

A PROOF THEORY OF (ω -)CONTEXT-FREE LANGUAGES, VIA NON-WELLFOUNDED PROOFS

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ABSTRACT. We investigate the proof theory of regular expressions with fixed points, construed as a notation for (ω -)context-free grammars. Starting with a hypersequential system for regular expressions due to Das and Pous [DP17], we define its extension by least fixed points and prove the soundness and completeness of its non-wellfounded proofs for the standard language model. From here we apply proof-theoretic techniques to recover an infinitary axiomatisation of the resulting equational theory, complete for inclusions of context-free languages. Finally, we extend our syntax by greatest fixed points, now computing ω -context-free languages. We show the soundness and completeness of the corresponding system using a mixture of proof-theoretic and game-theoretic techniques.

1. INTRODUCTION

The characterisation of context-free languages (CFLs) as the least solutions of algebraic inequalities, sometimes known as the *ALGOL-like theorem*, is a folklore result attributed to several luminaries of formal language theory including Ginsburg and Rice [GR62], Schutzenberger [Sch63], and Gruska [Gru71]. This induces a syntax for CFLs by adding least fixed point operators to regular expressions, as first noted by Salomaa [Sal73]. Leiß [Lei92] called these constructs “ μ -expressions” and defined an algebraic theory over them by appropriately extending Kleene algebras, which work over regular expressions. Notable recent developments include a generalisation of Antimirov’s partial derivatives to μ -expressions [Thi17] and criteria for identifying μ -expressions that can be parsed unambiguously [KY19].

Establishing axiomatisations and proof systems for classes of formal languages has been a difficult challenge. Many *theories* of regular expressions, such as Kleene algebras (KA) were proposed in the late 20th century (see, e.g., e.g. [Con71, Kle56, Koz94]). The completeness of KA for the (equational) theory of regular languages, due to Kozen [Koz94] and Krob [Kro90] independently, is a celebrated result that has led to several extensions and refinements, e.g. [KS97, KS12, CLS15, KS20]. More recently the proof theory of KA has been studied via *infinitary* systems. On one hand, [Pal07] proposed an ω -branching sequent calculus and on the other hand [DP17, DDP18, HK22] have studied *cyclic* ‘hypersequential’ calculi.

Inclusion of CFLs is Π_1^0 -complete, so any recursive (hence also cyclic) axiomatisation must necessarily be incomplete. Nonetheless various theories of μ -expressions

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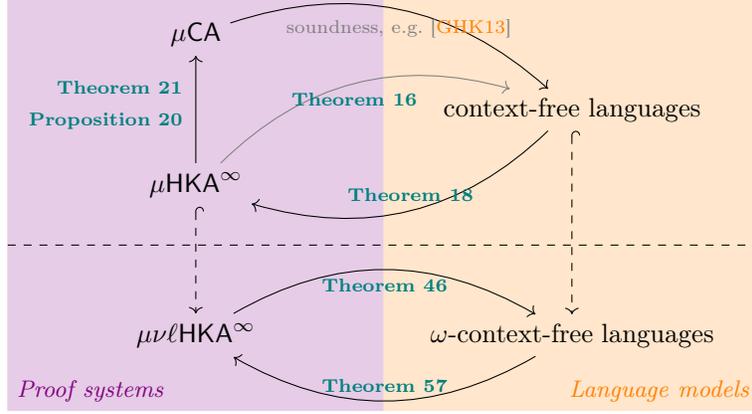


FIGURE 1. Summary of our main contributions. Each arrow \rightarrow denotes an inclusion of equational theories, over an appropriate language of μ -expressions. The gray arrow, Theorem 16, is also a consequence of the remaining black ones.

have been extensively studied, in particular *Chomsky algebras* and μ -semirings [ÉL02, EL05, Lei16, LH18], giving rise to a rich algebraic theory. Indeed Grathwohl, Henglein, and Kozen [GHK13] have given a complete (but infinitary) axiomatisation of the equational theory of μ -expressions, by extending these algebraic theories with a *continuity* principle for least fixed points.

Contributions. In this paper, we propose a *non-wellfounded* system μHKA^∞ for μ -expressions. It can be seen as an extension of the cyclic system of [DP17] for regular expressions. Our first main contribution is the adequacy of this system for CFLs: μHKA^∞ proves $e = f$ just if the CFLs computed by e and f , $\mathcal{L}(e)$ and $\mathcal{L}(f)$ respectively, are the same. We use this result to obtain alternative proof of completeness of the infinitary axiomatisation μCA of [GHK13], comprising our second main result. Our method is inspired by previous techniques in non-wellfounded proof-theory, namely [Stu08, DDS23], employing ‘projections’ to translate non-wellfounded proofs to wellfounded ones. Our result is actually somewhat stronger than that of [GHK13], since our wellfounded proofs are furthermore *cut-free*.

Finally we develop an extension $\mu\nu\ell\text{HKA}$ of (leftmost) μHKA by adding *greatest* fixed points, ν , for which $\mathcal{L}(\cdot)$ extends to a model of ω -context-free languages. Our third main contribution is the soundness and completeness of $\mu\nu\ell\text{HKA}$ for $\mathcal{L}(\cdot)$. Compared to μHKA , the difficulty for metalogical reasoning here is to control interleavings of μ and ν , both for soundness argument and in controlling proof search for completeness. To this end we employ *game theoretic* techniques to characterise word membership and control proof search.

All our main results are summarised in Fig. 1.

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2. A SYNTAX FOR CONTEXT-FREE GRAMMARS

Throughout this work we make use of a finite set \mathcal{A} (the **alphabet**) of **letters**, written a, b, \dots , and a countable set \mathcal{V} of **variables**, written X, Y, \dots . When speaking about context-free grammars (CFGs), we always assume non-terminals are from \mathcal{V} and the terminals are from \mathcal{A} .

We define (μ -)**expressions**, written e, f , etc., by:

$$e, f, \dots ::= 0 \mid 1 \mid X \mid a \mid e + f \mid e \cdot f \mid \mu X e \quad (1)$$

We usually simply write ef instead of $e \cdot f$. μ is considered a variable binder, with the *free variables* $\text{FV}(e)$ of an expression e defined as expected:

Definition 1 (Free variables). The set of **free variables** of an expression e , written $\text{FV}(e)$, is defined by:

- $\text{FV}(0) := \emptyset$
- $\text{FV}(1) := \emptyset$
- $\text{FV}(X) := \{X\}$
- $\text{FV}(e + f) := \text{FV}(e) \cup \text{FV}(f)$
- $\text{FV}(ae) := \text{FV}(e)$
- $\text{FV}(\mu X e) := \text{FV}(e) \setminus \{X\}$

We sometimes refer to expressions as *formulas*, and write \sqsubseteq for the subformula relation.

μ -expressions compute languages of finite words in the expected way:

Definition 2 (Language semantics). Let us temporarily expand the syntax of expressions to include each language $A \subseteq \mathcal{A}^*$ as a constant symbol. We interpret each closed expression (of this expanded language) as a subset of \mathcal{A}^* as follows:

- | | |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ul style="list-style-type: none"> • $\mathcal{L}(0) := \emptyset$ • $\mathcal{L}(1) := \{\varepsilon\}$ • $\mathcal{L}(a) := \{a\}$ • $\mathcal{L}(A) := A$ | <ul style="list-style-type: none"> • $\mathcal{L}(e + f) := \mathcal{L}(e) \cup \mathcal{L}(f)$ • $\mathcal{L}(ef) := \{vw : v \in \mathcal{L}(e), w \in \mathcal{L}(f)\}$ • $\mathcal{L}(\mu X e(X)) := \bigcap \{A \supseteq \mathcal{L}(e(A))\}$ |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

Note that all the operators of our syntax correspond to monotone operations on $\mathcal{P}(\mathcal{A}^*)$, with respect to \subseteq . Thus $\mathcal{L}(\mu X e(X))$ is just the least fixed point of the operation $A \mapsto \mathcal{L}(e(A))$, by the Knaster-Tarski fixed point theorem.

The **productive** expressions, written p, q etc. are generated by:

$$p, q, \dots ::= a \mid p + q \mid p \cdot e \mid e \cdot p \mid \mu X p \quad (2)$$

We say that an expression is **guarded** if each variable occurrence occurs free in a productive subexpression. **Left-productive** and **left-guarded** are defined in the same way, only omitting the clause $e \cdot p$ in the grammar above. For convenience of exposition we shall employ the following convention throughout:

Convention 3. Henceforth we assume all expressions are guarded.

Remark 4 (Why only guarded expressions?). There are several reasons for employing this convention. Most importantly, left-guardedness will be required for our treatment of ω -words later via greatest fixed points, where grammars naturally parse from the left via *leftmost derivations*. In the current setting, over finite words with only least fixed points, it makes little difference whether we use only guarded

expressions or not, nor whether we guard from the left or right. However our convention does simplify some proofs and change some statements; we will comment on such peculiarities when they are important.

Example 5 (Empty language). In the semantics above, note that the empty language \emptyset is computed by several expressions, not only 0 but also μXX and $\mu X(aX)$. Note that while the former is unguarded the latter is (left-)guarded. In this sense the inclusion of 0 is somewhat ‘syntactic sugar’, but it will facilitate some of our later development.

Example 6 (Kleene star and universal language). For any expression e we can compute its Kleene star $e^* := \mu X(1 + eX)$ or $e^* := \mu X(1 + Xe)$. These definitions are guarded just when e is productive. Now, note that we also have not included a symbol \top for the universal language \mathcal{A}^* . We can compute this by the expression $(\sum \mathcal{A})^*$, which is guarded as $\sum \mathcal{A}$ is productive.

