ON THE TEMPORAL ESTIMATES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS AND HALL-MAGNETOHYDRODYNAMIC EQUATIONS

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ABSTRACT. In this paper, we derive decay rates of the solutions to the incompressible Navier-Stokes equations and Hall-magnetohydrodynamic equations. We first improve the decay rate of weak solutions of these equations by refining the Fourier splitting method with initial data in the space of pseudo-measures. We also deal with these equations with initial data in Lei-Lin spaces and find decay rates of solutions in Lei-Lin spaces.

1. INTRODUCTION

In this paper, we study the large-time behavior of two parabolic systems: the incompressible Navier-Stokes equations and Hall-magnetohydrodynamic (in short, Hall-MHD) equations in $\mathbb{R}^3$ although the incompressible Navier-Stokes can be handled in $\mathbb{R}^2$. We begin with the incompressible Navier-Stokes equations and provide several results of decay rates of solutions under appropriate assumptions to initial data. We then turn our attention to Hall-MHD equations and do similar work.

Before proceeding further, we fix some notations. We assume that C_0 is a constant depending on various norms of initial data and the parameters in the statements of our results, but independent of time. Moreover, $f \in L^\infty([0,\infty); X)$ means $\sup\limits_{0\le t<\infty} \|f(t)\|_X \le C_0$, where X is a Banach space. There are some function spaces and inequalities holding in any dimensions, but we fix the spatial dimension to 3.

1.1. The incompressible Navier-Stokes equations.

The incompressible Navier-Stokes equations are given by

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla p - \mu\Delta u = 0,\\ &\operatorname{div} u = 0, \end{aligned} \tag{1.1}$$

where u is the fluid velocity and p is the pressure. $\mu > 0$ is a viscosity coefficient, and we set $\mu = 1$ for simplicity. We begin with a weak solution of (1.1) in $\mathbb{R}^3$. The existence of a global-in-time weak solution with a divergence-free initial datum $u_0 \in L^2$ is proved in [18] where u satisfies the energy inequality for all $t > 0$:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2\, d\tau \le \|u_0\|_{L^2}^2. \tag{1.2}$$

We also notice that the following question is raised in [18]: is $\|u(t)\|_{L^2}$ convergent to 0 as $t \to \infty$? This question is answered in [21]. Later, the decay rate of a weak solution is obtained in [23] by using the Fourier splitting method: if $u_0 \in L^2 \cap L^1$,

$$\|u(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}}. \tag{1.3}$$

We now define an invariant space of (1.1). We say $\mathbb{X}$ is an invariant space when $u_0 \in \mathbb{X}$ implies $u \in L^\infty([0,\infty);\mathbb{X})$. Since $u(t)$ is also in L^1 for all $t > 0$, which is proved in Appendix A, $\mathbb{X} = L^2 \cap L^1$ is an invariant space of (1.1). In this paper, we seek to find more invariant spaces which we employ to improve (1.3). When $u_0 \in L^1$,

$$\sup_{\xi\in\mathbb{R}^3} |\widehat{u}_0(\xi)| \le \|u_0\|_{L^1}.$$

So, to improve (1.3), it is natural to impose initial data in the space of pseudo-measures [7]:

$$\mathcal{Y}^\sigma := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \ : \ \sup_{\xi\in\mathbb{R}^3} ||\xi|^\sigma \widehat{f}(\xi)| < \infty \right\}.$$

Date: April 26, 2024.
2010 *Mathematics Subject Classification.* 35Q85, 35D35.
Key words and phrases. Navier-Stokes equations, Hall-MHD, Pseudo-measures, Lei-Lin spaces, Decay rates.

Theorem 1.1. Let $u_0 \in L^2 \cap \mathcal{Y}^\sigma$ with $\operatorname{div} u_0 = 0$ and $\sigma \in [-1,1]$. Let u be a weak solution of (1.1) satisfying (1.2). Then, $u \in L^\infty([0,\infty); L^2 \cap \mathcal{Y}^\sigma)$ and u decays in time as follows:

$$\|u(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma} \quad \text{for all } t>0. \tag{1.4}$$

We have the same decay rate of (1.3) when $\sigma = 0$, but $L^1 \subset \mathcal{Y}^0$. Moreover, we improve (1.3) because σ can be negative. Theorem 1.1 can be proved by using the approach in [30]: see Appendix B. However, we could not use the same approach to Hall-MHD equations: instead, we take another method that treat (1.1) and Hall-MHD equations in the same way.

When $u_0 \in L^2 \cap L^1$, we already mention that $\|u(t)\|_{L^1} \le C_0$ for all $t > 0$. By combining this with (1.3), the decay rate of L^p norms for $p \in (1,2)$ is given by

$$\|u(t)\|_{L^p} \le \|u(t)\|_{L^1}^{\frac{2}{p}-1} \|u(t)\|_{L^2}^{2(1-\frac{1}{p})} \le C_0 \|u(t)\|_{L^2}^{2(1-\frac{1}{p})} \le C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}. \tag{1.5}$$

Although we do not have the uniform L^1 bound of u when $u_0 \in L^2 \cap \mathcal{Y}^\sigma$, we utilize the decay rate in Theorem 1.1 to bound u in negative Sobolev spaces.

Corollary 1.1. If $\sigma \in [-1,1]$ and $0 < \delta < \frac{3}{2} - \sigma$,

$$\|u(t)\|_{\dot{H}^{-\delta}}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma+\delta}. \tag{1.6}$$

In particular, (1.5) with $p \in (1,2)$ and (1.6) with $\sigma = 0$ and $\delta \in (0, \frac{3}{2})$ have the same decay rate since $L^p \subset \dot{H}^{-\delta}$ with $1 - \frac{1}{p} = \frac{1}{2} - \frac{\delta}{3}$. In [24], decay of higher-order norms of the solutions is derived using the decay rate of $\|u(t)\|_{L^2}$. As a corollary of Theorem 1.1 and [24], we can verify the following decay rates and we skip the proof of it.

