CONVERGENCE OF STOCHASTIC INTEGRALS WITH APPLICATIONS TO TRANSPORT EQUATIONS AND CONSERVATION LAWS WITH NOISE

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Abstract. Convergence of stochastic integrals driven by Wiener processes W_n, with $W_n \to W$ almost surely in C_t, is crucial in analyzing SPDEs. Our focus is on the convergence of the form $\int_0^T V_n \,\mathrm{d}W_n \to \int_0^T V \,\mathrm{d}W$, where $\{V_n\}$ is bounded in $L^p(\Omega \times [0,T]; X)$ for a Banach space X and some finite $p > 2$. This is challenging when V_n converges to V *weakly* in the temporal variable. We supply convergence results to handle stochastic integral limits when strong temporal convergence is lacking. A key tool is a uniform mean L^1 time translation estimate on V_n, an estimate that is easily verified in many SPDEs. However, this estimate alone does not guarantee strong compactness of $(\omega, t) \mapsto V_n(\omega, t)$. Our findings, especially pertinent to equations exhibiting singular behavior, are substantiated by establishing several stability results for stochastic transport equations and conservation laws.

Contents

1. Introduction

We revisit the classical problem of proving convergence of stochastic integrals. Let W_n and W be Wiener processes, and V_n and V be predictable processes taking values in a Banach space, defined in respective filtrations for each n and at the limit. Suppose $V_n \to V$ and $W_n \to W$ in some topologies. We shall consider the question of the convergence

$$\int_0^T V_n \,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^T V \,\mathrm{d}W. \tag{1.1}$$

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The convergence of these integrals is essential in the stochastic compactness method, which is a fundamental component of many existence results for stochastic partial differential equations (SPDEs), see, e.g., [1, 4, 5]. Our goal is to refine some established results [1, 5] by focusing on scenarios where V_n does not exhibit strong temporal L^2 convergence. To achieve this, we introduce a critical assumption: *a uniform mean L^1 temporal translation estimate for* V_n, see (1.8) and (2.3). This assumption effectively compensates for the lack of strong convergence of $(\omega,t) \mapsto V_n(\omega,t)$. The proposed refinements are particularly relevant in the context of equations characterized by singular coefficients and/or solutions. As we will see, the L^1 translation estimate is straightforward to verify for many SPDEs.

The classical Skorokhod approach, which is based on nontrivial results concerning the tightness of probability measures and the almost sure representations of random variables, can assist in confirming strong temporal convergence of V_n. Subsequently, Lemma 2.1 from [5] can be applied to pass to the limit in the stochastic integrals. Let $\mathfrak{U}$, X be Hilbert spaces, and denote by $L_2(\mathfrak{U},X)$ the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to X. Let $\mathfrak{U}_1$ be a Hilbert space into which $\mathfrak{U}$ embeds, such that the embedding is Hilbert–Schmidt, and under which the cylindrical Wiener process becomes a Hilbert space valued Wiener process the sense of [4, Section 4.1.1]. Assuming that

$$\begin{aligned} &\text{(i)} \quad V_n \to V \text{ in probability in } L^2([0,T];L_2(\mathfrak{U},X)),\\ &\text{(ii)} \quad W_n \to W \text{ in probability in } C([0,T];\mathfrak{U}_1), \end{aligned} \tag{1.2}$$

Lemma 2.1 of [5] states that

$$\int_0^\cdot V_n \,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^\cdot V\,\mathrm{d}W \text{ in probability in } L^2([0,T];X). \tag{1.3}$$

In practice, V_n is often a deterministic map $G(u_n)$ of a solution u_n to the SPDE of which the stochastic integral is a term. The strong temporal convergence assumption (1.2)-(i) can often be obtained via the Skorokhod approach, if uniform statistical bounds can be obtained for u_n in spaces like $L^p([0,T];B_1)\cap W^{s,p}([0,T];B_{-1})$, where B_{-1} and B_1 are reflexive Banach spaces for which there is a Banach space B_0 giving $B_1 \Subset B_0 \subset B_{-1}$, and $p\in(1,\infty)$, $s\in(0,1)$ (see, e.g., [11, Theorem 2.1]).

The "martingale identification" approach offers an alternative to passing to the limit in SPDE sequences, as detailed in [4, Pages 229–231] (for example). This approach, differing from the one based on [5, Lemma 2.1], bypasses convergence theorems for stochastic integrals. Instead, it focuses on computing the limits of the deterministic integral terms in the equations and the quadratic variation of the stochastic integrals, leveraging a representation theorem for martingales to establish the existence of a probabilistic weak solution. However, a martingale representation theorem is not always available in the SPDE context. A more recent method [2, 21] utilizes only basic martingale and stochastic integral properties, avoiding the martingale representation theorem. This method is particularly effective in "quasi-Polish" spaces like $C([0,T];X-w)$ (see [17]) and has been much applied in subsequent research. Nonetheless, while these spaces adopt a weak spatial topology, their temporal topology remains strong (see also Remark 4.4 herein).

While the Skorokhod approach is widely adopted, it can be technically challenging and often necessitates detailed and lengthy proofs of convergence. Moreover, there are examples where the practicality of this approach might be less evident. This may occur with SPDEs that are too singular, hindering solutions from being uniformly bounded or converging in the more traditional spaces aforementioned. For some examples, see [13, 18].

The key observation of our paper is that intricate Skorokhod-type approaches can be bypassed when the primary goal is simply to pass to the limit in the governing SPDEs. This will be achieved through refinements of [5, Lemma 2.1], using a mean L^1 temporal translation estimate—see (1.8) and (2.3)—as a substitute for the strong temporal convergence condition (1.2)-(i).

To provide context for our results, we will explore two examples. Variations of these examples will be examined in detail in Sections 5 and 6, where they will serve as applications of our findings. We begin by considering a sequence of transport (continuity) equations characterized by multiplicative k-valued Brownian noise:

$$\mathrm{d}u_n + \operatorname{div}(bu_n)\,\mathrm{d}t = f\,\mathrm{d}t + \sigma(u_n)\,\mathrm{d}W_n, \quad u_n(0) = u_{0,n}, \tag{1.4}$$

where the equations are interpreted in the weak sense with respect to the spatial variable x and are evaluated pointwise with respect to the temporal variable t, being formulated as Itô integral equations.

The strong stability of weak solutions to deterministic transport equations was first addressed in the seminal work of DiPerna and Lions [7]. A natural question is whether strong stability results à la [7] can be established for stochastic transport equations such as (1.4). To keep the presentation simple and focused on the core issue, we will assume that the "data" $(b, \operatorname{div} b, f, u_{0,n})$ of (1.4) possess sufficiently high integrability. See [13] for a more "singular" example. We assume that $u_{0,n}$ converges strongly to u_0 in $L^p_{\omega,t,x}$ for some $p > 2$. The primary point of departure is the assumption that the solutions u_n of the stochastic transport equations (1.4) converge weakly in $L^p_{\omega,t,x}$ to a limit u.

In the deterministic scenario (where $\sigma \equiv 0$), the definition of weak convergence would directly allow us to formulate a transport equation for the limit object u. This forms the starting point for any strong stability analysis. However, this trivial step becomes less straightforward in the stochastic case (even when the noise coefficient σ is a linear function). If σ is globally Lipschitz, then the L^p bound on u_n allows us to infer that there exists a non-relabelled subsequence such that $\sigma(u_n) \rightharpoonup \overline{\sigma}$ in $L^p_{\omega,t,x}$, converging weakly to some limit $\overline{\sigma}$. However, even when simply attempting to confirm that the weak limit u solves the limiting equation (replacing $\sigma(u)$ with $\overline{\sigma}$), it becomes crucial to understand the effects of this weak convergence on the behavior of the associated stochastic integrals.

As in [7], our aim extends beyond merely establishing weak convergence; we seek to demonstrate that u_n converges strongly, thereby ensuring that $\overline{\sigma} = \sigma(u)$. To achieve this transition from weak to strong convergence, we employ a well-known propagation of compactness strategy. This involves formulating equations for nonlinear compositions $\eta(u_n)$ of u_n, as well as for their respective weak limits $\overline{\eta}$. This leads to us to stochastic integrals of the form

$$\int_0^T V_n\,\mathrm{d}W_n, \quad V_n := \int \varphi(x)\vartheta(u_n)\,\mathrm{d}x, \quad \varphi \in C_c^\infty, \tag{1.5}$$

where $\vartheta(u)$ is a nonlinear function given by $\vartheta(u) = \eta'(u)\sigma(u)$. From the uniform L^p bound on u_n, we may assume that $\vartheta(u_n)$ converges weakly in $L^{p_\vartheta}_{\omega,t,x}$, for some exponent p_ϑ dependent on p and the growth of ϑ. Consequently, our analysis is directed towards examining the convergence of the stochastic integrals (1.5), under the condition of weak convergence $V_n \rightharpoonup V$ in the space $L^{p_\vartheta}_{\omega,t}$. Even when we replace

$\sigma(u_n)$ with a strongly converging additive coefficient σ_n, proving convergence is still problematic. The issue lies with the nonlinear term $\vartheta(u_n) = \eta'(u_n)\sigma_n$, which remains weakly convergent.

Our second example entails a sequence of stochastic conservation laws of the following form:

$$du_n + \nabla \cdot F_n(u_n)\,\mathrm{d}t = \sigma_n(u_n)\,\mathrm{d}W_n. \tag{1.6}$$

In the so-called kinetic formulation [20], adapted to the stochastic setting in [6], the subgraphs $\chi_n(t,x,\xi) = \mathbb{1}_{\{\xi < u_n(t,x)\}}$ of u_n, where u_n are entropy solutions in the sense of Kružkov, satisfy the following kinetic equations:

$$\begin{aligned} &\mathrm{d}\chi_n + F_n'(\xi)\cdot\nabla_x \chi_n\,\mathrm{d}t - \sigma_n(\xi)\,\partial_\xi \chi_n\,\mathrm{d}W_n \\ &\qquad - \frac{1}{2}\partial_\xi \left(|\sigma_n(\xi)|^2\,\partial_\xi \chi_n\right)\mathrm{d}t = \partial_\xi m_n\,\mathrm{d}t, \quad \chi_n(0) = \chi_{n,0}, \end{aligned} \tag{1.7}$$

where the kinetic defects m_n are random variables taking values in the space of measures. The objective is to demonstrate that a $L^\infty_{\omega,t,x,\xi}$ weak-$\star$ limit χ of χ_n serves as a weak solution to the kinetic equation in its limiting form. This equation is formally deduced by omitting the n subscripts in (1.7). Again this necessitates the examination of stochastic integrals, where the integrands exhibit weak convergence in (ω,t). This issue, when the Wiener process W_n varies with the index n, is relevant in stochastic conservation laws with discontinuous flux, where there is a need to establish strong compactness through microlocal defect measures [18]. Furthermore, this problem has been encountered in the exploration of numerical approximations for stochastic conservation laws [8, 9].

Motivated by these two examples, we are therefore particularly interested in the general situation described by (1.1) when there is a lack of temporal compactness, i.e., when there is no strong topology serving the temporal variable in which the convergence $V_n \to V$ occurs.

Without an assumption such as (1.2)-(i), it is still feasible to use the SPDEs to derive a *mean L^1-translation estimate in the time variable t*, weak in the spatial variable. With $v_n = \sigma(u_n)$, where u_n solves (1.4), we have that for any $\beta \in C_c^\infty$,

$$\mathbb{E}\int_h^T \left| \int \beta(x)\left(v_n(t,x) - v_n(t-h,x)\right)\mathrm{d}x \right| \mathrm{d}t = o_{h\downarrow 0}(1), \tag{1.8}$$

uniformly in n. Similarly, for $v_n = \sigma_n(\xi)\,\partial_\xi \chi_n$ in (1.7), we have (1.8) with the spatial variable x replaced by the spatio-kinetic variable (x,ξ). Without ω-dependence in v_n, the estimate (1.8) implies strong L^1 compactness of $t \mapsto V_n = \int \beta(x) v_n(t)\,\mathrm{d}x$ (see (1.5) with $\eta' = 1$) via the Kolmogorov–Riesz compactness theorem. Of course with ω present, (1.8) does not imply strong compactness of $(\omega,t) \mapsto V_n(\omega,t)$.

The key point here is that the mean L^1 temporal translation estimate (1.8) is readily available for many SPDEs. For (1.4) and (1.7), this can be shown by deriving the equation for $\mathrm{d}v_n$ from the equation for $\mathrm{d}u_n$ or $\mathrm{d}\chi_n$ and integrating temporally over $[t-h,t]$ and then once again over $[h,T]$ after taking absolute values (see Lemma 5.3 and Sections 5 and 6). This double integration in time heuristically recovers some "temporal compactness" in the sense of (1.8). The estimate (1.8) remains valid even when the source term f in (1.4) is an L^1 function, or in cases where it is a measure, such as the defect term $\partial_\xi m_n$ in (1.7) (see [13, 18]).

Now assume we have the L^1 translation estimate as specified in (1.8), coupled with the weak convergence $v_n \rightharpoonup v$ in $L^p_{\omega,t,x}$ for some $p > 2$. The latter implies that for $\varphi \in C_c^\infty$, the processes $V_n = \int \varphi v_n \,\mathrm{d}x$ converge weakly in $L^p(\Omega \times [0,T])$. Our primary result (Theorem 2.1) then implies the convergence

$$\int_0^T \int \varphi(x) v_n \,\mathrm{d}x \,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^T \int \varphi(x) v \,\mathrm{d}x \,\mathrm{d}W \quad \text{in } L^2(\Omega). \tag{1.9}$$

Note that the mode of convergence in (1.3) is *weak* in $L^2(\Omega)$. This is often sufficient when studying SPDEs, as weak convergence in $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbb{P})$ allows us to integrate a sequence of SPDEs with respect to $\mathbb{P}$ against any element of $L^2(\Omega)$ and conclude that the limit SPDE holds $\mathbb{P}$–a.s., which is generally the desired outcome. We will also consider the stronger assumption $v_n \rightharpoonup v$ a.s. in $L^p_{t,x}$, in which case the convergence in (1.9) becomes strong in $L^2(\Omega)$. We will demonstrate (1.9) by carefully modifying the proof of [5, Lemma 2.1].

The remaining part of the paper is organised as follows: In Section 2, we prove our main result (Theorem 2.1). This is done in the setting of Hilbert space-valued stochastic integrals. In Section 3, we provide an example and adapt Theorem 2.1 slightly to integrals taking values in a non-reflexive Banach space. In Section 4, we present corollaries in which the assumptions on the convergence of $V_n \to V$ are further refined. In the last two sections, we give applications of our convergence theorems to stochastic transport equations (Section 5) and stochastic conservation laws (Section 6).

2. Main convergence result

In this section, we will prove our primary convergence theorem.

Theorem 2.1 (Weak $L^2(\Omega)$ convergence of stochastic integrals)**.** *Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a real, separable Hilbert space. For each $n \in \mathbb{N}$, let W_n be an $\mathbb{R}^k$-valued Wiener process on the filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t\in[0,T]}, \mathbb{P}\big)$, and W an $\mathbb{R}^k$-valued Wiener process on $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\big)$.*

Fix $p > 2$. For each $n \in \mathbb{N}$, let V_n be an $X^{m\times k}$-valued $\{\mathcal{F}_t^n\}$-predictable process, and V be an $X^{m\times k}$-valued $\{\mathcal{F}_t\}$-predictable process. Suppose

$$V_n \xrightharpoonup{n\uparrow\infty} V \quad \text{in } L^p(\Omega \times [0,T]; X^{m\times k}). \tag{2.1}$$

We further require that

$$W_n \xrightarrow{n\uparrow\infty} W \quad \text{in } C([0,T]; \mathbb{R}^k),\ \mathbb{P}\text{–a.s.} \tag{2.2}$$

Suppose for a given $\beta \in X^*$ that the following mean L^1 temporal translation estimate holds:

$$\mathbb{E} \int_h^T |\langle \beta, V_n\rangle_{X^*,X}(t) - \langle \beta, V_n\rangle_{X^*,X}(t-h)| \,\mathrm{d}t \xrightarrow{h\downarrow 0} 0 \quad \text{uniformly in } n. \tag{2.3}$$

Then for any $t \in [0,T]$,

$$\int_0^t \big\langle \beta, V_n \big\rangle_{X^*,X} \,\mathrm{d}W_n \xrightharpoonup{n\uparrow\infty} \int_0^t \langle \beta, V\rangle_{X^*,X} \,\mathrm{d}W \quad \text{in } L^2(\Omega; \mathbb{R}^m).$$

Proof. We carry out the proof for $t = T$ only, remarking that each step is valid for any $t \in [0,T]$ in place of T. Without loss of generality, we identify X with its dual X^* and shall often drop the subscript on its angled duality bracket, which is applied component-wise to V_n and V. We also use single line delimiters (absolute value delimiters) for the Euclidean norm of any finite dimension-valued object. We begin by noting that the weak convergence assumption (2.1), implies

$$\langle \beta, V_n \rangle \xrightharpoonup{n\uparrow\infty} \langle \beta, V \rangle \quad \text{in } L^p(\Omega \times [0,T]; \mathbb{R}^{m\times k}). \tag{2.4}$$

Moreover, we get the uniform bound:

$$\begin{aligned} \|\langle \beta, V_n \rangle\|_{L^p(\Omega\times[0,T];\mathbb{R}^{m\times k})}^p &= \mathbb{E} \int_0^T |\langle \beta, V_n(t)\rangle|^p \,\mathrm{d}t \\ &\le \|\beta\|_X^p \,\mathbb{E} \int_0^T \|V_n(t)\|_{X^{m\times k}}^p \,\mathrm{d}t \\ &= \|\beta\|_X^p \,\mathbb{E}\, \|V_n\|_{L^p([0,T];X^{m\times k})}^p \lesssim 1. \end{aligned} \tag{2.5}$$

This uniform bound in turn gives us

$$\mathbb{E}\, \|\langle \beta, V \rangle\|_{L^p([0,T];\mathbb{R}^{m\times k})}^p \lesssim 1. \tag{2.6}$$

Obviously, (2.5) and (2.6) also imply similar uniform bounds for $\langle \beta, V_n\rangle$ and $\langle \beta, V\rangle$ in $L^{p_\omega}(\Omega; L^{p_t}([0,T];\mathbb{R}^{m\times k}))$ with $p_\omega, p_t \le p$. We shall often use these bounds with $p_t = 2$.

