 3.28. (a) Let us write down the minimal polynomial of the generator $\overline{y} = [y]_{\lambda(x_0)}$ considered in Lemma 3.27. It is

$$\overline{G}(T) = T^p + \overline{g}_{p-1}T^{p-1} + \ldots + \overline{g}_1 T + \overline{g}_0 = 0, \qquad \overline{g}_i = \overline{g_i \pi^{-(p-i)s}}, \quad (3.12)$$

where $s = v_\eta(y) = v_{\lambda(x_0)}(g_0)/p$. To see this, note that $N_G(\lambda(x_0))$ being a straight line implies $v_{\lambda(x_0)}(g_i) \geqslant (p-i)s$ for $i = 1, \ldots, p-1$. Thus it makes sense to define the reductions $\overline{g_i \pi^{-(p-i)s}}$. Moreover, it is clear that (3.12) is a monic polynomial satisfied by $\overline{y} = \overline{y\pi^{-s}}$.

(b) The discriminant $\Delta_{\overline{G}}$ of the minimal polynomial $\overline{G}$ of the element $\overline{y}$ is a constant. Indeed, we have

$$\Delta_{\overline{G}} = \prod_{i<j}(\overline{y}_i - \overline{y}_j)^2,$$

where the $\overline{y}_i = [y_i]_{\lambda(x_0)}$, $i = 1, \ldots, p$, are the reductions of the power series y_i, as considered in Lemma 3.25. Since $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ is separable, the $\overline{y}_i$ are pairwise distinct, so by Lemma 3.7(b) we have $\overline{y}_i - \overline{y}_j = [y_i - y_j]_{\lambda(x_0)}$ for all $i < j$. It follows from Lemma 3.7(a) that

$$\Delta_{\overline{G}} = \prod_{i<j}[y_i - y_j]^2_{\lambda(x_0)} = \Big[\prod_{i<j}(y_i - y_j)^2\Big]_{\lambda(x_0)} = [\Delta_G]_{\lambda(x_0)}$$

is a constant.

Lemma 3.29. *Assume that $\lambda(x_0) > \mu(x_0)$ and that Assumption 3.23 is satisfied. Then we have*

$$\mathrm{ord}_{\lambda(x_0)}(g_1) = 0,$$

and the Newton polygon $N_G(\lambda(x_0))$ contains the point

$$(1, v_{\lambda(x_0)}(g_1)).$$

Proof. We again write

$$\overline{y}_i := [y_i]_{\lambda(x_0)} \in \kappa[\![\overline{t}]\!], \qquad i = 1, \ldots, p.$$

It follows from Corollary 3.26 that $N_G(\lambda(x_0))$ is a straight line, so we have

$$v_{\lambda(x_0)}(y_1) = \ldots = v_{\lambda(x_0)}(y_p).$$

As in the proof of Lemma 3.25, the reduction of the discriminant Δ_G,

$$[\Delta_G]_{\lambda(x_0)} = \prod_{i<j} [y_i - y_j]^2_{\lambda(x_0)}$$

and the factors $[y_i - y_j]_{\lambda(x_0)}$ are constant. (In Lemma 3.25, we considered reduction at a radius $r > \lambda(x_0)$, but the only thing that matters for this argument is that the radius be greater than $\mu(x_0)$, which is true here as it was there.)

By Lemma 3.24 and Lemma 3.8, there exists an $i_1 \in \{1, \ldots, p\}$ for which $\overline{y}_{i_1}$ vanishes at $\overline{t} = 0$. We saw in Remark 3.28(b) that the $\overline{y}_i$ are pairwise distinct, so it follows from Lemma 3.7(b) that the $\overline{y}_i$, $i \neq i_1$, do not vanish at $\overline{t} = 0$. Now Lemma 3.8 and Lemma 3.11 imply that the right derivatives of the functions $r \mapsto v_r(y_i)$ are zero at $r = \lambda(x_0)$ for $i \neq i_1$, while the right derivative of $r \mapsto v_r(y_{i_1})$ is positive at $r = \lambda(x_0)$. This readily implies the statements of the lemma. ∎

We now introduce notation for the coefficients of the power series $g_0, g_1, \ldots, g_{p-1}$, writing

$$g_i = \sum_{k=0}^{\infty} a_{i,k} t^k, \qquad i = 0, \ldots, p-1.$$

Our main theorems of this chapter, Theorem 3.30 below and Theorem 3.33 in the following section, give formulas for $\lambda(x_0)$ only in terms of valuations of these coefficients, that is, in terms of certain elements of K depending on x_0.

Theorem 3.30. *If $\lambda(x_0) > \mu(x_0)$ and Assumption 3.23 is satisfied for $m \geqslant 1$, then we have*

$$\lambda(x_0) = \max_{m \leqslant k < mp} \frac{p v_K(a_{1,0}) - (p-1) v_K(a_{0,k})}{(p-1)k}.$$

Proof. By Lemma 3.29, the Newton polygon $N_G(\lambda(x_0))$ is a straight line on which lies the vertex

$$(1, v_{\lambda(x_0)}(g_1)) = (1, v_K(a_{1,0})).$$

Thus for every $k \geqslant 0$ we have

$$\frac{v_K(a_{1,0})}{p-1} = \frac{v_{\lambda(x_0)}(g_0)}{p} \leqslant \frac{v_K(a_{0,k}) + k\lambda(x_0)}{p}.$$

with equality for k with $v_{\lambda(x_0)}(f_0) = v_K(a_{0,k}) + k\lambda(x_0)$. This readily implies

$$\lambda(x_0) \geqslant \frac{pv_K(a_{1,0}) - (p-1)v_K(a_{0,k})}{(p-1)k}$$

for all $k \geqslant 0$, again with equality for k with $v_{\lambda(x_0)}(f_0) = v_K(a_{0,k}) + k\lambda(x_0)$. To finish the proof, simply notice that because of Assumption 3.23 we have

$$m \leqslant \mathrm{ord}_\infty(g_0) \leqslant \mathrm{ord}_{\lambda(x_0)}(g_0) \leqslant \mathrm{ord}_{\mu(x_0)}(g_0) < mp.$$

Thus $v_{\lambda(x_0)}(g_0) = v_K(a_{0,k}) + k\lambda$ for certain k with $m \leqslant k < mp$. ∎

Remark 3.31. Let us record the following facts implicit in the proof of Theorem 3.30:

(a) The maximum in the statement of Theorem 3.30 is achieved for precisely those $k \in \{m, \ldots, mp-1\}$ that satisfy $v_{\lambda(x_0)}(g_0) = v_K(a_{0,k}) + k\lambda(x_0)$.

(b) We have $v_{\lambda(x_0)}(g_1) = v_K(a_{1,0})$. This in contrast with the statement of Theorem 3.33 below, where we may need to consider the minimum over various $v_K(a_{1,\ell})$, $\ell \geqslant 0$.

3.4 An upper bound for λ: general case

As announced in the previous section, the main result of this section, Theorem 3.33, is not dependent on the assumption $\lambda(x_0) > \mu(x_0)$. We retain the notation from the previous section; in particular G still denotes the minimal polynomial of the generator y, the power series $g_0, \ldots, g_{p-1}$ are the coefficients of G, and the coefficients of the g_i are denoted $a_{i,k}$, $k \geqslant 0$.

Lemma 3.32. *Suppose that Assumption 3.23 holds for some $m \geqslant 1$. Then for any $r \geqslant \lambda(x_0)$, we have the following:*

(a) The right derivative of the function θ_G (Definition 3.16) satisfies

$$(\partial_+\theta_G)(r) \geqslant m.$$

(b) The function $\beta\colon [\lambda(x_0), \infty)$ given by $r \mapsto \theta_G(r) - \frac{v_r(g_0)}{p}$ is monotonically increasing, and has monotonically decreasing right derivative.

Proof. By the same argument as in the proof of Lemma 3.24 we see that one of the $y_1, \dots, y_p$, say y_{i_1}, has a zero of order $\geqslant m$ in x_0, while the y_i, $i \neq i_1$, do not have a zero in x_0. By Lemma 3.11, the right derivative at $r \geqslant \lambda(x_0)$ of $v_r(y_i)$ is the same as the order $\mathrm{ord}_r(y_i)$, for $i = 1, \dots, p$. Thus we have $v_r(y_{i_1}) > v_r(y_i)$ and

$$\big(\partial_+\theta_G\big)(r) = \big(\partial_+ v_r(y_{i_1})\big)(r) \geqslant m$$

for $i \neq i_1$ and $r \gg \lambda(x_0)$. To prove (a), we show that $v_r(y_{i_1}) \geqslant v_r(y_i)$, where $i \neq i_1$, holds for *all* $r \geqslant \lambda(x_0)$.

If it were not true that $v_r(y_{i_1}) \geqslant v_r(y_i)$ for all $r \geqslant \lambda(x_0)$, we could find $r > \lambda(x_0)$ and $i_2 \neq i_1$ with

$$v_r(y_{i_1}) = v_r(y_{i_2}), \qquad (\partial_+ v_r(y_{i_2}))(r) = \mathrm{ord}_r(y_{i_2}) < (\partial_+ v_r(y_{i_1}))(r) = \mathrm{ord}_r(y_{i_1}).$$

The reductions $\overline{y}_{i_1} := [y_{i_1}]_r$ and $\overline{y}_{i_2} := [y_{i_2}]_r$ are polynomials by Lemma 3.9. By Lemma 3.8, $\overline{y}_{i_1}$ has larger order of vanishing at $\overline{t} = 0$ than $\overline{y}_{i_2}$. As in the proof of Lemma 3.25 we conclude from the fact that

$$[\Delta_G]_r = \prod_{i<j}[y_i - y_j]_r^2$$

is a unit that $v_r(y_{i_1}) \neq v_r(y_{i_2})$. We have arrived at a contradiction, which finishes the proof of (a).

To prove (b), note that we have

$$\beta(r) = \theta_G(r) - \frac{v_r(g_0)}{p} = \frac{(p-1)v_r(y_{i_1}) - \sum_{i \neq i_1} v_r(y_i)}{p}.$$

We just saw that $v_r(y_{i_1}) > v_r(y_i)$ for $i \neq i_1$ and $r > \lambda(x_0)$. Hence

$$[y_i]_r = [y_{i_1} - y_i]_r$$

must be constant for $r > \lambda(x_0)$ and $i \neq i_1$. Thus $v_r(y_i)$, where $i \neq i_1$, is constant for $r > \lambda(x_0)$, so that

$$\big(\partial_+\beta\big)(r) = \frac{p-1}{p}\,\mathrm{ord}_r(y_{i_1})$$

is positive, but monotonically decreasing as a function in r. This proves (b). ∎

Theorem 3.33. *Assume that Assumption 3.23 is satisfied for $m \geqslant 1$. Then we have*

$$\lambda(x_0) = \max\{\mu(x_0), \tilde{\lambda}\},$$

where

$$\tilde{\lambda} = \max_{m \leqslant k < mp} \min_{0<i<p} \min_{0 \leqslant p\ell < (p-i)k} \frac{p v_K(a_{i,\ell}) - (p-i)v_K(a_{0,k})}{(p-i)k - p\ell}.$$

Proof. Suppose that $k \in \{m, \ldots, mp-1\}$. In this proof, we will use the terminology of calling a pair $(i, \ell) \in \mathbb{Z}^2$ a *k-admissible* pair if $i \in \{1, \ldots, p-1\}$, $0 \leqslant p\ell < (p-i)k$, and $a_{i,\ell} \neq 0$. The significance of this is that for every k-admissible pair (i, ℓ), the two affine functions

$$r \mapsto \frac{v_K(a_{i,\ell}) + \ell r}{p-i}, \qquad r \mapsto \frac{v_K(a_{0,k}) + kr}{p}$$

coincide for a unique $r(k, i, \ell) \in \mathbb{Q}$, namely

$$r(k, i, \ell) = \frac{pv_K(a_{i,\ell}) - (p-i)v_K(a_{0,k})}{(p-i)k - p\ell}.$$

(If $a_{0,k} = 0$, then $r(k, i, \ell) = -\infty$.) For $r < r(k, i, \ell)$, the first function has larger value than the second, and for values $r > r(k, i, \ell)$, it has smaller value. Said differently, for any $r \geqslant \lambda(x_0)$ we have

$$\frac{v_K(a_{i,\ell}) + \ell r}{p-i} \leqslant \frac{v_K(a_{0,k}) + kr}{p} \tag{3.13}$$

if and only if

$$r(k, i, \ell) = \frac{pv_K(a_{i,\ell}) - (p-i)v_K(a_{0,k})}{(p-i)k - p\ell} \leqslant r, \tag{3.14}$$

and similarly if we replace the two "$\leqslant$" with "$\geqslant$".

To prove the theorem it is enough to show

$$\min\{r_{k,i,\ell} \mid (i, \ell) \text{ is } k\text{-admissible}\} \leqslant \lambda(x_0) \tag{3.15}$$

for all $k \in \{m, \ldots, mp-1\}$. Indeed, this implies $\tilde{\lambda} \leqslant \lambda$ from which follows the statement of the theorem in case $\lambda(x_0) = \mu(x_0)$. And in case $\lambda(x_0) > \mu(x_0)$, it follows from Theorem 3.30 that the Newton polygon $N_G(\lambda(x_0))$ is a straight line. The equivalence of (3.13) and (3.14) (with "$\geqslant$" instead of "$\leqslant$") then shows $\tilde{\lambda} \geqslant \lambda(x_0)$. Together with the inequality $\tilde{\lambda} \leqslant \lambda(x_0)$, this shows $\tilde{\lambda} = \lambda(x_0) = \max\{\lambda(x_0), \mu(x_0)\}$, as desired.

Before proving that (3.15) holds for all $k \in \{m, \ldots, mp-1\}$, we need another piece of preparation. Let $s \geqslant \lambda(x_0)$ be a real number. We will attach to s an element $k_s \in \{m, \ldots, mp-1\}$ and a k_s-admissible pair (i_s, ℓ_s) as follows. For $i \in \{1, \ldots, p-1\}$, the point $(i, v_s(g_i))$ lies on the steepest slope of the Newton polygon $N_G(s)$ if and only if

$$\theta_G(s) = \max_{1 \leqslant j \leqslant p} \frac{v_s(g_0) - v_s(g_j)}{j} = \frac{v_s(g_0) - v_s(g_i)}{i}. \tag{3.16}$$

There exists at least one $i \in \{1, \ldots, p-1\}$ with this property because of Lemma 3.25.

Among those $i \in \{1, \ldots, p-1\}$ such that $(i, v_s(g_i))$ lies on the steepest slope of the Newton polygon $N_G(s)$, choose one for which the quotient $\mathrm{ord}_s(g_i)/i$ is minimal, and denote it by i_s. Lemma 3.11 shows that $\mathrm{ord}_s(g_{i_s})$ is the right derivative of $v_s(g_{i_s})$ at s. Thus for $r = s + \varepsilon$ with $\varepsilon > 0$ small enough, we still have

$$\theta_G(r) = \max_{1 \leqslant j \leqslant p} \frac{v_r(g_0) - v_r(g_j)}{j} = \frac{v_r(g_0) - v_r(g_{i_s})}{i_s}, \tag{3.17}$$

that is, $(i_s, v_r(g_{i_s}))$ lies on the steepest slope of $N_r(G)$. Write $k_s := \mathrm{ord}_s(g_0)$ and $\ell_s := \mathrm{ord}_s(g_{i_s})$. Then we have

$$m \leqslant \mathrm{ord}_\infty(g_0) \leqslant k_s \leqslant \mathrm{ord}_{\mu(x_0)}(g_0) < mp.$$

It follows from Lemma 3.32(a) and from differentiating (3.17) that

$$\frac{k_s - \ell_s}{i_s} = \big(\partial_+ \theta_G\big)(s) \geqslant m > \frac{k_s}{p}.$$

Rearranging this gives

$$(p - i_s)k_s > p\ell_s,$$

that is, (i_s, ℓ_s) is k_s-admissible. Next, let us denote by $r_0, r_1, r_2, \ldots$ the kinks of the piecewise affine function

$$[\lambda(x_0), \infty) \to \mathbb{R}, \qquad r \mapsto v_r(g_0),$$

where we consider $r_0 = \lambda(x_0)$ as the first kink. Suppose that s is one of these kinks. We now show (3.15) for $k = k_s$. In the following estimation, the second "$\leqslant$" is true because of Lemma 3.32(b), and the first "$\leqslant$" is true because the negative of the steepest slope $\theta_G(\lambda(x_0))$ of $N_G(\lambda(x_0))$ must be greater than $v_{\lambda(x_0)}(g_0)/p$. We have:

$$\begin{aligned}
0 &\leqslant \frac{i_s}{p - i_s}\Big(\theta_G(\lambda(x_0)) - \frac{v_{\lambda(x_0)}(g_0)}{p}\Big) \\
&= \frac{i_s}{p - i_s}\Big(\theta_G(s) - \frac{v_s(g_0)}{p} - \int_{\lambda(x_0)}^{s} (\partial_+\theta_G)(r) - \frac{\mathrm{ord}_r(g_0)}{p}\, dr\Big) \\
&\leqslant \frac{i_s}{p - i_s}\Big(\theta_G(s) - \frac{v_s(g_0)}{p} - \int_{\lambda(x_0)}^{s} (\partial_+\theta_G)(s) - \frac{\mathrm{ord}_s(g_0)}{p}\, dr\Big) \\
&= \frac{i_s}{p - i_s}\Big(\frac{v_s(g_0) - v_s(g_{i_s})}{i_s} - \frac{v_s(g_0)}{p} - (s - \lambda(x_0))\big(\frac{k_s - \ell_s}{i_s} - \frac{k_s}{p}\big)\Big) \\
&= \frac{v_s(g_0)}{p} - \frac{v_s(g_{i_s})}{p - i_s} - (s - \lambda(x_0))\big(\frac{k_s}{p} - \frac{\ell_s}{p - i_s}\big) \\
&= \frac{v_K(a_{0,k_s}) + k_s s}{p} - \frac{v_K(a_{i_s,\ell_s}) + \ell_s s}{p - i_s} - (s - \lambda(x_0))\big(\frac{k_s}{p} - \frac{\ell_s}{p - i_s}\big) \\
&= \frac{v_K(a_{0,k_s}) + k_s\lambda(x_0)}{p} - \frac{v_K(a_{i_s,\ell_s}) + \ell_s\lambda(x_0)}{p - i_s}
\end{aligned}$$

Hence from the equivalence of (3.13) and (3.14) we get $\lambda(x_0) \geqslant r(k_s, i_s, \ell_s)$, which proves (3.15) for k_s.

Finally, suppose that $k \in \{m, \dots, mp-1\}$ is not of the form k_s, where s is one of the kinks $r_0, r_1, r_2, \dots$. Then there is a kink s for which $k \geqslant k_s$, and so (i_s, ℓ_s) being k_s-admissible is also k-admissible. We have

$$v_K(a_{0,k}) + kr \geqslant v_r(g_0) \geqslant v_K(a_{0,k_s}) + k_s r$$

for all $r \geqslant \lambda(x_0)$, so that $r(k, i_s, \ell_s) \leqslant r(k_s, i_s, \ell_s)$. It follows that (3.15) holds for k, finishing the proof of the theorem. ∎

We end this section with an example illustrating that Theorem 3.30 is wrong without the assumption $\lambda(x_0) > \mu(x_0)$.

Example 3.34. Consider the plane quartic curve

$$Y\colon \quad y^3 + y^2 + (3x^3 + x^2 + 3)y + x^4 - 3x^2 = 0$$

over $K = \mathbb{C}_3$ and the degree-3 morphism $\varphi\colon Y \to X$ given birationally by $(x, y) \mapsto x$. As in Example 3.13, we take the complement of the branch locus of φ for the chart U. Thus Γ_0 is the tree spanned by the branch points of φ, which include ∞.

We fix the pair of closed points $p_0 = (0, 0)$, $x_0 := \varphi(p_0) = 0$. Observe that Assumption 3.23 is satisfied for $m = 2$. The non-zero coefficients used to define the quantity $\tilde{\lambda}$ from Theorem 3.33 are

$$a_{0,2} = -3, \quad a_{0,4} = 1, \qquad a_{1,0} = 3, \quad a_{1,2} = 1, \quad a_{1,3} = 3, \qquad a_{2,0} = 1.$$

We first consider $k = 2$. The 2-admissible pairs (see the proof of Theorem 3.33 for this terminology) are $(i, \ell) = (1, 0), (2, 0)$. We have

$$\min_{0<i<3} \min_{0 \leqslant 3\ell < 2(3-i)} \left\{ \frac{3v_K(a_{i,\ell}) - (3-i)v_K(a_{0,2})}{(3-i)\cdot 2 - 3\ell} \right\} = \min\left\{ \frac{1}{4}, -\frac{1}{2} \right\} = -\frac{1}{2}.$$

Next, we consider $k = 4$. The 4-admissible pairs are $(i, \ell) = (1, 0), (1, 2), (2, 0)$. We have

$$\min_{0<i<3} \min_{0 \leqslant 3\ell < 4(3-i)} \left\{ \frac{3v_K(a_{i,\ell}) - (3-i)v_K(a_{0,4})}{(3-i)\cdot 4 - 3\ell} \right\} = \min\left\{ \frac{3}{8}, 0, 0 \right\} = 0.$$

It follows that $\tilde{\lambda} = \max\{-1/2, 0\} = 0$. The Newton polygon of the discriminant of the polynomial $y^3 + y^2 + (3x^3 + x^2 + 3)y + x^4 - 3x^2$ defining Y shows that φ has a branch point of valuation $1/3$. Thus $\mu(x_0) = 1/3$ and it follows from Theorem 3.33 that

$$\lambda(x_0) = \max\{\mu(x_0), \tilde{\lambda}\} = \mu(x_0) = \frac{1}{3}.$$

The assumption $\lambda(x_0) > \mu(x_0)$ of Theorem 3.30 is not satisfied, and indeed we have

$$\max_{k \in \{2,4\}} \left\{ \frac{3v_K(a_{1,0}) - 2v_K(a_{0,k})}{2k} \right\} = \max\left\{ -\frac{1}{2}, \frac{3}{8} \right\} = \frac{3}{8},$$

which is strictly larger than $\lambda(x_0)$.

3.5 Guaranteeing that Assumption 3.23 is satisfied

The main results of this chapter, Theorem 3.30 and Theorem 3.33 only hold under the additional Assumption 3.23 on the generator y that is part of the étale chart (U, y) fixed in Section 3.1. In this section we explain how to modify y so that Assumption 3.23 is guaranteed to be satisfied.

Write $A := \mathcal{O}_X(U)$, where U is the open subscheme that is part of the chart (U, y). A non-zero function $f \in A$ may be regarded as a rational function on X. Let us denote by v_P the valuation on $K(x)$ attached to a closed point $P \in X$. If $v_P(f) = m < 0$, then f is said to have a *pole* of order m at P.

The valuations v_P should not be confused with the pseudovaluations attached to closed points that we introduced in Section 1.3. The latter are not actually valuations (taking the values $\pm\infty$ on various functions) and extend the valuation v_K, while the v_P are honest valuations that are trivial on K.

Definition 3.35. Suppose that f is a non-zero regular function on an affine open $W \subset X$ containing U, that is, $f \in \mathcal{O}_X(W)$. The *degree* of f on W is defined to be the product

$$\deg_W(f) = \#(X \backslash W) \cdot \big(\text{highest order of a pole of } f\big).$$

Example 3.36. Consider the elliptic curve E over $K = \mathbb{C}_2$ with Weierstrass equation

$$y^2 = x^3 - x.$$

Then (U_1, y) is an affine étale chart for Y, where $U_1 = \mathbb{P}^1_K \backslash \{x = \infty, 0, 1, -1\}$ (see Example 3.2). The only pole of the function $f = x^3 - x$ is ∞. It is a pole of order 3, so we have $\deg_{W_1}(f) = 3$, where $W_1 = \mathbb{P}^1_K \backslash \{x = \infty\}$. The curve E may also be described by the equation

$$z^2 = \frac{u^3}{1 - u^2}, \qquad \text{where} \qquad z = 1/y, \quad u = 1/x.$$

Thus (U_2, z) is an affine étale chart for E, where $U_2 = \mathbb{P}^1_K \backslash \{u = \infty, 0, 1, -1\}$. The function $g = u^3/(1 - u^2)$ has three poles, 1, -1, and ∞, each of order 1. Thus we find $\deg_{W_2}(g) = 3$ as well, where $W_2 = \mathbb{P}^1_K \backslash \{u = 1, -1, \infty\}$.

Remark 3.37. In Chapter 4, all functions $f \in A \subset K(x)$ whose degree we consider will actually be polynomials in x. Thus they are regular functions on $W := \mathbb{A}^1_K = \mathbb{P}^1_K \backslash \{\infty\}$, with at most a pole at ∞. In this case, the function $\deg_W$ coincides with the usual degree function on the polynomial ring $K[x]$.

It is also in this light that the properties proved in the following lemma should be viewed.

Lemma 3.38. *For $f, g \in \mathcal{O}_X(W) \backslash \{0\}$ we have:*

(a) $\deg_W(fg) \leqslant \deg_W(f) + \deg_W(g)$

(b) If $f+g \neq 0$*, then* $\deg_W(f+g) \leqslant \max\big(\deg_W(f), \deg_W(g)\big)$

(c) If $\mathsf{D} \subset W^{\mathrm{an}}$ *is an open disk, then* $\mathrm{ord}_{\mathsf{D}}(f) \leqslant \deg_W(f)$

Proof. (a) Consider a closed point $P \in X\backslash W$. We have $v_P(fg) = v_P(f) + v_P(g)$, so the highest possible order of a pole of fg at P is the order of a pole of f at P plus the order of a pole of g at P. This immediately implies (a).

(b) The inequality $v_P(f+g) \geqslant \min(v_P(f), v_P(g))$ shows that the order of a pole of $f+g$ at P can not exceed the order of a pole of f at P nor the order of a pole of g at P. This immediately implies (b).

(c) The function f can have no more than $\deg_W(f)$ zeros on W, counted with multiplicity (since the degree of a principal divisor is 0, [Har77, Corollary II.6.10]). Thus (c) results from Lemma 3.8.

