An Overpartition Companion of Andrews and Keith's 2-colored q-series Identity

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Abstract

Andrews and Keith recently produced a general Schmidt type partition theorem using a novel interpretation of Stockhofe's bijection, which they used to find new q-series identities. This includes an identity for a trivariate 2-colored partition generating function. In this paper, their Schmidt type theorem is further generalized akin to how Franklin classically extended Glaisher's theorem. As a consequence, we obtain a companion to Andrews and Keith's 2-colored identity for overpartitions. These identities appear to be special cases of a much more general result.

1 Schmidt Type Partitions

There has been a growing interest (see [2, 4, 6, 10, 11, 14, 15, 21, 22]) following the expository work of Andrews and Paule [5] in researching the combinatorics of Schmidt type partitions, which are partitions weighted by the sum of parts having only indices from some strict subset of the positive integers. Their namesake Schmidt was the first to observe Theorem 1, which has since witnessed numerous proofs (notably, Mork produced a bijection [17] using diagonal hooks, similar to a variant of Sylvester's classic bijection [7]).

Theorem 1 (Schmidt [18])**.** *For all $n \geq 1$, the number of partitions λ with distinct parts such that $\lambda_1 + \lambda_3 + \lambda_5 + \cdots = n$ is equal to the number of partitions of size n.*

For the Schmidt weight to be defined, we must take any partition λ to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ of $\ell(\lambda)$ positive integers with $\lambda_k = 0$ whenever $k > \ell(\lambda)$, a convention used implicitly throughout this paper. The parts of λ are the positive λ_k, the size is the sum of all parts $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \cdots$, and the length is the number of parts $\ell(\lambda)$.

The first generalization of Theorem 1 was found by Uncu [20] as a special case of an identity using Boulet-Stanley weights [8]. Uncu's theorem can be stated as follows.

Theorem 2 (Uncu [20])**.** *For all $n \geq 1$, the number of partitions λ such that $\lambda_1 + \lambda_3 + \lambda_5 + \cdots = n$ is equal to the number of 2-colored partitions of n.*

Remark 1. Bowman and Alladi also each found results before Schmidt that can be considered equivalent to Theorem 1 (see [1, 2, 9]).

Here we use the language of colored partitions, where a part size is said to appear in t colors, which we take to be some fixed positive integers $c_1, \ldots, c_t$, if distinct markings of all such parts using these colors as subscripts in weakly decreasing order are distinguished. A t-colored partition has all part sizes appearing in t colors. For example, with colors $\{1,2\}$ the 2-colored partitions of 3 are (3_2), (3_1), $(2_2,1_2)$, $(2_2,1_1)$, $(2_1,1_2)$, $(2_2,1_2)$, $(1_2,1_2,1_2)$, $(1_2,1_2,1_1)$, $(1_2,1_1,1_1)$, and $(1_1,1_1,1_1)$.

Let $\mathcal{P}$ be the set of all partitions, and for given $m \geq 2$, let $\mathcal{D}_m \subset \mathcal{P}$ contain only the partitions where each part size appears fewer than m times. Schmidt weights appear to be intimately connected to colored partitions. This is exemplified by Andrews and Keith [4] in Theorem 3, which generalizes Theorem 1 from $\mathcal{D}_2$ (which is the set of partitions with distinct parts) to $\mathcal{D}_m$, along with a far more general Schmidt weight. Li and Yee also generalized Uncu's theorem to t-colored partitions in for all of $\mathcal{P}$ [15], which they call multipartitions.

Theorem 3 (Andrews-Keith [4])**.** *Fix $m \geq 2$ and $S = \{s_1, \ldots, s_i\} \subseteq \{1, 2, \ldots, m-1\}$ with $1 \in S$. For all $n \geq 1$, The partitions $\lambda \in \mathcal{D}_m$ such that*

$$n = \sum_{\substack{k \equiv S \\ (\mathrm{mod}\ m)}} \lambda_k = \lambda_{s_1} + \cdots + \lambda_{s_i} + \lambda_{s_1+m} + \cdots + \lambda_{s_i+m} + \cdots$$

$$\rho_j = \sum_{k \geq 0} (\lambda_{mk+j} - \lambda_{mk+j+1}) = \lambda_j - \lambda_{j+1} + \lambda_{j+m} - \lambda_{j+m+1} + \cdots$$

for $1 \leq j < m$ are equinumerous with the partitions of size n in $\mathcal{P}$ where any parts congruent to k modulo i appears in the $s_{k+1} - s_k$ colors $\{s_k, \ldots, s_{k+1}-1\}$ where we take $s_{i+1} = m$, and parts of color j appear ρ_j times.

