# Cluster Monomials in Graph Laurent Phenomenon Algebras 

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#### Abstract

Laurent Phenomenon algebras, first introduced by Lam and Pylyavskyy, are a generalization of cluster algebras that still possess many salient features of cluster algebras. Linear Laurent Phenomenon algebras, defined by Lam and Pylyavskyy, are a subclass of Laurent Phenomenon algebras whose structure is given by the data of a directed graph. The main result of this paper is that the cluster monomials of a linear Laurent Phenomenon algebra form a linear basis, conjectured by Lam and Pylyavskyy and analogous to a result for cluster algebras by Caldero and Keller.


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## 1 Introduction

Fomin and Zelevinsky [FZ02] introduced cluster algebras, which are commutative algebras with generators grouped into sets called clusters, and they have applications in various mathematical contexts such as Lie theory, triangulations of surfaces, and Teichmüller theory.

The core idea of cluster algebras is that the generators, known as cluster variables, are organized into sets called clusters, and these clusters are related by exchange binomials, which allow for the exchange of variables within a cluster according to the rule:

$$
\text { old variable } \times \text { new variable }=\text { exchange binomial }
$$

The Laurent Phenomenon is a remarkable property of cluster algebras, stating that any cluster variable can be expressed as a Laurent polynomial when written as a rational function in terms of any other cluster. This property, along with other salient features of cluster algebras, holds in more general settings as well. Laurent Phenomenon algebras, introduced by Lam and Pylyavskyy [LP16a], are a generalization of cluster algebras that retain many of these salient features, including the Laurent Phenomenon. In Laurent Phenomenon algebras, the exchange polynomials can be irreducible polynomials, rather than just binomials.

Lam and Pylyavskyy [LP16b] defined a specific class of Laurent Phenomenon algebras called graph Laurent Phenomenon algebras, which are characterized by the structure of a directed graph $\Gamma$ and provide an explicit description of the cluster variables and clusters of a graph Laurent Phenomenon algebra $\mathcal{A}_{\Gamma}$. About this structure, Lam and Pylyavskyy [LP16b, Conjecture 7.3a] conjectured Theorem 1.
Theorem 1. Let $\Gamma$ be a graph, and let $\mathcal{A}_{\Gamma}$ be the associated LP algebra with coefficient ring $R$. Then, the cluster monomials of $\mathcal{A}_{\Gamma}$ form a basis over $R$.

Theorem 1 parallels the case for cluster algebras, where Caldero and Keller [CK08] proved that cluster monomials form a linear basis in finite type cluster algebras.

The main result of this paper is a proof of Theorem 1, which is split into two parts. First, we establish the linear independence of cluster monomials in a graph LP algebra, as stated in Theorem 6. Second, we demonstrate that cluster monomials in a graph LP algebra form an $R$-linear spanning set, as stated in Theorem 14.

In Section 2, we recall and expand upon definitions and results from [LP16b]. In Section 3, we prove that cluster monomials in a graph LP algebra are linearly independent over $R$. In Section 4, we prove that cluster monomials in a graph LP algebra form an $R$-linear spanning set. In Section 5, we present partial results on a positivity conjecture [LP16b, Conjecture 7.3b].

### 1.1 Acknowledgments

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## 2 Preliminaries and Notational Conventions

### 2.1 Multisets

A multiset $S$ is a sequence of sets $S^{(1)} \supset S^{(2)} \supset \cdots$ indexed by the positive integers. We say that $x$ has multiplicity $m$ in $S$ if $m+1$ is the smallest positive integer such that $x \notin S^{(m+1)}$. In this case, if $m>0$, it follows that $x \in S^{(m)}$. Therefore, $S^{(i)}$ is the set of elements with multiplicity at least $i$ in $S$. A set $S$ is naturally identified with the multiset given by the sequence $S \supset \varnothing \supset \varnothing \supset \cdots$, that is, we interpret a set as a multiset with all multiplicities equal to 1 .

Given multisets $S$ and $T$, we say that $S$ and $T$ are disjoint if $S^{(1)}$ and $T^{(1)}$ are disjoint. We say that $S$ is contained in $T$, denoted by $S \subset T$, if $S^{(i)} \subset T^{(i)}$ for all $i$. The sum of two multisets $S$ and $T$ is the multiset $S+T$ with the multiplicity of each element $x$ being the sum of its multiplicities in $S$ and $T$. If $S$ is contained in $T$, the subtraction of $S$ from $T$ is the multiset $T-S$ with the multiplicity of each element $x$ being the subtraction of its multiplicity in $S$ from its multiplicity in $T$.

We may define a multiset by simply listing its elements, with the multiplicity of each element being the number of times it appears in the list. For example, $S=\{1,1,2,3\}$ is the multiset with elements $1,2,3$ and multiplicities $2,1,1$, respectively, in which case we have $S^{(1)}=\{1,2,3\}, S^{(2)}=\{1\}$, and $S^{(i)}=\varnothing$ for all $i \geq 3$.

When summations, multiplications, and other operations are applied to multisets, the result is obtained by applying the operation to the results obtained by applying the operation to each set in the sequence. In other words,

$$
\sum_{x \in S} f(x)=\sum_{m \in \mathbb{Z}>0} \sum_{x \in S^{(m)}} f(x),
$$

and similarly for products and other operations.

### 2.2 Directed graph

Let $\Gamma$ be a directed graph with vertex set $V$ and edge set $E$. We maintain this notation throughout the document. Examples of directed graphs are given in Figure 1.

(a) A directed graph with four vertices, $1,2,3$, and 4 , and ten directed edges, $12,13,14,21$, $23,31,32,34,41$, and 43.

(b) A directed graph with four vertices, 1, 2, 3 , and 4 , and six directed edges, $12,13,23,31$, 34 , and 41.

Figure 1: Two directed graphs. Each solid edge represents two directed edges, one in each direction.

### 2.3 Nested collections

A subset $I \subset V$ is strongly connected if the induced subgraph on $I$ is strongly connected, that is, if for all vertices $v, u \in I$, there is some directed path contained in $I$ from $v$ to $u$.

For example, taking $\Gamma$ to be the directed graph in Figure 1a, the strongly connected sets of vertices are $\varnothing,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,2,3\},\{1,2,4\}$, $\{1,3,4\},\{2,3,4\}$, and $\{1,2,3,4\}$. Taking $\Gamma$ to be the directed graph in Figure 1b, the strongly connected sets of vertices are $\varnothing,\{1\},\{2\},\{3\},\{4\},\{1,3\},\{1,2,3\},\{1,3,4\}$, and $\{1,2,3,4\}$.

Note that any subset $U \subset V$ is uniquely partitioned into maximal strongly connected subsets, called strongly connected components of $U$.

