When does a bent concatenation not belong to the completed Maiorana-McFarland class?

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***Abstract*—Every Boolean bent function f can be written either as a concatenation $f = f_1||f_2$ of two complementary semi-bent functions f_1, f_2; or as a concatenation $f = f_1||f_2||f_3||f_4$ of four Boolean functions f_1, f_2, f_3, f_4, all of which are simultaneously bent, semi-bent, or 5-valued spectra-functions. In this context, it is essential to ask: When does a bent concatenation f (not) belong to the completed Maiorana-McFarland class $\mathcal{M}^{\#}$? In this article, we answer this question completely by providing a full characterization of the structure of $\mathcal{M}$-subspaces for the concatenation of the form $f = f_1||f_2$ and $f = f_1||f_2||f_3||f_4$, which allows us to specify the necessary and sufficient conditions so that f is outside $\mathcal{M}^{\#}$. Based on these conditions, we propose several explicit design methods of specifying bent functions outside $\mathcal{M}^{\#}$ in the special case when $f = g||h||g||(h+1)$, where g and h are bent functions.**

I. Preliminaries

Let $\mathbb{F}_2^n$ be the vector space of all n-tuples $x = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{F}_2$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{F}_2^n$, the usual scalar product over $\mathbb{F}_2$ is defined as $x \cdot y = x_1y_1 + \cdots + x_ny_n$. By 0_n we denote the all-zero vector of $\mathbb{F}_2^n$. Every Boolean function $f\colon \mathbb{F}_2^n \to \mathbb{F}_2$ can be uniquely represented by its associated algebraic normal form (ANF) in the form $f(x_1, \ldots, x_n) = \sum_{u \in \mathbb{F}_2^n} \lambda_u (\prod_{i=1}^n x_i^{u_i})$, where $x_i, \lambda_u \in \mathbb{F}_2$ and $u = (u_1, \ldots, u_n) \in \mathbb{F}_2^n$. The algebraic degree of f, denoted by $\deg(f)$, is equal to the maximum Hamming weight of $u \in \mathbb{F}_2^n$ for which $\lambda_u \neq 0$.

The *first-order derivative* of a function f in the direction $a \in \mathbb{F}_2^n$ is given by $D_a f(x) = f(x) + f(x + a)$. Derivatives of higher orders are defined recursively, i.e., the *k-th order derivative* of a function $f \in \mathcal{B}_n$ is defined by $D_V f(x) = D_{a_k} D_{a_{k-1}} \ldots D_{a_1} f(x) = D_{a_k}(D_{a_{k-1}} \ldots D_{a_1} f)(x)$, where $V = \langle a_1, \ldots, a_k \rangle$ is a vector subspace of $\mathbb{F}_2^n$ spanned by elements $a_1, \ldots, a_k \in \mathbb{F}_2^n$. Note that if $a_1, \ldots, a_k \in \mathbb{F}_2^n$ are linearly dependent, then $D_{a_k} D_{a_{k-1}} \ldots D_{a_1} f = 0$. The *Walsh-Hadamard transform* of $f \in \mathcal{B}_n$ at any point $\omega \in \mathbb{F}_2^n$ is defined $W_f(\omega) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus \omega \cdot x}$. A function $f \in \mathcal{B}_n$, for even n, is called *bent* if $|W_f(u)| = 2^{\frac{n}{2}}$, for all $u \in \mathbb{F}_2^n$. Its unique *dual* function f^* is defined as $W_f(u) = 2^{\frac{n}{2}}(-1)^{f^*(u)}$, which is also bent. Two Boolean functions $f, f' \in \mathcal{B}_n$ are called *extended-affine equivalent*, if there exists an affine permutation A of $\mathbb{F}_2^n$ and affine function $l \in \mathcal{B}_n$, such that $f \circ A + l = f'$. It is well known, that extended-affine (EA) equivalence preserves the bent property.

The *completed Maiorana-McFarland class* $\mathcal{M}^{\#}$ [6] is the set of n-variable ($n = 2m$) Boolean bent functions, which are EA-equivalent to the functions of the form

$$f(x, y) = x \cdot \pi(y) + g(y), \text{ for all } x, y \in \mathbb{F}_2^m, \tag{1}$$

where π is a permutation on $\mathbb{F}_2^m$, and g is an arbitrary Boolean function on $\mathbb{F}_2^m$. It is well-known from Dillon's thesis [3] that a bent function $f \in \mathcal{B}_n$ belongs to $\mathcal{M}^{\#}$ iff there exists a vector space V of dimension m, such that $D_a D_b f = 0$ for all $a, b \in V$. This characterization motivates the following definition:

Definition 1. *[11] Let $f \in \mathcal{B}_n$ be a Boolean function. We call a vector subspace V of $\mathbb{F}_2^n$ an $\mathcal{M}$-subspace of f, if we have that $D_a D_b f = 0$, for any $a, b \in V$.*

Further, we will investigate $\mathcal{M}$-subspaces of the Boolean functions of the form $f = f_1||f_2$ or $f = f_1||f_2||f_3||f_4$, which are defined as follows. We define the concatenation $f_1||f_2 : \mathbb{F}_2^{n+1} \to \mathbb{F}_2$ of the two functions as:

$$\begin{aligned} &f_1||f_2(z, z_{n+1}) = f_1(z) + z_{n+1}(f_1(z) + f_2(z)), \\ &\text{for all } z \in \mathbb{F}_2^n,\ z_{n+1} \in \mathbb{F}_2, \end{aligned} \tag{2}$$

that is, $f_1||f_2(z, 0) = f_1(z)$, and $f_1||f_2(z, 1) = f_2(z)$.