As an application of Theorem 3, Andrews and Keith obtained new q-series sum-product identities, including the trivariate identity shown here in Theorem 4 that has a product side well-known to be a generating function for 2-colored partitions.

Theorem 4 (Andrews-Keith [4])**.** *We have the equality*

$$\sum_{n \geq 0} \sum_{\substack{j+k \geq n \\ j,k \leq n}} \frac{(-1)^{j+k+n} t_1^j t_2^k q^{\binom{n}{2} + \binom{j+1}{2} + \binom{k+1}{2}} \begin{bmatrix} n \\ n-j, n-k, j+k-n \end{bmatrix}_q}{(t_1 q; q)_n (t_2 q; q)_n (q; q)_n} = \frac{1}{(t_1 q; q)_\infty (t_2 q; q)_\infty}.$$

This is written as an equality of formal power series (here, in the ring $k[[q, t_1, t_2]]$ for any characteristic zero field k) using the standard notation for the q-Pochhammer symbol $(z;q)_0 = 1$ and $(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k)$ for $n > 0$, with the limiting case as $n \to \infty$ written $(z;q)_\infty$. The q-multinomial coefficient is given below on the left for $n = k_1 + \cdots + k_t$ with each $k_i \geq 0$

$$\begin{bmatrix} n \\ k_1, \ldots, k_t \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{k_1} \cdots (q;q)_{k_t}} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

and the q-binomial coefficient is the special case where $t = 2$, which can unambiguously be written as shown above on the right.

2 New Results

The 2-colored partitions are related to the well-studied overpartitions $\overline{\mathcal{P}}$, introduced by Corteel and Lovejoy in [16], which are partitions in $\mathcal{P}$ where the first occurrence of any part size may be distinguished with an overline. Overpartitions can be identified with 2-colored partitions where a fixed color may appear at most once per part size. For example, the overpartitions of 3 are $(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}),\ \ (1,1,1)$, and $(\overline{1},1,1)$.

The equivalent of the product side in Theorem 4 for overpartitions is $(-qt_1;q)_\infty/(qt_2;q)_\infty$. Andrews and Keith found a relationship between Schmidt type partitions and overpartitions in [4], although their result is inadequate to imply a companion identity for Theorem 4 with all three variables present. The main result in this paper is such an identity, which is presented here.

Theorem 5. *We have the equality*

$$\sum_{n\geq 0}\sum_{\substack{j+k\geq n\\ j,k\leq n}} \frac{(-1)^{j+k+n}t_1^j t_2^k q^{\binom{n}{2}+\binom{k+1}{2}+j^2-nj+j}\begin{bmatrix} n \\ n-j,n-k,j+k-n\end{bmatrix}_q}{(t_2q;q)_n(q;q)_n} = \frac{(-t_1q;q)_\infty}{(t_2q;q)_\infty}.$$

Ultimately, the proof of Theorem 5 given in this paper uses q-series techniques. However the identity was first conjectured by the author after further generalizing Theorem 3 to allow partition parts to appear freely, rather than being restricted to $\mathcal{D}_m$, with excessive repetition being tracked.

Theorem 6. *Fix m and S as in Theorem 3. The partitions in $\mathcal{P}$ with the same conditions on n and the ρ_j, having exactly the parts $\alpha_1, \ldots, \alpha_t$ appearing $p_1, \ldots, p_t \geq m$ times respectively, are equinumerous with the partitions of n in $\mathcal{P}$ with the same conditions on the colors, with in addition the parts $i\alpha_1, \ldots, i\alpha_t$ appearing $\lfloor p_1/m\rfloor, \ldots, \lfloor p_t/m\rfloor$ times respectively, in only the color $\{m\}$.*

Remark 2. The generalization of Theorem 3 to Theorem 6 is similar to how Franklin [12] classically extended Glaisher's theorem [13].

An explanation of how Theorem 5 was conjectured is provided here. In the limiting case of Theorem 6 with $m = 2$ and $S = \{1\}$ where all parts are allowed to repeat fewer than 4 times, we obtain the following equality, expressed in terms of generating functions.