A (multi)set $\mathcal{N}$ with elements from $\mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the powerset of $V$, is a nested (multi)collection if
(N1) for every pair $I, J \in \mathcal{N}$, either $I \subset J, J \subset I$, or $I \cap J=\varnothing$, and
(N2) for any $\mathcal{R} \subset \mathcal{N}$ such that $I \cap J=\varnothing$ for all distinct $I, J \in \mathcal{R}$, each $I \in \mathcal{R}$ is a strongly connected component of the subgraph induced by $\bigcup_{J \in \mathcal{R}} J$.

Note that (N2), applied to singletons $\mathcal{R}$, implies that each $I \in \mathcal{N}$ is strongly connected. Note that a multiset $\mathcal{N}$ is a nested multicollection if and only if the set $\mathcal{N}^{(1)}$ is a nested collection. We usually disregard the empty set when considering nested collections.

For example, consider $\Gamma$ to be the directed graph in Figure 1a. The set

$$
\{\{2\},\{4\},\{2,3,4\},\{1,2,3,4\}\}
$$

is a nested collection. The set

$$
\{\{1\},\{1,3\},\{1,4\},\}
$$

is not a nested collection because $\{1,3\}$ and $\{1,4\}$ do not satisfy (N1). The set

$$
\{\{2,4\},\{2,3,4\},\{1,2,3,4\}\}
$$

is not a nested collection because $\mathcal{R}=\{\{2,4\}\}$ does not satisfy (N2), since the strongly connected components of $\{2,4\}$ are $\{2\}$ and $\{4\}$. The set

$$
\{\{1\},\{3\},\{1,2,3\},\{1,2,3,4\}\}
$$

is not a nested collection because $\mathcal{R}=\{\{1\},\{3\}\}$ does not satisfy (N2), since the only strongly connected component of $\{1,3\}$ is $\{1,3\}$.

Lemma 2. The map $\mathcal{N} \mapsto+_{I \in \mathcal{N}} I$ is a bijection between the set of nested multicollections $\mathcal{N}$ and the set of multisets $T$ with elements from $V$.

The proof of Lemma 2 is split into two parts: surjectivity and injectivity.
Proof (surjectivity). Let $T$ be a multiset with elements from $V$. We construct a nested multicollection $\mathcal{N}$ such that $T=+_{I \in \mathcal{N}} I$.

Recall that $T^{(i)}$ denotes the set of elements of $T$ with multiplicity at least $i$. Note that $T^{(1)} \supset T^{(2)} \supset \cdots$. Let $\mathcal{N}_{i}$ be the set of strongly connected components of the subgraph induced by $T^{(i)}$. Let $\mathcal{N}=+_{i \in \mathbb{Z}_{>0}} \mathcal{N}_{i}$. Then,

$$
T={\underset{i \in \mathbb{Z}}{>0}} T^{(i)}={\underset{i \in \mathbb{Z}}{>0}}^{{\underset{V}{x}}} I={\underset{I \in \mathcal{N}_{i}}{ }} I
$$

First, we show that $\mathcal{N}$ satisfies (N1). Let $I, J \in \mathcal{N}$. Then, $I \in \mathcal{N}_{i}$ and $J \in \mathcal{N} \mathcal{N}_{j}$ for some $i, j \in \mathbb{Z}_{>0}$. Recall that $I$ is a strongly connected component of the subgraph induced by $T^{(i)}$, and $J$ is a strongly connected component of the subgraph induced by $T^{(j)}$. Without loss of generality, $i \leq j$, therefore, $T^{(i)} \supset T^{(j)}$. Therefore, there exists a strongly connected component $K$ of the subgraph induced by $T^{(i)}$ such that $J \subset K$. Moreover, since $I$ and $K$ are strongly connected components of the subgraph induced by $T^{(i)}, I=K$ or $I \cap K=\varnothing$. In the first case, $I \subset J$, while in the second case, $I \cap J=\varnothing$, as desired.

Now we show that $\mathcal{N}$ satisfies (N2), that is, for any $\mathcal{R} \subset \mathcal{N}$ such that $I \cup J=\varnothing$ for all distinct $I, J \in \mathcal{R}$, each $I \in \mathcal{R}$ is a strongly connected component of the subgraph induced by $R=\bigcup_{J \in \mathcal{R}} J$. We prove by induction on the number of elements of $\mathcal{R}$. If $\mathcal{R}=\varnothing$, then the statement is vacuously true. Suppose that $\mathcal{R}$ has at least one element. Let $i \in \mathbb{Z}_{>0}$ be the minimum index such that $\mathcal{R} \cap \mathcal{N}_{i} \neq \varnothing$, and let $I \in \mathcal{R} \cap \mathcal{N}_{i}$. Then, all $J \in \mathcal{R}$ are subsets of $T^{(i)}$, and consequently, $R=\bigcup_{J \in \mathcal{R}} J$ is a subset of $T^{(i)}$. Therefore, $I \subset R \subset T^{(i)}$. Since $I$ is a strongly connected component of the subgraph induced by $T^{(i)}$, it follows that $I$ is a strongly connected component of the subgraph induced by $R$. The connected components of the subgraph induced by $R$ different from $I$ are the connected components of $R \backslash I$. Applying the induction hypothesis to $\mathcal{R} \backslash\{I\}$, we obtain that each $J \in \mathcal{R} \backslash\{I\}$ is a strongly connected component of the subgraph induced by $R \backslash I$, and therefore, each $J \in \mathcal{R}$ is a strongly connected component of the subgraph induced by $R$, as desired.

Therefore, $\mathcal{N}$ is a nested multicollection and $T=+_{I \in \mathcal{N}} I$, as desired.
Proof (injectivity). Let $T$ be a multiset with elements from $V$. We prove that there is a unique nested multicollection $\mathcal{N}$ such that $T=+_{I \in \mathcal{N}} I$. The proof is by induction on the
number of elements of $T$. If $T=\varnothing$, then $\mathcal{N}=\varnothing$ is the unique nested multicollection such that $T=+_{I \in \mathcal{N}} I$.

Suppose that $T$ has at least one element. Let $\mathcal{N}$ be a nested multicollection such that $T=+_{I \in \mathcal{N}} I$. Let $T^{(1)}$ be the set of elements of $T$ with multiplicity at least 1 . Let $\mathcal{R}$ be the set of maximal elements of $\mathcal{N}$. Note that, for all $v \in T^{(1)}$, there exists a unique $I \in \mathcal{R}$ such that $v \in I$. Therefore, $T^{(1)}=+_{I \in \mathcal{R}} I$. Since $\mathcal{N}$ is nested and $I \cup J=\varnothing$ for all distinct $I, J \in \mathcal{R}$, it follows that each $I \in \mathcal{R}$ is a strongly connected component of the subgraph induced by $T^{(1)}$. Hence, $\mathcal{R}$ is the set of strongly connected components of the subgraph induced by $T^{(1)}$.