■

As in Section 3.1, write

$$G(T) = T^p + g_{p-1}T^{p-1} + \ldots + g_1T + g_0$$

for the minimal polynomial of y over the subfield $K(t)$, where $t = x - x_0$ is the parameter associated to the closed point $x_0 \in U$. We have studied the formal solutions $y_1, \ldots, y_p$ to the equation $G(T) = 0$; they are elements of $K[\![t]\!]$. Let us now choose and fix one of these, say y_1, and write

$$y_1 = \sum_{i=0}^{\infty} b_i t^i \in K[\![t]\!].$$

Let $m \geqslant 1$ be an integer. We denote by u_m the truncation of y_1 consisting of the first m monomials,

$$u_m = b_0 + b_1t + \ldots + b_{m-1}t^{m-1},$$

and consider the generator $w := y - u_m$ of the field extension F_Y/F_X. Let us write

$$H(T) = \sum_{i=0}^{p} h_iT^i = G(T + u_m) \tag{3.18}$$

for the minimal polynomial of w. We call the integer m the *order of approximation* of w. The following lemma will help us decide what order of approximation to choose.

Lemma 3.39. *Let* $W \subseteq \mathbb{A}^1_K$ *be an affine open subscheme containing* U *such that* $g_0, g_1, \ldots, g_{p-1} \in \mathcal{O}_X(W)$*. If the integer* m *satisfies*

$$m > \frac{\deg_W(g_i) - i}{p - i} \tag{3.19}$$

for all $0 \leqslant i \leqslant p-1$*, then*

$$\mathrm{ord}_{\mathsf{D}(\mu(x_0))}(h_0) < mp.$$

Proof. We have

$$h_0 = G(u_m) = u_m^p + g_{p-1}u_m^{p-1} + \ldots + g_1 u_m + g_0.$$

It is convenient, to also set $g_p := 1$. Since the $g_0, \ldots, g_p$ and u_m lie in $\mathcal{O}_X(W)$ (the latter even in $K[t] = \mathcal{O}_X(\mathbb{A}_K^1)$), we obtain the desired estimate as follows:

$$\begin{aligned}
\mathrm{ord}_{\mathsf{D}(\mu(x_0))}(h_0) &\leqslant \deg_W(h_0) && \text{(by Lemma 3.38(c))}\\
&\leqslant \max_{0\leqslant i\leqslant p}\{\deg_W(u_m^i g_i)\} && \text{(by Lemma 3.38(a))}\\
&\leqslant \max_{0\leqslant i\leqslant p}\{i(m-1)+\deg_W(g_i)\} && \text{(by Lemma 3.38(b)}\\
&< mp && \text{(by (3.19))}
\end{aligned}$$

■

Corollary 3.40. *Suppose that there exist an affine open subscheme $W \subseteq \mathbb{A}_K^1$ containing U with $g_0, g_1, \ldots, g_{p-1} \in \mathcal{O}_X(W)$ and an integer $m \geqslant 1$ satisfying* (3.19). *Then (U, w) is a chart satisfying Assumption 3.23 for this integer m.*

Proof. Since (U, y) is an affine étale chart, and $w = y - u_m$ for a function $u_m \in \mathcal{O}_X(U)$, it follows that (U, w) is an affine étale chart as well.

Since u_m is the truncation of a formal solution of the minimal polynomial G, it follows that $h_0 = G(u_m)$ has a zero of order at least m at $t = 0$. Thus the first inequality in Assumption 3.23 holds. The second inequality holds by Lemma 3.39. ■

Recall the discussion surrounding Lemma 3.14 in Section 3.1. We considered a "generic" version of the minimal polynomial of y,

$$\tilde{F}(T) = T^p + \tilde{f}_{p-1}T^{p-1} + \ldots + \tilde{f}_1 T + \tilde{f}_0 \in A[\![t]\!][T],$$

from which the minimal polynomial G of y over the subfield $K(t)$ (where $t = x - x_0$) is obtained by evaluating the coefficients at x_0. Let us also now denote by

$$\tilde{y} = y + b_1 t + b_2 t^2 + b_3 t^3 + \ldots \in B[\![t]\!]$$

the "generic" solution to $\tilde{F}$, from which y_1 is obtained by evaluation at the corresponding point p_0 lying above x_0. Using these ingredients, we may also define a "generic" version of the minimal polynomial $H(T)$ of w. Namely, first take the truncation of $\tilde{y}$ of level m,

$$\tilde{u}_m := y + b_1 t + \ldots + b_{m-1}t^{m-1} \in B[t],$$

and then define

$$\tilde{H}(T) = \sum_{i=0}^{p} \tilde{h}_i T^i := \tilde{F}(\tilde{u}_m + T).$$

Now $H(T)$ is obtained from $\tilde{H}(T)$ by evaluating at x_0. That is, if we write

$$\tilde{h}_i = \sum_{\ell=0}^{\infty} c_{i,\ell} t^\ell, \qquad c_{i,\ell} \in B,$$

then we have

$$h_i = \sum_{\ell=0}^{\infty} c_{i,\ell}(p_0) t^\ell.$$

As usual, we denote by $\widehat{c}_{i,\ell}$ the associated valuative function $Y^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$. We may then rephrase Theorems 3.30 and 3.33, as follows:

Corollary 3.41. *If $\lambda(x_0) > \mu(x_0)$, then we have*

$$\lambda(x_0) = \max_{m \leqslant k < mp} \frac{p\widehat{c}_{1,0}(p_0) - (p-1)\widehat{c}_{0,k}(p_0)}{(p-1)k}.$$

Corollary 3.42. *We have $\lambda(x_0) = \max\{\mu(x_0), \tilde{\lambda}(p_0)\}$, where*

$$\tilde{\lambda}(p_0) = \max_{m \leqslant k < mp} \min_{0 < i < p} \min_{0 \leqslant p\ell < (p-i)k} \frac{p\widehat{c}_{i,\ell}(p_0) - (p-i)\widehat{c}_{0,k}(p_0)}{(p-i)k - p\ell}.$$

Here p_0 is any closed point in $\varphi^{-1}(x_0)$.

It should be stressed that we just took an important conceptual step. Where Theorems 3.30 and 3.33 were concerned with describing the value of λ at a fixed point x_0 in terms of certain constants derived from x_0, the preceding two corollaries describe the value of λ at *any* point x_0 in terms of valuative functions independent of x_0. Crucially, the functions $c_{i,\ell}$ yield an equation satisfying Assumption 3.23 with respect to *any* closed point $x_0 \in U$, by evaluation at a closed point p_0 above x_0.

Example 3.43. Consider a superelliptic curve of degree p, say

$$Y\colon \quad F(y) = y^p - f = 0, \qquad f = \sum_{i=0}^{d} a_i x^i \in K[x].$$

The corresponding cover $Y \to X = \mathbb{P}^1_K$ is Galois, since K is algebraically closed. We take the chart $U = \operatorname{Spec} A$, where $A = K[x]_f$ (cf. Example 3.2). The "generic" minimal polynomial of y is

$$\tilde{F}(T) = T^p - \sum_{i=0}^{d} \frac{f^{(i)}}{i!} t^i \in A[\![t]\!][T].$$

It will be useful to divide the constant coefficient of $\tilde{F}$ by $f \in A^\times$, yielding

$$\tilde{G}(T) = T^p - \sum_{i=0}^{d} \frac{f^{(i)}}{i!f} t^i \in A[\![t]\!][T].$$

Indeed, this has the effect that there is a formal solution

$$\tilde{y} = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots \in A[\![t]\!]$$

to $\tilde{G}$ provided by Lemma 3.14 beginning with the constant 1. It is crucial that $\tilde{y} \in A[\![t]\!]$ — compare the general case of (3.9), where $\tilde{y} \in B[\![t]\!]$. Following Lemma 3.39, we define

$$m := \left\lfloor \frac{\deg(f)}{p} \right\rfloor + 1$$

and write

$$\tilde{u}_m = 1 + b_1 t + \ldots + b_{m-1} t^{m-1} \in A[t]$$

for the truncation of $\tilde{y}$ of order m. Moreover, as usual we write

$$\tilde{H}(T) := \tilde{G}(\tilde{u}_m + T) = T^p + p\tilde{u}_m T^{p-1} + \ldots + p\tilde{u}_m^{p-1} T + \tilde{u}_m^p - \sum_{i=0}^{d} \frac{f^{(i)}}{i! f} t^i,$$

and for a fixed closed point x_0 consider the polynomial obtained from $\tilde{H}$ by evaluating coefficients at x_0,

$$H(T) = T^p + p u_m T^{p-1} + \ldots + p u_m^{p-1} T + u_m^p - \sum_{i=0}^{d} \frac{f^{(i)}(x_0)}{i! f(x_0)} t^i,$$

$$\text{where} \quad u_m = 1 + b_1(x_0) t + \ldots + b_{m-1}(x_0) t^{m-1}.$$

The generator w that is a solution to $H(T)$ is related to the original generator y by

$$w = \frac{y}{f(x_0)^{1/p}} - u_m.$$

It follows from Corollary 3.40 that the chart (U, w) satisfies Assumption 3.23.

In a certain sense we have now reached our original goal of determining the semistable reduction of curves Y equipped with a degree-p morphism $\varphi\colon Y \to X$. Corollary 3.42 completely determines the function $\lambda = \lambda_{\varphi,\Gamma_0}$. And according to Theorem 2.34, if we can enlarge the skeleton Γ_0 of X to a skeleton Γ trivializing λ, then $\Sigma := \varphi^{-1}(\Gamma)$ is a skeleton of Y. Or, using the language of models, if $\mathcal{X}$ is the semistable $\mathcal{O}_K$-model of X obtained from Γ as in Remark 1.32, then the normalization $\mathcal{Y}$ of $\mathcal{X}$ in the function field F_Y is a semistable model of $\mathcal{Y}$.

However, this solution is not quite satisfactory for two reasons:

- The general formula for λ of Corollary 3.42 is quite unwieldy, involving many terms.
- Our formulas for λ are in terms of admissible functions on Y^{an}. For working with concrete examples and for implementation it is much more advantageous to work with admissible functions on X^{an}.

In the next section, we explain an approach bypassing these issues. When studying plane quartics in the next chapter, we will only appeal to the simpler formula for λ of Corollary 3.41. And we will see how to describe λ using admissible functions on X^{an}.

3.6 Tame locus

We denote by Σ the φ-minimal skeleton of Y with respect to Γ_0 (Definition 2.25). It is minimal among all skeletons of Y containing the inverse image $\varphi^{-1}(\Gamma_0)$. Attached to Σ is the canonical retraction map of Definition 1.37,

$$\mathrm{ret}_\Sigma \colon Y^{\mathrm{an}} \to \Sigma.$$

Definition 3.44. The *tame locus* associated to φ and Γ_0 is the subset

$$Y^{\mathrm{tame}} := Y^{\mathrm{tame}}_{\varphi,\Gamma_0} := \mathrm{ret}_\Sigma^{-1}\big(\{\eta \in \Sigma \mid \delta(\eta) = 0\}\big).$$

If $\eta \in Y^{\mathrm{an}}$ is a Type II point that is a leaf of Σ (Definition 1.35) with $\delta(\eta) = 0$, we call $\mathrm{ret}_\Sigma^{-1}(\eta)$ a *tail component* of Y^{tame}. The union of all the tail components is called the *tail locus*, denoted Y^{tail}. The complement

$$Y^{\mathrm{int}} := Y^{\mathrm{tame}} \backslash Y^{\mathrm{tail}}$$

is called the *interior tame locus*.

Note that the tame locus is an affinoid subdomain of Y^{an} (or equals Y^{an}) by the same argument as the one used in the proof of Lemma 2.6: Its complement consists of finitely many open disks and anuli, so it is an affinoid subdomain by [BPR13, Lemma 4.12]. Likewise, the interior tame locus, the tail locus, and each tail component are obtained from Y^{tame} by removing certain connected components. Thus they are themselves affinoid subdomains (or equal to Y^{an}).

Lemma 3.45. *Let $p_0 \in Y \backslash \Sigma$ be a closed point with image $x_0 = \varphi(p_0)$ in X. Then we have the following:*

(a) If $p_0 \in Y^{\mathrm{int}}$, then $\lambda(x_0) = \mu(x_0)$.

(b) If $p_0 \notin Y^{\mathrm{int}}$, then $\lambda(x_0) > \mu(x_0)$.

Proof. Since Σ is a skeleton, there exists a unique path $[p_0, \eta]$ with $[p_0, \eta] \cap \Sigma = \{\eta\}$. The image $\varphi([p_0, \eta])$ is the unique path connecting x_0 and $\xi := \varphi(\eta)$. The point p_0 lies in Y^{tame} if and only if $\delta(\eta) = 0$ and lies in Y^{tail} if and only if $\delta(\eta) = 0$ and η is a leaf of Σ.

According to the construction of Σ (see Proposition 2.24), if $\delta(\eta) = 0$ and η is a leaf of Σ, then η does not lie on the graph $\varphi^{-1}(\Gamma_0)$. Moreover, δ is strictly positive on the interval connecting ξ and Γ. Thus it follows from Proposition 2.31 that $\lambda(x_0) > \mu(x_0)$.

Similarly, if $\delta(\eta) = 0$, but η is not a leaf of Σ, then η lies on the graph $\varphi^{-1}(\Gamma_0)$. Thus $\xi \in \Gamma_0$ and so $\lambda(x_0) = \mu(x_0)$.

Finally, in the case that $\delta(\eta) = \delta(\xi) > 0$, it follows immediately from Proposition 2.31 that $\lambda(x_0) > \mu(x_0)$. ■

The following proposition shows that the tame locus, which was defined using a φ-minimal skeleton Σ of Y, retains the information contained in Σ and thus controls the semistable reduction of Y.

Proposition 3.46. *Let $\Sigma_1 \subset Y^{\mathrm{an}}$ be a graph containing $\varphi^{-1}(\Gamma_0)$ whose vertex set $S(\Sigma_1)$ contains all boundary points of Y^{tame}. Then Σ_1 is a skeleton of Y.*

Proof. We check that Σ_1 contains all vertices of the φ-minimal model Σ. Recall its construction in Section 2.5. It is obtained from the graph $\Sigma_0 = \varphi^{-1}(\Gamma_0)$ by adding to the vertex set $S(\Sigma_0)$ the points of positive genus on Σ_0 and attaching to Σ_0 all paths to points of positive genus in the complement of Σ_0.

Consider first a point η of positive genus on Σ_0. Clearly η is a topological ramification point with $\delta(\eta) = 0$. If η is not among the vertices of $\varphi^{-1}(\Gamma_0)$, then there are only two branches at η contained in Σ_0, which we denote v_1, v_2. The Riemann-Hurwitz Formula [CTT, Theorem 4.5.4] applied to η reads

$$2g(\eta) - 2 = -2n_\eta + n_{v_1} + \mathrm{slope}_{v_1}(\delta) + n_{v_2} + \mathrm{slope}_{v_2}(\delta) - 2.$$

If η is a tame topological ramification point, then $\delta = 0$ in a neighborhood of η, so we get

$$2g(\eta) = -2n_\eta + n_{v_1} + n_{v_2} \leqslant 0,$$

contradicting the assumption that η has positive genus. If η is a wild topological ramification point, a similar argument shows that we must have $\mathrm{slope}_{v_1}(\delta) > 0$ or $\mathrm{slope}_{v_2}(\delta) > 0$. In this case, η is a boundary point of the interior tame locus.

Finally consider a point η of positive genus not on Σ_0. Again we have $\delta(\eta) = 0$, while the path connecting η and Σ_0 consists by Proposition 2.22 of wild topological ramification points. Thus η is a boundary point of the tail locus Y^{tail}. ■

Our eventual goal is to compute for plane quartic curves Y a skeleton from knowledge of the tame locus Y^{an}. For illustrative purposes, we assume conversely that the φ-minimal skeleton $\Sigma \subset Y^{\mathrm{an}}$ is known in advance in the following examples. We will explain how to compute Σ (and the restriction of δ to Σ) in Chapter 4. In particular, we will revisit the following examples in Section 4.7.

Example 3.47. We consider the plane quartic curve

$$Y: \quad y^3 + 3xy^2 - 3y - 2x^4 - x^2 - 1 = 0$$

over $K = \mathbb{C}_3$, previously studied in Example 3.13. As usual, φ denotes the associated cover $Y \to \mathbb{P}^1_K$. We take for Γ_0 the skeleton spanned by the branch points of φ, which include ∞ (as we saw in Example 3.13). We will see in Example 4.17 that the φ-minimal skeleton Σ of Y with respect to Γ_0 looks as depicted in Figure 3.1.

The leaves labeled "1" are Type II points of genus 1. The other leaves lie above the branch points of φ. It follows from Lemma 4.1 below that φ is totally ramified above ∞, while the remaining eight branch points are ordinary. Above each of the ordinary branch points lies a ramification point and an unramified point. In Figure 3.1, these ordinary ramification points are colored blue, while the locus where δ is positive is colored red. The latter includes the ramification point above ∞, labeled "∞".

Thus the tame locus associated to φ is the disjoint union of eleven components. Eight of them are closed disks, containing one ramification point each. The other three are tail components whose unique boundary point has genus 1.

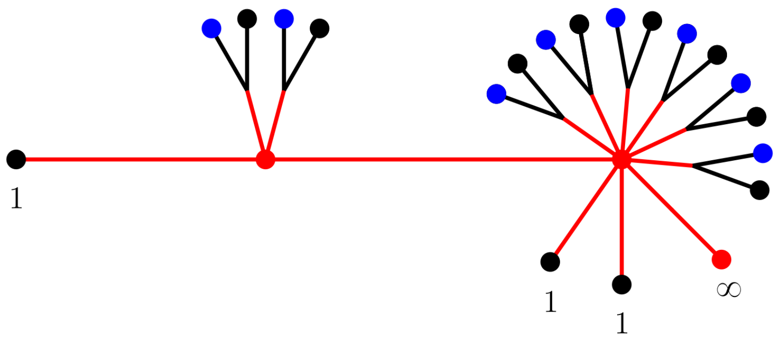


Figure 3.1: The φ-minimal skeleton of the curve considered in Example 3.47, with the locus where $\delta > 0$ in red

Example 3.48. Consider the plane quartic curve

$$Y: \quad y^3 - y^2 + (3x^3 + 1)y + 3x^4 = 0$$

over $K = \mathbb{C}_3$. Again we take for Γ_0 the skeleton spanned by the branch points of $\varphi\colon Y \to \mathbb{P}^1_K$, which again include ∞. We will see in Example 4.18 that the φ-minimal skeleton Σ of Y with respect to Γ_0 looks as depicted in Figure 3.2.

Again, the vertices labeled "1" are Type II points of genus 1. This time, there is only two of these, but the skeleton has a loop as well. All leaves lie above branch points of φ, so $Y^{\mathrm{tail}} = \varnothing$. There are ten branch points, all ordinary, among them ∞. Thus above each of them lie two points, one a ramification point, the other not. The ramification points are colored blue as in Example 3.47, while the locus where $\delta > 0$ is colored red.

The interior locus has seven components of which only the one on the left is important, containing the loop and the two points of positive genus. We note that the left point of positive genus has topological ramification index 2 (the point right above it has the same image), while the right point of positive genus is a wild topological branch point. It is part of the wild topological branch locus, but still we have $\delta = 0$ there (the case of Lemma 2.17(b)).

An illustration of Y^{an} is drawn on the cover of this thesis[1], also depicting the locus where $\delta > 0$ in red. For artistic reasons, and to visualize the myriad options to add Type II points to a skeleton of Y^{an}, it is drawn with much branching and squiggliness.

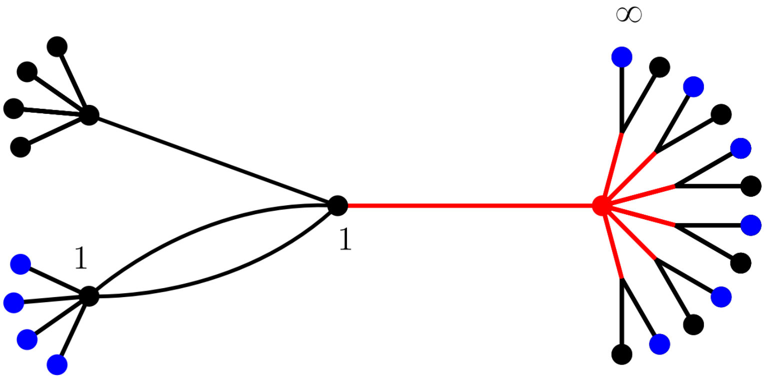


Figure 3.2: The φ-minimal skeleton of the curve considered in Example 3.48, with the locus where $\delta > 0$ in red

Example 3.49. We consider again the superelliptic curve Y given by $y^p = f$ of Example 3.43, keeping all notation introduced there. We will give a criterion for a closed point $x_0 \in U$ to lie in Y^{tame} dependent on the coefficients of the minimal polynomial

$$H(T) = T^p + pu_m T^{p-1} + \ldots + pu_m^{p-1} T + u_m^p - \sum_{i=0}^{d} \frac{f^{(i)}(x_0)}{i! f(x_0)} t^i$$

belong to the chart (U, w) associated to x_0. We write

$$g := u_m^p - \sum_{i=0}^{d} \frac{f^{(i)}(x_0)}{i! f(x_0)} t^i = \sum_{i=m}^{mp-1} c_i t^i.$$

Suppose first that $\lambda(x_0) > \mu(x_0)$. Recall from Lemma 3.27 that there is a unique point $\eta \in Y^{\mathrm{an}}$ above $\xi_{\lambda(x_0)}$, whose reduction curve we now investigate. According to Lemma 3.29, $(1, v_{\lambda(x_0)}(pu_m^{p-1}))$ is a vertex of the Newton polygon of $H(T)$ with respect to $v_{\lambda(x_0)}$, which is a straight line. It follows that

$$\frac{v_{\lambda(x_0)}(pu_m^{p-i})}{p-i} = \frac{1}{p-i} + v_{\lambda(x_0)}(u_m) > \frac{1}{p-1} + v_{\lambda(x_0)}(u_m) = \frac{v_{\lambda(x_0)}(pu_m^{p-1})}{p-1}$$

[1]Not included in the electronic version. You can take a look at my GitHub page.

for $i = 2, \dots p-1$, so that $(i, v_{\lambda(x_0)}(pu_m^{p-i}))$ does not lie on $N_H(\lambda(x_0))$ for $i = 2, \dots, p-1$. By Remark 3.28(a), the minimal polynomial of $\overline{w} := [w]_\eta$ is then given by

$$\overline{H}(T) = T^p + [pu_m^{p-1}]_{\lambda(x_0)} T + [g]_{\lambda(x_0)}.$$

The discriminant of this polynomial is

$$\Delta_{\overline{H}} = \det \begin{pmatrix} [pu_m^{p-1}]_{\lambda(x_0)} & & & \\ & [pu_m^{p-1}]_{\lambda(x_0)} & & \\ & & \ddots & \\ & & & [pu_m^{p-1}]_{\lambda(x_0)} \end{pmatrix} = [pu_m^{p-1}]^p_{\lambda(x_0)}$$

(the matrix is the Sylvester matrix associated to $\overline{H}$ and $\overline{H}'$ as defined in [Stacks, Tag 00UA]; it is a p-by-p diagonal matrix). By Remark 3.28(b), this discriminant $\Delta_{\overline{H}}$ is a unit. Thus the reduction curve at η is given by an Artin-Schreier equation

$$T^p + \overline{c}T + \overline{g}, \qquad \overline{c} \in \kappa, \quad \overline{g} = [g]_{\lambda(x_0)}.$$

This equation defines a curve of positive genus if and only if $\overline{g}$ has a non-zero coefficient of index different from the unique power q of p contained in the set $\{m, \dots, mp-1\}$. (See Section 4.2 below and in particular Lemma 4.2 for this.) By Remark 3.31(a), this is the case if and only if there exists a $k \in \{m, \dots, mp-1\} \backslash \{q\}$ with

$$\frac{p - (p-1)v_K(c_k)}{(p-1)k} \geqslant \frac{p - (p-1)v_K(c_q)}{(p-1)q}. \tag{3.20}$$

Now suppose that $\lambda(x_0) = \mu(x_0)$. By Lemma 3.45, we then have $x_0 \in Y^{\mathrm{int}}$. Unless x_0 already satisfies (3.20), we have

$$\mu(x_0) \geqslant \frac{p - (p-1)v_K(c_q)}{(p-1)q}. \tag{3.21}$$

In summary, we have $x_0 \in Y^{\mathrm{tame}}$ if and only if one of the inequalities (3.20) or (3.21) holds. These inequalities are the same as the inequalities used to compute the *étale locus* associated to Y (cf. [BW15, Section 4.2]) in the MCLF class `SuperpModel`. Thus the étale locus associated to a superelliptic curve of degree p coincides with its tame locus defined in this section.

Chapter 4

Smooth plane quartics

In this chapter, we apply the theory developed in Chapter 3 to smooth plane quartic curves Y, or just plane quartics for short. In Section 4.1, we introduce a normal form for plane quartics that is available whenever the curve Y in question has a rational point.

Our main goal is to gain a better understanding of the tame locus associated to Y. In Sections 4.5 and 4.6, we study the image of the tame locus in the projective line $\mathbb{P}^1_K$. The advantage of this approach is that we have a concrete understanding of the analytification $(\mathbb{P}^1_K)^{\mathrm{an}}$, which allows us to compute this image, and hence the semistable reduction of Y, for concrete examples.

Every curve Y of genus 3 over K is either hyperelliptic or else canonically embedded as a quartic plane curve in $\mathbb{P}^2_K$ ([Vak23, Section 19.7]). It is known how to compute the semistable reduction of any hyperelliptic K-curve (in the case that $p \neq 2$, using the theory of admissible reduction as explained in Section 1.8; in the case $p = 2$ as explained in [BW15, Section 4]; see also Example 3.49). As we will recall below, plane quartics are *trigonal* curves, so if $p > 3$, then the method explained in Section 1.8 will work for them. In this chapter, we explain one of the remaining cases, that of reduction at $p = 3$.

4.1 Normal form

In this section, we allow K to be either algebraically closed or discretely valued.

A plane quartic is, by definition, a closed subscheme $Y \subset \mathbb{P}^2_K$ cut out by a homogeneous polynomial

$$\sum_{i+j+k=4} a_{ijk} x^i y^j z^k, \qquad a_{ijk} \in K.$$

We shall assume that Y has a K-rational point. Without loss of generality we may take this rational point to be the point $[0 : 1 : 0]$. Then we must have $a_{0,4,0} = 0$. We may also assume that $a_{1,3,0} = 0$. Indeed, if $a_{0,3,1} \neq 0$ we may replace z with $z - \frac{a_{1,3,0}}{a_{0,3,1}} x$ to eliminate the xy^3-term. And if $a_{0,3,1} = 0$, but $a_{1,3,0} \neq 0$ we can simply swap x and z.