We divide the remainder of the proof into five steps.

1. Temporal regularisation.

For $\rho > 0$, let $\mathcal{R}(t) \ge 0$ be a smooth function supported on $\mathbb{R}_{\ge 0}$ such that $\int_0^\infty \mathcal{R}(t)\,\mathrm{d}t = 1$. Set $\mathcal{R}_\rho(t) = \rho^{-1}\mathcal{R}_\rho(t/\rho)$. From the properties of $\mathcal{R}_\rho(t)$,

$$\forall \delta > 0, \qquad \int_0^\delta \mathcal{R}_\rho(t)\,\mathrm{d}t \xrightarrow{\rho\downarrow 0} 1. \tag{2.7}$$

We will also use $\mathcal{R}_\rho$ to denote the (one-sided) temporal regularisation operator

$$\mathcal{R}_\rho f(t) = \int_0^t \mathcal{R}_\rho(t-s) f(s)\,\mathrm{d}s, \quad f \in L^1([0,T]), \quad \rho > 0, \tag{2.8}$$

for $t \in [0,T]$. Set $\widetilde{\mathcal{R}}_\rho(t) := \mathcal{R}_\rho(-t)$ and also

$$\widetilde{\mathcal{R}}_\rho f(s) := \int_s^T \widetilde{\mathcal{R}}_\rho(s-t) f(t)\,\mathrm{d}t, \quad s \in [0,T].$$

Observe that for $f, g \in L^1([0,T])$,

$$\int_0^T \mathcal{R}_\rho f(t)\, g(t)\,\mathrm{d}t = \int_0^T f(t)\, \widetilde{\mathcal{R}}_\rho g(t)\,\mathrm{d}t,$$

which follows from Fubini's theorem. We can also derive

$$\int_0^T \partial_t \mathcal{R}_\rho f(t)\, g(t)\,\mathrm{d}t = -\int_0^T f(t)\, \partial_t \widetilde{\mathcal{R}}_\rho g(t)\,\mathrm{d}t. \tag{2.9}$$

By standard convolution arguments,

$$\begin{aligned} &\|\mathcal{R}_\rho f\|_{L^r([0,T])} \le \|f\|_{L^r([0,T])}, \quad \|f - \mathcal{R}_\rho f\|_{L^r([0,T])} \xrightarrow{\rho\downarrow 0} 0, \\ &\left\|\widetilde{\mathcal{R}}_\rho f\right\|_{L^r([0,T])} \le \|f\|_{L^r([0,T])}, \quad \left\|f - \widetilde{\mathcal{R}}_\rho f\right\|_{L^r([0,T])} \xrightarrow{\rho\downarrow 0} 0, \end{aligned} \tag{2.10}$$

for any $f \in L^r([0,T])$ with $r \in [1,\infty)$.

For any fixed $\rho > 0$, and any $\ell \in \mathbb{N}$, convolution against $\partial_t^\ell \mathcal{R}_\rho$ is also a bounded opeator $L_t^p \to L_t^p$ by the smoothness of $\mathcal{R}_\rho$ and by standard convolution arguments.

Therefore $\mathcal{R}_\rho\langle\beta, V_n\rangle$ additionally lies in the space $L^p(\Omega; C^\infty([0,T];\mathbb{R}^{m\times k}))$ (and similarly for $\tilde{\mathcal{R}}_\rho$ in place of $\mathcal{R}_\rho$).

We now consider the sequence $\int_0^T \langle\beta, V_n\rangle \,\mathrm{d}W_n \in L^p(\Omega;\mathbb{R}^m)$ and its proposed limit by decomposing the difference between any element in the sequence and the proposed limit as follows:

$$I(n) = \int_0^T \langle\beta, V_n\rangle \,\mathrm{d}W_n - \int_0^T \langle\beta, V\rangle \,\mathrm{d}W = I_1(\rho,n) + I_2(n,\rho) + I_3(\rho), \tag{2.11}$$

where

$$\begin{aligned} I_1(\rho,n) &= \int_0^T \langle\beta, V_n\rangle \,\mathrm{d}W_n - \int_0^T \mathcal{R}_\rho\langle\beta, V_n\rangle \,\mathrm{d}W_n, \\ I_2(n,\rho) &= \int_0^T \mathcal{R}_\rho\langle\beta, V_n\rangle \,\mathrm{d}W_n - \int_0^T \mathcal{R}_\rho\langle\beta, V\rangle \,\mathrm{d}W, \\ I_3(\rho) &= \int_0^T \mathcal{R}_\rho\langle\beta, V\rangle \,\mathrm{d}W - \int_0^T \langle\beta, V\rangle \,\mathrm{d}W. \end{aligned}$$

2. The strong regularisation limit of $I_1(\rho,n)$ as $\rho\downarrow 0$.
By the Itô isometry,

$$\begin{aligned} \mathbb{E}I_1^2 &= \mathbb{E}\int_0^T |\langle\beta, V_n\rangle - \mathcal{R}_\rho\langle\beta, V_n\rangle|^2 \,\mathrm{d}t \\ &= \mathbb{E}\int_0^T \left|\langle\beta, V_n\rangle(t) - \int_0^t \mathcal{R}_\rho(t-s)\langle\beta, V_n\rangle(s)\,\mathrm{d}s\right|^2 \,\mathrm{d}t \\ &\le \mathbb{E}\int_0^T \left|\int_0^t \mathcal{R}_\rho(t-s)\big(\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(s)\big)\,\mathrm{d}s\right|^2 \,\mathrm{d}t \\ &\quad + \mathbb{E}\int_0^T \left(1 - \int_0^t \mathcal{R}_\rho(t-s)\,\mathrm{d}s\right)^2 |\langle\beta, V_n(t)\rangle|^2 \,\mathrm{d}t \\ &=: J_1 + J_2. \end{aligned} \tag{2.12}$$

First, by a change of variable $s\mapsto t-s$, then by Jensen's/Hölder's inequality,

$$\begin{aligned} J_1 &= \mathbb{E}\int_0^T \left|\int_0^t \mathcal{R}_\rho(s)\big(\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(t-s)\big)\,\mathrm{d}s\right|^2 \,\mathrm{d}t \\ &\le \mathbb{E}\int_0^T \underbrace{\|\mathcal{R}_\rho\|_{L^1([0,t])}}_{\le 1} \int_0^t \mathcal{R}_\rho(s)\,|\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(t-s)|^2 \,\mathrm{d}s\,\mathrm{d}t \\ &\le \mathbb{E}\int_0^T \int_0^T \mathbb{1}_{\{t\ge\rho u\}}\mathcal{R}_\rho(\rho u)\,|\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(t-\rho u)\rangle|^2 \,\mathrm{d}\,(\rho u)\,\mathrm{d}t \\ &= \int_0^\infty \mathbb{1}_{\{u\le T/\rho\}}\rho\mathcal{R}_\rho(\rho u)\,\mathbb{E}\int_{\rho u}^T |\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(t-\rho u)|^2 \,\mathrm{d}t\,\mathrm{d}u. \end{aligned}$$

By construction, $\rho\mathcal{R}_\rho(\rho u) = \mathcal{R}(u) \lesssim 1$ as $\rho\downarrow 0$. By the L^1 translation estimate (2.3), the integrand of the $\mathrm{d}u$ integral above tends to zero a.e. in u, uniformly in n, via the interpolation estimate $\|F\|^2_{L^2([0,T];\mathbb{R}^{m\times k})} \le \|F\|^{p/(p-1)}_{L^p([0,T];\mathbb{R}^{m\times k})}\,\|F\|^{(p-2)/(p-1)}_{L^1([0,T];\mathbb{R}^{m\times k})}$:

$$\mathbb{E}\int_{\rho u}^T |\langle\beta, V_n\rangle(t) - \langle\beta, V_n\rangle(t-\rho u)|^2 \,\mathrm{d}t$$

$$\leq \Big[\left(\mathbb{E}\,\|\langle\beta,V_n\rangle\|^p_{L^p([0,T];\mathbb{R}^{m\times k})}\right)^{1/(p-1)} + \left(\mathbb{E}\,\|\langle\beta,V\rangle\|^p_{L^p([0,T];\mathbb{R}^{m\times k})}\right)^{1/(p-1)} \Big]$$
$$\times \left(\mathbb{E}\int_{\rho u}^T |\langle\beta,V_n\rangle(t) - \langle\beta,V_n\rangle(t-\rho u)|\,\mathrm{d}t\right)^{(p-2)/(p-1)} \xrightarrow{\rho\downarrow 0} 0,$$

where we have used the bounds (2.5) and (2.6). Moreover, for any $\theta > 1$,

$$\int_0^{T/\rho} \left|\rho\mathcal{R}_\rho(\rho u)\mathbb{E}\int_{\rho u}^T |\langle\beta,V_n\rangle(t) - \langle\beta,V_n\rangle(t-\rho u)|^2\,\mathrm{d}t\right|^\theta \mathrm{d}u$$
$$= \int_0^{T/\rho} \rho^\theta \mathcal{R}_\rho^\theta(\rho u)\left(\mathbb{E}\int_0^T |\langle\beta,V_n\rangle(t) - \langle\beta,V_n\rangle(t-\rho u)|^2\,\mathrm{d}t\right)^\theta \mathrm{d}u$$
$$\lesssim_\theta \int_0^{T/\rho} \mathcal{R}^\theta(u)\,\mathrm{d}u \left(\mathbb{E}\int_0^T |\langle\beta,V_n\rangle(t)|^2\,\mathrm{d}t\right)^\theta \overset{(2.5)}{\lesssim} 1.$$

Therefore, by the Vitali convergence theorem (see, e.g., [24, page 94]), we have

$$J_1 \xrightarrow{\rho\downarrow 0} 0, \quad \text{uniformly in } n.$$

For any $t > \delta > 0$, by (2.7), we can estimate

$$1 - \int_0^t \mathcal{R}_\rho(t-s)\,\mathrm{d}s \leq 1 - \int_{t-\delta}^t \mathcal{R}_\rho(t-s)\,\mathrm{d}s = 1 - \int_0^\delta \mathcal{R}_\rho(s)\,\mathrm{d}s = o_{\rho\downarrow 0}(1),$$

and so

$$\left(\int_0^T \left|1 - \int_0^t \mathcal{R}_\rho(t-s)\,\mathrm{d}s\right|^{2p/(p-2)} \mathrm{d}t\right)^{(p-2)/p}$$
$$\leq \left(\delta^{2p/(p-2)} + \int_\delta^T + \left|1 - \int_0^t \mathcal{R}_\rho(t-s)\,\mathrm{d}s\right|^{2p/(p-2)} \mathrm{d}t\right)^{(p-2)/p} \lesssim \left(\delta^2 + o_{\rho\downarrow 0}(1)\right).$$

Using Hölder's inequality, then,

$$J_2 \lesssim_T \left(\delta^2 + o_{\rho\downarrow 0}(1)\right)\left(\mathbb{E}\,\|\langle\beta,V_n\rangle\|^p_{L^p([0,T];\mathbb{R}^{m\times k})}\right)^{2/p},$$

which tends to 0 as $\rho \downarrow 0$ by choosing $\delta = \rho$ and using the uniform bound implied by (2.5). And we find that $I_1(\rho,n) \xrightarrow{\rho\downarrow 0} 0$ in $L^2(\Omega;\mathbb{R}^m)$ as $\rho \downarrow 0$, uniformly in n.

3. *The strong regularisation limit of $I_3(\rho)$ as $\rho \downarrow 0$.*
By (2.6), $\langle\beta,V\rangle \in L^2([0,T];\mathbb{R}^{m\times k})$, $\mathbb{P}$–a.s. (since $p > 2$). Hence,

$$\int_0^T |\langle\beta,V\rangle - \mathcal{R}_\rho\langle\beta,V\rangle|^2\,\mathrm{d}t \xrightarrow{\rho\downarrow 0} 0, \quad \mathbb{P}\text{–a.s.}$$

By Hölder's inequality and Young's convolution inequality on Bochner spaces (see, e.g., [16, Proposition 1.2.5, Lemma 1.2.30]), , we obtain

$$\begin{aligned}\int_0^T |\langle\beta, V\rangle - \mathcal{R}_\rho\langle\beta, V\rangle|^2 \,\mathrm{d}t &\lesssim \int_0^T |\langle\beta, V\rangle|^2 + |\mathcal{R}_\rho\langle\beta, V\rangle|^2 \,\mathrm{d}t \\ &\leq \left(1 + \|\mathcal{R}_\rho\|^2_{L^1([0,T])}\right) \int_0^T |\langle\beta, V\rangle|^2 \,\mathrm{d}t \\ &\lesssim \int_0^T |\langle\beta, V\rangle|^2 \,\mathrm{d}t \overset{(2.6)}{\in} L^1(\Omega).\end{aligned} \tag{2.13}$$

Therefore, by Lebesgue's dominated convergence theorem (in the ω variable),

$$\mathbb{E}\int_0^T |\langle\beta, V\rangle - \mathcal{R}_\rho\langle\beta, V\rangle|^2 \,\mathrm{d}t \xrightarrow{\rho\downarrow 0} 0. \tag{2.14}$$

By the Itô isometry, the strong convergence $I_3(\rho) \xrightarrow{\rho\downarrow 0} 0$ in $L^2(\Omega;\mathbb{R}^m)$ then follows.

4. The weak limit of $I_2(n,\rho)$ as $n \uparrow \infty$.

Since $\mathcal{R}_\rho\big\langle\beta, V_n\big\rangle$ and $\mathcal{R}_\rho\langle\beta, V\rangle$ are almost surely smooth in time (for each fixed $\rho > 0$), they have zero quadratic variation. Moreover, their cross-variation with any process with finite quadratic variation must also be zero. This allows us to apply integration by parts to each integral in $I_2(n,\rho)$ to obtain:

$$I_2(n,\rho) = I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4},$$

where

$$\begin{aligned} I_{2,1} &= -\int_0^T \left(\partial_t\mathcal{R}_\rho\big\langle\beta, V_n\big\rangle - \partial_t\mathcal{R}_\rho\langle\beta, V\rangle\right) W_n \,\mathrm{d}t, \\ I_{2,2} &= \mathcal{R}_\rho\big\langle\beta, V_n - V\big\rangle(T)\, W_n(T), \\ I_{2,3} &= \int_0^T \partial_t\mathcal{R}_\rho\langle\beta, V\rangle \left(W - W_n\right) \mathrm{d}t, \\ I_{2,4} &= \mathcal{R}_\rho\langle\beta, V\rangle(T) \left(W_n(T) - W(T)\right). \end{aligned} \tag{2.15}$$

We aim to show that for any $Y \in L^2(\Omega)$, the $n \to \infty$ limit of $\mathbb{E}[Y I_{2,1}]$ is zero. (It is sufficient to take a scalar-valued Y because weak convergence in $\mathbb{R}^m$ is equivalent to element-wise convergence.) Using the property (2.9), we can write $I_{2,1}$ as

$$I_{2,1} = \int_0^T \big\langle\beta, V - V_n\big\rangle \partial_t\widetilde{\mathcal{R}}_\rho W_n \,\mathrm{d}t.$$

We first recall two facts. One, using a maximal inequality (see, e.g., [23, Proposition II.1.8]), one obtains the moment estimate $\mathbb{E}\sup_{t\in[0,T]} |W_n(t)|^q \lesssim_{T,q} 1$, for any finite q. Therefore, by the a.s. convergence (2.2) and Vitali's convergence theorem,

$$\mathbb{E}\sup_{t\in[0,T]} |W(t) - W_n(t)|^q \xrightarrow{n\uparrow\infty} 0, \quad \text{for any finite } q. \tag{2.16}$$

Two, the product of weakly and strongly converging sequences converge weakly.