Let $T^{\prime}=T-T^{(1)}$, and let $\mathcal{N}^{\prime}=\mathcal{N}-\mathcal{R}$. Since $T=+_{I \in \mathcal{N}} I$ and $T^{(1)}=+_{I \in \mathcal{R}} I$, it follows that $T^{\prime}=+_{I \in \mathcal{N}^{\prime}} I$. Since $\mathcal{N}$ is nested, $\mathcal{N}^{\prime}$ is nested. By the induction hypothesis, $\mathcal{N}^{\prime}$ is the unique, and consequently, $\mathcal{N}$ is the unique.

### 2.4 Laurent polynomial ring

Let the coefficient ring $R$ be a ring over $\mathbb{Z}$ containing elements $A_{v}$ for each $v \in V$ which are algebraically independent. For example, $R$ could be $\mathbb{Z}\left[A_{v}: v \in V\right]$. Let ambient ring $\mathcal{F}$ be the rational function field over $R$ in the independent variables $X_{v}$ for $v \in V$. Let $\mathcal{L} \subset \mathcal{F}$ denote the Laurent polynomial ring over $R$ in the independent variables $X_{v}$ for $v \in V$, that is, $\mathcal{L}=R\left[X_{v}^{ \pm 1}: v \in V\right]$.

The monomials in $\mathcal{L}$ in the variables $X_{v}$ and $X_{v}^{-1}$ for $v \in V$ are called Laurent monomials. Any Laurent monomial can be written as

$$
\ell(U, T)=\prod_{v \in U} X_{v} / \prod_{v \in T} X_{v}
$$

where $U$ and $T$ are disjoint multisets with elements in $V$. As a module over $R$, the Laurent polynomial ring $\mathcal{L}$ has a basis consisting of all Laurent monomials.

### 2.5 Multifunctions

Let $I$ be a multiset with elements from $V$. A multifunction $f$ (of $\Gamma$ ) on $I$ is a directed multigraph with vertex set $V$ and edge multiset $E_{f}$ such that
(F1) for each vertex $v \in V$, the outdegree of $v$ in $f$ is its multiplicity in $I$, and
(F2) each edge in $E_{f}$ is either a loop or an edge in $E$.
If $I$ is a set, then a multifunction on $I$ is naturally identified with a function from $I$ to $V$, explaining the chosen nomenclature.

A multifunction $f$ is acyclic if the only cycles in $f$ are loops. Given two multifunctions $f$ and $g$ on multisets $I$ and $J$, respectively, we define their sum $f+g$ as the multifunction on $I+J$ obtained by taking the sum of the edge multisets of $f$ and $g$.

For example, take $\Gamma$ to be the directed graph in Figure 1a. Examples of multifunctions of $\Gamma$ are given in Figure 2. The multifunctions in Figure 2a and Figure 2b are acyclic, and the multifunction in Figure 2c is not acyclic.

(a) A multifunction on $\{1$, $1,1,2\}$ with edge multiset $\{(1,1),(1,2),(1,2),(2,3)\}$.
(b) A multifunction on $\{3,4\}$ with edge multiset $\{(3,4),(4,1)\}$.
(c) The sum of the multifunctions in Figures 2a and 2 b .

Figure 2: Multifunctions of the directed graph in Figure 1a.

### 2.6 Weight of a multifunction

The weight of a multifunction $f$ on $I$, denoted by $\operatorname{wt}(f)$, is the non-Laurent polynomial in $\mathcal{L}$ given by

$$
\mathrm{wt}(f)=\prod_{(v, w) \in E_{f}} \tilde{X}_{(v, w)}
$$

where

$$
\tilde{X}_{(v, w)}= \begin{cases}X_{w} & \text { if } w \neq v \\ A_{v} & \text { if } w=v\end{cases}
$$

For example, the weights of the multifunctions in Figures 2a, 2b, and 2c are, respectively,

$$
A_{1} X_{2}^{2} X_{3}, \quad X_{1} X_{4}, \quad \text { and } \quad A_{1} X_{1} X_{2}^{2} X_{3} X_{4}
$$

The normalized weight of a multifunction $f$ on $I$, denoted by nwt $(f)$, is the Laurent polynomial in $\mathcal{L}$ given by

$$
\operatorname{nwt}(f)=\frac{\mathrm{wt}(f)}{\prod_{v \in I} X_{v}}=\prod_{(v, w) \in E_{f}} \frac{\tilde{X}_{(v, w)}}{X_{v}} .
$$

For example, the normalized weights of the multifunctions in Figures 2a, 2b, and 2c are, respectively,

$$
\frac{A_{1} X_{2}^{2} X_{3}}{X_{1}^{3} X_{2}}=\frac{A_{1} X_{2} X_{3}}{X_{1}^{3}}, \quad \frac{X_{1} X_{4}}{X_{3} X_{4}}=\frac{X_{1}}{X_{3}}, \quad \text { and } \quad \frac{A_{1} X_{1} X_{2}^{2} X_{3} X_{4}}{X_{1}^{3} X_{2} X_{3} X_{4}}=\frac{A_{1} X_{2}}{X_{1}^{2}}
$$

We remark that the nomenclature of "weights" and "normalized weights" is not used by Lam and Pylyavskyy [LP16b], although the concept is present in their work.

Note that both weights and normalized weights are products over edges of the multifunction, and therefore, if $f$ and $g$ are multifunctions, then

$$
\mathrm{wt}(f+g)=\mathrm{wt}(f) \mathrm{wt}(g) \quad \text { and } \quad \operatorname{nwt}(f+g)=\operatorname{nwt}(f) \operatorname{nwt}(g) .
$$

Note that the normalized weight of a cycle is 1 . This observation will be used in the proof of Lemma 11 in Section 4.

### 2.7 The $Y$ Laurent polynomials

Let $I$ be a subset of $V$. We define the Laurent polynomial $Y_{I} \in \mathcal{L}$ by

$$
Y_{I}=\frac{\sum_{f: I}^{\mathrm{acy}} \mathrm{wt}(f)}{\prod_{i \in I} X_{i}}
$$

where the sum is over all acyclic multifunctions $f$ on $I$. Recall that, since $I$ is a set, a multifunction on $I$ is naturally identified with a function from $I$ to $V$. We may rewrite $Y_{I}$ in terms of normalized weights as $Y_{I}=\sum_{f: I}^{\text {acy }} \operatorname{nwt}(f)$.