For $i = 1, \ldots, 4$, let $f_i \in \mathcal{B}_n$. The formula for the concatenation $f = f_1||f_2||f_3||f_4 \in \mathcal{B}_{n+2}$ of the four functions is given by:

$$\begin{aligned} f(z, z_{n+1}, z_{n+2}) =& f_1(z) + z_{n+1}z_{n+2}(f_1 + f_2 + f_3 + f_4)(z) \\ &+ z_{n+1}(f_1 + f_2)(z) + z_{n+2}(f_1 + f_3)(z), \end{aligned} \tag{3}$$

for all $z \in \mathbb{F}_2^n$ and $z_{n+1}, z_{n+2} \in \mathbb{F}_2$, that is, $f(z, 0, 0) = f_1(z)$, $f(z, 1, 0) = f_2(z)$, $f(z, 0, 1) = f_3(z)$ and $f(z, 1, 1) = f_4(z)$. Throughout this article, we will call bent functions of the form (2) and (3) *bent concatenations*.

The main aim of this article is to develop further a theory of $\mathcal{M}$-subspaces for bent concatenations initially analyzed in [11] and recently considered in [9]. For a more detailed treatment of bent functions we refer to [2], [7], and for their designs outside $\mathcal{M}^{\#}$ to [8], [10]. The rest of the paper is organized in the following way. In Sections II and III, we provide a full characterization of the structure of $\mathcal{M}$-subspaces for the concatenation of the form $f = f_1||f_2$ and

$f = f_1||f_2||f_3||f_4$, respectively. Consequently, we specify the necessary and sufficient conditions so that f is outside $\mathcal{M}^\#$. Based on these conditions, we propose in Section IV several explicit design methods of specifying bent functions outside $\mathcal{M}^\#$ in the special case when $f = g||h||g||(h+1)$.

II. Concatenation of two Functions

Let $a, b \in \mathbb{F}_2^n$. From Eq. (2), we deduce that the second-order derivative of the concatenation $f = f_1||f_2 : \mathbb{F}_2^{n+1} \to \mathbb{F}_2$, with respect to $(a,0)$ and $(b,0)$ has the following form

$$D_{(a,0)}D_{(b,0)}f = D_{(a,0)}D_{(b,0)}f_1||f_2 = D_aD_bf_1||D_aD_bf_2. \quad (4)$$

Similarly, from Eq. (2), the second-order derivative of $f = f_1||f_2$ w.r.t. $(a,0)$ and $(b,1)$, at the point $(z, z_{n+1}) \in \mathbb{F}_2^{n+1}$, can be computed as

$$\begin{aligned} &D_{(a,0)}D_{(b,1)}f = D_{(b,1)}(D_af_1||D_af_2) = g_1||g_2, \text{ where}\\ &g_1(z) = D_af_1(z) + D_af_2(z+b) \text{ and} \\ &g_2(z) = D_af_2(z) + D_af_1(z+b), \text{ for all } z \in \mathbb{F}_2^n. \end{aligned} \quad (5)$$

Since $D_{(a,a_{n+1})}D_{(b,b_{n+1})}f = D_{(b,b_{n+1})}D_{(a,a_{n+1})}f = D_{(a+b,a_{n+1}+b_{n+1})}D_{(b,b_{n+1})}f$, for all $a, b \in \mathbb{F}_2^n$ and $a_{n+1}, b_{n+1} \in \mathbb{F}_2$, the rest of the cases can also be computed with (4) and (5). Using these expressions, we relate $\mathcal{M}$-subspaces of f to $\mathcal{M}$-subspaces of f_1 and f_2 as follows:

Theorem 2. *Let $f_1, f_2 \in \mathcal{B}_n$ and let $k \in \{1, \ldots, n\}$. The function $f = f_1||f_2 \in \mathcal{B}_{n+1}$ has no $(k+1)$-dimensional $\mathcal{M}$-subspaces if and only if the following conditions hold:*

a) The functions f_1 and f_2 do not share a common $(k+1)$-dimensional $\mathcal{M}$-subspace;

b) For every vector $u \in \mathbb{F}_2^n$ and every k-dimensional $\mathcal{M}$-subspace $V \subset \mathbb{F}_2^n$ of both f_1 and f_2, there is $a \in V$ such that

$$D_af_1(z) + D_af_2(z+u) \neq 0, \textit{ for some } z \in \mathbb{F}_2^n. \quad (6)$$