It is well-known that μ -expressions compute just the context-free (CF) languages [GR62, Sch63, Gru71]. In fact this holds even under the restriction to left-guarded expressions, by simulating the *Greibach normal form*:

Theorem 7 (Adequacy, see, e.g., [ÉL02, EL05]). *L is context-free (and $\varepsilon \notin L$) $\iff L = \mathcal{L}(e)$ for some e left-guarded (and left-productive, respectively).*

While this argument is known, it is pertinent to recall it as we are working with only guarded expressions, and as some of the intermediate concepts will be useful to us later. First we will need to define a notion of subformula peculiar to fixed point expressions:

Definition 8 (Fisher-Ladner (FL)). The **Fischer-Ladner (FL) closure** of an expression e , written $\text{FL}(e)$, is the smallest set of expressions closed under closed subformulas and, whenever $\mu X f(X) \in \text{FL}(e)$ then $f(\mu X f(X)) \in \text{FL}(e)$.

It is well-known that $\text{FL}(e)$ is finite, and in fact has size linear in that of e . Now the \Leftarrow direction of the Adequacy theorem, Theorem 7, can be proved by construing $\text{FL}(e)$ as the non-terminals of an appropriate CFG. Formally, this is broken up into the following two Propositions.

Proposition 9. *$\mathcal{L}(e)$ is context-free, for any closed expression e .*

Proof of Proposition 9. We construct a CF grammar with nonterminals X_f for each $f \in \text{FL}(e)$, starting nonterminal X_e , and all productions of form:

$$\begin{aligned} X_1 &\rightarrow \varepsilon \\ X_a &\rightarrow a \\ X_{f+g} &\rightarrow X_f \mid X_g \\ X_{fg} &\rightarrow X_f X_g \\ X_{\mu X f(X)} &\rightarrow X_{f(\mu X f(X))} \quad \square \end{aligned} \tag{3}$$

For the \implies direction of Theorem 7, since we assume only guarded expressions, we work with grammars in *Greibach normal form* (GNF). Recall that a GNF grammar is one for which each production has form $X \rightarrow a\vec{X}$ or $X \rightarrow \varepsilon$, for X, \vec{X} non-terminals and $a \in \mathcal{A}$. It is well-known that such grammars exhaust all CF languages [JEH01]. Thus we obtain a slightly stronger result:

Proposition 10 (see, e.g., [ÉL02, EL05]). *L is context-free $\implies L = \mathcal{L}(e)$ for some left-guarded expression e . Moreover if $\varepsilon \notin L$ then e is left-productive.*

Proof sketch. We expand the statement to grammars where each non-terminal has a unique production whose RHS is an arbitrary left-guarded μ -expression, i.e. of the form $\{X_i \rightarrow e_i(\vec{X})\}_{i < n}$ for non-terminals $\vec{X} = X_0, \dots, X_{n-1}$. Note that this exhausts all context-free languages by (a) assuming Greibach normal form for left-guardedness; and (b) using $+$ to combine multiple productions from the same non-terminal. From here we proceed by induction on n , the number of non-terminals, using *Bekić's Theorem* for resolving equational systems. Namely from $\{X_i \rightarrow e_i(\vec{X}, X_n)\}_{i < n}$ we set $e'_n(\vec{X}) := \mu X_n e_n(\vec{X}, X_n)$ and first find solutions \vec{f} to the grammar $\{X_i \rightarrow e_i(\vec{X}, e'_n(\vec{X}))\}_{i < n}$, by inductive hypothesis. Now we set the solution for X_n to be $f_n := e'_n(\vec{f})$. Note that, since we did not introduce identities, the solutions contain 1 just if there is an ε production for some non-terminal, by Greibach normal form. \square

Example 11. Consider the left-guarded expressions $\text{Dyck}_1 := \mu X(1 + \langle X \rangle X)$ and $\{a^n b^n\}_n := \mu X(1 + aXb)$. As suggested, Dyck_1 indeed computes the language of well-bracketed words over alphabet $\{\langle, \rangle\}$, whereas $\{a^n b^n\}_n$ computes the set of words $\vec{a}\vec{b}$ with $|\vec{a}| = |\vec{b}|$. We can also write $(a^* b^*) := \mu X(1 + aX + Xb)$, which is guarded but not left-guarded. However, if we define Kleene $*$ as in Example 6, then we can write a^* and b^* as left-guarded expressions and then take their product for an alternative representation of $(a^* b^*)$. Note that the empty language \emptyset is computed by the left-guarded expression $\mu X(aX)$, cf. Example 5.

3. A NON-WELLFOUNDED PROOF SYSTEM

In this section we extend a calculus HKA from [DP17] for regular expressions to all μ -expressions, and prove soundness and completeness of its non-wellfounded proofs for the language model $\mathcal{L}(\cdot)$. We shall apply this result in the next section to deduce completeness of an infinitary axiomatisation for $\mathcal{L}(\cdot)$, before considering the extension to *greatest* fixed points later.

A **hypersequent** has the form $\Gamma \rightarrow S$ where Γ (the LHS) is a list of expressions (a **cedent**) and S (the RHS) is a set of such lists. We interpret lists by the product of their elements, and sets by the sum of their elements. Thus we extend our notation for language semantics by $\mathcal{L}(\Gamma) := \mathcal{L}(\prod \Gamma)$ and $\mathcal{L}(S) := \bigcup_{\Gamma \in S} \mathcal{L}(\Gamma)$.

The system μHKA is given by the rules in Fig. 2. Here we use commas to delimit elements of a list or set and square brackets $[,]$ to delimit lists in a set. In the k rules, we write $aS := \{[a, \Gamma] : \Gamma \in S\}$ and $Sa := \{[\Gamma, a] : \Gamma \in S\}$.

For each inference step, as typeset in Fig. 2, the **principal** formula is the distinguished **magenta** formula occurrence in the lower sequent, while any distinguished **magenta** formula occurrences in upper sequents are **auxiliary**. (Other colours may be safely ignored for now).

Our system differs from the original presentation of HKA in [DP17] as (a) we have general fixed point rules, not just for the Kleene $*$; and (b) we have included both left and right versions of the k rule, for symmetry. We extend the corresponding notions of non-wellfounded proof appropriately:

Definition 12 (Non-wellfounded proofs). A **preproof** (of μHKA) is generated *coinductively* from the rules of μHKA i.e. it is a possibly infinite tree of sequents (of height $\leq \omega$) generated by the rules of μHKA . A preproof is **regular** or **cyclic** if it has only finitely many distinct subproofs. An infinite branch of a preproof is

$$\begin{array}{l}
\text{Non-logical} \\
\text{rules:} \\
\text{Left logical} \\
\text{rules:} \\
\text{Right logical} \\
\text{rules:}
\end{array}
\begin{array}{l}
\text{init} \frac{}{\rightarrow []} \quad \text{w-r} \frac{\Gamma \rightarrow S}{\Gamma \rightarrow S, [\Delta]} \quad \text{k}_a^l \frac{\Gamma \rightarrow S}{a, \Gamma \rightarrow aS} \quad \text{k}_a^r \frac{\Gamma \rightarrow S}{\Gamma, a \rightarrow Sa} \\
\text{0-l} \frac{}{\Gamma, 0, \Gamma' \rightarrow S} \quad \text{1-l} \frac{\Gamma, \Gamma' \rightarrow S}{\Gamma, 1, \Gamma' \rightarrow S} \quad \text{..l} \frac{\Gamma, e, f, \Gamma' \rightarrow S}{\Gamma, ef, \Gamma' \rightarrow S} \\
\text{+l} \frac{\Gamma, e, \Gamma' \rightarrow S \quad \Gamma, f, \Gamma' \rightarrow S}{\Gamma, e + f, \Gamma' \rightarrow S} \quad \text{\(\mu\)-l} \frac{\Gamma, e(\mu Xe(X)), \Gamma' \rightarrow S}{\Gamma, \mu Xe(X), \Gamma' \rightarrow S} \\
\text{0-r} \frac{\Gamma \rightarrow S}{\Gamma \rightarrow S, [\Delta, 0, \Delta']} \quad \text{1-r} \frac{\Gamma \rightarrow S, [\Delta, \Delta']}{\Gamma \rightarrow S, [\Delta, 1, \Delta']} \quad \text{..r} \frac{\Gamma \rightarrow S, [\Delta, e, f, \Delta']}{\Gamma \rightarrow S, [\Delta, ef, \Delta']} \\
\text{+r} \frac{\Gamma \rightarrow S, [\Delta, e, \Delta'], [\Delta, f, \Delta']}{\Gamma \rightarrow S, [\Delta, e + f, \Delta']} \quad \text{\(\mu\)-r} \frac{\Gamma \rightarrow S, [\Delta, e(\mu Xe(X)), \Delta']}{\Gamma \rightarrow S, [\Delta, \mu Xe(X), \Delta']}
\end{array}$$

FIGURE 2. Rules of the system μHKA .

progressing if it has infinitely many μ -l steps. A preproof is progressing, or a ∞ -**proof**, if all its infinite branches are progressing. We write $\mu\text{HKA} \vdash^\infty \Gamma \rightarrow S$ if $\Gamma \rightarrow S$ has a ∞ -proof in μHKA , and sometimes write μHKA^∞ for the class of ∞ -proofs of μHKA .