Due to the strong convergence (2.16), for fixed $\rho > 0$, $\partial_t\widetilde{\mathcal{R}}_\rho W_n \xrightarrow{n\uparrow\infty} \partial_t\widetilde{\mathcal{R}}_\rho W$ in $L^q\big(\Omega; C([0,T];\mathbb{R}^k)\big)$ for any finite q. Therefore, the product $Y\partial_t\widetilde{\mathcal{R}}_\rho W_n$ converges strongly in $L^{p'}\big(\Omega\times[0,T];\mathbb{R}^k\big)$ for any $p' < 2$. Choosing p' as the Hölder conjugate of p, $p' = \frac{p}{p-1} < 2$, which is possible as $p > 2$, the weak convergence (2.4) implies that $\mathbb{E}\left[Y I_{2,1}\right]$ converges to zero as $n \to \infty$.

We will now examine the term $I_{2,2}$ of (2.15). By (2.4), the weak convergence of $\big\langle \beta, V_n - V\big\rangle$ in the temporal variable t implies pointwise convergence of the temporally regularised object

$$\mathcal{R}_\rho\big\langle \beta, V_n - V\big\rangle \xrightarrow{n\uparrow\infty} 0 \quad \text{in } L^p(\Omega;\mathbb{R}^{m\times k}), \text{ pointwise in } t.$$

Additionally, as noted in (2.16), W_n converges strongly to W in $L^q(\Omega;C([0,T];\mathbb{R}^{m\times k}))$ for any finite q. Therefore, we can conclude that

$$I_{2,2} = \mathcal{R}_\rho\big\langle \beta, V_n - V\big\rangle(T)W_n(T) \xrightarrow{n\uparrow\infty} 0,$$

weakly in $L^{p-\kappa}(\Omega;\mathbb{R}^m)$ for any $0<\kappa\le p-2$, and hence weakly in $L^2(\Omega;\mathbb{R}^m)$ since $p>2$.

Recall the bound (2.6). For any $2<r<p$, let $\bar p<\infty$ denote the Hölder conjugate of $r/2$. By Young's convolution inequality and Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left|I_{2,3}\right|^2 &\le \left\|\partial_t\mathcal{R}_\rho\right\|^2_{L^1([0,T])}\left(\mathbb{E}\left\|\langle\beta,V\rangle\right\|^r_{L^1([0,T];\mathbb{R}^{m\times k})}\right)^{2/r}\left(\mathbb{E}\left\|W-W_n\right\|^{2\bar p}_{C([0,T];\mathbb{R}^k)}\right)^{1/\bar p}\\
&\overset{(2.6)}{\lesssim_\rho}\left(\mathbb{E}\left\|W-W_n\right\|^{2\bar p}_{C([0,T];\mathbb{R}^k)}\right)^{1/\bar p}\xrightarrow{n\uparrow\infty} 0.
\end{aligned}\tag{2.17}$$

Similarly, using

$$\begin{aligned}
\left|\mathcal{R}_\rho\langle\beta,V\rangle(T)\right|^r &= \left|\int_0^T \mathcal{R}_\rho(T-s)\langle\beta,V\rangle(s)\,\mathrm{d}s\right|^r\\
&\le \left\|\mathcal{R}_\rho\right\|^r_{L^\infty([0,T])}\left\|\langle\beta,V\rangle\right\|^r_{L^1([0,T];\mathbb{R}^{m\times k})},
\end{aligned}$$

we find that

$$\begin{aligned}
\mathbb{E}\left|I_{2,4}\right|^2 &\le \left\|\mathcal{R}_\rho\right\|^r_{L^\infty([0,T])}\left(\mathbb{E}\left\|\langle\beta,V\rangle\right\|^r_{L^1([0,T];\mathbb{R}^{m\times k})}\right)^{2/r}\left(\mathbb{E}\left\|W-W_n\right\|^{2\bar p}_{C([0,T];\mathbb{R}^k)}\right)^{1/\bar p}\\
&\overset{(2.6)}{\lesssim_\rho}\left(\mathbb{E}\left\|W-W_n\right\|^{2\bar p}_{C([0,T];\mathbb{R}^k)}\right)^{1/\bar p}\xrightarrow{n\uparrow\infty} 0.
\end{aligned}\tag{2.18}$$

The strong convergences (2.17) and (2.18) imply that for $Y\in L^2(\Omega)$,

$$\mathbb{E}\big[YI_{2,3}\big],\ \mathbb{E}\big[YI_{2,4}\big]\xrightarrow{n\uparrow\infty} 0.$$

And in summary, our findings show that

$$I_2(n,\rho)\xrightarrow{n\uparrow\infty} 0 \quad \text{in } L^2(\Omega;\mathbb{R}^m), \text{ for any fixed } \rho>0.$$

5. Conclusion.

Returning to (2.11), testing against an arbitrary but fixed $Y\in L^2(\Omega)$, we can make

$$\mathbb{E}\big[YI(n)\big]=\mathbb{E}\Big[Y\left(I_1(\rho,n)+I_2(n,\rho)+I_3(\rho)\right)\Big]$$

arbitrarily small by first picking a sufficiently small ρ (uniformly in n) and then selecting a sufficiently large n (depending on the chosen ρ). More precisely, we pick some small ρ_0 such that for any $\rho\le\rho_0$ the terms $I_1(\rho,n)$ and $I_3(\rho)$ become small, uniformly in n. Then we select an integer $N=N(\rho_0,Y)$ such that for any $n\ge N$ the term $I_2(n,\rho_0)$ becomes small. □

Remark 2.2. *We exhibit two examples to show the significance of the uniform L^1 temporal translation estimate condition (2.3) and the optimality of $p > 2$.*

Example 2.2.1. Define the probability space to be $\Omega = [0,1]$ with the Lebesgue measure. Set $F_n = \sin(2\pi n\omega)\sin(2\pi nt)$, with $T = 1$. Assume that F_n and W_n are adapted to the same filtration. The set of functions $\{F_n\}$ is bounded, but no uniform temporal translation estimate (2.3) is available. Though

$$F_n \xrightarrow{n\uparrow\infty} 0 \quad \text{in } L^p(\Omega\times[0,T]) \text{ for any finite } p,$$

we have

$$\mathbb{E}\left|\int_0^T F_n \, \mathrm{d}W_n\right|^2 = \mathbb{E}\int_0^T F_n^2 \, \mathrm{d}t = \frac{1}{4}.$$

Example 2.2.2. Ignoring the translation estimate (2.3), the range $p > 2$ is optimal. For each $n \geq 3$, let $f_n = \sqrt{n}\mathbb{1}_{[0,1/n]}$, and let W_n be a standard, 1-dimensional Brownian motion. We have $\|f_n\|^2_{L^2([0,1])} \lesssim 1$, and $f_n \rightharpoonup 0$ in $L^2([0,1])$. However,

$$\int_0^1 f_n \, \mathrm{d}W_n = \sqrt{n}W_n(1/n).$$

By the scaling property of the Brownian motion, this stochastic integral is distributed as $W_n(1)$, which by assumption tends to $W(1)$ a.s.— and not zero.

3. An example with a non-reflexive space X

This section presents a mild extension of Theorem 2.1 to a non-reflexive Banach space X. We examine the convergence of stochastic integrals with integrands V_n that take values in $X^{m\times k} = \big(L^1(\mathbb{T}^d)\big)^{m\times k} = L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})$, and with $\mathbb{R}^k$-valued Brownian motions W_n. These integrands satisfy uniform bounds in $L^p\big(\Omega\times[0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})\big)$, for some $p > 2$, and weakly converge in $L^r(\Omega\times[0,T]\times\mathbb{T}^d;\mathbb{R}^{m\times k})$, for some $r \geq 1$ (possibly $r < p$ or even $r < 2$). The proof of the convergence theorem below involves addressing the discrepancy between these two spaces (cf. Theorem 2.1). This extension is useful in the analysis of the stochastic Camassa–Holm equation (see [13, 15]).

Theorem 3.1 ($L^2(\Omega)$ convergence of stochastic integrals, $X = L^1(\mathbb{T}^d)$)**.** *Fix a probability space $\big(\Omega,\mathcal{F},\mathbb{P}\big)$. For $n \in \mathbb{N}$, let W_n be an $\mathbb{R}^k$-valued Wiener process on a filtered probability space $\big(\Omega,\mathcal{F},\{\mathcal{F}^n_t\}_{t\in[0,T]},\mathbb{P}\big)$, and W be an $\mathbb{R}^k$-valued Wiener process on a filtered probability space $\big(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},\mathbb{P}\big)$.*

Fix $p > 2$ and $r \geq 1$. For $n \in \mathbb{N}$, let V_n be an $L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})$-valued $\{\mathcal{F}^n_t\}$-predictable process, and let V be an $L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})$-valued $\{\mathcal{F}_t\}$-predictable process, such that

$$\begin{aligned} &V_n \xrightarrow{n\uparrow\infty} V \quad \text{in } L^r(\Omega\times[0,T]\times\mathbb{T}^d;\mathbb{R}^{m\times k}),\\ &\mathbb{E}\,\|V_n\|^p_{L^p([0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k}))} \lesssim 1. \end{aligned} \tag{3.1}$$

We further require that

$$W_n \xrightarrow{n\uparrow\infty} W \quad \text{in } C([0,T];\mathbb{R}^k),\ \mathbb{P}\text{–a.s.}$$

Fix $q > d/2$. Suppose that for a $\beta \in H^q(\mathbb{T}^d)$ the following mean L^1 temporal translation estimate holds:

$$\mathbb{E}\int_h^T \left| \int_{\mathbb{T}^d} \beta(x)\big(V_n(t) - V_n(t-h)\big)\,\mathrm{d}x \right| \mathrm{d}t \xrightarrow{h\downarrow 0} 0 \quad \textit{uniformly in } n. \tag{3.2}$$

Then for any $t \in [0,T]$,

$$\int_0^t \int_{\mathbb{T}^d} \beta\, V_n \,\mathrm{d}x\,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^t \int_{\mathbb{T}^d} \beta\, V \,\mathrm{d}x\,\mathrm{d}W \quad \textit{in } L^2(\Omega;\mathbb{R}^m). \tag{3.3}$$

Proof. We will focus on the part of the proof of Theorem 2.1 that does not apply in the case where $X = L^1(\mathbb{T}^d)$, which are Steps 3 and 4. As in Theorem 2.1, we begin by observing the following bounds and convergences on $\int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x$ and $\int_{\mathbb{T}^d} \beta V \,\mathrm{d}x$:

(i) Using the embedding $H^q(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, we have the uniform bound

$$\mathbb{E}\left\| \int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x \right\|_{L_t^p}^p \le \|\beta\|_{L_x^\infty}^p \,\mathbb{E}\,\|V_n\|_{L_t^p L_x^1}^p \lesssim 1. \tag{3.4}$$

This implies the bound (2.5). We shall establish the same bound for the limit $\int_{\mathbb{T}^d} \beta V \,\mathrm{d}x$ in (3.8) below.

(ii) By the definition of weak limits, (3.1) implies $\int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x \xrightarrow{n\uparrow\infty} \int_{\mathbb{T}^d} \beta V \,\mathrm{d}x$ in $L^r(\Omega \times [0,T];\mathbb{R}^{m\times k})$. But (3.4) implies that along a subsequence,

$$\int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x \xrightarrow{n\uparrow\infty} \overline{\Phi} \quad \text{ in } L^p(\Omega \times [0,T];\mathbb{R}^{m\times k}) \tag{3.5}$$

to a limit $\overline{\Phi}$. By the uniqueness of weak limits, each subsequence in fact tends to the same limit $\overline{\Phi} = \int_{\mathbb{T}^d} \beta V \,\mathrm{d}x$, $\mathbb{P}\otimes \mathrm{d}t$-a.e.

With the regularisation operator $\mathcal{R}_\rho$ given by (2.8), it is again possible to split the difference

$$I(n) = \int_0^T \int_{\mathbb{T}^d} \beta\, V_n \,\mathrm{d}x\,\mathrm{d}W_n - \int_0^T \int_{\mathbb{T}^d} \beta\, V \,\mathrm{d}x\,\mathrm{d}W$$

as in (2.11) into $I_1(n,\rho) + I_2(n,\rho) + I_3(\rho)$. Retaining that decompostion, we will compute the regularisation limit of $I_3(\rho)$ as $\rho \downarrow 0$, the $\rho \downarrow 0$ limit of $I_1(n,\rho)$ uniformly in n, and the limit of $I_2(n,\rho)$ as $n \uparrow \infty$.

1. The strong regularisation limits of $I_1(n,\rho)$ and $I_3(\rho)$ as $\rho \downarrow 0$.

First, we embed the non-reflexive space in which we have uniform bounds (3.1) into a larger reflexive space:

$$L^p(\Omega\times[0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})) \hookrightarrow L^p(\Omega\times[0,T];H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})), \tag{3.6}$$

for a sufficiently large q. Since V_n is uniformly bounded in the space $L^p(\Omega \times [0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k}))$, we deduce that

$$\mathbb{E}\int_0^T \sup_{\|\varphi\|_{H^q(\mathbb{T}^d;\mathbb{R}^{m\times k})}=1} \left| \int_{\mathbb{T}^d} \varphi : V_n \,\mathrm{d}x \right|^p \mathrm{d}t \le \mathbb{E}\int_0^T \|V_n\|_{L^1(\mathbb{T}^d;\mathbb{R}^{m\times k})}^p \,\mathrm{d}t \lesssim 1, \tag{3.7}$$

where we have used the Sobolev embedding $\|\varphi\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^k)} \lesssim \|\varphi\|_{H^q(\mathbb{T}^d;\mathbb{R}^{m\times k})} = 1$ for any $q > d/2$. We also used the colon to denote the scalar product between the matrix-valued objects φ and V_n. And so V_n is uniformly bounded in $L^p\big(\Omega \times$

$[0,T]; H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})\big)$. As this is a reflexive Banach space, we can use a weak compactness argument to assume that

$$V_n \xrightharpoonup{n\uparrow\infty} \overline{V} \quad \text{in } L^p\big(\Omega\times[0,T]; H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})\big),$$

for some limit $\overline{V} \in L^p_{\omega,t}H^{-q}_x$. Since $L^\infty(\Omega; C^\infty([0,T]\times\mathbb{T}^d;\mathbb{R}^{m\times k}))$ is dense in the dual space of both $L^p_{\omega,t}H^{-q}_x$ and $L^r_{\omega,t,x}$, by the weak limit assumption (3.1), we find that $\overline{V}=V$, $\mathbb{P}\otimes \mathrm{d}t\otimes\mathrm{d}x$–a.e.

As we now know that the limit V of (3.1) satisfies

$$V \in L^p\big(\Omega\times[0,T]; H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})\big),$$

it follows that (see also (2.5) and (2.6)),

$$\mathbb{E}\int_0^T \left|\int_{\mathbb{T}^d} \beta\, V\,\mathrm{d}x\right|^p \mathrm{d}t \le \|\beta\|^p_{H^q_x}\,\mathbb{E}\,\|V\|^p_{L^p_t H^{-q}_x} \lesssim 1, \tag{3.8}$$

so in particular, $\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x \in L^2(\Omega\times[0,T];\mathbb{R}^{m\times k})$ and hence $\in L^2([0,T];\mathbb{R}^{m\times k})$ almost surely. Along with (2.10), we can conclude that

$$\int_0^T \left|\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x - \mathcal{R}_\rho\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x\right|^2 \mathrm{d}t \xrightarrow{\rho\downarrow 0} 0, \quad \mathbb{P}\text{–a.s.}$$

Moreover, proceeding as in (2.13),

$$\int_0^T \left|\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x - \mathcal{R}_\rho\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x\right|^2 \mathrm{d}t \lesssim \int_0^T \left|\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x\right|^2 \mathrm{d}t \overset{(3.8)}{\in} L^1(\Omega).$$

Therfore, by Lebesgue's dominated convergence theorem,

$$\mathbb{E}\int_0^T \left|\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x - \mathcal{R}_\rho\int_{\mathbb{T}^d}\beta\,V\,\mathrm{d}x\right|^2 \mathrm{d}t \xrightarrow{\rho\downarrow 0} 0,$$

and, via the Itô isometry, we conclude that $I_3(\rho) \xrightarrow{\rho\downarrow 0} 0$ in $L^2(\Omega;\mathbb{R}^m)$.