Corollary 1. *Let $e(\lambda)$ be the number of part sizes for $\lambda \in \mathcal{D}_4$ that appear 2 or 3 times and $o(\mu)$ be the number of parts that are overlined for $\mu \in \overline{\mathcal{P}}$. Then*

$$\sum_{\lambda\in\mathcal{D}_4} t_1^{e(\lambda)} t_2^{\lambda_1-\lambda_2+\lambda_3-\lambda_4+\cdots} q^{\lambda_1+\lambda_3+\lambda_5+\cdots} = \sum_{\mu\in\overline{\mathcal{P}}} t_1^{o(\mu)} t_2^{\ell(\mu)-o(\mu)} q^{|\mu|}.$$

The right hand side has the desired product form of Theorem 5, whereas developing the left hand side using a recurrence like those in [4] gives the terms of the sum. Define for $k \geq 0$

$$Q_{2k} = q^k$$
$$Q_{2k+1} = t_2 q^{k+1}$$

and let L_n for $n \geq 0$ be the generating function for the Schmidt type partitions having exactly n parts. Then, with the base cases $L_0 = 1, L_1 = Q_1/(1-Q_1)$, and

$$L_2 = \frac{Q_2 t_1}{1 - Q2} + \frac{Q_1 Q_2}{(1-Q_1)(1-Q_2)}$$

the recurrence

$$L_n = \frac{Q_n}{1-Q_n}\left(L_{n-1} + t_1 L_{n-2} + t_1 L_{n-3}\right)$$

is straightforward to establish.

An example of the partitions being enumerated in Corollary 1 is provided here. The coefficient of $t_1 t_2^2 q^6$ is 6 since the relevant Schmidt type partitions are $(5,3,1,1)$, $(4,3,1,1,1)$, $(3,3,3,1)$, $(3,3,2,1,1)$, $(3,2,2,2,1)$, and $(4,2,2,2)$. The overpartitions are $(\overline{4},1,1)$, $(4,\overline{1},1)$, $(\overline{2},2,2)$, $(\overline{3},2,1)$, $(3,\overline{2},1)$, and $(3,2,\overline{1})$.

Given the sheer generality of Theorem 6, there is a strong reason to expect that Theorem 4 and Theorem 5 are the $t = 2$ cases of infinite families of identities, one for each $t \geq 2$ associated to t-colored partitions, and a related family of Schmidt type partitions. The recurrences for larger t are no more difficult to establish, but even conjecturing the identity becomes considerably more difficult for $t = 3$.

Question. Can Theorem 4 and Theorem 5 be generalized for t-colored partitions, for each $t \geq 2$?

We finish this section with the proof of Theorem 5. In section 3, the proof of Theorem 6 is given, as well as a discussion of some of the associated generating functions.

Proof of Theorem 5. Using well-known generating function identities, we can write

$$\frac{(-t_1 q;q)_\infty}{(t_2 q;q)_\infty} = \sum_{n\geq 0} \frac{t_1^n q^{\binom{n+1}{2}}}{(q;q)_n} \times \sum_{m \geq 0} \frac{t_2^m q^{m^2}}{(q;q)_m (t_2 q;q)_m}$$

and so for any fixed $J \geq 0$, the coefficient of t_1^J on the product side of Theorem 5 is

$$\frac{q^{\binom{J+1}{2}}}{(q;q)_J} \sum_{m\geq 0} \frac{t_2^m q^{m^2}}{(q;q)_m (qt_2;q)_m}.$$

We proceed now to show that the coefficient of t_1^J is the same on the sum side. Extracting the coefficient of t_1^J, we obtain

$$
\begin{aligned}
&(-1)^J q^{J^2+J}\sum_{n\ge J}\sum_{\substack{J+k\ge n\\ k\le n}}\frac{(-1)^{k+n}t_2^k q^{\binom{n}{2}+\binom{k+1}{2}-nJ}\left[{n \atop n-J,n-k,J+k-n}\right]_q}{(t_2q;q)_n(q;q)_n}\\
=&q^{\binom{J+1}{2}}\sum_{n\ge 0}\sum_{k=n}^{n+J}\frac{(-1)^{k+n}t_2^k q^{\binom{n}{2}+\binom{k+1}{2}}\left[{n+J \atop n,n+J-k,k-n}\right]_q}{(t_2q;q)_{n+J}(q;q)_{n+J}}\\
=&q^{\binom{J+1}{2}}\sum_{n\ge 0}\sum_{k=0}^{J}\frac{(-1)^{k}t_2^{n+k} q^{n^2+\binom{k+1}{2}+nk}\left[{n+J \atop n,J-k,k}\right]_q}{(t_2q;q)_{n+J}(q;q)_{n+J}}\\
=&\frac{q^{\binom{J+1}{2}}}{(q;q)_J}\sum_{n\ge 0}\frac{t_2^n q^{n^2}}{(t_2q;q)_{n+J}(q;q)_n}\sum_{k=0}^{J}(-1)^k t_2^k q^{\binom{k}{2}+(n+1)k}\left[{J \atop k}\right]_q
\end{aligned}
$$