Note that our progress condition on preproofs is equivalent to simply checking that every infinite branch has infinitely many left-logical or k steps, as μ -l is the only rule among these that does not decrease the size of the LHS. This is simpler than usual conditions from non-wellfounded proof theory, as we do not have any alternations between the least and greatest fixed points. Indeed we shall require a more complex criterion later when dealing with ω -languages. Note that, as regular preproofs may be written naturally as finite graphs, checking progressiveness for them is efficiently decidable (even in **NL**, see e.g. [DP17, CD22]).

The need for such a complex hypersequential line structure is justified in [DP17] by the desideratum of *regular* completeness for the theory of regular expressions: intuitionistic ‘Lambek-like’ systems, cf. e.g. [Jip04, Pal07, DP18] are incomplete (wrt regular cut-free proofs). The complexity of the RHS of sequents in HKA is justified by consideration of proof search for, say, $a^* \rightarrow (aa)^* + a(aa)^*$ and $(a + b)^* \rightarrow a^*(ba^*)^*$, requiring reasoning under sums and products, respectively.

In our extended system we actually gain *more* regular proofs of inclusions between context-free languages. For instance:

Example 13. Recall the guarded expressions $\{a^n b^n\}_n$ and $(a^* b^*)$ from Example 11. We have the regular ∞ -proof R in Fig. 3 of $\{a^n b^n\}_n \rightarrow [(a^* b^*)]$, where \bullet marks roots of identical subproofs. Note that indeed the only infinite branch, looping on \bullet , has infinitely many μ -l steps.

Remark 14 (Impossibility of general regular completeness). At this juncture let us make an important point: it is impossible to have any (sound) recursively enumerable system, let alone regular cut-free proofs, complete for context-free inclusions, since this problem is Π_1^0 -complete (see e.g. [JEH01]). In this sense examples of regular proofs are somewhat coincidental.

It is not hard to see that each rule of μHKA is sound for language semantics:

$$\begin{array}{c}
 \vdots \\
 \frac{\mu-l, \mu-r}{\bullet} \frac{\{a^n b^n\}_n \rightarrow (a^* b^*)}{\bullet} \\
 \frac{k_b^r}{\bullet} \frac{\{a^n b^n\}_n, b \rightarrow [(a^* b^*), b]}{\bullet} \\
 \frac{\cdot-l, \cdot-r}{\bullet} \frac{\{a^n b^n\}_n b \rightarrow [(a^* b^*) b]}{\bullet} \\
 \frac{\mu-r, +r, w-r}{\bullet} \frac{\{a^n b^n\}_n b \rightarrow [(a^* b^*)]}{\bullet} \\
 \frac{k_a^l}{\bullet} \frac{a, \{a^n b^n\}_n b \rightarrow [a, (a^* b^*)]}{\bullet} \\
 \frac{\cdot-l, \cdot-r}{\bullet} \frac{a \{a^n b^n\}_n b \rightarrow [a(a^* b^*)]}{\bullet} \\
 \frac{w-r, +r}{\bullet} \frac{1 \rightarrow [1 + a(a^* b^*) + (a^* b^*) b]}{\bullet} \quad \frac{w-r, +r}{\bullet} \frac{a \{a^n b^n\}_n b \rightarrow [1 + a(a^* b^*) + (a^* b^*) b]}{\bullet} \\
 \frac{+l}{\bullet} \frac{1 \rightarrow [1 + a(a^* b^*) + (a^* b^*) b]}{\bullet} \\
 \frac{\mu-l, \mu-r}{\bullet} \frac{1 + a \{a^n b^n\}_n b \rightarrow [1 + a(a^* b^*) + (a^* b^*) b]}{\bullet} \\
 \frac{\mu-l, \mu-r}{\bullet} \frac{\{a^n b^n\}_n \rightarrow [(a^* b^*)]}{\bullet}
 \end{array}$$

 FIGURE 3. A regular ∞ -proof R of $\{a^n b^n\}_n \rightarrow [(a^* b^*)]$.

Lemma 15 (Local soundness). *For each inference step,*

$$\frac{\Gamma_0 \rightarrow S_0 \quad \cdots \quad \Gamma_{k-1} \rightarrow S_{k-1}}{r \quad \Gamma \rightarrow S} \quad (4)$$

for some $k \leq 2$, we have: $\forall i < k \mathcal{L}(\Gamma_i) \subseteq \mathcal{L}(S_i) \implies \mathcal{L}(\Gamma) \subseteq \mathcal{L}(S)$.

Consequently finite proofs are also sound, by induction on their structure. For non-wellfounded proofs, we must employ a less constructive argument, typical of non-wellfounded proof theory:

Theorem 16 (Soundness). $\mu\text{HKA} \vdash^\infty \Gamma \rightarrow S \implies \mathcal{L}(\Gamma) \subseteq \mathcal{L}(S)$.

Proof of Theorem 16. For each sequent $\mathcal{S} = \Gamma \rightarrow S$, define $n_{\mathcal{S}} \in \mathbb{N} \cup \{\infty\}$ for the least length of word $w \in \mathcal{L}(\Gamma) \setminus \mathcal{L}(S)$ (if there is no such word then $n_{\mathcal{S}} = \infty$). Now suppose, for contradiction, that P is a ∞ -proof of $\mathcal{S} = \Gamma \rightarrow S$, but $\mathcal{L}(\Gamma) \setminus \mathcal{L}(S) \neq \emptyset$, and so $n_{\mathcal{S}} \in \mathbb{N}$. By (the contraposition of) Lemma 15 we may continually choose invalid premisses of rules to build an infinite branch $(\mathcal{S}_i = \Gamma_i \rightarrow S_i)_{i < \omega}$ s.t. $\mathcal{L}(\Gamma_i) \setminus \mathcal{L}(S_i) \neq \emptyset$ for all $i < \omega$. Moreover, we can guarantee that the sequence $(n_{\mathcal{S}_i})_{i < \omega}$ is monotone non-increasing. For this note that for each inference step $\frac{\mathcal{S}_0 \quad \cdots \quad \mathcal{S}_{k-1}}{r \quad \mathcal{S}}$ of form as in (4) we have:

- (1) If r is a k step, then $n_{\mathcal{S}} > n_{\mathcal{S}_0}$; and,
- (2) Otherwise there is some $i < k$ with $n_{\mathcal{S}} = n_{\mathcal{S}_i}$.

In particular, if r is a $+l$ step, we should choose the \mathcal{S}_i admitting the smallest $n_{\mathcal{S}_i}$. Now, since P is a ∞ -proof, $(\mathcal{S}_i)_{i < \omega}$ must be progressing and has infinitely many $\mu-l$ steps. We have two cases:

- If $(\mathcal{S}_i)_{i < \omega}$ has infinitely many k steps then case 1 happens infinitely often along $(n_{\mathcal{S}_i})_{i < \omega}$, and so it is a monotone non-increasing sequent of natural numbers that does not converge. Contradiction.
- Otherwise $(\mathcal{S}_i)_{i \geq k}$ is k -step-free, for some $k < \omega$, and so the number of letters in the LHS of the sequent is monotone non-decreasing in $i \geq k$. Since there are infinitely many $\mu-l$ steps, by guardedness the number of producing expressions (whose languages necessarily are nonempty and do

not contain ε) is strictly increasing, and eventually dominates even $n_0 \geq n_i$.
 Contradiction. \square

By inspection of the rules of μHKA we have:

Lemma 17 (Invertibility). *Let r be a logical step as in (4). $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(S) \implies \mathcal{L}(\Gamma_i) \subseteq \mathcal{L}(S_i)$, for each $i < k$.*

Theorem 18 (Completeness). $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(S) \implies \mu\text{HKA} \vdash^\infty \Gamma \rightarrow S$.

Proof sketch. We describe a bottom-up proof search strategy:

- (1) Apply left logical rules maximally, preserving validity by Lemma 17. Any infinite branch is necessarily progressing.
- (2) This can only terminate at a sequent of the form $a_1, \dots, a_n \rightarrow S$ with $\vec{a} \in \mathcal{L}(S)$, whence we mimic a ‘leftmost’ parsing derivation for \vec{a} wrt S . \square

To flesh out a bit more the Item 2 of the argument above, let us state:

Lemma 19 (Membership). $a_1 \cdots a_n \in \mathcal{L}(S) \implies \mu\text{HKA} \vdash^\infty a_1, \dots, a_n \rightarrow S$.

Proof idea. Recall the ‘canonical’ grammar as constructed in (3). We proceed by induction on a leftmost (or rightmost) derivation (as defined for Muller CFGs in Section 6) of \vec{a} according to this canonical grammar. \square

4. COMPLETENESS OF AN INFINITARY CUT-FREE AXIOMATISATION

While our completeness result above was relatively simple to establish, we can use it, along with proof theoretic techniques, to deduce completeness of an infinitary axiomatisation of the theory of μ -expressions. In fact we obtain an alternative proof of the result of [GHK13], strengthening it to a ‘cut-free’ calculus μHKA_ω .

Write μCA for the set of axioms consisting of:

- $(0, 1, +, \cdot)$ forms an idempotent semiring (aka a *dioid*).
- (μ -continuity) $e\mu X f(X)g = \sum_{n < \omega} e f^n(0)g$.