For example, taking $\Gamma$ to be the directed graph in Figure 1a, the Laurent polynomial $Y_{\{1,2\}}$ is

$$
\frac{A_{1} X_{1}+X_{3} X_{1}+X_{4} X_{1}+A_{1} A_{2}+X_{2} A_{2}+X_{3} A_{2}+X_{4} A_{2}+A_{1} X_{3}+X_{2} X_{3}+X_{3}^{2}+X_{4} X_{3}}{X_{1} X_{2}}
$$

Note that the eleven terms in the numerator correspond to the eleven acyclic multifunctions on $\{1,2\}$, which are all twelve multifunctions on $\{1,2\}$ by assigning one of the possible four edges to 1 , and one of the possible three edges to 2 , except for the assignment of the edge $(1,2)$ to 1 and the edge $(2,1)$ to 2 which is not acyclic.

### 2.8 Graph LP algebra and clusters

The central algebraic structure of our research is the graph Laurent phenomenon algebra $\mathcal{A}_{\Gamma}$ over $R$ associated to $\Gamma$, defined by Lam and Pylyavskyy [LP16b]. Lam and Pylyavskyy [LP16b] proved that $\mathcal{A}_{\Gamma}$ is the algebra over $R$ generated by $X_{v}$ for $v \in V$ and $Y_{I}$ for strongly connected $I \subset V$. The reader who is new to graph LP algebras can simply take this as the definition of $\mathcal{A}_{\Gamma}$.

Although only the Laurent polynomials $Y_{I}$ for strongly connected $I \subset V$ are generators of $\mathcal{A}_{\Gamma}$, the Laurent polynomials $Y_{I}$ for $I \subset V$ are in $\mathcal{A}_{\Gamma}$ as well, as guaranteed by Lemma 3 .

Lemma 3 ([LP16b, Lemma 4.2]). Let $I$ be a subset of $V$. Then, $Y_{I}=\prod_{J} Y_{J}$, where the product is over the strongly connected components $J$ of the subgraph of $\Gamma$ induced by $I$.

The monomials in $\mathcal{A}_{\Gamma}$ in the elements $X_{v}$ for $v \in v$ and $Y_{I}$ for $I \subset V$ are simply called monomials. Any monomial in $\mathcal{A}_{\Gamma}$ can be written as

$$
m(U, \mathcal{S})=\prod_{v \in U} X_{v} \prod_{I \in \mathcal{S}} Y_{I}
$$

where $U$ is a multiset with elements in $V$ and $\mathcal{S}$ is a multiset with elements in $\mathcal{P}(V)$. The set of monomials in $\mathcal{A}_{\Gamma}$ is a spanning set of $\mathcal{A}_{\Gamma}$ as a module over $R$. The monomials in $\mathcal{A}_{\Gamma}$ in the elements $Y_{I}$ for $I \subset V$ are called $Y$-monomials, which can be written as $m(\varnothing, \mathcal{S})=\prod_{I \in \mathcal{S}} Y_{I}$ where $\mathcal{S}$ is a multiset with elements in $\mathcal{P}(V)$.

The elements $X_{v}$ for $v \in V$ and $Y_{I}$ for strongly connected $I \subset V$ are grouped into sets called clusters. We refer to [LP16b] for the definition of clusters. Lam and Pylyavskyy [LP16b] proved that the clusters of $\mathcal{A}_{\Gamma}$ are the sets of the form

$$
\left\{X_{v}: v \in U\right\} \cup\left\{Y_{I}: I \in \mathcal{N}\right\}
$$

where $U \subset V$ and $\mathcal{N}$ is a maximal nested collection on $V \backslash U$. The reader who is new to graph LP algebras can simply take this as the definition of the clusters of $\mathcal{A}_{\Gamma}$.

The monomials in $\mathcal{A}_{\Gamma}$ in the elements of a given cluster are called cluster monomials. Any cluster monomial can be written as

$$
m(U, \mathcal{N})=\prod_{v \in U} X_{v} \prod_{I \in \mathcal{N}} Y_{I}
$$

where $U$ is a multiset with elements in $V$ and $\mathcal{N}$ is a nested collection on $V \backslash U^{(1)}$. The monomials in $\mathcal{A}_{\Gamma}$ in the elements $Y_{I}$ for $I \subset V$ of a given cluster are called cluster $Y$ monomials, which can be written as $m(\varnothing, \mathcal{N})=\prod_{I \in \mathcal{N}} Y_{I}$ where $\mathcal{N}$ is a nested collection on $V$.

## 3 Linear independence

In this section, we show that the cluster monomials in a graph LP algebra are linearly independent.

By Lemma 2, we can reindex the cluster monomials by a pair $(U, T)$ where $U$ and $T$ are disjoint multisets with vertices from $V$, and the cluster monomial indexed by $(U, T)$ is $m(U, T)=m(U, \mathcal{N})$, where $\mathcal{N}$ is the unique nested multicollection such that $T=+_{I \in \mathcal{N}} I$.

Lemma 4. Let $U_{1}, U_{2}, T_{1}, T_{2}$ be multisets with vertices from $V$ such that $U_{1} \cap T_{1}=U_{2} \cap$ $T_{2}=\varnothing$. If $U_{1} \not \subset U_{2}$, then, when expanding $m\left(U_{1}, T_{1}\right)$ as a linear combination of Laurent monomials over $R$, the coefficient of $\ell\left(U_{2}, T_{2}\right)$ is 0 . If $T_{1} \not \supset T_{2}$, then, when expanding $m\left(U_{1}, T_{1}\right)$ as a linear combination of Laurent monomials over $R$, the coefficient of $\ell\left(U_{2}, T_{2}\right)$ is 0 .

Proof. Let $\mathcal{N}_{1}$ be the unique nested multicollection such that $T_{1}=+_{I \in \mathcal{N}_{1}} I$. Recall that

$$
m\left(U_{1}, T_{1}\right)=\prod_{v \in U_{1}} X_{v} \prod_{I \in \mathcal{N}_{1}} Y_{I}=\frac{\prod_{v \in U_{1}} X_{v}}{\prod_{v \in T_{1}} X_{v}}\left(\prod_{I \in \mathcal{N}_{1}} \sum_{f: I}^{\text {acy }} w t(f)\right) .
$$

Since $\prod_{I \in \mathcal{N}_{1}} \sum_{f: I}^{a c y} w t(f)$ is a polynomial, the numerator of any term of $m\left(U_{1}, T_{1}\right)$ is a multiple of $\prod_{v \in U_{1}} X_{v}$. Hence, if $U_{1} \not \subset U_{2}$, then the coefficient of

$$
\ell\left(U_{2}, T_{2}\right)=\frac{\prod_{v \in U_{2}} X_{v}}{\prod_{v \in T_{2}} X_{v}}
$$

in $m\left(U_{1}, T_{1}\right)$ must be 0.
Since $\prod_{I \in \mathcal{N}_{1}} \sum_{f: I}^{\text {acy }} w t(f)$ is a polynomial, the denominator of any term of $m\left(U_{1}, T_{1}\right)$ is a divisor of $\prod_{v \in T_{1}} X_{v}$. Hence, if $T_{1} \not \supset T_{2}$, then the coefficient of

$$
\ell\left(U_{2}, T_{2}\right)=\frac{\prod_{v \in U_{2}} X_{v}}{\prod_{v \in T_{2}} X_{v}}
$$

in $m\left(U_{1}, T_{1}\right)$ must be 0.
Lemma 5. Let $U, T$ be multisets with vertices from $V$. When expanding $m(U, T)$ as a linear combination of Laurent monomials over $R$, the coefficient of $\ell(U, T)$ is $\prod_{v \in T} A_{v}$.