Proof. (Sketch) Assume that W is an $\mathcal{M}$-subspace of f, with $\dim(W) = k+1$. Consider the projection $P : W \to \mathbb{F}_2$ given by $P(z, z_{n+1}) = z_{n+1}$, for all $(z, z_{n+1}) \in W$, where $z \in \mathbb{F}_2^n$ and $z_{n+1} \in \mathbb{F}_2$. Then, $\dim(\ker(P)) \geq k$ (by rank-nullity theorem). If $\dim(\ker(P)) = k+1$, then Eq. (4) implies that f_1 and f_2 share a common $(k+1)$-dimensional $\mathcal{M}$-subspace. Similarly, when $\dim(\ker(P)) = k$, define V through $\{(v,0) : v \in V\} = \ker(P)$. Then, taking $u \in \mathbb{F}_2^n$ be such that $(u,1) \in W \setminus \ker(P)$, by Eqs. (4) and (5) one deduces Eq. (6). In the other direction, it can be shown that assuming that f_1 and f_2 do not share a common $(k+1)$-dimensional $\mathcal{M}$-subspace leads to a contradiction. ☐

Using the fact that a bent function $f \in \mathcal{B}_t$ is in the $\mathcal{M}^\#$ class if and only if it has a $t/2$-dimensional $\mathcal{M}$-subspace, from Theorem 2 we deduce the following result.

Corollary 3. *Let $f_1, f_2 \in \mathcal{B}_n$, $n = 2k+1$, be Boolean functions such that $f = f_1||f_2 \in \mathcal{B}_{n+1}$ is a bent function. Then, the function f is outside the $\mathcal{M}^\#$ class if and only if the following conditions hold:*

1) The functions f_1 and f_2 do not share a common $(k+1)$-dimensional $\mathcal{M}$-subspace;

2) For every vector $u \in \mathbb{F}_2^n$ and every k-dimensional $\mathcal{M}$-subspace $V \subset \mathbb{F}_2^n$ of both f_1 and f_2, there is $a \in V$ such that $D_af_1(z) + D_af_2(z+u) \neq 0$, for some $z \in \mathbb{F}_2^n$.

It is well-known that in the above concatenation $f = f_1||f_2$, the function f is bent if and only if f_1 and f_2 are disjoint spectra semi-bent functions; see [14, Theorem 6]. In particular, when $f_i : \mathbb{F}_2^{2k+1} \to \mathbb{F}_2$ are represented in the form $f_i(x,y) = x \cdot \phi_i(y) + h_i(y)$, for $x \in \mathbb{F}_2^{k+1}$, $y \in \mathbb{F}_2^k$, where $\phi : \mathbb{F}_2^k \to \mathbb{F}_2^{k+1}$ and $h_i : \mathbb{F}_2^k \to \mathbb{F}_2$, then the properties of ϕ_i are essential in defining disjoint spectra semi-bent functions f_1 and f_2.

Theorem 4. *Let f_1 and f_2 defined as $f_i(x,y) = x \cdot \pi_i(y) + h_i(y)$, with $x \in \mathbb{F}_2^{k+1}$ and $y \in \mathbb{F}_2^k$ and h_i are arbitrary Boolean functions on $\mathbb{F}_2^k$. Then, the concatenation $f = f_1||f_2$ is a bent function on $\mathbb{F}_2^{2k+2}$ if and only if $\mathrm{im}(\pi_1) \cap \mathrm{im}(\pi_2) = \varnothing$ and π_i are injective mappings.*

Proof. Notice that $f = f_1||f_2 : \mathbb{F}_2^{k+1} \times \mathbb{F}_2^{k+1} \to \mathbb{F}_2$ is the function defined by $f(x,y) = x \cdot \pi(y, y_{k+1}) + h(y, y_{k+1})$, for all $x \in \mathbb{F}_2^{k+1}$, $y \in \mathbb{F}_2^k$ and $y_{n+1} \in \mathbb{F}_2$, where π is defined by $\pi(y,0) = \pi_1(y)$ and $\pi(y,1) = \pi_2(y)$, and similarly $h(y,0) = h_1(y)$ and $h(y,1) = h_2(y)$, for all $y \in \mathbb{F}_2^k$. We know that f is bent if and only if π is a permutation, and π is a permutation if and only if $\mathrm{im}(\pi_1) \cap \mathrm{im}(\pi_2) = \varnothing$ and π_1 and π_2 are injective mappings. ☐

However, it turns out that $f = f_1||f_2 \in \mathcal{M}^\#$ since f_1 and f_2 share an $\mathcal{M}$-subspace of maximal dimension.