We are using the notation $f^n(0)$ defined by $f^0(0) := 0$ and $f^{n+1}(0) := f(f^n(0))$. We also write $e \leq f$ for the natural order given by $e + f = f$. Now, define μHKA_ω to be the extension of μHKA by the ‘ ω -rule’:

$$\omega \frac{\{\Gamma, e^n(0), \Gamma' \rightarrow S\}_{n < \omega}}{\Gamma, \mu X e(X), \Gamma' \rightarrow S}$$

By inspection of the rules we have soundness of μHKA_ω for μCA :

Proposition 20. $\mu\text{HKA}_\omega \vdash \Gamma \rightarrow S \implies \mu\text{CA} \vdash \prod \Gamma \leq \sum_{\Delta \in S} \prod \Delta$.

Here the soundness of the ω -rule above is immediate from μ -continuity in μCA . Note, in particular, that μCA already proves that $\mu X e(X)$ is indeed a fixed point of $e(\cdot)$, i.e. $e(\mu X e(X)) = \mu X e(X)$ [GHK13]. The main result of this section is:

Theorem 21. $\mu\text{HKA} \vdash^\infty e \rightarrow f \implies \mu\text{HKA}_\omega \vdash e \leq f$

Note that, immediately from Theorem 18 and Proposition 20, we obtain:

Corollary 22. $\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \mu\text{HKA}_\omega \vdash e \leq f \implies \mu\text{CA} \vdash e \leq f$

To prove Theorem 21 we employ similar techniques to those used for an extension of *linear logic* with least and greatest fixed points [DDS23], only specialised to the current setting. We only sketch the ideas here, referencing the analogous definitions and theorems from that work at the appropriate moments.

4.1. Projections. Let us consider cedents $\Gamma = \Gamma(f_1, \dots, f_k)$ where some occurrences of f_1, \dots, f_k are distinguished. Note that the distinguished occurrences of each f_i may include some, none or all of the occurrences of f_i in Γ , including as subexpressions of expressions in Γ . We allow distinct f_i and f_j to be the same formula, as long as they distinguish a disjoint set of occurrences.

When $\vec{f} = (\mu X_1 g_1(X_1), \dots, \mu X_k g_k(X_k))$ and $\vec{n} \in \mathbb{N}^k$, we write $\vec{f}^{\vec{n}} := (g_1^{n_1}(0), \dots, g_k^{n_k}(0))$, the list obtained by **assigning** \vec{n} to \vec{f} .

Let us briefly recap the definition of *projection* in [DDS23, Definition 15], specialised to our setting:

Definition 23 (Projections). For each μ HKA preproof P of $\Gamma(\vec{f}) \rightarrow S$ and $\vec{n} \in \mathbb{N}^k$ we define $P(\vec{n})$ a preproof of $\Gamma(\vec{f}^{\vec{n}}) \rightarrow S$ by coinduction on P :

- The definition of $P(\vec{n})$ commutes with any step for which a distinguished formula is not principal for a μ -l step.
- If P ends with a step for which a distinguished formula is μ -l principal,

$$\mu\text{-l} \frac{\Gamma(\mu X e(X)), e(\mu X e(X)), \Gamma'(\mu X e(X)) \rightarrow S}{\Gamma(\mu X e(X)), \mu X e(X), \Gamma'(\mu X e(X)) \rightarrow S} \quad (5)$$

then we proceed by case analysis on the number assigned:

$$P(0, \vec{n}) := \text{0-l} \frac{}{\Gamma(0), 0, \Gamma'(0) \rightarrow S}$$

$$P(n+1, \vec{n}) := \frac{\Gamma(e^{n+1}(0)), e(e^n(0)), \Gamma'(e^{n+1}(0)) \rightarrow S}{\Gamma(e^{n+1}(0)), e^{n+1}(0), \Gamma'(e^{n+1}(0)) \rightarrow S}$$

where in the second case we have further distinguished the occurrences of $\mu X e(X)$ indicated in the auxiliary formula in (5).

Note that we have technically used a ‘repetition’ rule = in the translation above to ensure productivity of the translation. However it turns out this is unnecessary, as ∞ -proofs are indeed closed under taking projections:

Proposition 24. *If P is a μ HKA ∞ -proof, then so is $P(\vec{n})$.*

The proof of this result follows the same argument as [DDS23, Proposition 18]. To briefly recall the idea:

Proof sketch. Each maximal branch B of $P(\vec{n})$ is a prefix of some branch B' of P , by inspection of the translation, only with some μ -l steps replaced by ‘=’ steps or, when B is finite, a 0-l step. Thus if B is infinite then it must have infinitely many k steps, since B' must be progressing, and so must have infinitely many μ -l steps too. \square

In particular we have:

Corollary 25 (Projection). *For each ∞ -proof P of $\Gamma, \mu X e(X), \Gamma' \rightarrow S$, $P(n)$ is an ∞ -proof of $\Gamma, e^n(0), \Gamma' \rightarrow S$, for each $n < \omega$.*

From here it is simple to provide a translation from μHKA ∞ -proofs to μHKA_ω preproofs, as in Definition 30 shortly. However, to prove the image of the translation is *wellfounded*, we shall need some structural proof theoretic machinery, which will also serve later use when dealing with greatest fixed points in Sections 5 and 6.

4.2. Intermezzo: ancestry and threads. Given an inference step r , as typeset in Fig. 2, we say a formula occurrence f in an upper sequent is an **immediate ancestor** of a formula occurrence e in the lower sequent if they have the same colour; furthermore if e and f occur in a cedent $\Gamma, \Gamma', \Delta, \Delta'$, they must be the matching occurrences of the same formula (i.e. at the same position in the cedent); similarly if e and f occur in the RHS context S , they must be matching occurrences in matching lists.

Construing immediate ancestry as a directed graph allows us to characterise progress by consideration of its paths:

Definition 26 ((Progressing) threads). Fix a preproof P . A **thread** is a maximal path in the graph of immediate ancestry. An infinite thread on the LHS is **progressing** if it is infinitely often principal (i.o.p.) for a μ -l step.

Our overloading of terminology is suggestive:

Proposition 27. *P is progressing \Leftrightarrow each branch of P has a progressing thread.*

Proof sketch. The \Leftarrow direction is trivial. For \Rightarrow direction we appeal to König's lemma. Fix a branch B and take the subtree of its immediate ancestry graph with nodes the principal formula occurrences along B , and edges given by reachability ('direct ancestry'). By progressiveness this tree is infinite, and by inspection of the rules it is finitely branching, thus it must have an infinite path by König's Lemma. This path induces an infinitely often principal thread along B , which in turn must be infinitely often μ -l principal as every other LHS rule strictly decreases the size of formula. \square

Example 28. Recall the ∞ -proof in Example 13. The only infinite branch, looping on \bullet , has a progressing thread indicated in **magenta**.

Fact 29 (See, e.g., [Koz83, KMV22]). *Any i.o.p. thread has a unique smallest i.o.p. formula, under the subformula relation. This formula must be a fixed point formula.*

4.3. Translation to ω -branching system. We are now ready to give a translation from μHKA^∞ to μHKA_ω .

Definition 30 (ω -translation). For preproofs P define P^ω by coinduction:

- $^\omega$ commutes with any step not a μ -l.

$$\bullet \left(\frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ P \\ \text{---} \\ \Gamma, e(\mu X e(X)), \Gamma' \rightarrow S \end{array}}{\mu\text{-l} \frac{\Gamma, \mu X e(X), \Gamma' \rightarrow S}{\Gamma, \mu X e(X), \Gamma' \rightarrow S}} \right)^\omega := \frac{\left\{ \frac{\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ P(n)^\omega \\ \text{---} \\ \Gamma, e^n(0), \Gamma' \rightarrow S \end{array}}{\Gamma, e^n(0), \Gamma' \rightarrow S} \right\}_{n < \omega}}{\omega \frac{\Gamma, \mu X e(X), \Gamma' \rightarrow S}{\Gamma, \mu X e(X), \Gamma' \rightarrow S}}$$

Theorem 21 now follows immediately from the following result, obtained by analysis of progressing threads in the image of the ω -translation:

Lemma 31. P is progressing $\implies P^\omega$ is wellfounded.

The proof of Lemma 31 follows the same argument as for the analogous result in [DDS23, Lemma 23].

Proof sketch. Each branch of P^ω can be specified by a branch B of P and a (possibly infinite) sequence of natural numbers $\vec{n} \in \omega^{\leq \omega}$, specifying at each ω -step which premiss to follow. Call this $B^{\vec{n}}$. Now, take a progressing thread along B and consider its smallest i.o.p. μ -formula, say $\mu X e(X)$. If $B^{\vec{n}}$ follows, say the k^{th} premiss at the first corresponding principal step for $\mu X e(X)$ in B , write K for height of the $(k+1)^{\text{th}}$ unfolding of $\mu X e(X)$ in B . It follows from inspection of the ω -translation and projections that $B^{\vec{n}}$ has height $\leq K$. \square