Dehomogenizing by putting $z = 1$, an affine chart of Y is cut out of $\mathbb{A}^2_K$ by the polynomial

$$F = y^3 + Ay^2 + By + C,$$

where $A, B, C \in K[x]$ are polynomials of degree at most 2, at most 3, and at most 4 respectively. In fact, it is possible to eliminate the term Ay^2, which yields the normal form of [Shi93, Section 1], but we do not need this. In what follows, the discriminant of F,

$$\Delta_F = A^2B^2 - 4B^3 - 4A^3C - 27C^2 + 18ABC,$$

will play an important role.

From now on we assume that Y is smooth and geometrically connected. Then F is irreducible and the function field of Y is $F_Y = K(x, y)$, a cubic extension of the rational function field $K(x)$, whose generator y has minimal polynomial F. The corresponding morphism

$$\varphi\colon Y \to \mathbb{P}^1_K$$

is a degree-3 cover of the projective line. Geometrically, it is given by "projection to the x-axis". That is, points (x, y) in $Y \cap \mathbb{A}^2_K$ are mapped to their x-coordinate under φ.

The following lemma describes the branch locus of φ. Recall from Section 3.5 that to every closed point P on $X = \mathbb{P}^1_K$, there is an associated valuation v_P on the function field $F_X = K(x)$, trivial on K. It is the valuation attached to the local ring $\mathcal{O}_{X,P}$, which is a discrete valuation ring. Similarly, there is a valuation v_Q on the function field F_Y for every closed point Q on Y.

A closed point $P \in \mathbb{P}^1_K$ is called *unramified* with respect to φ if $e_Q = 1$ for every point Q above P. It is called *totally ramified* with respect to φ if there is a unique point Q above P with $e_Q = 3$. It is called an *ordinary branch point* with respect to φ if there exists a point Q above P with $e_Q = 2$. In the last case, there necessarily is another point Q' above P, with $e_{Q'} = 1$.

Lemma 4.1. *Suppose that* $\operatorname{char}(K) = 0$. *Let* $P \in \operatorname{Spec} K[x]$ *be a closed point and let* ∞ *denote the point at infinity in* $\mathbb{P}^1_K$.

(a) *P is unramified with respect to φ if and only if $v_P(\Delta_F) = 0$, is an ordinary branch point if and only if $v_P(\Delta_F) = 1$, and is totally ramified if and only if $v_P(\Delta_F) = 2$.*

(b) *∞ is unramified with respect to φ if and only if* $\deg(\Delta_F) = 10$, *is an ordinary branch point if and only if* $\deg(\Delta_F) = 9$, *and is totally ramified if and only if* $\deg(\Delta_F) = 8$.

Proof. The inverse image of $\mathbb{A}^1_K = \operatorname{Spec} K[x]$ in Y is the affine chart $V = \operatorname{Spec} R$, where $R = K[x, y]/(F)$. The different of the ring extension $R/K[x]$ is the principal ideal $(F'(y))$. Let $Q \in V$ be a closed point. Then according to [Neu99, Theorem III.2.6], we have $e_Q - 1 = v_Q(F'(y))$.

Part (a) now quickly follows from the fact that $(\Delta_F) = (\mathrm{Nm}(F'(y))$ ([Neu99, Theorem III.2.9]). Indeed, if P is an ordinary branch point, there are two points Q_1, Q_2 above P, say with $v_{Q_1}(F'(y)) = 1$ and $v_{Q_2}(F'(y)) = 0$. It follows that

$$v_P(\Delta_F) = v_P(\mathrm{Nm}(F'(y)) = v_{Q_1}(F'(y)) + v_{Q_2}(F'(y)) = 1.$$

The rest of (a) is proved in the same way.

For (b), we use the Riemann-Hurwitz formula. In the present situation it reads

$$4 = 2g_Y - 2 = \deg(\varphi)(2g_X - 2) + \sum_Q (e_Q - 1) = -6 + \sum_Q (e_Q - 1),$$

where the sums are taken over the closed points of Y. It follows from (a) that

$$\sum_{Q \in Y\setminus V} (e_Q - 1) = 10 - \sum_{P \in \mathrm{Spec}\, K[x]} v_P(\Delta_F) = 10 - \deg(\Delta_F). \tag{4.1}$$

Depending on whether ∞ is unramified, an ordinary branch point, or totally ramified, the sum on the left-hand side of (4.1) is equal to 0, 1, or 2 respectively. Thus we get (b). ■

4.2 Degree-p covers in characteristic p

Before continuing, we collect some facts about covers of curves of degree p over a field k of characteristic p. The crucial fact here is that there exist non-trivial étale covers of the affine line $\mathbb{A}^1_k$, in contrast to the situation for fields of characteristic 0.

The most important class of such covers is *Artin-Schreier covers*, which we discuss first.

Let k be an algebraically closed field of characteristic p, let $f = \sum_i a_i x^i \in k[x]$ be a polynomial of positive degree, and let $c \in k^\times$ be any element. Let us further suppose that f is not of the form $f = g^p - cg$ for some $g \in k[x]$. Then the polynomial

$$y^p - cy - f$$

is an irreducible element of the polynomial ring $k(x)[y]$ over the rational function field, and therefore defines a field extension of degree p. Let us denote by $\varphi\colon Y \to \mathbb{P}^1_k$ the corresponding morphism of curves.

Lemma 4.2. *If* $\deg(f)$ *is not a multiple of* p*, then*

$$g = \frac{(p-1)(\deg(f) - 1)}{2}.$$

Proof. We may assume that $c = 1$, by replacing y with $yc^{1/(p-1)}$ if necessary. Now this follows immediately from [Sti09, Theorem 3.7.8(d)]. ■

If $d := \deg(f)$ is a multiple of p, Lemma 4.2 is not directly applicable. But we may proceed as follows to compute the genus of Y. Write $d = p\ell$ and choose a p-th root $a_d^{1/p}$ of the leading coefficient a_d of f. Then substituting $y + a_d^{1/p} x^\ell$ for y replaces f with a new polynomial of lower degree. Continuing this procedure until the degree of f is no longer divisible by p, we may then apply Lemma 4.2.

It might happen that we end up with a polynomial f of degree 1; in this case, Lemma 4.2 tells us that Y has genus 0.

This is essentially the procedure of [Sti09, Lemma 3.7.7]. Compare also [Far10, Proposition 2.1.1], where the above change of variables is used to derive a normal form $y^p - y = f$ for Artin-Schreier covers such that $\deg(f)$ is coprime to p.

Now we consider the case that $p = 3$. Let k be an algebraically closed field of characteristic 3 and let $A, B, C \in k[x]$ be polynomials of degree $\leqslant 2$, $\leqslant 3$, and $\leqslant 4$ respectively. Let us further suppose that the polynomial

$$f = y^3 + Ay^2 + By + C$$

is an irreducible element of the polynomial ring $k(x)[y]$, so defines a field extension of degree 3. Let us denote again by $\varphi\colon Y \to \mathbb{P}^1_k$ the corresponding morphism of curves.

Lemma 4.3. *Assume that the discriminant Δ_f of the polynomial f is a unit (i. e. an element of $k^\times$). Then $A = 0$ and B is a constant in $k^\times$.*

Proof. Note that we have

$$\Delta_f = A^2B^2 - B^3 - A^3C.$$

Let us first suppose that $A \neq 0$. Then we can define

$$z := \frac{\sqrt{\Delta_f}}{A^2y - AB},$$

where $\sqrt{\Delta_f}$ is a choice of square root of the constant Δ_f. An easy calculation shows that this element satisfies the Artin-Schreier equation

$$z^3 - z - \frac{\sqrt{\Delta_f}}{A^3}.$$

If A is a constant, then we see that z and hence y are contained in $k(x)$, which is impossible. If A has degree $\geqslant 1$, then φ is ramified at the roots of A by [Sti09, Proposition 3.7.8(b)], contradicting that Δ_f is a unit.

Thus we find that $A = 0$. In this case Δ_f can only be a constant if B is a constant, and we are done. ∎

In the situation of Lemma 4.3, the discriminant Δ_f is in particular a square. Thus the Galois group of the polynomial f is contained in A_3. Said differently, the cover $Y \to \mathbb{P}^1_k$ is Galois.

Remark 4.4. (a) In Lemma 4.3, it is not enough to assume that $\varphi\colon Y \to \mathbb{P}^1_k$ is unramified outside of ∞. For example, the cover $Y \to \mathbb{P}^1_k$, where Y over $k = \overline{\mathbb{F}}_3$ is given by

$$f(y) = y^3 + y^2 + xy - x^4 = 0,$$

is only ramified at ∞. But $\Delta_f = x^2(x+1)^2$ is not a unit. The problem is that f does not describe Y as a smooth plane curve.

(b) It seems likely that Lemma 4.3 has the following generalization to algebraically closed fields k of arbitrary positive characteristic p: Suppose that

$$f(y) = y^p + f_{p-1}y^{p-1} + \ldots + f_1 y + f_0 \in k[x][y]$$

is an irreducible polynomial with $\Delta_f \in k^\times$. Then f is an Artin-Schreier polynomial, that is, $f_{p-1} = \ldots = f_2 = 0$ and $f_1 \in k^\times$. Said differently, a degree-p étale cover of $\mathbb{A}^1_k$ by a smooth plane curve is necessarily Artin-Schreier.

If this statement were true, it should allow us to generalize much of the content of this chapter to arbitrary residue characteristic. But as of now we could not prove it.

4.3 Setup

For the rest of this chapter, K is assumed to be algebraically closed of residue characteristic 3.

We have seen in Example 3.3 how to define an étale chart for a smooth plane curve. We now apply this to a smooth plane quartic as considered in Section 4.1. In particular we have at our disposal the equation for Y,

$$0 = F(T) = T^3 + AT^2 + BT + C,$$

and the discriminant Δ_F of F. The affine étale chart we use is then (U, y), where

$$U = D(\Delta_F) \subseteq \mathbb{A}^1_K,$$

and where $y \in F_Y$ is a solution to F. As in Section 3.1, we define Γ_0 to be the tree spanned by the complement $\mathbb{P}^1_K \backslash U$, which is a finite set of closed points. Note that Γ_0 always includes ∞, which may or may not be a branch point of φ. We write $R := \mathcal{O}_X(U)$ and $S := \mathcal{O}_Y(V)$, where $V = \varphi^{-1}(U)$.

To begin, we modify the generator y so that Assumption 3.23 is satisfied, following the strategy outlined in Section 3.5. Note that the functions A, B, C are polynomials, so according to Remark 3.37 we have

$$\deg_W(A) \leqslant 2, \qquad \deg_W(B) \leqslant 3, \qquad \deg_W(C) \leqslant 4,$$

where $W = \mathbb{A}^1_K$. It follows from Corollary 3.40 that order of approximation

$$m > \frac{4}{3},$$

will suffice. Thus we fix order of approximation $m = 2$.

Choose a closed point $x_0 \in U$ and write $t = x - x_0$ for the associated parameter. Since $A, B, C \in K[x]$ are polynomials, the Taylor expansions introduced in equation (3.7) are just polynomials as well; we write

$$\tilde{F}(T) = T^3 + \tilde{A}T^2 + \tilde{B}T + \tilde{C}, \tag{4.2}$$

where

$$\begin{aligned} \tilde{A} &= \sum_{\ell=0}^{2} a_\ell t^\ell, \qquad & a_\ell &= \frac{A^{(\ell)}}{\ell!}, \\ \tilde{B} &= \sum_{\ell=0}^{3} b_\ell t^\ell, \qquad & b_\ell &= \frac{B^{(\ell)}}{\ell!}, \\ \tilde{C} &= \sum_{\ell=0}^{4} c_\ell t^\ell, \qquad & c_\ell &= \frac{C^{(\ell)}}{\ell!}. \end{aligned}$$

Note that we have $a_0 = A$, $b_0 = B$, and $c_0 = C$. In particular, if $p_0 \in Y$ is a closed point with $\varphi(p_0) = x_0$, then $y(p_0)$ satisfies

$$y(p_0)^3 + a_0(x_0)y(p_0)^2 + b_0(x_0)y(p_0) + c_0(x_0) = 0.$$

The minimal polynomial of y over the subfield $K(t)$ of F_Y is obtained from $\tilde{F}$ by evaluation at x_0. That is, the minimal polynomial of y over $K(t)$ is

$$G(T) = T^3 + \Big(\sum_{\ell=0}^{2} a_\ell(x_0)t^\ell\Big)T^2 + \Big(\sum_{\ell=0}^{3} b_\ell(x_0)t^\ell\Big)T + \sum_{\ell=0}^{4} c_\ell(x_0)t^\ell.$$

Next, we bring into play the formal solution

$$\tilde{y} = y + v_1 t + v_2 t^2 + v_3 t^3 + \ldots \in S[\![t]\!].$$

To construct it, we follow the recipe explained in the proof of Proposition 3.14. As mentioned above, we have fixed order of approximation $m = 2$, so we will only need the truncation $\tilde{u}_2 = y + v_1 t$.

Plugging the ansatz $\tilde{u}_2 = y + v_1 t$, $v_1 \in S$, into $\tilde{F}$ and collecting all terms of degree 1 in t shows that v_1 should satisfy

$$3y^2 v_1 + y^2 a_1 + 2y v_1 a_0 + y b_1 + v_1 b_0 + c_1 = 0.$$

Thus our approximation is

$$\tilde{u}_2 = y - \frac{y^2 a_1 + y b_1 + c_1}{3y^2 + 2y a_0 + b_0} t \in S[t]. \tag{4.3}$$

Given a closed point $p_0 \in \varphi^{-1}(x_0)$, a generator w of F_Y for which (U, w) satisfies Assumption 3.23 is given by

$$w = y - \tilde{u}_2(p_0) = y - y(p_0) + \frac{y(p_0)^2 a_1(x_0) + y(p_0) b_1(x_0) + c_1(x_0)}{3y(p_0)^2 + 2y(p_0)a_0(x_0) + b_0(x_0)} t.$$

The coefficients of the polynomial $\tilde{H} := \tilde{F}(T + \tilde{u}_2)$ clearly lie in $S[t]$ again. It will be convenient to "forget" about the polynomial $\tilde{F}$ and change notation, now denoting the coefficients of $\tilde{H}$ by $\tilde{A}, \tilde{B}, \tilde{C}$. Thus we have

$$\tilde{H}(T) = T^3 + \tilde{A}T^2 + \tilde{B}T + \tilde{C}.$$

Correspondingly we write

$$\tilde{A} = \sum_{i=0}^{2} a_i t^i, \qquad \tilde{B} = \sum_{i=0}^{3} b_i t^i, \qquad \tilde{C} = \sum_{i=2}^{4} c_i t^i.$$

Similarly, we denote the coefficients of the minimal polynomial $H(T)$ of w over $K(t)$ by A, B, C. It is obtained from $\tilde{H}$ by evaluating at p_0. Thus we have

$$\begin{aligned} H(T) &= T^3 + AT^2 + BT + C \\ &= T^3 + \Big(\sum_{\ell=0}^{2} a_\ell(p_0)t^\ell\Big)T^2 + \Big(\sum_{\ell=0}^{3} b_\ell(p_0)t^\ell\Big)T + \sum_{\ell=0}^{4} c_\ell(p_0)t^\ell. \end{aligned}$$

Remark 4.5. Assume that $\lambda(x_0) > \mu(x_0)$. Then according to Lemma 3.27, there is a unique point $\eta \in Y^{\mathrm{an}}$ above $\xi_{\lambda(x_0)}$, and the extension of residue fields $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ is generated by $\overline{w} := [w]_\eta \in \kappa(\eta)$. Let us repeat how to compute the minimal polynomial of $\overline{w}$, following Remark 3.28.

Set $s := v_\eta(w) = v_{\lambda(x_0)}(C)/3$. Then the minimal polynomial of $\overline{w}$ is

$$T^3 + \overline{A}T^2 + \overline{B}T + \overline{C}, \qquad \text{where}$$

$$\overline{A} = \overline{A\pi^{-s}}, \qquad \overline{B} = \overline{B\pi^{-2s}}, \qquad \overline{C} = \overline{C\pi^{-3s}}.$$

Here the bar over $A\pi^{-s}$ and the other functions denotes reduction to the residue field $\kappa(\xi_{\lambda(x_0)})$. Note that as in Remark 3.28 this is well-defined because the Newton polygon of H with respect to $v_{\lambda(x_0)}$ is a straight line.

4.4 Tame locus

Let Γ_0 be the tree defined in Section 4.3. As in Section 3.6 we denote by Σ the φ-minimal skeleton of Y with respect to Γ_0. In Definition 3.44 we introduced the tame locus Y^{tame} associated to φ and Γ_0, as well as the tail locus Y^{tail} and the interior tame locus Y^{int}. The goal of this section is to describe the tame locus using certain valuative functions on Y^{an}.

Let $p_0 \in Y$ be a closed point in the complement of Σ and denote its image $\varphi(p_0)$ in $\mathbb{P}^1_K$ by x_0. As before, we consider the families of open disks

$$\mathsf{D}(r) := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x - x_0) > r\}$$

and closed disks

$$\mathsf{D}[r] := \{\xi \in (\mathbb{P}^1_K)^{\mathrm{an}} \mid v_\xi(x - x_0) \geqslant r\}$$

around x_0, denote the boundary point of $\mathsf{D}[r]$ by ξ_r, and the corresponding valuation by v_r. Below, we will use the affine étale chart (U, w) constructed in the previous section as well as the various coefficients of the polynomials $\tilde{A}, \tilde{B}, \tilde{C}$.

We define a subset $\mathsf{V} \subseteq Y^{\mathrm{an}}$ by

$$\mathsf{V} = \mathsf{V}_2 \cup \mathsf{V}_4, \qquad \text{where}$$

$$\mathsf{V}_2 = \{\eta \in Y^{\mathrm{an}} \mid 3\widehat{b}_0(\eta) + 4\widehat{c}_3(\eta) \geqslant 6\widehat{c}_2(\eta)\},$$
$$\mathsf{V}_4 = \{\eta \in Y^{\mathrm{an}} \mid 8\widehat{c}_3(\eta) \geqslant 3\widehat{b}_0(\eta) + 6\widehat{c}_4(\eta)\}.$$

It is an affinoid subdomain — or equals Y^{an} — by Lemma 2.6.

Theorem 4.6. *We have the following inclusions:*

$$Y^{\mathrm{tail}} \subseteq \mathsf{V} \subseteq Y^{\mathrm{tame}}$$

Proof. It suffices to show these inclusions for the underlying sets of closed points, and indeed for closed points p_0 not contained in Σ (the set of these points being dense in Y^{an}). Thus let $p_0 \in Y^{\mathrm{an}} \backslash \Sigma$ be an arbitrary closed point with image $x_0 := \varphi(p_0)$ in $X^{\mathrm{an}} \backslash \Gamma_0$.

Let us assume that $p_0 \notin Y^{\mathrm{int}}$. Because $Y^{\mathrm{int}} = Y^{\mathrm{tame}} \backslash Y^{\mathrm{tail}}$, it suffices to show that $p_0 \in Y^{\mathrm{tail}}$ if and only if $p_0 \in \mathsf{V}$. We know from Lemma 3.45 that $\lambda(x_0) > \mu(x_0)$. Thus it follows from Theorem 3.30 that

$$\lambda(x_0) = \max_{k \in \{2,3,4\}} \frac{3v_K(b_0(p_0)) - 2v_K(c_k(p_0))}{2k}. \tag{4.4}$$

Following Remark 4.5 we write $s := v_\eta(w) = v_{\lambda(x_0)}(C)/3$, where η is the unique point in the preimage of $\xi_{\lambda(x_0)}$; then we have

$$\overline{w} := [w]_\eta = \overline{w\pi^{-s}} \in \kappa(\eta).$$

The element $\overline{w}$ generates the field extension $\kappa(\eta)/\kappa(\xi_{\lambda(x_0)})$ and satisfies the equation

$$\overline{H}(\overline{w}) = \overline{w}^3 + \overline{A}\overline{w}^2 + \overline{B}\overline{w} + \overline{C} = 0,$$

where $\overline{A} = \overline{A\pi^{-s}}$, $\overline{B} = \overline{B}\pi^{-2s}$, $\overline{C} = \overline{C\pi^{-3s}}$. We have $p_0 \in \mathsf{V}$ if and only if one of the conditions

$$3v_K(b_0(p_0)) + 4v_K(c_3(p_0)) \geqslant 6v_K(c_2(p_0)),$$

$$8v_K(c_3(p_0)) \geqslant 3v_K(b_0(p_0)) + 6v_K(c_4(p_0))$$

holds. An easy calculation shows that the first condition holds (meaning $p_0 \in \mathsf{V}_2$) if and only if in the maximum in (4.4) the term involving c_3 is not greater than the term involving c_2, that is,

$$\frac{3v_K(b_0(p_0)) - 2v_K(c_3(p_0))}{6} \leqslant \frac{3v_K(b_0(p_0)) - 2v_K(c_2(p_0))}{4}.$$

Similarly the second condition holds (meaning $p_0 \in \mathsf{V}_4$) if and only if the term involving c_3 in (4.4) is not greater than the term involving c_4, that is,

$$\frac{3v_K(b_0(p_0)) - 2v_K(c_3(p_0))}{6} \leqslant \frac{3v_K(b_0(p_0)) - 2v_K(c_4(p_0))}{8}.$$

Thus we have $p_0 \in \mathsf{V}$ if and only if the maximum in (4.4) is assumed for the term involving c_2 or the term involving c_4. Using Remark 3.31 we find that this means

$$\frac{v_{\lambda(x_0)}(C)}{3} = \frac{v_K(c_k(p_0)) + k\lambda(x_0)}{3} = \frac{v_K(b_0(p_0))}{2} = \frac{v_{\lambda(x_0)}(B)}{2}$$

for $k = 2$ or $k = 4$. It follows that $p_0 \in \mathsf{V}$ if and only if in $\overline{C} = \overline{c}_4\overline{t}^4 + \overline{c}_3\overline{t}^3 + \overline{c}_2\overline{t}^2$ not both of $\overline{c}_2, \overline{c}_4$ vanish.

Moreover, by Remark 3.28(b), the discriminant $\Delta_{\overline{H}}$ of the polynomial $\overline{H}$ is constant. It follows from Lemma 4.3 that $\overline{A} = 0$ and that $\overline{B} = \overline{b}_0$ is constant, that is, $\overline{H}$ is Artin-Schreier. By Lemma 4.2, the genus of the reduction curve at η is positive if and only if $p_0 \in \mathsf{V}$.

Said differently, if $p_0 \in \mathsf{V}$, then the Type II point η is contained in the φ-minimal skeleton Σ; it is a leaf, since $\lambda(x_0) > \mu(x_0)$. Thus p_0 is contained in a tail component with boundary point η.

If on the other hand $p_0 \notin \mathsf{V}$, then the reduction curve at η has genus 0, so η is not contained in the φ-minimal skeleton Σ. Thus p_0 is not contained in the tame locus.

This concludes the proof that the point p_0 (with $p_0 \notin Y^{\mathrm{int}}$) is contained in V if and only if it is contained in Y^{tail} and thus the proof of the theorem. ■

4.5 Norm trick

In this section, we study the image $\varphi(Y^{\mathrm{tame}})$ of the tame locus. Note that we have

$$\varphi(Y^{\mathrm{tame}}) = \mathrm{ret}_\Gamma^{-1}\left(\{\xi \in \Gamma \mid \delta(\xi) = 0\}\right),$$

where $\Gamma = \varphi(\Sigma)$. It is the skeleton of $\mathbb{P}^1_K$ obtained from Γ_0 by adding paths to all points $\xi = \varphi(\eta)$, where $\eta \in Y^{\mathrm{an}}$ has positive genus (cf. Proposition 2.22). In particular, $\varphi(Y^{\mathrm{tame}})$ is an affinoid subdomain of X^{an}, or equals X^{an}.

The structure of affinoid subdomains of $(\mathbb{P}^1_K)^{\mathrm{an}}$ is well-understood. They are disjoint unions of so-called *standard affinoid subdomains* ([Ber90, p. 78]), which are obtained from closed disks by removing a finite number of open disks. In particular, closed disks and anuli are standard affinoid subdomains. The images of the tail components are closed disks, which we call *tail disks*. Indeed, they are given by retractions

$$\mathrm{ret}_\Gamma^{-1}(\xi),$$

where $\xi = \varphi(\eta)$ for a point $\eta \in Y^{\mathrm{an}}$ with positive genus.

To study $\varphi(Y^{\mathrm{tame}})$, we will see how to describe the image of the tail locus using admissible functions on X^{an}. Combining this with the results of the next section, wherein we explain how to compute the image of the interior locus, we will be able to describe the entire tame locus Y^{tame} using admissible functions on X^{an}. In this way, determining Y^{tame}, and hence the semistable reduction of Y, becomes computationally accessible in concrete examples.

The "norm trick" is named for the norm map of the extension of function fields F_Y/F_X,

$$\mathrm{Nm}\colon F_Y \to F_X.$$

Simply put, Corollary 4.8 below states that for computing the image of the tail locus, we may replace all rational functions appearing in the definition of the affinoid subdomain V from the previous section with their norms. Thus we are led to $\mathsf{U} \subseteq X^{\mathrm{an}}$ defined as

$$\mathsf{U} = \mathsf{U}_2 \cup \mathsf{U}_4, \qquad \text{where}$$

$$\mathsf{U}_2 = \{\xi \in X^{\mathrm{an}} \mid 3\widehat{\mathrm{Nm}\, b_0}(\xi) + 4\widehat{\mathrm{Nm}\, c_3}(\xi) \geqslant 6\widehat{\mathrm{Nm}\, c_2}(\xi)\} \qquad \text{and}$$
$$\mathsf{U}_4 = \{\xi \in X^{\mathrm{an}} \mid 8\widehat{\mathrm{Nm}\, c_3}(\xi) \geqslant 3\widehat{\mathrm{Nm}\, b_0}(\xi) + 6\widehat{\mathrm{Nm}\, c_4}(\xi)\}.$$

Recall that given a closed point $x_0 \in X^{\mathrm{an}}\backslash\Gamma_0$ with $\lambda(x_0) > \mu(x_0)$ we have by Theorem 3.30

$$\lambda(x_0) = \max_{k\in\{2,3,4\}} \frac{3\hat{b}_0(p_0) - 2\hat{c}_k(p_0)}{2k}, \tag{4.5}$$

where $p_0 \in Y$ is any point in the preimage of x_0 under φ. Let us denote the valuative functions appearing in this equation by

$$h_k := \frac{3\hat{b}_0 - 2\hat{c}_k}{2k}, \qquad k \in \{2,3,4\}.$$

Thus we have $\lambda(x_0) = \max_{k\in\{2,3,4\}} h_k(p_0)$. The norms of these valuative functions (see Definition 2.11), the functions

$$\mathrm{Nm}\, h_k = \frac{3\widehat{\mathrm{Nm}\, b_0} - 2\widehat{\mathrm{Nm}\, c_k}}{2k}, \qquad k \in \{2,3,4\},$$

will be important below.

