L^p-REGULARITY OF A GEOMETRICALLY NONLINEAR FLAT COSSERAT MICROPOLAR MODEL IN SUPERCRITICAL DIMENSIONS

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ABSTRACT. In a recent work [Ann. Inst. H. Poincaré C Anal. Non Linéaire 2024], Gastel and Neff introduced an interesting system from a geometrically nonlinear flat cosserat micropolar model and established interior regularity in the critical dimension. Motived by this work, in this article, we establish both interior regularity and sharp L^p regularity for their system in supercritical dimensions.

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1. INTRODUCTION

Motivated by applications to a geometrically nonlinear flat Cosserat shell model in continuum mechanics, Gastel and Neff [7] introduced the following interesting system:

$$\mathrm{Div} S(\nabla m, R) = 0, \tag{1.1}$$

$$\Delta R - \Omega_R \cdot \nabla R - \mathrm{skew}\,(\nabla m \circ S(\nabla m, R))\, R = 0, \tag{1.2}$$

where the unknown functions $(m, R) \in W^{1,2}(B^n, \mathbb{R}^N \times SO(N))$ with $\nabla m, \nabla R \in M^{2,n-2}(B^n)$,

$$\Omega_R = -R\nabla R^T \in M^{2,n-2}(B^n, \mathbb{R}^n \otimes so(N)),$$

$S : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times n}$ is a mapping given in (2.1) and skew $(\nabla m \circ S(\nabla m, R))$ is defined by (2.2). Here, B^n is the unit ball in $\mathbb{R}^n$ and $N > n$.

This system couples a harmonic map type equation (1.2) with a uniformly elliptic equation (1.1). In their main result, Gastel and Neff obtained the following interior regularity result in the planar case, that is, $n = 2$ and $N = 3$.

Theorem A (Theorem 1.1, [7]). Every weak solution $(m, R) \in W^{1,2}(B^2, \mathbb{R}^3 \times SO(3))$ of (1.1)-(1.2) (with $n = 2, N = 3$) is smooth.

As one easily observes, the function Ω_R from (1.2) is the same as that in

$$\Delta R - \Omega_R \cdot \nabla R = 0, \tag{1.3}$$

which models harmonic mappings from B^n to $SO(N) \subset \mathbb{R}^{N\times N}$. In the planar case, that is when $n = 2$, the regularity of harmonic mappings into manifolds was first obtained by Morrey in his seminal work [18] on Plateau's problem in Riemannian manifolds. In particular, he showed that minimizing harmonic mappings are locally Hölder continuous and thus are smooth when the Riemannian metric is smooth. This regularity result was later extended to weakly harmonic mappings by Heléin in his celebrated work; see [12]

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for a comprehensive introduction on it. Partially building on Hélein's idea, in another significant work, Rivère [20] successfully rewrote the harmonic mapping equation (1.3) as a conservation law, from which regularity and compactness follow routinely. An important discovery of Rivière [20] is that the specific form of Ω_R (as that of harmonic mappings) is not really essential. The algebraic anti-symmetry of Ω_R is sufficient for finding the conservation law, based on earlier seminal work of Uhlenbeck [30]. We recommend the interested readers to [21] for a comprehensive exploration of Rivière's conservation law approach.

Unfortunately, because of the appearance of an extra term $\operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R$ in (1.2), the powerful conservation law approach of Heléin and Rivière does not work. The key observation by Gastel and Neff [7] is that this extra term is indeed the product of a gradient term ∇m, a divergence free vector field $S(\nabla m, R)$ and a bounded term R, which makes it enjoy better property than merely being in L^1. Adapting the method of Rivière and Struwe [22], Gastel and Neff succeeded in deriving local Hölder regularity and thus also smoothness of weak solutions to (1.1)-(1.2).

As for the harmonic mapping equation (1.3), partial regularity results were known since the fundamental work of Schoen and Uhlenebck [25], where an important ϵ-regularity was established for minimizing harmonic mappings into manifolds. This work was later extended to the case of stationary harmonic mappings into spheres by Evans [5] and into general manifolds by Bethuel [2]. An alternative (but slightly more general) proof was obtained later by Rivière and Struwe [22], partially based on observations from [20]. It remains, however, open whether one can extend the conservation law approach of Rivière [20] to study (partial) regularity of harmonic mappings in supercritical dimensions; see [10] for some partial progress in this direction. It is then natural to ask whether one can derive partial regularity or even sharp L^p regularity theory for weak solutions of (1.1)-(1.2) in supercritical dimensions. Notice that, in [7, Remark 6.8], Gastel and Neff pointed out that the proof for the critical dimension case does not directly extend to higher dimensions:

It is essential that we are working in the critical dimension $n = 2$ here, $\cdots$, we would not succeed in finding similarly good estimates in the corresponding Morrey spaces.

Note that in the case of (stationary) harmonic mappings, due to the monotonicity formular, the gradient of a harmonic map lies in the Morrey space $M^{2,n-2}$; see for instance [22]. Thus for the latter question above, the natural Morrey spaces for a partial regularity theory would be $\nabla m, \nabla R \in M^{2,n-2}$.

Our main motivation of this paper is to provide an affirmative answer to the above question. Our first main result concerns interior regularity of weak solutions.

Theorem 1.1 (Interior regularity)**.** There exists some $\epsilon = \epsilon(n, N) > 0$ such that if $(m, R) \in W^{1,2}(B^n, \mathbb{R}^N \times SO(N))$ is a weak solution of (1.1)-(1.2) with

$$\|\nabla m\|_{M^{2,n-2}(B^n)} \leq \varepsilon \quad \text{and} \quad \|\nabla R\|_{M^{2,n-2}(B^n)} \leq \varepsilon,$$

then it is smooth in $B_{\frac{1}{2}}$.

It should be noticed that Theorem 1.1 was not the first partial regularity result for weak solutions of the nonlinear flat Cosserat micropolar model. Indeed, Gastel [6] already found interesting connections between Cosserat and harmonic maps and did partial regularity for minimizers in dimension 3. Later, Li and Wang [16] established the partial regularity theory for stationary solutions in dimension 3 using the classical regularity methods for stationary harmonic mappings (other than the method of Rivière and Struwe [22]). The monotonicity formula there implies that for stationary solutions, the Morrey regularity assumption as in Theorem 1.1 is satisfied. As was pointed out in [7], the main difficulty to establish Theorem 1.1 is the (local) Hölder continuity of weak solutions. In the proof of Theorem 1.1, we follow closely the idea of Gaste and Neff [7] and thus relies on ideas of Rivière and Struwe [22], except that we refine some of the estimates using Hardy-BMO inequalities (see Lemma 2.4 below). Once the Hölder continuity were established, smoothness follows routinely.

Motivated by applications in the associated heat flow and energy identity, L^p-regularity theory for harmonic mapping type equations has attracted great interest in the past decades; see for instance [4, 14, 15, 17, 29, 24, 27, 26, 19, 32, 31] and the references therein. Motivated by these works and potential applications, we aim at obtaining optimal L^p-regularity estimates for the corresponding inhomogeneous system:

$$\mathrm{Div} S(\nabla m, R) = 0, \tag{1.4}$$

$$\Delta R - \Omega_R \cdot \nabla R - \mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right) R = f, \tag{1.5}$$

where the inhomogeneous term $f \in L^p(B^n, \mathbb{R}^{N\times N})$.

Our second main result provides optimal L^p-regularity estimates for weak solutions of (1.4)-(1.5).

Theorem 1.2 (L^p-regularity)**.** Suppose $f \in L^p(B^n, \mathbb{R}^{N\times N})$ for some $p \in (\frac{n}{2}, \infty)$. There exists some $\epsilon = \epsilon(n, N) > 0$ such that if $(m, R) \in W^{1,2}(B^n, \mathbb{R}^N \times SO(N))$ is a weak solution of (1.4)-(1.5) with

$$\|\nabla m\|_{M^{2,n-2}(B^n)} \leq \varepsilon \quad \text{and} \quad \|\nabla R\|_{M^{2,n-2}(B^n)} \leq \varepsilon,$$

then $(m, R) \in W^{2,p}(B_{\frac{1}{2}})$. Furthermore, there exists some $C = C(n, N, p) > 0$ such that

$$\|m\|_{W^{2,p}(B_{\frac{1}{2}})} + \|R\|_{W^{2,p}(B_{\frac{1}{2}})} \leq C\left(\|f\|_{L^p(B_1)} + 1\right)^2. \tag{1.6}$$

As an immediate consequence of Theorem 1.2, we know that if $f = 0$, then the solution (m, R) of the system (1.1)-(1.2) belongs to $W^{2,p}_{\mathrm{loc}}(B^n)$ for any $p \in (1, \infty)$.

The idea for the proof of Theorem 1.2 dates back to Sharp and Topping [27], but with extra modifications from the recent works [26, 11, 9]. We will follow closely the presentation by Guo-Wang-Xiang [9] using a finitely iteration method. The extra constant 1 appearing on the right hand side of (1.6) comes from the fact that

$$|\nabla m| - 1 \lesssim |S(\nabla m, R)| \lesssim |\nabla m| + 1$$

and thus it cannot be removed.

This paper is organized as follows. After the introduction, we collect all the necesssary auxillary results in Section 2. In Section 3, we prove Theorem 1.1 and in Section 4, we prove Theorem 1.2.

Our notations are standard. By $A \lesssim B$ we mean there exists a universal constant $C > 0$ such that $A \leq CB$.

2. Preliminaries

2.1. **Operators on matrices.** For a matrix $A = (a_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$, we denote $A = (A_1|\cdots|A_N)$, where A_i are the column vectors. The projection operator $\pi_n : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times n}$ (on the first n columns) is defined by

$$\pi_n(A) := \pi(A_1|\cdots|A_N) = (A_1|\cdots|A_n) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{Nn} \end{pmatrix}_{N\times n}.$$

The operator $\mathbb{P} : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is defined by

$$\mathbb{P}(A) := \sqrt{\mu_1}\mathrm{devsym}(A) + \sqrt{\mu_2}\mathrm{skew}(A) + \sqrt{\kappa}\frac{\mathrm{tr}(A)}{N}\mathbb{1}_N,$$

where μ_1, μ_2 and κ are some positive constants, and we use

$$\mathrm{sym}(A) = \frac{1}{2}(A + A^T), \qquad \mathrm{skew}(A) = \frac{1}{2}(A - A^T)$$

to represent the symmetric and skew-symmetric parts of A, respectively; the first term $\mathrm{devsym}(A)$ in the definition of $\mathbb{P}(A)$ is thus defined as

$$\mathrm{devsym}(A) = \frac{1}{2}(A + A^T) - \frac{\mathrm{tr}(A)}{N}\mathbb{1}_N = \mathrm{sym}(A) - \frac{\mathrm{tr}(A)}{N}\mathbb{1}_N$$

so as to denote the trace free deviatoric part. As a result, there holds

$$A = \mathrm{devsym}(A) + \mathrm{skew}(A) + \frac{\mathrm{tr}(A)}{N}\mathbb{1}_N = \mathrm{sym}(A) + \mathrm{skew}(A),$$

which is an orthonormal decomposition for A.