by shifting n to $n+J$, k to $n+k$, and then finally breaking apart the q-multinomial coefficient and rewriting. Now we can evoke Cauchy's q-binomial theorem (see [3], Theorem 3.3)

$$(z;q)_N=\sum_{k=0}^{N}(-1)^k z^k q^{\binom{k}{2}}\left[{N \atop k}\right]_q$$

with $z=t_2q^{n+1}$ to write this in the desired form

$$
\begin{aligned}
&\frac{q^{\binom{J+1}{2}}}{(q;q)_J}\sum_{n\ge 0}\frac{t_2^n q^{n^2}}{(t_2q;q)_{n+J}(q;q)_n}(t_2q^{n+1};q)_J\\
=&\frac{q^{\binom{J+1}{2}}}{(q;q)_J}\sum_{n\ge 0}\frac{t_2^n q^{n^2}}{(t_2q;q)_{n}(q;q)_n}.
\end{aligned}
$$

□

3 The Proof of Theorem 6.

We start this section by introducing some notation and background. The reader is refered to [3] for fundamental definitions related to visualizing partitions, including the Ferrers diagram and the conjugate λ' of a partition $\lambda\in\mathcal{P}$.

For $m\ge 2$, let $\mathcal{R}_m\subset\mathcal{P}$ contain only the partitions with all parts divisble by m, and $\mathcal{F}_m$ be the set of conjugates of the partitions in $\mathcal{D}_m$. Equivalently, $\lambda\in\mathcal{F}_m$ if $\lambda_k-\lambda_{k+1}<m$ holds for all $k\ge 1$. The partitions of a fixed size in $\mathcal{R}_m$ and $\mathcal{D}_m$ are famously equinumerous [13], as are those in $\mathcal{D}_m$ and $\mathcal{F}_m$ since conjugation is a bijection. Stockhofe ([19], in German) produced a family of bijections from $\mathcal{P}$ to $\mathcal{P}$ that generalize conjugation, and can be restricted to a map $\phi_m\colon\mathcal{F}_m\to\mathcal{R}_m$ that Andrews and Keith showed implies Theorem 3.

To prove Theorem 6, we will extend ϕ_m to a larger map $\Phi_m\colon\mathcal{P}\to\mathcal{P}$ that has the desired properties. We note here that Stockhofe's full maps may very well also imply the theorem, but the proof given here is also likely much less sophisticated. We will also need the the map $\Psi_{m,S}\colon \mathcal{P} \to \mathcal{C}_S$ where $S = \{s_1, \ldots, s_i\} \subseteq \{1, 2, \ldots, m\}$ with $1 \in S$ is given, and $\mathcal{C}_S$ is the set of colored partitions with parts congruent to k modulo i appearing in the colors $\{s_k, \ldots, s_{k+1} - 1\}$ taking $s_{i+1} = m + 1$.

Definition of $\Psi_{m,S}$. For each $\lambda \in \mathcal{P}$, the parts of the colored partition $\Psi_{m,S}(\lambda) = \mu$ are obtained by marking the parts of λ counted by the Schmidt weight

$$n = \sum_{\substack{k \equiv S \\ (\mathrm{mod}\ m)}} \lambda_k = \lambda_{s_1} + \cdots + \lambda_{s_i} + \lambda_{s_1+m} + \cdots + \lambda_{s_i+m} + \cdots$$

and then deleting all others, and finally conjugating. Any part of μ that is congruent to k modulo i recieves the color $s_k + j$ where j is the number of squares at the bottom of the corresponding column of λ's Ferrers diagram that were not marked.