Corollary 1.2. Let $u_0 \in L^2 \cap \mathcal{Y}^\sigma$ with $\operatorname{div} u_0 = 0$ and $\sigma \in [-1,1]$. For each $k \in \mathbb{N}$, there exist $T_0 > 0$ and a constant C_k depending on u_0 and k, but independent of T_0, such that

$$\left\|\nabla^k u(t)\right\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\sigma-k} \quad \text{for all } t > T_0.$$

If we take u_0 in a function space different from L^2, we normally take a smallness condition of u_0 which comes from the scaling-invariant property of u_0: $u_0(x) \longmapsto \lambda u_0(\lambda x)$. Along this direction, the best result is [16] with initial data in BMO^{-1}. However, since we want to restrict the function spaces defined in Fourier variables, we investigate (1.1) with initial data in Lei-Lin spaces $\mathcal{X}^\sigma$:

$$\mathcal{X}^\sigma := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |\xi|^\sigma |\widehat{f}(\xi)|\, d\xi < \infty \right\}, \quad \sigma \in \mathbb{R}.$$

In [17], the global well-posedness of (1.1) is established with a small $u_0 \in \mathcal{X}^{-1}$ where $\mathcal{X}^{-1} \subset \mathrm{BMO}^{-1}$. Moreover, the spatial analyticity of the solution in [17] is established in [2, 3] which provides decay rates of derivatives of u. The decay of $\|u(t)\|_{\mathcal{X}^{-1}}$ when $u_0 \in \mathcal{X}^{-1} \cap L^2$ is studied in [6] as well. We here provide a decay rate of $\|u(t)\|_{\mathcal{X}^{k-1}}$ when $\|u_0\|_{\mathcal{X}^{-1}}$ is sufficiently small.

Theorem 1.2. Let $k \ge 0$ and $\sigma \in [-1,1]$. Let $u_0 \in \mathcal{X}^{k-1} \cap \mathcal{X}^{-1} \cap \mathcal{Y}^\sigma$ with $\operatorname{div} u_0 = 0$. There exists $\epsilon > 0$ such that if $\|u_0\|_{\mathcal{X}^{-1}} \le \epsilon$, then (1.1) admits a unique solution $u \in L^\infty([0,\infty); \mathcal{X}^{k-1} \cap \mathcal{X}^{-1} \cap \mathcal{Y}^\sigma) \cap L^1((0,\infty); \mathcal{X}^{k+1} \cap \mathcal{X}^1)$. Furthermore,

$$\|u(t)\|_{\mathcal{X}^{k-1}} \le C_k(1+t)^{-1+\frac{\sigma}{2}-\frac{k}{2}} \quad \text{for all } t > 0.$$

The smallness condition is given by $\|u_0\|_{\mathcal{X}^{-1}} < 1$ (since we set $\mu = 1$) in [17]. However, we need $\|u_0\|_{\mathcal{X}^{-1}} \le \epsilon$ with ϵ possibly smaller than 1 in the proof of Theorem 1.2. The case $k = 0$ shows the decay rate of the solution in [17], and our paper goes further and covers the case $k > 0$ as well. The same argument is applied to Theorem 1.4 below.

1.2. **Hall-magnetohydrodynamic equations.** We observe that the methodology developed to (1.1) can be utilized with the incompressible and resistive Hall-MHD equations:

$$u_t + u\cdot\nabla u + \nabla p - \mu\Delta u = (\nabla\times B)\times B, \tag{1.7a}$$

$$B_t - \nabla\times(u\times B) - \nu\Delta B + \nabla\times((\nabla\times B)\times B) = 0, \tag{1.7b}$$

$$\operatorname{div} u = 0, \quad \operatorname{div} B = 0, \tag{1.7c}$$

where u is the plasma velocity field, p is the pressure, and B is the magnetic field, respectively. $\mu,\nu>0$ are viscosity and resistivity coefficients, and we set $\mu=\nu=1$ without loss of generality. $\nabla\times((\nabla\times B)\times B)$ is called the Hall term. (1.7) is important in describing many physical phenomena [5, 13, 14, 19, 22, 25, 26, 29]. Moreover, (1.7) has been actively studied mathematically after [1, 8, 9, 10]: see [4] for a list of known results.

We begin with a weak solution of (1.7) with divergence-free initial data $(u_0,B_0)\in L^2$. The existence of a weak solution satisfying the following energy inequality is proved in [8]:

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2\,d\tau + 2\int_0^t \|\nabla B(\tau)\|_{L^2}^2\,d\tau \le \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 \quad \text{for all } t>0. \tag{1.8}$$

Moreover, the decay of a weak solution is established in [10] using the Fourier splitting method: if $(u_0,B_0)\in L^2\cap L^1$, then (u,B) decays in time as follows:

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}}. \tag{1.9}$$

As in the case of (1.1), we give a result improving (1.9).

