An Analog of the Rothe Method for Some Ill-Posed Problems for Parabolic Equations

Michael V. Klibanov *

Abstract

The classical method of Rothe proves existence and uniqueness theorems for initial boundary value problems for parabolic equations using the explicit finite difference scheme with respect to time. In this method, an elliptic boundary value problem is investigated on each time step. On the other hand, time dependent experimental data are always collected on discrete time grids, and the grid step size cannot be arranged to be infinitely small. The same is true for numerical studies. Therefore, it makes an applied sense to consider both unique continuation problems and coefficient inverse problems for parabolic equations, which are written in the form of finite differences with respect to time and without allowing the grid step size to tend to zero. This leads to a boundary value problem for a coupled system of elliptic equations with both Dirichlet and Neumann boundary data, which is somewhat similar to the Rothe's method. Dissimilarities are named as well. Two long standing open questions are addressed within this framework. A specific applied example of monitoring epidemics is discussed. In particular, a numerical method for this problem is constructed and its global convergence analysis is provided.

Key Words: Rothe's method, unique continuation problems, coefficient inverse problems, $t-$finite differences, parabolic equations, Carleman estimates, Hölder and Lipschitz stability estimates, numerical method

2020 MSC codes: 35R30

1 Introduction

The method of Rothe uses the explicit finite difference scheme with respect to the time $t \in (0, T)$ for proofs of existence and uniqueness theorems for the initial boundary value problems for parabolic Partial Differential Equations with the Dirichlet boundary condition. For each grid point $t_i \in (0, T)$ one solves the resulting Dirichlet boundary value problem for the corresponding elliptic equation. Then the grid step size is allowed to tend to zero. Since the initial condition is known, then these problems are solved sequentially starting from $\{t = 0\}$. This method was originally proposed by Rothe for the $1-$D parabolic equation [29]. We refer to the works of Ladyzhenskaya [21, section 1 of Chapter 2] and Ventzel [32], in which the method of Rothe was applied to proofs of existence and

*Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC, 28223, USA, mklibanv@charlotte.edu.

uniqueness results for the initial boundary value problem with the Dirichlet boundary condition for the quasiliniear parabolic equation in the $n-$D case, also, see [22, section 16 of chapter 3] for a short explanation. The Rothe's method was successfully utilized by Timonov [31] to numerically solve a coupled physics (hybrid) inverse problem of conductivity imaging, which, however, is significantly different from problems considered in this paper, and this causes a significant difference between [31] and our approach.

Problems considered in this paper have applications in, e.g. heat conduction [1], medical optical imaging [9] and monitoring of epidemics [20]. We consider the unique continuation problem (UCP) for a nonlinear integral differential equation with a parabolic operator in it and a coefficient inverse problem (CIP) for a parabolic Partial Differential Equation (PDE). It is well known that any CIP is nonlinear. It is also well known that both these problems are ill-posed in the sense of Hadamard. Furthermore, unlike the conventional case of the UCP for a parabolic PDE, Volterra integrals

$$\int_{0}^{t} (...)\, d\tau \tag{1.1}$$

are involved in our UCP, and the initial condition at $\{t = 0\}$ is unknown. The question of stability estimates for our CIP is reduced to that UCP, and the presence of integrals (1.1) is essential in this case. The absence of the initial condition being combined with the presence of the Volterra integrals (1.1) in our UCP causes a long standing open question, which is not addressed at the moment. Our goal here is to address this question within our approximate framework. This framework has a clear applied sense.

We are motivated by an observation that the time dependent experimental data are always collected at discrete grid points of time $t \in (0, T)$ with a fixed grid step size h, which cannot be made too small. The same observation is true in the numerical studies. Therefore, it makes an applied sense to consider UCPs and CIPs for parabolic equations, assuming that the conventional $t-$derivative is replaced with finite differences, in which

$$h \geq h_0 > 0, \tag{1.2}$$

where an arbitrary number h_0 is fixed. We call this "$t-$finite differences" (TFD) framework. We address two long standing open questions for the above problems within the TFD framework. We point out that all stability estimates of this paper as well as the numerical method are obtained only within the TFD framework.

A similarity between the TFD framework and the method of Rothe is that the TFD framework reduces the corresponding UCP/CIP for the parabolic equation to a set of boundary value problems (BVPs) for a system of coupled elliptic equations. However, the problem arising in the TFD framework is more complicated that the one in the Rothe's method. Indeed, since the initial condition is unknown in the TFD case, then one cannot solve those elliptic BVPs sequentially. The second dissimilarity is that boundary conditions for those BVPs are both Dirichlet and Neumann boundary conditions on a part of the boundary rather than only the Dirichlet boundary condition of the method of Rothe. The third dissimilarity is that Volterra integrals (1.1) are not involved in the original Rothe's method. The fourth dissimilarity is that our condition (1.2) is not imposed in that method.

Since the initial condition is unknown in our case, then we treat all elliptic equations of the obtained BVP simultaneously rather than sequentially, as in the original Rothe's method. More precisely, first, we apply to each equation of our system a Carleman estimate for the elliptic, rather than for the parabolic operator. Then we sum up resulting estimates. This way we obtain our desired results.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Let $\Gamma \subseteq \partial\Omega$ be a part of this boundary, $\Gamma \in C^3$. Let $x \in \Omega$ be the spatial variable. Denote

$$\Omega_T = \Omega \times (0,T)\,,\ S_T = \partial\Omega \times (0,T)\,,\ \Gamma_T = \Gamma \times (0,T)\,. \tag{1.3}$$

Hence, Ω_T is the time cylinder, and S_T is its lateral boundary.

The UCP for a parabolic PDE is the problem of finding the solution of this equation, assuming that both Dirichlet and Neumann boundary conditions at a part Γ_T of the lateral boundary S_T are known. These are the so-called "lateral Cauchy data". Currently Hölder [12, 16, 18, 24] and Lipschitz [34] stability estimates are obtained for this problem only in the domains like $\Omega' \times (\sigma, T-\sigma)$, where $\sigma \in (0, T/2)$ is an arbitrary number and $\Omega' \subseteq \Omega$ is a subdomain, although the Hölder stability estimate of [33, Theorem 3.5.1] does include $\{t = T\}$. In [7, 15] logarithmic stability estimates in the entire domain Ω_T are obtained: in [15] the lateral Cauchy data are given on the entire lateral boundary S_T, and in [7] they are given only on Γ_T.

However, all results mentioned in the previous paragraph are obtained only in the case when Volterra integrals (1.1) are absent. On the other hand, if they are present, and the $t-$derivative is regular, then the questions of uniqueness and stability for corresponding UCPs remain open even for the domains like $\Omega' \times (\sigma, T-\sigma)$ for quite a long time since the publication [8]. We note that the presence of these integrals is important for CIPs, see sections 4,7-9.

The first long standing open question we address here within the TFD framework is the question about Hölder and Lipschitz stability estimates for the above nonlinear UCP in the semidiscrete domains of the form $\Omega' \times (0, T]$, in which only grid points of the interval $[0, T]$ with the grid step size h are counted. This question is addressed for a general nonlinear integral differential equation, in which a nonlinear parabolic operator is present along with the integrals (1.1). The initial condition is unknown, however.

A typical CIP for a parabolic PDE is the problem of the recovery of an unknown $x-$dependent coefficient of this equation, given the lateral Cauchy data as well as the solution $u(x,t)$ of this equation at a certain moment of time $t_0 \in [0, T)$. There are many known results about stability estimates and uniqueness of these CIPs. We refer to, e.g. [12, 16, 18, 34] for some of them. These publications use various modifications of the method of the paper [8], in which Carleman estimates were introduced in the field of CIPs. However, it is assumed in all these works that $t_0 \neq 0$. The single exception is a special case when the CIP can be reduced to a similar CIP for an associated hyperbolic equation [16, 18], and then the "hyperbolic version" of the technique of [8] can be applied, see, e.g. [12, 16, 18] for this version. However, since that reduction requires the inversion of a modified Laplace transform [12, formula 9.2.1], [16, formula (3.36)], [24, formula (7.131)] and since this inversion is highly unstable, then only uniqueness theorems are obtained this way, rather than stability estimates.

Therefore, the second long standing open question we address here within the TFD framework is the question about Hölder and Lipschitz stability estimates for CIPs for parabolic PDEs in the case when the lateral Cauchy data are combined with the initial condition at $\{t = 0\}$. We reduce this question to a linear version of the above UCP and then derive our estimates from those for the UCP.

Finally, we demonstrate the performance of our technique for a specific applied example. More precisely, we use the TFD framework to prove Hölder and Lipschitz stability estimates for a CIP of monitoring epidemics using boundary measurements. A system of three nonlinear coupled parabolic equations is involved. We provide both Lipschitz stability estimate and a construction of a globally convergent numerical method for this problem. This is a version of the so-called convexification method. Another version of the convexification for this problem was developed in [20], where the lateral Cauchy data were combined with the data at $t = t_0 > 0$, which is unlike our case of $t = t_0 = 0$. In this paper, however, we use the TFD framework to develop a new version of the convexification method for that CIP with $t_0 = 0$.

The convexification method allows one to avoid the well known phenomenon of multiple local minima and ravines of conventional least squares cost functionals for CIPs [3, 10, **?**, 27]. More precisely, the convexification replaces the local convergence of the conventional numerical methods for CIPs with the global convergence, see Remark 9.1 about the global convergence property. The development of the convexification was originated in [**?**, 14] and continued since then, see, e.g. [2, 4, 5, 6, **?**, 17, 18, 19, 20] for some references.

In particular, some versions of the convexification method for CIPs for parabolic PDEs were constructed in [17, 20], [18, chapter 9] for the case when the lateral Cauchy data are complemented with the data at $t = t_0 > 0$. As mentioned above, in this paper $t_0 = 0$. A notable feature of the convexification method of this paper, which was not the case of other above cited references, is that the penalty regularization term is not involved here. We carry out convergence analysis of the version of the convexification method, which is presented here.

All functions considered below are real valued ones. In section 2 we introduce the $t-$finite differences. In section 3 we introduce statements of our UCP and the first CIP, which we call CIP1. In section 4 we present the TFD framework for CIP1. In section 5 we formulate two Carleman estimates and prove one of them. In section 6 we prove Hölder and Lipschitz stability estimates for our UCP. In section 7 we formulate Hölder and Lipschitz stability estimates for CIP1. Actually theorems of section 7 easily follow from theorems of section 6. In section 8 we first formulate our CIP of monitoring of epidemics using boundary measurements. This is CIP2. Next, we reformulate CIP2 in the TFD framework and formulate Lipschitz stability estimate for so reformulated CIP2. That estimate easily follows from one of theorems of section 6. In section 9, we formulate the convexification method for the TFD framework for CIP2 and provide its convergence analysis.

2 $t-$Finite Differences (TFD)

2.1 Sets and spaces

Consider the partition of the interval $[0, T]$ in $k \geq 3$ subintervals,

$$0 = t_0 < t_1 < ... < t_{k-1} < t_k = T,\ t_i - t_{i-1} = h,\ i = 1, ..., k. \tag{2.1}$$

Denote

$$Y = \{t_i\}_{i=0}^{k}. \tag{2.2}$$

Define semi-discrete analogs of sets (1.3) are:

$$\Omega_{h,T} = \Omega \times Y, \; S_{h,T} = \partial\Omega \times Y, \; \Gamma_{h,T} = \Gamma \times Y. \tag{2.3}$$

Similarly, for any set $\Psi \subset \mathbb{R}^n$

$$\Psi_{h,T} = \Psi \times Y. \tag{2.4}$$

Therefore, any function defined on $\Psi_{h,T}$ is actually a $(k+1)$ −dimensional vector function,

$$u(x,t) = (u(x,t_0), u(x,t_1), ..., u(x,t_k))^T \text{ for } (x,t) \in \Psi_{h,T}. \tag{2.5}$$

Let $m \in [0,3]$ be an integer. We now define some spaces associated with vector functions (2.5). Let $\Psi \subset \mathbb{R}^n$ be either a bounded domain or a hypersurface such that $\Psi \in C^3$. Then, using (2.4), we set

$$\begin{aligned} C^m\left(\overline{\Psi}_{h,T}\right) &= \left\{ u : \|u\|_{C^m\left(\overline{\Psi}_{h,T}\right)} = \left(\sum_{i=0}^{k} \|u(x,t_i)\|^2_{C^m\left(\overline{\Psi}\right)} \right)^{1/2} < \infty \right\}, \\ H^m(\Psi_{h,T}) &= \left\{ u : \|u\|_{H^m\left(\Psi_{h,T}\right)} = \left(\sum_{i=0}^{k} \|u(x,t_i)\|^2_{H^m(\Psi)} \right)^{1/2} < \infty \right\}. \end{aligned} \tag{2.6}$$

In particular, $C^0\left(\overline{\Psi}_{h,T}\right) = C\left(\overline{\Psi}_{h,T}\right)$ and $H^0(\Psi_{h,T}) = L_2(\Psi_{h,T})$. Recall that $\Gamma \subseteq \partial\Omega, \Gamma \in C^3$. We set

$$H^m(\Gamma_{h,T}) = \left\{ u : \|u\|_{H^m\left(\Gamma_{h,T}\right)} = \left(\sum_{i=0}^{k} \|u(x,t_i)\|^2_{H^m(\Gamma)} \right)^{1/2} < \infty \right\}, \tag{2.7}$$

We set

$$\partial\Omega = \Gamma \cup_{j=1}^{s} \partial_j\Omega, \; \partial_j\Omega \in C^3, \; \partial_j\Omega_{h,T} = \partial_j\Omega \times Y. \tag{2.8}$$

Using (2.3) and (2.4), we set similarly with (2.7)

$$H^m(S_{h,T}) = \left\{ \begin{array}{c} u : \|u\|_{H^m\left(S_{h,T}\right)} = \\ = \left(\|u\|^2_{H^m\left(\Gamma_{h,T}\right)} + \sum_{j=1}^{s} \sum_{i=0}^{k} \|u(x,t_i)\|^2_{H^m\left(\partial_j\Omega_{h,T}\right)} \right)^{1/2} < \infty \end{array} \right\}. \tag{2.9}$$

Remarks 2.1.

1. *It is well known that when discussing questions of existence and uniqueness of initial boundary value problems for parabolic equations, spaces like $C^{2r,r}, C^{2r+\rho,r+\rho/2}, H^{2r,r}$ are conventionally used, where $r \geq 1$ is an integer and $\rho \in (0,1)$, see, e.g. [22]. However, since we do not discuss these questions here and also since we are interested in stability estimates for our ill-posed problems within the TFD framework, then we will use spaces C^r in the continuos cases and spaces (2.6), (2.7), (2.9) in the semi-discrete cases. Furthermore, even though in some of our stability estimates we will use $H^2(\Omega_{h,T})$ of solutions of our resulting BVPs, we will still assume that the lateral Cauchy data belong to the spaces $H^2(\Gamma_{h,T}), H^2(S_{h,T})$. By trace theorem, the latter can be guaranteed if requiring the $H^3(\Omega_{h,T})$ −smoothness of those solutions.*

2. *It is well known that minimal smoothness assumptions are minor concerns in the field of CIPs, see, e.g. [26] and [28, Theorem 4.1]. Therefore, they are also minor concerns here.*

2.2 $t-$finite differences and discrete Volterra integrals

We now define $t-$finite differences $f_h'(t)$ of the function $f(t) \in C^3[0,T]$ on the discrete set (2.1). We set

$$\partial_{h,t} f(t_i) = \frac{f(t_{i+1}) - f(t_{i-1})}{2h}, \ i = 1, ..., k-1. \tag{2.10}$$

Expression (2.10) is valid for interior points t_i of the grid (2.1). Temporary allowing $h \to 0^+$, we note that it is well known that the second order approximation accuracy $O(h^2)$ is delivered by (2.10). In addition, we also need the finite difference derivatives at the edge points $t_0 = 0$ and $t_k = T$. It is easy to verify that the following two formulas deliver the second order accuracy at $t_0 = 0$ and at $t_k = T$:

$$\partial_{h,t} f(t_0) = \frac{3f(t_0) - 4f(t_1) + f(t_2)}{2h}, \tag{2.11}$$

$$\partial_{h,t} f(t_k) = \frac{3f(t_k) - 4f(t_{k-1}) + f(t_{k-2})}{2h}. \tag{2.12}$$

Remark 2.2. *It is an important observation for our method that values $f(t_i)$ at all points $t_i \in Y$ are involved in the right hand sides of (2.10)-(2.12).*

We also define the discrete Volterra integral of the function $f(t)$ via the trapezoidal rule, which gives the $O(h^2)$ accuracy,

$$\left(\int_0^{t_i} f d\tau \right)_h = \frac{h}{2} \sum_{j=1}^{i} \left(f(t_{j-1}) + f(t_j) \right), \ t_i \in Y, \ i = 1, ..., k, \tag{2.13}$$

$$\left(\int_0^{t_i} f d\tau \right)_h = 0, \ i = 0. \tag{2.14}$$

Thus, formulas (2.13), (2.14) define discrete Volterra integrals for all $t_i \in Y$, where the set Y is defined in (2.2).