Remark 5. *Any construction method employing the functions $f_i(x,y) = x \cdot \phi_i(y) + h_i(y)$, where $x \in \mathbb{F}_2^{k+1}$ and $y \in \mathbb{F}_2^k$ (consequently $\phi_i : \mathbb{F}_2^k \to \mathbb{F}_2^{k+1}$), will only provide a function f which belongs to $\mathcal{M}^\#$. This is due to Corollary 3 and the fact that $\mathbb{F}_2^{k+1} \times \{0_k\}$ is a canonical $\mathcal{M}$-subspace of dimension $k+1$ which is shared by f_1 and f_2.*

III. Concatenation of four Functions

Similarly as in the case of two functions concatenation, we derive the following formulas for the second-order derivatives of $f = f_1||f_2||f_3||f_4$ (where f_i are suitable bent, semi-bent or five-valued spectra functions) if f is bent [1]). For a function $h : \mathbb{F}_2^m \to \mathbb{F}_2$ and $r \in \mathbb{F}_2^m$ by h^r, we denote the translation of h by r, that is $h^r(x) = h(x+r)$, for all $x \in \mathbb{F}_2^m$. In the following formulas, a and b are two arbitrary elements from $\mathbb{F}_2^n$, not necessarily different.

$$\begin{aligned} D_{(a,0,0)}D_{(b,0,0)}f &= D_{(a,0,0)}D_{(b,0,0)}(f_1||f_2||f_3||f_4) \\ &= D_aD_bf_1||D_aD_bf_2||D_aD_bf_3||D_aD_bf_4 \end{aligned} \quad (7)$$

$$\begin{aligned} &D_{(a,1,0)}D_{(b,0,0)}f = (D_bf_1 + D_bf_2^a)|| \\ &(D_bf_1 + D_bf_2^a)^a||(D_bf_3 + D_bf_4^a)||(D_bf_3 + D_bf_4^a)^a \end{aligned} \quad (8)$$

$$\begin{aligned} &D_{(a,0,1)}D_{(b,0,0)}f = (D_bf_1 + D_bf_3^a)|| \\ &(D_bf_2 + D_bf_4^a)||(D_bf_1 + D_bf_3^a)^a||(D_bf_2 + D_bf_4^a)^a \end{aligned} \quad (9)$$

$$\begin{aligned} &D_{(a,1,1)}D_{(b,0,0)}f = (D_bf_1 + D_bf_4^a)|| \\ &(D_bf_2 + D_bf_3^a)||(D_bf_2 + D_bf_3^a)^a||(D_bf_1 + D_bf_4^a)^a \end{aligned} \quad (10)$$

$$D_{(a,0,1)}D_{(b,1,0)}f = (f_1+f_2^b+f_3^a+f_4^{a+b})|| \\ (f_1+f_2^b+f_3^a+f_4^{a+b})^b||(f_1+f_2^b+f_3^a+f_4^{a+b})^a|| \\ (f_1+f_2^b+f_3^a+f_4^{a+b})^{a+b}. \tag{11}$$

Compared to Proposition V.2 in [9], the result below gives the most general structure of $\mathcal{M}$-subspaces of varying dimension for a 4-concatenation of not necessarily bent functions.

Theorem 6. *Let $f = f_1||f_2||f_3||f_4\colon \mathbb{F}_2^{n+2} \to \mathbb{F}_2$ be the concatenation of arbitrary Boolean functions $f_1,\dots,f_4 \in \mathcal{B}_n$ and let W be a $(k+2)$-dimensional subspace of $\mathbb{F}_2^{n+2}$, $k \in \{0,\dots,n\}$. Then, W is an $\mathcal{M}$-subspace of f if and only if W has one of the following forms:*

a) $W = V \times \{(0,0)\}$, where $V \subset \mathbb{F}_2^n$ is a common $(k+2)$-dimensional $\mathcal{M}$-subspace of $f_1,\dots,f_4$.

b) $W = \langle V \times \{(0,0)\}, (a,1,0)\rangle$, where V is a common $(k+1)$-dimensional $\mathcal{M}$-subspace of $f_1,\dots,f_4$, and $a \in \mathbb{F}_2^n$ is such that

$$D_vf_1 + D_vf_2^a = D_vf_3 + D_vf_4^a = 0, \text{ for all } v \in V.$$

c) $W = \langle V \times \{(0,0)\}, (a,0,1)\rangle$, where V is a common $(k+1)$-dimensional $\mathcal{M}$-subspace of $f_1,\dots,f_4$, and $a \in \mathbb{F}_2^n$ is such that

$$D_vf_1 + D_vf_3^a = D_vf_2 + D_vf_4^a = 0, \text{ for all } v \in V.$$

d) $W = \langle V \times \{(0,0)\}, (a,1,1)\rangle$, where V is a common $(k+1)$-dimensional $\mathcal{M}$-subspace of $f_1,\dots,f_4$, and $a \in \mathbb{F}_2^n$ is such that

$$D_vf_1 + D_vf_4^a = D_vf_2 + D_vf_3^a = 0, \text{ for all } v \in V.$$

e) $W = \langle V \times \{(0,0)\}, (a,0,1), (b,1,0)\rangle$, where V is a common k-dimensional $\mathcal{M}$-subspace of $f_1,\dots,f_4$, and $a,b \in \mathbb{F}_2^n$ are such that $D_vf_1 + D_vf_3^a = D_vf_2 + D_vf_4^a = D_vf_1 + D_vf_2^b = D_vf_3 + D_vf_4^b = 0$, for all $v \in V$, and $f_1(x) + f_2(x+b) + f_3(x+a) + f_4(x+a+b) = 0$, for all $x \in \mathbb{F}_2^n$.