Proof. Let $\mathcal{N}$ be the unique nested multicollection such that $T=+_{I \in \mathcal{N}} I$. Recall that

$$
m(U, T)=\prod_{v \in U} X_{v} \prod_{I \in \mathcal{N}} Y_{I}=\frac{\prod_{v \in U} X_{v}}{\prod_{v \in T} X_{v}}\left(\prod_{I \in \mathcal{N}} \sum_{f: I}^{\text {acy }} w t(f)\right)
$$

Note that $\prod_{I \in \mathcal{N}} \sum_{f: I}^{a c y} w t(f)$ is a polynomial with constant term $\prod_{v \in T} A_{v}$, obtained from the constant acyclic function on each $I \in \mathcal{N}$. Therefore, the coefficient of $\ell(U, T)$ in $m(U, T)$ is $\prod_{v \in T} A_{v}$.
Theorem 6. The set of cluster monomials of a graph LP algebra is linearly independent over $R$.

Proof. Suppose that there exists a nontrivial linear combination of cluster monomials that equals 0 . Explicitly, suppose that there exists coefficients $c(U, T)$, not all zero, such that

$$
\sum_{U, T} c(U, T) m(U, T)=0
$$

Pick $\left(U_{0}, T_{0}\right)$ with $c\left(U_{0}, T_{0}\right) \neq 0$ such that $\left|U_{0}\right|-\left|T_{0}\right|$ is minimized. This implies that, for all $(U, T) \neq\left(U_{0}, T_{0}\right)$ such that $c(U, T) \neq 0$, it holds that $|U|-|T| \geq\left|U_{0}\right|-\left|T_{0}\right|$, and consequently, $U \not \subset U_{0}$ or $T \not \supset T_{0}$.

Hence, by Lemma 4, the coefficient of $\ell\left(U_{0}, T_{0}\right)$ in $c(U, T) m(U, T)$ is 0 for all $(U, T) \neq$ $\left(U_{0}, T_{0}\right)$ such that $c(U, T) \neq 0$; and, by Lemma 5 , the coefficient of $\ell\left(U_{0}, T_{0}\right)$ in the expression $c\left(U_{0}, T_{0}\right) m\left(U_{0}, T_{0}\right)$ is $c\left(U_{0}, T_{0}\right) \prod_{v \in T_{0}} A_{v}$. Therefore, the coefficient of $\ell\left(U_{0}, T_{0}\right)$ in $\sum_{U, T} c(U, T) m(U, T)$ is $c\left(U_{0}, T_{0}\right) \prod_{v \in T_{0}} A_{v} \neq 0$, a contradiction.

## 4 Spanning set

In this section, we show that the cluster monomials in a graph LP algebra form a spanning set.

### 4.1 Previous results

Let $p: v \rightarrow_{I} w$ denote the statement that $p$ is a vertex non-repeating directed path from $v$ to $w$ with intermediary vertices in $I$. Let $I \backslash p$ denote the set of vertices in $I$ that are not in $p$. Lemma 7 allows us to rewrite the product of $X_{v}$ and $Y_{I}$ whenever $v \in I$.

Lemma 7 ([LP16b, inferred from Lemma 4.7]). Let $I \in \mathcal{P}(V)$ and $v \in I$. Then,

$$
X_{v} Y_{I}=\sum_{w \in I} \sum_{p}^{v \rightarrow_{I} w} Y_{I \backslash p} X_{w}+\sum_{w \notin I} \sum_{p}^{v \rightarrow_{I} w} Y_{I \backslash p} A_{w} .
$$

### 4.2 Cluster $Y$-monomials span $Y$-monomials

Given a tuple of multifunctions, we define its weight as the product of the weights of its elements.

Lemma 8. Let $\mathcal{F}, \mathcal{G}$ be sets of tuples of multifunctions. If $\phi: \mathcal{F} \rightarrow \mathcal{G}$ preserves weights, that is, $\operatorname{wt}(f)=\operatorname{wt}(\phi(f))$ for all $f \in \mathcal{F}$, then

$$
\sum_{f \in \mathcal{F}} \mathrm{wt}(f)=\sum_{g \in \mathcal{G}}\left|\phi^{-1}(g)\right| \mathrm{wt}(g) .
$$

Proof. Apply the weight-preserving property and double-count pairs $(f, g) \in \mathcal{F} \times \mathcal{G}$ such that $\phi(f)=g$ to obtain

$$
\sum_{f \in \mathcal{F}} \mathrm{wt}(f)=\sum_{f \in \mathcal{F}} \mathrm{wt}(\phi(f))=\sum_{g \in \mathcal{G}}\left|\phi^{-1}(g)\right| \mathrm{wt}(g) .
$$

Lemma 9 (Preimages Lemma). Let $\mathcal{F}_{1}, \mathcal{F}_{2}, \mathcal{G}$ be sets of tuples of multifunctions. If $\phi_{1}: \mathcal{F}_{1} \rightarrow$ $\mathcal{G}$ and $\phi_{2}: \mathcal{F}_{2} \rightarrow \mathcal{G}$ preserve weights, and $\left|\phi_{1}^{-1}(g)\right|=\left|\phi_{2}^{-1}(g)\right|$ for all $g \in \mathcal{G}$, then

$$
\sum_{f \in \mathcal{F}_{1}} \mathrm{wt}(f)=\sum_{f \in \mathcal{F}_{2}} \mathrm{wt}(f) .
$$

Proof. Apply Lemma 8 on $\phi_{1}$ and on $\phi_{2}$ to obtain

$$
\sum_{f \in \mathcal{F}_{1}} \mathrm{wt}(f)=\sum_{g \in \mathcal{G}}\left|\phi_{1}^{-1}(g)\right| \mathrm{wt}(g)=\sum_{g \in \mathcal{G}}\left|\phi_{2}^{-1}(g)\right| \mathrm{wt}(g)=\sum_{f \in \mathcal{F}_{2}} \mathrm{wt}(f) .
$$