3 Statements of Problems

3.1 Unique continuation problem for a nonlinear integral differential equation

We now formulate the unique continuation problem for a nonlinear integral differential equation in the TFD framework. Let the vector function $u \in C^2\left(\overline{\Omega}_{h,T}\right)$. Introduce the vector function $P(u, x, t_s)$,

$$P(u, x, t_s) =$$
$$= \left(\left(\int_0^{t_s} \nabla_x u(x, \tau) \, d\tau d\tau \right)_h, \left(\int_0^{t_s} u(x, \tau) \, d\tau \right)_h, \left(\int_0^{t_s} \partial_{h,t} u(x, \tau) \, d\tau \right)_h \right), \tag{3.1}$$
$$x \in \Omega, \ t_s \in Y, s = 0, ..., k.$$

The nonlinear integral differential equation with discrete Volterra integrals (2.13), (2.14) in it is :

$$
\begin{gathered}
\partial_{h,t} u\left(x, t_s\right) = \\
= F\left(u_{x_i x_j}, \nabla_x u, u, P\left(u, x, t_s\right), x, t_s\right), \\
i, j = 1, ..., n;\ x \in \Omega,\ t_s \in Y,
\end{gathered} \tag{3.2}
$$

where the function $F\left(y, x, t_s\right) \in C^2\left(\mathbb{R}^N \times \overline{\Omega}_{h,T}\right)$, where N is the total number of arguments in the vector y, which includes all terms with all components of the vector function $u \in C^2\left(\overline{\Omega}_{h,T}\right)$, its derivatives and discrete integrals in (3.1), (3.2).

Let $\mu > 0$ be a number. We impose the following conditions on the function F:

$$
\frac{\partial F\left(y, x, t_s\right)}{\partial u_{x_i x_j}\left(x, t_s\right)} = \frac{\partial F\left(y, x, t_s\right)}{\partial u_{x_j x_i}\left(x, t_s\right)}, \forall y \in \mathbb{R}^N,\ \forall\left(x, t_s\right) \in \Omega_{h,T}, \forall i, j = 1, ..., n, \tag{3.3}
$$

$$
\mu\left|\xi\right|^2 \leq \sum_{i,j=1}^{n} \frac{\partial F\left(y, x, t_s\right)}{\partial u_{x_i x_j}\left(x, t_s\right)} \xi_i \xi_j,\ \forall \xi \in \mathbb{R}^n,\ \forall y \in \mathbb{R}^N, \forall\left(x, t_s\right) \in \Omega_{h,T}. \tag{3.4}
$$

In the specific case of (3.3), (3.4), $u_{x_i x_j}\left(x, t_s\right)$ means the coordinate of the vector $y \in \mathbb{R}^N$, which corresponds to $u_{x_i x_j}\left(x, t_s\right)$ in (3.2). Obviously conditions 3.3), (3.4) are analogs of the ellipticity conditions. Hence, we call the operator F in (3.2) "nonlinear elliptic operator in $\Omega_{h,T}$". Problem (3.1)-(3.5) is exactly the Unique Continuation Problem. We are interested in stability estimates for this problem.

Stability Estimates for UCP (3.1)-(3.5). *Given notation (3.1), suppose that two vector functions $u_1\left(x, t_s\right), u_2\left(x, t_s\right) \in C^2\left(\overline{\Omega}_{h,T}\right)$ (see Remarks 2.1) are solutions of equation (3.2) with the following lateral Cauchy data at $\Gamma_{h,T}$:*

$$
u_i \mid_{\Gamma_{h,T}} = g_{i,0}\left(x, t_s\right),\ \partial_l u_i \mid_{\Gamma_{h,T}} = g_{i,1}\left(x, t_s\right),\ t_s \in Y,\ i = 1, 2, \tag{3.5}
$$

where l is the outward looking unit normal vector at Γ. Assume that conditions (3.3) and (3.4) hold. Estimate an appropriate norm of the difference $u_1 - u_2$ via certain norms of differences $g_{1,0} - g_{2,0}$ and $g_{1,1} - g_{2,1}$.

3.2 A Coefficient Inverse Problem

Let $\alpha = \left(\alpha_1, ..., \alpha_n\right)$ be a multii-index with non-negative integer coordinates. Denote $|\alpha| = \alpha_1 + ... + \alpha_n$. Let $D_x^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} ... D_{x_n}^{\alpha_n}$ be the differential operator. For any vector $\xi = \left(\xi_1, ..., \xi_n\right)^T \in \mathbb{R}^n$ set $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$. Consider functions $a_\alpha(x)$ for $|\alpha| = 2$ and $a_\alpha(x, t)$ for $|\alpha| \leq 1$ satisfying the following conditions

$$
\begin{gathered}
a_{ij}(x) = a_{ji}(x),\ i, j = 1, ..., n;\ x \in \Omega, \\
\mu\left|\xi\right|^2 \leq \sum_{|\alpha|=2}^{n} a_{ij}(x) \xi_i \xi_j,\ \forall x \in \overline{\Omega},\ \forall \xi \in \mathbb{R}^n, \\
a_\alpha \in C^1\left(\overline{\Omega}\right) \text{ for } |\alpha| = 2,\ \left\|a_\alpha\right\|_{C^1\left(\overline{\Omega}\right)} \leq a^0,
\end{gathered} \tag{3.6}
$$

$$
\begin{gathered}
a_\alpha, \partial_t a_\alpha \in C\left(\overline{\Omega}_T\right) \text{ for } |\alpha| \leq 1, \\
\left\|a_\alpha\right\|_{C\left(\overline{\Omega}_T\right)} \leq D,\ \left\|\partial_t a_\alpha\right\|_{C\left(\overline{\Omega}_T\right)} \leq D \text{ for } |\alpha| \leq 1.
\end{gathered} \tag{3.7}
$$

Here $a^0 > 0$ and $D > 0$ are certain numbers and $a_{ij}(x)$ are just different notations of $a_\alpha(x)$ with $|\alpha| = 2$. The first two lines of (3.6) are linear analogs of conditions (3.3), (3.4). Consider the linear elliptic operator of the second order L with its principal part L_0,

$$\begin{gathered} Lu = \sum_{|\alpha|\leq 2} a_\alpha(x,t) D_x^\alpha u = L_0 u + L_1 u, \\ L_0 u = \sum_{|\alpha|=2} a_\alpha(x) D_x^\alpha u, \\ L_1 u = \sum_{|\alpha|\leq 1} a_\alpha(x,t) D_x^\alpha u. \end{gathered} \tag{3.8}$$

Consider the parabolic equation with the initial condition at $\{t = 0\}$ and lateral Cauchy data at Γ_T,

$$u_t = Lu, \ \ (x,t) \in \Omega_T, \tag{3.9}$$

$$u(x,0) = f(x), \tag{3.10}$$

$$u\mid_{\Gamma_T} = p_0(x), \ \partial_l u\mid_{\Gamma_T} = p_1(x). \tag{3.11}$$

Coefficient Inverse Problem 1 (CIP1). *Let coefficients of the operator L satisfy conditions (3.6). Let α_0 be a fixed multi-index, $|\alpha_0| \leq 2$. Suppose that a coefficient $a_{\alpha_0} = a_{\alpha_0}(x)$ of the operator L in (3.8) is unknown, whereas other coefficients as well as right hand sides in (3.10) and (3.11) are known. Suppose that we have two pairs of functions (u_1, a_{1,α_0}) and (u_2, a_{2,α_0}) satisfying equation (3.9) with the same initial condition (3.10), where functions $u_1, u_2 \in C^3\left(\overline{\Omega}_T\right)$ (see Remarks 2.1). Assume that*

$$u_i\mid_{\Gamma_T} = p_{i,0}(x,t), \ \partial_l u_i\mid_{\Gamma_T} = p_{i,1}(x,t), \ i = 1,2. \tag{3.12}$$

Also, assume that

$$|D_x^{\alpha_0} f(x)| \geq c_1 \text{ in } \Omega, \tag{3.13}$$

where $c_1 > 0$ is a number. Estimate an appropriate norm of the differences $\widetilde{u} = u_1 - u_2, \widetilde{a}_{\alpha_0} = a_{1,\alpha_0} - a_{2,\alpha_0}$ via certain norms of differences $\widetilde{p}_0 = p_{1,0} - p_{2,0}$ and $\widetilde{p}_1 = p_{1,1} - p_{2,1}$.

4 The TFD Framework for CIP1 (3.9)-(3.13)

For any two pairs of numbers $(b_1, d_1), (b_2, d_2)$

$$\begin{gathered} b_1 d_1 - b_2 d_2 = \widetilde{b} d_1 + \widetilde{d} b_2, \\ \widetilde{b} = b_1 - b_2, \ \widetilde{d} = d_1 - d_2. \end{gathered} \tag{4.1}$$

Hence, using (3.9)-(3.12), (4.1) and notations of the formulation of CIP1, we obtain

$$\begin{gathered} \widetilde{u}_t = L^{(1)}\widetilde{u} + \widetilde{a}_{\alpha_0}(x) D_x^{\alpha_0} u_2, \text{ in } \Omega_T, \\ \widetilde{u}(x,0) = 0, \\ \widetilde{u}\mid_{\Gamma_T} = \widetilde{p}_0(x,t), \ \partial_l \widetilde{u}\mid_{\Gamma_T} = \widetilde{p}_1(x,t), \end{gathered} \tag{4.2}$$

where $L^{(1)}$ is the operator L in (3.8), in which the coefficient $a_{\alpha_0}(x)$ is replaced with $a_{1,\alpha_0}(x)$. By (3.10), (3.13) and (4.2)

$$\widetilde{a}_{\alpha_0}(x) = \frac{\widetilde{u}_t(x,0)}{D_x^{\alpha_0} f(x)}. \tag{4.3}$$

Denote

$$w(x,t) = \widetilde{u}_t(x,t). \tag{4.4}$$

Then $w \in H^2(\Omega_T)$. The second line of (4.2) implies:

$$\widetilde{u}(x,t) = \int_0^t w(x,\tau)\,d\tau. \tag{4.5}$$

Next,

$$\widetilde{u}_t(x,0) = w(x,0) = w(x,t) - \int_0^t w_t(x,\tau)\,d\tau. \tag{4.6}$$

Thus, (3.8) and (4.2)-(4.6) lead to the following BVP with the lateral Cauchy data:

$$w_t = L^{(1)}w + \sum_{|\alpha|\leq 1} \partial_t a_\alpha(x,t) \int_0^t D_x^\alpha w(x,\tau)\,d\tau +$$

$$+\frac{D_x^{\alpha_0}\partial_t u_2(x,t)}{D_x^{\alpha_0} f(x)} \left(w(x,t) - \int_0^t w_t(x,\tau)\,d\tau \right), \quad (x,t) \in Q_T, \tag{4.7}$$

$$w \mid_{\Gamma_T} = \partial_t \widetilde{p}_0(x,t), \ \partial_l w \mid_{\Gamma_T} = \partial_t \widetilde{p}_1(x,t). \tag{4.8}$$

Thus, we have transformed problem (4.2) in BVP (4.7), (4.8), which does not contain the unknown function $\widetilde{a}_{\alpha_0}(x)$ but has Volterra integrals. Likewise integral differential equation (4.7) does not have an initial condition at $\{t=0\}$. We are ready now to rewrite (4.7), (4.8) in the TFD framework.

The TFD Framework for the Problem of Stability Estimates for CIP1: *Let condition (3.13) be valid. Estimate the vector function $v \in H^2(\Omega_{h,T})$ via vector functions $\partial_{h,t}\widetilde{p}_0(x,t_s)$ and $\partial_{h,t}\widetilde{p}_1(x,t_s)$, assuming that the following conditions (4.9), (4.10) hold:*

$$\begin{gathered}\partial_{h,t} w(x,t_s) = \left(L^{(1)}w\right)(x,t_s) + \sum_{|\alpha|\leq 1} \left(\partial_{h,t} a_\alpha(x,t_s)\right) \left(\int_0^{t_s} D_x^\alpha w(x,\tau)\,d\tau \right)_h + \\ + \left(D_x^{\alpha_0}\left(\partial_{h,t} u_2(x,t_s)\right)\right)\left(D_x^{\alpha_0} f(x)\right)^{-1} \left[w(x,t_s) - \left(\int_0^{t_s} \partial_{h,\tau} w(x,\tau)\,d\tau \right)_h \right], \\ x \in \Omega, \ t_s \in Y,\end{gathered} \tag{4.9}$$

$$w \mid_{\Gamma_{h,T}} = \partial_{h,t}\widetilde{p}_0(x,t_s), \ \partial_l w \mid_{\Gamma_{h,T}} = \partial_{h,t}\widetilde{p}_1(x,t_s), \ t_s \in Y, \tag{4.10}$$

where Volterra integrals are understood as in (2.13) and (2.14), see Remark 2.2. Thus, BVP (4.9), (4.10) is a special linear version of UCP (3.2)-(3.5). In addition, estimate the function $\widetilde{a}_{\alpha_0}(x)$

$$\widetilde{a}_{\alpha_0}(x) = \frac{w(x,0)}{D_x^{\alpha_0} f(x)}, \ x \in \Omega. \tag{4.11}$$

Formula (4.11) is obtained from $t-$finite differences analogs of formulas (4.3) and (4.4). The boundary data (4.10) are incomplete since one might have $\Gamma \neq \partial\Omega$. Hence, we will use (4.10) for the Hölder stability estimate. However, for the Lipschitz stability estimate we need complete data at $S_{h,T}$. Thus, along with (4.10), we will also consider the following boundary data:

$$w\mid_{S_{h,T}}=\partial_{h,t}\widetilde{p}_0\left(x,t_s\right),\ \partial_l w\mid_{S_{h,T}}=\partial_{h,t}\widetilde{p}_1\left(x,t_s\right). \tag{4.12}$$

5 Two Carleman Estimates for Elliptic Operators

5.1 Domains

Let $A_1, A_2 > 0$ be two numbers. Denote $\overline{x} = (x_2, x_3, ..., x_n)$. Suppose that the part $\Gamma \in C^3$ of the boundary $\partial\Omega$ is:

$$\begin{gathered}\Gamma = \{x : x_1 = \omega\left(\overline{x}\right)\} \subset \partial\Omega,\ \text{ where } \overline{x} \in \{|\overline{x}| < A_1\}, \\ \omega \in C^3\left(|\overline{x}| \leq A_1\right) \text{ and let } \Omega \subset \{x_1 > \omega\left(\overline{x}\right),\ \overline{x} \in \{|\overline{x}| < A_1\}\}.\end{gathered} \tag{5.1}$$

Change variables as

$$(x_1, \overline{x}) \Leftrightarrow \left(x_1' = x_1 - \omega\left(\overline{x}/A_1\right), \overline{x}' = \overline{x}/A_1\right). \tag{5.2}$$

For brevity, we keep the same notation for $(x_1', \overline{x}')$ and Ω. Then by (5.1) Γ becomes: $\Gamma = \{x : x_1 = 0, |\overline{x}| < 1\}$. Since the domain Ω is bounded, then it follows from (5.1) and (5.2) that we can assume that $\Omega \subset \{x_1 \in (0, A_2), |\overline{x}| < 1\}$. Changing variables again $x_1 \Leftrightarrow x_1' = x_1/(2A_2)$ and again keeping the same notation for x_1 for brevity, we assume below that

$$\begin{gathered}\Omega \subseteq \{x : x_1 \in (0, 1/2),\ \ |\overline{x}| < 1\}, \\ \Gamma = \{x : x_1 = 0, |\overline{x}| < 1\} \subset \partial\Omega.\end{gathered} \tag{5.3}$$

Define the function $\psi(x)$ as

$$\psi(x) = x_1 + \frac{|\overline{x}|^2}{2} + \frac{1}{4}. \tag{5.4}$$

Let $\lambda \geq 1$ and $\nu \geq 1$ be two large parameters. Define the Carleman Weight Function $\varphi_{\lambda,\nu}(x)$ as [24, section 1 of chapter 4], [18, formula (2.66)]

$$\varphi_{\lambda,\nu}(x) = \exp\left(2\lambda\psi^{-\nu}(x)\right). \tag{5.5}$$

Let $\gamma \in [0, 1/2)$ be an arbitrary number from this interval. Define the domains G, G_α as:

$$G = \left\{x : x_1 > 0, x_1 + \frac{|\overline{x}|^2}{2} + \frac{1}{4} < \frac{3}{4}\right\} = \left\{x_1 > 0, \psi(x) < \frac{3}{4}\right\}, \tag{5.6}$$

$$G_\gamma = \left\{x : x_1 > 0, x_1 + \frac{|\overline{x}|^2}{2} + \frac{1}{4} < \frac{3}{4} - \gamma\right\} = \left\{x_1 > 0, \psi(x) < \frac{3}{4} - \gamma\right\}. \tag{5.7}$$

By (5.3), (5.6) and (5.7)

$$\begin{gathered}G_\gamma \subset G \text{ for } \gamma \in (0, 1/2), G_0 = G, \\ G \subseteq \Omega.\end{gathered} \tag{5.8}$$

It follows from (5.6) and (5.7) that the boundaries ∂G and ∂G_α of the domains G and G_α are

$$\begin{gathered}\partial G=\partial_1 G\cup\partial_2 G,\\ \partial_1 G=\{x_1=0,|\overline{x}|<1\}=\Gamma,\\ \partial_2 G=\{x_1>0,\psi(x)=3/4\},\\ \partial G_\gamma=\partial_1 G_\gamma\cup\partial_2 G_\gamma,\\ \partial_1 G_\gamma=\left\{x_1=0,|\overline{x}|^2/2<1/2-\gamma\right\},\\ \partial_2 G_\gamma=\{x_1>0,\psi(x)=3/4-\gamma\}.\end{gathered}\tag{5.9}$$

Hence, $\partial_2 G_\gamma$ is such a part of the level surface of the function $\psi(x)$, which is located in the half space $\{x_1>0\}$.