Proof. (Sketch) Assume first that W is an $\mathcal{M}$-subspace of f. Let $P\colon W \to \mathbb{F}_2^2$ be the projection on the last two coordinates, i.e., $P((w_1,\dots,w_{n+1},w_{n+2})) = (w_{n+1},w_{n+2})$, for all $(w_1,\dots,w_{n+1},w_{n+2}) \in W$. There are 5 subspaces of $\mathbb{F}_2^2$, and depending on which subspace $\mathrm{im}(P)$ is equal to, we obtain the five corresponding forms a) - e) of the subspace W. The proof follows by applying Eqs. (7) - (11). The other direction is proved similarly. □

Remark 7. *Proposition V.2 in [9] specifies the structure of $\mathcal{M}$-subspaces of maximal dimension $m+1$ for $f = f_1||f_2||f_3||f_4$, where both f and $f_i \in \mathcal{B}_{2m}$ are bent and additionally at least one f_i admits the canonical $\mathcal{M}$-subspace $U = \mathbb{F}_2^m \times \{0_m\}$. Thus, it is a special case of Theorem 6.*

From Theorem 6, we obtain the following full characterization of the class inclusion of $f = f_1||f_2||f_3||f_4$ in the $\mathcal{M}^\#$ class in terms of properties of $f_1,\dots,f_4$.

Corollary 8. *Let $f = f_1||f_2||f_3||f_4\colon \mathbb{F}_2^{n+2} \to \mathbb{F}_2$ be the concatenation of $f_1,\dots,f_4 \in \mathcal{B}_n$ and assume that f is bent; thus f_i are bent, semi-bent or five-valued spectra functions. Then, f is outside of the $\mathcal{M}^\#$ class if and only if the following conditions hold:*

a) The functions $f_1,\dots,f_4$ do not share a common $(n/2+1)$-dimensional $\mathcal{M}$-subspace;

b) There are no common $(n/2)$-dimensional $\mathcal{M}$-subspaces $V \subset \mathbb{F}_2^n$ of $f_1,\dots,f_4$ such that there is an element $a \in \mathbb{F}_2^n$ for which

$$\begin{aligned} D_vf_1 + D_vf_2^a = D_vf_3 + D_vf_4^a = 0, \text{ for all } v \in V, \text{ or} \\ D_vf_1 + D_vf_3^a = D_vf_2 + D_vf_4^a = 0, \text{ for all } v \in V, \text{ or} \\ D_vf_1 + D_vf_4^a = D_vf_2 + D_vf_3^a = 0, \text{ for all } v \in V. \end{aligned} \tag{12}$$

c) There are no common $(n/2-1)$-dimensional $\mathcal{M}$-subspaces $V \subset \mathbb{F}_2^n$ of $f_1,\dots,f_4$ such that there are elements $a,b \in \mathbb{F}_2^n$ (not necessarily different), for which

$$\begin{aligned} &D_vf_1 + D_vf_3^a = D_vf_2 + D_vf_4^a = D_vf_1 + D_vf_2^b \\ &= D_vf_3 + D_vf_4^b = 0, \text{ for all } v \in V, \text{ and} \\ &f_1(x) + f_2(x+b) + f_3(x+a) \\ &+ f_4(x+a+b) = 0, \text{ for all } x \in \mathbb{F}_2^n. \end{aligned} \tag{13}$$

Proof. The result follows directly from Theorem 6, by setting $k+2 = n/2+1$, and the fact that a bent function $f \in \mathcal{B}_{n+2}$ is in the $\mathcal{M}^\#$ class if and only if it has an $(n/2+1)$-dimensional $\mathcal{M}$-subspace. □

Notice that when f_i are bent in Corollary 8, then the item $a)$ is automatically satisfied since none of the functions f_i admits an $\mathcal{M}$-subspace of dimension $n/2+1$. The condition in $b)$ was recently deduced in [9, Corollary V.11] for a special case when f_i are bent functions on $\mathbb{F}_2^n$ that share an $\mathcal{M}$-subspace of maximal dimension $n/2$.

Open Problem 9. *Is the condition $c)$ in Corollary 8 independent of conditions $a), b)$? Particularly, the existence of bent functions $f = f_1||f_2||f_3||f_4$ on $\mathbb{F}_2^{n+2}$ in $\mathcal{M}^\#$, where all $f_i \in \mathcal{B}_n$ are bent and outside $\mathcal{M}^\#$, is hard to establish.*

Notice that, when $f = f_1||f_1||f_1||f_1 + 1$ so that $f(x,y_1,y_2) = f_1(x) + y_1y_2$, where f_1 is a bent function on $\mathbb{F}_2^n$, it was deduced [13] that f is outside $\mathcal{M}^\#$ if and only if f_1 is outside $\mathcal{M}^\#$. This result also follows from Theorem 10 below, as we show in the next section.