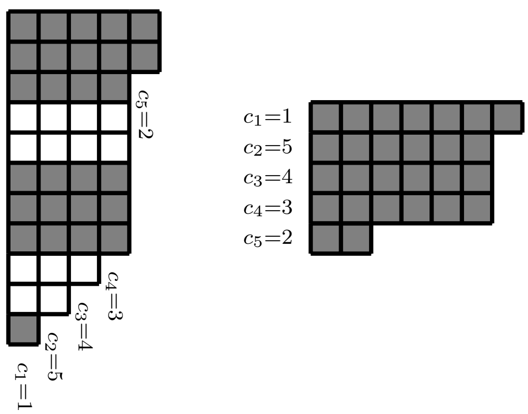

Figure 1: An example of $\Psi_{m,S}$ with $m = 5$ and $S = \{1, 2, 3\}$. In the image of $\Psi_{m,S}$, parts congruent to 1 or 2 modulo 3 can appear in the colors 1 or 2 respectively, and parts 0 modulo 3 can appear in the colors 3,4, or 5. As shown in the figure, $\Psi_{5,\{1,2,3\}}(5, 5, 4, 4, 4, 4, 4, 4, 3, 2, 1) = (7_1, 6_3, 6_2, 6_1, 2_1)$

The fact that $\Psi_{m,S}$ is invertible is clear since the columns of λ's Ferrers diagram can be determined from the corresponding part in μ by using the part's color and the repeating pattern of the Schmidt weight to add the appropriate number of uncounted squares. The Schmidt weight of λ becomes the size of μ by construction, and the alternating sums

$$\sum_{k \geq 0} (\lambda_{mk+j} - \lambda_{mk+j+1}) = \lambda_j - \lambda_{j+1} + \lambda_{j+m} - \lambda_{j+m+1} + \cdots$$

for $1 \leq j \leq m$ are separating the columns of λ by their residue modulo m, counting the parts of μ in each color.

Remark 3. From the properties of $\Psi_{m,S}$ a statement analogous to Theorem 3 holds with $\mathcal{D}_m$

replaced by $\mathcal{P}$, which is also a limiting case of Theorem 6 obtained by allowing all parts to repeat without restriction.

Below, we use ϕ_m in the definition of Φ_m. Since we only rely on the properties of ϕ_m which are already established, we do not provide a definition in this paper. The interested reader can find that in [4], along with a detailed example.

Definition of Φ_m. For each $\lambda \in \mathcal{P}$, $\Phi_m(\lambda) = \mu$ is given by the following process. From λ's Ferrers diagram, remove any columns of width m (corresponding to parts repeating m times in λ') until a partition $\lambda^f \in \mathcal{F}_m$ remains. Also, build the partition λ^c by taking any removed column with k parts and inserting the part km into λ^c. Next, let $\lambda^r = \phi_m(\lambda^f)$, and insert the parts of λ^c into λ^r to form μ.

In the case that $\lambda \in \mathcal{F}_m$, this will leave $\lambda^f = \lambda$ and $\lambda^c = ()$, so $\Phi_m(\lambda) = \phi_m(\lambda)$ which shows that Φ_m is indeed an extension of ϕ_m. Since $\lambda^r \in \mathcal{R}_m$ and all the parts of λ^c are divisible by m, λ^c and λ^r are easily recovered from μ, and λ is in turn recovered by taking the parts of λ^c to form the missing columns, so Φ_m is a bijection. Moreover, since λ is only being modified in multiples of m (including the application of ϕ_m) the Schmidt weight of λ' is preserved by Φ_m, as are the alternating sums ρ_j for $j \neq m$.

We now have all the tools needed to proceed with the the proof.

Proof of Theorem 6. The actual bijection between the two families of partitions in Theorem 3 is given by $\lambda \mapsto \Psi_{m,S}(\phi_m(\lambda')')$. We need only show that the map $\lambda \mapsto \Psi_{m,S}(\Phi_m(\lambda')')$ correctly handles the parts $\alpha_1, \ldots, \alpha_t$ that appear $p_1, \ldots, p_t \geq m$ times, respectively. These repetitions become columns in λ', which are removed in multiples of m until none more can be removed, each time becoming a single part that $\Psi_{m,S}$ will assign the color m to. $\square$

3.1 On The Generating Functions of Schmidt Type Partitions

We finish with some discussion of the generating functions associated to Schmidt type partitions. Many examples of these can be found in the references with various statistics attached, although none of them consider both the Schmidt weight and the size of the same partition. The question of what q-series identities are associated to this combination of statistics is, as such, unexplored, and may be another direction for q-series research.