Finally, we define

$$S(\nabla m, R) := \pi_n\left(2R\mathbb{P}^2\left(R^T(\nabla m|0) - (\mathbb{1}_n|0)\right)\right) \tag{2.1}$$

and introduce an operation $\circ : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times N}$ by

$$A_{N\times n} \circ B_{N\times n} := \frac{1}{2}AB^T \in \mathbb{R}^{N\times N}.$$

Then we have

$$\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right) := \frac{1}{4}\left(\nabla m\left(S(\nabla m, R)\right)^T - S(\nabla m, R)\nabla m^T\right). \tag{2.2}$$

2.2. **Morrey spaces and Riesz operators.** Let $1 \le p < \infty$ and $0 \le s \le n$. The Morrey space $M^{p,s}(U)$ consists of functions $u \in L^p(U)$ such that

$$\|u\|_{M^{p,s}(U)} \equiv \sup_{x\in U, 0<r<\mathrm{diam}(U)} r^{-s/p}\|u\|_{L^p(B_r(x)\cap U)} < \infty.$$

The weak Morrey space $M^{p,s}_*(U)$ consists of functions $f \in L^p_*(U)$ such that

$$\|f\|_{M^{p,s}_*(U)} \equiv \sup_{x\in U, 0<r<\mathrm{diam}(U)} r^{-s/p}\|f\|_{L^p_*(B_r(x)\cap U)} < \infty.$$

The space $M^{p,s}_1(U)$ consists of functions in $M^{p,s}(U)$ whose weak gradient belongs to $M^{p,s}(U)$.

We need the following Hölder's inequalities in Morrey spaces; see [9].

Proposition 2.1. Let $1 \le p_1, p_2 \le \infty$ and $0 \le q_1, q_2 \le n$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1 \quad \text{and} \quad q = \frac{p}{p_1} q_1 + \frac{p}{p_2} q_2.$$

Then, there hold

$$\|fg\|_{M^{p,q}(U)} \le \|f\|_{M^{p_1,q_1}(U)} \|g\|_{M^{p_2,q_2}(U)}. \tag{2.3}$$

and

$$\|fg\|_{M_*^{p,q}(U)} \le \|f\|_{M_*^{p_1,q_1}(U)} \|g\|_{M_*^{p_2,q_2}(U)}. \tag{2.4}$$

Let $I_\alpha(x) = c_{\alpha,n}|x|^{\alpha-n}$, $0 < \alpha < n$, be the standard Riesz potentials in $\mathbb{R}^n$. The following two propositions are well-known; see Theorem 3.1, Proposition 3.2 and Proposition 3.1 of Adams [1].

Proposition 2.2 ([1])**.** Let $0 < \alpha < n$ and $0 \le \lambda < n$. For $1 \le p < (n-\lambda)/\alpha$, $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, we have

(1) For every $1 < p < (n-\lambda)/\alpha$,

$$I_\alpha : M^{p,\lambda}(\mathbb{R}^n) \to M^{\tilde{p},\lambda}(\mathbb{R}^n)$$

is a bounded linear operator.

(2) For $p = 1$,

$$I_\alpha : M^{1,\lambda}(\mathbb{R}^n) \to M_*^{\tilde{p},\lambda}(\mathbb{R}^n)$$

is also a bounded linear operator.

Proposition 2.3 ([1])**.** Let $0 < \alpha < \beta \le n$ and $1 < p < \infty$. There exists a constant $C = C(\alpha, \beta, n, p) > 0$ such that for $f \in M^{1,n-\beta}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, there holds

$$\|I_\alpha f\|_{L^{\frac{p\beta}{\beta-\alpha}}(\mathbb{R}^n)} \le C \|f\|_{M^{1,n-\beta}(\mathbb{R}^n)}^{\frac{\alpha}{\beta}} \|f\|_{L^p(\mathbb{R}^n)}^{1-\frac{\alpha}{\beta}}.$$

2.3. **Hardy-BMO inequality.** As an application of the Hardy-BMO duality and div-curl lemma (see [3, 5, 23]), we have the following Hardy-BMO inequality.

Lemma 2.4 (Hardy-BMO inequality)**.** For any $p \in (1, \infty)$ and $\alpha \in (1, n)$, there exists a constant $C = C(n, p, \alpha) > 0$ such that the following holds:

(1). For all balls $B_r(x_0) \subset \mathbb{R}^n$, and functions $a \in M_1^{\alpha,n-\alpha}(B_{2r}(x_0))$, $\Gamma \in L^q(B_r(x_0), \mathbb{R}^n)$, $b \in W_0^{1,p} \cap L^\infty(B_r(x_0))$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathrm{Div}(\Gamma) = 0$ in the weak sense on $B_r(x_0)$, we have

$$\left| \int_{B_r(x_0)} \langle \nabla a, \Gamma \rangle b dx \right| \le C \, \|\Gamma\|_{L^q(B_r(x_0))} \, \|\nabla b\|_{L^p(B_r(x_0))} \|\nabla a\|_{M^{\alpha,n-\alpha}(B_{2r}(x_0))}.$$

(2). For all balls $B_r(x_0) \subset \mathbb{R}^n$, and functions $\varphi \in C_0^\infty(B_r(x_0))$, $\Gamma \in L^q(B_r(x_0), \mathbb{R}^n)$ and $b \in M_1^{\alpha,n-\alpha} \cap L^\infty(B_{2r}(x_0))$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathrm{Div}(\Gamma) = 0$ in the weak sense on $B_r(x_0)$, we have

$$\left| \int_{B_r(x_0)} \langle \nabla \varphi, \Gamma \rangle b dx \right| \le C \, \|\Gamma\|_{L^q(B_r(x_0))} \, \|\nabla \varphi\|_{L^p(B_r(x_0))} \|\nabla b\|_{M^{\alpha,n-\alpha}(B_{2r}(x_0))}.$$

We shall use the following technical inequality.

Lemma 2.5. If $F \in M^{2,n-2}(\mathbb{R}^n, \mathbb{R}^n)$ is a divergence free vector field, $g \in W^{1,2}(\mathbb{R}^n)$ and $\nabla g \in M^{2,n-2}(\mathbb{R}^n)$, then for each compact $K \subset \mathbb{R}^n$, we have

$$\|F \cdot \nabla g\|_{H^{-1}(K)}^2 \le C \|F\|_{L^2} \|F\|_{M^{2,n-2}} \|\nabla g\|_{L^2} \|\nabla g\|_{M^{2,n-2}}.$$

Proof. By Corollary 1.8 in [26], we have

$$\|F\cdot\nabla g\|^2_{H^{-1}(K)} \le C\|F\cdot\nabla g\|^2_{h^1}\|F\cdot\nabla g\|^2_{M^{1,n-2}}.$$

The result follows then from the div-curl lemma and Hölder's inequality in Morrey spaces. □

2.4. **Hodge decomposition.** We shall use the following well-known Hodge decomposition for L^p integrable vector fields; see for instance [13, Chapter 10.5].

Lemma 2.6 ([13])**.** Let $p\in(1,\infty)$. Every vector field $V\in L^p(B_r(x_0),\mathbb{R}^n)$, $B_r(x_0)\subset\mathbb{R}^n$, can be uniquely decomposed as

$$V=\nabla a+\nabla^\perp b+h,$$

where $a\in W^{1,p}(B_r(x_0)), b\in W^{1,p}_0(B_r(x_0),\bigwedge^2\mathbb{R}^n)$ with $db=0$, and $h\in C^\infty(B_r(x_0),\mathbb{R}^n)$ is harmonic. Moreover, we have the estimate

$$\|a\|_{W^{1,p}(B_r(x_0))}+\|b\|_{W^{1,p}(B_r(x_0))}+\|h\|_{L^p(B_r(x_0))}\le C\|V\|_{L^p(B_r(x_0))}.$$

Here $\nabla^\perp b:=(\delta b)^\sharp\in L^p(B_r(x_0),\mathbb{R}^n)$, where δ is the formal conjugate operator of d and $\sharp$ is the sharp operator from $\bigwedge^1\mathbb{R}^n$ to $\mathbb{R}^n$.

We will also use the following "nonlinear Hodge decomposition" for a connection matrix in certain Morrey space, proved by Rivière and Struwe [22].

Lemma 2.7 ([22])**.** There exists $\varepsilon(n,N)>0$ such that for every $\Omega\in M^{2,n-2}(B^n,\mathbb{R}^N\otimes so(N))$ with

$$\|\Omega\|^2_{M^{2,n-2}(B^n)}<\varepsilon(n,N),$$

there exist $P\in W^{1,2}(B^n,SO(N))$, $\xi\in W^{1,2}(B^n,so(N)\otimes\bigwedge^2\mathbb{R}^n)$ such that

$$-P^{-1}\nabla P+P^{-1}\Omega P=\nabla^\perp\xi\quad\text{in } B^n$$

and

$$\|\nabla P\|^2_{M^{2,n-2}(B^n)}+\|\nabla\xi\|^2_{M^{2,n-2}(B^n)}\le C\|\Omega\|^2_{M^{2,n-2}(B^n)}\le C\varepsilon(n,N).$$

3. Interior regularity

In this section, we shall prove Theorem 1.1. The key step towards it is the following Hölder continuity for weak solutions of (1.1)-(1.2).

Theorem 3.1. There exists some $\epsilon=\epsilon(n,N)>0$ such that if $(m,R)\in W^{1,2}(B^n,\mathbb{R}^N\times SO(N))$ is a weak solution of (1.4)-(1.5) with

$$\|\nabla m\|_{M^{2,n-2}(B^n)}\le\varepsilon\quad\text{and}\quad\|\nabla R\|_{M^{2,n-2}(B^n)}\le\varepsilon,$$

then there exists $\beta>0$ such that m and R are $C^{0,\beta}$-Hölder continuous on $B_{1/2}$.

Proof. Fix any $x_0\in B^n$ and write B_r for the ball $B_r(x_0)\subset B^n$, where r is small enough such that $B_{2r}(x_0)\subset B^n$. Assume $\varepsilon\in(0,\varepsilon_0)$ for some $\varepsilon_0=\varepsilon_0(n,N)>0$ to be determined later.