Remark 5.1. *Thus, we assume below that (5.3) holds and keep notations (5.4)-(5.9). It follows from (5.1) and the transformation (5.2) that the results below are applicable to quite general domains Ω.*

5.2 Carleman estimates

There exist two methods of proofs of Carleman estimates. The first one is using symbols of partial differential operators, see, e.g. [11, sections 8.3 and 8.4], [12, Theorem 3.2.1]. It comes up with formulations of Carleman estimates, which include only certain integrals. Although this method is both elegant and short, it assumes zero boundary conditions of involved functions. Hence, it does not allow to incorporate non zero boundary conditions. The second method uses space consuming derivations of pointwise Carleman estimates . As soon as a pointwise estimate is derived, it is integrated over an appropriate domain, Gauss formula results in boundary integrals, which, in turn allow to use non-zero boundary conditions, see, e.g. [16], [18, sections 2.3, 2.4], [24, section 1 of chapter 4], [34]. Hence, we formulate pointwise Carleman estimates first and integrate them then.

Carleman estimates for elliptic operators are usually derived from their counterparts for parabolic operators under the assumption of the independence on t of all involved functions. Keeping this in mind, there are two types of Carleman estimates for elliptic operators. In the first type only the tested function and its gradient are estimated. This kind of estimates allow one to work with non-zero Dirichlet and Neumann boundary conditions. In the second type, estimates of second order $x-$derivatives are incorporated as well, although with a small parameter $1/\lambda$. In this case one can work only with the zero boundary conditions. These two types of Carleman estimates are formulated below. It is well known that Carleman estimates are formulated only for principal parts of Partial Differential Operators and do not depend on their low order terms, see, e.g. [18, Lemma 2.1.1].

Theorem 5.1 (the Carleman estimate of the first type) [24, Lemma 3 in section 1 of chapter 4], [18, Theorem 2.4.1]. *Let the domain Ω be as in (5.3). Let L_0 be the principal part of the elliptic operator in (3.8). Assume that conditions (3.6) are satisfied. Then there exist sufficiently large numbers $\lambda_0=\lambda_0(\Omega,a^0,\mu)\geq 1$ and $\nu_0=\nu_0(G,a^0,\mu)\geq 1$ and a number $C=C(\Omega,a^0,\mu)>0$, all three numbers depending only on listed parameters, such that the following pointwise Carleman estimate is valid with the CWF $\varphi_{\lambda,\nu}(x)$ defined in (5.5):*

$$\begin{gathered}(L_0u)^2\varphi_{\lambda,\nu}\geq C\lambda\nu|\nabla u|^2\varphi_{\lambda,\nu}+C\lambda^3\nu^4\psi^{-2\nu-2}u^2\varphi_{\lambda,\nu}+\operatorname{div}U_1,\\ |U_1|\leq C\lambda^3\nu^4\psi^{-2\nu-2}\left(|\nabla u|^2+u^2\right)\varphi_{\lambda,\nu},\\ \forall\lambda\geq\lambda_0,\ \forall\nu\geq\nu_0,\ \forall x\in\overline{G},\ \forall u\in C^2\left(\overline{\Omega}\right).\end{gathered}\tag{5.10}$$

In Theorem 5.2 we incorporate second order derivatives of the function u in estimate (5.10). It is convenient to set in this theorem $\nu = \nu_0$.

Theorem 5.2 (the Carleman estimate of the second type). *Assume that conditions of Theorem 5.1 hold. Fix $\nu = \nu_0 (\Omega, a^0, \mu)$. Then the Carleman estimate (5.10) can be modified as:*

$$\begin{gathered}
(L_0 u)^2 \varphi_{\lambda,\nu_0} \geq (C/\lambda) \sum_{i,j=1}^{n} u_{x_i x_j}^2 \varphi_{\lambda,\nu_0} + \\
+ C\lambda |\nabla u|^2 \varphi_{\lambda,\nu_0} + C\lambda^3 u^2 \varphi_{\lambda,\nu_0} + \operatorname{div} U_1 + \operatorname{div} U_2, \\
|U_1| \leq C\lambda^3 \left(|\nabla u|^2 + u^2\right) \varphi_{\lambda,\nu_0}, \\
|U_2| \leq (C/\lambda) |\nabla u| \sum_{i,j=1}^{n} \left|u_{x_i x_j}\right| \varphi_{\lambda,\nu_0}, \\
\forall \lambda \geq \lambda_0, \ \forall x \in \overline{\Omega}, \ \forall u \in C^3 \left(\overline{\Omega}\right).
\end{gathered} \tag{5.11}$$

Proof. By Theorem 5.1 we need to prove only the involvement of the second derivatives and the estimate in the fourth line of (5.11). We have:

$$(L_0 u)^2 \varphi_{\lambda,\nu_0}^2 = \sum_{i,j=1}^{n} \sum_{k,s=1}^{n} a_{ij} a_{ks} u_{x_i x_j} u_{x_k x_s} \varphi_{\lambda,\nu_0}^2. \tag{5.12}$$

In addition, we have:

$$\begin{gathered}
a_{ij} a_{ks} u_{x_i x_j} u_{x_k x_s} \varphi_{\lambda,\nu_0}^2 = \\
= \left(a_{ij} a_{ks} u_{x_i x_j} u_{x_k} \varphi_{\lambda,\nu_0}^2\right)_{x_s} - a_{ij} a_{ks} u_{x_i x_j x_s} u_{x_k} \varphi_{\lambda,\nu_0}^2 - \left(a_{ij} a_{ks} \varphi_{\lambda,\nu_0}^2\right)_{x_s} u_{x_i x_j} u_{x_k} = \\
= \left(a_{ij} a_{ks} u_{x_i x_j} u_{x_k} \varphi_{\lambda,\nu_0}^2\right)_{x_s} - \left(a_{ij} a_{ks} \varphi_{\lambda,\nu_0}^2\right)_{x_s} u_{x_i x_j} u_{x_k} + \\
+ \left(-a_{ij} a_{ks} u_{x_i x_s} u_{x_k} \varphi_{\lambda,\nu_0}^2\right)_{x_j} + \left(a_{ij} a_{ks} u_{x_i x_s} u_{x_k} \varphi_{\lambda,\nu_0}^2\right)_{x_j} + a_{ij} a_{ks} u_{x_i x_s} u_{x_k x_j} \varphi_{\lambda,\nu_0}^2.
\end{gathered} \tag{5.13}$$

It was proven in [23, formula (6.12) of Chapter 2] that

$$\sum_{i,j=1}^{n} \sum_{k,s=1}^{n} a_{ij} a_{ks} u_{x_i x_s} u_{x_k x_j} \geq \mu^2 \sum_{i,j=1}^{n} u_{x_i x_j}^2. \tag{5.14}$$

Hence, (5.4), (5.5), (5.12)-(5.14) and Cauchy-Schwarz inequality imply:

$$\begin{gathered}
(L_0 u)^2 \varphi_{\lambda,\nu_0} \geq C \sum_{i,j=1}^{n} u_{x_i x_j}^2 \varphi_{\lambda,\nu_0} - C\lambda^2 |\nabla u|^2 \varphi_{\lambda,\nu_0} + \operatorname{div} \widetilde{U}_2, \\
\left|\widetilde{U}_2\right| \leq C |\nabla u| \sum_{i,j=1}^{n} \left|u_{x_i x_j}\right| \varphi_{\lambda,\nu_0}.
\end{gathered} \tag{5.15}$$

Divide both formulas (5.15) by $d\lambda$ with an appropriate number $d = d (\Omega, a^0, \mu) > 0$ and sum up with (5.10) setting there $\nu = \nu_0$. Then we obtain (5.11). $\square$

Remark 5.2. *It is clear from (5.10) and density arguments that the integration of these pointwise Carleman estimates over the domain G makes the resulting integral estimate valid for all functions $u \in H^2 (G)$, and in the case of (5.11) for all functions $u \in \{u \in H^2 (G) : \nabla u = 0$ on $\partial G\}$.*

6 Hölder and Lipschitz Stability Estimates for the Unique Continuation Problem (3.1)-(3.5)

Theorems of this section address, within the TFD framework, the first long standing open question formulated in Introduction.

6.1 The first Hölder stability estimate

Theorem 6.1 (the first Hölder stability estimate). *Let $M_1, M_2 > 0$ be two numbers. Consider the ball*

$$D(M_1) = \left\{ y \in \mathbb{R}^N : |y_i| < M_1, i = 1, ..., N \right\}, \tag{6.1}$$

where a special role of the vector $y \in \mathbb{R}^N$ is explained in the paragraph below (3.2). Let

$$\|F(y,x,t)\|_{C^2\left(\overline{D(M_1)}\right)\times C\left(\overline{\Omega}_{h,T}\right)} \leq M_2. \tag{6.2}$$

Let condition (1.2) be valid. Let the domain G be the one defined in (5.6) and the domain Ω be the one in (5.3). Assume that conditions (3.3) and (3.4) hold, where Ω is replaced with G. Given notation (3.1), let two vector functions

$$u_1(x,t), u_2(x,t) \in C^2\left(\overline{G}_{h,T}\right) \tag{6.3}$$

are solutions of equation (3.2) in $G_{h,T}$ with the lateral Cauchy data (3.5) at $\Gamma_{h,T}$. Assume that

$$\|u_i\|_{C^2\left(\overline{G}_{h,T}\right)} \leq M_1, \ i = 1, 2. \tag{6.4}$$

Denote

$$\overline{u} = u_1 - u_2, \ \overline{g}_0 = g_{1,0} - g_{2,0}, \ \overline{g}_1 = g_{1,1} - g_{2,1}. \tag{6.5}$$

Let $\delta \in (0,1)$ be a number. Assume that

$$\|\overline{g}_0\|_{H^1\left(\Gamma_{h,T}\right)} < \delta \text{ and } \|\overline{g}_1\|_{L_2\left(\Gamma_{h,T}\right)} < \delta. \tag{6.6}$$

Norms in (6.6) are understood as in (2.7). Choose an arbitrary number $\varepsilon \in (0, 1/6)$. Then there exists a sufficiently small number $\delta_1 = \delta_1(M_1, M_2, \mu, G, T, h_0, \varepsilon) \in (0,1)$ such that the following Hölder stability estimate holds:

$$\|\overline{u}\|_{H^1\left(G_{3\varepsilon,h,T}\right)} \leq C_1 \left(1 + \|\overline{u}\|_{H^1\left(G_{h,T}\right)}\right) \delta^{\rho_1}, \ \forall \delta \in (0, \delta_1), \tag{6.7}$$

where the number $\rho_1 = \rho_1(M_1, M_2, \mu, G, T, h_0, \varepsilon) \in (0,1)$ and the number $C_1 = C_1(M_1, M_2, \mu, G, T, h_0, \varepsilon) > 0$. Numbers δ_1, ρ_1 and C_1 depend only on listed parameters. In addition, problem (3.1), (3.2), (3.5) has at most one solution $u \in C^3\left(\overline{G}_{h,T}\right)$.

Remark 6.1. *Below $C_1 > 0$ denotes different numbers depending only on the above listed parameters.*

Proof of Theorem 6.1. It follows from (3.5), (6.3) and (6.5) that norms in (6.6) make sense. Using (2.13), (2.14) and (6.4), we obtain

$$
\begin{gathered}
\max_{t_s \in Y} \left\| \left(\int_0^{t_s} \left| \nabla u_i (x,\tau) \right| d\tau \right)_h \right\|_{C(\overline{\Omega})} \leq C_1 M_1, \\
\max_{t_s \in Y} \left\| \left(\int_0^{t_s} \left| u_i (x,\tau) \right| d\tau \right)_h \right\|_{C(\overline{\Omega})} \leq C_1 M_1 \\
\max_{t_s \in Y} \left\| \left(\int_0^{t_s} \left| \partial_{h,t} u_i (x,\tau) \right| d\tau \right)_h \right\|_{C(\overline{\Omega})} \leq C_1 M_1, \\
i = 1,2,
\end{gathered} \tag{6.8}
$$

where integrals are understood in terms of (2.13) and (2.14). Note that by (5.4), (5.6) and (5.7)

$$
G_{3\varepsilon} \neq \varnothing \text{ and } \psi (x) \in (1/4, 3/4 - 3\varepsilon) \text{ in } G_{3\varepsilon}. \tag{6.9}
$$

Subtract equation (3.2) for the vector u_2 from the same equation for u_1. Using (3.1) and the finite increment formula, we obtain for the vector function $\overline{u}$:

$$
\begin{gathered}
\partial_{h,t} \overline{u} (x,t_s) = \sum_{i,j=1}^{n} b_{i,j} (x,t_s) \overline{u}_{x_i x_j} (x,t_s) + \\
+ \sum_{j=1}^{n} d_j^{(1)} (x,t_s) \overline{u}_{x_j} (x,t_s) + d_0^{(1)} (x,t_s) \overline{u} (x,t_s) + \\
+ \sum_{j=1}^{n} d_j^{(2)} (x,t_s) \left(\int_0^{t_s} \overline{u}_{x_j} (x,\tau) d\tau \right)_h + d_0^{(2)} (x,t_s) \left(\int_0^{t_s} \overline{u} (x,\tau) d\tau \right)_h + \\
+ d_0^{(3)} (x,t_s) \left(\int_0^{t_s} \partial_{h,t} \overline{u} (x,\tau) d\tau \right)_h, \ x \in G, t_s \in Y, \\
\overline{u} \mid_{\Gamma_{h,T}} = \overline{g}_0 (x,t_s), \ \overline{u}_{x_1} \mid_{\Gamma_{h,T}} = -\overline{g}_1 (x,t_s), \ t_s \in Y, ,
\end{gathered} \tag{6.10}
$$

where Γ is given in (5.3). It follows from (6.1)-(6.4) that in (6.10) vector functions

$$
\begin{gathered}
b_{i,j}, d_r^{(s)} \in C^1 \left(\overline{\Omega}_{h,T} \right); \ \left\| b_{i,j} \right\|_{C^1 \left(\overline{\Omega}_{h,T} \right)}, \left\| d_r^{(s)} \right\|_{C^1 \left(\overline{\Omega}_{h,T} \right)} \leq M_1, \\
i,j = 1,...,n; \ r = 0,...,n; \ s = 1,2,3.
\end{gathered} \tag{6.11}
$$

In addition, it follows from (3.3) and (3.4) that

$$
\begin{gathered}
b_{i,j} (x,t_s) = b_{j,i} (x,t_s), \ (x,t_s) \in G_{h,T}, \\
\mu |\xi|^2 \leq \sum_{i,j=1}^{n} b_{i,j} (x,t_s) \xi_i \xi_j, \ (x,t_s) \in G_{h,T}.
\end{gathered} \tag{6.12}
$$

Denote

$$
L^{(s)} \left(\overline{u} (x,t_s) \right) = \sum_{i,j=1}^{n} b_{i,j} (x,t_s) \overline{u}_{x_i x_j} (x,t_s), \ s = 0,...,k, x \in \Omega. \tag{6.13}
$$

It follows from (2.10)-(2.12) that the function $\overline{u}(x,t_s)/(2h)$ is involved two times in the left hand side of (6.10) for each $s=0,...,k,$ also, see Remark 2.1. Hence, using (2.10)-(2.14), (6.8), (6.10), (6.11) and (6.13), we obtain the following inequalities:

$$\left|L^{(s)}\left(\overline{u}\left(x,t_s\right)\right)\right| \leq C_1 \sum_{j=0}^{k}\left(\left|\nabla\overline{u}\left(x,t_j\right)\right| + \left|\overline{u}\left(x,t_j\right)\right|\right),\ x\in G;\ t_j,t_s\in Y. \tag{6.14}$$

Consider a cut-off function $\chi_{\varepsilon}(x)$ such that

$$\chi_{\varepsilon}(x)\in C^2\left(\overline{G}\right),\ \chi_{\varepsilon}(x)=\left\{\begin{array}{c} 1 \text{ in } G_{2\varepsilon}, \\ 0 \text{ in } G\diagdown G_{\varepsilon}, \\ \text{between 0 and 1 in } G_{\varepsilon}\diagdown G_{2\varepsilon}. \end{array}\right. \tag{6.15}$$

The existence of such functions is well known from the Analysis course. Denote

$$v\left(x,t_i\right)=\chi_{\varepsilon}(x)\,\overline{u}\left(x,t_i\right),\ \ (x,t_i)\in G_{h,T}. \tag{6.16}$$

It follows from (5.9) and (6.15) and (6.16) that

$$v\left(x,t_i\right)=0 \text{ in a small neighborhood of } \partial_2 G. \tag{6.17}$$