The bounds (3.4) and (3.8) also allow us to get $I_1(n,\rho) \xrightarrow{\rho\downarrow 0} 0$ in $L^2(\Omega;\mathbb{R}^m)$, uniformly in n, as in Step 2 of Theorem 2.1.

2. The weak limit of $I_2(n,\rho)$ as $n\uparrow\infty$.

In Step 4 of the proof of Theorem 2.1, where the convergence of I_2 was analysed, we only used the bound (2.6) and the weak convergence (2.4) for the integrals $\int_{\mathbb{T}^d}\beta V_n\,\mathrm{d}x$ and $\int_{\mathbb{T}^d}\beta V\,\mathrm{d}x$. These are respectively the conditions (3.8) and (3.5). And we get $I_2(n,\rho) \xrightharpoonup{n\uparrow\infty} 0$ in $L^2(\Omega;\mathbb{R}^m)$ for any fixed $\rho>0$, thereby allowing us to conclude (3.3) as in Step 5 of Theorem 2.1. □

4. Assuming the almost sure convergence $V_n \to V$

In this section, we present two corollaries, strengthening the weak convergence V_n to V in all three variables with the assumption of weak convergence a.s. This is a situation that often arise after an application of the Skorokhod representation theorem, in which convergence in law is converted into a.s. convergence on a different probability space. We shall discover that under analogous assumptions, the resultant convergence of stochastic integrals is *strong* in $L^2(\Omega;\mathbb{R}^m)$.

Consider therefore the following assumption instead of (2.1):

$$\begin{aligned} &V_n \xrightharpoonup{n\uparrow\infty} V \quad \text{in } L^p([0,T];X^{m\times k}) \text{ a.s.},\\ &\mathbb{E}\,\|V_n\|^p_{L^p([0,T];X^{m\times k})} \lesssim 1, \end{aligned} \tag{4.1}$$

for some $p>2$.

We first observe that the bound (2.5) for $\langle\beta,V_n\rangle$ in $L^p(\Omega\times[0,T];\mathbb{R}^{m\times k})$ still holds using only the boundedness condition in (4.1), i.e.,

$$\mathbb{E}\,\|\langle\beta,V_n\rangle\|^p_{L^p([0,T];\mathbb{R}^{m\times k})} \lesssim 1. \tag{4.2}$$

For any $\beta\in X$ and $\zeta\in L^{p'}([0,T];\mathbb{R}^{m\times k})$, the weak convergence assumption in (4.1) implies that almost surely

$$\int_0^T \zeta : \langle\beta,V_n\rangle\,\mathrm{d}t \to \int_0^T \zeta : \langle\beta,V\rangle\,\mathrm{d}t, \tag{4.3}$$

where we use the colon to denote the scalar product between the matrix-valued objects ζ and $\langle\beta,V_n\rangle$. From the uniform bound in (4.1), we have

$$\mathbb{E}\left|\int_0^T \zeta : \langle\beta,V_n\rangle\,\mathrm{d}t\right|^p \le \|\beta\|_X^p\,\|\zeta\|^p_{L^{p'}([0,T];\mathbb{R}^{m\times k})}\,\mathbb{E}\,\|V_n\|^p_{L^p([0,T];X^{m\times k})} \lesssim 1.$$

Almost sure convergence and a uniform bound implies the weak convergence (4.3) ([24, Chapter 8, Theorem 12]). The arbitrariness of ζ and β in turn recovers the condition (2.1) that $V_n \rightharpoonup V$ in $L^p\big(\Omega\times[0,T];X^{m\times k}\big)\big)$, so we also have

$$\mathbb{E}\,\|\langle\beta,V\rangle\|^p_{L^p([0,T];\mathbb{R}^{m\times k})} \lesssim 1, \tag{4.4}$$

which is the bound (2.6). Therefore, Steps 2 and 3 of the proof of Theorem 2.1, demonstrating the (strong) $L^2(\Omega;\mathbb{R}^m)$ convergence of the integrals $I_1(n,\rho)$ and $I_3(\rho)$ of (2.11), carry through under (4.2) and (4.4).

We now show that the convergence of $I_2(n,\rho)$ (Step 4 of the proof of Theorem 2.1) can be improved in the following lemma:

Lemma 4.1. *Suppose $p>2$. Let V_n, V, W_n, W, X, and β satisfy the assumptions of Theorem 2.1, with the exception that* (2.1) *be replaced by* (4.1)*. Let $I_2(n,\rho)$ be given by* (2.11)*. Then*

$$I_2(n,\rho) \xrightarrow{n\uparrow\infty} 0 \quad \textit{in } L^2(\Omega;\mathbb{R}^m), \textit{ for any fixed } \rho>0.$$

Proof. We reprise the decomposition (2.15) for I_2 into $I_{2,1}$ to $I_{2,4}$. We further recall that by (2.17) and (2.18), $I_{2,3}, I_{2,4} \xrightarrow{n\uparrow\infty} 0$ in $L^2(\Omega;\mathbb{R}^m)$ already, using only (4.2) and (4.4). This leaves us with arguing for the strong convergences of $I_{2,1}$ and $I_{2,2}$. We do so by arguing for their a.s. convergence and providing a pth moment bound. Vitali's convergence theorem then implies convergence in L^2 as long as $p>2$, which we assume.

We can use (2.9) to obtain

$$\begin{aligned} I_{2,1} = -\int_0^T \big\langle\beta,V_n-V\big\rangle\,\partial_t\widetilde{\mathcal{R}}_\rho W_n\,\mathrm{d}t &= -\int_0^T \big\langle\beta,V_n-V\big\rangle\,\partial_t\widetilde{\mathcal{R}}_\rho W\,\mathrm{d}t\\ &\quad -\int_0^T \big\langle\beta,V_n-V\big\rangle\Big(\partial_t\widetilde{\mathcal{R}}_\rho W_n - \partial_t\widetilde{\mathcal{R}}_\rho W\Big)\,\mathrm{d}t =: I_{2,1}^{(1)} + I_{2,1}^{(2)}. \end{aligned}$$

By the a.s. weak convergence $V_n \rightharpoonup V$ in $L^p([0,T];X^{m\times k})$ (see (4.1)), and thus (weakly) in $L^1([0,T];X^{m\times k})$, we conclude that

$$\big\langle \beta, V_n - V \big\rangle \rightharpoonup 0 \quad \text{a.s. in } L^1([0,T];\mathbb{R}^{m\times k}). \tag{4.5}$$

At the same time, $\partial_t \widetilde{\mathcal{R}}_\rho W \in L^\infty([0,T];\mathbb{R}^k)$ a.s., so that $I_{2,1}^{(1)} \xrightarrow{n\uparrow\infty} 0$.

On the other hand, given (2.2), $\partial_t \widetilde{\mathcal{R}}_\rho W_n$ converges a.s. to $\partial_t \widetilde{\mathcal{R}}_\rho W$ strongly in $C([0,T];\mathbb{R}^k)$, while $\big\langle \beta, V_n - V \big\rangle$ are a.s. bounded in $L^1([0,T];\mathbb{R}^{m\times k})$, uniformly in n. This implies that $I_{2,1}^{(2)} \xrightarrow{n\uparrow\infty} 0$. Therefore, $I_{2,1}$ converges a.s. to 0 as $n\uparrow\infty$ for a fixed $\rho > 0$.

Next we show that

$$I_{2,2} = \mathcal{R}_\rho \big\langle \beta, V_n - V \big\rangle(T) W_n(T) \to 0 \quad \text{a.s.}$$

By (4.5), $\mathcal{R}_\rho\langle \beta, V_n - V\rangle(t) \rightharpoonup 0$ a.s. pointwise in t. Meanwhile $W_n(T) \to W$ a.s. by assumption. Therefore $I_{2,2} \to 0$ a.s.

Since $I_{2,1} + I_{2,2} \to 0$ a.s., it converges weakly in L^p if it is bounded in L^p for $p > 2$. We already know that $I_{2,3} + I_{2,4} \to 0$ strongly in $L^2(\Omega;\mathbb{R}^m)$. Therefore, it is necessary only to produce a pth moment bound ($p > 2$) for the entirety of I_2. The Itô isometry and the convolution inequality (2.10) supply us with the promised pth moment bound:

$$\begin{aligned}
&\mathbb{E}\big|I_2(n,\rho)\big|^p \\
&\lesssim \mathbb{E}\left|\int_0^T \mathcal{R}_\rho\big\langle \beta, V_n\big\rangle \,\mathrm{d}W_n\right|^p + \mathbb{E}\left|\int_0^T \mathcal{R}_\rho\langle \beta, V\rangle \,\mathrm{d}W\right|^p \\
&= \mathbb{E}\left(\int_0^T \big\|\mathcal{R}_\rho\big\langle \beta, V_n\big\rangle\big\|^2 \,\mathrm{d}t\right)^{p/2} + \mathbb{E}\left(\int_0^T \|\mathcal{R}_\rho\langle \beta, V\rangle\|^2 \,\mathrm{d}t\right)^{p/2} \\
&\lesssim \|\mathcal{R}_\rho\|_{L^1_t}^p \left(\mathbb{E}\,\big\|\big\langle \beta, V_n\big\rangle\big\|_{L^2([0,T];\mathbb{R}^{m\times k})}^p + \mathbb{E}\,\|\langle \beta, V\rangle\|_{L^2([0,T];\mathbb{R}^{m\times k})}^p\right) \overset{(4.2),(4.4)}{\lesssim} 1.
\end{aligned} \tag{4.6}$$

This concludes the proof of the lemma. ☐

The following corollary of Theorem 2.1 then follows from Lemma 4.1.

Corollary 4.2. *Suppose $p > 2$. Let V_n, V, W_n, W, X, and β satisfy the assumptions of Theorem 2.1, with the exception that* (2.1) *be replaced by* (4.1).

Then for any $t \in [0,T]$,

$$\int_0^t \big\langle \beta, V_n\big\rangle \,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^t \langle \beta, V\rangle \,\mathrm{d}W \quad \textit{in } L^2(\Omega;\mathbb{R}^m). \tag{4.7}$$

Motivated as in Theorem 3.1, we can trade spatial integrability for temporal integrability in the uniform bound:

Corollary 4.3. *Suppose $p > 2$ and $1 \le r < p$. Let V_n, V, W_n, W, and β satisfy the assumptions of Theorem 3.1, with the exception that* (3.1) *be replaced by*

$$\begin{aligned}
&V_n \xrightarrow{n\uparrow\infty} V \quad \textit{in } L^r([0,T]\times\mathbb{T}^d;\mathbb{R}^{m\times k}),\ \textit{a.s.}, \\
&\mathbb{E}\,\|V_n\|_{L^p([0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k}))}^p \lesssim 1.
\end{aligned} \tag{4.8}$$

Then for any $t \in [0,T]$,

$$\int_0^t \int_{\mathbb{T}^d} \beta\, V_n \,\mathrm{d}x\,\mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^t \int_{\mathbb{T}^d} \beta\, V \,\mathrm{d}x\,\mathrm{d}W \quad \textit{in } L^2(\Omega;\mathbb{R}^m).$$

Proof. We first observe that using the bound in (4.8), we retain the uniform bound (2.5), or equivalently (3.4), on $\int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x$ in $L^p(\Omega\times[0,T];\mathbb{R}^{m\times k})$.We will next show that $V \in L^p(\Omega\times[0,T];H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k}))$, and therefore

$$\mathbb{E}\int_0^T \left|\int_{\mathbb{T}^d} \beta\, V\,\mathrm{d}x\right|^p \mathrm{d}t \le \|\beta\|_{H^q_x}^p\, \mathbb{E}\,\|V\|^p_{L^p_t H^{-q}_x} \lesssim 1. \tag{4.9}$$

As in (3.7), the uniform bound $V_n \in_b L^p(\Omega\times[0,T];L^1(\mathbb{T}^d;\mathbb{R}^{m\times k}))$ and the inclusion (3.6) implies a uniform bound $V_n \in_b L^p(\Omega\times[0,T];H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k}))$ (where "$\in_b$" refers to uniformly bounded inclusion). Therefore, we may assume that

$$V_n \xrightharpoonup{n\uparrow\infty} \overline{V} \quad \text{in } L^p\big(\Omega\times[0,T];H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})\big). \tag{4.10}$$

We now argue that $\overline{V} = V$, $\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x$–a.e., where V is the a.s. weak limit (4.8) of the theorem. For any $\psi \in C^\infty([0,T])$, $\zeta \in C^\infty(\mathbb{T}^d;\mathbb{R}^{m\times k})$, and measurable set $B \subset \Omega$,

$$\mathbb{E}\int_0^T \mathbb{1}_B\, \psi \left\langle V_n,\zeta\right\rangle_{H^{-q},H^q} \mathrm{d}t \xrightarrow{n\uparrow\infty} \mathbb{E}\int_0^T \mathbb{1}_B\,\psi\left\langle \overline{V},\zeta\right\rangle_{H^{-q},H^q}\mathrm{d}t, \tag{4.11}$$

by the weak convergence (4.10). On the other hand, by the a.s. weak convergence in (4.8), with ζ, ψ, and B as above,

$$\mathbb{1}_B \int_0^T \int_{\mathbb{T}^d} \psi\,\zeta : V_n\,\mathrm{d}x\,\mathrm{d}t \xrightarrow{n\uparrow\infty} \mathbb{1}_B \int_0^T\int_{\mathbb{T}^d} \psi\,\zeta : V\,\mathrm{d}x\,\mathrm{d}t, \quad \mathbb{P}\text{–a.s.},$$

where we again use the colon to denote the scalar product between the matrix-valued objects ζ and V_n. Besides, we have the higher moment bound ($p>2$)

$$\mathbb{E}\left|\int_0^T\int_{\mathbb{T}^d} \mathbb{1}_B\,\psi\,\zeta : V_n\,\mathrm{d}x\,\mathrm{d}t\right|^p \le \|\zeta\psi\|^p_{L^\infty_{t,x}}\,\mathbb{E}\,\|V_n\|^p_{L^p_{\omega,t}L^1_x} \overset{(4.8)}{\lesssim} 1.$$

Hence, by Vitali's convergence theorem,

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^d} \mathbb{1}_B\,\psi\,\zeta : V_n\,\mathrm{d}x\,\mathrm{d}t \xrightarrow{n\uparrow\infty} \mathbb{E}\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_B\,\psi\,\zeta : V\,\mathrm{d}x\,\mathrm{d}t.$$

Upon comparison with (4.11), we arrive at $\left\langle \overline{V}-V,\zeta\right\rangle_{H^{-q},H^q} = 0$, $\mathbb{P}\otimes\mathrm{d}t$–a.e., for any $\zeta \in C^\infty(\mathbb{T}^d;\mathbb{R}^{m\times k})$ and, via an approximation argument, for any $\zeta \in H^q(\mathbb{T}^d;\mathbb{R}^{m\times k})$. Taking the supremum over ζ with $\|\zeta\|_{H^q(\mathbb{T}^d;\mathbb{R}^{m\times k})} \le 1$, we have $\left\|\overline{V}-V\right\|_{H^{-q}(\mathbb{T}^d;\mathbb{R}^{m\times k})} = 0$, $\mathbb{P}\otimes\mathrm{d}t$–a.e. This allows us to conclude that $V = \overline{V}$, $\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x$–a.e.

The bounds (3.4) and (4.9) allows us to argue as in Step 1 of the proof of Theorem 3.1 to conclude that in the decomposition (2.11), $I_1(n,\rho), I_3(\rho) \xrightarrow{\rho\downarrow 0} 0$ in $L^2(\Omega;\mathbb{R}^m)$ uniformly in n.

To conclude the strong convergrence in $L^2(\Omega;\mathbb{R}^m)$ of $I_2(n,\rho)$ of the decomposition (2.11) as $n\uparrow\infty$, we argue as in Lemma 4.1. Again taking up the decomposition (2.15) for I_2 into $I_{2,1}$ to $I_{2,4}$, by (2.17) and (2.18), $I_{2,3}, I_{2,4} \xrightarrow{n\uparrow\infty} 0$ in $L^2(\Omega;\mathbb{R}^m)$ already, using only (3.4) and (4.9).