Theorem 1.3. Let $(u_0,B_0)\in(L^2\cap\mathcal{Y}^{\sigma_1})\times(L^2\cap\mathcal{Y}^{\sigma_2})$ with $\operatorname{div} u_0=\operatorname{div} B_0=0$, $\sigma_1\in[-1,1]$ and $\sigma_2\in[-1,0]$. Let (u,B) be a weak solution of (1.7) satisfying (1.8). Then, $(u,B)\in L^\infty([0,\infty);\mathcal{Y}^{\sigma_1}\times\mathcal{Y}^{\sigma_2})$ and

$$\|u(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma_1}, \quad \|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma_2} \quad \text{for all } t>0.$$

In addition to improving [10], our result also shows that u_0 and B_0 may not be in the same space. Due to the Hall term, the range of σ_2 is set to be 1 smaller than that of σ_1. The L^1 space is still an invariant space of u which can be shown as in Appendix A using (1.8) and (1.9), but the Hall term prevents us from proving that L^1 is an invariant space of B. However, at least, $(L^2\cap\mathcal{Y}^{\sigma_1})\times(L^2\cap\mathcal{Y}^{\sigma_2})$ is an invariant space of (1.7). Using the argument in [24] applied to (1.7), decay rates of higher-order norms of the solutions are derived in [10]. We here take the same approach to Theorem 1.3.

Corollary 1.3. Under the assumptions in Theorem 1.3, for each $k\in\mathbb{N}$, there exist $T_0>0$ and a constant C_k depending on (u_0,B_0) and k, but independent of T_0, such that

$$\|\nabla^k u(t)\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\sigma_1-k}, \quad \|\nabla^k B(t)\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\sigma_2-k} \quad \text{for all } t>T_0.$$

We also deal with (1.7) in Lei-Lin spaces $\mathcal{X}^\sigma$. The global well-posedness of (1.7) is established with when (u_0,B_0) is sufficiently small in $\mathcal{X}^{-1}\cap\mathcal{X}^0$ [15] and $(u_0,B_0,\nabla\times B_0)$ is sufficiently small in $\mathcal{X}^{-1}$ [20]. We here establish the global well-posendess and derive the decay rate of (u,B) using $\mathcal{Y}^\sigma$.

Theorem 1.4. Let $k\ge 0$ and $\sigma\in[-1,1]$. Let $u_0\in\mathcal{X}^{k-1}\cap\mathcal{X}^{-1}\cap\mathcal{Y}^\sigma$ and $B_0\in\mathcal{X}^{k-1}\cap\mathcal{X}^{-1}\cap\mathcal{X}^0\cap\mathcal{Y}^\sigma$ with $\operatorname{div} u_0=\operatorname{div} B_0=0$. There exists $\epsilon>0$ such that if $\|u_0\|_{\mathcal{X}^{-1}}+\|B_0\|_{\mathcal{X}^{-1}}+\|B_0\|_{\mathcal{X}^0}\le\epsilon$, then (1.7) admits a unique solution $u\in L^\infty([0,\infty);\mathcal{X}^{k-1}\cap\mathcal{X}^{-1}\cap\mathcal{Y}^\sigma)\cap L^1((0,\infty);\mathcal{X}^{k+1}\cap\mathcal{X}^1)$, $B\in L^\infty([0,\infty);\mathcal{X}^{k-1}\cap\mathcal{X}^{-1}\cap\mathcal{X}^0\cap\mathcal{Y}^\sigma)\cap L^1((0,\infty);\mathcal{X}^{k+1}\cap\mathcal{X}^1\cap\mathcal{X}^2)$. Furthermore,

$$\|u(t)\|_{\mathcal{X}^{k-1}} + \|B(t)\|_{\mathcal{X}^{k-1}} \le C_k(1+t)^{-1+\frac{\sigma}{2}-\frac{k}{2}} \quad \text{for all } t>0.$$

2. Preliminaries

All constants will be denoted by C and we follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression. We also use the simplified form of the integral of the spatial variables:

$$\int =: \int_{\mathbb{R}^3} dx.$$

We begin with 2 inequalities:

(1) For all $x > 0$ and $p > 0$

$$|x^p e^{-ax^2}| \le |x^p e^{-ax^2}|\Big|_{x=\sqrt{\frac{p}{2a}}} = \left(\frac{p}{2a}\right)^{\frac{p}{2}} e^{-\frac{p}{2}}. \tag{2.1}$$

(2) For $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$

$$\int_0^t (t-\tau)^{-\alpha}\tau^{-\beta} d\tau = \int_0^1 (1-\theta)^{-\alpha}\theta^{-\beta}\, d\theta = \mathcal{B}(\alpha, \beta),$$

where $\mathcal{B}(\cdot,\cdot)$ is the beta function.

We give the following Sobolev inequalities in 3D:

$$\|f\|_{L^3} \le C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}, \quad \|f\|_{L^6} \le C\|\nabla f\|_{L^2},$$

and a product estimate which is called Leibniz rule [11], [28, Page 105]: for $1 < p < \infty$ and $1 < p_1, p_2, p_3, p_4 \le \infty$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p},$$

$$\left\|\nabla^k (fg)\right\|_{L^p} \le C\left\|\nabla^k f\right\|_{L^{p_1}}\|g\|_{L^{p_2}} + C\|f\|_{L^{p_3}}\left\|\nabla^k g\right\|_{L^{p_4}}. \tag{2.2}$$

We finally present 3 interpolation results in $\mathcal{X}^\sigma$:

$$\sigma \in (-1,1) : \|f\|_{\mathcal{X}^{-\sigma}} \le C\|f\|_{\mathcal{X}^{-1}}^{\frac{1+\sigma}{2}}\|f\|_{\mathcal{X}^1}^{\frac{1-\sigma}{2}}, \tag{2.3a}$$

$$\sigma \in (-1,0) : \|f\|_{\mathcal{X}^{-\sigma}} \le C\|f\|_{\mathcal{X}^0}^{\frac{2+\sigma}{2}}\|f\|_{\mathcal{X}^2}^{\frac{-\sigma}{2}}, \tag{2.3b}$$

$$\sigma \in [0,k) : \|f\|_{\mathcal{X}^{\sigma}} \le C\|f\|_{\mathcal{X}^{-1}}^{\frac{k-\sigma}{k+1}}\|f\|_{\mathcal{X}^k}^{\frac{\sigma+1}{k+1}}. \tag{2.3c}$$

2.1. **Main Lemmas.** We now provide two lemmas being central to the proofs of our decay rates results.