Multiply both sides of (6.14) by $\chi_{\varepsilon}(x)$, keeping in mind that by (6.15) $\chi_{\varepsilon}(x)\geq 0$ in G. Use:

$$\begin{gathered}\chi_{\varepsilon}\overline{u}_{x_i}=v_{x_ix_i}-\left(\chi_{\varepsilon}\right)_{x_i}\overline{u}=v_{x_i}-\left(\chi_{\varepsilon}\right)_{x_i}\overline{u}, \\ \chi_{\varepsilon}\overline{u}_{x_ix_j}=v_{x_ix_j}-\left(\chi_{\varepsilon}\right)_{x_j}\overline{u}_{x_i}-\left(\chi_{\varepsilon}\right)_{x_i}\overline{u}_{x_j}-\left(\chi_{\varepsilon}\right)_{x_ix_j}\overline{u}.\end{gathered}$$

Hence, using the last line of (6.10) and (6.16), we obtain

$$\begin{gathered}\left|L^{(s)}\left(v\left(x,t_s\right)\right)\right| \leq C_1\sum_{j=0}^{k}\left(\left|\nabla v\left(x,t_j\right)\right|+\left|v\left(x,t_j\right)\right|\right) \\ +C_1\left(1-\chi_{\varepsilon}(x)\right)\left(\sum_{j=0}^{s}\left|\nabla\overline{u}\left(x,t_j\right)\right|+\sum_{j=0}^{k}\left|\overline{u}\left(x,t_j\right)\right|\right),\ x\in G;\ t_j,t_s\in Y, \\ v\mid_{\Gamma_{h,T}}=\chi_{\varepsilon}(x)\,\overline{g}_0\left(x,t_s\right),\ v_{x_1}\mid_{\Gamma_{h,T}}=-\left(\left(\chi_{\varepsilon}\right)_{x_1}\overline{g}_0+\chi_{\varepsilon}\overline{g}_1\right)\left(x,t_s\right).\end{gathered} \tag{6.18}$$

Square both sides of each inequality (6.18), use Cauchy-Schwarz inequality and then multiply both sides of the resulting inequality by the function $\varphi_{\lambda,\nu_0}(x)$, where $\varphi_{\lambda,\nu_0}(x)$ is the CWF (5.5) at $\nu=\nu_0$, where ν_0 is the number chosen in Theorem 5.1. We obtain

$$\begin{gathered}\left[L^{(s)}\left(v\left(x,t_s\right)\right)\right]^2\varphi_{\lambda,\nu_0}(x)\leq \\ \leq C_1\sum_{i=0}^{k}\left(\left|\nabla v\left(x,t_j\right)\right|^2+\left|v\left(x,t_j\right)\right|^2\right)\varphi_{\lambda,\nu_0}(x)+ \\ +C_1\left(1-\chi_{\varepsilon}(x)\right)\sum_{i=0}^{k}\left(\left|\nabla u\left(x,t_j\right)\right|^2+\left|u\left(x,t_j\right)\right|^2\right)\varphi_{\lambda,\nu_0}(x), \\ x\in G;\ t_j,t_s\in Y.\end{gathered} \tag{6.19}$$

It follows from (6.11)-(6.13) that we can apply the pointwise Carleman estimate (5.10) to the left hand side of (6.19). Integrate each of those estimates over the domain G, using

Gauss formula, the second line of (5.10) and (6.17). Then sum up the resulting estimates with respect to the index $s = 0, ..., k$. We also note that by (5.3)-(5.6)

$$\max_{\overline{G}} \varphi_{\lambda,\nu_0}(x) = \exp\left(2\lambda \cdot 4^{\nu_0}\right). \tag{6.20}$$

Let λ_0 be the number of Theorem 5.1. We obtain for all $\lambda \geq \lambda_0$

$$\begin{gathered}
\exp\left(2\lambda \cdot 4^{\nu_0}\right)\left(\|\overline{g}_0\|^2_{H^1\left(\Gamma_{h,T}\right)} + \|\overline{g}_1\|^2_{L_2\left(\Gamma_{h,T}\right)}\right) + \\
+ \int_G \left(1 - \chi_\varepsilon(x)\right)\left(\sum_{j=0}^{k}\left(|\nabla \overline{u}(x,t_j)|^2 + |\overline{u}(x,t_j)|^2\right)\right)\varphi_{\lambda,\nu_0}(x)\,dx + \\
+ \int_G \sum_{j=0}^{k}\left(|\nabla v(x,t_j)|^2 + |v(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx \geq \\
\geq C_1\lambda \int_G \sum_{j=0}^{k}\left(|\nabla v(x,t_j)|^2 + \lambda^2|v(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx.
\end{gathered} \tag{6.21}$$

Note that by (6.15) $1 - \chi_\varepsilon(x) = 0$ in $G_{2\varepsilon}$ and by (6.6) $\|\overline{g}_0\|^2_{H^1\left(\Gamma_{h,T}\right)} + \|\overline{g}_1\|^2_{L_2\left(\Gamma_{h,T}\right)} < 2\delta^2$. Choose $\lambda_1 = \lambda_1(M_1, M_2, \mu, \Omega, T, h_0, \varepsilon) \geq \lambda_0$ depending only on listed parameters such that $C_1\lambda_1/2 > 1$. Hence, using (6.21), we obtain

$$\begin{gathered}
\exp\left(2 \cdot 4^{\nu_0} \cdot \lambda\right)\delta^2 + \\
+ \int_{G \diagdown G_{2\varepsilon}} \sum_{j=0}^{k}\left(|\nabla \overline{u}(x,t_j)|^2 + |\overline{u}(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx \geq \\
\geq C_1\lambda \int_G \sum_{j=0}^{k}\left(|\nabla v(x,t_j)|^2 + \lambda^2|v(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx \geq \\
\geq C_1\lambda \int_{G_{3\varepsilon}} \sum_{j=0}^{k}\left(|\nabla \overline{u}(x,t_j)|^2 + \lambda^2|\overline{u}(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx, \ \forall \lambda \geq \lambda_1.
\end{gathered} \tag{6.22}$$

The last line of (6.22) is less or equal than its third line since the first line of (5.8) implies that $G_{3\varepsilon} \subset G$. Also, we use $\overline{u}$ instead of v in the last line of (6.22) due to Next, by (5.6) and (5.7)

$$\begin{gathered}
1/4 < \psi(x) < 3/4 - 3\varepsilon \text{ in } G_{3\varepsilon}, \\
3/4 - 2\varepsilon < \psi(x) < 3/4, \text{ in } G \diagdown G_{2\varepsilon}.
\end{gathered}$$

Hence, using (5.5), we obtain

$$\begin{gathered}
\varphi_{\lambda,\nu_0}(x) \geq \exp\left[2\lambda\left(3/4 - 3\varepsilon\right)^{-\nu_0}\right] \text{ in } G_{3\varepsilon}, \\
\varphi_{\lambda,\nu_0}(x) \leq \exp\left[2\lambda\left(3/4 - 2\varepsilon\right)^{-\nu_0}\right] \text{ in } G \diagdown G_{2\varepsilon}.
\end{gathered} \tag{6.23}$$

Denote

$$\xi = \xi(M_1, M_2, \mu, \Omega, T, h_0, \varepsilon) = \left[\left(3/4 - 3\varepsilon\right)^{-\nu_0} - \left(3/4 - 2\varepsilon\right)^{-\nu_0}\right] > 0.$$

Hence, (6.22) and (6.23) lead to

$$\|\overline{u}\|^2_{H^1\left(G_{3\varepsilon,h,T}\right)} \leq C_1 \exp\left(2\lambda \cdot 4^{\nu_0}\right)\delta^2 + C_1 e^{-2\lambda\xi}\|\overline{u}\|^2_{H^1\left(G_{h,T}\right)}, \ \forall \lambda \geq \lambda_1. \tag{6.24}$$

Choose $\lambda = \lambda(\delta)$ such that

$$\exp\left(2\lambda \cdot 4^{\nu_0}\right)\delta^2 = e^{-2\lambda\xi}.$$

Hence,

$$\lambda = \ln\left(\frac{1}{\delta^{\theta}}\right), \ \theta = \frac{1}{4^{\nu_0} + \xi}. \tag{6.25}$$

Hence, we should have $\delta \in (0, \delta_1)$, where the number $\delta_1 = \delta_1(M_1, M_2, \mu, \Omega, T, h_0, \varepsilon) \in (0,1)$ is so small that $\ln\left(\delta_1^{-\gamma}\right) \geq \lambda_1$. Using (6.23) and (6.23), we obtain

$$\|\overline{u}\|_{H^1\left(G_{3\varepsilon,h,T}\right)} \leq C_1\left(1 + \|\overline{u}\|_{H^1\left(G_{h,T}\right)}\right)\delta^{\rho_1}, \ \forall \delta \in (0, \delta_1), \tag{6.26}$$

$$\rho_1 = \frac{\xi}{\left(4^{\nu_0} + \xi\right)} \in (0,1). \tag{6.27}$$

The target Hölder stability estimate (6.7) follows immediately from (6.26) and (6.27).

To prove uniqueness, we set $\delta = 0$. Since the number $\varepsilon \in (0, 1/6)$ is an arbitrary one, then (6.7) implies that $\overline{u}(x,t) = 0$ in the entire domain $G_{h,T}$. $\square$

6.2 The second Hölder stability estimate

While Theorem 6.1 is about an estimate of the $H^1(G_{3\varepsilon,h,T})$ −norm of the function $\overline{u}(x,t)$, we now estimate in this subsection a stronger $H^2(G_{3\varepsilon,h,T})$ −norm of this function. To do this, we use the Carleman estimate of Theorem 5.2 instead of Theorem 5.1.

Theorem 6.2 (the second Hölder stability estimate)**.** *Assume that all conditions of Theorem 6.1 hold, except that now (6.3) is replaced with*

$$u_1(x,t), u_2(x,t) \in C^2\left(\overline{G}_{h,T}\right) \tag{6.28}$$

and (6.6) is replaced with estimates in stronger norms

$$\|\overline{g}_0\|_{H^2\left(\Gamma_{h,T}\right)} < \delta \text{ and } \|\overline{g}_1\|_{H^2\left(\Gamma_{h,T}\right)} < \delta, \tag{6.29}$$

see Remarks 2.1. Norms in (6.29) are understood as in (2.7). Keep notations (6.5). Choose an arbitrary number $\varepsilon \in (0, 1/6)$. *Then there exists a sufficiently small number* $\delta_2 = \delta_2(M_1, M_2, \mu, G, T, h_0, \varepsilon) \in (0,1)$ *and a number* $\rho_2 = \rho_2(M_1, M_2, \mu, G, T, h_0, \varepsilon) \in (0,1)$ *such that the following Hölder stability estimate holds:*

$$\|\overline{u}\|_{H^2\left(G_{3\varepsilon,h,T}\right)} \leq C_1\left(1 + \|\overline{u}\|_{H^1\left(G_{h,T}\right)}\right)\delta^{\rho_2}, \ \forall \delta \in (0, \delta_2), \tag{6.30}$$

where the number $C_1 = C_1(M_1, M_2, \mu, G, T, h_0, \varepsilon) > 0$. *Numbers* δ_2, ρ_2 *and* C_1 *depend only on listed parameters.*

Proof. By (6.28) norms in (6.29) make sense. Keeping in mind (5.3) and (5.9), consider the following function $\widetilde{u}(x,t)$ in $G_{h,T}$:

$$\widetilde{u}(x,t_s) = \overline{u}(x,t_s) - \overline{g}_0(\overline{x},t_s) + x_1\overline{g}_1(\overline{x},t_s), (x,t_s) \in G_{h,T}. \tag{6.31}$$

Then the last line of (6.10) implies:

$$\widetilde{u} \mid_{\Gamma_{h,T}} = \widetilde{u}_{x_1} \mid_{\Gamma_{h,T}} = 0. \tag{6.32}$$

Taking from (6.31) $\overline{u}=\widetilde{u}+\overline{g}_0+x_1\overline{g}_1$, substituting this in the first four lines of (6.10) and then turning the resulting equation in the inequality, we obtain similarly with (6.14)

$$\begin{aligned}&\left|L^{(s)}\left(\widetilde{u}\left(x,t_s\right)\right)\right|\leq C_1\sum_{j=0}^{k}\left(\left|\nabla\widetilde{u}\left(x,t_j\right)\right|+\left|\widetilde{u}\left(x,t_j\right)\right|\right)+\\&+C_1\sum_{|\alpha|\leq 2}\sum_{j=0}^{k}\sum_{i=0}^{1}\left|D_x^{\alpha}\overline{g}_i\left(x,t_j\right)\right|,\ x\in G;\ t_j,t_s\in Y.\end{aligned}\tag{6.33}$$

Next, we proceed similarly with the proof of Theorem 6.1, except that we apply the Carleman estimate (5.11) of Theorem 5.2 instead of (5.10) of Theorem 5.1. It follows

from the third and fourth lines of (5.11), (6.17) and (6.32) that boundary terms will not occur when using Gauss formula, also, see Remark 5.2. Then, using (6.29), we obtain a direct analog of (6.26), (6.27), in which, however, norms $\|\overline{u}\|_{H^1\left(G_{h,T}\right)}$ and $\|\overline{u}\|_{H^1\left(G_{3\varepsilon,h,T}\right)}$ are replaced with $\|\widetilde{u}\|_{H^2\left(G_{h,T}\right)}$ and $\|\widetilde{u}\|_{H^2\left(G_{3\varepsilon,h,T}\right)}$ respectively. Next, the use of (6.31) leads to (6.30). $\square$

6.3 Lipschitz stability estimates

We now replace the domain G of Theorems 6.1 and 6.2 with the domain Ω and recall that by (5.8) $G\subseteq\Omega$. We also assume now that the lateral Cauchy data are given on the entire lateral boundary $S_{h,T}$ of the semi-discrete time cylinder $\Omega_{h,T}$ instead of the part $\Gamma_{h,T}$ of $S_{h,T}$, see (2.1)-(2.4) and (5.3).

Theorem 6.3 (the first Lipschitz stability estimate). *Assume that conditions (1.2), (2.8), (5.3), (6.1) and (6.2) hold. Given notation (3.1), let two vector functions*

$$u_1\left(x,t\right),u_2\left(x,t\right)\in C^2\left(\overline{\Omega}_{h,T}\right)\tag{6.34}$$

are solutions of equation (3.2) in $\Omega_{h,T}$ and $\Gamma_{h,T}$ in (3.5) is replaced with $S_{h,T}$. Assume that conditions (6.4) hold, in which $G_{h,T}$ is replaced with $\Omega_{h,T}$. Keep notations (6.5). Then the following Lipschitz stability estimate is valid:

$$\|\overline{u}\|_{H^1\left(\Omega_{h,T}\right)}\leq C_2\left(\|\overline{g}_0\|_{H^1\left(S_{h,T}\right)}+\|\overline{g}_1\|_{L_2\left(S_{h,T}\right)}\right),\tag{6.35}$$

where norms in $H^1\left(S_{h,T}\right)$, $L_2\left(S_{h,T}\right)$ are understood as in (2.6)-(2.9). The number $C_2=C_2\left(M_1,M_2,\mu,\Omega,T,h_0\right)>0$ depends only on listed parameters. In addition, problem

Proof. By (6.34), (2.8) and (2.9) norms in the right hand side of (6.35) make sense. Below $C_2>0$ denotes different numbers depending only on the above listed parameters. Similarly with the proof of Theorem 6.1, we obtain the following analog of inequality (6.14):

$$\left|L^{(s)}\left(\overline{u}\left(x,t_s\right)\right)\right|\leq C_2\sum_{j=0}^{k}\left(\left|\nabla\overline{u}\left(x,t_j\right)\right|+\left|\overline{u}\left(x,t_j\right)\right|\right),\ x\in\Omega;\ t_j,t_s\in Y.\tag{6.36}$$

In addition,

$$\overline{u}\mid_{S_{h,T}}=\overline{g}_0,\ \partial_l\overline{u}\mid_{S_{h,T}}=\overline{g}_1.\tag{6.37}$$

Square both sides of each inequality (6.36), apply Cauchy-Schwarz inequality, then multiply by the function $\varphi^2_{\lambda,\nu_0}(x)$, where ν_0 is the number of Theorem 5.1, then integrate over the domain Ω and apply Carleman estimate (5.10). In doing the latter, use Gauss formula to $\operatorname{div} U_1$ and use the estimate for $|U_1|$ in the second line of (5.10). Then sum up resulting inequalities with respect to $s = 0, ..., k$. Using (6.20) we obtain for all $\lambda \geq \lambda_0$

$$\begin{gathered} \exp\left(2\lambda \cdot 4^{\nu_0}\right)\left(\|\overline{g}_0\|^2_{H^1(S_{h,T})} + \|\overline{g}_1\|^2_{L_2(S_{h,T})}\right) + \\ + \int_\Omega \sum_{j=0}^{k} \left(|\nabla \overline{u}(x,t_j)|^2 + |\overline{u}(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx \geq \\ \geq C_2\lambda \int_\Omega \sum_{j=0}^{k} \left(|\nabla \overline{u}(x,t_j)|^2 + \lambda^2 |\overline{u}(x,t_j)|^2\right)\varphi_{\lambda,\nu_0}(x)\,dx, \ \forall \lambda \geq \lambda_0. \end{gathered} \tag{6.38}$$

Choose $\lambda_2 = \lambda_2(M_1, M_2, \mu, \Omega, T, h_0) \geq \lambda_0$ depending only on listed parameters such that $C_2\lambda_2/2 > 1$. Hence, using (6.38), we obtain

$$\begin{gathered} \exp\left(2 \cdot 4^{\nu_0} \cdot \lambda\right)\left(\|\overline{g}_0\|^2_{H^1(S_{h,T})} + \|\overline{g}_1\|^2_{L_2(S_{h,T})}\right) \geq \\ \geq C_2 \int_\Omega \sum_{j=0}^{k} \left(|\nabla \overline{u}(x,t_j)|^2 + |\overline{u}(x,t_j)|^2\right)\varphi^2_{\lambda,\nu_0}(x)\,dx, \ \forall \lambda \geq \lambda_2. \end{gathered} \tag{6.39}$$

Next, it follows from (5.3), (5.4) and (5.5) that $\varphi^2_{\lambda,\nu_0}(x) \geq \exp\left(2\lambda (4/5)^{\nu_0}\right)$ in Ω. Hence, (6.39) implies

$$C_2 \|\overline{u}\|_{H^1(\Omega_{h,T})} \leq \exp\left[2\lambda\left(4^{\nu_0} - (4/5)^{\nu_0}\right)\right]\left(\|\overline{g}_0\|^2_{H^1(S_{h,T})} + \|\overline{g}_1\|^2_{L_2(S_{h,T})}\right), \ \forall \lambda \geq \lambda_2.$$

Setting here $\lambda = \lambda_2 = \lambda_2(M_1, M_2, \mu, \Omega, T, h_0)$ and recalling Remark 6.2, we obtain the target estimate (6.35). $\square$

Theorem 6.3 ensures the Lipschitz stability estimate (6.35) in the $H^1(\Omega_{h,T})-$ norm. We prove in Theorem 6.4 a similar estimate in the stronger $H^2(\Omega_{h,T})-$norm. Similarly with Theorem 6.2, we should use in this case Carleman estimate (5.11) instead of (5.10). This is because (5.11) contains derivatives of the second order.