The following lemma states in particular that given a closed point x_0 with $\lambda(x_0) > \mu(x_0)$, the function λ agrees with a valuative function on the closed disk $\mathsf{D}[\lambda(x_0)]$: The restriction $\lambda|_{\mathsf{D}[\lambda(x_0)]}$ is equal to one of the functions $(\mathrm{Nm}\, h_2)|_{\mathsf{D}[\lambda(x_0)]}$, $(\mathrm{Nm}\, h_3)|_{\mathsf{D}[\lambda(x_0)]}$, or $(\mathrm{Nm}\, h_4)|_{\mathsf{D}[\lambda(x_0)]}$.

Lemma 4.7. *Suppose that $\lambda(x_0) > \mu(x_0)$. Then there exists an index $k_0 \in \{2,3,4\}$ such that for all $\xi \in \mathsf{D}[\lambda(x_0)]$ and $\eta \in \varphi^{-1}(\xi)$ the following statements hold:*

(a) $h_{k_0}(\eta) = \max_{k\in\{2,3,4\}} h_k(\eta)$

(b) $(\mathrm{Nm}\, h_{k_0})(\xi) = \max_{k\in\{2,3,4\}}(\mathrm{Nm}\, h_k)(\xi)$

(c) The functions h_{k_0} *and* $\mathrm{Nm}\, h_{k_0}$ *are constant on* $\varphi^{-1}(\mathsf{D}[\lambda(x_0)])$ *and* $\mathsf{D}[\lambda(x_0)]$ *respectively, with value* $\lambda(x_0)$ *and* $3\lambda(x_0)$ *respectively*

(d) If (a)–(c) are only true for $k_0 = 3$*, then the inequalities*

$$h_3(\eta) \geqslant h_k(\eta), \qquad (\mathrm{Nm}\, h_3)(\xi) \geqslant (\mathrm{Nm}\, h_k)(\xi), \qquad k = 2, 4,$$

are strict

Proof. Suppose first that the maximum in (4.5) is assumed for $k_0 = 4$. In the proof of Theorem 4.6 we have verified that in this case the reduction curve at $\eta_{\lambda(x_0)} \in Y^{\mathrm{an}}$ (where $\eta_{\lambda(x_0)}$ is the unique preimage of $\xi_{\lambda(x_0)}$ under φ) is given by an Artin-Schreier equation

$$y^3 + \bar{b}y + \overline{C} = 0,$$

for a polynomial $\overline{C}$ of degree 4 and $\bar{b} \in \kappa^\times$. By Lemma 4.2, the genus of this curve is 3. Now let $x_1 \in \mathsf{D}[\lambda(x_0)]$ be any closed point; choose a preimage $p_1 \in \varphi^{-1}(x_1)$. Then (4.5) holds with x_1 and p_1 in place of x_0 and p_0, that is,

$$\lambda(x_1) = \max_{k\in\{2,3,4\}} \frac{3\widehat{b}_0(p_1) - 2\widehat{c}_k(p_1)}{2k} = \max_{k\in\{2,3,4\}} h_k(p_1). \tag{4.6}$$

Repeating the above argument shows that here too the maximum must be assumed for $k_0 = 4$. Indeed, otherwise the reduction curve at the point $\eta_{\lambda(x_0)}$ above $\xi_{\lambda(x_0)} = \xi_{\lambda(x_1)}$ could not be of genus 3.

Thus we have shown that $\lambda(x_1) = h_4(p_1)$ for every closed point $x_1 \in \mathsf{D}[\lambda(x_0)]$ and every choice of preimage $p_1 \in \varphi^{-1}(x_1)$.

We proceed similarly if the maximum in (4.5) is not assumed for $k_0 = 4$, but is assumed for $k_0 = 2$. Then the reduction curve at $\eta_{\lambda(x_0)}$ is of genus 1. The maximum in (4.6) cannot be assumed for $k_0 = 4$ (or the reduction curve at $\eta_{\lambda(x_0)}$ would have genus 3), but has to be assumed for $k_0 = 2$ (or the reduction curve at $\eta_{\lambda(x_0)}$ would have genus 0). Thus we deduce that $\lambda(x_1) = h_2(p_1)$ for every closed point $x_1 \in \mathsf{D}[\lambda(x_0)]$ and every choice of preimage $p_1 \in \varphi^{-1}(x_1)$.

Finally, if the maximum in (4.5) is only assumed for $k_0 = 3$, the reduction curve at $\eta_{\lambda(x_0)}$ is of genus 0. The maximum in (4.6) can also only be assumed for $k_0 = 3$ (or else the reduction curve at $\eta_{\lambda(x_0)}$ would have positive genus). Thus $\lambda(x_1) = h_3(p_1)$ for every closed point $x_1 \in \mathsf{D}[\lambda(x_0)]$ and choice of preimage $p_1 \in \varphi^{-1}(x_1)$.

We have seen that there exists a $k_0 \in \{2, 3, 4\}$ such that $h_{k_0}(p_1)$ agrees with the maximum $\lambda(x_1)$ over $k \in \{2, 3, 4\}$ in (4.6) for every closed point in $\varphi^{-1}(\mathsf{D}[(\lambda(x_0)])$, and hence for every point $\eta \in \varphi^{-1}(\mathsf{D}[(\lambda(x_0)])$ outright. This means that part (a) holds. Moreover, we have seen that if the maximum in (4.5) is only assumed for $k_0 = 3$, then the same is true for the maximum in (4.6). It follows that the first inequality in (d) is indeed strict.

Next, note that by Corollary 2.33 the function λ is constant on $\mathsf{D}[\lambda(x_0)]$ with value $\lambda(x_0) = \lambda(\xi_{\lambda(x_0)})$. This also shows that h_{k_0} is constant on $\varphi^{-1}(\mathsf{D}[\lambda(x_0)])$, which is the first part of (c). The second statement in (c) follows from Lemma 2.13.

To prove part (b), fix a point $\xi \in \mathsf{D}[\lambda(x_0)]$, which we may take to be different from $\xi_{\lambda(x_0)}$. As a consequence of Proposition 2.32 we have $\varphi^{-1}(\xi) = \{\eta_1, \eta_2, \eta_3\}$, and these three points correspond to the three extensions of the valuation v_ξ to the function field F_Y. As we have remarked above, we have

$$h_k(\eta_j) \leqslant h_{k_0}(\eta_j) = \frac{(\mathrm{Nm}\, h_{k_0})(\xi)}{3}, \qquad k \in \{2,3,4\}, \quad j \in \{1,2,3\}. \tag{4.7}$$

Another application of Lemma 2.13 shows

$$\frac{(\mathrm{Nm}\, h_k)(\xi)}{3} \leqslant \frac{(\mathrm{Nm}\, h_{k_0})(\xi)}{3}, \qquad k \in \{2,3,4\}. \tag{4.8}$$

Thus we have proven (b).

Finally note that in the case that (a) only holds for $k_0 = 3$, the inequalities in (4.7) are strict for $k \neq 3$ by the first inequality in (d) which we have already proven. Thus the inequality in (4.8) is also strict for $k \neq 3$. This proves that the second inequality in (d) is strict and finishes the proof of the lemma. ∎

Corollary 4.8. *We have the following inclusions:*

$$\varphi(Y^{\mathrm{tail}}) \subseteq \mathsf{U} \subseteq \varphi(Y^{\mathrm{tame}}).$$

Proof. It suffices to show these inclusions for the underlying sets of closed points in the complement of Γ_0.

Thus let $x_0 \in X^{\mathrm{an}} \backslash \Gamma_0$ be a closed point. Because of Theorem 4.6 it suffices to show that $x_0 \in \mathsf{U}$ if and only if there exists a preimage $p_0 \in \varphi^{-1}(x_0)$ with $p_0 \in \mathsf{V}$.

Let us assume first that the statement of Lemma 4.7 is true for $k_0 = 4$. Part (a) of Lemma 4.7 then implies that

$$\frac{3\widehat{b}_0(p_0) - 2\widehat{c}_4(p_0)}{8} = h_4(p_0) \geqslant h_3(p_0) = \frac{3\widehat{b}_0(p_0) - 2\widehat{c}_3(p_0)}{6}$$

for any preimage $p_0 \in \varphi^{-1}(x_0)$. This means that $p_0 \in \mathsf{V}_4$. Similarly, part (b) of Lemma 4.7 implies that

$$\begin{aligned}\frac{3\widehat{\mathrm{Nm}\, b_0}(x_0) - 2\widehat{\mathrm{Nm}\, c_4}(x_0)}{8} &= (\mathrm{Nm}\, h_4)(x_0) \\ &\geqslant (\mathrm{Nm}\, h_3)(x_0) = \frac{3\widehat{\mathrm{Nm}\, b_0}(x_0) - 2\widehat{\mathrm{Nm}\, c_3}(x_0)}{6}.\end{aligned}$$

This means that $x_0 \in \mathsf{U}_4$. Thus the claim that $x_0 \in \mathsf{U}$ if and only if there exists a preimage $p_0 \in \varphi^{-1}(x_0)$ with $p_0 \in \mathsf{V}$ holds in the case that the statement of Lemma 4.7 holds for $k_0 = 4$.

In the case that the statement of Lemma 4.7 holds for $k_0 = 2$, we conclude completely analogously that $x_0 \in \mathsf{U}_2$ and $p_0 \in \mathsf{V}_2$.

Finally we consider the case that the statement of Lemma 4.7 only holds for $k_0 = 3$. Then part (d) of that lemma shows that we have the opposite and strict inequalities

$$h_3(p_0) > h_2(p_0), h_4(p_0), \qquad (\operatorname{Nm} h_3)(x_0) > (\operatorname{Nm} h_2)(x_0), (\operatorname{Nm} h_4)(x_0).$$

This means that $x_0 \notin \mathsf{U}$ and $p_0 \notin \mathsf{V}$. Thus the claim is also true in this last case, completing the proof of the corollary. ∎

4.6 Computing δ directly

Let $x_0 \in X^{\mathrm{an}}$ be a closed point different from ∞. In this section, we explain a method to directly compute the function δ_φ on the interval $[x_0, \infty] \subset X^{\mathrm{an}}$. The approach presented here cannot replace the one developed in the previous sections, because the point x_0 has to stay fixed and we can only consider "one interval $[x_0, \infty]$ at a time". But analyzing finitely many intervals is enough to pin down the interior locus.

Consider the coordinate $t = x - x_0$, which is a parameter for the interval $[x_0, \infty]$. Our point of departure is the minimal polynomial G of the generator y of F_Y over $K(t)$, considered for example in Section 4.3. However, in this section we begin not by modifying y using an approximation of level $m = 2$, but using an approximation of level $m = 1$.

Thus choose a point $p_0 \in \varphi^{-1}(x_0)$ and write $y_0 := y(p_0)$. Then the generator $y - y_0$ of F_Y has minimal polynomial of the form

$$T^3 + (a_2t^2 + a_1t + a_0)T^2 + (b_3t^3 + b_2t^2 + b_1t + b_0)T + c_4t^4 + c_3t^3 + c_2t^2 + c_1t. \quad (4.9)$$

Note that all the coefficients $a_0, a_1, a_2,\ b_0, \ldots, b_3,$ and $c_0, \ldots, c_4$ are simply elements of K, depending on x_0; we do not view them as functions in p_0 here. (In other words, the "generic polynomials", which we have indicated with tildes, are absent from this section.) Finally, we consider the generator $z := y + vt$, where v is a solution to the equation

$$v^3 + a_1v^2 + b_2v + c_3 = 0. \quad (4.10)$$

The minimal polynomial of z is then by construction of the form

$$T^3 + (a_2t^2 + a_1t + a_0)T^2 + (b_3t^3 + b_2t^2 + b_1t + b_0)T + c_4t^4 + c_2t^2 + c_1t. \quad (4.11)$$

(Of course, the coefficients are not the same as the ones in (4.9), we just reuse the notation for convenience.) We write $A = a_2t^2 + a_1t + a_0$, $B = b_3t^3 + b_2t^2 + b_1t + b_0$, $C = c_4t^4 + c_2t^2 + c_1t$.

Let $\xi \in [x_0, \infty]$ be a point of Type II. Our goal is to compute $\delta(\xi)$. For the moment we assume that ξ is a wild topological branch point so that

$\varphi^{-1}(\xi) = \{\eta\}$ for a Type II point $\eta \in Y^{\mathrm{an}}$. The completed residue fields are *one-dimensional analytic fields* over K, meaning that they are complete valued fields over K which are finite over a subfield of the form $\widehat{K(x)}$, $x \notin K$. In [CTT, Section 2.4], a *different*

$$\delta^{\mathrm{CTT}}(\mathcal{H}_2/\mathcal{H}_1)$$

is attached to every separable extension $\mathcal{H}_2/\mathcal{H}_1$ of one-dimensional analytic fields over K. In parallel with our normalization of δ in Section 2.4, we define

$$\delta(\mathcal{H}_2/\mathcal{H}_1) := \delta^{\mathrm{add}}(\mathcal{H}_2/\mathcal{H}_1) := -\frac{3}{2}\log\big(\delta^{\mathrm{CTT}}(\mathcal{H}_2/\mathcal{H}_1)\big).$$

By the definition of δ in [CTT, Section 4.1] we then have

$$\delta(\xi) = \delta(\eta) = \delta\big(\mathcal{H}(\eta)/\mathcal{H}(\xi)\big).$$

We will need the following definitions.

Definition 4.9. Let $\mathcal{H}$ be a one-dimensional analytic field over K with valuation $v_{\mathcal{H}}$. An element $x \in \mathcal{H}\backslash K$ such that $\mathcal{H}/\widehat{K(x)}$ is a finite separable extension is called a *parameter* for $\mathcal{H}$.

(a) If $x \in \mathcal{H}\backslash K$ is a parameter such that $\mathcal{H}/\widehat{K(x)}$ is a tame extension of valued fields, then x is called a *tame parameter* for $\mathcal{H}$.

(b) If $x \in \mathcal{H}\backslash K$ is a parameter such that we have

$$v_{\mathcal{H}}\Big(\sum_i a_i x^i\Big) = \min_i\big(v_K(a_i) + i v_{\mathcal{H}}(x)\big), \qquad a_i \in K,$$

then x is called a *monomial parameter* for $\mathcal{H}$.

Our key tool for computing δ is the following:

Proposition 4.10. *Let $\mathcal{H}_2/\mathcal{H}_1$ be a finite separable extension of one-dimensional analytic fields over K. Suppose that x_1 and x_2 are tame monomial parameters of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then we have*

$$\frac{2}{3}\delta(\mathcal{H}_2/\mathcal{H}_1) = v_{\mathcal{H}_2}(\tfrac{dx_1}{dx_2}) + v_{\mathcal{H}_2}(x_2) - v_{\mathcal{H}_1}(x_1),$$

where $\frac{dx_1}{dx_2}$ is the unique element of $\mathcal{H}_2$ for which $\frac{dx_1}{dx_2}dx_2 = dx_1$ in the module of differentials $\Omega_{\mathcal{H}_2/K}$.

Proof. Combine [CTT, Corollary 2.4.6(ii)] and [CTT, Lemma 2.1.6(ii)]. ∎

Lemma 4.11. *Let ξ be a wild topological branch point with $\varphi^{-1}(\xi) = \{\eta\}$. Then the elements $t = x - x_0 \in \mathcal{H}(\xi)$ and $z \in \mathcal{H}(\eta)$ (constructed at the beginning of this section, with minimal polynomial (4.11)) are tame monomial parameters.*

Proof. For t this is clear. Indeed, $\mathcal{H}(\xi)$ actually equals $\widehat{K(t)}$, so t is in particular a tame parameter. And since v_ξ is a Gauss valuation centered at x_0, it follows that $t = x - x_0$ is a monomial parameter.

Let us write

$$\overline{z} := [z]_\eta = \overline{z\pi^{-s}}, \qquad s = \frac{v_\xi(C)}{3}.$$

Exactly as in the proof of Proposition 3.21 (the key assumption is that $\delta(\xi) > 0$), it follows that the Newton polygon of the minimal polynomial (4.11) of z is inseparable, so $\overline{z}$ satisfies the equation

$$T^3 + \overline{C} = 0, \qquad \overline{C} = \overline{C\pi^{-3s}}. \tag{4.12}$$

However, in contrast to the situation in Proposition 3.21, we now know that $\overline{C}$ is not a third power (because by construction, $C = c_4t^4 + c_2t^2 + c_1t$). Thus the extension of residue fields $\kappa(\eta)/\kappa(\xi)$ is purely inseparable of degree 3 and is generated by $\overline{z}$.

To show that z is a monomial parameter, we study the extension of v_ξ to F_Y. In the following we use MacLane's theory of inductive valuations. (See [Mac36a] or [Rüt14, Chapter 4]. We give a brief introduction to it in Section 5.2 below.) The Newton polygon of the minimal polynomial of $z\pi^{-s}$,

$$G = T^3 + A\pi^{-s}T^2 + B\pi^{-2s}T + C\pi^{-3s},$$

shows that $z\pi^{-s}$ has valuation 0. Thus the Gauss valuation

$$v_0\colon \sum_i a_iT^i \mapsto \min_i \big(v_\xi(a_i)\big), \qquad a_i \in F_X, \tag{4.13}$$

is a first approximant of v_η. Since the polynomial (4.12) is irreducible, it follows from [Rüt14, Lemma 4.8] that G is a key polynomial over the first approximant (4.13). Thus v_η corresponds to the pseudo-valuation $[v_0, v(G) = \infty]$. This shows that v_η is simply the Gauss valuation with respect to $z\pi^{-s}$,

$$\sum_i a_i(z\pi^{-s})^i \mapsto \min_i \big(v_K(a_i)\big).$$

Thus z is a monomial parameter as was to be shown.

To show that z is a tame parameter, it suffices to show that the degree of $\mathcal{H}(\eta)/\widehat{K(z)}$ is coprime to 3. Note that the residue field $\kappa(\eta)$ is generated over the residue field κ of K by $\overline{z} = [z]_\eta$ and $\overline{t} = [t]_\eta$. The relation

$$\overline{z}^3 = \overline{c}_4\overline{t}^4 + \overline{c}_2\overline{t}^2 + \overline{c}_1\overline{t}$$

shows that the field extension $\kappa(\eta)/k(\overline{z})$ has degree dividing 4. By Remark 1.26, this degree equals the degree $\mathcal{H}(\eta)/\widehat{K(z)}$. Thus z is a tame parameter as well. ∎

Remark 4.12. The key step in the proof of Lemma 4.11 is that by construction, the element $\overline{z}$ is a generator of the field extension $\kappa(\eta)/\kappa(\xi)$. The importance of ensuring this was already foreshadowed in Remark 3.22. We now provide some additional context for this result.

Suppose that $(L, v_L)/(F, v_F)$ is a finite extension of valued fields of residue characteristic $p > 0$, say of ramification index $e \geqslant 1$. The extension of valuations $v_L|v_F$ is called *weakly unramified* if $e = 1$. Thus if $v_L|v_F$ is weakly unramified and has separable extension of residue fields, then it is unramified. We say that a finite extension F'/F *eliminates the ramification* of $v_L|v_F$ if each extension $v_{L'}$ of v_L to the compositum $L' := LF'$ is weakly unramified over its restriction to F'. The situation is summarized in the following diagram of field and valuation extensions:

$$\begin{tikzcd}
& L' \arrow[dd, dash] \\
L \arrow[ur, dash] \arrow[dd, dash] & \\
& F' \\
F \arrow[ur, dash] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& v_{L'} \arrow[dd, dash, "e=1"] \\
v_L \arrow[ur, dash] \arrow[dd, dash] & \\
& v_{L'}|_{F'} \\
v_F \arrow[ur, dash] &
\end{tikzcd}$$

If the ramification index e of $v_L|v_F$ is coprime to p, then it follows from Abhyankar's Lemma that there exists a finite extension F'/F such that each extension $v_{L'}$ of v_L to L' is unramified over its restriction to F'. This result is false if p divides e. However, under a mild assumption on the associated extension of residue fields, a theorem of Epp ([Epp73], [Stacks, Tag 09F9]) shows that there still exists a finite extension F'/F eliminating the ramification of $v_L|v_F$.

Epp's result explains why we found that the extension of residue fields $\kappa(\eta)/\kappa(\xi)$ was purely inseparable of degree 3. Indeed, according to Epp's result, the extension of valuations $v_\eta|v_K$ becomes weakly unramified after a finite extension of K. Since K is algebraically closed, it already is weakly unramified. This is only possible if $v_\eta|v_\xi$ has ramification index 1, that is, if $\kappa(\eta)/\kappa(\xi)$ has degree 3. Note the following points however.

- We get by using only one generator z to compute δ at each point in the interval $[x_0, \infty]$; we only appropriately scale z before reducing.
- If the curve Y/K is defined over some non-algebraically closed subfield $K_0 \subset K$, then our generator is defined over a finite extension K'/K_0. In fact, if x_0 is a K_0-rational point, then K' is obtained from K_0 by solving two cubic equations: the equation $T^3 + a_0T^2 + b_0T + c_0 = 0$ to eliminate the constant term of C, and the equation (4.10) to eliminate the degree-3 term of C.

In the next chapter, we will use the fact that we can compute K' explicitly to compute the semistable reduction of plane quartics defined over K_0. See [Rüt14, Section 6.2.1] for a discussion of why Epp's proof is not constructive enough to determine K' in general.

Corollary 4.13. *We have*

$$\delta(\xi) = \min\big(\frac{3}{2}, \frac{3v_\xi(A)}{2} - \frac{v_\xi(C)}{2}, \frac{3v_\xi(B)}{2} - v_\xi(C)\big), \tag{4.14}$$

where A, B, C denote the coefficients of the minimal polynomial (4.11) *of the parameter z.*

Proof. This is now simply a calculation. We first collect all the ingredients we need.

(i) It follows from Proposition 4.10 and Lemma 4.11 that

$$\frac{2}{3}\delta(\xi) = v_\eta(\tfrac{dt}{dz}) + v_\eta(z) - v_\xi(t).$$

(ii) Differentiating the equation $0 = F(z)$ shows

$$0 = dF = F_z dz + F_t dt = (3z^2 + 2Az + B)dz + (A'z^2 + B'z + C')dt,$$

so

$$\frac{dt}{dz} = -\frac{3z^2 + 2Az + B}{A'z^2 + B'z + C'}.$$

(iii) Since t is a monomial parameter we have

$$v_\xi(A't) \geqslant v_\xi(A), \qquad v_\xi(B't) \geqslant v_\xi(B), \qquad v_\xi(C't) = v_\xi(C).$$

To illustrate the reason that the first two inequalities are not equalities in general, suppose that $v_\xi(B) = v_\xi(b_i t^i)$ for $i = 0$ or $i = 3$, while $v_\xi(B) < v_\xi(b_j t^j)$ for $j \neq i$. Then

$$B't = 3b_3t^3 + 2b_2t^2 + b_1t$$

has strictly larger valuation than B. The same reasoning applies to A, but not to $C = c_4t^4 + c_2t^2 + c_1t$, whence the equality $v_\xi(Ct) = v_\xi(C)$.

(iv) Since v_η is an infinite inductive valuation obtained from a Gauss valuation (see the proof of Lemma 4.11), we have

$$v_\eta\big(\sum_i A_i z^i\big) = \min_i \big(v_\xi(A_i) + i v_\eta(z)\big), \qquad A_i \in K(t).$$

(v) Because the Newton polygon of the minimal polynomial of z is inseparable, we have

$$v_\eta(z) = \frac{v_\xi(C)}{3} < \frac{v_\xi(B)}{2}, \qquad v_\eta(z) = \frac{v_\xi(C)}{3} < v_\xi(A).$$

(vi) Combining (iii) and (v) shows

$$\frac{v_\xi(C't)}{3} = \frac{v_\xi(C)}{3} < \frac{v_\xi(B)}{2} \leqslant \frac{v_\xi(B't)}{2},$$

and similarly $v_\xi(C't)/3 = v_\xi(C)/3 < v_\xi(A't)$.

Putting all this together, we can compute:

$$\begin{aligned}
&\frac{2}{3}\delta(\xi) \\
&= v_\eta(\tfrac{dt}{dz}) + v_\eta(z) - v_\eta(t) && \text{(by (i))} \\
&= v_\eta(3z^2 + 2Az + B) - v_\eta(A'z^2 + B'z + C') + v_\eta(z) - v_\xi(t) && \text{(by (ii))} \\
&= v_\eta(3z^2 + 2Az + B) - v_\eta(A'tz^2 + B'tz + C't) + v_\eta(z) \\
&= \min(v_\eta(3z^2), v_\eta(2Az), v_\eta(B)) \\
&\quad - \min(v_\eta(A'tz^2), v_\eta(B'tz), v_\eta(C't)) + v_\eta(z) && \text{(by (iv))} \\
&= \min\big(1 + \frac{2v_\xi(C)}{3}, v_\xi(A) + \frac{v_\xi(C)}{3}, v_\xi(B)\big) && \text{(by (v))} \\
&\quad - \min\big(v_\xi(A't) + \frac{2v_\xi(C)}{3}, v_\xi(B't) + \frac{v_\xi(C)}{3}, v_\xi(C't)\big) + \frac{v_\xi(C)}{3} \\
&= \min\big(1 + \frac{2v_\xi(C)}{3}, v_\xi(A) + \frac{v_\xi(C)}{3}, v_\xi(B)\big) - v_\xi(C) + \frac{v_\xi(C)}{3} && \text{(by (vi))} \\
&= \min\big(1, v_\xi(A) - \frac{v_\xi(C)}{3}, v_\xi(B) - \frac{2v_\xi(C)}{3}\big)
\end{aligned}$$

■

Remark 4.14. We have derived Corollary 4.13 under the assumption that $\xi \in X^{\mathrm{an}}$ is a wild topological branch point. In particular, formula (4.14) is true for all ξ with $\delta(\xi) > 0$.