Lemma 2.1. Let f be a smooth function satisfying

$$\frac{d}{dt}\|f(t)\|_{L^2}^2 + \|\nabla f(t)\|_{L^2}^2 \le 0$$

for all $t > 0$. Suppose there exists a positive constant $C_* > 0$ and $\sigma < \frac{3}{2}$ such that

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1} |\xi|^\sigma \left|\widehat{f}(t,\xi)\right| \le C_*.$$

Then,

$$\|f(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}+\sigma} \quad \text{for all } t \ge N-1,$$

where $N-1$ is a non-negative constant with $N > \frac{3}{2} - \sigma$.

Proof. By using the Plancherel's theorem,

$$\begin{aligned}\frac{d}{dt}\|f(t)\|_{L^2}^2 &\le -\int |\xi|^2\left|\widehat{f}(t,\xi)\right|^2 d\xi \le -\int_{\{|\xi|^2 > \frac{N}{1+t}\}} |\xi|^2 \left|\widehat{f}(t,\xi)\right|^2 d\xi \\ &\le -\frac{N}{1+t}\int_{\{|\xi|^2 > \frac{N}{1+t}\}}\left|\widehat{f}(t,\xi)\right|^2 d\xi \\ &= -\frac{N}{1+t}\|f(t)\|_{L^2}^2 + \frac{N}{1+t}\int_{\{|\xi|^2 \le \frac{N}{1+t}\}}\left|\widehat{f}(t,\xi)\right|^2 d\xi.\end{aligned}$$

Hence, we attain

$$\frac{d}{dt}\|f(t)\|_{L^2}^2 + \frac{N}{1+t}\|f(t)\|_{L^2}^2 \le \frac{N}{1+t}\int_{\{|\xi|^2 \le \frac{N}{1+t}\}}\left|\widehat{f}(t,\xi)\right|^2 d\xi,$$

which implies

$$\frac{d}{dt}\left[(1+t)^N\|f(t)\|_{L^2}^2\right] \le N(1+t)^{N-1}\int_{\{|\xi|^2 \le \frac{N}{1+t}\}}\left|\widehat{f}(t,\xi)\right|^2 d\xi. \tag{2.4}$$

Since $\left\{\xi : |\xi|^2 \leq N/(1+t)\right\} \subseteq \{\xi : |\xi| \leq 1\}$ for all $t \geq N-1$, we use our assumption to bound the right-hand side of (2.4) as follows:

$$\begin{aligned}\frac{d}{dt}\left[(1+t)^N \|f(t)\|_{L^2}^2\right] &\leq NC_*(1+t)^{N-1}\int_{\{|\xi|^2\leq \frac{N}{1+t}\}} |\xi|^{-2\sigma}\, d\xi \\ &\leq C(1+t)^{N-1}\int_0^{\sqrt{N/(1+t)}} r^{2-2\sigma}\, dr \leq C(1+t)^{N-1+\sigma-\frac{3}{2}}\end{aligned}$$

when $\sigma < \frac{3}{2}$. Finally, we integrate the inequality with respect to time and obtain

$$\begin{aligned}\|f(t)\|_{L^2}^2 &\leq \frac{\|f(N-1)\|_{L^2}^2}{(1+t)^N} + \frac{C}{N-\frac{3}{2}+\sigma}\frac{(1+t)^{N-\frac{3}{2}+\sigma} - N^{N-\frac{3}{2}+\sigma}}{(1+t)^N} \\ &\leq C_0(1+t)^{-\frac{3}{2}+\sigma}\end{aligned}$$

for $t \geq N-1$, where we use $N > \frac{3}{2} - \sigma$. This complete the proof of Lemma 2.1. □

Lemma 2.2. Let $k \geq 0$ and $\theta > 0$. Let f be a smooth function satisfying the following inequality

$$\frac{d}{dt}\|f(t)\|_{\mathcal{X}^{k-1}} + \theta\|f(t)\|_{\mathcal{X}^{k+1}} \leq 0. \tag{2.5}$$

for all $t > 0$. Suppose there exists a positive constant $C_* > 0$ and $\sigma < k+2$ such that

$$\sup_{0\leq t<\infty}\sup_{|\xi|\leq 1} |\xi|^\sigma \left|\widehat{f}(t,\xi)\right| \leq C_*. \tag{2.6}$$

Then,

$$\|f(t)\|_{\mathcal{X}^{k-1}} \leq C_0(1+t)^{-1+\frac{\sigma}{2}-\frac{k}{2}} \quad \text{for all } t \geq N/\theta - 1,$$

where $N/\theta - 1$ is a non-negative constant with $N > 1 + \frac{k}{2} - \frac{\sigma}{2}$.