Lemma 10. Let $\mathcal{S}$ be a multiset with elements in $\mathcal{P}(V)$. Let $T=+_{I \in \mathcal{S}} I$ be the multiset containing the vertices that appear in the sets of $\mathcal{S}$, counting multiplicities. Let $\mathcal{T}$ be the multiset whose elements are $T^{(1)}, T^{(2)}, \cdots \in \mathcal{P}(V)$. Then,

$$
\begin{equation*}
\prod_{S \in \mathcal{S}} \sum_{f: S} \mathrm{wt}(f)=\prod_{T^{\prime} \in \mathcal{T}} \sum_{f: T^{\prime}} \mathrm{wt}(f), \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\prod_{S \in \mathcal{S}} \sum_{f: S} \operatorname{nwt}(f)=\prod_{T^{\prime} \in \mathcal{T}} \sum_{f: T^{\prime}} \operatorname{nwt}(f) . \tag{2}
\end{equation*}
$$

Proof. Let $S_{1}, S_{2}, \ldots$ be the elements of $\mathcal{S}$. Let $T_{1}, T_{2}, \ldots$ be the elements of $\mathcal{T}$. Note that $T=+_{i} S_{i}=+_{j} T_{j}$. Let

$$
\begin{aligned}
& \mathcal{F}_{1}=\left\{\left(f_{1}, f_{2}, \ldots\right): f_{i} \text { is a multifunction on } S_{i} \text { for each } i \in \mathbb{Z}_{>0}\right\}, \\
& \mathcal{F}_{2}=\left\{\left(f_{1}, f_{2}, \ldots\right): f_{i} \text { is a multifunction on } T_{i} \text { for each } i \in \mathbb{Z}_{>0}\right\}, \\
& \qquad \mathcal{G}=\{g: g \text { is a multifunction on } T\},
\end{aligned}
$$

and consider the functions $\phi_{1}: \mathcal{F}_{1} \rightarrow \mathcal{G}$ and $\phi_{2}: \mathcal{F}_{2} \rightarrow \mathcal{G}$ where both functions send a tuple of multifunctions to the multifunction obtained by taking the sum of the edges of each multifunction in the tuple.

Let $g \in \mathcal{G}$. Let's count the number of $\left(f_{1}, f_{2}, \ldots\right)$ in $\mathcal{F}_{1}$ such that $\phi_{1}\left(f_{1}, f_{2}, \ldots\right)=g$. Fix $v \in V$. Let $g_{v}$ denote the multiset of edges in $g$ from $v$. Note that $\left|g_{v}\right|$ is the multiplicity of $v$ in $T$, which is the number of sets $T_{i}$ in $\mathcal{T}$ that contain $v$. Therefore, the number of ways to assign the edges of $g_{v}$ to the appropriate functions $f_{i}$ is

$$
\frac{\left|g_{v}\right|!}{1^{g_{v}^{(1)} \mid} \cdot 2^{\left|g_{v}^{(2)}\right|} \cdot 3^{\left|g_{v}^{(3)}\right|} \ldots}
$$

Therefore, the number of $\left(f_{1}, f_{2}, \ldots\right)$ in $\mathcal{F}_{1}$ such that $\phi_{1}\left(f_{1}, f_{2}, \ldots\right)=g$ is

$$
\left|\phi_{1}^{-1}(g)\right|=\prod_{v \in V} \frac{\left|g_{v}\right|!}{1^{\left|g_{v}^{(1)}\right|} \cdot 2^{\left|g_{v}^{(2)}\right|} \cdot 3^{\left|g_{v}^{(3)}\right|} \ldots}
$$

The same argument applies to $\phi_{2}$, and we obtain, for all $g \in \mathcal{G}$,

$$
\left|\phi_{1}^{-1}(g)\right|=\left|\phi_{2}^{-1}(g)\right| .
$$

Hence, by Lemma 9, we obtain

$$
\sum_{f \in \mathcal{F}_{1}} \mathrm{wt}(f)=\sum_{f \in \mathcal{F}_{2}} \mathrm{wt}(f),
$$

which implies Equation (1). The result in Equation (2) follows by dividing both sides of Equation (1) by $\prod_{v \in T} X_{v}$.

Lemma 11. Let $S \in \mathcal{P}(V)$. Then,

$$
\sum_{f: S} \operatorname{nwt}(f)=\sum_{C \in \mathcal{C}_{S}} Y_{S \backslash C},
$$

where $C$ ranges over the set $\mathcal{C}_{S}$ of all families of vertex-disjoint cycles in the restriction of $\Gamma$ to $S$.

Proof. Given a function $f: S$, a vertex can be in at most one cycle, since the only possible cycle containing $v$ must be the periodic part of $v, f(v), f^{2}(v), \ldots$. Therefore, each function $f: S$ can be uniquely decomposed into a family of vertex-disjoint cycles $C$ and an acyclic
function $g: S \backslash C$ such that $f$ is the sum of $g$ and the cycles in $C$. Similarly, given a family of vertex-disjoint cycles $C$ and an acyclic function $g: S \backslash C$, a function $f: S$ can be obtained by taking the sum of $g$ and the cycles in $C$. Thus, there is a bijection between the set of functions $f: S$ and the set of pairs $(C, g)$ where $C$ is a family of vertex-disjoint cycles and $g: S \backslash C$.

Moreover, if this bijection associates $f$ to $(C, g)$, then the normalized weight of $f$ is the product of the normalized weight of $g$ and the normalized weight of the cycles in $C$. Since the normalized weigh of a cycle is 1 , it follows that $\operatorname{nwt}(f)=\operatorname{nwt}(g)$.

Finally,

$$
\sum_{f: S} \operatorname{nwt}(f)=\sum_{C \in \mathcal{C}_{S}} \sum_{g: S \backslash C}^{\text {acy }} \operatorname{nwt}(g)=\sum_{C \in \mathcal{C}_{S}} Y_{S \backslash C} .
$$

Lemma 12. Let $T$ be a multiset of vertices. Let $\mathcal{T}$ be the multiset whose elements are $T^{(1)}, T^{(2)}, \cdots \in \mathcal{P}(V)$. Then, the $Y$-monomial $m(\varnothing, \mathcal{T})$ is a cluster $Y$-monomial.