Note that

$$\|\nabla m\|_{M^{2,n-2}(B^n)}\le\varepsilon\quad\text{and}\quad\|\Omega_R\|_{M^{2,n-2}(B^n)}\approx\|\nabla R\|_{M^{2,n-2}(B^n)}\le\varepsilon.$$

According to Lemma 2.7, there exist $P \in W^{1,2}(B^n, SO(N))$, $\xi \in W^{1,2}(B^n, so(N) \otimes \bigwedge^2 \mathbb{R}^n)$ such that

$$-P^{-1}\nabla P + P^{-1}\Omega_R P = \nabla^\perp \xi \quad \text{in } B^n$$

and

$$\|\nabla P\|^2_{M^{2,n-2}(B^n)} + \|\nabla \xi\|^2_{M^{2,n-2}(B^n)} \le C\|\Omega_R\|^2_{M^{2,n-2}(B^n)} \le C\varepsilon(n, N).$$

Then direct computation shows

$$\begin{aligned} \operatorname{Div}(P^{-1}\nabla R) &= \nabla(P^{-1}) \cdot \nabla R + P^{-1}\Delta R \\ &= -P^{-1}(\nabla P)P^{-1} \cdot \nabla R + P^{-1}\Omega_R \cdot \nabla R + P^{-1}\text{skew}\,(\nabla m \circ S(\nabla m, R))\, R \\ &= P^{-1}\text{skew}\,(\nabla m \circ S(\nabla m, R))\, R + (\nabla^\perp \xi)P^{-1}\nabla R. \end{aligned} \tag{3.1}$$

By the Hodge decomposition, Lemma 2.6, we may find functions $a \in W^{1,2}(B_r, \mathbb{R}^{N\times N})$, $b \in W^{1,2}_0(B_r, \mathbb{R}^{N\times N} \otimes \bigwedge^2 \mathbb{R}^n)$ and a component-wise harmonic $h \in C^\infty(B_r, \mathbb{R}^{N\times N} \otimes \mathbb{R}^n)$ such that

$$P^{-1}\nabla R = \nabla a + \nabla^\perp b + h \quad \text{in } B_r.$$

Define an operator $\operatorname{Div}^\perp : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n$ as $\operatorname{Div}^\perp V := d(V^\flat)$, where $\flat$ is the flat operator from $\mathbb{R}^n$ to $\bigwedge^1 \mathbb{R}^n$. Then for any $b \in \bigwedge^2 \mathbb{R}^n$ with $db = 0$ we have

$$\operatorname{Div}^\perp \nabla^\perp b = d(\nabla^\perp b)^\flat = d * d * b = \Delta_{\mathbb{R}^n} b,$$

$$\operatorname{Div}^\perp \nabla = d \circ \flat \circ \sharp \circ d = d \circ d = 0$$

and

$$\operatorname{Div}\nabla^\perp = \operatorname{Div} \circ (\sharp * d*) = *d * \flat\sharp * d* = 0.$$

Using operators Div and $\operatorname{Div}^\perp$ to act on both sides of the equation, we obtain

$$\Delta a = \operatorname{Div}(P^{-1}\nabla R) = P^{-1}\text{skew}\,(\nabla m \circ S(\nabla m, R))\, R + (\nabla^\perp \xi)P^{-1}\nabla R \tag{3.2}$$

and

$$\Delta b = d(P^{-1}dR) = dP^{-1} \wedge d(R - R_0), \tag{3.3}$$

for any constant $R_0 \in \mathbb{R}^{N\times N}$.

To ease our notation, we set $\lambda = \frac{n+1}{n}$, whose Hölder conjugate index is $n+1$. In order to estimate $\|\nabla a\|_{L^\lambda(B_r)}$ and $\|\nabla^\perp b\|_{L^\lambda(B_r)}$, we define

$$T := \left\{\varphi \in C^\infty_0(B_r, \mathbb{R}^{N\times N}) : \|\nabla \varphi\|_{L^{n+1}(B_r)} \le 1\right\}.$$

Then

$$\begin{aligned} \|\nabla a\|_{L^\lambda(B_r)} &\lesssim \sup_{\varphi\in T} \int_{B_r} \langle \nabla a, \nabla \varphi\rangle \, dx \\ &\lesssim \sup_{\varphi\in T} \int_{B_r} \left\langle P^{-1}\text{skew}\,(\nabla m \circ S(\nabla m, R))\, R + (\nabla^\perp \xi)P^{-1}\nabla R, -\varphi\right\rangle dx. \end{aligned} \tag{3.4}$$

Note that $\frac{n-\lambda}{\lambda} = \frac{n-\frac{n+1}{n}}{\frac{n+1}{n}} = \frac{n^2-n-1}{n+1}$, $\frac{2-\lambda}{2\lambda} = \frac{n-1}{2n+2}$. Write $\xi = (\xi_{ij})$, $P^{-1} = (P^{-1}_{ij})$, $R = (R_{ij})$ and $\varphi = (\varphi_{ij})$ and observe that

$$\left\langle (\nabla^\perp \xi)P^{-1}\nabla R, \varphi\right\rangle = \left\langle \nabla^\perp \xi_{ij}, \nabla R_{kl}\right\rangle P^{-1}_{jk}\varphi_{il}.$$

Since $\mathrm{Div}(\nabla^\perp \xi_{ij}) = 0$, we may apply Lemma 2.4 (1) with $\Gamma = \nabla^\perp \xi_{ij} \in L^2$, $a = R_{kl} \in M_1^{\lambda,n-\lambda}$ and $b = P^{-1}_{jk}\varphi_{il} \in W^{1,2}_0 \cap L^\infty$ to obtain

$$
\begin{aligned}
&\int_{B_r} \langle (\nabla^\perp \xi) P^{-1} \nabla R, \varphi \rangle \, dx \\
&\lesssim \|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \|\nabla^\perp \xi\|_{L^2(B_r)} \|\nabla(P^{-1})\|_{L^2(B_r)} \|\varphi\|_{L^\infty(B_r)} \\
&\quad + \|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \|\nabla^\perp \xi\|_{L^2(B_r)} \|\nabla \varphi\|_{L^{n+1}(B_r)} \|1\|_{L^{\frac{2n+2}{n-1}}(B_r)} \|P^{-1}\|_{L^\infty(B_r)} \\
&\lesssim \|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \cdot r^{\frac{n-2}{2}} \varepsilon \cdot r^{\frac{n-2}{2}} \varepsilon \cdot \|\nabla \varphi\|_{L^{n+1}(B_r)} r^{1-\frac{n}{n+1}} \\
&\quad + \|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \cdot r^{\frac{n-2}{2}} \varepsilon \cdot r^{\frac{n(2-\lambda)}{2\lambda}} \|\nabla \varphi\|_{L^{n+1}(B_r)} \\
&\lesssim \varepsilon r^{\frac{n-\lambda}{\lambda}} \|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \|\nabla \varphi\|_{L^{n+1}(B_r)},
\end{aligned}
\tag{3.5}
$$

where in the second inequality we used the estimates

$$
\|\nabla^\perp \xi\|_{L^2(B_r)} \lesssim r^{\frac{n-2}{2}} \|\nabla \xi\|_{M^{2,n-2}(B_r)} \lesssim \varepsilon r^{\frac{n-2}{2}},
$$
$$
\|\nabla(P^{-1})\|_{L^2(B_r)} \lesssim \|\nabla P\|_{L^2(B_r)} \lesssim r^{\frac{n-2}{2}} \|\nabla P\|_{M^{2,n-2}(B_r)} \lesssim \varepsilon r^{\frac{n-2}{2}}.
$$

Similarly, since $\mathrm{Div}(S(\nabla m, R)) = 0$, we may apply Lemma 2.4 (1) with $\Gamma = S(\nabla m, R)_{ks} \in L^2$, $a = m_{js} \in M_1^{\lambda,n-\lambda}$ and $b = P^{-1}_{ij} R_{kl} \varphi_{il} \in W^{1,2}_0 \cap L^\infty$ to obtain

$$
\begin{aligned}
&\int_{B_r} \langle P^{-1} \mathrm{skew}\, (\nabla m \circ S(\nabla m, R))\, R, \varphi \rangle \, dx \\
&\lesssim \|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \|S(\nabla m, R)\|_{L^2(B_r)} \|\nabla P\|_{L^2(B_r)} \|\varphi\|_{L^\infty(B_r)} \\
&\quad + \|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \|S(\nabla m, R)\|_{L^2(B_r)} \|\nabla \varphi\|_{L^2(B_r)} \\
&\quad + \|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \|S(\nabla m, R)\|_{L^2(B_r)} \|\nabla R\|_{L^2(B_r)} \|\varphi\|_{L^\infty(B_r)} \\
&\lesssim r^{\frac{n(2-\lambda)}{2\lambda}} \|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \|S(\nabla m, R)\|_{L^2(B_r)} \|\nabla \varphi\|_{L^{n+1}(B_r)}.
\end{aligned}
\tag{3.6}
$$

Note that the definition of $S(\nabla m, R)$ implies that there exists a constant C such that

$$
C^{-1} (|\nabla m| - 1) \le |S(\nabla m, R)| \le C (|\nabla m| + 1). \tag{3.7}
$$

Then it follows

$$
\begin{aligned}
\|S(\nabla m, R)\|_{L^2(B_r)} &\le C \left(\|\nabla m\|_{L^2(B_r)} + \|\mathbb{1}\|_{L^2(B_r)} \right) \\
&\le C \left(r^{\frac{n-2}{2}} \|\nabla m\|_{M^{2,n-2}(B_{2r})} + r^{\frac{n}{2}} \right) \\
&\le C r^{\frac{n-2}{2}} (r + \varepsilon),
\end{aligned}
\tag{3.8}
$$

and

$$
C^{-1} \|\nabla m\|_{L^\lambda(B_r)} - C r^{\frac{n}{\lambda}} \le \|S(\nabla m, R)\|_{L^\lambda(B_r)} \le C \|\nabla m\|_{L^\lambda(B_r)} + C r^{\frac{n}{\lambda}}. \tag{3.9}
$$

Substituting (3.8) into (3.6) gives

$$
\begin{aligned}
&\int_{B_r} \langle P^{-1} \mathrm{skew}\, (\nabla m \circ S(\nabla m, R))\, R, \varphi \rangle \, dx \\
&\le C r^{\frac{n-\lambda}{\lambda}} (r + \varepsilon) \|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \|\nabla \varphi\|_{L^{n+1}(B_r)}.
\end{aligned}
\tag{3.10}
$$

For convenience, we define

$$
\Phi(x_0, s) := \|\nabla m\|_{M^{\lambda,n-\lambda}(B_s(x_0))} + \|\nabla R\|_{M^{\lambda,n-\lambda}(B_s(x_0))}.
$$

Combining (3.4), (3.5) with (3.10), we conclude

$$\|\nabla a\|_{L^\lambda(B_r)} \le Cr^{\frac{n-\lambda}{\lambda}}(r+\varepsilon)\Phi(x_0,2r). \tag{3.11}$$

Now we estimate the L^{n+1}-norm of $\nabla^\perp b$ using (3.3). Suppose $b=\sum_{1\le s<t\le n} b_{st}dx^s\wedge dx^t$, $dP^{-1}=\partial_s P^{-1}dx^s$, $dR=\partial_t R dx^t$. Then we have

$$\Delta b_{st}=\partial_s P^{-1}\partial_t R-\partial_t P^{-1}\partial_s R=\nabla^\perp_{st}P^{-1}\cdot\nabla R,$$

where $\nabla^\perp_{st}:=(0,\cdots,0,-\partial_t,0,\cdots,0,\partial_s,0,\cdots,0)$ for $1\le s<t\le n$. Note that $\mathrm{Div}\nabla^\perp_{st}P^{-1}=0$ for any $1\le s<t\le n$. Then we use Lemma 2.4 (1) with $b=\varphi\in C_0^\infty$, $\Gamma=\nabla^\perp_{st}(P^{-1}_{ij})\in L^\lambda$ and $a=R_{jk}\in M_1^{\lambda,n-\lambda}\cap L^\infty$ as follows:

$$\begin{aligned}
\|\nabla b_{st}\|_{L^\lambda(B_r)} &\lesssim \sup_{\varphi\in T}\int_{B_r}\langle\nabla b_{st},\nabla\varphi\rangle\,dx\\
&\lesssim \sup_{\varphi\in T}\int_{B_r}-\Delta b_{st}\varphi dx\\
&\lesssim \sup_{\varphi\in T}\int_{B_r}-\nabla^\perp_{st}P^{-1}\cdot\nabla R\varphi dx\\
&\lesssim \sup_{\varphi\in T}\|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})}\|\nabla\varphi\|_{L^{n+1}(B_r)}\|\nabla P\|_{L^\lambda(B_r)}\\
&\lesssim \varepsilon r^{\frac{n-\lambda}{\lambda}}\|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})}.
\end{aligned}$$

where in the last inequality we used the estimate

$$\|\nabla P\|_{L^\lambda(B_r)}\lesssim r^{\frac{n(2-\lambda)}{2\lambda}}\|\nabla P\|_{L^2(B_r)}\lesssim r^{\frac{n(2-\lambda)}{2\lambda}+\frac{n-2}{2}}\|\nabla P\|_{M^{2,n-2}(B_r)}\lesssim\varepsilon r^{\frac{n-\lambda}{\lambda}}.$$