Proof. From (2.5),

$$\begin{aligned}\frac{d}{dt}\|f(t)\|_{\mathcal{X}^{k-1}} &\leq -\theta\int_{\{\theta|\xi|^2 > \frac{N}{1+t}\}} |\xi|^{k+1}\left|\widehat{f}(t,\xi)\right| d\xi \\ &\leq -\frac{N}{1+t}\|f(t)\|_{\mathcal{X}^{k-1}} + \frac{N}{1+t}\int_{\{\theta|\xi|^2\leq\frac{N}{1+t}\}} |\xi|^{k-1}\left|\widehat{f}(t,\xi)\right| d\xi.\end{aligned}$$

Since $\left\{\xi : \theta|\xi|^2 \leq N/(1+t)\right\} \subseteq \{\xi : |\xi| \leq 1\}$ for $t \geq N/\theta - 1$, we obtain

$$\begin{aligned}\frac{d}{dt}\left[(1+t)^N\|f(t)\|_{\mathcal{X}^{k-1}}\right] &\leq NC_*(1+t)^{N-1}\int_{\{\theta|\xi|^2\leq\frac{N}{1+t}\}} |\xi|^{k-\sigma-1}\, d\xi \\ &\leq C(1+t)^{N-1}\int_0^{\sqrt{N/(\theta(1+t))}} r^{k-\sigma+1}\, dr \leq C(1+t)^{N-1-1+\frac{\sigma}{2}-\frac{k}{2}}\end{aligned}$$

from this we deduce that

$$\begin{aligned}\|f(t)\|_{\mathcal{X}^{k-1}} &\leq \frac{\|f(N/\theta-1)\|_{\mathcal{X}^{k-1}}}{(1+t)^N} + \frac{C}{N-1+\frac{\sigma}{2}-\frac{k}{2}}\frac{(1+t)^{N-1+\frac{\sigma}{2}-\frac{k}{2}} - (N/\theta)^{N-1+\frac{\sigma}{2}-\frac{k}{2}}}{(1+t)^N} \\ &\leq C_0(1+t)^{-1+\frac{\sigma}{2}-\frac{k}{2}}\end{aligned}$$

for $t \geq N/\theta - 1$, where we use $N > 1 + \frac{k}{2} - \frac{\sigma}{2}$. This completes the proof of Lemma 2.2. □

3. Incompressible Naiver-Stokes equations

In this section, we prove the decay rates results for (1.1). Let

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\left(\operatorname{div}(u\otimes u)\right)(\tau)\, d\tau, \tag{3.1}$$

where $\mathbb{P}$ is the Leray projection operator with its matrix valued Fourier multiplier $m(\xi)$: for a vector field v,

$$\mathbb{P}v = v - \nabla\Delta^{-1}\operatorname{div} v, \quad m_{ij}(\xi) = \delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}. \tag{3.2}$$

By taking the Fourier transform to (3.1), we have

$$\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0(\xi) - \int_0^t e^{-(t-\tau)|\xi|^2} m(\xi) i\xi \cdot \left(\widehat{u\otimes u}\right)(\tau,\xi)d\tau. \tag{3.3}$$

Since $|m_{ij}(\xi)| \leq 1$, we may proceed to bound u as if m is absent. We also bound $\|\widehat{fg}\|_{L^\infty_\xi}$ by Young's inequality and the Plancherel's theorem:

$$\|\widehat{fg}\|_{L^\infty_\xi} \leq \|f\|_{L^2}\|g\|_{L^2}.$$

3.1. **Proof of Theorem 1.1.** Since we already have (1.2), we only need to show $u \in L^\infty([0,\infty);\mathcal{Y}^\sigma)$. Then, Lemma 2.1 gives the desired decay rate in Theorem 1.1. To prove $u \in L^\infty([0,\infty);\mathcal{Y}^\sigma)$, we divide the range of σ into 3 cases.

► (**Case 1**: $\sigma = 1$): By multiplying (3.3) by $|\xi|$

$$\begin{aligned}|\xi||\widehat{u}(t,\xi)| &\leq |\xi|e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|\widehat{u\otimes u}(\tau)\|_{L^\infty_\xi}\,d\tau \\ &\leq \|u_0\|_{\mathcal{Y}^1} + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \leq \|u_0\|_{\mathcal{Y}^1} + \|u_0\|_{L^2}^2 \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\,d\tau \leq C_0\end{aligned}$$

from which we deduce that

$$\sup_{0\leq t<\infty} \|u(t)\|_{\mathcal{Y}^1} \leq C_0.$$

► (**Case 2**: $\sigma \in [0,1)$). For $|\xi| \leq 1$, we obtain

$$|\xi||\widehat{u}(t,\xi)| \leq |\xi|^{1-\sigma}|\xi|^\sigma e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \leq \|u_0\|_{\mathcal{Y}^\sigma} + \|u_0\|_{L^2}^2 \leq C_0.$$

Then, Lemma 2.1 brings us

$$\|u(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{1}{2}} \quad \text{for all } t > 0.$$

By (2.1),

$$\begin{aligned}|\xi|^\sigma|\widehat{u}(t,\xi)| &\leq |\xi|^\sigma e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^{1+\sigma}e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \\ &\leq \|u_0\|_{\mathcal{Y}^\sigma} + C_0\int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau \leq C_0,\end{aligned} \tag{3.4}$$

where we use

$$\begin{aligned}\int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau &\leq \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}(1+\tau)^{-\frac{1-\sigma}{2}}\,d\tau \leq \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}\tau^{-\frac{1-\sigma}{2}}\,d\tau \\ &= \int_0^1 (1-\theta)^{-\frac{1+\sigma}{2}}\theta^{-\frac{1-\sigma}{2}}\,d\theta = \mathcal{B}\left(\frac{1-\sigma}{2},\frac{1+\sigma}{2}\right).\end{aligned} \tag{3.5}$$

► (**Case 3**: $\sigma \in [-1,0]$) When $\sigma \in [-1,0]$, we have

$$\sup_{0\leq t<\infty}\sup_{|\xi|\leq 1}|\xi||\widehat{u}(t,\xi)| \leq C_0, \quad \text{and} \quad \|u(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{1}{2}} \quad \text{for all } t > 0$$

as in **Case 2**. For $|\xi| \leq 1$ we obtain

$$|\widehat{u}(t,\xi)| \leq e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \leq \|u_0\|_{\mathcal{Y}^\sigma} + C_0\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau \leq C_0,$$

which implies

$$\|u(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{3}{2}} \quad \text{for all } t > 0.$$

So, we bound u as

$$\begin{aligned}|\xi|^\sigma|\widehat{u}(t,\xi)| &\leq |\xi|^\sigma e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^{1+\sigma}e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \\ &\leq \|u_0\|_{\mathcal{Y}^\sigma} + C_0\int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}(1+\tau)^{-\frac{3}{2}}\,d\tau \leq C_0.\end{aligned} \tag{3.6}$$

Thus, we use Lemma 2.1 to obtain the desired temporal decay and this completes the proof of Theorem 1.1.