If ξ is not a wild topological branch point, then the right-hand side of (4.14) is $\leqslant 0$. Indeed, the Newton polygon of F with respect to v_ξ cannot be inseparable, since this would imply that ξ is a wild topological branch point (compare the proof of Lemma 4.11). Thus we have

$$v_\xi(A) \leqslant \frac{v_\xi(C)}{3} \qquad \text{or} \qquad \frac{v_\xi(B)}{2} \leqslant \frac{v_\xi(C)}{3},$$

which implies that the right-hand side of (4.14) is $\leqslant 0$. All in all we thus see that for *any* $\xi \in [x_0, \infty]$ we have

$$\delta(\xi) = \max\Big(0, \min\big(\frac{3}{2}, \frac{3v_\xi(A)}{2} - \frac{v_\xi(C)}{2}, \frac{3v_\xi(B)}{2} - v_\xi(C)\big)\Big).$$

Example 4.15. Let us consider the example of the smooth plane quartic curve over $\mathbb{C}_3$ with equation

$$Y\colon \quad F(y) = y^3 - 3y^2 + (-3x^2 - 2x)y + 3x^4 - 3x - 1 = 0.$$

The discriminant Δ_F of F is a polynomial of degree 8; by Lemma 4.1, the point ∞ is a branch point of the morphism $\varphi\colon Y \to X = \mathbb{P}^1_K$ of ramification index 3. The other branch points, each of ramification index 2, are the eight zeros of Δ_F in K. Using the technique outlined in [Rüt14, Section 4.8], or by applying the MCLF-function `BerkovichTree.adapt_to_function` to Δ_F, it is not hard too see that the skeleton of X spanned by the branch points of φ has the shape indicated in Figure 4.1.

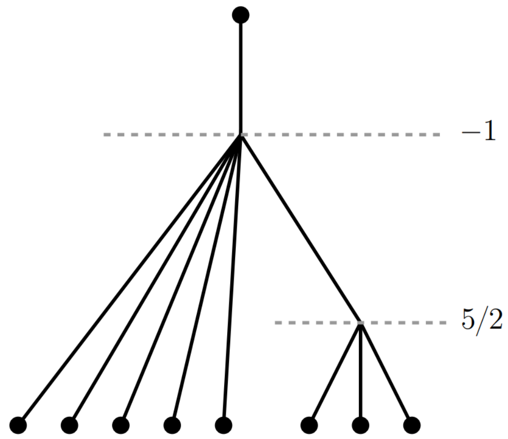


Figure 4.1: The configuration of the branch locus of the morphism φ considered in Example 4.15. Recall our conventions for visualizing the tree spanned by a finite set of points (Section 1.6 and particularly Example 1.39). The point ∞ is at the top, the finite branch points are at the bottom. The dashed lines represent clusters of roots of the indicated radius

We will describe the function δ on the interval $[x_0, \infty]$, where x_0 is one of the three branch points contained in the cluster of radius $5/2$. As usual, we write ξ_r for the boundary point of the disk $\mathsf{D}(r)$ of radius $r \in \mathbb{Q}$ centered at x_0. The first step is to rewrite F in terms of the parameter $t = x - x_0$ and to obtain a new generator z as explained in the beginning of this section. Then z has minimal polynomial of the form

$$T^3 + (a_2t^2 + a_1t + a_0)T^2 + (b_3t^3 + b_2t^2 + b_1t + b_0)T + c_4t^4 + c_2t^2 + c_1t.$$

All the coefficients lie in a finite extension of $\mathbb{Q}_3$, and it is not hard to compute their valuation (see Code Listing A.3). We have

$$v_K(a_0) = \frac{5}{3}, \quad v_K(a_1) = 2, \quad a_2 = 0,$$

$$v_K(b_0) = \frac{10}{3}, \quad v_K(b_1) = 0, \quad v_K(b_2) = 1, \quad b_3 = 0,$$

$$v_K(c_1) = 0, \quad v_K(c_2) = \frac{5}{3}, \quad v_K(c_4) = 1.$$

It follows from Remark 4.14 that

$$\delta(\xi_r) = \max\Big(0, \min\big(\frac{3}{2}, \frac{3A(r)}{2} - \frac{C(r)}{2}, \frac{3B(r)}{2} - C(r)\big)\Big).$$

where A, B, C are the piecewise affine functions

$$A(r) = \min\big(\frac{5}{3}, r+1\big), \qquad B(r) = \min\big(\frac{10}{3}, r, 2r+1\big),$$

$$C(r) = \min\big(r, 2r + \frac{5}{3}, 4r+1\big).$$

The graph of the function $r \mapsto \delta(\xi_r)$ is plotted in Figure 4.2 — for $r \leqslant -1$ and $r \geqslant 5$, the function is constant.

Recall that in Example 2.21 we have related the slope of δ at a point $\xi \in [x_0, \infty]$ to the number and type of branch points contained in $\mathsf{D}(r)$ and the genus of the inverse image $\mathsf{C} := \varphi^{-1}(\mathsf{D}(r))$.

The kinks at $r = -1$ and $r = 5/2$ are because of the branch points. Indeed, the slope of δ changes by $-5/2$ at the radius $r = -1$, since $\mathsf{D}(-1)$ contains five fewer branch points than $\mathsf{D}(r)$ for $r < -1$. And the slope changes by -1 at the radius $r = 5/2$, since $\mathsf{D}(5/2)$ contains two fewer branch points than $\mathsf{D}(r)$ for $-1 < r < 5/2$.

Let us highlight in particular the radius $r = -2/5$. Following Remark 4.5, we compute the minimal polynomial of $\overline{z} := \overline{z\pi^{-s}}$, where $s = v_r(C)/3$. It is the polynomial

$$T^3 + \overline{b}tT + \overline{c}t^4 \in k[T],$$

where $\overline{b}, \overline{c} \in \kappa^\times$ are certain non-zero elements we don't need to compute here. This shows that the reduction curve at the point η above ξ_r is of genus 2, totally ramified above ∞ and with ordinary ramification above $t = 0$.

This explains the other kinks in the graph of δ. The slope at radii $0 < r < 5/2$ is $1/2$, an increase of $3 = 6/2$ over the slope at radii $-1 < r < -2/5$. This is because the genus of $\mathsf{C} = \varphi^{-1}(\mathsf{D}(r))$ is 0 for $r > 0$, while it is 3 for $r < -2/5$. Indeed, C contains a point of genus 2 and a loop (lying above the interval $[-2/5, 0]$) if $r < -2/5$. We will also re-recover these insights in Example 5.16, where we return to this example.

Now that we have explained how to compute δ on intervals of the form $[x_0, \infty]$, the following algorithm to compute the image of the tame locus presents itself.

Algorithm 4.16. Input: A smooth plane quartic curve Y over K, equipped with a degree-3 morphism $\varphi\colon Y \to X = \mathbb{P}^1_K$, given in the normal form discussed in Section 4.1.

(1) For each closed point $x_0 \in \Gamma_0$ different from ∞, determine the restriction of δ to $[x_0, \infty]$ using the formula in Remark 4.14.

(2) For each x_0, define U_{x_0} to be the inverse image under the canonical retraction map $\mathrm{ret}_{\Gamma_0}\colon X^{\mathrm{an}} \to \Gamma_0$ of $\{\xi \in [x_0, \infty] \mid \delta(\xi) = 0\}$. It is a disjoint union of closed anuli and closed disks.

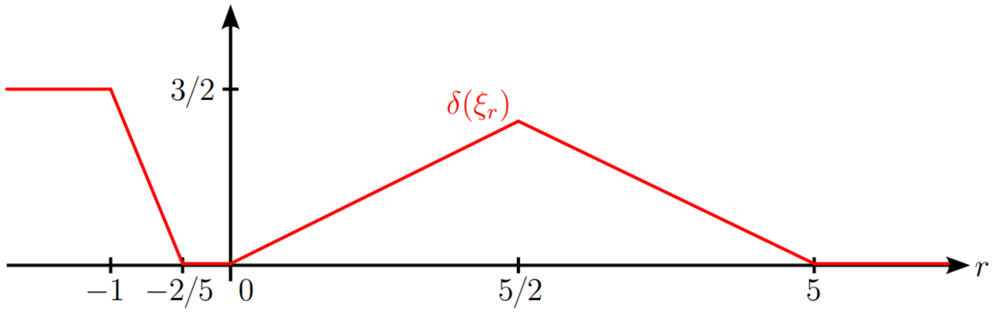


Figure 4.2: The graph of $\delta(\xi_r)$ as a function of r

(3) Compute the affinoid subdomain $\mathsf{U} = \mathsf{U}_2 \cup \mathsf{U}_4$ defined in Section 4.5.

Output: The image $\varphi(Y^{\mathrm{int}})$ of the interior locus is given by $\bigcup_{x_0} \mathsf{U}_{x_0}$ and the image of the tail locus $\varphi(Y^{\mathrm{tail}})$ by $\mathsf{U}\backslash\varphi(Y^{\mathrm{int}})$.

Proof of correctness of the algorithm. By Corollary 4.8, it suffices to show that $\varphi(Y^{\mathrm{int}}) = \bigcup_{x_0} \mathsf{U}_{x_0}$. Let $\eta \in Y^{\mathrm{an}}$ be a point with retraction η_1 onto Σ, the φ-minimal skeleton of Y^{an} with respect to Γ_0. To say that $\eta \in Y^{\mathrm{int}}$ is to say that η_1 is not a leaf of Σ and $\delta(\eta_1) = 0$. Equivalently, $\xi_1 := \varphi(\eta_1)$ lies in Γ_0 and $\delta(\xi_1) = 0$. If this is the case, ξ_1 lies on one of the paths $[x_0, \infty]$; then $\xi \in \mathsf{U}_{x_0}$. Conversely, if $\delta(\xi_1) > 0$ or $\xi_1 \notin \Gamma_0$, then the retraction of ξ onto Γ_0 does not lie in the locus $\{\xi \in \Gamma_0 \mid \delta(\xi) = 0\}$, so ξ does not lie in any of the U_{x_0}. ■

4.7 Examples

In this section, we compute the tame locus in two examples. For illustration, we do this in relative detail, even though there is nothing here that our implementation (see Section A.1 in the appendix and the examples in Section 5.5) cannot do on its own.

Example 4.17. We revisit Example 3.47 and compute the tame locus associated to the plane quartic over $K = \mathbb{C}_3$

$$Y: \quad F(y) = y^3 + 3xy^2 - 3y - 2x^4 - x^2 - 1 = 0.$$

The code used for several computations in this example — the computation of the degree-27 numerator of $\mathrm{Nm}(c_3)$, the clustering behavior of the roots of $\mathrm{Nm}(c_3)$, the radii $\lambda(x_0)$, and the reductions of $\tilde{H}$ — may be found in Code Listing A.4.

Following Section 4.3, we first compute the polynomial $\tilde{F}(T)$ defined in (4.2). It is

$$\begin{aligned}\tilde{F}(T) = {} & T^3 + (3t + 3x)T^2 - 3T \\ & -2t^4 - 8xt^3 + (-12x^2 - 1)t^2 + (-8x^3 - 2x)t - 2x^4 - x^2 - 1.\end{aligned}$$

Note that its coefficients are just Taylor expansions of the coefficients of F. Next, the approximation of order 2 as defined in (4.3) is

$$\tilde{u}_2 = y - \frac{3y^2 - 8x^3 - 2x}{3y^2 + 6xy - 3} t.$$

This leads to $\tilde{H}(T) = \tilde{F}(T + \tilde{u}_2)$, a polynomial whose coefficients we denote

$$\tilde{H}(T) = T^3 + \tilde{A}T^2 + \tilde{B}T + \tilde{C}.$$

Moreover, we use the usual notations $\tilde{C} = \sum_i c_i t^i$ and $\tilde{B} = \sum_i b_i t^i$ for the coefficients of $\tilde{C}$ and $\tilde{B}$.

To find the three points of positive genus indicated in Figure 3.1, we make an educated guess. Corollary 4.8 shows that on every tail disk the function $\mathrm{Nm}(c_3)$ has high valuation compared to $\mathrm{Nm}(c_2)$ or $\mathrm{Nm}(c_4)$. For this reason we consider the numerator of the rational function $\mathrm{Nm}(c_3)$, suspecting that the tail disks are centered at its roots. It is (up to multiplication with a constant in K) the degree-27 polynomial

$$\begin{aligned} x^{27} &- \tfrac{27}{4}x^{26} + \tfrac{410067}{21296}x^{25} - \tfrac{4362147}{85184}x^{24} + \tfrac{5803011}{42592}x^{23} - \tfrac{43814169}{170368}x^{22} \\ &+ \tfrac{159975987}{340736}x^{21} - \tfrac{67087575}{85184}x^{20} + \tfrac{760899573}{681472}x^{19} - \tfrac{1082838429}{681472}x^{18} \\ &+ \tfrac{2331229365}{1362944}x^{17} - \tfrac{5848321365}{2725888}x^{16} + \tfrac{2354253261}{1362944}x^{15} - \tfrac{4942346193}{2725888}x^{14} \\ &+ \tfrac{5475421305}{5451776}x^{13} - \tfrac{5607571491}{5451776}x^{12} + \tfrac{1571650461}{10903552}x^{11} - \tfrac{3708417465}{10903552}x^{10} \\ &- \tfrac{2896673413}{21807104}x^{9} - \tfrac{2860325271}{43614208}x^{8} - \tfrac{14807853}{123904}x^{7} - \tfrac{360410715}{10903552}x^{6} - \tfrac{10916775}{5451776}x^{5} \\ &+ \tfrac{351645543}{21807104}x^{4} + \tfrac{180770859}{10903552}x^{3} + \tfrac{1673055}{10903552}x^{2} - \tfrac{28048275}{21807104}x + \tfrac{373977}{43614208}. \end{aligned}$$

The denominator of $\mathrm{Nm}(c_3)$ serves a similar role to the *monodromy polynomial* associated to a superelliptic curve defined in [LM06], which is also a polynomial at whose roots the tail disks are centered. See [BW15, Section 4] for more background on the link between monodromy polynomials and tail disks. The zeros of $\mathrm{Nm}(c_3)$ cluster as visualized in Figure 4.3. That is, there exist three closed disks $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, each of radius 1, each containing nine roots of $\mathrm{Nm}(c_3)$. We do not claim however that these disks cut out the only proper clusters among the roots, or that the clusters cut out by $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ have depth 1.

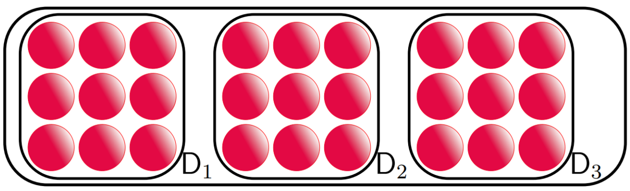


Figure 4.3: The clustering behavior of the roots of $\mathrm{Nm}(c_3)$

If the disks $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ are concentric with three tail disks, then by Lemma 4.7(c) we have for each Type I center x_0 of one of them the equality

$$\lambda(x_0) = \frac{3\widehat{\mathrm{Nm}\, b_0}(x_0) - 2\widehat{\mathrm{Nm}\, c_2}}{3 \cdot 4}.$$

For two of the disks $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, the resulting radius is $\lambda(x_0) = 3/4$, and for the third it is $\lambda(x_0) = 1$. Now using Remark 4.5 it is easy to compute the minimal polynomials of the generators $\overline{w}$ of the extension of residue fields $\kappa(\eta_i)/\kappa(\xi_i)$, where ξ_i is the boundary point of D_i and $\varphi^{-1}(\xi_i) = \{\eta_i\}$. In each case, it is of the form

$$T^3 + \bar{b}T + \bar{c}\bar{t}^2, \qquad \bar{b}, \bar{c} \in \kappa. \tag{4.15}$$

These are Artin-Schreier polynomials defining function fields of genus 1 (Lemma 4.2). As mentioned above, we refer to Code Listing A.4 for details on the computation of $\lambda(x_0)$ and (4.15).

This essentially shows that the tame locus has the structure claimed in Example 3.47, except for the position of the ramification points. We omit this point for now, but we will once more return to this curve in Example 5.16.

Example 4.18. Now we revisit Example 3.48, computing the tame locus associated to the plane quartic over $K = \mathbb{C}_3$

$$Y: \quad F(y) = y^3 - y^2 + (3x^3 + 1)y + 3x^4 = 0.$$

The discriminant of F,

$$\Delta_F = -108x^9 - 243x^8 - 162x^7 - 99x^6 - 42x^4 - 30x^3 - 3,$$

has four zeros of valuation 0 and five zeros of valuation $-2/5$. In particular, $\varphi\colon Y \to \mathbb{P}^1_K$ has ordinary ramification, one branch point being ∞. Among the zeros of Δ_F, there is only one proper cluster, which is cut out by a closed disk of radius 0 and contains the four zeros of Δ_F of valuation 0. As in Example 4.15, we compute the function δ on the interval $[x_0, \infty]$, where x_0 is one of the four zeros of valuation 0. The resulting graph is shown in Figure 4.4; see Code Listing A.5 for this computation.

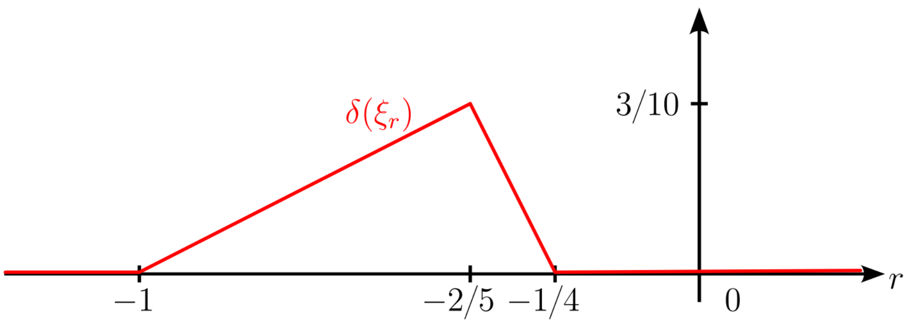


Figure 4.4: The graph of $\delta(\xi_r)$ as a function of r

In Code Listing A.5, we furthermore compute the inverse images of the two points ξ_0 and $\xi_{-1/4}$. As can be seen from Figure 4.4, the latter is a wild topological branch point; its unique preimage has residue field given by

$$\bar{z}^3 - \bar{z}^2 - \bar{z} + 1/\bar{t}^4 = 0.$$

This defines a curve of genus 1, ramified only above ∞ (corresponding to the fact that $\delta > 0$ for $-1 < r < -1/4$). Two points lie above ξ_0, one not a topological ramification point, the other a tame topological ramification point with residue field given by

$$\overline{z}^2 = \overline{t}^4 + \overline{t}^3 + 1.$$

This defines a curve of genus 1, with cover to $\mathbb{P}^1_\kappa$ branched at four finite points (corresponding to the cluster of four branch points of φ).

The above shows that the tame locus indeed looks as claimed in Example 3.48. Indeed, we have recovered the two points of positive genus. They must be connected by a loop as in Figure 3.2, since two branches lie above the branch pointing from ξ_0 towards ∞.

Chapter 5

Examples and implementation

Throughout most of Chapters 2, 3, and 4 we have assumed that our base field K is algebraically closed. However, we are ultimately interested in studying curves over discretely valued fields. In this chapter, we will explain how to adapt our main results to this goal. We will continue to assume that K is algebraically closed, but make the following assumption (essentially the option (b) in the section on notations and conventions):

Assumption 5.1. *There exists a discretely valued subfield $K_0 \subset K$ such that K is the completion of an algebraic closure of K_0.*

In Chapter 4, we related the tail locus associated to a plane quartic Y over K to admissible functions on Y^{an}, the main result being Theorem 4.6. In the following Section 5.1, we will see that in the case that Y is defined over K_0 and equipped with a morphism $Y \to \mathbb{P}^1_{K_0}$ over K_0, the construction of these valuative functions and the statement of Theorem 4.6 remains valid while working over K_0.

In Sections 5.2 and 5.3 we will describe the structure of the projective line $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ over K_0. It mirrors the structure of $(\mathbb{P}^1_K)^{\mathrm{an}}$ explained in Section 1.4, with *discoids* taking the place of closed disks. This enables us to adapt Algorithm 4.16 to the case of discretely valued ground field. The resulting Algorithm 5.9 is implemented in Sage. We illustrate it with various examples in Section 5.5. Its output is a set of Type II valuations on the rational function field $K_0(x)$, or equivalently (using Proposition 1.6) a normal model $\mathcal{X}_0$ of $\mathbb{P}^1_{K_0}$. By construction, the normalization of $\mathcal{X}_0$ in the function field F_{Y_K} is a semistable model of Y_K. In Section 5.4, we discuss the problem of finding a finite field extension L/K_0 such that the normalization of $\mathcal{X}_0$ in F_{Y_L} has semistable reduction.

5.1 Discretely valued base field

Suppose that we are in the situation considered in Section 3.1, but over the discretely valued field K_0. That is, we have a degree-p cover

$$\varphi\colon Y \to X = \mathbb{P}^1_{K_0}$$

of smooth projective and geometrically connected curves over K_0. Let us moreover suppose that we have an étale affine chart (U, y) for φ. This is defined just as it was in Definition 3.1, with K replaced by K_0. Equivalently, $U \subseteq \mathbb{P}^1_{K_0}\backslash\{\infty\}$ is a non-empty open subscheme and y a generator of the field extension of function fields F_Y/F_X such that the pair

$$(U_K, y \otimes 1)$$

obtained from (U, y) by applying the base change functor $\cdot \otimes_{K_0} K$ is an étale affine chart in the previous sense.

We will now repeat the constructions of Sections 3.1 and 3.5, but working with objects over K_0 rather than over the algebraically closed field K. We will see that we can describe the function λ and the tame locus working only over K_0. Indeed, this is necessary for working out concrete examples and implementing our algorithms.

As before, write $A := \mathcal{O}_X(U)$ and $B := \mathcal{O}_Y(V)$, where $V = \varphi^{-1}(U)$. Then A and B are finite type K_0-algebras. Furthermore write

$$F(T) = T^p + f_{p-1}T^{p-1} + \ldots + f_1 T + f_0$$

for the minimal polynomial of y over F_X. The coefficients $f_0, \ldots, f_{p-1}$ lie in A (compare Lemma 3.4), hence we have

$$\tilde{f}_i(t) := \sum_{\ell=0}^{\infty} a_{i,\ell} t^\ell \in A[\![t]\!], \qquad \text{where} \quad a_{i,\ell} = \frac{f_i^{(\ell)}}{\ell!}, \quad 0 \leqslant i \leqslant p-1.$$

The $\tilde{f}_i$ are the coefficients of the "generic" minimal polynomial of y,

$$\tilde{F}(T) = T^p + \tilde{f}_{p-1}T^{p-1} + \ldots + \tilde{f}_1 T + \tilde{f}_0 \in A[\![t]\!][T].$$

The minimal polynomial

$$G(T) = T^p + g_{p-1}T^{p-1} + \ldots + g_1 T + g_0$$

of y over the subfield $K_0(t)$, where $t = x - x_0$ is the parameter associated to a fixed K_0-rational point x_0, is obtained from $\tilde{F}$ by evaluation at x_0; we have $g_i = \sum_\ell a_{i,\ell}(x_0)t^\ell$.

Next, we repeat the application of Lemma 3.14 to the polynomial $\tilde{F}$ and the zero $y \in B$ of $\tilde{F}$, resulting in a formal solution

$$\tilde{y} = y + \tilde{b}_1 t + \tilde{b}_2 t^2 + \tilde{b}_3 t^3 + \ldots \in B[\![t]\!].$$

Consider the truncation $\tilde{u}_m = \tilde{b}_0 + \tilde{b}_1 t + \ldots + \tilde{b}_{m-1}t^{m-1}$ of order m, where $m \geqslant 1$ is some integer. Then the coefficients $\tilde{h}_0, \ldots, \tilde{h}_p$ of the polynomial

$$\tilde{H}(T) = \tilde{F}(T + \tilde{u}_m) = \sum_{i=0}^{p} \tilde{h}_i T^i$$

also lie in $B[\![t]\!]$. As in Section 3.5, we write

$$\tilde{h}_i = \sum_{\ell=0}^{\infty} c_{i,\ell} t^\ell.$$

The $c_{i,\ell}$, $0 \leqslant i < p$, $\ell \geqslant 0$, lie in F_Y, so define valuative functions

$$\widehat{c}_{i,\ell} \colon Y^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}, \qquad \xi \mapsto v_\xi(c_{i,\ell}).$$

While we assumed that the base field was algebraically closed in Chapter 2 (because we treated skeletons, and by extension admissible functions, only in the context of algebraically closed fields), this is not necessary for the above definition of $\widehat{c}_{i,\ell}$. In the present context, Y^{an} is an analytic curve over the discretely valued field K_0, and its underlying set may be identified with pseudovaluations on the function field F_Y.

Proceeding as in Lemma 2.2 and Remark 2.3, we may then define functions

$$\tilde{\lambda} = \max_{m \leqslant k < mp} \min_{0<i<p} \min_{0 \leqslant p\ell < (p-i)k} \frac{p\widehat{c}_{i,\ell} - (p-i)\widehat{c}_{0,k}}{(p-i)k - p\ell},$$

$$\max_{m \leqslant k < mp} \frac{p\widehat{c}_{1,0} - (p-1)\widehat{c}_{0,k}}{(p-1)k} \tag{5.1}$$

on Y^{an}. The subtlety of defining these at the closed points where the constituent valuative functions have zeros or poles may be dealt with in the same manner as in the proof of Lemma 2.2, namely by studying the limiting behavior of as one approaches such closed points. We do not discuss this in detail, since ultimately we need only know the values of these functions at Type II points.

Consider the morphism

$$\pi \colon Y_K^{\mathrm{an}} \to Y^{\mathrm{an}}$$

induced by the base change morphism $Y_K \to Y$. The points lying over a point $\eta_0 \in Y^{\mathrm{an}}$ correspond to the pseudovaluations on F_{Y_K} extending the pseudovaluation v_{η_0} on F_Y. Thus for every $\eta \in Y_K^{\mathrm{an}}$ and every valuative function $h \colon Y^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$, say $h = \widehat{f}$ for $f \in F_Y$, we have

$$(\widehat{f \otimes 1})(\eta) = \widehat{f}(\pi(\eta)). \tag{5.2}$$

There is a similar morphism $X_K^{\mathrm{an}} \to X^{\mathrm{an}}$, which we also denote by π.