Thus

$$\|\nabla^\perp b\|_{L^\lambda(B_r)}\le\varepsilon r^{\frac{n-\lambda}{\lambda}}\|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})} \tag{3.12}$$

For the harmonic term h, standard estimate for harmonic functions implies that for any $0<\rho<r$, there holds

$$\int_{B_\rho}|h|^\lambda dx\le C\left(\frac{\rho}{r}\right)^n\int_{B_r}|h|^\lambda dx. \tag{3.13}$$

Combining (3.11), (3.12) with (3.13), we infer

$$\begin{aligned}
\|\nabla R\|_{L^\lambda(B_\rho)} &\lesssim \|P^{-1}\nabla R\|_{L^\lambda(B_\rho)}\\
&\lesssim \|h\|_{L^\lambda(B_\rho)}+\|\nabla a\|_{L^\lambda(B_\rho)}+\|\nabla^\perp b\|_{L^\lambda(B_\rho)}\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}}\|h\|_{L^\lambda(B_r)}+\|\nabla a\|_{L^\lambda(B_r)}+\|\nabla^\perp b\|_{L^\lambda(B_r)}\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}}\|\nabla R\|_{L^\lambda(B_r)}+r^{\frac{n-\lambda}{\lambda}}(r+\varepsilon)\Phi(x_0,2r)\\
&\quad+\varepsilon r^{\frac{n-\lambda}{\lambda}}\|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})}\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}}\|\nabla R\|_{L^\lambda(B_r)}+(r+\varepsilon)r^{\frac{n-\lambda}{\lambda}}\Phi(x_0,2r).
\end{aligned} \tag{3.14}$$

It remains to estimate $\|\nabla m\|_{L^\lambda(B_r)}$. Applying the Hodge decomposition to the divergence free matrix $S(\nabla m,R)\in\mathbb{R}^{N\times 1}\otimes\mathbb{R}^n$, we obtain

$$\pi_n\left(2R\mathbb{P}^2\left(R^T(\nabla m|0)-(\mathbb{1}_n|0)\right)\right)=\nabla^\perp\alpha+\chi, \tag{3.15}$$

where $\alpha \in W_0^{1,2}(B_r, \mathbb{R}^{N\times 1} \otimes \bigwedge^2 \mathbb{R}^n)$ and $\chi \in C^\infty(B_r, \mathbb{R}^{N\times 1} \otimes \mathbb{R}^n)$ is (component-wise) harmonic. Due to (1.1), we do not have terms of the form $\nabla\zeta$ on the right side.

For convenience, we define a linear map $\mathbb{P}_R$ as $\xi \mapsto 2R\mathbb{P}(R^T\xi)$. Using operator $\mathrm{Div}^\perp$ to act on both sides of the equation, we obtain

$$\begin{aligned}
\Delta\alpha &= \mathrm{Div}^\perp\nabla^\perp\alpha \\
&= \mathrm{Div}^\perp\left[\pi_n(\mathbb{P}_R(\nabla m|0) - 2R\mathbb{P}^2(\mathbb{1}_n|0))\right] \\
&= d\mathbb{P}_R \wedge dm - \mathrm{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right].
\end{aligned} \tag{3.16}$$

Similar as the previous case, set $U := \left\{\psi \in C_0^\infty(B_r, \mathbb{R}^{N\times 1} \otimes \bigwedge^2 \mathbb{R}^n) : \|\nabla^\perp\psi\|_{L^{n+1}(B_r)} \le 1\right\}$. It follows then from Lemma 2.4 (2) and Poincaré's inequality that

$$\begin{aligned}
&\|\nabla^\perp\alpha\|_{L^\lambda(B_r)} \lesssim \sup_{\psi\in U}\int_{B_r}\left\langle\nabla^\perp\alpha, \nabla^\perp\psi\right\rangle dx \\
&= \sup_{\psi\in U}\int_{B_r}\left(\left\langle(d\mathbb{P}_R)m, \nabla^\perp\psi\right\rangle - \left\langle\pi_n(2(R-R_{B_r})\mathbb{P}^2(\mathbb{1}_n|0)), \nabla^\perp\psi\right\rangle\right) dx \\
&\lesssim \sup_{\psi\in U}\|\nabla^\perp\psi\|_{L^{n+1}(B_r)}\|d\mathbb{P}_R\|_{L^\lambda(B_r)}\|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} \\
&\quad + \sup_{\psi\in U}\|\nabla^\perp\psi\|_{L^{n+1}(B_r)}\|R - R_{B_r}\|_{L^\lambda(B_r)} \\
&\lesssim \sup_{\psi\in U}\left[\|\nabla^\perp\psi\|_{L^{n+1}(B_r)}\left(\|\nabla R\|_{L^\lambda(B_r)}\|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} + r\|\nabla R\|_{L^\lambda(B_r)}\right)\right] \\
&\lesssim \left(\varepsilon r^{\frac{n-\lambda}{\lambda}}\|\nabla m\|_{M^{\lambda,n-\lambda}(B_{2r})} + r^{\frac{n}{\lambda}}\|\nabla R\|_{M^{\lambda,n-\lambda}(B_{2r})}\right) \\
&\lesssim (r+\varepsilon)r^{\frac{n-\lambda}{\lambda}}\Phi(x_0, 2r),
\end{aligned} \tag{3.17}$$

where in the third inequality we used the estimate

$$\|\nabla R\|_{L^\lambda(B_r)} \lesssim r^{\frac{n(2-\lambda)}{2\lambda}}\|\nabla R\|_{L^2(B_r)} \lesssim r^{\frac{n(2-\lambda)}{2\lambda}+\frac{n-2}{2}}\|\nabla R\|_{M^{2,n-2}(B_r)} \lesssim \varepsilon r^{\frac{n-\lambda}{\lambda}}.$$

Returning to (3.9), by (3.17) we have

$$\begin{aligned}
\|\nabla m\|_{L^\lambda(B_\rho)} &\lesssim \|\chi\|_{L^\lambda(B_\rho)} + \|\nabla^\perp\alpha\|_{L^\lambda(B_\rho)} + \rho^{\frac{n}{\lambda}} \\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}}\|\chi\|_{L^\lambda(B_r)} + \|\nabla^\perp\alpha\|_{L^\lambda(B_r)} + \rho^{\frac{n}{\lambda}} \\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}}\|\nabla m\|_{L^\lambda(B_r)} + (\varepsilon + r)r^{\frac{n-\lambda}{\lambda}}\Phi(x_0, 2r) + \rho^{\frac{n}{\lambda}}.
\end{aligned} \tag{3.18}$$

Combining (3.14) with (3.18), we conclude

$$\begin{aligned}
&\rho^{-\frac{n-\lambda}{\lambda}}\left(\|\nabla m\|_{L^\lambda(B_\rho)} + |\nabla R\|_{L^\lambda(B_\rho)}\right) \\
&\lesssim \rho r^{-\frac{n}{\lambda}}\left(\|\nabla m\|_{L^\lambda(B_r)} + |\nabla R\|_{L^\lambda(B_r)}\right) + (\varepsilon + r)\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}}\Phi(x_0, 2r) + \rho \\
&\lesssim \left(\frac{\rho}{r} + (\varepsilon + r)\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}}\right)\Phi(x_0, 2r) + \rho.
\end{aligned}$$

We may assume $r \le \varepsilon \le \varepsilon_0$, where ε_0 will be determined soon. Adding ρ on both sides leads to the estimate

$$\begin{aligned}&\rho^{-\frac{n-\lambda}{\lambda}}\left(\|\nabla m\|_{L^\lambda(B_\rho)}+|\nabla R\|_{L^\lambda(B_\rho)}\right)+\rho\\ &\le C_0\left(\frac{\rho}{r}+(\varepsilon+r)\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}}\right)(\Phi(x_0,2r)+2r)\end{aligned} \tag{3.19}$$

for suitable positive constant C_0 that depends only on the data.

Now we fix $\rho=\theta r$, $\theta=\frac{1}{8C_0}$, and $\varepsilon_0=\frac{1}{2}(8C_0)^{-\frac{n}{\lambda}}$. Then (3.19) gives

$$(\theta r)^{-\frac{n-\lambda}{\lambda}}\left(\|\nabla m\|_{L^\lambda(B_{\theta r})}+|\nabla R\|_{L^\lambda(B_{\theta r})}\right)+\theta r\le\frac{1}{4}\left(\Phi(x_0,2r)+2r\right), \tag{3.20}$$

which holds for all $B_{\theta r}(x_0)$ and $B_{2r}(x_0)\subset B^n$. Clearly we can replace $B_{2r}(x_0)$ with any $B_s(y_0)\subset B^n$ containing $B_{2r}(x_0)$, provided that $s\le\varepsilon$. Thus (3.20) implies

$$(\theta r)^{-\frac{n-\lambda}{\lambda}}\left(\|\nabla m\|_{L^\lambda(B_{\theta r}(x_0))}+|\nabla R\|_{L^\lambda(B_{\theta r}(x_0))}\right)+\theta r\le\frac{1}{4}\left(\Phi(y_0,s)+s\right),$$

which is valid for all r,s,x_0,y_0 such that $B_{2r}(x_0)\subset B_s(y_0)\subset B^n$. Note that the family of balls $\{B_{\theta r}(x_0)\}$ forms an open cover of $B_{\frac{\theta s}{2}}(y_0)$. Thus we can take the supremum over all admissible $B_r(x_0)$ to find

$$\Phi(y_0,\frac{\theta s}{2})+\frac{\theta s}{2}\le\frac{1}{2}\left(\Phi(y_0,s)+s\right). \tag{3.21}$$

Setting $\Psi(y_0,r):=\Phi(y_0,r)+r$ and then iterating (3.21), we obtain

$$\Psi(y_0,\left(\frac{\theta}{2}\right)^k s)\le 2^{-k}\Psi(y_0,s)\qquad\text{for all } k\in\mathbb{N}.$$

For $r\approx\left(\frac{\theta}{2}\right)^k s$, we select $k\approx\frac{\log(r/s)}{\log(\theta/2)}$. Then $2^{-k}\approx(r/s)^{\frac{\log 2}{\log(2/\theta)}}=:(r/s)^\beta$. This implies that for all $r\le s\le\varepsilon_0$, we have the estimate

$$\Psi(y_0,r)\le Cr^\beta s^{-\beta}\Psi(y_0,s).$$

We may choose $s=s_0>0$, depending only on the data, such that for all $r\le s_0/2$, there holds

$$\Phi(y_0,r)\le\Psi(y_0,r)\le Cr^\beta s_0^{-\beta}\Psi(y_0,s_0),$$

which gives

$$\nabla m,\nabla R\in M^{\lambda,n-\lambda+\lambda\beta}_{\rm loc}(B^n).$$

Finally, Morrey's Dirichlet growth theorem (see for instance [8]) implies $(m,R)\in C^{0,\beta}_{\rm loc}$. The proof of Theorem 3.1 is thus complete. □

Proof of Theorem 1.1. With Theorem 3.1 at hand, the proof follows directly from that of Gastel-Neff [7, Section 6.2]. □

4. L^p regularity theory

In this section, we shall prove Theorem 1.2. The proof relies on ideas from earlier works on similar problems in [27, 26, 11, 9]. In the first step, we show a quantitative Hölder continuity result for weak solutions of (1.4)-(1.5).