Example 32. Recalling Example 13, let us see the ω -translation of R in (3). First, let us (suggestively) write $\{a^k b^k\}_{k < n}$ for the n^{th} approximant of $\{a^n b^n\}_n$, i.e. $\{a^k b^k\}_{k < 0} := 0$ and $\{a^k b^k\}_{k < n+1} := 1 + a\{a^k b^k\}_{k < n}b$. Now R^ω is given below,

left, where recursively $R(0) := 0 \xrightarrow{-l} (a^* b^*)$ and $R(n+1)$ is given below, right:

$$\begin{array}{c}
 \left\{ \begin{array}{c} \triangle \\ R(n) \\ \{a^k b^k\}_{k < n} \rightarrow [(a^* b^*)] \end{array} \right\}_{n < \omega} \\
 \omega, \mu\text{-}r \frac{\quad}{\{a^n b^n\}_n \rightarrow [(a^* b^*)]}
 \end{array}
 \quad ; \quad
 \begin{array}{c}
 \triangle \\
 R(n) \\
 \frac{\{a^k b^k\}_{k < n} \rightarrow (a^* b^*)}{k_b^r} \\
 \frac{\{a^k b^k\}_{k < n}, b \rightarrow [(a^* b^*), b]}{-l, \text{-}r} \\
 \frac{\{a^k b^k\}_{k < n}b \rightarrow [(a^* b^*)b]}{\mu\text{-}r, +\text{-}r, w\text{-}r} \\
 \frac{\{a^k b^k\}_{k < n}b \rightarrow [(a^* b^*)]}{k_a^l} \\
 \frac{a, \{a^k b^k\}_{k < n}b \rightarrow [a, (a^* b^*)]}{-l, \text{-}r} \\
 \frac{a\{a^k b^k\}_{k < n}b \rightarrow [a(a^* b^*)]}{\quad} \\
 \frac{a\{a^k b^k\}_{k < n}b \rightarrow [(a^* b^*)]}{\quad} \\
 \frac{1 \rightarrow [(a^* b^*)]}{+l} \\
 \frac{1 \rightarrow [1]}{1\text{-}l, 1\text{-}r} \\
 \frac{1 \rightarrow [1]}{\text{init}} \\
 \frac{1 \rightarrow [(a^* b^*)]}{1 + a\{a^k b^k\}_{k < n}b \rightarrow [(a^* b^*)]}
 \end{array}$$

5. GREATEST FIXED POINTS AND ω -LANGUAGES

We extend the grammar of expressions from (1) by:

$$e, f \dots ::= \dots \mid \nu X e(X)$$

We call such expressions $\mu\nu$ -expressions when we need to distinguish them from ones without ν . The notions of a (left-)productive and (left-)guarded expression are defined in the same way, extending the grammar of (2) by the clause $\nu X p$.

As expected $\mu\nu$ -expressions denote languages of finite and infinite words:

Definition 33 (Intended semantics of $\mu\nu$ -expressions). We extend the notation vw to all $v, w \in \mathcal{A}^{\leq \omega}$ by setting $vw = v$ when $|v| = \omega$. We extend the definition of $\mathcal{L}(\cdot)$ from Definition 2 to all $\mu\nu$ -expressions by setting $\mathcal{L}(\nu X e(X)) := \bigcup \{A \subseteq \mathcal{L}(e(A))\}$ where now A varies over subsets of $\mathcal{A}^{\leq \omega}$.

Again, since all the operations are monotone, $\mathcal{L}(\nu X e(X))$ is indeed the greatest fixed point of the operation $A \mapsto \mathcal{L}(e(A))$, by the Knaster-Tarski theorem. In fact (ω -)languages computed by $\mu\nu$ -expressions are just the ‘ ω -context-free languages’ (ω -CFLs), cf. [CG77, Lin76], defined as the ‘Kleene closure’ of CFLs:

Definition 34 (ω -context-free languages). For $A \subseteq \mathcal{A}^+$ we write $A^\omega := \{w_0 w_1 w_2 \cdots : \forall i < \omega w_i \in A\}$. The class of ω -CFLs (CF^ω) is defined by:

$$\text{CF}^\omega := \left\{ \bigcup_{i < n} A_i B_i^\omega : n < \omega; A_i, B_i \text{ context-free and } \varepsilon \notin A_i, B_i, \forall i < n \right\}$$

It is not hard to see that each ω -CFL is computed by a $\mu\nu$ -expression, by noting that $\mathcal{L}(e)^\omega = \mathcal{L}(\nu X(eX))$:

Proposition 35. $L \in \text{CF}^\omega \implies L = \mathcal{L}(e)$ for some left-productive e .

Proof. Given $L = \bigcup_{i=1}^n A_i B_i^\omega$, by Proposition 10 let e_i and f_i be left-productive with $\mathcal{L}(e_i) = A_i$, $\mathcal{L}(f_i) = B_i$ for $i = 1, \dots, n$. Then, $L = \mathcal{L}(\sum_{i=1}^n e_i \nu X(f_i X))$. \square

We shall address the converse of this result later. First let us present our system for $\mu\nu$ -expressions, a natural extension of μHKA earlier:

Definition 36 (System). The system $\mu\nu\text{HKA}$ extends μHKA by the rules:

$$\nu\text{-l} \frac{\Gamma, e(\nu X e(X)), \Gamma' \rightarrow S}{\Gamma, \nu X e(X), \Gamma' \rightarrow S} \quad \nu\text{-r} \frac{\Gamma \rightarrow S, [\Delta, e(\nu X e(X), \Delta')]}{\Gamma \rightarrow S, [\Delta, \nu X e(X), \Delta']} \quad (6)$$

Preproofs for this system are defined just as for μHKA before. The definitions of *immediate ancestor* and *thread* for $\mu\nu\text{HKA}$ extends that of μHKA from Definition 26 according to the colouring above in (6).

However we must be more nuanced in defining progress, requiring a definition at the level of threads as in Section 4. Noting that Fact 29 holds for our extended language with νs as well as μs , we call an i.o.p. thread a μ -**thread** (or ν -**thread**) if its smallest i.o.p. formula is a μ -formula (or ν -formula, respectively).

Definition 37 (Progress). Fix a preproof P . We say that an infinite thread τ along a (infinite) branch B of P is **progressing** if it is i.o.p. and it is a μ -thread on the LHS or it is a ν -thread on the RHS. B is **progressing** if it has a progressing thread. P is a ∞ -**proof** of $\mu\nu\text{HKA}$ if each of its infinite branches has a progressing thread.

Example 38. Write $e := \nu Z(abZ)$ and $f := \mu Y(b + \nu X(aYX))$. The sequent $e \rightarrow [f]$ has a preproof given in Fig. 4. This preproof has just one infinite branch, looping on \bullet , which indeed has a progressing thread following the **magenta** formulas. The only fixed point infinitely often principal along this thread is $\nu X(aYX)$, which is principal at each \bullet . Thus this preproof is a proof and $e \rightarrow [f]$ is a theorem of $\mu\ell\text{HKA}^\infty$.

Note that, even though this preproof is progressing, the infinite branch’s smallest i.o.p. formula on the RHS is *not* a ν -formula, e.g. given by the **magenta** thread, as f is also i.o.p. Let us point out that (a) the progressiveness condition only requires *existence* of a progressing thread, even if other threads are not progressing (like the unique LHS thread above).

$$\begin{array}{c}
\vdots \\
\frac{\nu\text{-}r}{e \rightarrow [\nu X(afX)]} \bullet \\
\frac{k_b}{b, e \rightarrow [b, \nu X(afX)]} \\
\frac{w\text{-}r}{b, e \rightarrow [b, \nu X(afX)], [\nu X(aYX), \nu X(afX)]} \\
\frac{+\text{-}r}{b, e \rightarrow [b + \nu X(aYX), \nu X(afX)]} \\
\frac{\mu\text{-}r}{b, e \rightarrow [f, \nu X(afX)]} \\
\frac{k_a}{a, b, e \rightarrow [a, f, \nu X(afX)]} \\
\frac{\cdot\text{-}l, \cdot\text{-}r}{abe \rightarrow [af\nu X(afX)]} \\
\frac{\nu\text{-}r}{e \rightarrow [\nu X(afX)]} \bullet \\
\frac{w\text{-}r}{e \rightarrow [b], [\nu X(afX)]} \\
\frac{+\text{-}r}{e \rightarrow [b + \nu X(afX)]} \\
\frac{\mu\text{-}r}{e \rightarrow [f]}
\end{array}$$

FIGURE 4. A $\mu\nu\ell\text{HKA}$ ∞ -preproof of $e \rightarrow [f]$, where $e := \nu Z(abZ)$ and $f := \mu Y(b + \nu X(aYX))$.

Some necessary conventions: left-guarded and leftmost. Crucially, due to the asymmetry in the definition of the product of infinite words, we must employ further conventions to ensure soundness and completeness of ∞ -proofs for $\mathcal{L}(\cdot)$. Our choice of conventions is inspired by the usual ‘leftmost’ semantics of ‘ ω -CFGs’, which we shall see in the next section.

First, we shall henceforth work with a *leftmost* restriction of $\mu\nu\text{HKA}$ in order to maintain soundness for $\mathcal{L}(\cdot)$:

Definition 39. A $\mu\nu\text{HKA}$ preproof is **leftmost** if each logical step has principal formula the leftmost formula of its cedent, and there are no k^r -steps. Write $\mu\nu\ell\text{HKA}$ for the restriction of $\mu\nu\text{HKA}$ to only leftmost steps and $\mu\nu\ell\text{HKA}^\infty$ for the class of ∞ -proofs of $\mu\nu\ell\text{HKA}$.

We must also restrict ourselves to left-guarded expressions in the sequel:

Convention 40. Henceforth, all expressions are assumed to be left-guarded.

Let us justify both of these restrictions via some examples.