IV. An Application: Designing bent functions outside $\mathcal{M}^\#$ of the form $g||h||g||(h+1)$

The concatenation $f = g||h||g||h+1$ (where g and h are bent) is interesting in terms of the class inclusion, as the dual bent condition is automatically satisfied. Recall that when f_i are all bent, then $f = f_1||f_2||f_3||f_4$ is bent if and only if $f_1^* + f_2^* + f_3^* + f_4^* = 1$; see [4]. The analysis of structural properties of $\mathcal{M}$-subspaces presented in the previous section turns out to be useful when considering certain special cases of bent 4-concatenation.

A. The necessary and sufficient condition for $f = g||h||g||(h+1)$ to be outside $\mathcal{M}^{\#}$

Theorem 10. *Let h and g be two arbitrary bent functions in $\mathcal{B}_n$. Then, the function $f = f_1||f_2||f_3||f_4 \colon \mathbb{F}_2^{n+2} \to \mathbb{F}_2$, where $f_1 = f_3 = g$ and $f_2 = f_4 + 1 = h$ is a bent function in the $\mathcal{M}^{\#}$ class if and only if the functions g and h have a common $(n/2)$-dimensional $\mathcal{M}$-subspace, thus $g, h \in \mathcal{M}^{\#}$.*

Proof. We compute $f_1^* + f_2^* + f_3^* + f_4^* = g^* + h^* + g^* + h^* + 1 = 1$, hence f is a bent function. Let $V \subset \mathbb{F}_2^n$ be a common $(n/2)$-dimensional $\mathcal{M}$-subspace of g and h. Then, V is also a common $(n/2)$-dimensional $\mathcal{M}$-subspace of $f_1, \ldots, f_4$ and $D_v f_1 + D_v f_3 = D_v g + D_v g = 0$, $D_v f_2 + D_v f_4 = D_v h + D_v h = 0$, for all $v \in V$. Setting $a = 0_n$ in the item $b)$ of Corollary 8, we deduce that f is a bent function in $\mathcal{M}^{\#}$.

Assume now that g and h do not have a common $(n/2)$-dimensional $\mathcal{M}$-subspace, and that $f \in \mathcal{M}^{\#}$. Then, the cases $a)$ and $b)$ in Corollary 8 hold, hence it has to be the case $c)$ that fails. That is, there is a common $(n/2-1)$-dimensional $\mathcal{M}$-subspace $V \subset \mathbb{F}_2^n$ of $f_1, \ldots, f_4$, (i.e. of g and h) such that there are elements $a, b \in \mathbb{F}_2^n$ (not necessarily different), for which

$$\begin{aligned} & D_v f_1 + D_v f_3^a = D_v f_2 + D_v f_4^a = D_v f_1 + D_v f_2^b = \\ & D_v f_3 + D_v f_4^b = 0, \text{ for all } v \in V, \text{ and} \\ & f_1(x) + f_2(x+b) + f_3(x+a) + f_4(x+a+b) = 0, \\ & \text{ for all } x \in \mathbb{F}_2^n. \end{aligned}$$

From $D_v f_1 + D_v f_3^a = 0$, we get $D_v g + D_v g^a = D_a D_v g = 0$, for all $v \in V$. Similarly, $D_v f_2 + D_v f_4^a = 0$ implies $D_v h + D_v h^a = D_a D_v h = 0$, for all $v \in V$. This implies that a has to be in V, otherwise $\langle V, a \rangle$ would be a common $(n/2)$-dimensional $\mathcal{M}$-subspace of g and h. Setting $v = a$ in $D_v f_1 + D_v f_2^b = 0$, we get

$$\begin{aligned} & g(x) + g(x+a) + h(x+b) + h(x+a+b) = 0, \\ & \text{ for all } x \in \mathbb{F}_2^n. \end{aligned} \tag{14}$$

On the other hand, from $f_1(x) + f_2(x+b) + f_3(x+a) + f_4(x+a+b) = 0$ we have $g(x) + h(x+b) + g(x+a) + h(x+a+b) + 1 = 0$, that is $g(x) + g(x+a) + h(x+b) + h(x+a+b) = 1$, for all $x \in \mathbb{F}_2^n$. However, this is in contradiction with Eq. (14). We conclude that f is a bent function outside the $\mathcal{M}^{\#}$ class. ☐

Remark 11. *Notice that Theorem 10 answers negatively Open Problem 9 when a bent function $f \in \mathcal{B}_{n+2}$ is represented as $f = g||h||g||h+1$.*

However, Theorem 10 provides a very flexible method of constructing bent functions outside $\mathcal{M}^{\#}$ for $n \geq 10$.

Corollary 12. *Let $g \in \mathcal{B}_n$ be any bent function outside $\mathcal{M}^{\#}$, with $n \geq 8$, and h be any bent function on $\mathbb{F}_2^n$. Then, the bent function $f \in \mathcal{B}_{n+2}$ defined as $f = g||h||g||h+1$ is outside the $\mathcal{M}^{\#}$ class.*

Proof. By Theorem 10, $f \in \mathcal{M}^{\#}$ if and only if g and h share a common $(n/2)$-dimensional $\mathcal{M}$-subspace. But since g is outside $\mathcal{M}^{\#}$ it does not admit any $(n/2)$-dimensional $\mathcal{M}$-subspace, and therefore it cannot share with h regardless of h belongs to $\mathcal{M}^{\#}$ or not. Thus, $f \in \mathcal{B}_{n+2}$ is outside $\mathcal{M}^{\#}$. ☐

Another important consequence of Theorem 10 is the following result which also sheds more light on the existence of bent functions outside $\mathcal{M}^{\#}$, for the special case when $n = 8$.