Proposition 4.1. Under the same assumptions as Theorem 1.2, when $\frac{n}{2}<p<n$, we have $(m,R)\in C^{0,\gamma}(B_{1/2},\mathbb{R}^N\times SO(N))$, where $\gamma=2-n/p\in(0,1)$. Moreover, there exists

some $C = C(n,N,p) > 0$ such that

$$[m]_{C^{0,\gamma}(B_{1/2})} + [R]_{C^{0,\gamma}(B_{1/2})} \leq C\left(\varepsilon + \|f\|_{L^p(B_1)}\right). \tag{4.1}$$

Proof. Similar to the proof of Theorem 3.1, we write B_r for a fixed ball $B_r(x_0) \subset B_{1/2}$, where r is small enough such that $B_{2r}(x_0) \subset B^n$.

Note that

$$\|\nabla m\|_{M^{2,n-2}(B^n)} \leq \varepsilon \quad \text{and} \quad \|\Omega_R\|_{M^{2,n-2}(B^n)} \approx \|\nabla R\|_{M^{2,n-2}(B^n)} \leq \varepsilon.$$

According to Lemma 2.7, there exist $P \in W^{1,2}(B^n, SO(N))$, $\xi \in W^{1,2}(B^n, so(N) \otimes \bigwedge^2 \mathbb{R}^n)$ such that

$$-P^{-1}\nabla P + P^{-1}\Omega_R P = \nabla^\perp \xi \quad \text{in } B^n$$

and

$$\|\nabla P\|^2_{M^{2,n-2}(B^n)} + \|\nabla \xi\|^2_{M^{2,n-2}(B^n)} \leq C\|\Omega_R\|^2_{M^{2,n-2}(B^n)} \leq C\varepsilon(n,N).$$

A straightforward computation gives

$$\begin{aligned}
&\mathrm{Div}(P^{-1}\nabla R) = \nabla(P^{-1})\cdot\nabla R + P^{-1}\Delta R\\
&= -P^{-1}(\nabla P)P^{-1}\cdot\nabla R + P^{-1}\Omega_R\cdot\nabla R + P^{-1}\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right)R + P^{-1}f\\
&= P^{-1}\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right)R + (\nabla^\perp\xi)P^{-1}\nabla R + P^{-1}f
\end{aligned} \tag{4.2}$$

By the Hodge decomposition, Lemma 2.6, there exist $a \in W^{1,2}(B_r, \mathbb{R}^{N\times N})$, $b \in W^{1,2}_0(B_r, \mathbb{R}^{N\times N} \otimes \bigwedge^2 \mathbb{R}^n)$ and a component-wise harmonic $h \in C^\infty(B_r, \mathbb{R}^{N\times N} \otimes \mathbb{R}^n)$ such that

$$P^{-1}\nabla R = \nabla a + \nabla^\perp b + h \quad \text{in } B_r. \tag{4.3}$$

Then a and b satisfy

$$\Delta a = \mathrm{Div}(P^{-1}\nabla R) = P^{-1}\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right)R + (\nabla^\perp\xi)P^{-1}\nabla R + P^{-1}f, \tag{4.4}$$

and

$$\Delta b = \mathrm{Div}^\perp(P^{-1}\nabla R) = dP^{-1}\wedge dR = dP^{-1}\wedge d(R - R_0) \tag{4.5}$$

for any constant $R_0 \in \mathbb{R}^{N\times N}$.

As in the proof of Theorem 3.1, let $T := \left\{\varphi \in C^\infty_0(B_r, \mathbb{R}^{N\times N}) : \|\nabla\varphi\|_{L^{n+1}(B_r)} \leq 1\right\}$ and set $\lambda = \frac{n+1}{n}$. Then, we have

$$\begin{aligned}
&\|\nabla a\|_{L^\lambda(B_r)} \lesssim \sup_{\varphi\in T}\int_{B_r}\langle\nabla f, \nabla\varphi\rangle\, dx\\
&\lesssim \sup_{\varphi\in T}\int_{B_r}\left\langle P^{-1}\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right)R + (\nabla^\perp\xi)P^{-1}\nabla R + P^{-1}f, -\varphi\right\rangle dx.
\end{aligned}$$

Note that $n + \frac{1}{n+1} - \frac{n}{p} = \frac{n-\lambda}{\lambda} + 2 - \frac{n}{p} = \frac{n-\lambda}{\lambda} + \gamma$. Using Hölder's inequality and Sobolev's inequality, we deduce

$$\begin{aligned}
\int_{B_r}\left\langle P^{-1}f, \varphi\right\rangle dx &\lesssim \|P\|_{L^\infty(B_r)}\|f\|_{L^p(B_r)}\|1\|_{L^{\frac{p}{p-1}}(B_r)}\|\varphi\|_{L^\infty(B_r)}\\
&\lesssim \|f\|_{L^p(B_r)}\cdot r^{n\left(1-\frac{1}{p}\right)}\cdot r^{\frac{1}{n+1}}\|\nabla\varphi\|_{L^{n+1}(B_r)}\\
&\lesssim r^{\frac{n-\lambda}{\lambda}+\gamma}\|f\|_{L^p(B_r)}\|\nabla\varphi\|_{L^{n+1}(B_r)}.
\end{aligned} \tag{4.6}$$

As in the previous proof, we define

$$\Phi(x_0, s) := \|\nabla m\|_{M^{\lambda,n-\lambda}(B_s(x_0))} + \|\nabla R\|_{M^{\lambda,n-\lambda}(B_s(x_0))}.$$

Combining (4.6) with (3.5) and (3.10), we infer

$$\|\nabla a\|_{L^\lambda(B_r)} \le Cr^{\frac{n-\lambda}{\lambda}} \left((r+\varepsilon)\Phi(x_0, 2r) + r^\gamma \|f\|_{L^p(B_r)}\right). \tag{4.7}$$

Combining (4.7) with (3.12) and (3.13), we conclude

$$\|\nabla R\|_{L^\lambda(B_\rho)} \le C\left(\frac{\rho}{r}\right)^{\frac{n}{\lambda}} \|\nabla R\|_{L^\lambda(B_r)} + Cr^{\frac{n-\lambda}{\lambda}} \left((r+\varepsilon)\Phi(x_0, 2r) + r^\gamma \|f\|_{L^p(B_r)}\right).$$

This together with (3.18) gives

$$\begin{aligned}
&\rho^{-\frac{n-\lambda}{\lambda}} \left(\|\nabla m\|_{L^\lambda(B_\rho)} + |\nabla R\|_{L^\lambda(B_\rho)}\right) + \rho \\
\le& C\rho r^{-\frac{n}{\lambda}} \left(\|\nabla m\|_{L^\lambda(B_r)} + |\nabla R\|_{L^\lambda(B_r)}\right) \\
&+ C\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}} \left((r+\varepsilon)\Phi(x_0, 2r) + r^\gamma \|f\|_{L^p(B_r)}\right) + C\rho \\
\le& C_0\left(\frac{\rho}{r} + (\varepsilon + r)\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}}\right)\Phi(x_0, 2r) + C_0\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}} r^\gamma \|f\|_{L^p(B_1)} + C_0\rho
\end{aligned} \tag{4.8}$$

for some C_0 depending only on the data.

Now select ε small enough such that $(2\varepsilon)^{\frac{\lambda}{n}} = \frac{1}{2}(2C_0)^{\frac{2}{\gamma-1}}$. Then let $\rho = \theta r$, $r \le \varepsilon$ and $\theta \in \left((2\varepsilon)^{\frac{\lambda}{n}}, (2C_0)^{\frac{2}{\gamma-1}}\right)$. It follows

$$C_0\left(\frac{\rho}{r} + (\varepsilon + r)\left(\frac{r}{\rho}\right)^{\frac{n-\lambda}{\lambda}}\right) \le C_0\theta\left(1 + 2\varepsilon\theta^{-\frac{n}{\lambda}}\right) \le 2C_0\theta \le \theta^{\frac{\gamma+1}{2}}.$$

This together with (4.8) gives

$$\Phi(x_0, \theta r) + \theta r \le \theta^{\frac{\gamma+1}{2}} \left(\Phi(x_0, 2r) + 2r\right) + C_1 r^\gamma \|f\|_{L^p(B_1)},$$

where C_1 is a constant depending only on the data. By a standard iteration argument, we eventually obtain

$$\Phi(x_0, \theta r) + \theta r \le C\left(\theta^\gamma \Phi(x_0, 2r) + 2\theta^\gamma r + r^\gamma \|f\|_{L^p(B_1)}\right).$$

This implies

$$\Phi(x_0, r) \le Cr^\gamma \left(\varepsilon + \|f\|_{L^p(B_1)}\right),$$

from which we conclude that $\nabla m, \nabla R \in M^{\lambda, n-\lambda+\gamma\lambda}_{\mathrm{loc}}$. By Morrey's Dirichlet growth theorem, this further implies that $m, R \in C^{0,\gamma}_{\mathrm{loc}}$ together with the desired estimate (4.1).

□

Next, we prove an improved Morrey regularity estimate, in the spirit of [27, Lemma 7.3] (or [26, Proposition 2.1]).

Proposition 4.2. Under the same assumption as Theorem 1.2, when $\frac{n}{2} < p < n$, we have $(m, R) \in M^{2,n-2+2\gamma}_{\mathrm{loc}}(B^n, \mathbb{R}^N \times SO(N))$, where $\gamma = 2 - n/p \in (0, 1)$. Moreover, there exists some $C = C(n, N, p) > 0$ such that

$$\|\nabla m\|_{M^{2,n-2+2\gamma}(B_{1/2})} + \|\nabla R\|_{M^{2,n-2+2\gamma}(B_{1/2})} \le C\left(\|f\|_{L^p(B_1)} + \varepsilon\right). \tag{4.9}$$

Proof. We shall use (4.4), (4.5) and (3.16) to estimate $\|\nabla m\|_{M^{2,n-2+2\gamma}}$ and $\|\nabla R\|_{M^{2,n-2+2\gamma}}$. First of all, we extend m, R, P, ξ, f to $\mathbb{R}^n$ with compact support in a norm-bounded way.

Step 1. Estimate $\|\nabla R\|_{L^2(B_\rho)}$.