We now specialize to the case that Y is a plane quartic. Using the same notation as in Section 4.3, we then have

$$\tilde{H}(T) = T^3 + \tilde{A}T^2 + \tilde{B}T + \tilde{C},$$

$$\tilde{A} = \sum_{\ell=0}^{2} a_\ell t^\ell, \quad \tilde{B} = \sum_{\ell=0}^{3} b_\ell t^\ell, \quad \tilde{C} = \sum_{\ell=0}^{2} c_\ell t^\ell.$$

We define subsets $\mathsf{V}_{K_0} \subseteq Y^{\mathrm{an}}$ and $\mathsf{U}_{K_0} \subseteq X^{\mathrm{an}}$ by the same formulas as in Sections 4.4 and 4.5 respectively. That is, we have

$$\mathsf{V}_{K_0} = \{\eta \in Y^{\mathrm{an}} \mid 3\widehat{b}_0(\eta) + 4\widehat{c}_3(\eta) \geqslant 6\widehat{c}_2(\eta) \text{ or } 8\widehat{c}_3(\eta) \geqslant 3\widehat{b}_0(\eta) + 6\widehat{c}_4(\eta)\},$$

$$\begin{aligned}\mathsf{U}_{K_0} = \{\xi \in X^{\mathrm{an}} \mid & 3\widehat{\mathrm{Nm}\, b_0}(\xi) + 4\widehat{\mathrm{Nm}\, c_3}(\xi) \geqslant 6\widehat{\mathrm{Nm}\, c_2}(\xi) \\ \text{or} \quad & 8\widehat{\mathrm{Nm}\, c_3}(\xi) \geqslant 3\widehat{\mathrm{Nm}\, b_0}(\xi) + 6\widehat{\mathrm{Nm}\, c_4}(\xi)\}.\end{aligned}$$

Let us write

$$\mathsf{U}_K := \pi^{-1}(\mathsf{U}_{K_0}) \subseteq X_K^{\mathrm{an}}, \qquad \mathsf{V}_K := \pi^{-1}(\mathsf{V}_{K_0}) \subseteq Y_K^{\mathrm{an}}.$$

Corollary 5.2. *U_K and V_K are the same as the affinoid subdomains $\mathsf{U} \subset X_K^{\mathrm{an}}$ (defined in Section 4.5) and $\mathsf{V} \subset Y_K^{\mathrm{an}}$ (defined in Section 4.4) associated to the base change Y_K.*

Proof. This follows immediately from the discussion in this section. Namely, a point $\eta_0 \in Y^{\mathrm{an}}$ lies in V_{K_0} if and only if $3\widehat{b}_0(\eta_0) + 4\widehat{c}_3(\eta_0) \geqslant 6\widehat{c}_2(\eta_0)$ or $8\widehat{c}_3(\eta_0) \geqslant 3\widehat{b}_0(\eta_0) + 6\widehat{c}_4(\eta_0)$. The affinoid subdomain V of Section 4.4 is defined by the same equalities, except using the functions $b_0 \otimes 1$, $c_2 \otimes 1$, $c_3 \otimes 1$, and $c_4 \otimes 1$ instead of b_0, c_2, c_3, c_4. By (5.2), a point $\eta \in \pi^{-1}(\eta_0)$ lies in V if and only if it lies in V_K. The equality of U_K and U is proved in the same way. ■

5.2 Discoids

We have studied the structure of the Berkovich line $(\mathbb{P}^1_K)^{\mathrm{an}}$ (where K is algebraically closed) in Section 1.4. To derive a similar description for $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$, we need the notion of *discoids*. Building on work of MacLane ([Mac36a], [Mac36b]), their connection to models and valuations has been worked out in [Rüt14, Section 4.4]. However, in [Rüt14], discoids are by definition subsets of K; essentially, points that are not Type I are disregarded. Following [Mic20, Chapter 2], the following definition takes all points into account.

Definition 5.3. A *discoid* is a subset of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the form

$$\mathsf{D}[\psi, \rho] := \{\xi \in (\mathbb{P}^1_{K_0})^{\mathrm{an}} \mid v_\xi(\psi) \geqslant \rho\},$$

where $\rho \in \mathbb{R}$ and $\psi \in K_0[x]$ is a monic irreducible polynomial. The number ρ is called the *radius* of $\mathsf{D}[\psi, \rho]$. The *degree* of $\mathsf{D}[\psi, \rho]$ is simply the degree of ψ.

It will be convenient to extend the notation to the case $\rho = \infty$. Then $\mathsf{D}[\psi, \infty]$ is just the closed point associated to ψ. Note that if ψ is a constant polynomial, say $\psi = x - x_0$ for some $x_0 \in K_0$, then $\mathsf{D}[\psi, \rho]$ is simply the closed disk of radius ρ centered at x_0.

For a discoid $\mathsf{D} \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$, there is an associated Type II valuation v_{D} on $K_0(x)$ (compare [Rüt14, Lemma 4.50]), defined by

$$v_{\mathsf{D}}(f) = \inf\{v_\xi(f) \mid \xi \in \mathsf{D}\}, \qquad f \in K[x].$$

In fact, v_{D} is the valuation corresponding to the unique boundary point of D in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$. We denote this boundary point by ξ_{D} or, if $\mathsf{D} = \mathsf{D}[\psi, \rho]$, by $\xi_{\psi,\rho}$. If $\rho = \infty$, then we denote by $\xi_{\psi,\rho}$ the unique closed point in $\mathsf{D}[\psi, \infty]$.

Proposition 5.4. *The map $\mathsf{D} \mapsto v_{\mathsf{D}}$ is a bijection from the set of discoids $\mathsf{D} \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of rational radius to the set of Type II valuations on $K_0(x)$.*

Proof. This follows from [Rüt14, Theorem 4.56]. ∎

In Section 1.4, we saw how to describe all the points of $(\mathbb{P}^1_K)^{\mathrm{an}}$, where K was algebraically closed, in terms of nested families of disks (Definition 1.24). The analogue of this in the present context of discoids is given by MacLane's theory of inductive valuations. We only introduce these briefly here and refer the reader to [Rüt14] and [Mic20] for more information.

One begins with a Gauss valuation of a certain radius $\lambda_0 \in \mathbb{R}$,

$$v_0\colon K(x) \to \mathbb{R} \cup \{\infty\}, \qquad \sum_i a_i x^i \mapsto \min_i\{v_{K_0}(a_i) + i\lambda_0\}.$$

Then given so-called *key polynomials* $\psi_1, \ldots, \psi_n \in K[x]$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \cup \{\infty\}$, one inductively defines pseudovaluations

$$v_k = [v_0(x) = \lambda_0, v_1(\psi_1) = \lambda_1, \ldots, v_k(\psi_k) = \lambda_k], \qquad k = 1, \ldots, n,$$

on $K(x)$. As the notation suggests, the pseudovaluation v_k satisfies $v_k(\psi_k) = \lambda_k$; moreover, $v_k(\chi) = v_{k-1}(\chi)$ for $\chi \in K[x]$ of degree less than $\deg(\psi_k)$. The key fact ([Mic20, Proposition 1.110]) is then that every pseudovaluation v on $K(x)$ — with the exception of the pseudovaluation associated to the closed point $\infty \in \mathbb{P}^1_{K_0}$ — can be approximated by such inductive pseudovaluations. To be precise, we have the following ([Mic20, Corollary 1.117]):

- If $\lambda_n = \infty$, then $v_n = [v_0(x) = \lambda_0, \ldots, v_n(\psi_n) = \lambda_n]$ is a pseudo-valuation that is not a valuation, so is associated to a Type I point of X^{an}.
- If $\lambda_n \in \mathbb{Q}$, then v_n is a Type II valuation. In fact, it is the valuation v_{D}, where D is the discoid $\mathsf{D}[\psi_n, \lambda_n]$.
- If $\lambda_n \in \mathbb{R}\backslash\mathbb{Q}$, then v_n is associated to a Type III point of X^{an}.
- Given infinite sequences $(\psi_n)_{n\geqslant 1}$ and $(\lambda_n)_{n\geqslant 0}$, one may define a *limit valuation*
$$v_\infty := \lim_{n\to\infty}[v_0(x) = \lambda_0, \ldots, v_n(\psi_n) = \lambda_n].$$
All Type IV points arise in this way. Note however that there is a definitional difference regarding Type IV points between [Mic20] and our definition in Section 1.3: In [Mic20], only closed points are considered Type I, while for us, a point $\xi \in X^{\mathrm{an}}$ such that $\mathcal{H}(\xi)$ may be imbedded in K is considered Type I as well. Thus the statement [Mic20, Corollary 1.117(4)] is not quite correct if one follows our definition.

It is important to understand how discoids behave under extensions of the base field. What is the preimage of a discoid $\mathsf{D}[\psi, \rho]$ under the natural morphism

$$\pi \colon Y_K^{\mathrm{an}} \to Y^{\mathrm{an}}?$$

Phrased differently, which valuations on $F_{Y_K} = F_Y \otimes_{K_0} K$ extend the valuations $\{v_\eta \mid \eta \in \mathsf{D}[\psi, \rho]\}$ on F_Y? These questions are treated in [Mic20, Section 2.3].

Let $P \in \mathbb{P}^1_{K_0} \backslash \{\infty\}$ be a closed point. It corresponds to a maximal ideal in $K_0[x]$, say the one generated by the monic irreducible polynomial $\psi \in K_0[x]$. Then the points in $\mathbb{P}^1_K$ above P correspond to the $\deg(\psi)$ zeros of ψ in K. We write

$$\psi = \prod_{i=1}^{s} (x - \alpha_i), \qquad \alpha_1, \ldots, \alpha_s \in K, \tag{5.3}$$

and for $r \in \mathbb{R} \cup \{\infty\}$ define the sets

$$I_r := \{i \in \{1, \ldots, s\} \mid v_K(\alpha_1 - \alpha_i) \geqslant r\},$$

$$J_r := \{i \in \{1, \ldots, s\} \mid v_K(\alpha_1 - \alpha_i) < r\}.$$

Now following [Mic20, Definition 2.25], we define the function

$$\theta_\psi \colon \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}, \qquad r \mapsto r\,|I_r| + \sum_{i \in J_r} v_K(\alpha_1 - \alpha_i).$$

The function θ_ψ is independent of the numbering of the roots in (5.3) ([Mic20, Lemma 2.27]) and is strictly monotonously increasing ([Mic20, Lemma 2.28]).

Proposition 5.5. *For any $\rho \in \mathbb{R}$ we have*

$$\pi^{-1}(\mathsf{D}[\psi, \theta_\psi(\rho)]) = \bigcup_{i=1}^{r} \mathsf{D}[\alpha_i, \rho].$$

Proof. This is a special case of [Mic20, Theorem 2.29]. ∎

Note that the union in Proposition 5.5 need not be disjoint; if the radius ρ is chosen small enough, the disks $\mathsf{D}[\alpha_i, \rho]$ will coalesce.

5.3 The structure of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$

Recall that in Section 1.4 we have explained how $(\mathbb{P}^1_K)^{\mathrm{an}}$ may be thought of as an infinitely branching tree. Chapter 2 of [Mic20] leverages discoids to obtain a similar description of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$.

Crucially, it is still true that $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ is a simply connected special quasipolyhedron ([Ber90, Theorem 4.2.1]). Thus there exists a unique path between any pair of points in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$. These paths are explicitly described in

[Mic20, Section 2.7]. For example, given two monic irreducible polynomials $\psi_1, \psi_2 \in K_0[x]$, the unique path connecting the closed points $\xi_{\psi_1,\infty}$ and $\xi_{\psi_2,\infty}$ is

$$\{\xi_{\psi_1,r'} \mid r' \geqslant r\} \cup \{\xi_{\psi_2,r'} \mid r' \geqslant r\},$$

where r denotes the smallest radius for which $\xi_{\psi_2,\infty}$ is still contained in $\mathsf{D}[\psi_1, r]$. The path may be visualized in the same way as in the case of algebraically closed ground field, cf. Figure 1.2.

The description of the affinoid subdomains of $(\mathbb{P}^1_K)^{\mathrm{an}}$ in terms of standard affinoid subdomains that we gave in Section 4.5 remains valid over K_0. To be more precise, we define an *open discoid* to be a subset of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the form

$$\mathsf{D}(\psi, \rho) := \{\xi \in (\mathbb{P}^1_{K_0})^{\mathrm{an}} \mid v_\xi(\psi) > \rho\},$$

where $\psi \in K_0[x]$ is monic and irreducible and $\rho \in \mathbb{R}$, or of the form

$$\mathsf{D}(1/x, \rho) := \{\xi \in (\mathbb{P}^1_{K_0})^{\mathrm{an}} \mid v_\xi(x) < -\rho\},$$

where $\rho \in \mathbb{R}$. The latter type is just open disks centered at ∞. Then a *standard affinoid subdomain* of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ is a closed discoid, or a closed disk centered at ∞, with a finite number of open discoids removed.

Let $S \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ be a finite set of Type I and Type II points, satisfying $|S| \geqslant 3$ or containing at least one Type II point. Using the same procedure as in the proof of Proposition 1.38, we may construct the *tree spanned by S*. It is the union of all the paths connecting points $\xi, \xi' \in S$. (Really, this tree is a skeleton of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$, but we have only defined skeletons in the case that the ground field is algebraically closed.)

Let $T \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ be the tree spanned by a finite set S. Since $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ is uniquely path-connected, we may in the same way as in Definition 1.37 define a canonical retraction

$$\mathrm{ret}_T\colon (\mathbb{P}^1_{K_0})^{\mathrm{an}} \to T.$$

Now let $U \subset T$ be a closed subset. Then the subset

$$\mathrm{ret}_T^{-1}(U) \subseteq (\mathbb{P}^1_{K_0})^{\mathrm{an}}$$

is an affinoid subdomain, since it is a disjoint union of standard affinoid subdomains. This way of describing affinoid subdomains by means of the retraction to a tree underlies the implementation of affinoid subdomains of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ in the MCLF Sage package. For details, we refer the reader to the documentation of the MCLF classes `AffinoidTree` and `AffinoidDomainOnBerkovichLine`. We illustrate the concept in Example 5.7 below.

Remark 5.6. The MCLF package also is able to compute subsets like $\mathsf{U}_{K_0} \subseteq (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ defined in Section 5.1 as part of the functionality of the class `PiecewiseAffineFunction`. In fact, U_{K_0} is an affinoid subdomain. This is because the valuative functions $\widehat{\mathrm{Nm}\, b_0}, \widehat{\mathrm{Nm}\, c_2}, \widehat{\mathrm{Nm}\, c_3}, \widehat{\mathrm{Nm}\, c_4}$ used to define U_{K_0} are constant outside the tree spanned by the zeros and poles of $\mathrm{Nm}\, b_0, \mathrm{Nm}\, c_2, \mathrm{Nm}\, c_3, \mathrm{Nm}\, c_4$. In particular, it is clear how to represent U_{K_0} (and other affinoid subdomains like it) using retraction to a tree.

Example 5.7. Suppose that T is a tree contained in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ which looks like the one drawn in Figure 5.1. Let U be its closed subset that is colored black. (The black-red color scheme is so chosen because in the next section, closed subsets such as U will arise as the retractions of tame loci.)

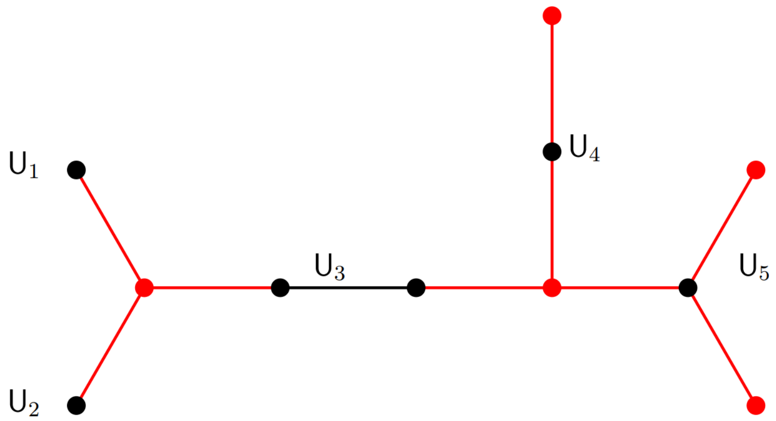


Figure 5.1: A tree within $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$

The inverse image of U under the retraction $\mathrm{ret}_T\colon (\mathbb{P}^1_{K_0})^{\mathrm{an}} \to T$ consists of five components (or standard affinoid subdomains), numbered $\mathsf{U}_1, \ldots, \mathsf{U}_5$. The components $\mathsf{U}_1, \mathsf{U}_2$ are discoids; they are the complements of open discoids. The component U_3 is the complement of two open discoids. We call such an affinoid subdomain an *anuloid* (cf. [Mic20, Definition 2.12]). The component U_4 is also an anuloid, but of radius 0. Finally, U_5 is the complement of three discoids. One may visualize it as a closed discoid punctured by two open discoids of equal radius.

Remark 5.8. Let $T \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ be a tree. It may happen that the inverse image of the vertex set of T under the map $\pi\colon Y_K^{\mathrm{an}} \to Y^{\mathrm{an}}$ is not the vertex set of a tree.

For example, consider the tree $T \subset (\mathbb{P}^1_{\mathbb{Q}_3})^{\mathrm{an}}$ spanned by the closed point ξ with equation $x^2 + 3$ and the Gauss point (the boundary point of the closed disk of radius 0 centered at ξ). The inverse image of this vertex set under π consists of the Gauss point and the two closed points $\pm\sqrt{-3}$. The tree spanned by these three points also includes the Type II point that is the boundary point of the disk of radius $1/2$ centered at the closed points, see Figure 5.2.
In practice, this problem will not concern us much, since MCLF has an inbuilt function `permanent_completion` that refines a tree $T \subset (\mathbb{P}^1_{K_0})^{\mathrm{an}}$ to a tree T' such that the inverse image of the vertex set of T' under π is again a tree. We refer to the documentation of this function for details.

As an application of our discussion of discoids, we can now explain how to modify Algorithm 4.16 to the case of the discretely valued ground field K_0.

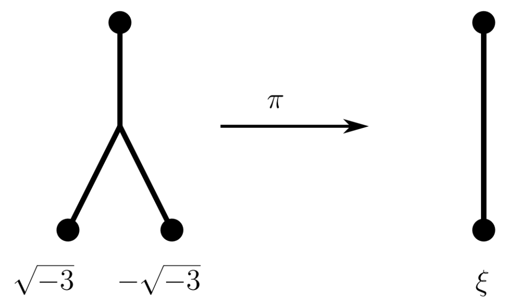


Figure 5.2: The trees mentioned in Remark 5.8, illustrating that one needs to add another vertex to the inverse image of T

Algorithm 5.9. Input: A smooth plane quartic curve Y over K_0, equipped with a degree-3 morphism $\varphi\colon Y \to X = \mathbb{P}^1_{K_0}$, given in the normal form discussed in Section 4.1.

(1) For each finite branch point $P \in X$ of φ, do the following steps to compute an affinoid subdomain U_P associated to P:

(1a) Choose a root $x_0 \in K$ of ψ, where $\psi \in K_0[x]$ is the monic irreducible polynomial generating the maximal ideal in $K_0[x]$ corresponding to P.

(1b) Write $t := x - x_0$. As in Section 4.6, find a generator of F_{Y_K} whose minimal polynomial over $K(t)$, say

$$T^3 + AT^2 + BT + C, \qquad A, B, C \in K[t],$$

has coefficients of the form $A = a_2t^2 + a_1t + a_0$, $B = b_3t^3 + b_2t^2 + b_1t + b_0$, and $C = c_4t^4 + c_2t^2 + c_1t$. Note that we may assume all coefficients of A, B, C to lie in a finite intermediate extension $K/L/K_0$.

(1c) Using the family of Gauss valuations

$$v_r\colon K(t) \to \mathbb{R}\cup\{\infty\}, \qquad \sum_i d_it^i \mapsto \min_i\{v_K(d_i)+ir\}, \qquad r \in \mathbb{R}\cup\{\infty\},$$

define a function $\delta\colon [-\infty, \infty] \to [0, \infty]$ by

$$r \mapsto \max\Big(0, \min\big(\frac{3}{2}, \frac{3A(r)}{2} - \frac{C(r)}{2}, \frac{3B(r)}{2} - C(r)\big)\Big).$$

(1d) Define

$$\mathsf{U}_P := \mathrm{red}_{\Gamma_0}^{-1}\big(\{\xi_{\psi,\theta(r)} \mid r \in [0,\infty] : \delta(r) = 0\}\big),$$

where $\Gamma_0 \subset X^{\mathrm{an}}$ is the tree spanned by the branch points of φ and ∞.

(2) Compute the affinoid subdomain U_{K_0} as explained in Remark 5.6.

(3) Put $\mathsf{U}_{K_0}^{\mathrm{int}} := \bigcup_P \mathsf{U}_P$ and $\mathsf{U}_{K_0}^{\mathrm{tail}} := \mathsf{U}_{K_0} \backslash \mathsf{U}_{K_0}^{\mathrm{int}}$.

Result: For each point $\xi \in X_K$, the point ξ lies in $\varphi(Y_K^{\mathrm{int}})$ if and only if $\pi(\xi) \in \mathsf{U}_{K_0}^{\mathrm{int}}$ and lies in $\varphi(Y_K^{\mathrm{tail}})$ if and only if $\pi(\xi) \in \mathsf{U}_{K_0}^{\mathrm{tail}}$.

Proof of correctness of the algorithm. This follows immediately from the correctness of Algorithm 4.16. Indeed, by Proposition 5.5, the inverse image under π of $\mathsf{U}_{K_0}^{\mathrm{int}}$ equals the union of the U_{x_0} considered in Algorithm 4.16, which equals the image of the interior locus $\varphi(Y_K^{\mathrm{int}})$. And by Corollary 5.2, the inverse image under π of $\mathsf{U}_{K_0} \backslash \mathsf{U}_{K_0}^{\mathrm{int}}$ equals $\mathsf{U}_K \backslash \varphi(Y_K^{\mathrm{int}})$, which is the same as the image of the tail locus $\varphi(Y_K^{\mathrm{tail}})$. ■

In summary, we have now found an affinoid subdomain

$$\mathsf{U}_{K_0}^{\mathrm{tail}} \cup \mathsf{U}_{K_0}^{\mathrm{int}} \subseteq (\mathbb{P}_{K_0}^1)^{\mathrm{an}}$$

whose inverse image in Y_K^{an} is the tame locus associated to $\varphi_K \colon Y_K \to X_K$. As explained in Remark 5.6, we may describe it using retraction to a tree $T \subseteq (\mathbb{P}_{K_0}^1)^{\mathrm{an}}$. Namely, we may take for T any tree containing all boundary points of $\mathsf{U}_{K_0}^{\mathrm{tail}} \cup \mathsf{U}_{K_0}^{\mathrm{int}}$. Now let T' be the permanent completion (cf. Remark 5.8) of the tree spanned by T, by the branch points of φ, and by ∞. Using the language of models, our main result is then the following.

Corollary 5.10. *Let $\mathcal{X}_0$ be the model of $\mathbb{P}_{K_0}^1$ that corresponds via Proposition 1.6 to the set of Type II vertices of the tree T'. Let $\mathcal{Y}_0$ denote the normalization of $\mathcal{X}_0$ in F_Y. Then the normalized base change $\mathcal{Y} := (\mathcal{Y}_0)_K$ is a semistable model of Y_K.*

Proof. The inverse image $\Gamma := \pi^{-1}(T') \subset (\mathbb{P}_K^1)^{\mathrm{an}}$ is a tree containing the tree Γ_0 spanned by the branch points of φ_K and the point ∞. Thus $\Sigma := \varphi_K^{-1}(\Gamma)$ is a graph containing $\varphi^{-1}(\Gamma_0)$ and all boundary points of the tame locus Y_K^{tame}. It follows from Proposition 3.46 that Σ is a skeleton of Y_K. Its set of Type II vertices corresponds via Proposition 1.6 to the model $\mathcal{Y}$, which is therefore semistable. ■

In the terminology of Section 1.2, the model $\mathcal{Y}$ is *potentially semistable.*

5.4 Finding a field extension over which Y has semistable reduction

In Corollary 5.10, we have described how to find an $\mathcal{O}_{K_0}$-model $\mathcal{Y}_0$ of Y such that the normalized base change $\mathcal{Y}_K$ is a semistable model of Y_K. In this section, we will explain how to find a finite field extension L/K_0 such that $\mathcal{Y}_L$ is semistable. For this, we need not restrict ourselves to the case of plane quartic curves. Our discussion will apply to any class of curves for which we have a way to find potentially semistable $\mathcal{O}_{K_0}$-models.

Let us write G_{K_0} for the absolute Galois group of K_0 and I_{K_0} for its inertia subgroup. Similarly we will write G_L and I_L for the absolute Galois group and inertia subgroup of a finite extension L/K_0.

The group G_{K_0} acts naturally on $Y_K = Y \otimes_{K_0} K$ via the second factor. This action extends to certain $\mathcal{O}_K$-models $\mathcal{Y}$ of Y_K. For example, if $g_Y \geqslant 2$, there exists a *stable model* $\mathcal{Y}^{\mathrm{stab}}$ of Y_K. It is the minimal semistable model of Y_K and is characterized by the fact that all rational components of $\mathcal{Y}_s^{\mathrm{stab}}$ intersect the rest of $\mathcal{Y}_s^{\mathrm{stab}}$ in at least three points (with self-intersection points counting as two points). Uniqueness of the stable model shows that the action of G_{K_0} on Y_K extends uniquely to an action on $\mathcal{Y}^{\mathrm{stab}}$; see [Liu02, Corollary 3.37] for details. A similar argument shows that given any $\mathcal{O}_{K_0}$-model $\mathcal{Y}_0$ of Y, the action of G_{K_0} on Y_K extends to the normalized base change $\mathcal{Y} := (\mathcal{Y}_0)_K$.

Lemma 5.11. *Let $\mathcal{Y}_0$ be a potentially semistable $\mathcal{O}_{K_0}$-model of a K_0-curve Y. If the inertia subgroup I_{K_0} acts trivially on $\mathcal{Y}_s$, where $\mathcal{Y} = (\mathcal{Y}_0)_K$, then Y has semistable reduction.*

Proof. We use the results collected in [Liu02, Theorem 4.44]: Denoting by K_0^{nr} the maximal unramified extension of K_0, there exists a minimal field extension L/K_0^{nr} such that Y_L has semistable reduction. Moreover, L/K_0^{nr} is Galois and the action of $\mathrm{Gal}(L/K_0^{nr})$ on $\mathcal{Y}_s^{\mathrm{stab}}$, and hence on $\mathcal{Y}_s$, is faithful.