Proof. Let $\mathcal{N}_{i}$ be the set of strongly connected components of the subgraph of $\Gamma$ induced by $T^{(i)}$ for each $i \in \mathbb{Z}_{>0}$, and let $\mathcal{N}=+_{i} \mathcal{N}$. Lemma 2 implies that $\mathcal{N}$ is a nested multicollection. Finally, Lemma 3 implies that $Y_{T^{(i)}}=\prod_{J \in \mathcal{N}_{i}} Y_{J}$, and consequently

$$
m(\varnothing, \mathcal{T})=\prod_{i} Y_{T^{(i)}}=\prod_{i} \prod_{J \in \mathcal{N}_{i}} Y_{J}=\prod_{J \in \mathcal{N}} Y_{J}=m(\varnothing, \mathcal{N})
$$

Theorem 13. Any $Y$-monomial is an integer linear combination of cluster $Y$-monomials. More precisely, for any multiset $\mathcal{S}$ with elements in $\mathcal{P}(V)$, the cluster $Y$-monomial $m(\varnothing, \mathcal{S})$ is an integer linear combination of cluster $Y$-monomials. Moreover, the cluster $Y$-monomials $m(\varnothing, \mathcal{R})$ in such a linear combination have that all vertices in $\mathcal{R}$ are vertices in $\mathcal{S}$.

Proof. We use induction on the number of vertices in $\mathcal{S}$. Assume, by induction hypothesis, that the result holds for any multiset $\mathcal{R}$ with elements in $\mathcal{P}(V)$ with a smaller number of vertices than $\mathcal{S}$.

Let $T=+_{I \in \mathcal{S}} I$, that is, the multiset containing the vertices that appear in the sets of $\mathcal{S}$, counting multiplicities. Let $\mathcal{T}$ be the multiset whose elements are $T^{(1)}, T^{(2)}, \cdots \in \mathcal{P}(V)$. By Lemma 12, the $Y$-monomial $m(\varnothing, \mathcal{T})$ is a cluster $Y$-monomial.

If $\mathcal{T}=\mathcal{S}$, then we are done because $m(\varnothing, \mathcal{S})=m(\varnothing, \mathcal{T})$ is a cluster $Y$-monomial. Otherwise, $\mathcal{T} \neq \mathcal{S}$. From Lemma 10, we know that

$$
\prod_{S \in \mathcal{S}} \sum_{f: S} \operatorname{nwt}(f)=\prod_{T \in \mathcal{T}} \sum_{f: T} \operatorname{nwt}(f) .
$$

Applying Lemma 11 to each term of both products, we obtain

$$
\prod_{S \in \mathcal{S}} \sum_{C_{S} \in \mathcal{C}_{S}} Y_{S \backslash C_{S}}=\prod_{T \in \mathcal{T}} \sum_{C_{T} \in \mathcal{C}_{T}} Y_{T \backslash C_{T}} .
$$

Rewriting the equation above in terms of cluster $Y$-monomials, we obtain

$$
\sum_{\substack{\left(C_{S}^{(1)}, C_{S}^{(2)}, \ldots\right) \\ \Psi \\ \mathcal{C}_{S}^{(1)} \times \mathcal{C}_{S}^{(2)} \times \ldots}} m\left(\varnothing,\left\{S^{(1)} \backslash C_{S}^{(1)}, \ldots\right\}\right)=\sum_{\substack{\left(C_{T}^{(1)}, C_{T}^{(2)}, \ldots\right) \\ \Psi}} m\left(\varnothing,\left\{T^{(1)} \backslash C_{T}^{(1)}, \ldots\right\}\right) .
$$

Note that when $\varnothing=C_{S}^{(1)}=C_{S}^{(2)}=\cdots$, the term in the sum in the left-hand side is the cluster $Y$-monomial $m(\varnothing, \mathcal{S})$. Therefore, the cluster $Y$-monomial $m(\varnothing, \mathcal{S})$ evaluates to

$$
\sum_{\substack{\left(C_{T}^{(1)}, C_{T}^{(2)}, \ldots\right)}} m\left(\varnothing,\left\{T^{(1)} \backslash C_{T}^{(1)}, \ldots\right\}\right)-\sum_{\substack{\left(C_{S}^{(1)}, C_{S}^{(2)}, \ldots\right) \\ \mathcal{C}_{T}^{(1)} \times \mathcal{C}_{T}^{(2)} \times \ldots}} m\left(\varnothing,\left\{S^{(1)} \backslash C_{S}^{(1)}, \ldots\right\}\right) .
$$

Note that each monomial in the expression above of the form $\pm m(\varnothing, \mathcal{R})$ satisfies either $\mathcal{R}=\mathcal{T}$ or $\mathcal{R}$ has a fewer number of vertices than $\mathcal{S}$. Hence, by Lemma 12 and by the induction hypothesis, each monomial $m(\varnothing, \mathcal{R})$ in the expression above is an integer linear combination of cluster $Y$-monomials, and therefore, the $Y$-monomial $m(\varnothing, \mathcal{S})$ is an integer combination of cluster $Y$-monomials.

Moreover, note that each monomial in the expression above of the form $\pm m(\varnothing, \mathcal{R})$ have the property that all vertices in $\mathcal{R}$ are vertices in $\mathcal{S}$. Consequently, by the induction hypothesis, the cluster $Y$-monomials $m(\varnothing, \mathcal{R})$ in the expansion of $m(\varnothing, \mathcal{S})$ are such that all vertices in $\mathcal{R}$ are vertices in $\mathcal{S}$.

Therefore, by induction, the result holds.

### 4.3 Cluster monomials span monomials

Theorem 14. Any monomial is a linear combination of cluster monomials over $R$. More precisely, for any multiset $U$ with elements in $V$ and any multiset $\mathcal{S}$ with elements in $\mathcal{P}(V)$, the cluster monomial $m(U, \mathcal{S})$ is a linear combination of cluster monomials over $R$.

Proof. We apply induction on the number of vertices in $\mathcal{S}$, counting multiplicities. Assume, by induction hypothesis, that the cluster monomial $m(T, \mathcal{R})$ is a linear combination over $R$ of cluster monomials whenever $\mathcal{R}$ has a smaller number of vertices than $\mathcal{S}$.

Assume there exists $v \in U$ such that $v \in S$ for some $S \in \mathcal{S}$. We can write

$$
m(U, \mathcal{S})=m(U-\{v\}, \mathcal{S}-\{S\}) \cdot X_{v} \cdot Y_{S}
$$

Apply Lemma 7 to $X_{v}$ and $Y_{S}$ to obtain

$$
X_{v} Y_{S}=\sum_{w \in S} \sum_{p}^{v \rightarrow s w} Y_{S \backslash p} X_{w}+\sum_{w \notin S} \sum_{p}^{v \rightarrow s w} Y_{S \backslash p} A_{w}
$$

Therefore,

$$
\begin{aligned}
m(U, \mathcal{S})= & \sum_{w \in S} \sum_{p}^{v \rightarrow s w} m(U-\{v\}+\{w\}, \mathcal{S}-\{S\}+\{S \backslash p\}) \\
& +\sum_{w \notin S} \sum_{p}^{v \rightarrow s w} A_{w} m(U-\{v\}, \mathcal{S}+\{S \backslash p\}-\{S\})
\end{aligned}
$$

This implies that we can write $m(U, \mathcal{S})$ as a linear combination over $R$ of cluster monomials of the form $m(T, \mathcal{R})$, where $\mathcal{R}$ has a smaller number of vertices than $\mathcal{S}$. Therefore, by the induction hypothesis, $m(U, \mathcal{S})$ is a linear combination over $R$ of cluster monomials.