Theorem 6.4 (the second Lipschitz stability estimate). *Assume that conditions of Theorem 6.3 hold. Suppose that boundary functions*

$$\overline{g}_0, \overline{g}_1 \in H^2(S_{h,T}) \tag{6.40}$$

(see Remarks 2.1) also that there exists a vector function $p(x,t) \in H^2(\Omega_{h,T})$ such that

$$\begin{gathered} p\mid_{S_{h,T}} = \overline{g}_0, \ \partial_l p\mid_{S_{h,T}} = \overline{g}_1, \\ \|p\|_{H^2(\Omega_{h,T})} \leq B_1\left(\|\overline{g}_0\|_{H^2(S_{h,T})} + \|\overline{g}_1\|_{H^2(S_{h,T})}\right), \end{gathered} \tag{6.41}$$

where the number $B_1 > 0$ is independent on functions $\overline{g}_0, \overline{g}_1$, and norms in (6.41) are understood as in (2.6)-(2.9). Keep notations (6.5). Then the following Lipschitz stability estimate holds:

$$\|\overline{u}\|_{H^2(\Omega_{h,T})} \leq C_3\left(\|\overline{g}_0\|_{H^2(S_{h,T})} + \|\overline{g}_1\|_{H^2(S_{h,T})}\right), \tag{6.42}$$

where the number $C_3 = C_3(M_1, M_2, B_1, \mu, \Omega, T, h_0) > 0$ depends only on listed parameters.

Proof. In this proof C_3 denotes different positive numbers depending on the above parameters. By (2.8), (2.9) and (6.40) norms in the right hand sides of (6.41) and (6.42) make sense. Using the first line of (6.41), and similarly with (6.31) consider the following function $\widetilde{u}$:

$$\widetilde{u}\left(x,t_s\right)=\overline{u}\left(x,t_s\right)-p\left(x,t_s\right),\left(x,t_s\right)\in\Omega_{h,T}. \tag{6.43}$$

Then the first line of (6.41) implies:

$$\widetilde{u}\mid_{S_{h,T}}=\partial_l\widetilde{u}\mid_{S_{h,T}}=0. \tag{6.44}$$

Substituting $\overline{u}=\widetilde{u}+p$ in the first four lines of (6.10) and turning the resulting equation in the inequality, we obtain the following analog of (6.33)

$$\begin{aligned}\left|L^{(s)}\left(\widetilde{u}\left(x,t_s\right)\right)\right|\leq C_3\sum_{j=0}^{k}\left(\left|\nabla\widetilde{u}\left(x,t_j\right)\right|+\left|\widetilde{u}\left(x,t_j\right)\right|\right)+\\ +C_3\sum_{|\alpha|\leq 2}\sum_{j=0}^{k}\left|D_x^\alpha p\left(x,t_j\right)\right|,\ x\in\Omega;\ t_j,t_s\in Y.\end{aligned} \tag{6.45}$$

Square both sides of each inequality (6.45), apply Cauchy-Schwarz inequality, then multiply by the CWF $\varphi_{\lambda,\nu_0}\left(x\right)$, apply Carleman estimate (5.11) and then integrate the resulting inequality over the domain Ω, using Gauss formula. It follows from the third and fourth lines of (5.11) and (6.44) that boundary integrals will not occur. Summing up resulting estimates with respect to $s=0,...,k$, using the second line of (6.41) and (6.43) and proceeding similarly with the part of the proof of Theorem 6.3, which starts from (6.38), we obtain the target estimate (6.42). $\square$

7 Stability Estimates for CIP1 in the TFD Framework (4.9)-(4.12)

Theorems of this section address, within the TFD framework, the second long standing question formulated in Introduction. We do not prove Theorems 7.1-7.4 since the TFD framework (4.9)-(4.12) is just a linear case of UCP (3.2)-(3.5). Following (4.10) and (4.12), denote

$$q_0\left(x,t\right)=\partial_{h,t}\widetilde{p}_0\left(x,t\right),\ q_1\left(x,t\right)=\partial_{h,t}\widetilde{p}_1\left(x,t\right). \tag{7.1}$$

Then

$$w\mid_{\Gamma_{h,T}}=q_0\left(x,t\right),\ \partial_l w\mid_{\Gamma_{h,T}}=q_1\left(x,t\right)\ \text{ in the case (4.10)}, \tag{7.2}$$

$$w\mid_{S_{h,T}}=q_0\left(x,t\right),\ \partial_l w\mid_{S_{h,T}}=q_1\left(x,t\right)\ \text{ in the case (4.12)}. \tag{7.3}$$

Theorem 7.1 is an analog of Theorem 6.1. Recall Remark 5.1.

Theorem 7.1. *Assume that in conditions (3.6), (3.13), (4.9)-(4.11), the domain Ω is replaced with the domain G* of *(5.6), and in so replaced conditions (4.9)-(4.11), the vector function $w\in H^2\left(G_{h,T}\right)$. Let conditions (1.2), (7.1) and (7.2) hold. Suppose that conditions (3.7) are replaced with*

$$\begin{aligned}a_\alpha,\partial_t a_\alpha\in C\left(\overline{G}_{h,T}\right)\ \text{ for }\ |\alpha|\leq 1,\\ \|a_\alpha\|_{C\left(\overline{G}_{h,T}\right)}\leq D,\ \ \|\partial_{h,t}a_\alpha\|_{C\left(\overline{G}_{h,T}\right)}\leq D\ \text{for}\ \ |\alpha|\leq 1.\end{aligned} \tag{7.4}$$

Let $\delta \in (0,1)$ *be a number. Assume that the following analog of (6.6) is valid:*

$$\|q_0\|_{H^1(\Gamma_{h,T})} < \delta \text{ and } \|q_1\|_{L_2(\Gamma_{h,T})} < \delta. \tag{7.5}$$

Choose an arbitrary number $\varepsilon \in (0,1/6)$. *Then there exists a sufficiently small number* $\delta_3 = \delta_3(a^0, D, \mu, G, T, c_1, h_0, \varepsilon) \in (0,1)$ *such that the following Hölder stability estimates hold:*

$$\|w\|_{H^1(G_{3\varepsilon,h,T})} \leq C_4 \left(1 + \|w\|_{H^1(G_{h,T})}\right) \delta^{\rho_3}, \ \forall \delta \in (0,\delta_3), \tag{7.6}$$

$$\|\widetilde{a}_{\alpha_0}\|_{L_2(G_{3\varepsilon,h,T})} \leq C_4 \left(1 + \|w\|_{H^1(G_{h,T})}\right) \delta^{\rho_3}, \ \forall \delta \in (0,\delta_3), \tag{7.7}$$

where the number $\rho_3 = \rho_3(a^0, D, \mu, G, T, c_1, h_0, \varepsilon) \in (0,1)$ *and the number* $C_4 = C_4(a^0, D, \mu, G, T, c_1,$ 0. *Numbers* δ_3, ρ_3 *and* C_4 *depend only on listed parameters. In addition, problem (4.9)-(4.11) has at most one solution* $(w, \widetilde{a}_{\alpha_0}) \in H^2(G_{h,T}) \times L_2(G)$. *Also, there exists at most one vector function* $(w, \widetilde{a}_{\alpha_0}) \in H^2(G_{h,T}) \times L_2(G)$ *conditions (4.9)-(4.11).*

Theorem 7.2 is an analog of Theorem 6.2.

Theorem 7.2. *Assume that all conditions of Theorem 7.1 hold, except that estimates (7.5) are replaced with estimates in stronger norms,*

$$\|q_0\|_{H^2(\Gamma_{h,T})} < \delta \text{ and } \|q_1\|_{H^2(\Gamma_{h,T})} < \delta,$$

see Remarks 2.1. Then estimates (7.6) and (7.7) become:

$$\|w\|_{H^2(G_{3\varepsilon,h,T})} \leq C_4 \left(1 + \|w\|_{H^1(G_{h,T})}\right) \delta^{\rho_4}, \ \forall \delta \in (0,\delta_3),$$

$$\|\widetilde{a}_{\alpha_0}\|_{L_2(G_{3\varepsilon,h,T})} \leq C_4 \left(1 + \|w\|_{H^1(G_{h,T})}\right) \delta^{\rho_4}, \ \forall \delta \in (0,\delta_3),$$

where the number $\rho_4 = \rho_4(a^0, D, \mu, G, T, c_1, h_0, \varepsilon) \in (0,1)$ *depends only on listed parameters.*

Theorem 7.3 is an analog of Theorem 6.3.

Theorem 7.3. *Assume that conditions (1.2), (2.8), (3.6), (3.13), (4.9), (4.11), (4.12), (5.3), (7.1), (7.3) and (7.4) hold and the vector function* $w \in H^2(\Omega_{h,T})$. *Then there exists a number* $C_5 = C_5(a^0, D, \mu, \Omega, T, c_1, h_0) > 0$ *depending only on listed parameters such that the following Lipschitz stability estimates hold:*

$$\|w\|_{H^1(\Omega_{h,T})} \leq C_5 \left(\|q_0\|_{H^1(S_{h,T})} + \|q_1\|_{L_2(S_{h,T})}\right), \tag{7.8}$$

$$\|\widetilde{a}_{\alpha_0}\|_{L_2(\Omega)} \leq C_5 \left(\|q_0\|_{H^1(S_{h,T})} + \|q_1\|_{L_2(S_{h,T})}\right). \tag{7.9}$$

In addition, problem (4.9), (4.11), (4.12) has at most one solution $(w, \widetilde{a}_{\alpha_0}) \in H^2(\Omega_{h,T}) \times L_2(\Omega)$.

Theorem 7.4 is an analog of Theorem 6.4.

Theorem 7.4. *Assume that all conditions of Theorem 7.3 hold. Suppose that functions* $q_0, q_1 \in H^2(S_{h,T})$ *(see Remarks 2.1). In addition, assume that there exists a vector function* $\widetilde{p}(x,t) \in H^2(\Omega_{h,T})$ *such that*

$$\widetilde{p}\mid_{S_{h,T}} = q_0, \ \partial_t \widetilde{p}\mid_{S_{h,T}} = q_1,$$
$$\|\widetilde{p}\|_{H^2(\Omega_{h,T})} \leq B_2 \left(\|q_0\|_{H^2(S_{h,T})} + \|q_1\|_{H^2(S_{h,T})}\right),$$

where the number $B_2 > 0$ is independent on functions q_0, q_1. Then estimates (7.8) and (7.9) become:

$$\|w\|_{H^2(\Omega_{h,T})} \leq C_6 \left(\|q_0\|_{H^2(S_{h,T})} + \|q_1\|_{H^2(S_{h,T})} \right),$$
$$\|\widetilde{a}_{\alpha_0}\|_{L_2(\Omega)} \leq C_6 \left(\|q_0\|_{H^2(S_{h,T})} + \|q_1\|_{H^2(S_{h,T})} \right),$$

where the number $C_6 = C_6 \left(a^0, D, \mu, \Omega, T, c_1, B_2, h_0\right) > 0$ depends only on listed parameters.

8 An Applied Example: Temporal and Spatial Monitoring of Epidemics

8.1 The CIP of monitoring of epidemics

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain modeling a city where an epidemic occurs. Let S be the number of susceptible patients, I be the number of infected patients and R be the number of recovered patients in this city. Traditionally this so-called SIR model is governed by a system of three coupled Ordinary Differential Equations (), which was originally proposed by Kermack and McKendrick in 1927 [13], also, see [25] for this system. However, this system describes only the total number of SIR patients in the affected city at any moment of time. To obtain both spatial and temporal distributions of SIR populations, Lee, Liu, Tembine, Li and Osher have proposed a new model, which governs hose distributions via a system of three coupled nonlinear parabolic PDEs [25].

It was noticed in [20] that coefficients of this system, which describe infection and recovery rates, are unknown. Hence, a CIP of the recovery of these coefficients was posed in [20] and a globally convergent numerical method for it, the so-called convexification method, was developed analytically and tested numerically in [20], also, see Introduction about the convexification method. The results of [20] indicated that if those coefficients are recovered, then it is possible to monitor spatial and temporal distributions of SIR populations inside of that city using temporal measurements of both SIR populations and their fluxes only at the boundary of the city. The latter is intriguing, since the cost of monitoring might be significantly decreased.

However, it was assumed in the CIP of [20] that the lateral Cauchy data for that system are complemented by the measurements of its solution at $t = t_0 \in (0, T)$ inside the domain Ω. Stability estimates for that CIP were not obtained in [20], although uniqueness theorem was proven. The goal of this section is to use the TFD framework to prove Lipschitz stability estimate for the direct analog of CIP of [20], assuming that $t_0 = 0$. In the next section we develop a new version of the convexification method for this CIP within the TFD framework.

8.2 The CIP of monitoring epidemics within the TFD framework

We keep notations (1.3), although, to shorten the presentation, we consider only the case of complete data given at the entire lateral boundary S_T, thus setting $\Gamma = \varnothing$. Let $u_S(x,t), u_I(x,t)$ and $u_R(x,t)$ be the numbers of S,I and R populations respectively at the point $(x,t) \in Q_T$. The SIR system of parabolic PDEs is [25, formulas (2.1)]:

$$
\begin{gathered}
\partial_t u_S - (\eta_S^2/2)\,\Delta u_S + \operatorname{div}(u_S q_S) + \beta(x)\, u_S u_I = 0, \text{ in } Q_T, \\
\partial_t u_I - (\eta_I^2/2)\,\Delta u_I + \operatorname{div}(u_I q_I) - \beta(x)\, u_S u_I + \gamma(x)\, u_I = 0, \text{ in } Q_T, \\
\partial_t u_R - (\eta_R^2/2)\,\Delta u_R + \operatorname{div}(u_R q_R) - \gamma(x)\, u_I = 0, \text{ in } Q_T.
\end{gathered} \tag{8.1}
$$

Initial and boundary data for system (8.1) are:

$$
u_S(x,0) = f_S(x), \ u_I(x,0) = f_I(x), \ u_R(x,0) = f_R(x), \text{ in } \Omega, \tag{8.2}
$$

$$
\partial_l u_S \mid_{S_T} = g_S(x,t), \ \partial_l u_I \mid_{S_T} = g_I(x,t), \ \partial_l u_R \mid_{S_T} = g_R(x,t). \tag{8.3}
$$

In (8.1), $\eta_S^2, \eta_R^2, \eta_R^2 > 0$ are the so-called viscosity terms, q_S, q_I and q_R are 2D vectors of velocities of propagations of S,I and R populations respectively. We assume solely for brevity that

$$
\frac{\eta_S^2}{2} \equiv \frac{\eta_I^2}{2} \equiv \frac{\eta_R^2}{2} \equiv 1, \tag{8.4}
$$

$$
q_S, q_I, q_R \in \left(C^1\left(\overline{\Omega}\right)\right)^2. \tag{8.5}
$$

Normal derivatives in (8.3) are fluxes of those populations through the boundary of that city. Next, $\beta(x)$ and $\gamma(x)$ are infection and recovery rates respectively.

System (8.1) is a nonlinear one. Therefore, unlike the well investigated linear case [23, Chapter 4], existence of a sufficiently smooth solution of problem (8.1)-(8.3) can be proven only under a certain assumption imposed on the length T of the interval $(0,T)$, see, e.g. [23, Theorem 7.1 of Chapter 7] for the case of the Dirichlet boundary condition for one nonlinear parabolic PDE. Uniqueness of the forward problem (8.1)-(8.3) can also be proven by one of classical techniques of [23]. However, we are not interested in these questions. Rather, we are interested in CIP2 which we pose below. Our Lipschitz stability estimate of Theorem 8.1 immediately implies uniqueness of that CIP within the TFD framework.