To show $I_{2,1}, I_{2,2} \xrightarrow{n\uparrow\infty} 0$ in $L^2(\Omega;\mathbb{R}^m)$, we only require (4.5), which implies a.s. convergence, and a uniform bound (4.6). The a.s. convergence (4.5) follows directly from the assumption (4.8). Meanwhile, the uniform bound (4.6) holds by (3.4) and (4.9). □

We end this section by making two remarks: the first pertains to the assumption of almost sure convergence (4.1), and the second addresses related convergence theorems for stochastic integrals.

Remark 4.4. *The time translation estimate* (2.3) *in the almost sure convergence context of* (4.1) *brings us very close to strong compactness (in ω, t). In fact, if $F_n = \int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x \xrightarrow{n\uparrow\infty} 0$ in $L^2(\Omega\times[0,T])$, then we see immediately that the strong $L^2(\Omega)$ convergence* (4.7) *implies*

$$\mathbb{E}\int_0^T F_n^2 \,\mathrm{d}t = \mathbb{E}\left|\int_0^T F_n \,\mathrm{d}W_n\right|^2 \to 0.$$

The convergence of F_n in $L^2_{\omega,t}$ cannot happen in the same way under the weak $L^2(\Omega)$ convergence provided by Theorems 2.1 and 3.1. The strong convergence of $\int_{\mathbb{T}^d} \beta V_n \,\mathrm{d}x$ suggests that in some situations, a suitable topology to consider the convergence of V_n is in fact in $L^p(\Omega\times[0,T]; L^2(\mathbb{T}^d)-w)$, with the toplogy of strong convergence in (ω, t) and weak convergence in the spatial variable. That is, for every $\beta \in L^2(\mathbb{T}^d)$,

$$\mathbb{E}\int_0^T \left|\int_{\mathbb{T}^d} \beta(x)\left(V_n(t) - V(t)\right)\mathrm{d}x\right|^p \mathrm{d}t \to 0.$$

This is the L^p analogue to the time-continuous space $C([0,T]; X-w)$ mentioned in Section 1 of our paper (see [2, 21]*), and of the space $\mathbb{D}([0,T]; L^2(\mathbb{T}^d)-w)$ which is Skorokhod (càdlàg) in time and weakly L^2 in space, used in* [3]*. Strong-weak spaces of the form $L^p([0,T]; X-w)$ featured saliently in the recent works* [13] *and* [18].

Remark 4.5. *If $W_n = W$ for all $n \in \mathbb{N}$, then a related convergence result for stochastic integrals is discussed in* [14, Remark 4 (iii)]*. There it is asserted that for $V_n \rightharpoonup V$ in $L^2(\Omega\times[0,T])$, which is the assumption* (2.1) *with $p = 2$ and $X = \mathbb{R}$, one also has $\int_0^t V_n \,\mathrm{d}W \rightharpoonup \int_0^t V \;dW$ in $L^2(\Omega\times[0,T])$ (but cf. Remark 2.2). Here the stochastic integral is understood as a process instead of being evaluated at $t = T$ and treated as a random variable. A similar assertion is made and proved in* [12, page 25] *(for finite dimensional noise). The assertions in* [12, 14] *follow from the fact that the Itô integral is linear and continuous from the space of adapted $L^2(\Omega\times[0,T])$ processes to $L^2(\Omega\times[0,T])$, and is thus also weakly continuous. In addition, since the space of adapted processes is a closed subspace of $L^2(\Omega\times[0,T])$, it is weakly closed, and so the limit V is adapted. It is not possible to use this simple convergence proof when the Wiener process and the stochastic integrand both depend on $n \in \mathbb{N}$, which is the situation covered by Theorem 2.1.*

5. Stochastic transport equations

In this and the next section, we consider two applications of the limit theorems of Sections 2 to 4. Here we establish a stability result for a scalar semilinear stochastic transport equation for which the nonlinearity is in the noise term. We use the term "transport equation" loosely to mean either the transport equation or the continuity equation, and in fact mostly work with the latter. Our results apply to either type of equation with minimal technical modifications between the proofs, including the presence or absence of an additional lower order term. In our equations, we also incorporate a vanishing viscosity term to address the numerical diffusion commonly encountered in numerical schemes.

More precisely, we wish to develop a stochastic analogue of the strong stability result [7, Theorem II.4]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space on which W is an $\mathbb{R}^k$-valued Brownian motion. A one-dimensional Brownian motion is employed here solely for the simplicity of presentation.

We say that an adapted element $u \in L^p(\Omega \times [0,T] \times \mathbb{T}^d)$ is a weak solution of

$$\mathrm{d}u + \operatorname{div}(bu)\,\mathrm{d}t = f\,\mathrm{d}t + \varepsilon\Delta u\,\mathrm{d}t + \sigma(u)\,\mathrm{d}W, \qquad u(0) = u_0 \tag{5.1}$$

if the equation holds a.s. weakly in x in the sense of Itô, i.e., for every $\varphi \in C^2(\mathbb{T}^d)$, a.s. for every $t \in (0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^d} u(t)\varphi\,\mathrm{d}x - \int_{\mathbb{T}^d} u_0\varphi\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^d} \nabla\varphi\cdot bu\,\mathrm{d}x\,\mathrm{d}s \\ = \int_0^t\int_{\mathbb{T}^d} f\varphi\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta\varphi u\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^d}\sigma(u)\varphi\,\mathrm{d}x\,\mathrm{d}W.\end{aligned} \tag{5.2}$$

Consider now solutions u_n to (5.1) above where b, f, σ, and ε are indexed with a parameter n. We assume that $u_n \rightharpoonup u$ in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$ for some $p > 2$. The strong convergence $u_n \to u$, where u is a weak solution of a limiting equation, will depend on a renormalisation procedure, in which a nonlinear function $\eta(u_n)$ of u_n is evaluated. In the expression for $\mathrm{d}\eta(u_n)$, another nonlinear function $\vartheta(u_n) := \eta'(u_n)\sigma(u_n)$ of u_n appears in the stochastic integral against $\mathrm{d}W_n$ (which are different for each n). Observe that even if $u_n \to u$ strongly in time and weakly in space (say), no strong convergence will be preserved *a priori* for the compostion $\vartheta(u_n)$, which will only satisfy L^p bounds in (ω, t, x) depending on its growth in terms of its argument. This leaves us with a *weak* limit $\vartheta(u_n) \rightharpoonup \overline{\vartheta}$ on $L^p_{\omega,t,x}$ for the integrand of the stochastic integral. The convergence of these stochastic integrals then does not seem to follow from the standard convergence lemma of, e.g., [5, Lemma 2.1] because of the lack of strong temporal compactness. However, we shall see from these examples that a temporal translation estimate of the form (2.3) is often available to solutions of even quite singular SPDEs (Lemma 5.3).

We shall establish the following analogue to [7, Theorem II.4]:

Theorem 5.1. *Fix $p > 2$. Set $p' = p/(p-1)$ and $p'' = p/(p-2)$, respectively the Hölder conjugates of p and $p/2$. Let $\{u_n\}_{n=1}^\infty$ be a sequence of weak solutions to*

$$\mathrm{d}u_n + \operatorname{div}(b_n u_n)\,\mathrm{d}t = f_n\,\mathrm{d}t + \frac{1}{n}\Delta u_n\,\mathrm{d}t + \sigma(u_n)\,\mathrm{d}W_n, \quad u_n(0) = u_{0,n} \tag{5.3}$$

for which $u_n \rightharpoonup u$ in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$. Suppose:

(i) *$\sigma \in C^{1,1}(\mathbb{R};\mathbb{R}^k)$ is globally Lipschitz on $\mathbb{R}$,*
(ii) *$\{b_n\}_{n\ge1} \subset_b L^1([0,T]; W^{1,p''}(\mathbb{T}^d;\mathbb{R}^d))$, and $\{\operatorname{div}(b_n)\}_{n\ge1} \subset_b L^1([0,T]; L^\infty(\mathbb{T}^d))$,*
(iii) *$b_n \to b$ in $L^1([0,T]; W^{1,p'}(\mathbb{T}^d;\mathbb{R}^d))$, $\operatorname{div}(b_n) \to \operatorname{div}(b)$ in $L^1([0,T]; L^{p''}(\mathbb{T}^d))$,*
(iv) *$f_n \to f$ in $L^1([0,T]; L^p(\mathbb{T}^d))$, $u_{0,n} \to u_0$ in $L^p(\Omega; L^p(\mathbb{T}^d))$, and*
(v) *$W_n \to W$ a.s. in $C([0,T];\mathbb{R}^k)$.*

Then u is a weak solution to (5.1) with $\varepsilon = 0$, and $u_n \to u$ in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$.

(We use "$\subset_b$" to indicate a subset is bounded.)

Remark 5.2. *We assumed weak convergence $u_n \rightharpoonup u$ in $L^{q_1}_{\omega,t}L^{q_2}_x$ in Theorem 5.1 with $q_1 = q_2 = p$. The $q_1 > 2$ integrability in time is necessary for applying the convergence theorems of Section 4. The spatial integrability restriction $q_2 > 2$ can be relaxed to $q_2 \geq 2$, however, with extra technical steps. See Remark 5.6. The resultant strong convergence would be in $L^{q_1}_{\omega,t}L^{q^*}_x$ with $q^* = q_2$ if $q_2 \leq q_1$ or any $q^* < q_2$ if $q_2 > q_1$.*

To establish this theorem, we shall repeatedly use the following result to get uniform time translation estimates for solutions to SPDEs. Let M_b denote the space of bounded (finite total variation) signed Radon measures.

Lemma 5.3. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}^n_t\}_{t\geq 0}, \mathbb{P})$ be a sequence of filtered probability spaces. Denote multi-indices by $\alpha \in \mathbb{N}^d$, and let $m_F, m_G \geq 0$ be three integers. Suppose*

(i) *for each $|\alpha| \leq m_F$, $\{F^{(\alpha)}_n\}_{n\geq 1} \subset_b L^1(\Omega; M_b([0,T]\times\mathbb{T}^d))$,*
(ii) *for each $|\alpha| \leq m_G$, $\{G^{(\alpha)}_n\}_{n\geq 1} \subset_b L^2(\Omega\times[0,T]; L^2(\mathbb{T}^d;\mathbb{R}^k))$ where $G^{(\alpha)}_n$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted,*
(iii) *$\{\varrho_{0,n}\}_{n\geq 1} \subset_b L^1(\Omega\times\mathbb{T}^d)$, and*
(iv) *for each $n \in \mathbb{N}$, W_n is an $\mathbb{R}^k$-valued a $\{\mathcal{F}^n_t\}_{t\geq 0}$-Wiener processes.*

Let $\{\varrho_n\}_{n\geq 1}$ be solutions to the equation

$$\mathrm{d}\varrho_n = \sum_{|\alpha|\leq m_F} \partial^\alpha_x F^{(\alpha)}_n \,\mathrm{d}t + \sum_{|\alpha|\leq m_G} \partial^\alpha_x G^{(\alpha)}_n \,\mathrm{d}W_n,$$

i.e., for any $\varphi \in C^\infty(\mathbb{T}^d)$, and a.s. for a.e. $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^d} \varrho_n(t)\varphi\,\mathrm{d}x - \int_{\mathbb{T}^d} \varrho_{0,n}\varphi\,\mathrm{d}x = \sum_{|\alpha|\leq m_F} (-1)^{|\alpha|} \int_0^T \int_{\mathbb{T}^d} \partial^\alpha_x\varphi \mathbb{1}_{[0,t]}\, F^{(\alpha)}_n(\mathrm{d}r,\mathrm{d}x) \\ + \sum_{|\alpha|\leq m_G} (-1)^{|\alpha|} \int_0^t \int_{\mathbb{T}^d} \partial^\alpha_x\varphi\, G^{(\alpha)}_n \,\mathrm{d}x\,\mathrm{d}W_n.\end{aligned} \tag{5.4}$$

Then for any $\psi \in C^\infty(\mathbb{T}^d)$, there is a constant $C_\psi > 0$, dependent on $\|\partial^\alpha_x\psi\|_{L^\infty(\mathbb{T}^d)}$ for every $|\alpha| < m_F \vee m_G$, but independent of n, such that

$$\mathbb{E}\int_h^T \left| \int_{\mathbb{T}^d} \psi(x)\big(\varrho_n(t) - \varrho_n(t-h)\big)\,\mathrm{d}x \right| \mathrm{d}t \leq C_\psi h^{1/2} \quad \textit{uniformly in } n.$$

Proof. Observe that the assumptions in this lemma are all boundedness assumptions rather than convergence assumptions. By subtracting (5.4) at $t-h$ from the same at $t-h$, we have

$$\mathbb{E}\int_h^T \left| \int_{\mathbb{T}^d} \varphi\ (\varrho_n(t) - \varrho_n(t-h))\,\mathrm{d}x \right| \mathrm{d}t \leq \sum_{j=2} I_j,$$

where

$$I_1 := \mathbb{E}\int_h^T \left| \sum_{|\alpha|\leq m_F} (-1)^{|\alpha|} \int_0^T \int_{\mathbb{T}^d} \partial^\alpha_x\varphi \mathbb{1}_{(t-h,t]}\, F^{(\alpha)}_n(\mathrm{d}r,\mathrm{d}x) \right|$$

$$I_2 := \mathbb{E}\int_h^T \left| \sum_{|\alpha|\leq m_G} (-1)^{|\alpha|} \int_{t-h}^t \left\langle G^{(\alpha)}_n, \partial^\alpha_x\varphi \right\rangle_{H^{-q},H^q} \mathrm{d}W_n \right| \mathrm{d}t.$$

We can estimate I_1 as follows:

$$I_1 \leq \sum_{|\alpha|\leq m_F} \mathbb{E}\int_0^T \int_h^T \mathbb{1}_{\{r<t<r+h\}}\,\mathrm{d}t\ \ \|\partial_x^\alpha \varphi\|_{L^\infty(\mathbb{T}^d)} \left\|F_n^{(\alpha)}(\mathrm{d}r)\right\|_{TV(\mathbb{T}^d)} \overset{\text{(i)}}{\lesssim}_{\psi,m_F,T} h,$$

where the implied constant does not depend on n because of the uniform bound on F_n in $M_b([0,T]\times\mathbb{T}^d)$. We have used the fact that F_n admits a disintegration (e.g., [10, Theorem 1.10]), so $\left\|F_n^{(\alpha)}\right\|_{TV(\mathbb{T}^d)} \in M_b([0,T])$. By the BDG inequality and then Jensen's inequality, we obtain the estimate

$$\begin{aligned} I_2 &\leq \sum_{|\alpha|\leq m_G} \int_h^T \mathbb{E}\left(\int_{t-h}^t \left|\left\langle G_n^{(\alpha)}, \partial_x^\alpha \varphi\right\rangle_{H^{-q},H^q}\right|^2 \mathrm{d}r\right)^{1/2} \mathrm{d}t \\ &\lesssim_T \sum_{|\alpha|\leq m_G} \left(\int_h^T \mathbb{E}\int_{t-h}^t \left|\left\langle G_n^{(\alpha)}, \partial_x^\alpha \varphi\right\rangle_{H^{-q},H^q}\right|^2 \mathrm{d}r\,\mathrm{d}t\right)^{1/2} \overset{\text{(ii)}}{\lesssim}_{\psi,T} h^{1/2}, \end{aligned}$$

which is uniform in n. The final inequality can be attained as for I_1, but within the square-root. This establishes the lemma. ☐

Remark 5.4. *Even though the statement of Lemma 5.3 is written for $x\in\mathbb{T}^d$, it is clear from the proof that it holds for $x\in\mathbb{R}^d$, under the qualification that the test functions φ must be compactly supported.*

We will repeatedly use the following simple fact about weak convergence.

Lemma 5.5. *Suppose $F_n = F_n(\omega,t)\leq 0$ for a.e. $(\omega,t)\in\Omega\times[0,T]$, for each $n\in\mathbb{N}$. Furthermore, suppose we have the weak convergence $F_n \rightharpoonup F$ in $L^p(\Omega\times[0,T])$, for some $p\in[1,\infty)$. Then $F\leq 0$ a.e. on $\Omega\times[0,T]$. The same statement holds with "$\leq$" replaced consistently by "$\geq$" or "$=$" throughout.*

Proof. Since $L^\infty(\Omega\times[0,T])\subset L^p(\Omega\times[0,T])$, by choosing a non-negative element $Y\in L^\infty(\Omega\times[0,T])$, we see immediately that $0\geq \mathbb{E}\int_0^T YF_n\,\mathrm{d}t \to \mathbb{E}\int_0^T YF\,\mathrm{d}t$. By the arbitrariness of $Y\geq 0$, it follows that $F\leq 0$ a.e. on $\Omega\times[0,T]$. ☐

Proof of Theorem 5.1. We prove this theorem with the additional technical assumptions that

$$u_{0,n}\to u_0 \quad \text{in } L^{2p}(\Omega;L^p(\mathbb{T}^d)), \qquad |\sigma''(v)|\lesssim 1\wedge |v|^{p-3}. \tag{5.5}$$

These can be removed — see Remark 5.6. Our emphasis will be on the convergence of stochastic integrals.