3.2. **Proof of Corollary 1.1.** We decompose $\|u\|_{\dot{H}^{-\delta}}^2$ as follows:

$$\|u\|_{\dot{H}^{-\delta}}^2 = \int_{|\xi|\le M} |\xi|^{-2\delta} \left|\widehat{u}(\xi)\right|^2 d\xi + \int_{|\xi|\ge M} |\xi|^{-2\delta} \left|\widehat{u}(\xi)\right|^2 d\xi \le (\mathrm{I}) + M^{-2\delta}\|u\|_{L^2}^2,$$

where M is decided below. We now bound (I): when $\delta < \frac{3}{2} - \sigma$

$$(\mathrm{I}) = \int_{|\xi|\le M} |\xi|^{-2\delta-2\sigma}|\xi|^{2\sigma} \left|\widehat{u}(\xi)\right|^2 d\xi \le C\|u\|_{\mathcal{Y}^\sigma}^2 \int_0^M r^{-2\delta-2\sigma+2} dr = C\|u\|_{\mathcal{Y}^\sigma}^2 M^{-2\delta-2\sigma+3}.$$

By taking M satisfying

$$M^{-2\delta}\|u\|_{L^2}^2 = \|u\|_{\mathcal{Y}^\sigma}^2 M^{-2\delta-2\sigma+3},$$

we have

$$\|u(t)\|_{\dot{H}^{-\delta}}^2 \le C\|u(t)\|_{\mathcal{Y}^\sigma}^{\frac{2\delta}{3/2-\sigma}} \|u(t)\|_{L^2}^{\frac{3-2\sigma-2\delta}{3/2-\sigma}} \le C_0(1+t)^{-\frac{3}{2}+\sigma+\delta},$$

which implies the desired estimate.

3.3. **Proof of Theorem 1.2.** From [17], we deduce that if $\|u_0\|_{\mathcal{X}^{-1}} \le \epsilon < 1$, then $u \in L^\infty([0,\infty);\mathcal{X}^{-1}) \cap L^1([0,\infty);\mathcal{X}^1)$ and

$$\|u(t)\|_{\mathcal{X}^{-1}} + (1-\epsilon)\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\, d\tau \le \epsilon \quad \text{for all } t > 0.$$

Moreover, u satisfies (2.5) with $k = 0$ and $\theta = 1 - \epsilon$.

We now show $u \in L^\infty([0,\infty);\mathcal{Y}^\sigma)$: in doing so, we use

$$1 \le \frac{|\eta|^\alpha}{2|\xi-\eta|^\alpha} + \frac{|\xi-\eta|^\alpha}{2|\eta|^\alpha}. \tag{3.7}$$

▶ (**Case 1**: $\sigma = 1$) By using (3.7) with $\alpha = 1$, we obtain

$$\begin{aligned}
|\xi||\widehat{u}(t,\xi)| &\le |\xi|e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \int |\widehat{u}(\tau,\xi-\eta)||\widehat{u}(\tau,\eta)|\, d\eta d\tau \\
&\le \|u_0\|_{\mathcal{Y}^1} + \sup_{0\le\tau\le t} (\|u(\tau)\|_{\mathcal{Y}^1}\|u(\tau)\|_{\mathcal{X}^{-1}}) \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\, d\tau \\
&\le \|u_0\|_{\mathcal{Y}^1} + C\epsilon \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^1}
\end{aligned}$$

and so we have

$$\sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^1} \le \|u_0\|_{\mathcal{Y}^1} + C\epsilon \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^1}.$$

▶ (**Case 2**: $\sigma \in [-1,1)$) When $\sigma \in (-1,1)$, we use (3.7) with $\alpha = \sigma$ to estimate u as

$$\begin{aligned}
|\xi|^\sigma|\widehat{u}(t,\xi)| &\le |\xi|^\sigma e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^{1+\sigma} e^{-(t-\tau)|\xi|^2} \int_{\mathbb{R}^3} |\widehat{u}(\tau,\xi-\eta)||\widehat{u}(\tau,\eta)|\, d\eta d\tau \\
&\le \|u_0\|_{\mathcal{Y}^\sigma} + \int_0^t |\xi|^{1+\sigma} e^{-(t-\tau)|\xi|^2} \|u(\tau)\|_{\mathcal{X}^{-\sigma}}\|u(\tau)\|_{\mathcal{Y}^\sigma}\, d\tau \\
&\le \|u_0\|_{\mathcal{Y}^\sigma} + \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^\sigma} \left(\int_0^t |\xi|^2 e^{-(t-\tau)\frac{2}{1+\sigma}|\xi|^2}\, d\tau\right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|u(\tau)\|_{\mathcal{X}^{-\sigma}}^{\frac{2}{1-\sigma}}\, d\tau\right)^{\frac{1-\sigma}{2}} \\
&\le \|u_0\|_{\mathcal{Y}^\sigma} + C \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^\sigma} \left(\sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{X}^{-1}}\right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\, d\tau\right)^{\frac{1-\sigma}{2}} \\
&\le \|u_0\|_{\mathcal{Y}^\sigma} + C\epsilon \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathcal{Y}^\sigma},
\end{aligned}$$

where we also use (2.3a) to handle $\|u(\tau)\|_{\mathcal{X}^{-\sigma}}$. When $\sigma=-1$, we use (3.7) with $\alpha=1$ to get

$$\begin{aligned}|\xi|^{-1}|\widehat{u}(t,\xi)| &\le |\xi|^{-1}e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t e^{-(t-\tau)|\xi|^2}\int |\widehat{u}(\tau,\xi-\eta)||\widehat{u}(\tau,\eta)|\,d\eta d\tau \\ &\le \|u_0\|_{\mathcal{Y}^{-1}} + \sup_{0\le\tau\le t}\|u(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\,d\tau.\end{aligned}$$