Corollary 13. *Let $g \in \mathcal{B}_n$ be an arbitrary bent function $n \geq 6$. Then, there exists a bent function $f \in \mathcal{B}_{n+2}$ outside the $\mathcal{M}^{\#}$ class such that $g(x) = f(x, 0, 0)$, for all $x \in \mathbb{F}_2^n$.*

Proof. Let h be a bent function in n variables with a unique $(n/2)$-dimensional $\mathcal{M}$-subspace V; see [9] for their existence. Since g is bent, thus not affine, there exist are two elements $a, b \in \mathbb{F}_2^n$ such that $D_a D_b g \neq 0$. Let A be any affine permutation of $\mathbb{F}_2^n$ such that $A^{-1}(\{a, b\}) \subset V$. Define $h' = h \circ A$. Then, by construction g and h' do not share an $(n/2)$-dimensional $\mathcal{M}$-subspace. Therefore, by Theorem 10, the function $f = g||h'||g||(h'+1)$ is a bent function outside the $\mathcal{M}^{\#}$ class, and the result follows. ☐

Note that certain design methods of constructing 8-variable bent functions outside $\mathcal{M}^{\#}$ using bent functions $f_1, \ldots, f_4 \in \mathcal{M}^{\#}$ were considered in [9], but Corollary 13 confirms this fact theoretically and thus excludes the case that bent functions outside $\mathcal{M}^{\#}$ originate from the 4-concatenation of semi-bent or five-valued spectra functions only. Moreover, it is always possible to find more than one permutation A (from the proof of Corollary 13). It means that for $n \geq 6$, the number of bent functions outside $\mathcal{M}^{\#}$ in $n+2$ variables is always strictly greater than the number of all bent functions in n variables.

Theorem 14. *Let n, k be two integers such that $k < n/2 - 1$. Let g, h be two bent functions in $\mathcal{B}_n$ whose $\mathcal{M}$-subspaces of maximal dimension k are mutually non-intersecting. Assume that for any subspace $\Lambda \subset \mathbb{F}_2^n$ with $\dim(\Lambda) = k-1$, there exists $a \in \Lambda$ such that $D_a g \neq D_a h$. Then, $f = f_1||f_2||f_3||f_4 \colon \mathbb{F}_2^{n+2} \to \mathbb{F}_2$, where $f_1 = f_3 = g$ and $f_2 = f_4 + 1 = h$, is a bent function whose $\mathcal{M}$-subspaces have dimension $< k+1$.*

Proof. (Sketch) By assumption, we have that $W = \langle V \times \{(0,0)\}, (a, i_1, i_2) \rangle$ is not an $\mathcal{M}$-subspace of f, where V is a k-dimensional $\mathcal{M}$-subspace of g (resp. h), $(i_1, i_2) \in \mathbb{F}_2^2$. Thus, let Δ be a common $(k-1)$-dimensional $\mathcal{M}$-subspace of g and h. Set $W = \langle \Delta \times \{(0,0)\}, (a, i_1, i_2), (b, i_3, i_4) \rangle$, where $(i_1, i_2), (i_3, i_4) \in \mathbb{F}_2^2$ and $(i_1, i_2) \neq (i_3, i_4)$. Then, there are two cases to be considered, namely 1) $a \notin \Delta$ or $b \notin \Delta$ and 2) $a, b \in \Delta$. It can be shown that the vector space W, with $\dim(W) = k+1$, is not an $\mathcal{M}$-subspace of f. The result follows then from Theorem 6. ☐

Corollary 15. *Let n be an even integer. Let π be a permutation such that $g = x \cdot \pi(y)$ has only one $(n/2)$-dimensional $\mathcal{M}$-subspace. Let A be an invertible matrix on $\mathbb{F}_2^n$ such that $I + A$ is also an invertible matrix on $\mathbb{F}_2^n$. Let $h = g \circ A$. Then, the function $f = f_1||f_2||f_3||f_4 \colon \mathbb{F}_2^{n+2} \to \mathbb{F}_2$, where $f_1 = f_3 = g$ and $f_2 = f_4 + 1 = h$, is a bent function outside $\mathcal{M}^{\#}$.*

Proof. Since g has only one $(n/2)$-dimensional $\mathcal{M}$-subspace, we have that g and h have no common $(n/2)$-dimensional $\mathcal{M}$-subspace. Since $I+A$ is also an invertible matrix on $\mathbb{F}_2^n$, we have $g+h$ is also a bent function, that is, for any nonzero vector $a \in \mathbb{F}_2^n$ we have $D_a g \neq D_a h$. From Theorem 14, we have the maximal dimension of $\mathcal{M}$-subspaces of f is $< n/2+1$, thus $f \notin \mathcal{M}^\#$. □

B. A special case of relating g and h in a linear manner

The following result, obtained in [5], provides two secondary constructions of bent functions in $n+2$ variables from bent functions in n variables. Notice that a version of the result is also stated as Theorem 45 in [12].