Coefficient Inverse Problem 2 (CIP2). *Assume that there exists the vector function* $\Phi(x,t) = (u_S, u_I, u_R)(x,t) \in \left(C^3\left(\overline{\Omega}_T\right)\right)^3$ *(see Remarks 2.1) satisfying conditions (8.1)-(8.3). Suppose that coefficients* $\beta(x), \gamma(x) \in C\left(\overline{\Omega}\right)$ *are unknown. Also, assume that in addition to the right hand sides of (8.2), (8.3), Dirichlet boundary conditions for functions* u_S, u_I, u_R *are known,*

$$
u_S \mid_{S_T} = p_S(x,t), \ u_I \mid_{S_T} = p_I(x,t), \ u_R \mid_{S_T} = p_R(x,t). \tag{8.6}
$$

Find functions $\beta(x)$ *and* $\gamma(x)$.

We assume the availability of the data (8.6) on the entire lateral boundary S_T for the sake of brevity only. In fact, results, similar with ones of Theorems 6.1 and 6.2 for incomplete data, also hold true. The knowledge of the right hand sides of (8.3) and (8.6) means that both fluxes and the numbers of SIR populations are measured at the boundary of the affected city. Our technique allows to find the vector function $\Phi(x,t)$ along with the unknown coefficients $\beta(x), \gamma(x)$. This means that our technique allows one to monitor spatial and temporal distributions of SIR populations inside of the affected city using boundary measurements.

We now reformulate CIP2 in the TFD framework. Denote:

$$
\begin{gathered}
v_1(x,t) = \partial_t u_S(x,t), v_2(x,t) = \partial_t u_I(x,t), v_3(x,t) = \partial_t u_R(x,t), \\
V(x,t) = (v_1, v_2, v_3)^T(x,t), \ V \in \left(C^3\left(\overline{\Omega}_T\right)\right)^3.
\end{gathered} \tag{8.7}
$$

By (8.2) and (8.7)

$$
\begin{gathered}
u_S(x,t)=\int\limits_0^t v_1(x,\tau)\,d\tau+f_S(x)\,,\ \ u_I(x,t)=\int\limits_0^t v_2(x,\tau)\,d\tau+f_I(x)\,,\\
u_R(x,t)=\int\limits_0^t v_3(x,\tau)\,d\tau+f_R(x)\,,
\end{gathered}
\tag{8.8}
$$

$$
v_i(x,0)=v_i(x,t)-\int\limits_0^t \partial_t v_i(x,\tau)\,d\tau,\ \ i=1,2,3. \tag{8.9}
$$

Assume that

$$
|f_S(x)|\,,\ \ |f_I(x)|\geq c_2>0, \tag{8.10}
$$

where $c_2>0$ is a positive number. Set $t=0$ in the first and third equations (8.1). Using (8.2), (8.4), (8.5) (8.7), (8.9) and (8.10), we obtain

$$
\beta(x)=\frac{1}{(f_S f_I)(x)}\left[-v_1(x,t)+\int\limits_0^t \partial_t v_1(x,\tau)\,d\tau+\Delta f_S(x)-\operatorname{div}\left(f_S(x)\,q_S(x)\right)\right], \tag{8.11}
$$

$$
\gamma(x)=\frac{1}{f_I(x)}\left[v_3(x,t)-\int\limits_0^t \partial_t v_3(x,\tau)\,d\tau-\Delta f_R(x)+\operatorname{div}\left(f_R(x)\,q_R(x)\right)\right]. \tag{8.12}
$$

Differentiate equations (8.1) with respect to t. Then, write the resulting equations in finite differences with respect to t, using also discrete versions (2.13), (2.14) of Volterra integrals and keeping the same notations for semi-discrete versions $v_1(x,t_s)\,,v_2(x,t_s)\,,v_3(x,t_s)$ of functions $v_1(x,t)\,,v_2(x,t)\,,v_3(x,t)$,

$$
V(x,t_s)=(v_1,v_2,v_3)^T(x,t_s)\in\left(H^2(\Omega_{h,T})\right)^3. \tag{8.13}
$$

The smoothness assumption in (8.13) is reasonable due to the second line of (8.7) and (2.6), also, see Remarks 2.1. In addition, we use (8.4), (8.5) and (8.7)-(8.12). We obtain three integral differential equations. The first equation is:

$$
\begin{gathered}
A_1(V)(x,t_s)=\partial_{h,t}v_1(x,t_s)-\Delta v_1(x,t_s)+\operatorname{div}\left(v_1(x,t_s)\,q_S(x)\right)+\\
+\left[(f_S f_I)(x)\right]^{-1}\times\\
\times\left[-v_1(x,t_s)+\left(\int\limits_0^{t_s}\partial_{h,t}v_1(x,\tau)\,d\tau\right)_h+\Delta f_S(x)-\operatorname{div}\left(f_S(x)\,q_S(x)\right)\right]\times\\
\times\left[v_1\left(\left(\int\limits_0^{t_s}v_2(x,\tau)\,d\tau\right)_h+f_I(x)\right)+v_2\left(\left(\int\limits_0^{t_s}v_1(x,\tau)\,d\tau\right)_h+f_S(x)\right)\right]=\\
=0,\ x\in\Omega,t_s\in Y.
\end{gathered}
\tag{8.14}
$$

The second equation is:

$$
\begin{gathered}
A_2(V)(x,t_s) = \partial_{h,t} v_2(x,t_s) - \Delta v_2(x,t_s) + \operatorname{div}(v_2(x,t_s)\, q_I(x)) - \\
- \left[(f_S f_I)(x)\right]^{-1} \times \\
\times \left[-v_1(x,t_s) + \left(\int_0^{t_s} \partial_{h,t} v_1(x,\tau)\, d\tau \right)_h + \Delta f_S(x) - \operatorname{div}(f_S(x)\, q_S(x)) \right] \times \\
\times \left[\begin{array}{c} v_1(x,t_s) \left(\left(\int_0^{t_s} v_2(x,\tau)\, d\tau \right)_h + f_I(x) \right) + \\ +v_2(x,t_s) \left(\left(\int_0^{t_s} v_1(x,\tau)\, d\tau \right)_h + f_S(x) \right) \end{array} \right] + \\
+ (f_I(x))^{-1} \left[\begin{array}{c} v_3(x,t_s) - \left(\int_0^{t_s} \partial_{h,t} v_3(x,\tau)\, d\tau \right)_h - \\ -\Delta f_R(x) + \operatorname{div}(f_R(x)\, q_R(x)) \end{array} \right] = \\
= 0, \; x \in \Omega, t_s \in Y.
\end{gathered} \tag{8.15}
$$

The third equation is:

$$
\begin{gathered}
A_3(V)(x,t_s) = \partial_{h,t} v_3(x,t_s) - \Delta v_3(x,t_s) + \operatorname{div}(v_3(x,t_s)\, q_R(x)) - \\
- (f_I(x))^{-1} \left[\begin{array}{c} v_3(x,t_s) - \left(\int_0^{t_s} \partial_{h,t} v_3(x,\tau)\, d\tau \right)_h - \\ -\Delta f_R(x) + \operatorname{div}(f_R(x)\, q_R(x)) \end{array} \right] v_2(x,t_s) = \\
= 0, \; x \in \Omega, \; t_s \in Y.
\end{gathered} \tag{8.16}
$$

Cauchy boundary data for system (8.14)-(8.16) are obtained from (8.3), (8.6) and (8.7),

$$
\begin{gathered}
Q = (v_1, v_2, v_3) \mid_{S_{h,T}} = (\partial_{h,t} p_S, \partial_{h,t} p_I, \partial_{h,t} p_R)(x,t_s), \; t_s \in Y, \\
Z = (\partial_l v_1, \partial_l v_2, \partial_l v_3) \mid_{S_{h,T}} = (\partial_{h,t} g_S, g_I, \partial_{h,t} g_R)(x,t_s), \; t_s \in Y.
\end{gathered} \tag{8.17}
$$

Suppose that BVP (8.14)-(8.17) is solved. Then, using (8.9), (8.11) and (8.12), we set

$$
\begin{gathered}
\beta(x) = ((f_S f_I)(x))^{-1} \left[-v_1(x,0) + \Delta f_S(x) - \operatorname{div}(f_S(x)\, q_S(x)) \right], \\
\gamma(x) = (f_I(x))^{-1} \left[v_3(x,0) - \Delta f_R(x) + \operatorname{div}(f_R(x)\, q_R(x)) \right].
\end{gathered} \tag{8.18}
$$

8.3 Lipschitz stability estimate for problem (8.14)-(8.18)

BVP (8.14)-(8.17) is the full analog of UCP (3.1)-(3.5), in which $\Gamma_{h,T}$ in (3.5) should be replaced with $S_{h,T}$. Hence, we will formulate an analog of Theorem 6.4. Recall that we took in section 8 the complete data at $S_{h,T}$ instead of the incomplete data at $\Gamma_{h,T}$ only for the sake of brevity. The second small difference with problem (3.1)-(3.5) is that we now have three equations (8.14)-(8.16) instead of one equation (3.2). It is clear, however, that Theorem 6.4 can easily be extended to this case. We omit the proof of Theorem 8.1 since it is very similar with the proof of Theorem 6.4.

First, we formulate analogs of conditions (6.2), (6.4) for our specific case of problem (8.14)-(8.18). Let $R > 0$ be a number. We assume that

$$
\|V\|_{(H^2(\Omega_{h,T}))^3} \leq R, \tag{8.19}
$$

$$\begin{gathered}\|q_S\|_{\left(C^1\left(\overline{\Omega}\right)\right)^2}, \ \ \|q_I\|_{\left(C^1\left(\overline{\Omega}\right)\right)^2}, \|q_R\|_{\left(C^1\left(\overline{\Omega}\right)\right)^2} \leq R, \\ \|f_S\|_{C^2\left(\overline{\Omega}\right)}, \ \ \|f_I\|_{C^2\left(\overline{\Omega}\right)}, \|f_R\|_{C^2\left(\overline{\Omega}\right)} \leq R.\end{gathered} \tag{8.20}$$

Assume that there exist two vector functions $\left(V_i\left(x,t_s\right),\beta_i\left(x\right),\gamma_i\left(x\right)\right),\ i=1,2$ satisfying conditions (8.13)-(8.19) as well as boundary conditions generated by (8.17):

$$\begin{gathered}Q_i = V_i\mid_{S_T} = \left(\partial_{h,t}p_{i,S}\left(x,t_s\right),\partial_{h,t}p_{i,I},\partial_{h,t}p_{i,R}\right)\left(x,t_s\right),\ t_s\in Y,\ i=1,2, \\ Z_i = \partial_l V_i\mid_{S_T} = \left(\partial_{h,t}g_{i,S},\partial_{h,t}g_{iI},\partial_{h,t}g_{i,R}\right)\left(x,t_s\right),\ t_s\in Y,\ i=1,2.\end{gathered} \tag{8.21}$$

Denote

$$\begin{gathered}\widetilde{Q}\left(x,t_s\right) = \left(\partial_{h,t}\left(p_{1,S}-p_{2,S}\right),\partial_{h,t}\left(p_{1,I}-p_{2,I}\right),\partial_{h,t}\left(p_{1,R}-p_{2,R}\right)\right)\left(x,t_s\right), \\ \widetilde{Z}\left(x,t_s\right) = \left(\partial_{h,t}\left(g_{1,S}-g_{2,S}\right),\partial_{h,t}\left(g_{1,I}-g_{2,I}\right),\partial_{h,t}\left(g_{1,R}-g_{2,R}\right)\right)\left(x,t_s\right), \\ \widetilde{V}\left(x,t_s\right) = V_1\left(x,t_s\right)-V_2\left(x,t_s\right),\ x\in\Omega, t_s\in Y.\end{gathered} \tag{8.22}$$

Using (8.13) and (8.21)-(8.22), we obtain

$$\widetilde{V}\mid_{S_{h,T}} = \widetilde{Q},\ \partial_l\widetilde{V}\mid_{S_{h,T}} = \widetilde{Z}. \tag{8.23}$$

Following Remarks 2.1, we assume that

$$\widetilde{Q},\widetilde{Z}\in\left(H^2\left(S_{h,T}\right)\right)^3. \tag{8.24}$$

Suppose that there exists a vector function $W\left(x,t_s\right)$ such that

$$\begin{gathered}W\in H^2\left(\Omega_{h,T}\right),\ W\mid_{S_{h,T}} = \widetilde{Q},\ \partial_l W\mid_{S_{h,T}} = \widetilde{Z}, \\ \|W\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} \leq B_3\left(\left\|\widetilde{Q}\right\|_{\left(H^2\left(S_{h,T}\right)\right)^3} + \left\|\widetilde{Z}\right\|_{\left(H^2\left(S_{h,T}\right)\right)^3}\right),\end{gathered} \tag{8.25}$$

where the number $B_3>0$ is independent on vector functions $\widetilde{Q}$ and $\widetilde{Z}$. In Theorem 8.1 we naturally mean the obvious 2D analog of condition (5.3).

Theorem 8.1. *Let the initial conditions $f_S,f_I,f_R\in C^2\left(\overline{\Omega}\right)$. Assume that conditions (1.2), (2.8), (5.3), (8.10) and (8.20) hold. Assume that there exist two vector functions $\left(V_i\left(x,t_s\right),\beta_i\left(x\right),\gamma_i\left(x\right)\right),\ i=1,2$ satisfying conditions (8.13)-(8.19) and (8.21)-(8.24). Denote $\widetilde{\beta}\left(x\right)=\beta_1\left(x\right)-\beta_2\left(x\right),\ \widetilde{\gamma}\left(x\right)=\gamma_1\left(x\right)-\gamma_2\left(x\right)$. In addition, assume that there exists a vector function W satisfying conditions (8.25) with the number $B_3>0$ independent on $\widetilde{Q}$ and $\widetilde{Z}$. Then the following Lipschitz stability estimate is valid:*

$$\left\|\widetilde{V}\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} + \left\|\widetilde{\beta}\right\|_{L_2(\Omega)} + \|\widetilde{\gamma}\|_{L_2(\Omega)} \leq C_7\left(\left\|\widetilde{Q}\right\|_{\left(H^2\left(S_{h,T}\right)\right)^3} + \left\|\widetilde{Z}\right\|_{\left(H^2\left(S_{h,T}\right)\right)^3}\right),$$

where the number $C_7=C_7\left(\Omega,R,T,c_2,B_3,h_0\right)>0$ depends only on listed parameters. Problem has at most one solution $\left(V,\beta,\gamma\right)\in\left(H^2\left(\Omega_{h,T}\right)\right)^3\times C\left(\overline{\Omega}\right)\times C\left(\overline{\Omega}\right)$.

9 Convexification Method for Problem (8.14)-(8.18)

9.1 The convexification functional

To numerically solve the problem (8.14)-(8.18), the convexification method constructs a globally strongly cost functional for finding the vector function $V\left(x,t_s\right)$. We want to be close to our previous numerical paper about this subject [20]. Recall that in [20] a different version of the convexification was constructed and numerically tested for the case $t_0 \neq 0$. Hence, it is convenient now to replace the domain Ω in (5.3) with the square

$$\Omega = \{x = (x_1, x_2) : a < x_1, x_2 < b\}, \tag{9.1}$$

where $0 < a < b$ are some numbers, which is similar with formula (2.1) of [20].