Rewriting (4.4) as before, we have

$$\begin{aligned}\Delta a = P^{-1}\text{skew}\left(\nabla m\circ S(\nabla m, R)\right)R + (\nabla^\perp\xi)\nabla\left(P^{-1}(R-R_{x_0,r})\right)\\ -(\nabla^\perp\xi)\nabla\left(P^{-1}\right)(R-R_{x_0,r}) + P^{-1}f,\end{aligned} \tag{4.10}$$

where $R_{x_0,r} = \fint_{B_r(x_0)} R dx$. Now we calculate H^{-1}-norm of the right hand side of (4.10).

We abbreviate $S(\nabla m, R)$ as S. For the first item on the right hand side of (4.10), we have

$$P^{-1}\text{skew}\left(\nabla m\circ S\right)R = \frac{1}{4}P^{-1}(\nabla m)S^TR - \frac{1}{4}P^{-1}S(\nabla m^T)R$$

and

$$\begin{aligned}P^{-1}(\nabla m)S^TR = \nabla\left(P^{-1}(m-m_{x_0,r})S^TR\right)\\ -(\nabla P^{-1})(m-m_{x_0,r})S^TR - P^{-1}(m-m_{x_0,r})S^T(\nabla R).\end{aligned}$$

Thus, according to (3.7), we may estimate the first term above as follows:

$$\begin{aligned}&\|\nabla\left(P^{-1}(m-m_{x_0,r})S^TR\right)\|_{H^{-1}(B_r)} \lesssim \|P^{-1}(m-m_{x_0,r})S^TR\|_{L^2(B_r)}\\ &\lesssim \|P^{-1}\|_{L^\infty(B_r)}\|(m-m_{x_0,r})\|_{L^\infty(B_r)}\|S\|_{L^2(B_r)}\|R\|_{L^\infty(B_r)}\\ &\lesssim r^\gamma[m]_{C^{0,\gamma}(B_r)}\|S\|_{L^2(B_r)} \lesssim r^{\frac{n-2}{2}+\gamma}[m]_{C^{0,\gamma}(B_r)}\left(r+\|\nabla m\|_{M^{2,n-2}(B_r)}\right).\end{aligned} \tag{4.11}$$

For the middle term, we may apply Lemma 2.5 to obtain

$$\begin{aligned}&\|(\nabla P^{-1})(m-m_{x_0,r})S^TR\|_{H^{-1}(B_r)}\\ &\lesssim \|R\|_{L^\infty(B_r)}\|(m-m_{x_0,r})\|_{L^\infty(B_r)}\|(\nabla P^{-1})S^T\|_{H^{-1}(B_r)}\\ &\lesssim r^\gamma[m]_{C^{0,\gamma}(B_r)}\|\nabla P^{-1}\|^{\frac12}_{L^2(B_r)}\|\nabla P^{-1}\|^{\frac12}_{M^{2,n-2}(B_r)}\|\nabla S\|^{\frac12}_{L^2(B_r)}\|\nabla S\|^{\frac12}_{M^{2,n-2}(B_r)}\\ &\lesssim r^{\frac{n-2}{2}+\gamma}[m]_{C^{0,\gamma}(B_r)}\|\nabla R\|_{M^{2,n-2}(B_r)}\left(r+\|\nabla m\|_{M^{2,n-2}(B_r)}\right)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\|P^{-1}(m-m_{x_0,r})S^T(\nabla R)\|_{H^{-1}(B_r)}\\ &\lesssim \|P^{-1}\|_{L^\infty(B_r)}\|(m-m_{x_0,r})\|_{L^\infty(B_r)}\|S^T(\nabla R)\|_{H^{-1}(B_r)}\\ &\lesssim r^\gamma[m]_{C^{0,\gamma}(B_r)}\|\nabla S\|^{\frac12}_{L^2(B_r)}\|\nabla S\|^{\frac12}_{M^{2,n-2}(B_r)}\|\nabla R\|^{\frac12}_{L^2(B_r)}\|\nabla R\|^{\frac12}_{M^{2,n-2}(B_r)}\\ &\lesssim r^{\frac{n-2}{2}+\gamma}[m]_{C^{0,\gamma}(B_r)}\|\nabla R\|_{M^{2,n-2}(B_r)}\left(r+\|\nabla m\|_{M^{2,n-2}(B_r)}\right).\end{aligned} \tag{4.13}$$

Combining (4.11), (4.12) with (4.13), we conclude

$$\begin{aligned}&\|P^{-1}\text{skew}\left(\nabla m\circ S\right)R\|_{H^{-1}(B_r)}\\ &\lesssim r^{\frac{n-2}{2}+\gamma}[m]_{C^{0,\gamma}(B_r)}\left(1+\|\nabla R\|_{M^{2,n-2}(B_r)}\right)\left(r+\|\nabla m\|_{M^{2,n-2}(B_r)}\right)\\ &\lesssim r^{\frac{n-2}{2}+\gamma}(r+\varepsilon)[m]_{C^{0,\gamma}(B_r)}.\end{aligned} \tag{4.14}$$

For the second item on the right hand side of (4.10), we introduce

$$T := \left\{\varphi\in C_0^\infty(B_r,\mathbb{R}^{N\times N}) : \|\nabla\varphi\|_{L^2(B_r)} \le 1\right\}.$$

Then we have

$$\begin{aligned}
&\left|\int_{B_r}(\nabla^\perp\xi)\nabla\left(P^{-1}(R-R_{x_0,r})\right)\varphi\right| = \left|-\int_{B_r}(\nabla^\perp\xi)P^{-1}(R-R_{x_0,r})\nabla\varphi\right|\\
&\lesssim \|P^{-1}\|_{L^\infty(B_r)}\|\nabla^\perp\xi\|_{L^2(B_r)}\|\nabla\varphi\|_{L^2(B_r)}\|R-R_{x_0,r}\|_{L^\infty(B_r)}\\
&\lesssim r^{\frac{n-2}{2}}\|\nabla^\perp\xi\|_{M^{2,n-2}(B_r)}\cdot r^\gamma[R]_{C^{0,\gamma}(B_r)}\cdot\|\nabla\varphi\|_{L^2(B_r)},
\end{aligned}$$

which implies

$$\|(\nabla^\perp\xi)\nabla\left(P^{-1}(R-R_{x_0,r})\right)\|_{H^{-1}(B_r)} \le Cr^{\frac{n-2}{2}+\gamma}\|\nabla R\|_{M^{2,n-2}(B_r)}[R]_{C^{0,\gamma}(B_r)}. \tag{4.15}$$

For the third term on the right hand side of (4.10), we have

$$\begin{aligned}
&\|(\nabla^\perp\xi)\nabla\left(P^{-1}\right)(R-R_{x_0,r})\|_{H^{-1}(B_r)}\\
&\lesssim \|(R-R_{x_0,r})\|_{L^\infty(B_r)}\|(\nabla^\perp\xi)\nabla\left(P^{-1}\right)\|_{H^{-1}(B_r)}\\
&\lesssim r^\gamma[R]_{C^{0,\gamma}(B_r)}\|\nabla^\perp\xi\|^{\frac12}_{L^2(B_r)}\|\nabla^\perp\xi\|^{\frac12}_{M^{2,n-2}(B_r)}\|\nabla P\|^{\frac12}_{L^2(B_r)}\|\nabla P\|^{\frac12}_{M^{2,n-2}(B_r)} \qquad (4.16)\\
&\lesssim r^{\frac{n-2}{2}+\gamma}[R]_{C^{0,\gamma}(B_r)}\|\nabla^\perp\xi\|_{M^{2,n-2}(B_r)}\|\nabla P\|_{M^{2,n-2}(B_r)}\\
&\lesssim r^{\frac{n-2}{2}+\gamma}[R]_{C^{0,\gamma}(B_r)}\|\nabla R\|^2_{M^{2,n-2}(B_r)}.
\end{aligned}$$

For the last term on the right hand side of (4.10), we simply apply the embedding $L^{\frac{2n}{n+2}}\hookrightarrow H^{-1}$ to derive

$$\|P^{-1}f\|_{H^{-1}(B_r)} \le C\|f\|_{L^{\frac{2n}{n+2}}(B_r)} \le Cr^{\frac n2+1-\frac np}\|f\|_{L^p(B_r)} \le Cr^{\frac{n-2}{2}+\gamma}\|f\|_{L^p(B_r)}. \tag{4.17}$$

Combining (4.14), (4.15), (4.16) with (4.17), we conclude

$$\|\nabla a\|_{L^2(B_r)} \le Cr^{\frac{n-2}{2}+\gamma}\left[\|f\|_{L^p(B_r)}+(r+\varepsilon)\left([m]_{C^{0,\gamma}(B_r)}+[R]_{C^{0,\gamma}(B_r)}\right)\right], \tag{4.18}$$

where we used the assumption

$$\|\Omega_R\|_{M^{2,n-2}(B_r)} \le C\|\nabla R\|_{M^{2,n-2}(B_r)} \le \varepsilon \quad \text{and} \quad \|\nabla m\|_{M^{2,n-2}(B_r)} \le \varepsilon.$$

Now we estimate ∇b via (4.5). For any 2-form φ whose component belongs to T, we have

$$\begin{aligned}
&\left|\int_{B_r}\langle d\left((dP^{-1})(R-R_{x_0,r})\right),\varphi\rangle\right| = \left|\int_{B_r}(\langle dP^{-1})(R-R_{x_0,r}),\nabla^\perp\varphi\rangle\right|\\
&\lesssim \|R-R_{x_0,r}\|_{L^\infty(B_r)}\|dP^{-1}\|_{L^2(B_r)}\|\nabla^\perp\varphi\|_{L^2(B_r)}\\
&\lesssim r^{\frac{n-2}{2}+\gamma}[R]_{C^{0,\gamma}(B_r)}\|\Omega_R\|_{M^{2,n-2}(B_r)}\|\varphi\|_{W^{1,2}(B_r)}.
\end{aligned}$$

This gives

$$\|\nabla b\|_{L^2(B_r)} \le C\varepsilon r^{\frac{n-2}{2}+\gamma}[R]_{C^{0,\gamma}(B_r)}. \tag{4.19}$$

Combining (4.18), (4.19) with the standard estimate for h, we conclude

$$\begin{aligned}
&\|\nabla R\|_{L^2(B_\rho)} \le \|h\|_{L^2B_\rho}+\|\nabla a\|_{L^2(B_\rho)}+\|\nabla^\perp b\|_{L^2(B_\rho)}\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac n2}\|h\|_{L^2(B_r)}+\|\nabla a\|_{L^2(B_\rho)}+\|\nabla^\perp b\|_{L^2(B_\rho)} \qquad (4.20)\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac n2}\|\nabla R\|_{L^2(B_r)}+C\rho^{\frac{n-2}{2}+\gamma}\left[\|f\|_{L^p(B_\rho)}+(\rho+\varepsilon)\left([m]_{C^{0,\gamma}(B_\rho)}+[R]_{C^{0,\gamma}(B_\rho)}\right)\right].
\end{aligned}$$

Step 2. Estimate $\|\nabla m\|_{L^2(B_\rho)}$.

Applying the Hodge decomposition to the divergence free matrix $S(\nabla m, R)$, we obtain (3.15) as in the proof of Theorem 3.1.