▶ By collecting all the bounds together, we arrive at

$$\sup_{0\le t<\infty}\|u(t)\|_{\mathcal{Y}^\sigma} \le \|u_0\|_{\mathcal{Y}^\sigma} + C\epsilon \sup_{0\le t<\infty}\|u(t)\|_{\mathcal{Y}^\sigma}$$

for all $\sigma\in[-1,1]$. By restricting the size of ϵ as $2C\epsilon<1$, we obtain $u\in L^\infty([0,\infty);\mathcal{Y}^\sigma)$. The desired decay rates follow by Lemma 2.2.

▶ Finally when $k>0$, we need to show that $u\in L^\infty([0,\infty);\mathcal{X}^{k-1})\cap L^1([0,\infty);\mathcal{X}^{k+1})$ and u satisfies (2.5) with $k>0$. By using (2.3c), we obtain

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{\mathcal{X}^{k-1}} + \|u(t)\|_{\mathcal{X}^{k+1}} &\le \int\int |\xi|^k|\widehat{u}(t,\xi-\eta)||\widehat{u}(t,\eta)|\,d\eta d\xi \le C\|u(t)\|_{\mathcal{X}^0}\|u(t)\|_{\mathcal{X}^k} \\ &\le C\|u(t)\|_{\mathcal{X}^{-1}}\|u(t)\|_{\mathcal{X}^{k+1}} \le C\epsilon\|u(t)\|_{\mathcal{X}^{k+1}}.\end{aligned}$$

Then we take a sufficiently small ϵ satisfying $\theta=1-C\epsilon>0$ to have (2.5) with $k>0$ and direct application of Lemma 2.2 completes the proof.

4. Hall-magnetohydrodynamic equations

In this section, we prove the decay rates results for Hall MHD (1.7). We first write

$$\begin{aligned}&(\nabla\times B)\times B = B\cdot\nabla B - \frac{1}{2}\nabla|B|^2 = \operatorname{div}(B\otimes B) - \frac{1}{2}\nabla|B|^2, \\ &\nabla\times((\nabla\times B)\times B) = \nabla\times\left(\operatorname{div}(B\otimes B) - \frac{1}{2}\nabla|B|^2\right) = \nabla\times\operatorname{div}(B\otimes B),\end{aligned}$$

and express (u,B) as the integral form

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\left(\operatorname{div}(u\otimes u) - \operatorname{div}(B\otimes B)\right)(\tau)\,d\tau, \tag{4.1a}$$

$$B(t) = e^{t\Delta}B_0 + \int_0^t e^{(t-\tau)\Delta}\left(\nabla\times(u\times B) - \nabla\times\operatorname{div}(B\otimes B)\right)(\tau)\,d\tau. \tag{4.1b}$$

By taking the Fourier transform to (4.1), we have

$$\begin{aligned}&\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0(\xi) - \int_0^t e^{-(t-\tau)|\xi|^2}m(\xi)i\xi\cdot\left(\widehat{u\otimes u} - \widehat{B\otimes B}\right)(\tau,\xi)d\tau, \\ &\widehat{B}(t,\xi) = e^{-t|\xi|^2}\widehat{B}_0(\xi) + \int_0^t e^{-(t-\tau)|\xi|^2}i\xi\times(\widehat{u\times B})(\tau,\xi)d\xi - \int_0^t e^{-(t-\tau)|\xi|^2}i\xi\times i\xi\cdot(\widehat{B\otimes B})(\tau,\xi)d\xi.\end{aligned}$$

4.1. Proof of Theorem 1.3.

As a preliminary step for the proof of Theorem 1.3, we give two lemmas.

Lemma 4.1. Under the assumptions in Theorem 1.3,

$$u\in L^\infty([0,\infty);\mathcal{Y}^{\sigma_1}) \quad \text{and} \quad \sup_{0\le t<\infty}\sup_{|\xi|\le 1}|\widehat{B}(t,\xi)|\le C_0.$$

Moreover,

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\max\{\sigma_1,\sigma_2\}} \quad \text{for all } t>0.$$

Proof of Lemma 4.1. Although (4.1a) contains $B\otimes B$, we can use (1.8) to bound u as in the case of (1.1). So, we just copy the bounds of u from Section 3.1 and focus on estimating B. During the proof of Lemma 4.1, we will repeatedly use Lemma 2.1 with $\|f\|_{L^2}^2=\|u\|_{L^2}^2+\|B\|_{L^2}^2$ and $\|\nabla f\|_{L^2}^2=\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2$.