Remark 41 (Unsound for non-leftmost). Unlike the μ -only setting it turns out that $\mu\nu\text{HKA}^\infty$ is unsound without the leftmost restriction, regardless of left-guardedness. For instance consider the preproof,

$$\begin{array}{c}
\vdots \\
\frac{\nu\text{-}r}{\rightarrow [a, \nu X(aX)]} a, \bullet \\
\frac{\cdot\text{-}r}{\rightarrow [a\nu X(aX)]} \\
\frac{\nu\text{-}r}{\rightarrow [\nu X(aX)]} \bullet
\end{array}$$

where a, \bullet roots the same subproof as \bullet , but for an extra a on the left of every RHS. Of course the endsequent is not valid, as the LHS denotes $\{\varepsilon\}$ while the

Theorem 45 (Evaluation). $w \in \mathcal{L}(\Gamma) \Leftrightarrow$ *there is a winning play from (w, Γ) .*

The proof is rather involved, employing the method of ‘signatures’ common in fixed point logics, cf. e.g. [NW96], which serve as ‘least witnesses’ to word membership via carefully managing *ordinal approximants* for fixed points. Here we must be somewhat more careful in the argument because positions of our puzzle include *cedents*, not single formulas: we must crucially assign signatures to *each* formula of a cedent. Working with cedents rather than formulas allows the evaluation puzzle to remain strictly single player. This is critical for expressivity: *alternating* context-free grammars and pushdown automata compute more than just CFLs [MH05, CKS81].

The next subsection is devoted to a proof of the Evaluation Theorem above. Before that, let us give an important consequence:

We can now prove the soundness of $\mu\nu\text{HKA}^\infty$ by reduction to Theorem 45:

Theorem 46 (Soundness). $\mu\nu\text{HKA} \vdash^\infty \Gamma \rightarrow S \implies \mathcal{L}(\Gamma) \subseteq \mathcal{L}(S)$.

Proof sketch. Let P be a ∞ -proof of $\Gamma \rightarrow S$ and $w \in \mathcal{L}(\Gamma)$. We show $w \in \mathcal{L}(S)$. First, since $w \in \mathcal{L}(\Gamma)$ there is a winning play π from (w, Γ) by Theorem 45, which induces a unique (maximal) branch B_π of P which must have a progressing thread τ . Now, since π is a *winning* play from (w, e) , τ cannot be on the LHS, so it is an RHS ν -thread following, say, a sequence of cedents $[\Gamma_i]_{i < \omega}$. By construction $[\Gamma_i]_{i < \omega}$ has an infinite subsequence, namely whenever it is principal, that forms (the right components of) a winning play from (w, Γ_0) , with $\Gamma_0 \in S$. Thus indeed $w \in \mathcal{L}(S)$ by Theorem 45. \square

6.2. Proof of the Evaluation Theorem. Let us now revisit the argument for Theorem 45 more formally. This subsection may be safely skipped by the reader comfortable with that result.

Write \sqsubseteq for the subformula relation. Recalling Definition 8, let us write $e \leq_{\text{FL}} f$ if $e \in \text{FL}(f)$, $e <_{\text{FL}} f$ if $e \leq_{\text{FL}} f \not\leq_{\text{FL}} e$ and $e =_{\text{FL}} f$ if $e \leq_{\text{FL}} f \leq_{\text{FL}} e$.

Definition 47 (Dependency order). Let the **dependency order** be $\preceq := \leq_{\text{FL}} \times \sqsubseteq$, i.e. $e \preceq f$ if either $e <_{\text{FL}} f$ or $e =_{\text{FL}} f$ and $f \sqsubseteq e$.

Note that, by the properties of FL closure, \preceq is a well partial order on expressions. In the sequel, we assume an arbitrary extension of \preceq to a total well-order \leq .

Definition 48 (Signatures). Let M and N be finite sets of μ -expressions $\{\mu X_0 e_0 \succ \dots \succ \mu X_{n-1} e_{n-1}\}$ and $\{\nu X_0 e_0 \succ \dots \succ \nu X_{n-1} e_{n-1}\}$ respectively. A μ -**signature** (respectively, a μ -**signature**) is a sequence $\vec{\alpha} = (\alpha_i)_{i=0}^{n-1}$ of ordinals indexed by M (respectively, N). Signatures are ordered by the lexicographical product order.

Let us temporarily expand the language of expressions by,

$$e, f, \dots ::= \dots \mid \mu^\alpha X e \mid \nu^\alpha X e$$

where α ranges over ordinals.

Definition 49. Fix a finite set of μ -formulas $\{\mu X_0 e_0 \succ \dots \succ \mu X_{n-1} e_{n-1}\}$ and a μ -signature $\vec{\alpha} = (\alpha_i)_{i=0}^{n-1}$. Given an expression e , its corresponding μ -**signed formula** $e^{\vec{\alpha}}$ is one where every occurrence $\mu X_i e_i$ has been replaced by $\mu^{\alpha_i} X_i e_i$.

Similarly, given a finite set of ν -formulas, a ν -signature $\vec{\alpha}$, and an expression e , its corresponding ν -**signed formula** $e_{\vec{\alpha}}$ is one where every occurrence of a ν -subformula has been replaced by its corresponding approximant.

We interpret such expressions by the inflationary and deflationary constructions respectively:

- $\mathcal{L}(\mu^0 X e(X)) := \emptyset$
- $\mathcal{L}(\mu^{\alpha+1} X e(X)) := \mathcal{L}(e(\mu^\alpha X e(X)))$
- $\mathcal{L}(\mu^\lambda X e(X)) := \bigcup_{\alpha < \lambda} \mathcal{L}(\mu^\alpha X e(X))$, when λ is a limit ordinal.

- $\mathcal{L}(\nu_0 X e(X)) := \mathcal{A}^{\leq \omega}$
- $\mathcal{L}(\nu_{\alpha+1} X e(X)) := \mathcal{L}(e(\nu^\alpha X e(X)))$
- $\mathcal{L}(\nu_\lambda X e(X)) := \bigcap_{\alpha < \lambda} \mathcal{L}(\nu^\alpha X e(X))$, when λ is a limit ordinal.

Finally, we extend the notion of μ and ν signatures to lists of expressions by writing $\Gamma^{\vec{\alpha}}$ and $\Gamma_{\vec{\alpha}}$ (parameterised by *lists* of vectors of ordinals now, by abuse of notation) for the corresponding μ and ν -signed lists of Γ . Spelt out, $[e_1, \dots, e_n]^{\vec{\alpha}_1, \dots, \vec{\alpha}_n}$ is a shorthand for $[e_1^{\vec{\alpha}_1}, \dots, e_n^{\vec{\alpha}_n}]$ (similarly for ν -signed). Lists of vectors are lexicographically ordered.

Recall that least and greatest fixed points can be computed as limits of approximants. In particular, we have,

- $\mathcal{L}(\mu X e) = \bigcup_{\alpha \in \text{Ord}} \mathcal{L}(\mu^\alpha X e)$
- $\mathcal{L}(\nu X e) = \bigcap_{\alpha \in \text{Ord}} \mathcal{L}(\nu^\alpha X e)$

where α ranges over ordinals. Thus we have immediately:

Proposition 50. *Suppose Γ is a list of expressions. We have:*

- *If $w \in \mathcal{L}(\Gamma)$ then there is a μ -signature $\vec{\alpha}$ such that $w \in \mathcal{L}(\Gamma^{\vec{\alpha}})$.*
- *If $w \notin \mathcal{L}(\Gamma)$ then there is a ν -signature $\vec{\alpha}$ such that $w \notin \mathcal{L}(\Gamma_{\vec{\alpha}})$.*

We are now ready to prove our characterisation of evaluation:

Proof of Theorem 45. (\implies) Suppose $w \in \mathcal{L}(\Gamma)$. By Proposition 50, there is a least μ -signature $\vec{\alpha}$ such that $w \in \mathcal{L}(\Gamma^{\vec{\alpha}})$. We will construct a winning play $(w_i, \Gamma_i)_{i \in \lambda}$ and sequence of signatures $(\vec{\alpha}^i)_{i \in \lambda}$ from (w, Γ) such that:

- $(w_0, \Gamma_0, \vec{\alpha}_0) = (w, \Gamma, \vec{\alpha})$;
- for all $i \in \lambda$, $\vec{\alpha}^i$ is a μ -signature such that $w_i \in \mathcal{L}(\Gamma^{\vec{\alpha}^i})$;

where $\lambda \in \omega + 1$. If the play is finite then it is winning by construction, so assume it is infinite i.e. $\lambda = \omega$.

We will construct it by induction on i . The base case is already defined. For the induction case, assume $\Gamma_i = f, \Delta$ and $\vec{\alpha}_i = \vec{\alpha} :: \vec{\alpha}$ and we will do a case-analysis on f .

- Suppose $f = f_0 + f_1$.

$$\begin{aligned}
 w_i &\in \mathcal{L}([f_0 + f_1, \Delta]^{\vec{\alpha}::\vec{\alpha}}) && \text{[By induction hypothesis]} \\
 \implies w_i &\in \mathcal{L}([f_0 + f_1^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \\
 \implies w_i &\in \mathcal{L}([f_0^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \text{ or } w_i \in \mathcal{L}([f_1^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \\
 \implies w_i &\in \mathcal{L}([f_0, \Delta]^{\vec{\alpha}^i}) \text{ or } w_i \in \mathcal{L}([f_1, \Delta]^{\vec{\alpha}^i})
 \end{aligned}$$

Wlog, assume $w_i \in \mathcal{L}([f_0, \Delta]^{\vec{\alpha}^i})$. Choose $(w_{i+1}, \Gamma_{i+1}, \overline{\vec{\alpha}^{i+1}}) = (w_i, [f_0, \Delta], \vec{\alpha}^i)$.