To estimate $\nabla^\perp \alpha$, we shall use the following equation for α?

$$\Delta\alpha = d\mathbb{P}_R \wedge d(m - m_{x_0,r}) - \mathrm{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]. \tag{4.21}$$

Let $U := \left\{\psi \in C_0^\infty(B_r, \mathbb{R}^{N\times n}\otimes \bigwedge^2\mathbb{R}^n) : \|\nabla^\perp\psi\|_{L^2(B_r)} \leq 1\right\}$. Then we have

$$\begin{aligned}
&\|\nabla^\perp\alpha\|_{L^2(B_r)} \lesssim \sup_{\psi\in U}\int_{B_r}\langle\nabla^\perp\alpha, \nabla^\perp\psi\rangle\,dx\\
&\lesssim \sup_{\psi\in U}\int_{B_r}\left(\langle(d\mathbb{P}_R)(m-m_{x_0,r}), \nabla^\perp\psi\rangle - \langle\pi_n(2(R-R_{x_0,r})\mathbb{P}^2(\mathbb{1}_n|0)), \nabla^\perp\psi\rangle\right)dx\\
&\lesssim \sup_{\psi\in U}\|\nabla^\perp\psi\|_{L^2(B_r)}\|d\mathbb{P}_R\|_{L^2(B_r)}\|m-m_{x_0,r}\|_{L^\infty(B_r)}\\
&\quad + \sup_{\psi\in U}\|\nabla^\perp\psi\|_{L^2(B_r)}\|R-R_{x_0,r}\|_{L^\infty(B_r)}\|1\|_{L^2(B_r)}\\
&\lesssim \|\nabla R\|_{L^2(B_r)}\cdot r^\gamma[m]_{C^{0,\gamma}(B_r)} + r^\gamma[R]_{C^{0,\gamma}(B_r)}\cdot r^{\frac{n}{2}}\\
&\lesssim r^{\frac{n-2}{2}+\gamma}(r+\varepsilon)\left([m]_{C^{0,\gamma}(B_r)} + [R]_{C^{0,\gamma}(B_r)}\right).
\end{aligned}$$

Finally, we may can estimate ∇m as follows:

$$\begin{aligned}
\|\nabla m\|_{L^2(B_\rho)} &\lesssim \left(\|\chi\|_{B_\rho} + \|\nabla^\perp\alpha\|_{L^2(B_\rho)} + \rho^{\frac{n}{2}}\right)\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2}}\|\chi\|_{L^2(B_r)} + \rho^{\frac{n-2}{2}+\gamma}(\rho+\varepsilon)\left([m]_{C^{0,\gamma}(B_\rho)} + [R]_{C^{0,\gamma}(B_\rho)}\right) + \rho^{\frac{n}{2}}\\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2}}\|\nabla m\|_{L^2(B_r)} + \rho^{\frac{n-2}{2}+\gamma}(\rho+\varepsilon)\left([m]_{C^{0,\gamma}(B_\rho)} + [R]_{C^{0,\gamma}(B_\rho)}\right) + \rho^{\frac{n}{2}}.
\end{aligned}\tag{4.22}$$

Step 3. Iteration.

Define $\Psi(x_0, r) = \|\nabla m\|_{L^2(B_r)} + \|\nabla R\|_{L^2(B_r)}$. Combining (4.20) with (4.22), we have

$$\Psi(x_0,\rho) \leq C_0\left(\frac{\rho}{r}\right)^{\frac{n}{2}}\Psi(x_0,r) + C_0\rho^{\frac{n-2}{2}+\gamma}\Gamma,$$

where

$$\begin{aligned}
\Gamma &= (\rho+\varepsilon)\left([m]_{C^{0,\gamma}(B_\rho)} + [R]_{C^{0,\gamma}(B_\rho)}\right) + \|f\|_{L^p(B_\rho)} + \rho^{1-\gamma}\\
&\leq C\left(\|f\|_{L^p(B_1)} + \varepsilon\right) =: \Gamma_0.
\end{aligned}$$

A standard iteration argument gives

$$\Psi(x_0, r) \leq r^{\frac{n-2}{2}+\gamma}C\left(\|f\|_{L^p(B_1)} + \varepsilon\right),$$

from which we obtain (4.9). The proof of Proposition 4.2 is thus complete. □

Proof of Theorem 1.2. We shall consider separately two cases.

Case 1. $p \in (\frac{n}{2}, n)$.

By Proposition 4.2 and Hölder's inequality (see Proposition 2.1), we have

$$\Omega_R\cdot\nabla R \in M^{1,n-2+2\gamma}_{\mathrm{loc}} \hookrightarrow M^{1,n-2+\gamma}_{\mathrm{loc}},$$

$$\mathrm{skew}\left(\nabla m \circ S(\nabla m, R)\right)R \in M^{1,n-2+2\gamma}_{\mathrm{loc}} \hookrightarrow M^{1,n-2+\gamma}_{\mathrm{loc}}.$$

Extend m, R and f from B^n into $\mathbb{R}^n$ with compact support in a norm-bounded way.

Let $I_\alpha = c|x|^{\alpha-n}$ be the standard Riesz potential. Set

$$R_1 = I_2\left(\Omega_R \cdot \nabla R + \operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R\right) \quad \text{and} \quad R_2 = I_2\left(f\right)$$

so that $R_3 = R - R_1 - R_2$ is harmonic.

Note that $\frac{2-\gamma}{1-\gamma} > 2$ and $\zeta = \frac{1}{2}\left(\frac{2-\gamma}{1-\gamma}\right) > 1$. Then Proposition 2.2, together with Propositions 2.1 and 4.2, implies that for any x_0, r such that $B_{2r}(x_0) \subset B_{1/2}$, we have

$$\begin{aligned}
\|\nabla R_1\|_{M_*^{2\zeta,n-2+\gamma}(B_r)} &\lesssim \|\Omega_R \cdot \nabla R + \operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R\|_{M^{1,n-2+\gamma}(B_r)} \\
&\lesssim \|\nabla R\|_{M^{2,n-2}(B_r)}\|\nabla R\|_{M^{2,n-2+2\gamma}(B_r)} \\
&\quad + \|\nabla m\|_{M^{2,n-2}(B_r)}\|S(\nabla m, R)\|_{M^{2,n-2+2\gamma}(B_r)} \\
&\overset{(3.7)}{\lesssim} \varepsilon\|\nabla R\|_{M^{2,n-2+2\gamma}(B_r)} + \varepsilon\|\nabla m\|_{M^{2,n-2+2\gamma}(B_r)} \\
&\quad + \|\nabla m\|_{M^{2,n-2}(B_r)}\|1\|_{M^{2,n-2+2\gamma}(B_r)} \\
&\lesssim \varepsilon\left(\|\nabla R\|_{M^{2,n-2+2\gamma}(B_r)} + \|\nabla m\|_{M^{2,n-2+2\gamma}(B_r)} + 1\right) \\
&\lesssim \varepsilon\left(\|f\|_{L^p(B_1)} + 1\right),
\end{aligned} \tag{4.23}$$

where in the fourth inequality we used the estimate

$$\|\nabla m\|_{M^{2,n-2}(B_r)}\|1\|_{M^{2,n-2+2\gamma}(B_r)} \le \varepsilon r^{1-\gamma} \le \varepsilon.$$

By standard elliptic regularity theory, we have $R_2 \in W^{2,p}(B_r)$ with

$$\|\nabla R_2\|_{L^{\frac{np}{n-p}}(B_r)} \le \|R_2\|_{W^{2,p}(B_r)} \le C\|f\|_{L^p(B_1)}.$$

Applying the Hodge decomposition to the divergence free matrix $S(\nabla m, R)$, there exist $\alpha \in W_0^{1,2}(B_r, \mathbb{R}^{N\times 1} \otimes \bigwedge^2 \mathbb{R}^n)$ and a harmonic $\chi \in C^\infty(B_r, \mathbb{R}^{N\times 1} \otimes \mathbb{R}^n)$ such that

$$S(\nabla m, R) = \nabla^\perp \alpha + \chi.$$

Moreover,

$$\Delta \alpha = d\mathbb{P}_R \wedge dm - \operatorname{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right],$$

where the linear map $\mathbb{P}_R$ is defined by $\xi \mapsto 2R\mathbb{P}(R^T\xi)$.

Similarly, $d\mathbb{P}_R \wedge dm \in M^{1,n-2+2\gamma}_{\text{loc}}$ and $\operatorname{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right] \in M^{2,n-2+2\gamma}_{\text{loc}}$. It follows again from Proposition 2.1 and Proposition 4.2 that

$$\begin{aligned}
\|\nabla \alpha\|_{M_*^{2\zeta,n-2+\gamma}(B_r)} &\lesssim \|d\mathbb{P}_R \wedge dm - \operatorname{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{M^{1,n-2+\gamma}(B_r)} \\
&\lesssim \|\nabla R\|_{M^{2,n-2}(B_r)}\|\nabla m\|_{M^{2,n-2+2\gamma}(B_r)} + \|\nabla R\|_{M^{1,n-2+\gamma}(B_r)} \\
&\lesssim \varepsilon\left(\|\nabla R\|_{M^{2,n-2+2\gamma}(B_r)} + \|\nabla m\|_{M^{2,n-2+2\gamma}(B_r)}\right) \\
&\lesssim \varepsilon\left(\|f\|_{L^p(B_1)} + 1\right).
\end{aligned} \tag{4.24}$$

Since $R_3, \chi \in C^\infty$, $\nabla m, \nabla R \in L^{2\zeta,\infty}(B_r)$. Using standard estimates for harmonic functions and (4.23), we infer that for all $0 < \rho < r \le \frac{1}{2}$, there holds

$$\begin{aligned}
&\|\nabla R\|_{L^{2\zeta,\infty}(B_\rho)} \lesssim \|\nabla R_3\|_{L^{2\zeta,\infty}(B_\rho)} + \|\nabla R_1\|_{L^{2\zeta,\infty}(B_\rho)} + \|\nabla R_2\|_{L^{2\zeta,\infty}(B_\rho)} \\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2\zeta}} \|\nabla R_3\|_{L^{2\zeta,\infty}(B_r)} + \|\nabla R_1\|_{L^{2\zeta,\infty}(B_r)} + \|\nabla R_2\|_{L^{\frac{np}{n-p}}(B_r)} \\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2\zeta}} \|\nabla R\|_{L^{2\zeta,\infty}(B_r)} + \varepsilon\left(\|f\|_{L^p(B_1)} + 1\right) + \|f\|_{L^p(B_1)} \\
&\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2\zeta}} \|\nabla R\|_{L^{2\zeta,\infty}(B_r)} + \|f\|_{L^p(B_1)} + \varepsilon.
\end{aligned}$$

Similarly, with (4.24), we derive

$$\begin{aligned}\|S(\nabla m,R)\|_{L^{2\zeta,\infty}(B_\rho)} &\lesssim \|\nabla\alpha\|_{L^{2\zeta,\infty}(B_\rho)}+\|\chi\|_{L^{2\zeta,\infty}(B_\rho)}\\ &\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2\zeta}}\|\chi\|_{L^{2\zeta,\infty}(B_r)}+\|\nabla\alpha\|_{L^{2\zeta,\infty}(B_r)}\\ &\lesssim \left(\frac{\rho}{r}\right)^{\frac{n}{2\zeta}}\|S(\nabla m,R)\|_{L^{2\zeta,\infty}(B_r)}+\varepsilon\|f\|_{L^p(B_1)}+\varepsilon.\end{aligned}$$