The strategy is to take a nonlinear function $\eta(v)=\frac{1}{2}v^2$, derive an inequality for the weak limit $\overline{\eta}$ of $\eta(u_n)$ and to derive an equation $\eta(u)$ in order to compare the two quantities. The $\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x$-a.e. coincidence $\eta(u)=\overline{\eta}$ will imply the strong convergence $u_n\to u$.

First we derive an equation for $\eta(u_n)$, whose weak limit will give an inequality for $\overline{\eta}$. Let J_δ be a standard mollifier on $\mathbb{T}^d$. For locally integrable functions g, let $g_\delta := J_\delta * g$. By testing (5.3) against J_δ, we have the pointwise-in-x equation

$$\begin{aligned} &u_{n,\delta}(t) - u_{0,n,\delta} - \int_0^t \operatorname{div}(b_{n,\delta}u_{n,\delta})\,\mathrm{d}s \\ &= \int_0^t f_{n,\delta}\,\mathrm{d}s + \frac{1}{n}\int_0^t \Delta u_{n,\delta}\,\mathrm{d}s + \sigma(u_n)_\delta\,\mathrm{d}W_n + E_{n,\delta}^{(1)}[u_n]. \end{aligned}$$

The term $E^{(1)}_{n,\delta}$ is the commutator $E^{(1)}_{n,\delta}[u] := \operatorname{div}(b_{n,\delta}u_\delta) - \operatorname{div} J_\delta * (b_n u)$. This is the classical commutator of DiPerna–Lions, so applying [19, Lemma II.1], and the dominated convergence theorem in the ω variable, this error tends to nought a.s. and in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$. We can now apply the Itô formula pointwise in x, to get that for any entropy $S = S(x,u) \in C(\mathbb{T}^d; C^{1,1}(\mathbb{R}))$, a.s. for all $s,t \in [0,T]$, $s \le t$,

$$
\begin{aligned}
&S(x,u_{n,\delta}(t)) - S(x,u_{n,\delta}(s)) + \int_s^t S'(x,u_{n,\delta}(r)) \operatorname{div}(b_{n,\delta}u_{n,\delta})\, \mathrm{d}r \\
&= \int_s^t S'(x,u_{n,\delta}(r)) f_{n,\delta} + \frac{1}{n}\int_s^t S'(x,u_{n,\delta}(r)) \Delta u_{n,\delta}\, \mathrm{d}r \\
&\quad + \frac{1}{2}\int_s^t S''(x,u_{n,\delta}(r)) \left|\sigma(u_n(r))_\delta\right|^2 \mathrm{d}r \\
&\quad + \int_s^t S'(x,u_{n,\delta}(r)) \sigma(u_n(r))_\delta \,\mathrm{d}W_n + \int_s^t S'(u_{n,\delta}(r)) E^{(1)}_{n,\delta}[u_n]\, \mathrm{d}r,
\end{aligned}
$$

where S' and S'' refer to derivatives in the second argument only. Suppose $S(x,v) = \psi(x)\vartheta(v)$, $\psi \in C^2(\mathbb{T}^d)$, and

$$
|\vartheta(v)| \lesssim 1 + |v|^p, \quad |\vartheta'(v)| \lesssim 1 + |v|^{p-1}, \quad |\vartheta''(v)| \lesssim 1 + |v|^{p-2}. \tag{5.6}
$$

Then we can integrate in x to get

$$
\begin{aligned}
&\int_{\mathbb{T}^d} \psi(x)\big(\vartheta(u_{n,\delta}(t)) - \vartheta(u_{n,\delta}(s))\big)\, \mathrm{d}x \\
&\quad + \int_s^t \int_{\mathbb{T}^d} \psi \operatorname{div}(b_{n,\delta}) \left(\vartheta'(u_{n,\delta})u_{n,\delta} - \vartheta(u_{n,\delta})\right) - b_{n,\delta} \cdot \nabla\psi \vartheta(u_{n,\delta})\, \mathrm{d}x\, \mathrm{d}r \\
&= \int_s^t \int_{\mathbb{T}^d} \psi\vartheta'(u_{n,\delta}) f_{n,\delta}\, \mathrm{d}x\, \mathrm{d}r + \int_s^t \int_{\mathbb{T}^d} \psi\vartheta'(u_{n,\delta}) E^{(1)}_\delta[u]\, \mathrm{d}x\, \mathrm{d}r \\
&\quad + \frac{1}{n}\int_s^t \int_{\mathbb{T}^d} \Delta\psi\vartheta(u_{n,\delta}) - \psi\vartheta''(u_{n,\delta}) \left|\nabla u_{n,\delta}\right|^2 \mathrm{d}x\, \mathrm{d}r \\
&\quad + \frac{1}{2}\int_s^t \int_{\mathbb{T}^d} \psi\vartheta''(u_{n,\delta}) \left|\sigma(u_n)_\delta\right|^2 \mathrm{d}x\, \mathrm{d}r + \int_s^t \int_{\mathbb{T}^d} \psi\vartheta'(u_{n,\delta})\sigma(u_n)_\delta\, \mathrm{d}x\, \mathrm{d}W_n.
\end{aligned} \tag{5.7}
$$

The global Lipschitz assumption on σ implies $|\sigma(v)| \lesssim 1+|v|$. Therefore, $\sigma(u_n) \in L^p(\Omega \times [0,T] \times \mathbb{T}^d; \mathbb{R}^k)$. Setting $S(v) = \frac{1}{p}|v|^p$, $\psi \equiv 1$, and $s = 0$, the dissipation is conveniently signed. We can take the supremum over $t \in [0,T]$, an expectation, and apply the BDG inequality and Gronwall's lemma. The result is

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} \|u_{n,\delta}(t)\|^p_{L^p(\mathbb{T}^d)} + \frac{1}{n}\mathbb{E}\int_0^T \int_{\mathbb{T}^d} |u_{n,\delta}|^{p-2} \left|\nabla u_{n,\delta}\right|^2 \mathrm{d}x\, \mathrm{d}t \\
\lesssim_{T, \|\operatorname{div}(b_n)\|_{L^1_t L^\infty_x}, \|f_n\|_{L^1_t L^p_x}} 1.
\end{aligned} \tag{5.8}
$$

This estimate is uniform in δ, and holds with $\delta = 0$.

Except in the dissipation term, by the boundedness assumptions (i) to (iv) in the theorem statement, using standard properties of the mollifier we are able to take an a.s. limit $\delta \downarrow 0$ in (5.7) above. In particular, the discussion on the convergence of the commutator $E^{(1)}_\delta$ ensures that the final integral tends to 0 as $\delta \downarrow 0$.

By the standard properties of mollifiers and the temporal integrability on $\vartheta'(u_n)$ and $\sigma(u_n)$ implied by (5.8), (5.6), and $|\sigma(v)| \lesssim 1 + |v|$, we have

$$\int_{\mathbb{T}^d} \psi \vartheta'(u_{n,\delta})\sigma(u_n)_\delta \,\mathrm{d}x \xrightarrow{\delta \downarrow 0} \int_{\mathbb{T}^d} \psi \vartheta'(u_n)\sigma(u_n)\,\mathrm{d}x \quad \text{in } L^2([0,T];\mathbb{R}^k) \text{ a.s.,} \tag{5.9}$$

so that there is strong temporal L^2 convergence of the stochastic integrands indexed by the mollification parameter $\delta > 0$. Thus, with reference to (1.2) – (1.3), the stochastic integral then tends to itself without the subscript δ by the standard convergence result of [5, Lemma 2.1].

Suppose that in addition to (5.6), we have

$$|\vartheta''(v)| \lesssim 1. \tag{5.10}$$

Using the fact that $a_n \in_b L^\infty$, $b_n \to b$ in L^1 imply $a_n b_n \to ab$ in L^1, we can derive the convergence of the dissipation term $\vartheta''(u_{n,\delta})\left|\nabla u_{n,\delta}\right|^2$ also. Finally, we get the following equation a.s. for every $t, s \in [0,T]$, $s \le t$:

$$\begin{aligned} &\int_{\mathbb{T}^d} \psi(x)\big(\vartheta(u_n(t)) - \vartheta(u_n(s))\big)\,\mathrm{d}x \\ &\qquad = \sum_{j=1}^4 \int_s^t \int_{\mathbb{T}^d} I_j(n)\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\mathbb{T}^d} I_5(n)\,\mathrm{d}x\,\mathrm{d}W_n, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} &I_1(n) := -\psi \operatorname{div}(b_n)\left(\vartheta'(u_n)u_n - \vartheta(u_n)\right) - b_n \cdot \nabla\psi\vartheta(u_n), \\ &I_2(n) := \psi\vartheta'(u_n) f_n, \qquad I_3(n) := \frac{1}{n}\Delta\psi\vartheta(u_n) - \frac{1}{n}\psi\vartheta''(u_n)\left|\nabla u_n\right|^2, \\ &I_4(n) := \frac{1}{2}\psi\vartheta''(u_n)\left|\sigma(u_n)\right|^2, \qquad I_5(n) := \psi\vartheta'(u_n)\sigma(u_n). \end{aligned}$$

With $\vartheta(v) = \eta(v) = \frac{1}{2}v^2$, and $\psi \equiv 1$, $I_3 \le 0$, (ω, t, x)-a.e. Therefore, we find:

$$\begin{aligned} &\int_{\mathbb{T}^d} \eta(u_n(t)) - \eta(u_{0,n})\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \left(\eta'(u_n)u_n - \eta(u_n)\right) \operatorname{div}(b_n)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \le \int_0^t \int_{\mathbb{T}^d} \eta'(u_n) f_n + \frac{1}{2}\eta''(u_n)\left|\sigma(u_n)\right|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_0^t \int_{\mathbb{T}^d} \eta'(u_n)\sigma(u_n)\,\mathrm{d}x\,\mathrm{d}W_n, \quad \text{a.s.} \end{aligned} \tag{5.12}$$

We will take limits of terms in this inequality to get an inequality for $\overline{\eta}$.

By the bound (5.8) on u_n,

$$\{\eta(u_n)\}, \{\eta'(u_n)u_n\}, \{\eta'(u_n)\} \subset_b L^{p/2}(\Omega \times [0,T] \times \mathbb{T}^d).$$

We deduce that along an unrelabelled subsequence, these three sequences tend respectively to weak limits $\overline{\eta}$, $\overline{\eta' u}$ and $\overline{\eta'}$ in $L^{p/2}(\Omega \times [0,T] \times \mathbb{T}^d)$. Additionally, by the assumption (i) on σ, for each $j \in \{1, \dots, k\}$,

$$\{\eta'(u_n)\sigma_j(u_n)\}, \{\eta''(u_n)\left|\sigma(u_n)\right|^2\} \subset_b L^{p/2}(\Omega \times [0,T] \times \mathbb{T}^d). \tag{5.13}$$

Let $\overline{\eta'\sigma}$ and $\overline{\eta''\left|\sigma\right|^2}$ respectively be the (component-wise) weak limits of $\eta'(u_n)\sigma(u_n)$ and $\eta''(u_n)\left|\sigma(u_n)\right|^2$ in this space. As $p > 2$, these limits are not singular measures.

We now take a closer look at the limit of the martingale term, whose convergence shall be a result of Theorem 3.1. Exploiting the higher moment assumption on the initial data $u_{0,n}$ in $L^p(\mathbb{T}^d)$, by a calculation similar to one that led to (5.8),

$$\mathbb{E} \sup_{t\in[0,T]} \|u_n(t)\|_{L^p(\mathbb{T}^d)}^{2p} \lesssim 1.$$

Hence,

$$\begin{aligned}\mathbb{E}\left\|\eta'(u_n)\sigma(u_n)\right\|_{L^p([0,T];L^1(\mathbb{T}^d;\mathbb{R}^k))}^p &\lesssim 1+\mathbb{E}\int_0^T\left(\int_{\mathbb{T}^d}\left|u_n\right|^2\,\mathrm{d}x\right)^p\mathrm{d}t \\ &\lesssim \mathbb{E}\sup_{t\in[0,T]}\|u_n(t)\|_{L^p(\mathbb{T}^d)}^{2p}\lesssim 1.\end{aligned} \tag{5.14}$$

Turning to the L^1 temporal translation estimate (3.2) of Theorem 3.1, we use (5.11) again, this time choosing

$$\psi\in C^2(\mathbb{T}^d),\quad \vartheta=\eta'\sigma,\quad \text{and}\quad s=t-h.$$

We have (cf. (5.6) and (5.10)):

$$|\vartheta(v)|\lesssim 1+|v|^2,\quad |\vartheta'(v)|\lesssim 1+|v|,\quad |\vartheta''(v)|\lesssim 1,$$

where we used the assumption (i) on σ in the theorem statement. We can now apply Lemma 5.3 to $\vartheta(u_n)$ in (5.11). In particular, with $\psi,\nabla\psi\in L_x^\infty$, $b_n,\operatorname{div}(b_n)\in_b L_t^1L_x^{p''}$, and $u_n\in_b L_\omega^2L_t^\infty L_x^p$,

$$I_1(n), I_2(n)\in_b L^1(\Omega\times[0,T]\times\mathbb{T}^d). \tag{5.15}$$

Similarly, using the bound (5.8),

$$|I_3(n)|\in_b L^1(\Omega;\times[0,T]\times\mathbb{T}^d). \tag{5.16}$$

For I_4, using the bounds $|\vartheta''(v)|\lesssim|v|^{p-2}$, $|\sigma(v)|\lesssim 1+|v|$,

$$I_4(n)\in_b L^1(\Omega\times[0,T]\times\mathbb{T}^d). \tag{5.17}$$

Again in light of the bounds $|\sigma(v)|,|\vartheta'(v)|\lesssim 1+|v|$,

$$I_5(n)\in_b L^2(\Omega\times[0,T];L^1(\mathbb{T}^d)). \tag{5.18}$$

Since (5.15) – (5.18) hold for each $\psi\in C^2(\mathbb{T}^d)$, we have by Lemma 5.3,

$$\mathbb{E}\int_h^T\left|\int_{\mathbb{T}^d}\psi(x)\left(\vartheta(u_n(t))-\vartheta(u_n(t-h))\right)\mathrm{d}x\right|\mathrm{d}t\lesssim_{p,T,\psi} h^{1/2},\quad \text{uniformly in } n.$$

Then by Theorem 3.1 for any $t\in[0,T]$, choosing $\psi\equiv 1$,

$$\int_0^t\int_{\mathbb{T}^d}\eta'(u_n)\sigma(u_n)\,\mathrm{d}x\,\mathrm{d}W_n\xrightarrow{n\uparrow\infty}\int_0^t\int_{\mathbb{T}^d}\overline{\eta'\sigma}\,\mathrm{d}x\,\mathrm{d}W\quad \text{in } L^2(\Omega).$$

In view of the assumed strong convergences of f_n, $\operatorname{div}(b_n)$, $u_{0,n}$, the remaining integrals of (5.12) converge appropriately, and by Lemma 5.5 we have the $\mathbb{P}\otimes\mathrm{d}t$-a.e. inequality:

$$\begin{aligned}&\int_{\mathbb{T}^d}\overline{\eta}(t)-\eta(u_0)\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^d}\overline{(\eta'u-\eta)}\operatorname{div}(b)\,\mathrm{d}x\,\mathrm{d}s\\ &\le\int_0^t\int_{\mathbb{T}^d}\overline{\eta'}f\,\mathrm{d}x\,\mathrm{d}s+\frac12\int_0^t\int_{\mathbb{T}^d}\overline{\eta''|\sigma|^2}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^d}\overline{\eta'\sigma}\,\mathrm{d}x\,\mathrm{d}W.\end{aligned} \tag{5.19}$$

Next, we derive an equation for $\eta(u)$, for which we first need an equation for u. To do this we take a weak limit of (5.3). On the assumptions of the theorem, except for the stochastic integral, the appropriate weak limit holds with (u, u_0, b, f) in place $(u_n, u_{0,n}, b_n, f_n)$ in (5.3).

Since $u_n \rightharpoonup u$ in $L^p(\Omega\times[0,T]\times\mathbb{T}^d)$ and hence is bounded in $L^p(\Omega\times[0,T];L^2(\mathbb{T}^d))$, and σ is of sublinear growth, we may assume that

$$\sigma(u_n) \rightharpoonup \overline{\sigma} \quad \text{in } L^p(\Omega\times[0,T];L^2(\mathbb{T}^d)).$$

We shall invoke Theorem 2.1 to prove the convergence of the stochastic integral.