- Suppose $f = f_0 \cdot f_1$.

$$\begin{aligned}
 w_i &\in \mathcal{L}([f_0 \cdot f_1, \Delta]^{\vec{\alpha}::\vec{\alpha}}) && \text{[By induction hypothesis]} \\
 \implies w_i &\in \mathcal{L}([f_0^{\vec{\alpha}}, f_1^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \\
 \implies w_i &\in \mathcal{L}([f_0, f_1, \Delta]^{\vec{\alpha}::\vec{\alpha}::\vec{\alpha}})
 \end{aligned}$$

Choose $(w_{i+1}, \Gamma_{i+1}, \overline{\vec{\alpha}^{i+1}}) = (w_i, [f_0, f_1, \Delta], \vec{\alpha} :: \vec{\alpha} :: \vec{\alpha})$.

- Suppose $f = a$ for some $a \in \mathcal{A}$.

$$\begin{aligned}
 w_i &\in \mathcal{L}([a, \Delta]^{\vec{\alpha}::\vec{\alpha}}) && \text{[By induction hypothesis]} \\
 \implies w_i &\in \mathcal{L}([a^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \\
 \implies w_i &= aw' \text{ and } w' \in \mathcal{L}(\Delta^{\vec{\alpha}})
 \end{aligned}$$

Choose $(w_{i+1}, \Gamma_{i+1}, \overline{\vec{\alpha}^{i+1}}) = (w', \Delta, \vec{\alpha})$.

- Suppose $f = \nu X f_0$.

$$\begin{aligned}
 w_i &\in \mathcal{L}([\nu X f_0, \Delta]^{\vec{\alpha}::\vec{\alpha}}) && \text{[By induction hypothesis]} \\
 \implies w_i &\in \mathcal{L}([\nu X f_0^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) \\
 \implies w_i &\in \mathcal{L}([f_0(\nu X f_0)^{\vec{\alpha}}, \Delta^{\vec{\alpha}}]) && \text{[}\mathcal{L}(\nu X e, \Gamma) = \mathcal{L}(e(\nu X e), \Gamma)\text{]} \\
 \implies w_i &\in \mathcal{L}([f_0(\nu X f_0), \Delta]^{\vec{\alpha}::\vec{\alpha}})
 \end{aligned}$$

Choose $(w_{i+1}, \Gamma_{i+1}, \overline{\vec{\alpha}^{i+1}}) = (w_i, [f_0(\nu X f_0), \Delta], \vec{\alpha}^i)$

- Suppose $f = \mu X f_0$.

$$\begin{aligned}
 w_i &\in \mathcal{L}([\mu X f_0, \Delta]^{\vec{\alpha}::\vec{\alpha}}) && \text{[By induction hypothesis]} \\
 \implies w_i &\in \mathcal{L}([\mu^{\alpha_j} X f_0, \Delta^{\vec{\alpha}}]) && \text{[where } \vec{\alpha} = (\alpha_i)\text{]} \\
 \implies w_i &\in \mathcal{L}([f_0(\mu^\beta X f_0), \Delta^{\vec{\alpha}}]) && \text{[for some } \beta < \alpha_j\text{]} \\
 \implies w_i &\in \mathcal{L}([f_0(\mu X f_0), \Delta]^{\vec{\alpha}'::\vec{\alpha}}) && \text{[}\vec{\alpha}' = \vec{\alpha}[\beta/\alpha_j]\text{]}
 \end{aligned}$$

Choose $(w_{i+1}, \Gamma_{i+1}, \overline{\vec{\alpha}^{i+1}}) = (w_i, [f_0(\mu X f_0), \Delta], \vec{\alpha}' :: \vec{\alpha})$ works.

We now claim that this play is winning. Suppose not. Then, the smallest formula principal infinitely often is μ . We follow its thread and obtain a strictly decreasing sequence of ordinals. Contradiction.

(\Leftarrow) For the converse direction, we will prove the contrapositive. Suppose $w \notin \mathcal{L}(\Gamma)$. Fix an arbitrary play $\pi = (w_i, \Gamma_i)_{i \in \lambda}$ from (w, Γ) for some $\lambda \in \omega + 1$. By inspection of the puzzle rules, non-membership is preserved i.e. we always have $w_i \notin \mathcal{L}(\Gamma_i)$. By Proposition 50, there are ν -signatures $\bar{\alpha}_i$ such that $w_i \notin \mathcal{L}(\Gamma_{\bar{\alpha}_i})$. Following an argument like above, the signature corresponding to a thread is a monotone non-increasing sequence. Moreover, if π is winning, it is strictly decreasing. Therefore, π is not winning. \square

6.3. ω -context-freeness via Muller grammars. We can now use the adequacy of the evaluation puzzle to recover a converse of Proposition 35. For this, we need to recall a grammar-formulation of CF^ω , due to Cohen and Gold [CG77] and independently Nivat [Niv77, Niv78].

A **Muller (ω -)CFG** (MCFG) is a CFG \mathcal{G} , equipped with a set $F \subseteq \mathcal{P}(\mathcal{V})$ of **acceptable** sets. We define a rewrite relation $\rightarrow_{\mathcal{G}} \subseteq (\mathcal{V} \cup \mathcal{A})^* \times (\mathcal{V} \cup \mathcal{A})^*$, **leftmost reduction**, by $\bar{a}Xv \rightarrow_{\mathcal{G}} \bar{a}uv$ whenever $\bar{a} \in \mathcal{A}^*$, $X \rightarrow u$ is a production of \mathcal{G} and $v \in (\mathcal{V} \cup \mathcal{A})^*$. A **leftmost derivation** is just a maximal (possibly infinite) sequence along $\rightarrow_{\mathcal{G}}$. We say \mathcal{G} **accepts** $w \in \mathcal{A}^{\leq \omega}$ if there is a leftmost derivation δ such that δ converges to w and the set of infinitely often occurring states that are LHSs of productions along δ is in F . We write $\mathcal{L}(\mathcal{G})$ for the set of words \mathcal{G} accepts.

Theorem 51 ([CG77, Niv77, Niv78]). *Let $L \subseteq \mathcal{A}^\omega$. $L \in \text{CF}^\omega \Leftrightarrow L = \mathcal{L}(\mathcal{G})$ for a MCFG \mathcal{G} .*

Now we have a converse of Proposition 35 by:

Proposition 52. *For each expression e there is a MCFG \mathcal{G} s.t. $\mathcal{L}(e) = \mathcal{L}(\mathcal{G})$.*

Proof sketch. Given a $\mu\nu$ -expression e , we construct a grammar just like in (3), but with extra clause $X_{\nu Xf(X)} \rightarrow X_{f(\nu Xf(X))}$. We maintain two copies of each non-terminal, one **magenta** and one normal, so that a derivation also ‘guesses’ a **ν -thread** ‘on the fly’. Formally, the **magenta** productions of our grammar are:

$$\begin{array}{lcl} X_1 & \rightarrow & \varepsilon \\ X_a & \rightarrow & a \\ X_{f+g} & \rightarrow & X_f \mid X_g \\ X_{fg} & \rightarrow & X_f X_g \mid X_f X_g \\ X_{\mu Xf(X)} & \rightarrow & X_{f(\mu Xf(X))} \\ X_{\nu Xf(X)} & \rightarrow & X_{f(\nu Xf(X))} \end{array}$$

Productions for normal non-terminals have only normal non-terminals on their RHSs.

Now set F , the set of acceptable sets, to include all sets extending some $\{X_f : f \in E\}$, for E with smallest expression a ν -formula, by normal non-terminals. Now any accepting leftmost derivation of a word w from X_e describes a winning play of the evaluation puzzle from (w, e) and vice-versa. \square

6.4. Proof search game and completeness. In order to prove completeness of $\mu\nu\ell\text{HKA}^\infty$, we need to introduce a game-theoretic mechanism for organising proof search, in particular so that we can rely on *determinacy* principles thereof.

Definition 53 (Proof search game). The *proof search game* (for $\mu\nu\ell\text{HKA}$) is a two-player game played between Prover (**P**), whose positions are inference steps of $\mu\nu\ell\text{HKA}$, and Denier (**D**), whose positions are sequents of $\mu\nu\ell\text{HKA}$. A **play** of the

game starts from a particular sequent: at each turn, **P** chooses an inference step with the current sequent as conclusion, and **D** chooses a premiss of that step; the process repeats from this sequent as long as possible.

An infinite play of the game is **won** by **P** (aka **lost** by **D**) if the branch constructed has a progressing thread; otherwise it is won by **D** (aka lost by **P**). In the case of deadlock, the player with no valid move loses.

Proposition 54 (Determinacy ($\exists 0\#$)). *The proof search game is determined, i.e. from any sequent $\Gamma \rightarrow S$, either **P** or **D** has a winning strategy.*

Note that the winning condition of the proof search game is (lightface) analytic, i.e. Σ_1^1 : “there *exists* a progressing thread”. Lightface analytic determinacy lies beyond ZFC, as indicated equivalent to the existence of $0\#$ [Har78]. Further consideration of our metatheory is beyond the scope of this work.

It is not hard to see that **P**-winning-strategies are ‘just’ ∞ -proofs. Our goal is to show a similar result for **D**, a sort of ‘countermodel construction’.

Lemma 55. ***D** has a winning strategy from $\Gamma \rightarrow S \implies \mathcal{L}(\Gamma) \setminus \mathcal{L}(S) \neq \emptyset$.*

Before proving this, let us point out that Lemma 17 applies equally to the system $\mu\nu\text{HKA}$. We also have the useful observation:

Proposition 56 (Modal). $\mathcal{L}(a\Gamma) \subseteq \{\varepsilon\} \cup \bigcup_{a \in \mathcal{A}} \mathcal{L}(aS_a) \implies \mathcal{L}(\Gamma) \subseteq \mathcal{L}(S_a)$.