Thus a standard iteration argument gives

$$\|\nabla R\|_{L^{2\zeta,\infty}(B_r)}+\|S(\nabla m,R)\|_{L^{2\zeta,\infty}(B_r)}\lesssim \|f\|_{L^p(B_1)}+\varepsilon.$$

Note that $\frac{n}{2\zeta}=n-p$. With (3.7) we have the estimate

$$\|\nabla m\|_{L^{2\zeta,\infty}(B_r)}+\|\nabla R\|_{L^{2\zeta,\infty}(B_r)}\lesssim \|f\|_{L^p(B_1)}+\varepsilon+r^{n-p}\lesssim \|f\|_{L^p(B_1)}+1. \tag{4.25}$$

If $2\zeta>\frac{np}{n-p}$, then Hölder's inequality implies $\nabla m,\nabla R\in L^{\frac{np}{n-p}}(B_r)$ with the estimate

$$\|\nabla m\|_{L^{\frac{np}{n-p}}(B_r)}+\|\nabla R\|_{L^{\frac{np}{n-p}}(B_r)}\lesssim \|f\|_{L^p(B_1)}+1.$$

If $2\zeta\le\frac{np}{n-p}$, then $\nabla m,\nabla R\in L^q_{\mathrm{loc}}$ for any $q\in(2,\frac{4\zeta}{1+\zeta})\subset(2,2\zeta)$. By the definition of Ω_R, $\mathbb{P}$, (3.7) and Hölder's inequality, we have

$$\Omega_R\cdot\nabla R\in M^{1,n-2+\gamma}_{\mathrm{loc}}\cap L^{\frac{q}{2}}_{\mathrm{loc}},$$

$$\mathrm{skew}\left(\nabla m\circ S(\nabla m,R)\right)R\in M^{1,n-2+\gamma}_{\mathrm{loc}}\cap L^{\frac{q}{2}}_{\mathrm{loc}},$$

and

$$d\mathbb{P}_R\wedge dm-\mathrm{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\in M^{1,n-2+\gamma}_{\mathrm{loc}}\cap L^{\frac{q}{2}}_{\mathrm{loc}}.$$

Next, we estimate the $L^{q\zeta}$ norm of ∇R_1 and $\nabla\alpha$. By (4.23) and (4.24), we have

$$\begin{aligned}&\|\Omega_R\cdot\nabla R+\mathrm{skew}\left(\nabla m\circ S(\nabla m,R)\right)R\|_{M^{1,n-2+\gamma}(B_r)}\\ &\lesssim\varepsilon\left(\|f\|_{L^p(B_1)}+1\right)\end{aligned}\tag{4.26}$$

and

$$\begin{aligned}&\|d\mathbb{P}_R\wedge dm-\mathrm{Div}^\perp\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{M^{1,n-2+\gamma}(B_r)}\\ &\lesssim\varepsilon\left(\|f\|_{L^p(B_1)}+1\right).\end{aligned}\tag{4.27}$$

Since $q/2>1$, using Hölder's inequality and (4.25), we have

$$\begin{aligned}&\|\Omega_R\cdot\nabla R+\mathrm{skew}\left(\nabla m\circ S(\nabla m,R)\right)R\|_{L^{\frac{q}{2}}(B_r)}\\ &\lesssim\|\Omega_R\cdot\nabla R+\mathrm{skew}\left(\nabla m\circ S(\nabla m,R)\right)R\|_{L^{\frac{2\zeta}{1+\zeta},\infty}(B_r)}\\ &\lesssim\|\nabla R\|_{L^2(B_r)}\|\nabla R\|_{L^{2\zeta,\infty}(B_r)}+\|S(\nabla m,R)\|_{L^2(B_r)}\|\nabla m\|_{L^{2\zeta,\infty}(B_r)}\\ &\lesssim\|\nabla R\|_{L^{2\zeta,\infty}(B_r)}+\|\nabla m\|_{L^{2\zeta,\infty}(B_r)}\\ &\lesssim\|f\|_{L^p(B_1)}+1.\end{aligned}\tag{4.28}$$

Similarly, using (4.25), we obtain

$$
\begin{aligned}
&\|d\mathbb{P}_R \wedge dm - \operatorname{Div}^{\perp}\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{L^{\frac{q}{2}}(B_r)} \\
\lesssim\, &\|d\mathbb{P}_R \wedge dm - \operatorname{Div}^{\perp}\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{L^{\frac{2\zeta}{1+\zeta},\infty}(B_r)} \\
\lesssim\, &\|\nabla R\|_{L^2(B_r)}\|\nabla m\|_{L^{2\zeta,\infty}(B_r)} + \|\nabla R\|_{L^2(B_r)}\|\nabla R\|_{L^{2\zeta,\infty}(B_r)} \\
\lesssim\, &\varepsilon\left(\|\nabla R\|_{L^{2\zeta,\infty}(B_r)} + \|\nabla m\|_{L^{2\zeta,\infty}(B_r)}\right) \\
\lesssim\, &\varepsilon\left(\|f\|_{L^p(B_1)} + 1\right).
\end{aligned} \tag{4.29}
$$

Applying Proposition 2.3, (4.26) and (4.28), we infer

$$
\begin{aligned}
\|\nabla R_1\|_{L^{q\zeta}(B_r)} &\lesssim \|\Omega_R \cdot \nabla R + \operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R\|_{M^{1,n-2+\gamma}(B_r)}^{\frac{1}{2-\gamma}} \\
&\quad \cdot \|\Omega_R \cdot \nabla R + \operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R\|_{L^{\frac{q}{2}}(B_r)}^{1-\frac{1}{2-\gamma}} \\
&\lesssim \|f\|_{L^p(B_1)} + 1.
\end{aligned}
$$

Similarly, using (4.27) and (4.29), we derive

$$
\begin{aligned}
\|\nabla \alpha\|_{L^{q\zeta}(B_r)} &\lesssim \|d\mathbb{P}_R \wedge dm - \operatorname{Div}^{\perp}\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{M^{1,n-2+\gamma}(B_r)}^{\frac{1}{2-2\gamma}} \\
&\quad \cdot \|d\mathbb{P}_R \wedge dm - \operatorname{Div}^{\perp}\left[\pi_n(2R\mathbb{P}^2(\mathbb{1}_n|0))\right]\|_{L^{\frac{q}{2}}(B_r)}^{1-\frac{1}{2-2\gamma}} \\
&\lesssim \varepsilon\left(\|f\|_{L^p(B_1)} + 1\right).
\end{aligned}
$$

Thus we find the following iteration:

$$
\nabla m, \nabla R \in L^q(B_r) \Longrightarrow \nabla m, \nabla R \in L^{q\zeta}(B_r)
$$

together with the estimate

$$
\|\nabla m\|_{L^{q\zeta}(B_r)} + \|\nabla R\|_{L^{q\zeta}(B_r)} \leq C_0\left(\|f\|_{L^p(B_1)} + 1\right),
$$

where C_0 is independent of q.

Since $\zeta > 1$, there exists some $k \geq 1$ such that $\zeta^k q \leq \frac{np}{n-p} < \zeta^{k+1}q$. After finitely many times iteration, we shall have $\nabla m, \nabla R \in L^{\frac{np}{n-p}}(B_r)$ with the estimate

$$
\|\nabla m\|_{L^{\frac{np}{n-p}}(B_r)} + \|\nabla R\|_{L^{\frac{np}{n-p}}(B_r)} \lesssim \|f\|_{L^p(B_1)} + 1.
$$

Now we can estimate the L^p norms of $\Omega_R \cdot \nabla R$ and $\operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R$ as follows:

$$
\begin{aligned}
\|\Omega_R \cdot \nabla R\|_{L^p(B_r)} &\lesssim \|\nabla R\|_{L^n(B_r)}\|\nabla R\|_{L^{\frac{np}{n-p}}(B_r)} \\
&\lesssim \|\nabla R\|^2_{L^{\frac{np}{n-p}}(B_r)} \lesssim \left(\|f\|_{L^p(B_1)} + 1\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|\operatorname{skew}\left(\nabla m \circ S(\nabla m, R)\right) R\|_{L^p(B_r)} &\lesssim \|\nabla m\|_{L^n(B_r)}\|S(\nabla m, R)\|_{L^{\frac{np}{n-p}}(B_r)} \\
&\overset{(3.7)}{\lesssim} \|\nabla m\|_{L^{\frac{np}{n-p}}(B_r)}\left(\|\nabla m\|_{L^{\frac{np}{n-p}}(B_r)} + \|1\|_{L^{\frac{np}{n-p}}(B_r)}\right) \\
&\lesssim \|\nabla m\|^2_{L^{\frac{np}{n-p}}(B_r)} + r^{\frac{2(n-p)}{p}} \lesssim \left(\|f\|_{L^p(B_1)} + 1\right)^2.
\end{aligned}
$$

Thus, we have $\Delta R \in L^p$ via (1.5). Furthermore, by the usual elliptic regularity theory, (4.9) and the above estimates, we have

$$\|R\|_{W^{2,p}(B_r)} \lesssim \left(\|f\|_{L^p(B_1)} + 1\right)^2. \tag{4.30}$$

By [7, Section 6.2], the linear operator $L_R : \xi \mapsto \pi_n(2R\mathbb{P}^2(R^T(\xi|0)))$ is uniformly positive with

$$\begin{aligned}\langle L_R(\xi), \xi\rangle &= \langle \pi_n(2R\mathbb{P}^2(R^T(\xi|0))), \xi\rangle = \langle 2R\mathbb{P}^2(R^T(\xi|0)), (\xi|0)\rangle \\ &= \langle 2\mathbb{P}(R^T(\xi|0)), \mathbb{P}(R^T(\xi|0))\rangle \geq 2\widehat{\lambda}|R^T(\xi|0)|^2 = 2\widehat{\lambda}|\xi|^2.\end{aligned}$$

Observe that (1.4) can be rewrirren as an elliptic equation

$$\mathrm{Div} L_R(\nabla m) = \mathrm{Div}(\pi_n(2R\mathbb{P}^2(R^T(\mathbb{1}_n|0)))).$$

Since the coefficients are Hölder continuous, (1.6) follows from the Calderon-Zygmund theory and (4.30).

Case 2. $p \geq n$.

In this case, $f \in L^q$ for any $q \in (\frac{n}{2}, n)$. Repeating the previous argument, we conclude that $\nabla m, \nabla R \in L^{\frac{nq}{n-q}}_{\mathrm{loc}}$ with the estimate

$$\|\nabla m\|_{L^{\frac{nq}{n-q}}(B_r)} + \|\nabla R\|_{L^{\frac{nq}{n-q}}(B_r)} \lesssim \|f\|_{L^p(B_1)} + 1.$$

This implies that $R_1, \alpha \in \bigcap_{1<s<\infty} W^{1,s}_{\mathrm{loc}}$ and so $m, R \in W^{2,p}_{\mathrm{loc}}$ with (1.6) by a similar argument as in Case 1. $\square$

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