Recall that σ satsifes

$$|\sigma(v)| \lesssim 1 + |v|\,, \quad |\sigma'(v)| \lesssim 1, \quad |\sigma''(v)| \lesssim 1 \wedge |v|^{p-3},$$

Setting $s = t - h$, $\vartheta = \sigma$ (cf. (5.6) and (5.10)), and $\psi = \varphi$, in (5.11), we can estimate as in (5.15) – (5.18) to get

$$\mathbb{E}\int_h^T \left|\int_{\mathbb{T}^d} \varphi(x)\left(\vartheta(u_n(t)) - \vartheta(u_n(t-h))\right)\,\mathrm{d}x\right|\mathrm{d}t = O(h^{1/2}), \quad \text{uniformly in } n,$$

from Lemma 5.3. Hence, Theorem 2.1 gives us

$$\int_0^t\int_{\mathbb{T}^d} \varphi\sigma(u_n)\,\mathrm{d}x\,\mathrm{d}W_n \xrightharpoonup{n\uparrow\infty} \int_0^t\int_{\mathbb{T}^d}\varphi\overline{\sigma}\,\mathrm{d}x\,\mathrm{d}W \quad \text{in } L^2(\Omega),$$

pointwise in t. We can easily turn this into weak convergence in $L^2(\Omega\times[0,T])$ by a uniform-in-n $L^2(\Omega\times[0,T])$ bound. By Lemma 5.5 we find that there is a version of u, still denoted by u such that the following equation holds weakly in x, (ω,t)-a.e.:

$$\mathrm{d}u + \operatorname{div}(bu)\;\mathrm{d}t = f\,\mathrm{d}t + \overline{\sigma}\,\mathrm{d}W. \tag{5.20}$$

In order to show that u is a weak solution, what is required now is that

$$\overline{\sigma} = \sigma(u), \text{ a.e. in } \Omega\times[0,T]\times\mathbb{T}^d.$$

This will be a by-product of showing that $\eta(u) = \overline{\eta}$ a.e. (and therefore of the strong convergence $u_n \to u$).

We can mollify (5.20) and derive an analogue of (5.7) where $\vartheta = \eta$, and the subscripts n and the dissipation terms are absent. The coefficients $(b, \operatorname{div}(b), f)$ reside in the spaces of convergence of each of the coefficients $(b_n, \operatorname{div}(b_n), f_n)$. It is then possible using these inclusions to take the $\delta\downarrow 0$ limit by the dominated convergence theorem to get the $\mathbb{P}\otimes\mathrm{d}t$-a.e. equality:

$$\begin{aligned}
&\int_{\mathbb{T}^d}\psi(x)\left(\eta(u(t)) - \eta(u_0)\right)\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^d}\psi\eta'(u)f\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + \int_0^t\int_{\mathbb{T}^d}\psi\big[\operatorname{div}(b)\left(\eta'(u)u - \eta(u)\right) + \operatorname{div}(bu)\big]\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{T}^d}\psi\eta'(u)\overline{\sigma}\,\mathrm{d}x\,\mathrm{d}W + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\psi\eta''(u)\left|\overline{\sigma}\right|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{5.21}$$

In particular, the stochastic integral term attains the appropriate limit by the standard [5, Lemma 2.1] as $\int_{\mathbb{T}^d}\psi\eta''(u_\delta)\left(|\overline{\sigma}|^2\right)_\delta\,\mathrm{d}x \to \int_{\mathbb{T}^d}\psi\eta''(u)\left|\overline{\sigma}\right|^2\,\mathrm{d}x$ in $L^2([0,T])$ a.s., as in (5.9) above.

We now compare $\overline{\eta}$ and $\eta(u)$ by subtracting the integrated form of (5.21) (with $\psi \equiv 1$) from (5.19). This gives us:

$$\begin{aligned}\int_{\mathbb{T}^d} \overline{\eta}(t) - \eta(u(t))\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \Big(\overline{(\eta' u - \eta)} - (\eta'(u)u - \eta(u))\Big)\,\mathrm{div}\,(b)\,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_0^t \int_{\mathbb{T}^d} \big(\overline{\eta'} - \eta'(u)\big)\, f\,\mathrm{d}x + \frac{1}{2}\int_0^t \int_{\mathbb{T}^d} \overline{\eta'' \left|\sigma\right|^2} - \eta''(u)\left|\overline{\sigma}\right|^2 \mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{T}^d} \overline{\eta'\sigma} - \eta'(u)\overline{\sigma}\,\mathrm{d}x\,\mathrm{d}W.\end{aligned} \tag{5.22}$$

Using $\eta(v) = \frac{1}{2}v^2$,

$$\overline{(\eta' u - \eta)} - (\eta'(u)u - \eta(u)) = \overline{\eta} - \eta(u), \qquad \overline{\eta'} - \eta'(u) = u - u = 0. \tag{5.23}$$

Since $\eta'' = 1$,

$$\overline{\eta'' \left|\sigma\right|^2} - \eta''(u)\left|\overline{\sigma}\right|^2 = \overline{\left|\sigma\right|^2} - \left|\overline{\sigma}\right|^2 = \int_{\mathbb{R}} \left|\sigma(v) - \overline{\sigma}\right|^2 \,\nu_{\omega,t,x}(\mathrm{d}v),$$

where $\nu_{\omega,t,x}$ is the Young's measure at (ω, t, x) characterising the weak convergence of $u_n \rightharpoonup u$ [22, Theorem 6.2]. For any $Z \in \big(L^2(\nu_{\omega,t,x})\big)^k$, we have the elementary inequality for the variance:

$$\begin{aligned}\int_{\mathbb{R}} \left|Z(v) - \int_{\mathbb{R}} Z(w)\,\nu_{\omega,t,x}(\mathrm{d}w)\right|^2 \,\nu_{\omega,t,x}(\mathrm{d}v) \\ = \int_{\mathbb{R}} \left|Z\right|^2(v)\nu_{\omega,t,x}(\mathrm{d}v) - \left|\int_{\mathbb{R}} Z(v)\nu_{\omega,t,x}(\mathrm{d}v)\right|^2 \leq \int_{\mathbb{R}} \left|Z\right|^2(v)\nu_{\omega,t,x}(\mathrm{d}v).\end{aligned}$$

Let L be the maximum (global) Lipschitz constant among σ. Applying the variance inequality with $Z(v) = \sigma(v) - \sigma(u(\omega,t,x))$, and using $\int_{\mathbb{R}} v\,\nu_{\omega,t,x}(\mathrm{d}v) = u(\omega,t,x)$, we get

$$\begin{aligned}\int_{\mathbb{R}} \left|\sigma(v) - \overline{\sigma}\right|^2 \nu_{\omega,t,x}(\mathrm{d}v) \leq \int_{\mathbb{R}} \left|\sigma(v) - \sigma(u(\omega,t,x))\right|^2 \,\nu_{\omega,t,x}(\mathrm{d}v) \\ \leq L^2 \int_{\mathbb{R}} \left|v - u(\omega,t,x)\right|^2 \,\nu_{\omega,t,x}(\mathrm{d}v) = 2L^2\big(\overline{\eta} - \eta(u)\big).\end{aligned} \tag{5.24}$$

Putting (5.23) and (5.24) back in (5.22), and taking an expectation, we find:

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^d} \overline{\eta}(t) - \eta(u(t))\,\mathrm{d}x + \mathbb{E}\int_0^t \int_{\mathbb{T}^d} \big(\overline{\eta} - \eta(u)\big)\,\mathrm{div}\,(b)\,\mathrm{d}x\,\mathrm{d}s \\ \leq L^2\mathbb{E}\int_0^t \int_{\mathbb{T}^d} \overline{\eta} - \eta(u)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.25}$$

By convexity, we already know that $\overline{\eta} - \eta(u) \geq 0$. Gronwall's lemma applied to the inequality above then tells us that $\overline{\eta} = \eta(u)$, $\mathbb{P} \otimes \mathrm{d}t \otimes \mathrm{d}x$-a.e.

All the analysis foregoing applies to $\eta_\lambda(v) := \eta(v - \lambda) = \frac{1}{2}\left(v - \lambda\right)^2$ in place of η, giving $\overline{\eta_\lambda} = \eta_\lambda(u)$, where $\overline{\eta_\lambda}$ is the weak limit of $\eta_\lambda(u_n)$ in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$. Using again the Young's measure $\{\nu_{\omega,t,x}\}$ charactersing the convergence of u_n, we find that

$$\int_{\mathbb{R}} \eta(v - \lambda)\nu_{\omega,t,x}(\mathrm{d}v) = \overline{\eta_\lambda} = \eta_\lambda(u(\omega,t,x)) \geq 0,$$

with equality when $\lambda = u(\omega,t,x)$. The integrand $\eta(v-\lambda)$ is strictly positive away from $v=\lambda$. Therefore, $\nu_{\omega,t,x}$ must be supported on $\{u(\omega,t,x)\}$, and is a Dirac mass. By, e.g., [22, Proposition 6.12], we conclude that $u_n \to u$ in $L^p(\Omega\times[0,T]\times\mathbb{T}^d)$. $\square$

Remark 5.6. *Suppose $p<3$ and $|\sigma''|\lesssim 1$ but (5.5) is unavailable. In order to get (5.6) and (5.10) with $\vartheta = \eta'\sigma$ as in (5.15) – (5.18), we would have needed to use a convex, linearly growing approximation η_ℓ with bounded first and second derivatives (see,e.g., [13, Equation (4.2)]) in place of η to derive inequalities for the weak limits $\overline{\eta_\ell}$. This function will give us $(\eta_\ell)'\sigma \in C^{1,1}(\mathbb{R};\mathbb{R}^k)$, which is needed in order to derive bounds analogous to (5.15) – (5.18). We can then derive the following in place of (5.22):*

$$\begin{aligned}
&\int_{\mathbb{T}^d} \overline{\eta_\ell}(t) - \eta(u(t))\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d}\Big(\overline{\left((\eta_\ell)'\,u-\eta_\ell\right)} - (\eta'(u)u-\eta(u))\Big)\operatorname{div}(b)\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_0^t\int_{\mathbb{T}^d}\Big(\overline{(\eta_\ell)'}-\eta'(u)\Big)f\,\mathrm{d}x + \frac12\int_0^t\int_{\mathbb{T}^d}\overline{(\eta_\ell)''\,|\sigma|^2} - \eta''(u)\,|\overline{\sigma}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + \int_0^t\int_{\mathbb{T}^d}\overline{(\eta_\ell)'\,\sigma} - \eta'(u)\overline{\sigma}\,\mathrm{d}x\,\mathrm{d}W.
\end{aligned}$$

It is possible to derive managable expressions for $\eta_\ell(u_n)-\eta(u_n)$, $(\eta_\ell)'(u_n)-\eta'(u_n)$, etc. (see [13, Remark 7.11]), which are supported on the small sets $\{|u_n|\ge \ell\}$. Isolating these as error terms appended onto (5.22), the uniform-in-n bound

$$(\mathbb{P}\otimes \mathrm{d}x\otimes\mathrm{d}t)(\{|u_n|\ge\ell\}) \le \ell^{-p}\mathbb{E}\,\|u_n\|^p_{L^p([0,T]\times\mathbb{T}^d)} \lesssim \ell^{-p}$$

gives us the a.e. equality $\eta(u)=\overline{\eta}$ in the $\ell\uparrow\infty$ limit, similar to the argument in [13, Proof of Theorem 7.1]. That will imply the strong convergence $u_n\to u$ in $L^p_{\omega,t,x}$ as in the proof of Theorem 5.1 above. Using this linearly growing nonlinearity η_ℓ, we can also relax the assumption (5.5) on the initial data to convergence in $L^p_{\omega,t}$, where this was used in (5.14).

The approximation η_ℓ will also allow us to relax to $q_2\ge 2$ in the assumption $u_n\rightharpoonup u$ in $L^{q_1}_{\omega,t}L^{q_2}_x$, from the present assumption of $q_2>2$, which is necessary to keep the bound (5.13) from being spatially in L^1_x, where a Young's measure representation may not exist. For the sake of expository clarity, we adopted $p>2$ to avoid keeping track of an extra parameter in order to take the $\ell\uparrow\infty$ limit.

The convergence theorems of Section 4 are important even in the linear, additive noise case. We consider a transport equation with noise $\sigma_n\,\mathrm{d}W_n$, where $\sigma_n\to\sigma$ in $L^p([0,T]\times\mathbb{T}^d;\mathbb{R}^k)$. Our convergence theorems are required in the renormalised equation for $\mathrm{d}\eta(u_n)$, $\eta(v)=\frac12 v^2$, to obtain the limit for $\int_0^t\eta'(u_n)\sigma_n\,\mathrm{d}W_n$ (cf. (5.12) above and see (5.26) below). Suitably modifying the definition of solutions to take into account the additive nature of the noise, we can establish the following theorem:

Theorem 5.7. *Fix $p>2$. Set $p'=p/(p-1)$ and $p''=p/(p-2)$, respectively the Hölder conjugates of p and $p/2$. Let $\{u_n\}_{n=1}^\infty$ be a sequence of weak solutions to*

$$\mathrm{d}u_n + \operatorname{div}(b_n u_n)\,\mathrm{d}t = f_n\,\mathrm{d}t + \frac1n\Delta u_n\,\mathrm{d}t + \sigma_n\,\mathrm{d}W_n,\quad u_n(0)=u_{0,n},$$

for which $u_n\rightharpoonup u$ in $L^p(\Omega\times[0,T]\times\mathbb{T}^d)$. Suppose

(i) $\sigma_n\to\sigma$ *in* $L^p([0,T]\times\mathbb{T}^d;\mathbb{R}^k)$,
(ii) $\{(\partial_t\sigma_n,\nabla\sigma_n)\}\subset_b L^1([0,T];L^{p'}(\mathbb{T}^d;\mathbb{R}^k))\times L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^{d\times k})$

(iii) $\{b_n\}_{n\geq 1} \subset_b L^1([0,T]; W^{1,p''}(\mathbb{T}^d;\mathbb{R}^d))$, *and* $\{\mathrm{div}\,(b_n)\}_{n\geq 1} \subset_b L^1([0,T]; L^\infty(\mathbb{T}^d))$,

(iv) $b_n \to b$ *in* $L^1([0,T]; L^{p'}(\mathbb{T}^d;\mathbb{R}^d))$, *and* $\mathrm{div}\,(b_n) \to \mathrm{div}\,(b)$, *in* $L^1([0,T]; W^{1,p''}(\mathbb{T}^d))$,

(v) $f_n \to f$ *in* $L^1([0,T]; L^p(\mathbb{T}^d))$, $u_{0,n} \to u_0$ *in* $L^p(\Omega\times\mathbb{T}^d)$, *and*

(vi) $W_n \to W$ *a.s. in* $C([0,T];\mathbb{R}^k)$.

Then u *is a weak solution of*

$$\mathrm{d}u + \mathrm{div}\,(bu)\,\mathrm{d}t = f\,\mathrm{d}t + \sigma\,\mathrm{d}W, \quad u(0) = u_{0,n},$$

and $u_n \to u$ *in* $L^p(\Omega\times[0,T]\times\mathbb{T}^d)$.

Proof. We shall avail ourselves of the same strategy we used in the proof of Theorem 5.1, and borrow heavily from the machinery described there. First, we argue as in (5.8) to achieve the same bound with $\sigma(u_n)_\delta$ replaced by $\sigma_{n,\delta}$ in the derivation. Let $\eta(v) = \frac{1}{2}v^2$ again. By arguing as in (5.12), we get the analogous inequality

$$\begin{aligned}\int_{\mathbb{T}^d} \eta(u_n(t)) - \eta(u_{0,n})\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d} \left(\eta'(u_n)u_n - \eta(u_n)\right)\,\mathrm{div}\,(b_n)\,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_0^t\int_{\mathbb{T}^d} \eta'(u_n(t))f_n + \frac{1}{2}\left|\sigma\right|_n^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^d} \eta'(u_n(t))\sigma_n\,\mathrm{d}x\,\mathrm{d}W_n.\end{aligned} \tag{5.26}$$

We now seek to take the limit as $n\to\infty$ to get the equation for $\overline{\eta}$. By the bound (5.8) on u_n, by the form of η and the uniqueness of weak limits, we have the following weak convergences as $n\uparrow\infty$:

$$\begin{aligned}(\eta(u_n), \eta'(u_n)u_n, \eta'(u_n)) \rightharpoonup (\overline{\eta}, 2\overline{\eta}, u) \text{ in } (L^{p/2}(\Omega\times[0,T]\times\mathbb{T}^d))^3,\\ \eta'(u_n)\sigma_n \rightharpoonup u\sigma \text{ in } L^{p/2}(\Omega\times[0,T]\times\mathbb{T}^d;\mathbb{R}^k),\end{aligned}$$

along an unrelabelled subsequence. The convergence of the final element of the tuple follows from the strong convergence of σ_n in $L^p([0,T]\times\mathbb{T}^d;\mathbb{R}^k)$, assumed in (i).