This follows directly from the definition of $\mathcal{L}(\cdot)$. Now we can carry out our ‘countermodel construction’ from **D**-winning-strategies:

Proof of Lemma 55. Construct a **P**-strategy **p** that is deadlock-free by always preserving validity, relying on Lemma 17 and Proposition 56. In more detail **p** does the following:

- (1) Apply leftmost logical steps (on LHS or RHS) as long as possible.
- (2) If the LHS is empty and the RHS contains an empty list, then weaken the remainder of the RHS and apply init.
- (3) Otherwise, if the LHS has form $a\Gamma$ and RHS has form aS_a, S , where S contains only lists that are empty or begin with some $b \neq a$, then apply w - r and k_a^l to obtain the sequent $\Gamma \rightarrow S_a$ and go back to 1.

Now each iteration of 1 must terminate by left-guardedness and leftmostness. This must end at a valid sequent, by Lemma 17, each of whose lists are either or begin with a letter, by inspection of the rules. Now, if the LHS is empty, then the RHS must contain an empty list, and so step 2 successfully terminates the preproof. If the LHS has form $a\Gamma$, then step 3 applies and preserves validity by Proposition 56. Note that any infinite play of **p** must repeat step 3 infinitely often, as each iteration of 1 terminates, and so has infinitely many k^l steps and is not ultimately stable.

Now, suppose **d** is a **D**-winning-strategy and play **p** against it to construct a play $B = (\mathcal{S}_i)_{i < \omega} = (\Gamma_i \rightarrow S_i)_{i < \omega}$. Note that indeed this play must be infinite since (a) **p** is deadlock-free; and (b) **d** is **D**-winning. Now, let $w = \prod_{k_a^l \in B} a$ be the

product of labels of k steps along B , in the order they appear bottom-up. We claim $w \in \mathcal{L}(\Gamma) \setminus \mathcal{L}(S)$:

- $w \in \mathcal{L}(\Gamma)$. By construction $[\Gamma_i]_i$ has a subsequence forming an infinite play π of the evaluation puzzle from (w, Γ) . Since the play B is won by **D**, B

cannot have a μ -thread so it must have a ν -thread (since it is i.o.p.), and so π is winning. Thus $w \in \mathcal{L}(\Gamma)$ by Theorem 45.

- $w \notin \mathcal{L}(S)$. Take an arbitrary play π of the evaluation puzzle from some (w, Δ) with $\Delta \in S$. This again induces an infinite sequence of cedents $[\Delta_i]_{i < \omega}$ along the RHSs of B . Now, $[\Delta_i]_{i < \omega}$ cannot have a ν -thread by assumption that B is winning for \mathbf{D} , and so π is not a winning play of the evaluation puzzle from (w, Δ) . Since the choices of $\Delta \in S$ and play π were arbitrary, indeed we have $w \notin \mathcal{L}(S)$ by Theorem 45.

□

Now from Proposition 54 and Lemma 55, observing that \mathbf{P} -winning-strategies are just ∞ -proofs, we conclude:

Theorem 57 (Completeness). $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(S) \implies \mu\nu\ell\text{HKA} \vdash^\infty \Gamma \rightarrow S$.

7. COMPLEXITY MATTERS AND FURTHER PERSPECTIVES

In this subsection we make further comments, in particular regarding the complexity of our systems, at the level of arithmetical and analytical hierarchies. These concepts are well-surveyed in standard textbooks, e.g. [MW85, Sac17], as well as various online resources.

Complexity and irregularity for finite words. The equational theory of μ -expressions in $\mathcal{L}(\cdot)$ is actually Π_1^0 -complete, i.e. co-recursively-enumerable, due to the same complexity of universality of context-free grammars (see, e.g., [JEH01]). In this sense there is no hope of attaining a finitely presentable (e.g. cyclic, inductive) system for the equational theory of μ -expressions in $\mathcal{L}(\cdot)$. However it is not hard to see that our wellfounded system μHKA_ω enjoys optimal Π_1^0 proof search, thanks to invertibility and termination of the rules, along with decidability of membership checking. Indeed a similar argument is used by Palka in [Pal07] for the theory of ‘*-continuous action lattices’. Furthermore let us point out that our non-wellfounded system also enjoys optimal proof search: $\mu\text{HKA} \vdash^\infty \Gamma \rightarrow S$ is equivalent, by invertibility, to checking that *every* sequent of form $\vec{a} \rightarrow S$ reachable by only left rules in bottom-up proof search has a polynomial-size proof (bound induced by length of leftmost derivations). This is a Π_1^0 property.

Complexity and inaxiomatisability for infinite words. It would be natural to wonder whether a similar argument to Section 4 gives rise to some infinitary axiomatisation of the equational theory of $\mu\nu$ -expressions in $\mathcal{L}(\cdot)$. In fact, it turns out this is impossible: the equational theory of ω -CFLs is Π_2^1 -complete [Fin09], so there is no hope of a Π_1^0 (or even Σ_2^1) axiomatisation. In particular, the projection argument of Section 4 cannot be scaled to the full system $\mu\nu\ell\text{HKA}$ because \cdot does not distribute over \bigcap in $\mathcal{L}(\cdot)$, for the corresponding putative ‘right ω steps’ for ν . For instance $0 = ((aa)^* \cap a(aa)^*)a^* \neq (aa)^*a^* \cap a(aa)^*a^* = aa^*$. Indeed let us point out that here it is crucial to use our hypersequential system HKA as a base rather than, say, the intuitionistic systems of other proof theoretic works for regular expressions (and friends) [Pal07, DP18]: the appropriate extension of those systems by μ s and ν s should indeed enjoy an ω -translation, due to only one formula on the right, rendering them incomplete.

Again let us point out that ∞ -provability in $\mu\nu\ell\text{HKA}$, in a sense, enjoys optimal complexity. By determinacy of the proof search game, $\mu\nu\ell\text{HKA} \vdash^\infty \Gamma \rightarrow S$ if and

only if there is *no* **D**-winning-strategy from $\Gamma \rightarrow S$. The latter is indeed a Π_2^1 statement: “*for every D-strategy, there exists a play along which there exists a progressing thread*”.

Comparison to [GHK13]. Our method for showing completeness of μHKA_ω is quite different from the analogous result of [GHK13] which uses the notion of ‘rank’ for μ -formulas, cf. [AKS14]. Our result is somewhat stronger, giving *cut-free* completeness, but it could be possible to use ranks directly to obtain such a result too. More interestingly, the notion of projections and ω -translation should be well-defined (for LHS μ formulas) even in the presence of ν s, cf. [DDS23], whereas the rank method apparently breaks down in such extensions. This means that our method should also scale to $\mu\nu\text{HKA}$ ∞ -proofs where, say, each infinite branch has a LHS μ -thread. It would be interesting to see if this method can be used to axiomatise some natural fragments of ω -context-free inclusions.

Note that, strictly speaking, our completeness result for μCA was only given for the guarded fragment. However it is known that μCA (and even weaker theories) already proves the equivalence of each expression to one that is even left-guarded, by formalising conversion to Greibach normal form [EL05].

8. CONCLUSIONS

In this work we investigated of the proof theory of context-free languages (CFLs) over a syntax of μ -expressions. We defined a non-wellfounded proof system μHKA^∞ and showed its soundness and completeness for the model $\mathcal{L}(\cdot)$ of context-free languages. We used this completeness result to recover the same for a cut-free ω -branching system μHKA_ω via proof-theoretic techniques. This gave an alternative proof of the completeness for the theory of μ -continuous Chomsky algebras from [GHK13]. We extended μ -expressions by *greatest* fixed points to obtain a syntax for ω -context-free languages. We studied an extension by *greatest* fixed points, $\mu\nu\ell\text{HKA}^\infty$ and showed its soundness and completeness for the model $\mathcal{L}(\cdot)$ of context-free languages, employing game theoretic techniques.

Since inclusion of CFLs is Π_1^0 -complete, no recursively enumerable (r.e.) system can be sound and complete for their equational theory. However, by restricting products to a letter on the left one can obtain a syntax for *right-linear grammars*. Indeed, for such a restriction complete cyclic systems can be duly obtained [DD24]. It would be interesting to investigate systems for related decidable or r.e. inclusion problems, e.g. inclusions of context-free languages in regular languages, and inclusions of *visibly pushdown* languages [AM04, AM09].

The positions of our evaluation puzzle for $\mu\nu$ -expressions use cedents to decompose products, similar to the stack of a pushdown automaton, rather than requiring an additional player. Previous works have similarly proposed model-checking games for (fragments/variations of) context-free expressions, cf. [Lan02, LMS04], where more complex winning conditions seem to be required. It would be interesting to compare our evaluation puzzle to those games in more detail.

Note that our completeness result, via determinacy of the proof search game, depends on the assumption of (lightface) analytic determinacy. It is natural to ask whether this is necessary, but this consideration is beyond the scope of this work. Let us point out, however, that even ω -context-free determinacy exceeds the capacity of ZFC [Fin13, LT17].

Finally, it would be interesting to study the *structural* proof theory arising from systems μHKA^∞ and $\mu\nu\text{HKA}^\infty$, cf. [DP18]. It would also be interesting to see if the restriction to leftmost ∞ -proofs can be replaced by stronger progress conditions, such as the ‘alternating threads’ from [DG22, DG23], in a similar hypersequential system for predicate logic. Note that the same leftmost constraint was employed in [HK22] for an extension of HKA to ω -regular languages.

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