Now if the action of I_{K_0} on $\mathcal{Y}_s$ is trivial, then the action of $\mathrm{Gal}(L/K_0^{nr})$ (which is a finite quotient of I_{K_0}) can only be faithful if $L = K_0^{nr}$. Thus $Y_{K_0}^{nr}$ has semistable reduction. In fact, because étale base change preserves normality ([Stacks, Tag 025P]), the curve Y itself must have semistable reduction: If $\mathcal{Y}_1$ is an $\mathcal{O}_{K_0}$-model of Y whose base change (with no normalization required!) $(\mathcal{Y}_1)_{K_0^{nr}}$ has semistable reduction, the model $\mathcal{Y}_1$ itself has semistable reduction. ∎

For the next proposition, we consider the familiar situation of a degree-p morphism of K_0-curves $\varphi\colon Y \to X$. Given an $\mathcal{O}_K$-model $\mathcal{X}$ of X_K, each component $Z \subseteq \mathcal{X}_s$ corresponds to a Type II valuation v_Z on F_{X_K}. We call the component Z *inseparable* if v_Z is wildly ramified in F_{Y_K}/F_{X_K}. Otherwise, we call Z *separable*.

Proposition 5.12. *Let $\mathcal{X}_0$ and $\mathcal{Y}_0$ be potentially semistable $\mathcal{O}_{K_0}$-models of X and Y respectively such that $\mathcal{Y}_0$ is the normalization of $\mathcal{X}_0$ in F_Y. Denote by $\mathcal{X} := (\mathcal{X}_0)_K$ and $\mathcal{Y} := (\mathcal{Y}_0)_K$ the normalized base changes to K.*

Suppose furthermore that L/K_0 is an extension with the following properties:

- *(a) The normalized base change $(\mathcal{X}_0)_L$ is semistable*
- *(b) For each separable irreducible component $Z \subseteq \mathcal{X}_s$ there exists an L-rational point x_0 on X_L specializing to a smooth non-branch point of $\mathcal{Y}_s \to \mathcal{X}_s$ such that the fiber $\varphi^{-1}(x_0)$ consists of L-rational points on Y_L*

Then Y_L has semistable reduction.

Proof. Since $(\mathcal{X}_0)_L$ is semistable, the special fiber of the normalized base change $\mathcal{X}$ is given simply by the base change $((\mathcal{X}_0)_L)_s \otimes_\lambda \kappa$ (here λ denotes the residue field of $\mathcal{O}_L$). The action of I_L on this special fiber is via the factor κ, so is trivial. Our goal is to show that I_L also acts trivially on $\mathcal{Y}_s$, which by Lemma 5.11 implies that Y_L has semistable reduction.

Let $Z \subseteq \mathcal{X}_s$ be an irreducible component. If Z is inseparable, then clearly the unique component $W \subseteq \mathcal{Y}_s$ above Z is fixed by the action of I_L. Otherwise, (b) shows that the components of $\mathcal{Y}_s$ above Z are not permuted by the action of I_L. Let $W \subseteq \mathcal{Y}_s$ be one such component. By (b), the action of I_L on W fixes a non-ramification point $\overline{y}$ of the map $W \to Z$.

Assume that there exists some $\sigma \in I_L$ acting non-trivially on W. Then the map $W \to Z$ factors through $W/\langle\sigma\rangle$, and $\overline{y}$ is a ramification point of $W \to W/\langle\sigma\rangle$. This contradicts $\overline{y}$ being a non-ramification point, proving the claim that I_L acts trivially on $\mathcal{Y}_s$ and finishing the proof. ∎

A word of caution: In the situation of Proposition 5.12, it is not necessarily the case that the model $(\mathcal{Y}_0)_L$ is semistable. This is because $\mathcal{Y}_s$ might contain more components than $\mathcal{Y}_s^{\mathrm{stab}}$, not all of which correspond to reduced components of the special fiber of $(\mathcal{Y}_0)_L$.

Example 5.13. Let us consider once again the plane quartic from Examples 3.13, 3.47, and 4.17,

$$Y/\mathbb{Q}_3\colon \quad y^3 + 3xy^2 - 3y - 2x^4 - x^2 - 1 = 0.$$

The $\mathcal{O}_{K_0}$-model $\mathcal{X}_0$ of Corollary 5.10 for which the normalization $\mathcal{Y}_0$ in F_Y is potentially semistable corresponds to the boundary points of the three discoids

$$\mathsf{D}[x,1], \quad \mathsf{D}[x,0], \quad \mathsf{D}[x^2+1,3/4]$$

(see also Example 5.17 below). Let us first define an extension L_0 of $K_0 = \mathbb{Q}_3$ by adjoining an element of valuation $1/4$, say $\sqrt[4]{3}$, and a square root i of -1. Over L_0, the discoid $\mathsf{D}[x^2+1,3/4]$ splits into the two disks $\mathsf{D}[x-i,3/4]$ and $\mathsf{D}[x+i,3/4]$, so $(\mathcal{X}_0)_L$ is the model corresponding to the boundary points of the four disks $\mathsf{D}[x,1]$, $\mathsf{D}[x,0]$, $\mathsf{D}[x-i,3/4]$, and $\mathsf{D}[x+i,3/4]$. Since $1/4 \in \Gamma_{L_0}$, the model $(\mathcal{X}_0)_L$ is permanent (cf. the proof of Lemma 1.12), and hence semistable. Thus we have dealt with Condition (a) of Proposition 5.12.

Next, we choose points x_0 as in Condition (b) of Proposition 5.12. Recall from Example 4.17 that above each component Z corresponding to a boundary point of $\mathsf{D}[x,1]$, $\mathsf{D}[x-i,3/4]$, and $\mathsf{D}[x+i,3/4]$ lies a single component W, given by an Artin-Schreier equation. In particular, $W \to Z$ is only ramified at ∞, and we may take for the x_0 points contained in the open disks $\mathsf{D}(x,1)$, $\mathsf{D}(x-i,3/4)$, and $\mathsf{D}(x-i,3/4)$. The simplest choice is the centers $x=0$, $x=i$, and $x=-i$.

A field extension L over which Y acquires semistable reduction is then obtained by adjoining to L_0 the coordinates of the fibers $\varphi^{-1}(0)$, $\varphi^{-1}(i)$, and $\varphi^{-1}(-i)$. Note that there is no condition for the disk $\mathsf{D}[x,0]$, since the corresponding component is inseparable.

Remark 5.14. To find a point $x_0 \in X_L(L)$ specializing to a smooth non-branch point on a given irreducible component $Z \subseteq \mathcal{X}_s$ as demanded by Proposition 5.12(b), one may use the theorem on *purity of the branch locus* [Stacks, Tag 0BMB].

Indeed, any point $x_0 \in X_L(L)$ whose reduction to the special fiber of $(\mathcal{X}_0)_L$ is a smooth point on Z and not the same as the reduction of a branch point of φ will work. To see this, denote by $\overline{x}$ the reduction of x_0 and let $\overline{y}$ be a preimage of $\overline{x}$ under the map $(\mathcal{Y}_0)_L \to (\mathcal{X}_0)_L$. The local rings $\mathcal{O}_{(\mathcal{X}_0)_L,\overline{x}}$ and $\mathcal{O}_{(\mathcal{Y}_0)_L,\overline{y}}$ are isomorphic to $\mathcal{O}_L[\![T]\!]$ (see for example [AW12, Proposition 3.4]); in particular, they are regular local rings of dimension 2. The prime ideals of $\mathcal{O}_L[\![T]\!]$ are classified in [Hel21, Lemma 3.1]. There are two types of codimension one prime ideals in $\mathcal{O}_{(\mathcal{X}_0)_L,\overline{x}}$: one corresponding to the generic point of Z, the others corresponding to closed points on X_L specializing to $\overline{x}$. Analogous considerations apply to $\mathcal{O}_{(\mathcal{Y}_0)_L,\overline{y}}$.

Since $\overline{x}$ is not the specialization of a branch point of φ and since the component Z is separable, the induced map $\mathcal{O}_{(\mathcal{Y}_0)_L,\overline{y}} \to \mathcal{O}_{(\mathcal{X}_0)_L,\overline{x}}$ is étale at codimension 1 primes. It follows from purity of the branch locus that it is outright étale.

The MCLF package is able to perform the steps outlined in this section automatically. We will make extensive use of this in the examples of the following section.

5.5 Examples

In this section we present a number of worked examples. In each, we explain how to compute the semistable reduction of a plane quartic Y over $\mathbb{Q}_3$. All the curves and the corresponding covers $\varphi\colon Y \to \mathbb{P}^1_{K_0}$ are in the normal form of Section 4.1. A variety of different reduction types and combinatorial configurations of the tame locus is included.

At last, we let the computer do as much of the work as possible. In practice this means that we use our implementation of Algorithm 5.9 to compute the image of the tame locus in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$. The boundary points of this affinoid subdomain then determine a potentially semistable model of Y. Afterwards, we use the existing functionality of MCLF to compute equations for the components corresponding to the valuations we have found. We refer to Section A.1 of the appendix for a brief overview of our implementation.

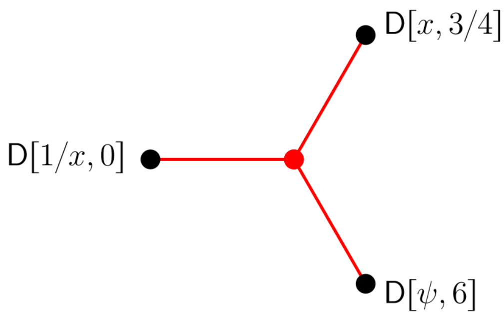
Example 5.15. The plane quartic

$$Y/\mathbb{Q}_3\colon \quad F(y) = y^3 + (2x^3 + 3x^2)y - 3x^4 - 2x^2 - 1 = 0.$$

We ask Sage to compute the image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the tame locus associated to Y, using the following commands:

```
sage: from mclf import *
sage: R.<x,y> = QQ[]
sage: v = QQ.valuation(3)
sage: F = y^3 + (2*x^3 + 3*x^2)*y - 3*x^4 - 2*x^2 - 1
sage: Y = SmoothProjectiveCurve(F)
sage: M = Quartic3Model(Y, v)
sage: M.tame_locus()
Affinoid with 3 components:
Elementary affinoid defined by
v(1/x) >= 0
Elementary affinoid defined by
v(x^9 - 9*x^8 + 27*x^7 + 54*x^6 + 81*x^5 + 54*x^4 - 27*x^2 + 27/5) >=
↪  6
Elementary affinoid defined by
v(x) >= 3/4
```

Recall Section 5.3 and Example 5.7 in particular for how we represent affinoid subdomains of $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ using retraction to a tree. In this case, the output is an affinoid subdomain consisting of three discoids, which may be described by retraction to the following tree:



where

$$\psi = x^9 - 9x^8 + 27x^7 + 54x^6 + 81x^5 + 54x^4 - 27x^2 + \frac{27}{5}.$$

The polynomial ψ is not equal to the discriminant Δ_F of F, but we have $\mathsf{D}[\psi, 6] = \mathsf{D}[\Delta_F, 6]$, and this discoid contains the branch locus except for the point ∞. The image of the tail locus is given by the disk of radius $3/4$:

```
sage: M.tail_locus()
Elementary affinoid defined by
v(x) >= 3/4
```

It follows easily from Proposition 5.5 that $\mathsf{D}[\psi, 6]$ splits into 9 disjoint disks over K. Thus the boundary points of these disks play no role in finding a semistable reduction of Y. We use MCLF to verify that the model corresponding to the boundary points of the two disks $\mathsf{D}[x, 0]$ and $\mathsf{D}[x, 3/4]$ indeed is potentially semistable. Essentially MCLF works by computing extensions of the involved valuations, that is, using Proposition 1.7. Continuing the above code fragment:

```
sage: X = M._X
sage: T = BerkovichTree(X)
sage: T.add_point(X.point_from_discoid(x, 0))
sage: T.add_point(X.point_from_discoid(x, 3/4))
sage: R = ReductionTree(Y, v, T)
sage: R.is_semistable()
True
```

MCLF also provides equations for the reduction curves:

```
sage: R.inertial_components()[0].upper_components()[0].component()
the smooth projective curve with Function field in u1 defined by u1^3
↪  + 2*x^3*u1 + x^2 + 2
sage: R.inertial_components()[1].upper_components()[0].component()
the smooth projective curve with Function field in u2 defined by u2^3
↪  + u2 + x^2
```

The reduction curve of the point above the boundary point of $\mathsf{D}[x, 3/4]$ is the Artin-Schreier curve of genus 1 with equation

$$y^3 + y + x^2 = 0.$$

The reduction curve above the boundary point of $\mathsf{D}[x, 0]$ is of genus 2 and is given by

$$y^3 - x^3 y + x^2 - 1 = 0.$$

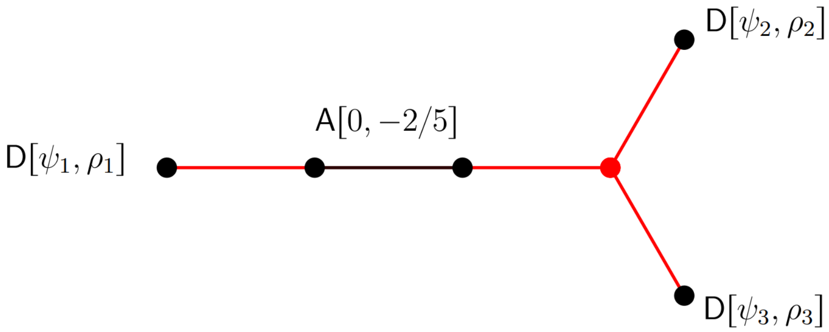
We note that it has ordinary ramification over $x = \infty$ (the specialization of the ordinary branch point ∞ of φ) and that it is totally ramified over $x = 0$ (reflecting the fact that φ is inseparable on the branch pointing towards $x = 0$).

In summary, we have found one component of genus 1 and one component of genus 2 on the special fiber of the model determined by the boundary points of the tame locus. To obtain the stable reduction, we can contract all other components. This corresponds to discarding the central red vertex and the boundary point of $\mathsf{D}[\psi, 6]$ in the tree above, keeping only the boundary points of $\mathsf{D}[1/x]$ and $\mathsf{D}[x, 3/4]$.

Example 5.16. The plane quartic

$$Y/\mathbb{Q}_3\colon \quad y^3 - 3y^2 + (-3x^2 - 2x)y + 3x^4 - 3x - 1 = 0,$$

previously considered in Example 4.15. Using the same commands as in the previous example, we find that the image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the tame locus may be described by retraction to the following tree:



Note that during instantiation of the `Quartic3Model` associated to Y, a coordinate change is performed replacing x with $3x$. However, the image of the tame locus with respect to the original normalization can be obtained using the method `rescaled_tame_locus`:

```
sage: from mclf import *
sage: R.<x,y> = QQ[]
sage: v = QQ.valuation(3)
sage: F = y^3 - 3*y^2 + (-3*x^2 - 2*x)*y + 3*x^4 - 3*x - 1
sage: Y = SmoothProjectiveCurve(F)
sage: M = Quartic3Model(Y, v)
We do a base change so that the branch locus is integral and now
↪  consider the smooth projective curve with Function field in y
↪  defined by y^3 - 27*y^2 + (-27*x^2 - 54*x)*y + 27*x^4 - 729*x -
↪  729
sage: M.tame_locus()
Affinoid with 4 components:
Elementary affinoid defined by
v(x) >= 3/5
v(1/x) >= -1
Elementary affinoid defined by
v(x^2 - x - 4) >= 3
Elementary affinoid defined by
v(x^3 + x^2 + 10*x + 8) >= 3
Elementary affinoid defined by
v(x^3 + 27/8*x^2 + 2673/2*x + 3645/22) >= 13
sage: M.rescaled_tame_locus()
Affinoid with 4 components:
Elementary affinoid defined by
v(1/x) >= 0
v(x) >= -2/5
Elementary affinoid defined by
v(x^3 - 90*x^2 - 54/7*x - 1485) >= 10
Elementary affinoid defined by
v((-63*x^2 + 3/4*x + 1)/x^2) >= 5
Elementary affinoid defined by
v((27/8*x^3 + 153*x^2 + 24*x + 1)/x^3) >= 6
```

There are no tail discoids. The discoids $\mathsf{D}[\psi_i, \rho_i]$, $i = 1, 2, 3$, each contain branch points of φ (reflecting the structure of the branch locus of φ as explained in Example 4.15). The anulus

$$\mathsf{A}[0, -2/5] = \{\xi \in (\mathbb{P}^1_{K_0})^{\mathrm{an}} \mid 0 \geqslant v_\xi(x) \geqslant -\frac{2}{5}\}$$

coincides with the interval in Figure 4.2 on which $\delta = 0$. It turns out that its boundary points are all that we need:

```
sage: FX.<x> = FunctionField(QQ)
sage: X = BerkovichLine(FX, v)
sage: T = BerkovichTree(X)
sage: T.add_point(X.point_from_discoid(x, 0))
sage: T.add_point(X.point_from_discoid(x, -2/5))
sage: R = ReductionTree(Y, v, T)
sage: R.inertial_components()[0].upper_components()[0].component()
the smooth projective curve with Function field in u1 defined by u1^3
↪  + x*u1 + 2
sage: R.inertial_components()[1].upper_components()[0].component()
the smooth projective curve with Function field in u2 defined by u2^3
↪  + 1/x*u2 + 1/x^4
sage: R.is_semistable()
True
```

Let us denote the boundary points of $\mathsf{A}[0, -2/5]$ by ξ_0 and $\xi_{-2/5}$. The reduction curve of the point above ξ_0, given by

$$y^3 + xy + 2 = 0,$$

is totally ramified at $x = 0$ and has ordinary ramification at ∞. Conversely, the reduction curve of the point above $\xi_{-2/5}$, given by

$$y^3 + xy + x^4 = 0,$$

is totally ramified at ∞ and has ordinary ramification at $x = 0$. This reflects the fact that the open interval $(\xi_0, \xi_{-2/5})$ in $\mathsf{A}[0, -2/5]$ consists of tame topological branch points, while the adjacent intervals consist of wild topological branch points. The reduction curve of the point above $\xi_{-2/5}$ has genus 2, while the reduction curve of the point above ξ_0 has genus 0. A loop lies above the interval $[\xi_0, \xi_{-2/5}]$. To double check this, we let MCLF verify that there are two points above $\xi_{-1/5}$:

```
sage: T = BerkovichTree(X)
sage: T.add_point(X.point_from_discoid(x, -1/5))
sage: R = ReductionTree(Y, v, T)
sage: for U in R.inertial_components()[1].upper_components():
....:   U.component()
```

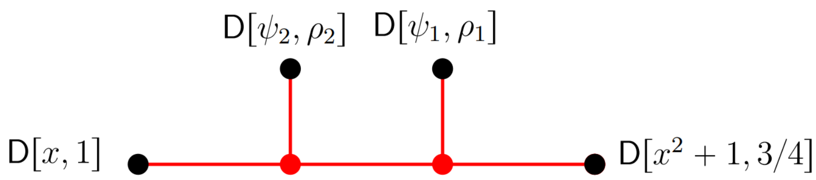
```
the smooth projective curve with Rational function field in x over
↪  Finite Field of size 3
the smooth projective curve with Function field in u2 defined by u2^2
↪  + 1/x
```

It follows that the stable reduction of Y has only one component, which has geometric genus 2 and one self-intersection point. Note that as announced in Remark 2.26, the point $\xi_{-2/5}$ is not a vertex of the tree spanned by the branch points of φ even though we need its preimage to obtain a skeleton of Y.

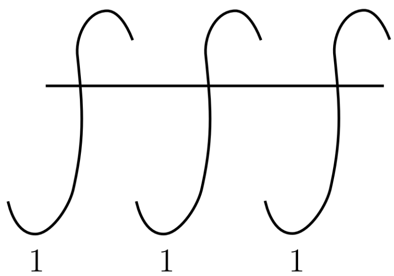
Example 5.17. The plane quartic

$$Y/\mathbb{Q}_3\colon \quad y^3 + 3xy^2 - 3y - 2x^4 - x^2 - 1 = 0,$$

previously considered in Examples 3.13, 3.47, 4.17, and 5.13. Again using the same commands as in Example 5.15, we find that the image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the tame locus may be described by retraction to the following tree:



The discoids on the left and right are tail discoids. The discoid $\mathsf{D}[x^2 + 1, 3/4]$ splits into two disks over K. From this the structure of the stable reduction of Y is immediately clear. Since each tail component has a boundary point of genus $\geqslant 1$, there can be no further points of positive genus nor any loops. Thus the stable reduction is a "comb" consisting of one inseparable rational component intersecting with three genus 1 curves, as follows:



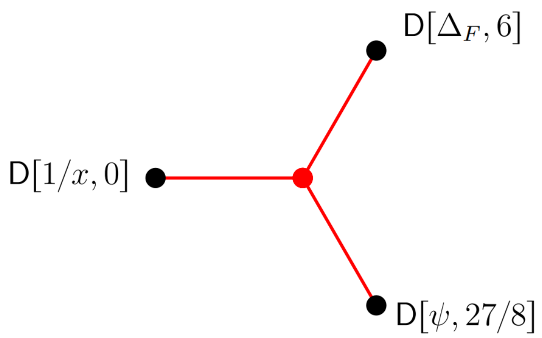
The discoids $\mathsf{D}[\psi_1, \rho_1]$ and $\mathsf{D}[\psi_2, \rho_2]$ contain branch points. In fact, we already determined the structure of the branch locus of φ in Example 3.13. The discoid $\mathsf{D}[\psi_1, \rho_1]$ has degree 6, while $\mathsf{D}[\psi_2, \rho_2]$ has degree 2.

We can now also check our experimental hypothesis from Example 3.13 that $\lambda(x_0) = 1$, where $x_0 = 0$. Indeed, x_0 is contained in $\mathsf{D}[x, 1]$ and it follows from the structure of the tree above and from Proposition 2.31 that $\lambda(x_0) = 1$.

Example 5.18. The plane quartic

$$Y/\mathbb{Q}_3\colon \quad y^3 + x^3 y + x^4 + 1 = 0,$$

previously considered in Example 1.49. The image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the tame locus may be described by retraction to the following tree:



The discoid $\mathsf{D}[\Delta_F, 6]$ contains the single finite branch point of φ, which is of degree 9. In fact, the (image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the) Type II point we used in Example 1.49 lies above the path connecting $\mathsf{D}[\Delta_F, 6]$ to the central red vertex.

The only important component in this example is the tail discoid $\mathsf{D}[\psi, 27/8]$. We have

$$\psi = x^9 + 27x + 54.$$

As expected, the reduction curve at the point above the boundary point of $\mathsf{D}[\psi, 27/8]$ is given by an Artin-Schreier equation

$$y^3 + y = x^4.$$

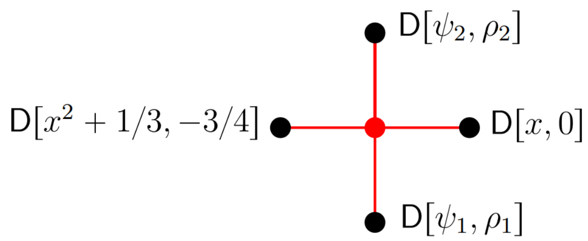
It has genus 3 and is only ramified at ∞. Thus Y has potentially good reduction.

Example 5.19. The plane quartic

$$Y/\mathbb{Q}_3\colon \quad F(y) = y^3 + y^2 + (x+1)y + 3x^4 + 2x^3 + 2x = 0.$$

The map $\varphi\colon Y \to \mathbb{P}^1_{K_0}$ is totally ramified above ∞. There are eight finite branch points corresponding to the eight zeros of the discriminant Δ_F. Two of these have valuation $1/2$, while the others have negative valuation.

The image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of the tame locus may be described by retraction to the following tree:



The discoid on the left is a tail discoid. Over K, it splits into two disks centered at $\pm 1/\sqrt{-3}$. The reduction curves at the points above the boundary points of these tail disks are the Artin-Schreier curves of genus 1 given by

$$y^2 + y + x^2 = 0.$$

The discoids $\mathsf{D}[\psi_1, \rho_1]$ and $\mathsf{D}[\psi_2, \rho_2]$ split into 4 and 2 disks over K respectively, so they cannot contribute to the stable reduction of Y. We need to keep the central red vertex however, since it is contained in the permanent completion (cf. Remark 5.8) of the tree spanned by $\mathsf{D}[x^2 + 1/3, -3/4]$ and $\mathsf{D}[x, 0]$. It is the boundary point of the disk $\mathsf{D}[x, -1/2]$.

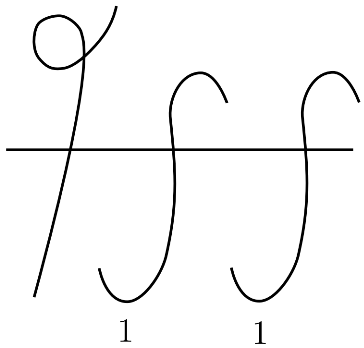
We furthermore add the boundary point of $\mathsf{D}[x, 1/2]$, which separates the two branch points of valuation 1/2. In general it is necessary to use a tree separating the branch points of φ — compare for example the hypotheses in Proposition 3.46, which demand that Σ_1 contain the boundary points of the tame locus *and the inverse image of the tree spanned by the branch points of* φ.

In summary, we take the model $\mathcal{X}$ corresponding to the boundary points of the four discoids $\mathsf{D}[x^2 + 1/3, -3/4]$, $\mathsf{D}[x, -1/2]$, $\mathsf{D}[x, 0]$, and $\mathsf{D}[x, 1/2]$. We check that these determine a potentially semistable model. Note that we add the boundary point of $\mathsf{D}[x, -1/2]$ using MCLF's function `permanent_completion`.

```
sage: from mclf import *
sage: R.<x,y> = QQ[]
sage: v = QQ.valuation(3)
sage: F = y^3 + y^2 + (x + 1)*y + 3*x^4 + 2*x^3 + 2*x
sage: Y = SmoothProjectiveCurve(F)
sage: FX.<x> = FunctionField(QQ)
sage: X = BerkovichLine(FX, v)
sage: T = BerkovichTree(X)
sage: T.add_point(X.point_from_discoid(x, 0))
sage: T.add_point(X.point_from_discoid(x^2 + 1/3, -3/4))
sage: T.add_point(X.point_from_discoid(x, 1/2))
sage: T.permanent_completion()
sage: T.vertices()
[Point of type II on Berkovich line, corresponding to v(x) >= 0,
 Point of type II on Berkovich line, corresponding to v(1/x) >= 1/2,
```

```
 Point of type II on Berkovich line, corresponding to v(3*x^2 + 1) >=
 ↪  1/4,
 Point of type II on Berkovich line, corresponding to v(x) >= 1/2]
sage: R = ReductionTree(Y, v, T)
sage: R.is_semistable()
 True
```

The reduction curves at the points above the boundaries of each of $\mathsf{D}[x,-1/2]$, $\mathsf{D}[x,0]$, and $\mathsf{D}[x,1/2]$ are all rational. The ones corresponding to the latter two disks intersect in two points. Thus the stable reduction of Y looks as follows:



5.6 An example over $\mathbb{Q}_3(\zeta_3)$

In this section we apply our results to a slightly more involved example. Namely, we compute the semistable reduction of a particular plane quartic curve that appears in the attempts of Rouse, Sutherland, and Zureick-Brown to compute the rational points on the non-split Cartan modular curve $X_{\mathrm{ns}}^+(27)$, see [RSZ22]. The semistable reduction was given in [Oss23], but the method of finding it (essentially Algorithm 5.9) was left out.