Otherwise, assume all $v \in U$ satisfy $v \notin S$ for all $S \in \mathcal{S}$. We can write $m(U, \mathcal{S})=$ $m(U, \varnothing) m(\varnothing, \mathcal{S})$. Apply Theorem 13 to obtain that $m(\varnothing, \mathcal{S})$ is a linear combination over $R$ of cluster $Y$-monomials of the form $m(\varnothing, \mathcal{R})$, where the vertices in sets of $\mathcal{R}$ are a subset of the vertices in sets of $\mathcal{S}$. Explicitly,

$$
m(\varnothing, \mathcal{S})=\sum_{\mathcal{R}} c(\mathcal{R}) m(\varnothing, \mathcal{R})
$$

where the vertices in sets of $\mathcal{R}$ are a subset of the vertices in sets of $\mathcal{S}$. Therefore,

$$
m(U, \mathcal{S})=\sum_{\mathcal{R}} c(\mathcal{R}) m(U, \mathcal{R})
$$

and hence, $m(U, \mathcal{S})$ is a linear combination over $R$ of cluster monomials.

## 5 Further Directions and Concluding Remarks

Lam and Pylyavskyy [LP16b] also propose Conjecture 15, a stronger version of Theorem 14 that states not only that any monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster monomials over $R$, but also that the coefficients are nonnegative. Conjecture 15 is related to a positivity result for cluster algebras.

Conjecture 15 ([LP16b, Conjecture 7.3b]). Each monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster monomials over $R$ with nonnegative coefficients.

We propose a slightly different conjecture, that is about $Y$-monomials instead of monomials, a stronger version of Theorem 13 that states not only that any $Y$-monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster $Y$-monomials over $\mathbb{Z}$, but also that the coefficients are nonnegative integers.

Conjecture 16. Each Y-monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster $Y$-monomials with nonnegative integer coefficients.

Conjecture 16 implies Conjecture 15, and the proof of this implication is analogous to the proof of Theorem 14.

The ideas from this paper might be useful to prove Conjecture 16 (and hence Conjecture 15). For example, we can solve the special case when $\Gamma$ is a tree, as shown in Proposition 17.

Proposition 17. If $\Gamma$ is a tree, then each $Y$-monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster $Y$-monomials over $\mathbb{Z}$ with nonnegative integer coefficients. Moreover, each monomial of $\mathcal{A}_{\Gamma}$ is a linear combination of cluster monomials over $R$ with nonnegative coefficients.

Lam and Pylyavskyy [LP16b, Theorem 6.1] give a Ptolemy-like formula for expanding $Y_{I} Y_{J}$ into a positive integer linear combination of cluster monomials in the case where $\Gamma$ is a path. It is possible to extend this formula to the case where $\Gamma$ is a tree, as shown in Proposition 18. Proposition 18 is the key lemma used in the proof of Proposition 17.

Proposition 18. Let $\Gamma$ be a tree. Let $I, J \subset V$. A path from $I$ to $J$ is a list of vertices $v_{1}, v_{2}, \ldots, v_{k}$ such that $v_{1} \in I \backslash J, v_{k} \in J \backslash I$, and all other vertices are in $I \cap J$. Then, the expansion of $Y_{I} Y_{J}$ into cluster monomials is given by

$$
Y_{I} Y_{J}=\sum_{P} Y_{I \cup J \backslash P} Y_{I \cap J \backslash P},
$$

where the sum is taken over all families $P$ of disjoint paths from I to J.
Proposition 18 can be used to recover the Ptolemy-like formula for paths given in Lam and Pylyavskyy [LP16b, Theorem 6.1].

A proof of Proposition 18 can be derived using Lemma 9; the reader can consult the authors' report [DTW23] for a full proof of Proposition 18, as well as a proof of Proposition 17. It would be interesting to further generalize this formula to all graphs, or to a larger class of graphs.

We illustrate Proposition 18 by considering two examples of choices of a tree $\Gamma$, illustrated in Figure 3, and choices of subsets $I, J \subset V$.

(a) The path graph $P_{6}$, with six vertices, 1,2 ,
 $3,4,5$, and 6 , and five edges, $12,23,34,45$, and 56 .
(b) The tree with six vertices, $1,2,3,4,5$, and 6 , and five edges, $23,12,45,56$, and 52 .

Figure 3: Two examples of trees.

Example 1. Let $\Gamma=P_{n}$, the path graph with vertices $\{1,2, \ldots, n\}$. Figure 3a shows the path graph $P_{6}$. Let $I=\{1,2, \ldots, k\}$ and $J=\{l, l+1, \ldots, n\}$. If $k<l-1$, then there is no path from $I$ to $J$ and Proposition 18 implies that $Y_{I} Y_{J}=Y_{I \cup J}$, which is a cluster monomial.

If $k \geq l-1$, then the only path from $I$ to $J$ is $\{l-1, l, \ldots, k, k+1\}$. Therefore, $Y_{I} Y_{J}$ is expanded into two cluster monomials: one corresponding to the empty family of paths, and the other corresponding to the family containing the only path from $I$ to $J$. Explicitly, we have

$$
Y_{\{1, \ldots, k\}} Y_{\{l, \ldots, n\}}=Y_{\{1, \ldots, m\}} Y_{\{k+1, \ldots, l-1\}}+Y_{\{1, \ldots, l-1, k+1, \ldots, n\}} .
$$

Example 2. Let $\Gamma$ be the tree with vertex set $V=\{1,2,3,4,5,6\}$ and edge set $E=\{23,12$, $45,56,52\}$, as shown in Figure 3b. Let $I=\{1,2,4,5\}$ and let $J=\{2,3,5,6\}$. There are six families of disjoint paths from $I$ to $J$,

$$
\varnothing, \quad\{(1,2,3)\}, \quad\{(4,5,6)\}, \quad\{(1,2,5,6)\}, \quad\{(4,5,2,3)\}, \quad\{(1,2,3),(4,5,6)\} .
$$

Therefore, Proposition 18 implies that $Y_{I} Y_{J}$ is expanded into six cluster monomials, one for each family of disjoint paths. Explicitly,

$$
Y_{\{1,2,4,5\}} Y_{\{2,3,5,6\}}=Y_{\{1,2,3,4,5,6\}} Y_{\{2,5\}}+Y_{\{4,5,6\}} Y_{\{5\}}+Y_{\{1,2,3\}} Y_{\{2\}}+Y_{\{4,3\}}+Y_{\{1,6\}}+1
$$

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