Mork's bijection [17] easily implies two identities with these statistics attached. We briefly describe the map here. Given a partition λ, place numbers in the diagonal squares from the top left of λ's Ferrers diagram, as well as on the diagonal directly above, which count the square they are in and all squares below or to the right. These quantities are called hook lengths. The Schmidt type partition μ is then constructed so that $\mu_1 + \mu_3 + \mu_5 + \cdots$ is the sum of the lower diagonal hook lengths, and $\mu_2 + \mu_4 + \mu_6 + \cdots$ is the sum of the upper diagonal hook lengths.

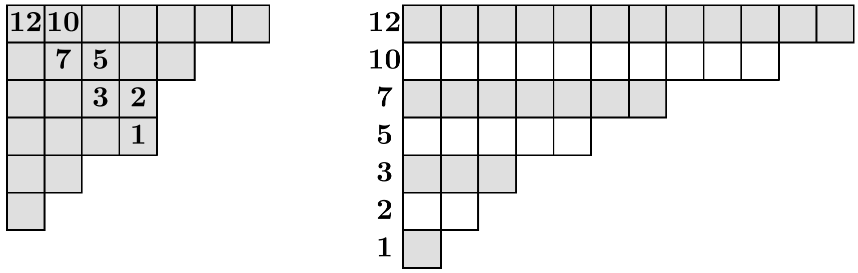

Figure 2: Mork's bijection maps the partition $(7,5,4,4,2,1)$ to $(12,10,7,5,3,2,1)$.

Since $|\lambda| = \mu_1 + \mu_3 + \mu_5 + \cdots$ and this map is invertible, Theorem 1 is implied. Observe that $|\mu| = 2|\lambda| - \ell(\lambda)$ since the sum of all hook lengths double counts each square, except those furthest to the left, and also that $\mu_2 + \mu_4 + \mu_6 + \cdots = |\lambda| - \ell(\lambda)$. These relationships readily imply:

$$\sum_{\lambda \in \mathcal{D}_2} s^{|\lambda|} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(qs; qs^2)_\infty}$$
$$\sum_{\lambda \in \mathcal{D}_2} s^{|\lambda|} q^{\lambda_2 + \lambda_4 + \lambda_6 + \cdots} = \frac{1}{(s; qs^2)_\infty}.$$

The second identity here is notable since there are infinitely many partitions λ such that $\lambda_2 + \lambda_4 + \lambda_6 + \cdots = n$ for each $n \geq 1$, so s cannot be excluded.

We can obtain much more general identities using the map $\Psi_{m,S}$ discussed above. Let $1 \leq i \leq m$ where $m \geq 2$. Then

$$\sum_{\lambda \in \mathcal{P}} s^{|\lambda|} q^{\sum_{k=0}^{\infty} \sum_{j=1}^{i} \lambda_{km+j}} = \frac{1}{(sq, \ldots, s^i q^i, s^{i+1} q^i, \ldots, s^m q^i; s^m q^i)_\infty}$$
$$\sum_{\lambda \in \mathcal{D}_m} s^{|\lambda|} q^{\sum_{k=0}^{\infty} \sum_{j=1}^{i} \lambda_{km+j}} = \frac{1}{(sq, \ldots, s^i q^i, s^{i+1} q^i, \ldots, s^{m-1} q^i; s^m q^i)_\infty}.$$

where we are using the notation for products of q-Pochhammer symbols

$$(z_1, \ldots, z_n; q)_\infty = (z_1; q)_\infty \cdots (z_n; q)_\infty.$$

There are a few ways to show this. The first identity is implied by the map $\Psi_{m,S}$ by setting $S = \{1, \ldots, i\}$ and tracking what happens to the uncounted parts. As for the second, the relationship

$$\frac{1}{(s^m q^i; s^m q^i)_\infty} \sum_{\lambda \in \mathcal{D}_m} s^{|\lambda|} q^{\sum_{k=0}^{\infty} \sum_{j=1}^{i} \lambda_{km+j}} = \sum_{\lambda \in \mathcal{P}} s^{|\lambda|} q^{\sum_{k=0}^{\infty} \sum_{j=1}^{i} \lambda_{km+j}}$$

can be established through the observation that any $\lambda \in \mathcal{P}$ may uniquely be decomposed into a partition in $\mathcal{D}_m$ and a partition with parts repeating in multiples of m, and that this decomposition preserves not just the Schmidt weight as similarly argued for the definition of Φ_m, but also the size (see also [8], where the same observation is applied to Stanley-Boulet weights).

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