The next question is about the choice of the CWF. The CWF (5.4), (5.5) is good for the Carleman estimate for a general elliptic operator L_0 in (3.8) with variable coefficients. However, due to its dependence on two large parameters $\lambda, \nu \geq 1$, this function changes too rapidly, which is inconvenient for computations. The same was also observed in [4, 5, 6] for the second generation of the convexification method for some CIPs for hyperbolic PDEs. Since the principal part of each elliptic operator in equations (8.14)-(8.17) is the Laplace operator, then a simpler CWF depending only on one large parameter can work. Thus, keeping in mind a possible future numerical implementation of the method introduced below, we use below a simpler CWF. This CWF as well as the subspace $H_0^2(\Omega) \subset H^2(\Omega)$ are:

$$\begin{gathered} \varphi_\lambda(x) = e^{2\lambda x_1^2}, \\ H_0^2(\Omega) = \{u \in H^2(\Omega) : u\mid_{\partial\Omega} = \partial_l u\mid_{\partial\Omega} = 0\}. \end{gathered} \tag{9.2}$$

Theorem 9.1 (Carleman estimate [**?**, Theorem 4.1], [18, Theorem 8.4.1])**.** *Let the domain Ω be as in (9.1) and the function $\varphi_\lambda(x)$ be as in (9.2). Then there exists a sufficiently large number $\lambda_0 = \lambda_0(\Omega) \geq 1$ and a number $C = C(\Omega) > 0$, both numbers depending only on Ω, such that the following Carleman estimate holds:*

$$\begin{gathered} \int_\Omega (\Delta u)^2 \varphi_\lambda(x)\, dx \geq (C/\lambda) \int_\Omega \left(\sum_{i,j=1}^2 u_{x_i x_j}^2 \right) \varphi_\lambda(x)\, dx + \\ + C\lambda \int_\Omega \left(|\nabla u|^2 + u^2 \right) \varphi_\lambda(x)\, dx,\ \forall \lambda \geq \lambda_0,\ \forall u \in H_0^2(\Omega). \end{gathered} \tag{9.3}$$

To avoid the use of the penalty regularization term, we now consider only the case $V \in (H^2(\Omega_{h,T}))^3$ instead of $V \in \left(C^4\left(\overline{\Omega}_{h,T}\right)\right)^3$ in (8.13). Embedding theorem implies:

$$\|V\|_{\left(C\left(\overline{\Omega}_{h,T}\right)\right)^3} \leq K \|V\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3},\ \forall V \in \left(H^2(\Omega_{h,T})\right)^3, \tag{9.4}$$

where the number $K = K(\Omega, h_0, T) > 0$ depends only on listed parameters. Let $R > 0$ be the number in (8.19), (8.20). Introduce the set of functions $B(R)$ as:

$$B(R) = \left\{ \begin{array}{c} V \in (H^2(\Omega_{h,T}))^3 : \ \|V\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} < R, \\ \text{boundary conditions (8.17) hold.} \end{array} \right\}. \tag{9.5}$$

Let $A_j(V)(x, t_s)$, $j = 1, 2, 3$ be the nonlinear integral differential operators in the left hand sides of equations (8.14)-(8.16). We consider

Minimization Problem. *Minimize the functional $J_\lambda(V)$ on the set $\overline{B(R)}$, where*

$$J_\lambda(V) = \int_\Omega \left[\sum_{j=1}^3 \sum_{s=0}^k (A_j(V)(x, t_s))^2 \right] \varphi_\lambda(x)\, dx. \tag{9.6}$$

Note that the penalty regularization term is not a part of this functional, unlike our previous works on the convexification for CIPs for parabolic PDEs [17, 20], [18, section 9.3.2]. Also, theorems formulated below claim that all required properties of $J_\lambda(V)$ take place only for sufficiently large values of the parameter λ. This seems to be inconvenient since the function $\varphi_\lambda(x)$ changes too rapidly then. Nevertheless, a rich computational experience of all above cited publications on the convexification shows that reasonable values $\lambda \in [1,5]$ are sufficient for obtaining accurate reconstructions of unknown coefficients. This can be explained by an analogy with any asymptotic theory. Indeed, such a theory typically states that if a certain parameter X is sufficiently large, then a certain formula Y is valid with a good accuracy. However, in specific numerical studies only computational results can typically show which exactly values of X are sufficiently large to ensure a good accuracy of Y. And quite often these values are reasonable ones. More precisely, in [20] the optimal value $\lambda = 3$. On the other hand, Figures 1,2 of [20] indicate that if $\lambda = 0$, i.e. in the case when the CWF $\varphi_0(x) \equiv 1$, then the computational results might likely be unsatisfactory.

9.2 Theorems

Denote $[,]$ the scalar product in $\left(H_0^2(\Omega)\right)^3$.

Theorem 9.2. *Let the initial conditions* $f_S, f_I, f_R \in C^2\left(\overline{\Omega}\right)$. *Let (1.2), (8.10) and (8.20) hold with certain constants* $h_0, c_2, M_3 > 0$. *Let the domain* Ω *be as in (9.1) and let* $\lambda_0 \geq 1$ *be the number of Theorem 9.1. Then:*

1. At each point $V \in \overline{B(R)}$, *the functional* $J_\lambda(V)$ *has the Fréchet derivative* $J'_\lambda(V) \in \left(H_0^2(\Omega_{h,T})\right)^3$. *Furthermore, this derivative is Lipschitz continuous, i.e.*

$$\left\|J'_\lambda(V_1) - J'_\lambda(V_2)\right\|_{\left(H^2(\Omega_{h,T})\right)^3} \leq M_3 \left\|V_1 - V_2\right\|_{\left(H^2(\Omega_{h,T})\right)^3}, \ \forall V_1, V_2 \in \overline{B(R)} \tag{9.7}$$

with a certain constant $M_3 = M_3(\lambda, \Omega, R, T, h_0) > 0$ *depending only on listed parameters.*

2. There exists a sufficiently large number $\lambda_1 = \lambda_1(\Omega, R, T, c_2, h_0) \geq \lambda_0$ *such that the functional* $J_\lambda(V)$ *is strongly convex on the set* $\overline{B(R)}$ *for all* $\lambda \geq \lambda_1$, *i.e. the following inequality holds*

$$\begin{gathered} J_\lambda(V_2) - J_\lambda(V_1) - \left[J'_\lambda(V_1), V_2 - V_1\right] \geq C_8 e^{2\lambda a^2} \left\|V_2 - V_1\right\|^2_{\left(H^2(\Omega_{h,T})\right)^3}, \\ \forall \lambda \geq \lambda_1, \ \forall V_1, V_2 \in \overline{B(R)}, \end{gathered} \tag{9.8}$$

where the number $C_8 = C_8(\Omega, R, T, c_2, h_0) > 0$ *is independent on* V_1, V_2. *Both numbers* λ_1 *and* C_8 *depend only on listed parameters.*

3. For each $\lambda \geq \lambda_1$ *there exists unique minimizer* $V_{\lambda,\min} \in \overline{B(R)}$ *of the functional* $J_\lambda(V)$ *on the set* $\overline{B(R)}$ *and the following inequality holds:*

$$\left[J'_\lambda(V_{\lambda,\min}), V - V_{\lambda,\min}\right] \geq 0, \ \forall V \in \overline{B(R)}. \tag{9.9}$$

Below $C_8 = C_8(\Omega, R, T, c_2, h_0) > 0$ denotes different numbers depending on the listed parameters. In practice, the boundary data (8.17) are always given with noise of a level $\delta \in (0,1)$. One of the main assumptions of the regularization theory is the assumption of the existence of the true solution for the ideal, noiseless data [30]. The vector function $V_{\lambda,\min}$ is called then the "regularized solution". The question of the $\delta-$dependent estimations of the accuracy of the regularized solutions is an important topic of the regularization theory. Theorem 9.3 addresses this question for our case.

Let the true solution of problem (8.14)-(8.18) be $(V^*, \beta^*, \gamma^*) \in (H^2(\Omega_{h,T}))^3 \times C(\overline{\Omega}) \times C(\overline{\Omega})$. The boundary data for V^* are vector functions $Q^*$and Z^*, which are full analogs of the boundary data (8.17). Thus,

$$V^* \mid_{S_{h,T}} = Q^*, \ \partial_l V^* \mid_{S_{h,T}} = Z^*. \tag{9.10}$$

Let $B^*(R)$ be the full analog of the set $B(R)$ in (9.5), in which, however, boundary conditions (8.17) are replaced with right hand sides of (9.10). Assume that there exist vector functions W, W^* such that

$$\begin{aligned} & W \in B(R), W^* \in B^*(R), \\ & \|W - W^*\|_{(H^2(\Omega_{h,T}))^3} < \delta. \end{aligned} \tag{9.11}$$

Theorem 9.3 (the accuracy of the regularized solution $V_{\lambda,\min}$). *Let*

$\lambda_1 = \lambda_1(\Omega, R, T, c_2, h_0) \geq 1$ be the number, which was found in Theorem 9.1. Assume that conditions (9.10), (9.11) hold. In the parameters for λ_1 replace R with $2R$ and let λ_2 be such that $\lambda_2 = \lambda_1(\Omega, 2R, T, c_2, h_0) \geq \lambda_1(\Omega, R, T, c_2, h_0) \geq 1$. For any $\lambda \geq \lambda_2$ choose the level of noise in the data $\delta(\lambda)$ so small

$$\left(\exp\left(\lambda\left(b^2 - a^2\right)\right) + 1\right)\delta(\lambda) = \widetilde{\delta}(\lambda) < R. \tag{9.12}$$

Let

$$V^* \in B^*\left(R - \widetilde{\delta}(\lambda)\right). \tag{9.13}$$

Fix a number $\lambda \geq \lambda_2$. Let $V_{\lambda,\min} \in \overline{B(R)}$ be the unique minimizer of the functional $J_\lambda(V)$ on this set, which was found in Theorem 9.2. Then the following accuracy estimates hold:

$$\begin{aligned} \|V_{\lambda,\min} - V^*\|_{(H^2(\Omega_{h,T}))^3} &\leq (1 + \exp(\lambda(b^2 - a^2)))\,\delta, \ \forall \delta \in \left(0, \widetilde{\delta}(\lambda)\right), \\ \left\|\beta_{\lambda,\min} - \beta^*\right\|_{L_2(\Omega)} + \left\|\gamma_{\lambda,\min} - \gamma^*\right\|_{L_2(\Omega)} &\leq C_8 \delta, \ \forall \delta \in \left(0, \widetilde{\delta}(\lambda)\right), \end{aligned} \tag{9.14}$$

where functions $\beta_{\lambda,\min}(x)$ and $\gamma_{\lambda,\min}(x)$ are found from the first and third components respectively of the vector function $V_{\lambda,\min}$ via full analogs of formulas (8.18).

Let $\lambda \geq \lambda_2$ and let the number $\widetilde{\delta}(\lambda)$ be the one defined in (9.12). Let $\widetilde{\delta}(\lambda)$ be so small that

$$\widetilde{\delta}(\lambda) \in \left(0, \frac{R}{3}\right). \tag{9.15}$$

Let

$$V_0 \in B\left(\frac{R}{3} - \widetilde{\delta}(\lambda)\right) \tag{9.16}$$

be an arbitrary point. The gradient descent method of the minimization of the functional $J_\lambda(V)$ constructs the following sequence:

$$V_n = V_{n-1} - \omega J'_\lambda(V_{n-1}), \ n = 1, 2, ..., \tag{9.17}$$

where the step size $\omega > 0$. Note that since by Theorem 9.1 $J'_\lambda(V_{n-1}) \in (H_0^2(\Omega_{h,T}))^3$, then the second line of (9.2) implies that all iterates V_n have the same boundary conditions (8.17).

Theorem 9.4 (global convergence of the method (9.17)). *Let $\lambda \geq \lambda_2$. Assume that conditions (9.10)-(9.12), (9.15) and (9.16) hold. Let (9.13) be replaced with*

$$V^* \in B^* \left(\frac{R}{3} - \widetilde{\delta} (\lambda) \right). \tag{9.18}$$

Then there exists a sufficiently small number $\omega_0 \in (0,1)$ such that for any $\omega \in (0, \omega_0)$ there exists *a number $\theta = \theta(\omega) \in (0,1)$ such that all iterates $V_n \in B(R)$ and the following convergence estimate for the sequence (9.17) is valid:*

$$\begin{gathered} \|V_n - V^*\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} + \|\beta_n - \beta^*\|_{L_2(\Omega)} + \|\gamma_n - \gamma^*\|_{L_2(\Omega)} \leq \\ \leq \left(1 + \exp\left(\lambda\left(b^2 - a^2\right)\right)\right)\delta + C_8 \theta^n \|V_{\lambda,\min} - V_0\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}, \ \forall \delta \in \left(0, \widetilde{\delta}(\lambda)\right). \end{gathered} \tag{9.19}$$

Remark 9.1. *We call a numerical method for a CIP globally convergent, if there is a theorem, which claims that it delivers at least one point in a sufficiently small neighborhood of the true solution without any advanced knowledge of this neighborhood. Thus, since $R > 0$ is an arbitrary number and since V_0 is an arbitrary point in (9.16), then (9.18) and (9.19) implies the global convergence of the sequence (9.17).*

9.3 Proof of Theorem 9.2

Let $V_1, V_2 \in \overline{B(R)}$ be two arbitrary vector functions and

$$\widehat{V} = V_2 - V_1. \tag{9.20}$$

Then triangle inequality, (9.2) and (9.5) imply:

$$\widehat{V} \in \overline{B_0(2R)} = \left\{ V \in \left(H_0^2(\Omega_{h,T})\right)^3 : \|V\| \leq 2R \right\}. \tag{9.21}$$

It follows from (9.6) that to prove (9.8), we need to analyze the differences

$$(A_j(V_2))^2 - (A_j(V_1))^2 = \left(A_j\left(V_1 + \widehat{V}\right)\right)^2 - (A_j(V_1))^2, \ j = 1,2,3, \tag{9.22}$$

where the operators A_j are defined in (8.14)-(8.16). Since the operator A_3 has the simplest form out of these three, then, to save space, we consider only the case of A_3. Two other cases are completely similar. By (8.16)

$$\left(A_3\left(V_1 + \widehat{V}\right)\right)^2 - (A_3(V_1))^2 = A_{3,\text{lin}}\left(\widehat{V}\right) + A_{3,\text{nonlin}}\left(\widehat{V}\right), \tag{9.23}$$

where $A_{3,\text{lin}}\left(\widehat{V}\right)$ and $A_{3,\text{nonlin}}\left(\widehat{V}\right)$ are linear and nonlinear operators respectively with respect to $\widehat{V}$. Let $V_1 = (v_{1,1}, v_{2,1}, v_{3,1})$ and $\widehat{V} = (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)$. The explicit form of $A_{3,\text{lin}}$ is:

$$\begin{gathered} A_{3,\text{lin}}\left(\widehat{V}\right)(x,t_s) = 2A_{3,\text{lin}}^{(1)}\left(\widehat{V}\right)(x,t_s) \cdot A_3(V_1)(x,t_s), \\ A_{3,\text{lin}}^{(1)}\left(\widehat{V}\right)(x,t_s) = \left(\partial_{h,t}\widehat{v}_3(x,t_s) - \Delta\widehat{v}_3(x,t_s) + \operatorname{div}\left(\widehat{v}_3(x,t_s)\, q_R(x)\right)\right) - \\ - (f_I(x))^{-1}\left[\widehat{v}_3(x,t_s) - \left(\int_0^{t_s} \partial_{h,t}\widehat{v}_3(x,\tau)\,d\tau\right)_h\right] v_{2,1}(x,t_s) - \\ - (f_I(x))^{-1}\left[v_{3,1}(x,t_s) - \left(\int_0^{t_s} \partial_{h,t} v_{3,1}(x,\tau)\,d\tau\right)_h\right] \widehat{v}_2(x,t_s) \\ - (f_I(x))^{-1}\left(-\Delta f_R(x) + \operatorname{div}\left(f_R(x)\, q_R(x)\right)\right)\widehat{v}_2(x,t_s), \ x \in \Omega, \ t_s \in Y. \end{gathered} \tag{9.24}$$

The explicit form of $A_{3,\text{nonlin}}\left(\widehat{V}\right)(x,t_s)$ is:

$$A_{3,\text{nonlin}}\left(\widehat{V}\right)(x,t_s)=\left[A_{3,\text{lin}}^{(1)}\left(\widehat{V}\right)\right]^2(x,t_s),\ x\in\Omega,\ t_s\in Y. \tag{9.25}$$

Since formulas, similar with ones of (9.23)-(9.25), are also valid for $j=1,2$ in (9.22), then (9.6) leads to:

$$\begin{gathered}J_\lambda\left(V_1+\widehat{V}\right)-J_\lambda(V_1)=\int_\Omega\left[\sum_{j=1}^3\sum_{s=0}^k A_{j,\text{lin}}\left(\widehat{V}\right)(x,t_s)\right]\varphi_\lambda(x)\,dx+\\+\int_\Omega\sum_{j=1}^3\sum_{s=0}^k\left[A_{j,\text{lin}}^{(1)}\left(\widehat{V}\right)\right]^2(x,t_s)\,\varphi_\lambda(x)\,dx,\end{gathered} \tag{9.26}$$

where $A_{j,\text{lin}}\left(\widehat{V}\right)(x,t_s)$ for $j=1,2$ depend linearly on $\widehat{V}$, and they are similar with $A_{3,\text{lin}}\left(\widehat{V}\right)(x,t_s)$. Also, $\left[A_{j,\text{lin}}^{(1)}\left(\widehat{V}\right)\right]^2(x)$ for $j=1,2$ are similar with (9.25).

Since $\widehat{V}\in\left(H_0^2(\Omega_{h,T})\right)^3$ by (9.21), then the integral term in the first line of (9.26) is a bounded linear functional $\widetilde{J}_\lambda\left(\widehat{V}\right):\left(H_0^2(\Omega_{h,T})\right)^3\to\mathbb{R}$. Next it follows from (9.26) that

$$\lim_{\|\widehat{V}\|_{\left(H^2(\Omega_{h,T})\right)^3}}\frac{J_\lambda\left(V_1+\widehat{V}\right)-J_\lambda(V_1)-\widetilde{J}_\lambda\left(\widehat{V}\right)}{\left\|\widehat{V}\right\|_{\left(H^2(\Omega_{h,T})\right)^3}}=0.$$

Hence, $\widetilde{J}_\lambda\left(\widehat{V}\right)$ is the Fréchet derivative of the functional $J_\lambda(V)$ at the point V_1. By Riesz theorem, there exists a point $J'_\lambda(V_1)\in\left(H_0^2(\Omega_{h,T})\right)^3$ such that $\widetilde{J}_\lambda\left(\widehat{V}\right)=\left[J'_\lambda(V_1),\widehat{V}\right]$ for all $\widehat{V}\in\left(H_0^2(\Omega_{h,T})\right)^3$. Thus, the existence of the Fréchet derivative of the functional $J_\lambda(V)$ is established. Its Lipschitz continuity property (9.7) can be proven similarly with [2, Theorem 3.1] and [18, Theorem 5.3.1]. Thus, we omit the proof of (9.7).