Let $K_0 = \mathbb{Q}_3(\zeta_3)$ be the field obtained by adjoining to $\mathbb{Q}_3$ a primitive third root of unity and let K be the completion of an algebraic closure of K_0. As always, we normalize the valuation $v_K\colon K \to \mathbb{Q} \cup \{\infty\}$ so that $v_K(3) = 1$, that is to say $v_K(\zeta_3 - 1) = 1/2$. The curve we are interested in is the plane quartic K_0-curve cut out of $\mathbb{P}^2_{K_0}$ by the equation

$$\begin{aligned} &x^4 + (\zeta_3 - 1)x^3y + (3\zeta_3 + 2)x^3z - 3x^2z^2 + (2\zeta_3 + 2)xy^3 \\ &\quad - 3\zeta_3xy^2z + 3\zeta_3xyz^2 - 2\zeta_3xz^3 - \zeta_3y^3z \\ &\quad + 3\zeta_3y^2z^2 + (-\zeta_3 + 1)yz^3 + (\zeta_3 + 1)z^4 = 0. \end{aligned}$$

Note that $[0:1:0]$ is a rational point on Y. We achieve the normal form discussed in Section 4.1 by replacing z with $z + x(2\zeta_3 + 2)/\zeta_3$. Dehomogenizing yields the affine equation

$$Y\colon \quad F(y) = y^3 + Ay^2 + By + C = 0,$$

where

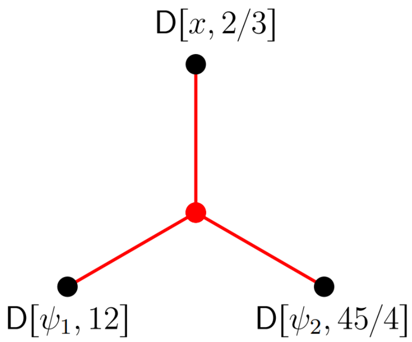
$$A = (6\zeta_3 + 12)x^2 + (36\zeta_3 + 9)x - 27,$$

$$B = (9\zeta_3 - 18)x^3 + (-108\zeta_3 - 108)x^2 + (-162\zeta_3 + 81)x + (81\zeta_3 + 162),$$

$$\begin{aligned} C = {} & (27\zeta_3 - 243)x^4 + (-1458\zeta_3 - 999)x^3 \\ & + (-1215\zeta_3 + 1701)x^2 + (1944\zeta_3 + 2430)x + 729\zeta_3. \end{aligned}$$

We use the usual projection map $\varphi\colon Y \to \mathbb{P}^1_{K_0}$. The discriminant Δ_F has degree 10, so by Lemma 4.1 the point ∞ is not a branch point of φ. The branch locus of φ consists of one point of degree 1 and one point of degree 9.

For convenience, the curve Y is defined in Code Listing A.6 and the image in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$ of its tame locus is computed. It may be described in terms of reduction to the following tree:



The disk $\mathsf{D}[x, 2/3]$ contains ∞ and the branch point of degree 1. The polynomial

$$\begin{aligned} \psi_1 = {} & x^9 + (9\zeta_3 + 18)x^8 + (54\zeta_3 - 27/2)x^7 + (54\zeta_3 + 108)x^6 \\ & + 486\zeta_3 x^5 + (-729\zeta_3 + 3645)x^4 + (729\zeta_3 + 13851)x^3 \\ & + (37179\zeta_3 + 15309)x^2 + (-6561\zeta_3/2 + 72171)x + 2187\zeta_3/13 + 15309/2 \end{aligned}$$

is the factor of Δ_F of degree 9, so $\mathsf{D}[\psi_1, 12]$ contains the other branch point of φ. The last component, the tail discoid $\mathsf{D}[\psi_2, 45/4]$, is the most important; we have

$$\begin{aligned} \psi_2 = {} & x^9 + (9\zeta_3 - 9)x^8 + (54\zeta_3 + 27)x^7 + (54\zeta_3 - 27/2)x^6 \\ & + (243\zeta_3 + 972)x^5 + 729\zeta_3 x^4 + (2916\zeta_3 - 1458)x^3 + (37179\zeta_3 \\ & + 41553)x^2 + (6561\zeta_3 + 6561/8)x - 63423\zeta_3 + 155277. \end{aligned}$$

We also note that the red vertex in the middle of the tree above is the boundary point of the discoid $\mathsf{D}[\psi_1, 11] = \mathsf{D}[\psi_2, 11]$. The splitting behavior of all these discoids is particularly interesting. While the discoids $\mathsf{D}[\psi_2, 45/4]$ and $\mathsf{D}[\psi_2, 11]$ split into three disks over K each containing three of the nine roots of ψ_2, the discoid $\mathsf{D}[\psi_1, 12]$ splits into nine disks, each containing one root of ψ_1.

The tree spanned by all the roots of Δ_F and ψ_2 is depicted in Figure 5.3. The tree spanned by the zeros of ψ_2 already appears in [Oss23, Figure 3]. As usual, the locus where $\delta > 0$ is depicted in red. The dashed gray lines represent disks of the indicated radius. For example, the smallest disk containing all roots of both ψ_1 and ψ_2 has radius 7/6. We see that the nine disks into which $\mathsf{D}[\psi_1, 12]$ splits have radius 2, while the three disks into which $\mathsf{D}[\psi_2, 45/4]$ splits have radius 17/12. Given a root of either ψ_1 or ψ_2, the closest other root has distance 3/2.

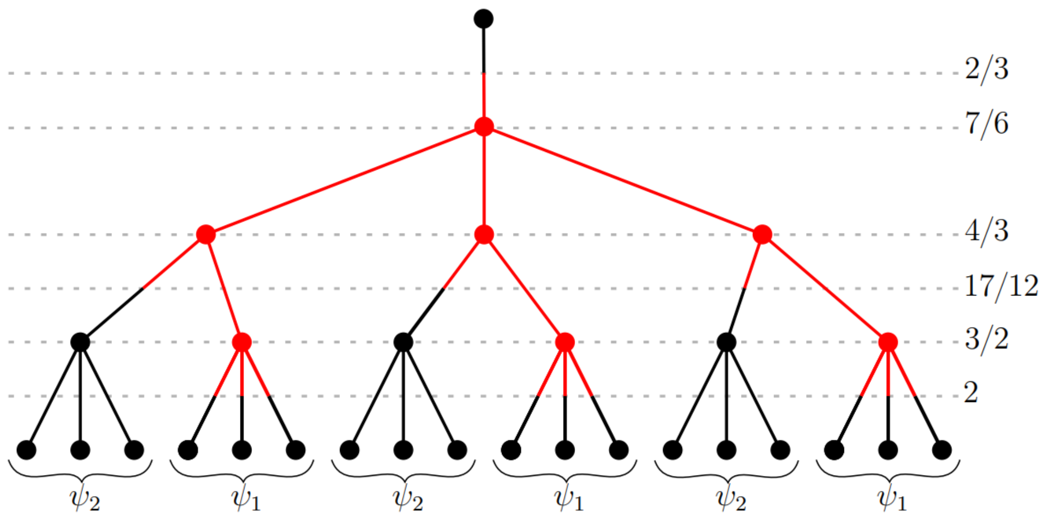


Figure 5.3: The tree in $(\mathbb{P}^1_K)^{\mathrm{an}}$ spanned by the zeros of Δ_F and ψ_2

Since there are three tail disks, the structure of the stable reduction of Y is clear: It must be a "comb" (the same as in Example 5.17) consisting of one rational component intersecting three components of genus 1, which are Artin-Schreier curves given by

$$y^3 - y = x^2.$$

Remark 5.20. (a) Following the strategy outlined in Section 5.4, a field extension of K_0 over which Y has semistable reduction is determined in [Oss23]. It is of ramification index 54 over K_0.

(b) In [Oss23, Section 3], a coordinate transformation is used to ensure the curve Y has an inflection point at infinity. It turns out that in these new coordinates, the strategy of separating branch points (Section 1.8) works to compute the semistable reduction of Y! We have no answer to the interesting question of whether such a coordinate transformation may be found in other examples as well; in this example, this transformation was found by accident.

Appendix A

Computer programs

In Section A.1, we give a short overview of our implementation of Algorithm 5.9. It is included in a branch of the MCLF Sage package available at

https://github.com/oossen/mclf/tree/plane-quartics

All Sage code provided in this thesis was tested on this version of MCLF with SageMath version 9.8.

Section A.2 provides various short programs that were used for calculations in examples throughout the thesis.

A.1 Implementation of Algorithm 5.9

The central part of our implementation of Algorithm 5.9 is the class `Quartic3Model`. It extends the MCLF class `SemistableModel`, like its siblings `AdmissibleModel` and `SuperpModel`, and provides book-keeping objects for the semistable reduction of plane quartics. For instantiation, use

```
sage: M = Quartic3Model(Y, vK)
```

where `Y` is a curve over a number field K and where `vK` is a valuation on K with residue characteristic 3. The two methods of `Quartic3Model` with the greatest theoretical importance are `interior_locus` and `potential_tail_locus`. The former computes the image of the tame locus Y_K^{an} in $(\mathbb{P}^1_{K_0})^{\mathrm{an}}$, that is, the affinoid subdomain $\mathsf{U}_{K_0}^{\mathrm{int}}$ from Algorithm 5.9. The second computes the affinoid subdomain U_{K_0} discussed in Section 5.1. The method `tail_locus` is available to compute the difference $\mathsf{U}_{K_0} \backslash \mathsf{U}_{K_0}^{\mathrm{int}}$, the image of the tail locus.

The class `Quartic3Model` implements the abstract method `reduction_tree`. The object returned essentially represents the pair of models $\mathcal{X}_0$ and $\mathcal{Y}_0$ discussed in Corollary 5.10. However, to save time in the computation of the normalization $(\mathcal{Y}_0)_K$, it is checked if certain vertices in the tree corresponding to $\mathcal{X}_0$ can be left out. At first, it is checked if the tail components alone account for the stable reduction of Y. If not, the remaining boundary points of the image of the tame locus are used, and only in the end the branch points of φ are separated if necessary.

If not all branch points of φ are integral, then during the instantiation of the model a change of coordinates is performed to ensure this. For convenience, the method `rescaled_tame_locus` gives the tame locus with respect to the original normalization.

Our implementation is based on elementary classes `AffineMap` and `PiecewiseAffineMap` that represent affine and piecewise affine maps $\mathbb{R} \to \mathbb{R}$ as defined at the beginning of Section 2.1.

A.2 Code fragments

In each of the following code listings, we note for what example the code was used; the code listing is also always mentioned in the corresponding example.

Code Listing A.1. The following code computes the valuations of the minimal polynomial of the generator z from Example 1.49.

```
R.<x> = QQ[]
K.<x0> = QQ.extension(x^9 + 27/4*x^8 + 27/2*x^4 + 27/4)

R.<x> = K[]
R.<y> = R[]
F = y^3 + x^3*y + x^4 + 1
y0 = F(x=x0).roots()[1][0]
G = F(x=x+x0, y=y+y0)

v = K.valuation(3)
# G is the minimal polynomial of (y-y0)
# to compute the minimal polynomial of z = (y-y0)^{1/2}, we subtract
↪  suitable values
print([v(ai) - 1/2 for ai in G[2]])
print([v(bi) - 1 for bi in G[1]])
print([v(ci) - 3/2 for ci in G[0]])
```

Code Listing A.2. The following code is used in Example 3.13 to compute a power series expansion and the valuations of its coefficients.

```
R.<a> = QQ[]
K.<a> = QQ.extension(a^3 - 3*a - 1)
v = K.valuation(3)
R.<x> = K[]
R.<y> = R[]
F = y^3 + 3*x*y^2 - 3*y - 2*x^4 - x^2 - 1
G = F(y=y+a)

approx = a

order = 100
for n in range(1, order + 1):
```

```
  next_term = -x^n*G[0][n]/G[1][0]
  G = G(y=y+next_term)
  approx += next_term

v = K.valuation(3)
for n in range(1, order + 1):
  print(n, v(approx[n]), v(approx[n])/n)
```

Code Listing A.3. The following code computes the valuations of the coefficients of the minimal polynomial of the generator z from Example 4.15. Note that we generate the field extension containing the point x_0 using the minimal polynomial of $3x_0$ instead, because Sage's functionality for extending valuations requires that the minimal polynomial be integral.

```
R.<x> = QQ[]
K.<a> = QQ.extension(x^8 - 22*x^6 - 114*x^5 - 237*x^4 + 346*x^3 + 2079*x^2
↪  + 5346*x + 3645)
x0 = a/3

R.<x> = K[]
R.<y> = R[]
F = y^3 - 3*y^2 + (-3*x^2 - 2*x)*y + 3*x^4 - 3*x - 1
y0 = F(x=x0).roots()[0][0]
G = F(x=x+x0, y=y+y0)
R.<u> = K[]
f = u^3 + G[2][1]*u^2 + G[1][2]*u + G[0][3]
L.<u> = K.extension(f)

R.<x> = L[]
R.<y> = R[]
H = F(x=x+x0, y=y+y0+x*u)
v = QQ.valuation(3)
w = v.extensions(L)[0]
for h in H:
  print([w(hi) for hi in h])
```

Code Listing A.4. The following code computes the following, all of which are used in Example 4.17:

- The "monodromy polynomial", which is the numerator of $\mathrm{Nm}(c_3)$
- The clustering behavior of the roots of the monodromy polynomial. For this, we use the strategy explained in [Rüt14, Section 4.8]. We compute three Newton polygons, each showing nine roots of valuation > 1 and eighteen roots of valuation 0
- For a center x_0 of each of the three disks $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, the radius $\lambda(x_0)$
- Following Remark 4.5, the valuations of each coefficient of the equation in whose reduction we are interested. The output shows that the constant coefficient of $\overline{B}$ and the degree-2 coefficient of $\overline{C}$ are the only ones not to vanish, yielding the reduction claimed in Example 4.17

```
from mclf import *

R.<x,y> = QQ[]
Y = SmoothProjectiveCurve(y^3 + 3*x*y^2 - 3*y - 2*x^4 - x^2 - 1)
FY = Y.function_field()
FX = Y.rational_function_field()
y = FY.gen()
x = FX.gen()

R.<t> = FY[]
R.<T> = R[]
A = 3*t + 3*x
B = -3
C = -2*t^4 - 8*x*t^3 + (-12*x^2 - 1)*t^2 + (-8*x^3 - 2*x)*t - 2*x^4 - x^2 -
↪  1
F = T^3 + A*T^2 + B*T + C
w = y - (3*y^2 - 8*x^3 - 2*x)/(3*y^2 + 6*x*y - 3)*t
H = F(T+w)

A = H[2]
B = H[1]
C = H[0]
monodromy_poly = C[3].norm().numerator().monic()
print(monodromy_poly)

K.<a> = QQ.extension(monodromy_poly)
R.<y> = K[]
b = (y^3 + 3*a*y^2 - 3*y - 2*a^4 - a^2 - 1).roots()[0][0]
w = QQ.valuation(3).extensions(K)
w.append(w[1])
centers = [a, a, -a]
for i in range(3):
  print(npolygon(monodromy_poly, centers[i], w[i]))
print(w[0](a), w[1](a))

lambdas = [(3*w[i](B[0].norm().numerator()(centers[i])) -
↪  2*w[i](C[2].norm().numerator()(centers[i])) +
↪  2*w[i](C[2].norm().denominator()(centers[i])))/12 for i in range(3)]
print(lambdas)

h = FY.hom([b, a])
s = [2*lambdas[i]/3 for i in range(2)]
for i in range(2):
  A_val = [w[i](h(A[l])) - s[i] + l*lambdas[i] for l in
   ↪  range(A.degree()+1)]
  B_val = [w[i](h(B[l])) - 2*s[i] + l*lambdas[i] for l in
   ↪  range(B.degree()+1)]
  C_val = [w[i](h(C[l])) - 3*s[i] + l*lambdas[i] for l in
   ↪  range(C.degree()+1)]
  print(A_val, B_val, C_val)
```

Code Listing A.5. This listing accompanies Example 4.18. We first construct an equation for Y as explained in Section 4.6. We then compute the valuations of its coefficients, from which the graph in Figure 4.4 results by the usual formula of Remark 4.14. We then use the functionality of MCLF to compute the points above ξ_0 and $\xi_{-1/4}$. Essentially this just amounts to computing extensions of the accompanying valuations to F_Y. But using the computer, we cannot work over the algebraically closed field $\mathbb{C}_3$. Conveniently, MCLF automatically computes the necessary finite extension of $\mathbb{Q}_3$.

```
R.<x> = QQ[]
K.<a> = QQ.extension(x^9 + 27/4*x^8 + 27/2*x^7 + 99/4*x^6 + 189/2*x^4 +
↪  405/2*x^3 + 2187/4)
x0 = a/3

R.<x> = K[]
R.<y> = R[]
F = y^3 - y^2 + (3*x^3 + 1)*y + 3*x^4
y0 = F(x=x0).roots()[0][0]
G = F(x=x+x0, y=y+y0)
R.<u> = K[]
f = u^3 + G[2][1]*u^2 + G[1][2]*u + G[0][3]
L.<u> = K.extension(f)

R.<x> = L[]
R.<y> = R[]
H = F(x=x+x0, y=y+y0+x*u)
v = QQ.valuation(3)
w = v.extensions(L)[0]
for h in H:
  print([w(hi) for hi in h])

from mclf import *

FX.<x> = FunctionField(QQ)
R.<y> = FX[]
F = y^3 - y^2 + (3*x^3 + 1)*y + 3*x^4
FY = FX.extension(F)
Y = SmoothProjectiveCurve(FY)
X = BerkovichLine(FX, v)
T = BerkovichTree(X)
T.add_point(X.gauss_point())
T.add_point(X.point_from_discoid(x, -1/4))
R = ReductionTree(Y, v, T)
print(R.inertial_components()[0].upper_components()[0].component())
print(R.inertial_components()[1].upper_components()[0].component())
```

Code Listing A.6. The following simply defines the curve studied in Section 5.6 and computes its tame locus.

```
from mclf import *

R.<x> = QQ[]
K.<zeta> = QQ.extension(x^2 + x + 1)
R.<x,y> = K[]
```

```
A = (6*zeta + 12)*x^2 + (36*zeta + 9)*x - 27
B = (9*zeta - 18)*x^3 + (-108*zeta - 108)*x^2 + (-162*zeta + 81)*x +
 ↪  (81*zeta + 162)
C = (27*zeta - 243)*x^4 + (-1458*zeta - 999)*x^3 + (-1215*zeta + 1701)*x^2
 ↪  + (1944*zeta + 2430)*x + 729*zeta
F = y^3 + A*y^2 + B*y + C
Y = SmoothProjectiveCurve(F)
v = K.valuation(3)
M = Quartic3Model(Y, v)
V = M.rescaled_tame_locus()
print(V)
```

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Zusammenfassung in deutscher Sprache

Es sei K ein Körper, der vollständig ist bezüglich einer nicht-trivialen nicht-archimedischen Bewertung mit Restklassencharakteristik $p > 0$. Diese Arbeit befasst sich mit der Berechnung der semistabilen Reduktion gewisser algebraischer Kurven über K. Es ist wohlbekannt, wie die semistabile Reduktion einer K-Kurve Y berechnet werden kann, wenn ein Morphismus $Y \to \mathbb{P}^1_K$ vom Grad n kleiner als p existiert. Wir behandeln aber nun den schwierigeren "wilden" Fall, in dem die Existenz eines Morphismus $Y \to \mathbb{P}^1_K$ vom Grad genau p vorausgesetzt wird.

Die analytische Geometrie über dem Körper K im Sinne von Berkovich ([Ber90]) ist eine nützliche Sprache für das Studium der Reduktion von Kurven über K. Zu jeder algebraischen K-Kurve Y mag man ihre *Analytifizierung* Y^{an} assoziieren, die die Reduktionen sämtlicher semistabiler Modelle von Y in sich vereint. Die Schnittgraphen solcher Reduktionen lassen sich nämlich mit Teilräumen von Y^{an} identifizieren, den sogenannten *Skeletten*. Somit verwandelt sich die ursprüngliche Problemstellung über die Bestimmung einer semistabilen Reduktion von Y in die Frage, wie man ein Skelett von Y finden kann.

Eine entscheidende Zutat ist die Arbeit [CTT] von Cohen, Temkin, und Trushin. Ihr Gegenstand ist die zu einer Überlagerung $\varphi\colon Y \to X$ assoziierte *Differentenfunktion* δ_φ auf Y^{an}. Ist $\varphi\colon Y \to \mathbb{P}^1_K$ nun eine Überlagerung vom Grad p, so definieren wir ausgehend von δ_φ eine Funktion λ auf $(\mathbb{P}^1_K)^{\mathrm{an}}$. Die Funktion λ ist außerhalb eines gewissen Skeletts von $\mathbb{P}^1_K$, dessen Urbild in Y^{an} ebenfalls ein Skelett ist, lokal konstant. Somit wird durch λ ein semistabiles Modell von Y bestimmt. Zu den Hauptresultaten dieser Arbeit zählen Formeln für λ in Termen von Koeffizienten einer geeigneten Gleichung von Y.

Besondere Aufmerksamkeit wird dem Fall ebener quartischer Kurven über einem bezüglich einer Bewertung mit Restklassencharakteristik 3 vollständigen Körper K geschenkt. Die ebenen Quartiken bilden die einfachste Klasse von Kurven, für die noch keine allgemeine Methode zur Berechnung ihrer semistablen Reduktion zu beliebiger Restklassencharakteristik bekannt ist. Sie sind trigonal, können also mit einem Morphismus $\varphi\colon Y \to \mathbb{P}^1_K$ vom Grad 3 ausgestattet werden, sodass unsere Ergebnisse auf sie angewandt werden können.

Um die Berechnung der semistabilen Reduktion ebener Quartiken in konkreten Beispielen fassbar zu machen, führen wir den Begriff des zu einer Überlagerung $\varphi\colon Y \to \mathbb{P}^1_K$ vom Grad 3 assoziierten *zahmen Ortes* ein. Seine Existenz entspringt den Eigenschaften der Funktion λ und er beherrscht in gleichem Maße die semistable Reduktion der Überlagerung φ.

Ein Beitrag dieser Arbeit ist auch eine Implementierung der Bestimmung des zahmen Ortes im Computeralgebrasystem SageMath, basierend auf dem MCLF-Paket ([MCLF]). Unter Zuhilfenahme bereits existierender MCLF-Methoden lässt sich so die semistabile Reduktion ebener Quartiken, die in einer geeigneten Normalform vorliegen, automatisch durchführen. Wir illustrieren dies in vielen Beispielen.

Breviārium Latīnē redditum

Sit K corpus explētum secundum aestimātiōnem nōn triviālem nec archimēdēam charactēristicae residuālis $p > 0$. Prōpositum praesentis dissertātiōnis est reductiōnem semistabilem quārundam K-curvārum computāre. Quae rēs quōmodo efficī potest, satis nōtum est, sī morphismus inveniātur $Y \to \mathbb{P}^1_K$ gradūs n ipsō p minōris. Nōs autem tractābimus quaestiōnem difficiliōrem et "feram" dē curvā, cui suppositus morphismus $Y \to \mathbb{P}^1_K$ gradūs ad amussim ipsō p aequālis.

Geōmetria analytica secundum illum Berkovich ([Ber90]) super corpus K instrūmentum est ūtilissimum ad K-curvārum reductiōnī studendum. Cuilibet K-curvae Y adiungere licet eius *analytificātiōnem* Y^{an}, quae omnium reductiōnēs exemplārium semistabilium ipsīus Y complectitur. Rētia intersectiōnālia enim eius modī reductiōnum subspatia ipsīus Y^{an} fingī possunt, quae *ossa* vocantur. Ita problēma dē comperiendā reductiōne semistabilī in quaestionem vertitur, quōmodo ossa ipsīus Y dēterminārī possint.

Magnī mōmentī est opus [CTT] illōrum Cohen, Temkin, Trushin, quī morphismīs curvārum $\varphi\colon Y \to X$ fīnītīs *fūnctiōne differentī* δ_φ ūtentēs student. Quodsī $\varphi\colon Y \to \mathbb{P}^1_K$ est gradūs p, differentem δ_φ adhibēmus ad fūnctiōnem λ super $(\mathbb{P}^1_K)^{\mathrm{an}}$ fabricandam, quae fūnctiō constans est extrā quaedam ossa ipsīus $\mathbb{P}^1_K$, quōrum imāgō reciproca in Y^{an} quoque ossa suppeditat. Ita exemplar curvae Y per fūnctiōnem λ constituitur, quam ut coefficientibus idōneae aequātiōnis ipsīus Y dēscrībāmus, in prīmīs successibus habēmus.

Animum advertimus praeciupē curvīs quārticīs plānīs super corpus explētum secundum aestimātiōnem charactēristicae residuālis 3. Quārticae plānae simplicissimae curvae sunt, quārum reductiō semistabilis ad quamlibet charactēristicam residuālem nōndum semper computārī potest. Cum quārticae trigōnālēs sint, id est morphismum $\varphi\colon Y \to \mathbb{P}^1_K$ admittant, quae repperimus adhibērī possunt.

Ut singulārum quārticārum reductiōnem rēvērā attrectēmus, ūtēmur *locō mītī* morphismī fīnītī $\varphi\colon Y \to \mathbb{P}^1_K$ gradūs 3. Cuius fōns proprietātēs fūnctiōnis λ sunt; reductiōnem semistabilem ipsīus φ eōdem modō regit ac ipsa λ.

Nec parva pars huius operis est illum sapientem mathēseōs computātōrium "SageMath" docēre locum mītem per sē sōlum dēterminā̄re, id quod fasciculō MCLF ([MCLF]) innītentēs efficimus. Auxiliāribus fūnctiōnibus veteribus fasciculī MCLF computātiō reductiōnis semistabilis plānārum quārticārum suā sponte fit, ut multīs exemplīs illūstrāmus.