The convergence of (5.26) to

$$\begin{aligned}\int_{\mathbb{T}^d} \overline{\eta}(t) - \eta(u_0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d} \overline{\eta' u - \eta}\,\mathrm{div}\,(b)\,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_0^t\int_{\mathbb{T}^d} u\,f + \frac{1}{2}\left|\sigma\right|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^d} u\sigma\,\mathrm{d}x\,\mathrm{d}W,\end{aligned} \tag{5.27}$$

follows from the assumed convergences on the coefficients b_n, f_n, σ_n and the weak convergence $u_n \rightharpoonup u$, except in the stochastic integral.

In the stochastic integral $\int_0^t\int_{\mathbb{T}^d} u_n\sigma_n\,\mathrm{d}x\,\mathrm{d}W_n$, $\int_{\mathbb{T}^d} u_n\sigma_n\,\mathrm{d}x$ converges only weakly in (ω,t) due to the assumed weak convergence of u_n. The lack of strong temporal compactness compels us to apply on of the variants of Theorem 2.1 here. Using the supremum bound (5.8) one u_n, we find that

$$\mathbb{E}\,\|\eta'(u_n)\sigma_n\|^q_{L^p_tL^1_x} \lesssim \|\sigma_n\|^p_{L^p_tL^{p'}_x} \lesssim 1, \quad \mathbb{E}\sup_{t\in[0,T]}\|u_n(t)\|^p_{L^p_x} \lesssim 1. \tag{5.28}$$

Next we deduce the temporal translation estimate using Lemma 5.3. By the product formula,

$$\mathrm{d}\,(u_n\sigma_n) = \partial_t\sigma_n\,u_n\,\mathrm{d}t + \sigma_n\,\mathrm{d}u_n.$$

We can integrate this against any $\psi \in C^1(\mathbb{T}^d)$ to get:

$$\begin{aligned}&\int_{\mathbb{T}^d} \psi(x)\big((u_n\sigma_n)(t) - (u_n\sigma_n)(t-h)\big)\,\mathrm{d}x \\ &= \int_{t-h}^t \int_{\mathbb{T}^d} \underbrace{\psi \partial_t \sigma_n u_n}_{\tilde{I}_1(n)} \,\mathrm{d}x\,\mathrm{d}r + \int_{t-h}^t \int_{\mathbb{T}^d} \underbrace{\nabla(\psi\sigma_n)\cdot b_n u_n + \psi\sigma_n f_n}_{\tilde{I}_2(n)} \,\mathrm{d}x\,\mathrm{d}r \\ &\quad - \frac{1}{n}\int_{t-h}^t \int_{\mathbb{T}^d} \underbrace{\nabla(\psi\sigma_n)\cdot\nabla u_n}_{-n\tilde{I}_3(n)} \,\mathrm{d}x\,\mathrm{d}r + \int_{t-h}^t \int_{\mathbb{T}^d} \underbrace{\psi\left|\sigma_n\right|^2}_{\tilde{I}_4(n)} \,\mathrm{d}x\,\mathrm{d}W_n.\end{aligned}$$

Using (5.8) and the assumed bounds in the theorem statement, we can show via Lemma 5.3 that

$$\mathbb{E}\int_h^T \left| \int_{\mathbb{T}^d} \psi(x)\big((u_n\sigma_n)(t) - (u_n\sigma_n)(t-h)\big)\,\mathrm{d}x \right| \lesssim h^{1/2}, \quad \text{uniformly in } n.$$

Indeed,

$$\begin{aligned}&\left\|\tilde{I}_1(n)\right\|_{L^1(\Omega\times[0,T]\times\mathbb{T}^d)} + \left\|\tilde{I}_2(n)\right\|_{L^1(\Omega\times[0,T]\times\mathbb{T}^d)} \\ &\quad\le \|\psi\|_{L^\infty_x}\Big(\|\partial_t\sigma_n\|_{L^1_tL^{p'}_x} + \|\nabla(\psi\sigma_n)\|_{L^1_tL^{p'}_x}\|b_n\|_{L^1_tL^{p''}_x}\Big)\mathbb{E}\|u_n\|_{L^\infty_tL^p_x} \\ &\qquad + \|\psi\|_{L^\infty_x}\|\sigma_n\|_{L^p_tL^2_x}\|f_n\|_{L^{p'}_tL^2_x} \lesssim 1,\end{aligned}$$

and

$$\left\|\tilde{I}_3(n)\right\|_{L^1(\Omega\times[0,T]\times\mathbb{T}^d)} \le \|\nabla(\psi\sigma_n)\|_{L^2([0,T]\times\mathbb{T}^d)}\frac{1}{n}\mathbb{E}\|\nabla u_n\|_{L^2([0,T]\times\mathbb{T}^d)} \to 0.$$

Since $\left|\sigma_n\right|^2 \in_b L^2([0,T];L^1(\mathbb{T}^d))$ are deterministic, $\left\|\tilde{I}_4(n)\right\|_{L^2(\Omega\times[0,T];L^1(\mathbb{T}^d))} \lesssim 1$. By choosing $\psi \equiv 1$, Theorem 3.1 then implies the convergence of the stochastic integral in (5.26) weakly in $L^2(\Omega)$ and we get (5.27).

With $\eta(v) = \frac{1}{2}v^2$, the derivation of the equation for $\eta(u)$ follows the same procedure as that leading up to (5.21), but is more straightforward. In particular, the convergence of stochastic integrals in this process only relies on [5, Lemma 2.1], seeing as $\int_{\mathbb{T}^d}\psi\sigma_\delta u_\delta\,\mathrm{d}x \to \int_{\mathbb{T}^d}\psi u\sigma\,\mathrm{d}x$ a.s. in $L^2([0,T])$ by the assumed inclusion $\sigma \in L^p([0,T]\times\mathbb{T}^d)$ and (5.28) (cf. (5.9)).

We have that for any $\psi \in C^2(\mathbb{T}^d)$,

$$\begin{aligned}&\int_{\mathbb{T}^d}\psi(x)\left(\eta(u)-\eta(u_0)\right)\,\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\Big[\psi\,\mathrm{div}(b)u^2 + \nabla\psi\cdot bu^2\Big]\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{T}^d}\psi uf\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^d}\psi u\sigma\,\mathrm{d}x\,\mathrm{d}W + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\psi\left|\sigma\right|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.29}$$

Subtracting (5.29) (with $\psi \equiv 1$) from (5.27), we find

$$\int_{\mathbb{T}^d}\overline{\eta}(t) - \eta(u(t))\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d}\left(\overline{\eta' u - \eta} - \frac{1}{2}u^2\right)\mathrm{div}\,(b)\,\mathrm{d}x\,\mathrm{d}s \le 0.$$

Since $\overline{\eta' u - \eta} = \frac{1}{2}\overline{u^2} = \overline{\eta}$, we can take an expectation above and apply Gronwall's inequality to get

$$\mathbb{E}\int_{\mathbb{T}^d}\overline{\eta} - \eta(u)\,\mathrm{d}x = 0, \quad \mathrm{d}t\text{-a.e.}$$

Since $\overline{\eta} \geq \eta(u)$ (ω, t, x)-a.e. by convexity, we arrive at the a.e. equality $\overline{\eta} = \eta(u)$. We can then argue as following (5.25) and conclude that $u_n \to u$ in $L^p(\Omega \times [0,T] \times \mathbb{T}^d)$. □

Remark 5.8. *Under suitable assumptions, our techniques are applicable to SPDEs with transport noise. Specifically, they can assist in facilitating the passage to the limit in stochastic integrals of the form* $\int_0^T \int_{\mathbb{T}^d} \sigma_n \nabla u_n \, \varphi \, \mathrm{d}x \circ \mathrm{d}W_n$, *where* $\circ$ *refers to the Stratonovich integral, where* σ_n *is* $\mathbb{R}^{d\times k}$*-valued*

6. Stochastic conservation laws

In this section, we focus on an application pertaining to sequences of stochastic conservation laws (1.6), as expressed in the form of kinetic equations (1.7). In (1.7), the driving noise processes W_n are $\mathbb{R}^k$-valued Brownian motions. The random defect measures m_n in (1.7) take values in $M_b^+([0,T] \times \mathbb{T}^d \times \mathbb{R})$, the space of non-negative Radon measures. Moreover, the solutions χ_n to (1.7) trivially satisfy $0 \leq \chi_n \leq 1$. The interpretation of the kinetic equations (1.7) is in the Itô sense and is considered a.s. in a weak formulation over the domain $\mathbb{T}^d \times \mathbb{R}$ [6].

By the Itô isometry, convergence of stochastic integrals in L^2_ω, assuming $W_n = W$ for each n, essentially translates to weak convergence of the integrands across the variables (ω, t, x). This specific case is straightforward and has been treated in, e.g., [6, 8, 9]. In contrast, our study extends to the scenario involving a sequence of potentially distinct Brownian motions $\{W_n\}_{n\geq 1}$.

Theorem 6.1. *Fix* $p > 2$. *Let* χ_n *be the kinetic solution to* (1.7) *and suppose that*

(i) σ_n *are bounded in* $W^{3,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$,
(ii) $F_n \to F$ *in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$, *and* $\sigma_n \to \sigma$ *in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$,
(iii) $m_n([0,t]) \rightharpoonup m([0,t])$ *in* $L^1(\Omega \times [0,T]; M_b^+(\mathbb{T}^d \times \mathbb{R}))$, *in the sense that for every* $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$ *and every* $Y \in L^\infty(\Omega \times [0,T])$,

$$\mathbb{E}\left[\int_0^T Y \int_{\mathbb{T}^d\times\mathbb{R}} \varphi \, m_n([0,t], \mathrm{d}x, \mathrm{d}\xi)\, \mathrm{d}t\right] \to \mathbb{E}\left[\int_0^T Y \int_{\mathbb{T}^d\times\mathbb{R}} \varphi \, m([0,t], \mathrm{d}x, \mathrm{d}\xi)\, \mathrm{d}t\right],$$

(iv) $\chi_{0,n} \rightharpoonup \chi_0$ *in* $L^p(\Omega \times \mathbb{T}^d; \mathcal{D}'(\mathbb{R}))$, *and*
(v) $W_n \to W$ *in* $C([0,T];\mathbb{R}^k)$ *a.s.*

Assume $\chi_n \overset{\star}{\rightharpoonup} \chi$ *in* $L^\infty(\Omega \times [0,T] \times \mathbb{T}^d \times \mathbb{R})$. *Then* χ *satisfies*

$$\mathrm{d}\chi + \left(F'(\xi)\cdot \nabla_x \chi - \partial_\xi m\right)\mathrm{d}t - \sigma(\xi)\partial_\xi \chi \, \mathrm{d}W - \frac{1}{2}\partial_\xi \left(|\sigma(\xi)|^2 \, \partial_\xi \chi\right)\mathrm{d}t = 0, \tag{6.1}$$

a.s. in the sense of Itô for a.e. $t \in [0,T]$, *and in* $\mathcal{D}'(\mathbb{T}^d \times \mathbb{R})$ *with* $\chi(0) = \chi_0$.

By $m_n([0,t])$, we mean $\int_{r=0}^t m_n(\mathrm{d}r, \mathrm{d}x, \mathrm{d}\xi)$, which remains a measure in the space $L^1_\omega(M_b^+)_{x,\xi}$ as m_n admits a disintegration [10, Theorem 1.10].

Proof. Let $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$ so that $\mathrm{supp}(\varphi) \subset K \Subset \mathbb{T}^d \times \mathbb{R}$. Using the definition of weak solutions to (1.7) tested against the φ, we see that each term of (1.7) tends weakly in (ω, t) to the appropriate limit by the assumptions of the theorem. It remains then to argue that the stochastic integral converges, whereupon we can invoke Lemma 5.5 to conclude.

Since χ_n converges only weakly-$\star$ in $L^\infty_{\omega,t,x,\xi}$, we will apply Theorem 2.1 to get weak convergence of the stochastic integral in $L^2(\Omega)$:

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_\xi \left(\varphi\sigma_n\right) \chi_n \, \mathrm{d}x \, \mathrm{d}\xi \, \mathrm{d}W_n \xrightarrow{n\uparrow\infty} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_\xi \left(\varphi\sigma\right) \chi \, \mathrm{d}x \, \mathrm{d}\xi \, \mathrm{d}W. \tag{6.2}$$

By $W^{3,1}_{\mathrm{loc}} \hookrightarrow L^\infty_{\mathrm{loc}}$ and the assumption on σ_n, and the $L^\infty_{\omega,t,x,\xi}$ bound on χ_n, we find

$$\mathbb{E} \left\| \partial_\xi \left(\varphi\sigma_n\right) \chi_n \right\|^p_{L^p([0,T];L^2(\mathbb{T}^d\times\mathbb{R};\mathbb{R}^k))} \lesssim 1. \tag{6.3}$$

The time translation estimate will follow from Lemma 5.3, which can be easily modified to accommodate integration on $\mathbb{T}^d \times \mathbb{R}$ instead of $\mathbb{T}^d$, against ξ-compactly supported test functions. Integrating (1.7) against $\psi\partial_\xi\left(\varphi\sigma_n\right)$, where $\psi \in C^2_c(\mathbb{T}^d \times \mathbb{R})$, set

$$I_1(n) := \nabla_x \otimes \big(\psi\partial_\xi\left(\varphi\sigma_n\right)\big) F_n'(\xi)\chi_n, \quad I_2(n) := \partial_\xi\left(\psi\partial_\xi\left(\varphi\sigma_n\right)\right) m_n,$$
$$I_3(n) := \partial_\xi \Big(\left|\sigma_n\right|^2 \partial_\xi\left(\psi\,\partial_\xi\left(\varphi\sigma_n\right)\right)\Big) \chi_n, \quad I_4(n) := \partial_\xi\big(\sigma_n \otimes \psi\,\partial_\xi\left(\varphi\sigma_n\right)\big)\chi_n.$$

As in (5.15) – (5.18), we can estimate these terms separately. We have

$$\left\| I_1(n) \right\|_{L^1_{\omega,t,x,\xi}} \le T \left\| \nabla \otimes \left(\psi\,\partial_\xi\left(\varphi\sigma_n\right)\right)\right\|_{L^\infty(\mathbb{T}^d\times\mathbb{R})} \left\| F_n' \right\|_{L^1(K)} \lesssim 1,$$

since $\sigma_n \in W^{3,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k) \hookrightarrow W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$, and $F_n \to F$ in $W^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$. Similarly,

$$\left\| I_2(n) \right\|_{L^1_\omega TV_{\xi,t,x}} \le \left\| \partial_\xi\left(\psi\partial_\xi(\varphi\sigma_n)\right)\right\|_{L^\infty_{t,x,\xi}} \mathbb{E}\, m_n([0,T]\times\mathbb{T}^d\times\mathbb{R}) \lesssim 1.$$

Next, using that $0 \le \chi_n \le 1$,

$$\left\| I_3(n) \right\|_{L^1_{\omega,t,x,\xi}} \le \left\| \partial_\xi \Big(\left|\sigma_n\right|^2 \partial_\xi\left(\psi\,\partial_\xi\left(\varphi\sigma_n\right)\right)\Big)\right\|_{L^1(K)} \lesssim 1.$$

Finally, we have

$$\left\| I_4 \right\|_{L^1_{\omega,t}L^2_{x,\xi}} \le T \left\| \partial_\xi\big(\sigma_n \otimes \psi\,\partial_\xi\left(\varphi\sigma_n\right)\big)\right\|_{L^1_{x,\xi}} \lesssim 1.$$

The bounds above are j-independent. Therefore, Lemma 5.3 gives us

$$\mathbb{E} \int_h^T \left| \int_{\mathbb{R}} \int_{\mathbb{T}^d} \psi\partial_\xi\left(\varphi\sigma_n\right) \big(\chi_n(t) - \chi_n(t-h)\big) \, \mathrm{d}x \, \mathrm{d}\xi \right| \mathrm{d}t \lesssim_{T,\psi} h^{1/2},$$

uniformly in n. Along with (6.3) and the weak $L^p_{\omega,t,x,\xi}$ convergence of χ_n, Theorem 2.1 allows us to conclude that the stochastic integral term converges (6.2).

It then follows that χ is a weak solution to the limiting equation (6.1). □

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