Theorem 16. *[5] Let g be a bent function in n variables. Then, the functions f and f' in $(n+2)$-variables defined by*

$$\begin{aligned} f(z,z_{n+1},z_{n+2}) =& g(z)+\sum_{i=1}^{n}\alpha_i z_i z_{n+1}+z_{n+1}z_{n+2}, \\ f'(z,z_{n+1},z_{n+2}) =& g(z)+\sum_{i=1}^{n}\alpha_i z_i (z_{n+1}+z_{n+2}) \\ & +z_{n+1}z_{n+2}, \end{aligned} \tag{15}$$

for all $z \in \mathbb{F}_2^n$ and $z_{n+1}, z_{n+2} \in \mathbb{F}_2$, are bent functions for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_2$.

Nevertheless, these methods fall under the concatenation framework given by $f = g||h||g||(h+1)$ and the class inclusion of these functions is then easily determined.

Proposition 17. *Let g be a bent function in n variables. Let f and f' be the bent functions in $(n+2)$-variables defined in Eq. (15). Then, the functions f and f' are in the $\mathcal{M}^\#$ class if and only if the function g is in the $\mathcal{M}^\#$ class.*

Proof. Note that f and f' are extended affine equivalent, hence it is enough to investigate the class inclusion for one of them; therefore, we will prove the result for f. By looking at $f(z,0,0)$, $f(z,1,0)$, $f(z,0,1)$ and $f(z,1,1)$, we see that $f = f_1||f_2||f_3||f_4$, where $f_1(z) = f_3(z) = g(z)$, $f_2(z) = f_4(z)+1 = g(z)+\sum_{i=1}^{n}\alpha_i z_i$. Since the functions $g(z)$ and $g(z)+\sum_{i=1}^{n}\alpha_i z_i$ have the same $\mathcal{M}$-subspaces, the result follows from Theorem 10. □

Consequently, we provide an alternative proof of existence of cubic bents functions outside $\mathcal{M}^\#$ on $\mathbb{F}_2^n$ for all $n \geq 10$.

Corollary 18. *Cubic bent functions on $\mathbb{F}_2^n$ outside $\mathcal{M}^\#$ exist for all $n \geq 10$.*

Proof. In Theorem 16, take $g \in \mathcal{B}_{10}$ as h_3^{10} or h_4^{10} from [11, Table 4], which are both outside $\mathcal{M}^\#$. □

C. Applying suitable affine transforms

The class inclusion properties are substantially affected by applying suitable affine transformations to bent functions used in 4-bent concatenation.

Theorem 19. *Let $g \in \mathcal{B}_n$ be a bent function, $n \geq 6$, in the $\mathcal{M}^\#$ class, and let $q \in \mathcal{B}_n$ be a bent function with a unique $n/2$-dimensional $\mathcal{M}$-subspace. Then, there exist two linear permutations A and B of $\mathbb{F}_2^n$ such that for $h = q \circ A$ and $h' = q \circ B$ in $\mathcal{B}_n$, the function $f \in \mathcal{B}_{n+2}$ defined by $f = g||h||g||(h+1)$ is a bent function inside the $\mathcal{M}^\#$ class, and the function $f' \in \mathcal{B}_{n+2}$ defined by $f' = g||h'||g||(h'+1)$ is a bent function outside the $\mathcal{M}^\#$ class.*

Proof. Let $a, b \in \mathbb{F}_2^n$ be two elements such that $D_a D_b g \neq 0$ (we know such two elements exist, otherwise g would be affine), and let V be the unique $(n/2)$-dimensional $\mathcal{M}$-subspace of q. Let B be any linear isomorphism such that $\{a,b\} \subset B^{-1}(V)$. The subspace $B^{-1}(V)$ is the unique $(n/2)$-dimensional $\mathcal{M}$-subspace of $q \circ B$. Since $\{a,b\} \subset B^{-1}(V)$ and $D_a D_b g \neq 0$, we deduce that g and $q \circ B$ do not share an $(n/2)$-dimensional $\mathcal{M}$-subspace. Setting $h' = q \circ B$, from Theorem 10, we deduce that $f' = g||h'||g||(h'+1) \notin \mathcal{M}^\#$. Notice that h' also admits a unique $(n/2)$-dimensional $\mathcal{M}$-subspace as h does.

On the other hand, since $g \in \mathcal{M}^\#$, it has at least one $(n/2)$-dimensional $\mathcal{M}$-subspace, denote it by W. Let A be any linear isomorphism such that $A(W) = V$, and set $h = q \circ A$. Then, W is an $(n/2)$-dimensional $\mathcal{M}$-subspace of both g and h. By Theorem 10, we have that $f = g||h||g||(h+1) \in \mathcal{M}^\#$. □

Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable comments, which helped to improve the presentation of the results.

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