Note that until now we have not used the Carleman estimate of Theorem 9.1. Now, however, we focus on the proof of the strong convexity property (9.8) and use, therefore, Theorem 9.1. Using (2.10)-(2.14), (8.19), Cauchy-Schwarz inequality, (9.24), (9.25) as well as obvious analogs of the two latter formulas for $A_{j,\text{nonlin}}\left(\widehat{V}\right)(x,t_s)$, $j=1,2$, we obtain

$$\begin{gathered}\left[A_{j,\text{lin}}^{(1)}\left(\widehat{V}\right)\right]^2(x,t_s)\geq(1/2)\left(\Delta\widehat{v}_j\right)^2(x,t_s)-C_8\sum_{m=1}^3\sum_{i=0}^k\left(\left|\nabla\widehat{v}_m\right|^2+\widehat{v}_m^2\right)(x,t_i),\\ j=1,2,3;\ x\in\Omega;\ t_i,t_s\in Y.\end{gathered} \tag{9.27}$$

Hence, using (9.26) and (9.27), we obtain

$$\begin{gathered}J_\lambda\left(V_1+\widehat{V}\right)-J_\lambda(V_1)-\left[J'_\lambda(V_1),\widehat{V}\right]\geq\\ \geq(1/2)\int_\Omega\sum_{j=1}^3\sum_{s=0}^k\left(\Delta\widehat{v}_j\right)^2(x,t_s)\,\varphi_\lambda(x)\,dx-C_8\int_\Omega\sum_{j=1}^3\sum_{s=0}^k\left(\left|\nabla\widehat{v}_j\right|^2+\widehat{v}_j^2\right)(x,t_s)\,\varphi_\lambda(x)\,dx.\end{gathered}$$

Applying Carleman estimate (9.3) to the second line of this inequality, we obtain

$$J_\lambda\left(V_1+\widehat{V}\right)-J_\lambda\left(V_1\right)-\left[J_\lambda'\left(V_1\right),\widehat{V}\right]\geq$$
$$\geq (C/\lambda)\int\limits_\Omega \sum_{m,r=1}^{2}\sum_{j=1}^{3}\sum_{s=0}^{k}\left(\left(\widehat{v}_j\right)_{x_m x_r}\right)^2(x,t_s)\,\varphi_\lambda(x)\,dx+$$
$$+C\lambda\int\limits_\Omega \sum_{j=1}^{3}\sum_{s=0}^{k}\left[\left(\nabla\widehat{v}_j\right)^2(x,t_s)+\lambda^2\nabla\widehat{v}_j^2(x,t_s)\right]\varphi_\lambda(x)\,dx-$$
$$-C_8\int\limits_\Omega \sum_{j=1}^{3}\sum_{s=0}^{k}\left(\left|\nabla\widehat{v}_j\right|^2+\widehat{v}_j^2\right)(x,t_s)\,\varphi_\lambda(x)\,dx,\ \forall\lambda\geq\lambda_0.$$

Hence, there exists a sufficiently large number $\lambda_1=\lambda_1\left(\Omega,T,M_3,c_2,h_0\right)\geq\lambda_0$ such that

$$J_\lambda\left(V_1+\widehat{V}\right)-J_\lambda\left(V_1\right)-\left[J_\lambda'\left(V_1\right),\widehat{V}\right]\geq$$
$$\geq C_8\int\limits_\Omega\left[\sum_{j=1}^{3}\sum_{s=0}^{k}\left(\sum_{m,r=1}^{2}\left(\left(\widehat{v}_j\right)_{x_m x_r}\right)^2+\left|\nabla\widehat{v}_j\right|^2+\widehat{v}_j^2\right)(x,t_s)\right]\varphi_\lambda(x)\,dx, \tag{9.28}$$
$$\forall\lambda\geq\lambda_1.$$

Since by (9.1) and (9.2) $\varphi_\lambda(x)\geq e^{2\lambda a^2}$ in Ω, then (9.28), (2.6) and (9.20) imply (9.8). Finally, we omit proofs of the existence and uniqueness of the minimizer $V_{\lambda,\min}\in\overline{B(R)}$ as well as of inequality (9.9) since these statements follow from a combination of (9.7) and (9.8) with two results of [2]: Lemma 2.1 and Theorem 2.1. □

9.4 Proof of Theorem 9.3

Let $V\in\overline{B(R)}$ be an arbitrary vector function. Denote

$$\overline{V}=V-W,\ \overline{V}^*=V^*-W^*. \tag{9.29}$$

Using the first line of (9.11), (9.13), (9.21) and (9.29), we obtain

$$\overline{V},\overline{V}^*\in\overline{B_0(2R)}. \tag{9.30}$$

Consider the functional I_λ,

$$I_\lambda:\overline{B_0(2R)}\to\mathbb{R},\ I_\lambda\left(\overline{V}\right)=J_\lambda\left(\overline{V}+W\right). \tag{9.31}$$

Then the full analog of Theorem 9.2 is valid for the functional $I_\lambda\left(\overline{V}\right)$ for all $\lambda\geq\lambda_2$. Let $\overline{V}_{\lambda,\min}\in\overline{B_0(2R)}$ be the unique minimizer of $I_\lambda\left(\overline{V}\right)$ on the set $\overline{B_0(2R)}$, which was found in that analog. Then by the analog of (9.9)

$$\left[I_\lambda'\left(\overline{V}_{\lambda,\min}\right),\overline{V}-\overline{V}_{\lambda,\min}\right]\geq 0,\ \forall\overline{V}\in\overline{B_0(2R)},\ \forall\lambda\geq\lambda_2. \tag{9.32}$$

By the analog of (9.8), (9.29) and (9.30)

$$\begin{gathered}I_\lambda\left(\overline{V}^*\right)-I_\lambda\left(\overline{V}_{\lambda,\min}\right)-\left[I_\lambda'\left(\overline{V}_{\lambda,\min}\right),\overline{V}^*-\overline{V}_{\lambda,\min}\right]\geq\\ \geq C_8e^{2\lambda a^2}\left\|\overline{V}^*-\overline{V}_{\lambda,\min}\right\|^2_{\left(H^2\left(\Omega_{h,T}\right)\right)^3},\ \forall\lambda\geq\lambda_2.\end{gathered} \tag{9.33}$$

Using (9.32), we obtain $-\left[I'_\lambda\left(\overline{V}_{\lambda,\min}\right),\overline{V}^*-\overline{V}_{\lambda,\min}\right]\leq 0$. Next, $-I_\lambda\left(\overline{V}_{\lambda,\min}\right)\leq 0$. Hence, (9.33) implies

$$\left\|\overline{V}^*-\overline{V}_{\lambda,\min}\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}\leq \frac{e^{-\lambda a^2}}{\sqrt{C_8}}\sqrt{I_\lambda\left(\overline{V}^*\right)}. \tag{9.34}$$

Estimate now $I_\lambda\left(\overline{V}^*\right)$. By (8.14)-(8.16), (9.1), (9.2), (9.11), (9.29) and (9.31)

$$\begin{gathered} I_\lambda\left(\overline{V}^*\right)=J_\lambda\left(\overline{V}^*+W\right)=J_\lambda\left(\left(\overline{V}^*+W^*\right)+\left(W-W^*\right)\right)\leq \\ \leq 2J_\lambda\left(V^*\right)+C_8\delta^2e^{2\lambda b^2}. \end{gathered} \tag{9.35}$$

However, $J_\lambda\left(V^*\right)=0$. Hence, using (9.34) and (9.35), we obtain

$$\left\|\overline{V}^*-\overline{V}_{\lambda,\min}\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}\leq e^{\lambda\left(b^2-a^2\right)}\delta,\ \forall\lambda\geq\lambda_2.$$

Hence, using triangle inequality, (9.11) and (9.29), we obtain

$$\left\|\left(\overline{V}_{\lambda,\min}+W\right)-V^*\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}\leq\left(e^{\lambda\left(b^2-a^2\right)}+1\right)\delta,\ \forall\lambda\geq\lambda_2. \tag{9.36}$$

Let $\widetilde{\delta}\left(\lambda\right)$ be the number defined in (9.12) and let $\delta\in\left(0,\widetilde{\delta}\left(\lambda\right)\right)$. Then it follows from (9.13), (9.36) and triangle inequality that

$$\left(\overline{V}_{\lambda,\min}+W\right)\in B\left(R\right). \tag{9.37}$$

Let $V_{\lambda,\min}\in\overline{B\left(R\right)}$ be the unique minimizer of the functional $J_\lambda\left(V\right)$ on the set $\overline{B\left(R\right)}$, which was found in Theorem 9.2. Then by (9.31) and (9.37)

$$J_\lambda\left(V_{\lambda,\min}\right)\leq J_\lambda\left(\overline{V}_{\lambda,\min}+W\right)=I_\lambda\left(\overline{V}_{\lambda,\min}\right).$$

This inequality is equivalent with

$$J_\lambda\left(V_{\lambda,\min}\right)=J_\lambda\left(\left(V_{\lambda,\min}-W\right)+W\right)=I_\lambda\left(V_{\lambda,\min}-W\right)\leq I_\lambda\left(\overline{V}_{\lambda,\min}\right). \tag{9.38}$$

Since $\left(V_{\lambda,\min}-W\right)\in\overline{B_0\left(2R\right)}$, then (9.38) implies that $\left(V_{\lambda,\min}-W\right)$ is another minimizer, in addition to $\overline{V}_{\lambda,\min}$, of the functional $I_\lambda\left(\overline{V}\right)$ on the set $\overline{B_0\left(2R\right)}$. However, since such a minimizer is unique, then $V_{\lambda,\min}-W=\overline{V}_{\lambda,\min}$. Hence, $V_{\lambda,\min}=\overline{V}_{\lambda,\min}+W$. This and (9.36) apply the estimate in the first line of (9.14) immediately. The second estimate (9.14) easily follows from a combination of the first one with full analogs of formulas (8.18). $\square$

9.5 Proof of Theorem 9.4

Recall that $\lambda\geq\lambda_2$. Let again $V_{\lambda,\min}\in\overline{B\left(R\right)}$ be the unique minimizer of the functional $J_\lambda\left(V\right)$ on this set, which was found in Theorem 9.2. By (9.12), the first line of (9.14), (9.18) and triangle inequality

$$\left\|V_{\lambda,\min}\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}\leq\left\|V^*\right\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}+\widetilde{\delta}\left(\lambda\right)<\frac{R}{3},\ \forall\delta\in\left(0,\widetilde{\delta}\left(\lambda\right)\right).$$

Hence, by (9.5) $V_{\lambda,\min} \in B\left(R/3\right)$. Hence, applying Theorem 6 of [19], we obtain the existence of a sufficiently small number $\omega_0 \in (0,1)$ such that for any $\omega \in (0,\omega_0)$ there exists a number $\theta = \theta(\omega) \in (0,1)$ such that $V_n \in B(R)$ for all $n = 0, 1, ...$ and also

$$\|V_{\lambda,\min} - V_n\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} \leq C_8 \theta^n \|V_{\lambda,\min} - V_0\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}, \ n = 1, ..., \tag{9.39}$$

for all $\delta \in \left(0, \widetilde{\delta}(\lambda)\right)$. Next, using the first line of (9.14), (9.39) and triangle inequality, we obtain

$$\begin{gathered}\|V_n - V^*\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3} \leq \left(1 + \exp\left(\lambda\left(b^2 - a^2\right)\right)\right)\delta + \\ +C_8 \theta^n \|V_{\lambda,\min} - V_0\|_{\left(H^2\left(\Omega_{h,T}\right)\right)^3}, \ \forall \delta \in \left(0, \widetilde{\delta}(\lambda)\right), \ n = 1, ...\end{gathered} \tag{9.40}$$

The similar estimate for $\|\beta_n - \beta^*\|_{L_2(\Omega)} + \|\gamma_n - \gamma^*\|_{L_2(\Omega)}$ follows immediately from (9.40) and full analogs of formulas (8.18). $\square$

References

[1] O.M. Alifanov, *Inverse Heat Conduction Problems*, Springer, New York, 1994.

[2] A. B. Bakushinskii, M. V. Klibanov, and N. A. Koshev, Carleman weight functions for a globally convergent numerical method for ill-posed Cauchy problems for some quasilinear PDEs, *Nonlinear Analysis: Real World Applications*, 34, 201–224, 2017.

[3] L. Beilina, Domain decomposition finite element/finite difference method for the conductivity reconstruction in a hyperbolic equation, *Communications in Nonlinear Science and Numerical Simulation*, 37, 222-237, 2016.

[4] L. Baudouin, M. de Buhan and S. Ervedoza, Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation, *SIAM J. Numer. Anal.* 55, 1578–1613, 2017.

[5] L. Baudouin, M. de Buhan, S. Ervedoza, and A. Osses, Carleman-based reconstruction algorithm for waves, *SIAM J. Numer. Anal.*, 59, 998-1039, 2021.

[6] L. Baudouin, M. de Buhan, E. Crépeau and J. Valein, Carleman-based reconstruction Algorithm on a wave network, available online, hal-04361363, 2023.

[7] L. Bourgeois, Quantification of the unique continuation property for the heat equation, *Mathematical Control and Related Fields*, 7, 347-367, 2017.

[8] A. L. Bukhgeim and M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, *Sov. Math. Doklady*, 17, 244–247, 1981.

[9] B.B. Das, F. Liu and R. R. Alfano, Time-resolved fluorescence and photon migration studies in biomedical and model random media, *Reports on Progress in Physics*, 60, 227-292, 1997.

[10] A.V. Goncharsky, S.Y. Romanov and S.Y. Seryozhnikov, On mathematical problems of two-coefficient inverse problems of ultrasonic tomography, *Inverse Problems*, 40, 045026, 2024

[11] L. Hörmander, *Linear Partial Differential Operators*, Springer Verlag, 1963.

[12] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, 2006.

[13] W. O. Kermack and A. G. McKendrick. A contribution to the mathematical theory of epidemics. *Proc. Roy. Soc. London Ser. A*, 115, 700–721, 1927.

[14] M.V. Klibanov, Global convexity in a three-dimensional inverse acoustic problem, *SIAM J. Math Anal.*, 28 (1997), pp. 1371–1388.

[15] M.V. Klibanov, Estimates of initial conditions of parabolic equations and inequalities via lateral Cauchy data, *Inverse Problems*, 22, 495–514, 2006.

[16] Klibanov M.V., Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, *J. of Inverse and Ill-Posed Problems*, 21, 477-510, 2013.

[17] M.V. Klibanov, J. Li and W. Zhang, Convexification for an inverse parabolic problem, *Inverse Problems*, 36, 085008, 2020.

[18] Klibanov M.V. and Li J., *Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data*, De Gruyter, 2021.

[19] M. V. Klibanov, V. A. Khoa, A. V. Smirnov, L. H. Nguyen, G. W. Bidney, L. Nguyen, A. Sullivan, and V. N. Astratov, Convexification inversion method for nonlinear SAR imaging with experimentally collected data, *J. Appl. Industrial Math.*, 15, 413–436, 2021.

[20] M.V. Klibanov, J. Li and Z. Yang, Spatiotemporal monitoring of epidemics via solution of a coefficient inverse problem, *arXiv*: 2401.02070, 2024.

[21] O. A. Ladyzhenskaya, Solution of the first boundary problem in the large for quasilinear parabolic equations, *Trudy Moscow Math. Obs.*, 7, 149–177, 1958.

[22] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, vol. 23, AMS, Providence, R.I., 1968.

[23] O.A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics*, Springer, 1985.

[24] M.M. Lavrentiev, V.G. Romanov and S.P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis*, AMS, Providence, RI, 1986.

[25] W. Lee, S. Liu, H. Tembine, W. Li and S. Osher, Controlling propagation of epidemics via mean-field control, SIAM J. Appl. Math., 81, 190–207, 2021.

[26] R. G. Novikov, The $\overline{\partial}$-approach to approximate inverse scattering at fixed energy in three dimensions, *International Math. Research Peports*, 6, 287–349, 2005.

[27] G. Rizutti and A. Gisolf, An iterative method for 2D inverse scattering problems by alternating reconstruction of medium properties and wavelets: theory and application to the inversion of elastic waveforms, *Inverse Problems*, 33, 035003, 2017.

[28] V.G. Romanov, *Inverse Probems of Mathematical Physics*, VNU Press, Utrecht, The Netherlands, 1986.

[29] E. Rothe, Wärmeleitungsgleichung mit nichtconstanten koeffizinten, *Math. Ann.*, 104, 340-362, 1930.

[30] A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov, and A. G. Yagola, *Numerical Methods for the Solutions of Ill-posed Problems*, Kluwer, London, 1995.

[31] A. Timonov, Numerical solution of a regularized weighted mean curvature flow problem for electrical conductivity imaging, *SIAM J.Sci.Comput.*, 41, B1137-B1154, 2019.

[32] T. D. Ventzel, The first boundary problem and the Cauchy problem for quasi-linear parabolic equations with several space variables, *Math. Sb. (N. S.),* 41, 499–520, 1957.

[33] S. Vessella, Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates, *Inverse Problems*, 24, 023001, 2008.

[34] M. Yamamoto, Carleman estimates for parabolic equations. Topical Review, *Inverse Problems*, 25, 123013, 2009