▶ (**Case 1**: $\sigma_1\in[0,1]$, $\sigma_2\in[-1,0]$) Using (1.8) we first have

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1}|\xi||\widehat{u}(t,\xi)|\le C_0. \tag{4.2}$$

For B, we note that for $|\xi| \le 1$,

$$|\xi||\widehat{B}(t,\xi)| \le |\xi|e^{-t|\xi|^2}|\widehat{B}_0(\xi)| + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}\|B(\tau)\|_{L^2}\,d\tau + \int_0^t |\xi|^3 e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{L^2}^2\,d\tau$$
$$\le \|B_0\|_{\mathcal{Y}^{\sigma_2}} + 2(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2)\int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\,d\tau \le C_0.$$

By combining this with (4.2),

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1}\Big(|\xi||\widehat{u}(t,\xi)| + |\xi||\widehat{B}(t,\xi)|\Big) \le C_0.$$

Hence, we apply $\sigma = 1$ to Lemma 2.1 to get

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{1}{2}} \quad \text{for all } t>0. \tag{4.3}$$

Then we use (4.3) and proceed similarly to (3.4) to obtain

$$\sup_{0\le t<\infty}\|u(t)\|_{\mathcal{Y}^{\sigma_1}} \le C_0.$$

This also implies that for $|\xi| \le 1$

$$|\widehat{B}(t,\xi)| \le |\xi|^{-\sigma_2}|\xi|^{\sigma_2}e^{-t|\xi|^2}|\widehat{B}_0(\xi)| + \int_0^t |\xi|e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}\|B(\tau)\|_{L^2}\,d\tau + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{L^2}^2\,d\tau$$
$$\le \|B_0\|_{\mathcal{Y}^{\sigma_2}} + C_0\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau + C_0 \le C_0.$$

Since $\sigma_1 \in [0,1]$, we arrive at

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1}\Big(|\xi|^{\sigma_1}|\widehat{u}(t,\xi)| + |\xi|^{\sigma_1}|\widehat{B}(t,\xi)|\Big) \le C_0,$$

and by Lemma 2.1,

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma_1} \quad \text{for all } t>0.$$

► (**Case 2**: $\sigma_1 \in [-1,0]$, $\sigma_2 \in [-1,0]$): Here, we proceed as **Case 1** so that

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1}\Big(|\widehat{u}(t,\xi)| + |\widehat{B}(t,\xi)|\Big) < \infty,$$

and hence

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}} \quad \text{for all } t>0.$$

Then we use the above inequality and argue similarly to (3.6) to yield

$$\sup_{0\le t<\infty}\|u(t)\|_{\mathcal{Y}^{\sigma_1}} \le C_0$$

and for B,

$$\begin{aligned}|\xi|^{\sigma_2}|\widehat{B}(t,\xi)| &\le |\xi|^{\sigma_2}e^{-t|\xi|^2}|\widehat{B}_0(\xi)| + \int_0^t |\xi|^{1+\sigma_2}e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}\|B(\tau)\|_{L^2}\,d\tau\\ &\quad + \int_0^t |\xi|^{2+\sigma_2}e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{L^2}^2\,d\tau\\ &\le \|B_0\|_{\mathcal{Y}^{\sigma_2}} + C_0\int_0^t (t-\tau)^{-\frac{1+\sigma_2}{2}}(1+\tau)^{-\frac{3}{2}}\,d\tau + C_0\int_0^t (t-\tau)^{-\frac{2+\sigma_2}{2}}(1+\tau)^{-\frac{3}{2}}\,d\tau \le C_0.\end{aligned} \tag{4.4}$$

Hence we arrive at

$$\sup_{0\le t<\infty}\sup_{|\xi|\le 1}|\xi|^{\max\{\sigma_1,\sigma_2\}}\Big(|\widehat{u}(t,\xi)| + |\widehat{B}(t,\xi)|\Big) \le C_0$$

from which we obtain the desired temporal decay by Lemma 2.1. This completes the proof of Lemma 4.1. □

Lemma 4.2. Under the assumptions in Theorem 1.3, for each $k \in \mathbb{N}$ there exists $T_0 > 0$ and a constant C_k depending on (u_0, B_0) and k, but independent of T_0, such that

$$\|\nabla^k u(t)\|_{L^2}^2 + \|\nabla^k B(t)\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\max\{\sigma_1,\sigma_2\}-k} \quad \text{for all } t>T_0.$$

Proof of Lemma 4.2. Lemma 4.2 can be proved by simply modifying the argument in [10] with Lemma 4.1, so we omit the proof. □

Now we are ready to prove Theorem 1.3 by estimating u and B separately.

4.1.1. **Decay rate of u.** When $\sigma_1 \geq \sigma_2$, $\|u(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}+\sigma_1}$ from Lemma 4.1. So, we only consider $\sigma_1 < \sigma_2 \leq 0$. Then, by Lemma 4.1 and Lemma 4.2, we have

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}+\sigma_2}, \quad \|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{5}{2}+\sigma_2} \quad \text{for all } t > T_0.$$

By taking L^2 inner product of (1.7a) with u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &= \int (B\cdot\nabla B)\cdot u \leq C\|\nabla u\|_{L^2}\|B\|_{L^6}\|B\|_{L^3} \\ &\leq \frac{1}{2}\|\nabla u\|_{L^2}^2 + C\|B\|_{L^2}\|\nabla B\|_{L^2}^3.\end{aligned}$$

Hence, for $t > T_0$

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C_0(1+t)^{-\frac{9}{2}+2\sigma_2}.$$

By modifying Lemma 2.1, we discover

$$\begin{aligned}\frac{d}{dt}\left[(1+t)^N\|u(t)\|_{L^2}^2\right] &\leq N(1+t)^{N-1}\int_{\{|\xi|^2\leq\frac{N}{1+t}\}} |\widehat{u}(t,\xi)|^2 d\xi + C(1+t)^{N-\frac{9}{2}+2\sigma_2} \\ &\leq C(1+t)^{N-1-\frac{3}{2}+\sigma_1} + C(1+t)^{N-\frac{9}{2}+2\sigma_2} \leq C(1+t)^{N-1-\frac{3}{2}+\sigma_1}.\end{aligned}$$

By integrating this in time, we derive

$$\|u(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}+\sigma_1} \quad \text{for all } t > T_0.$$

The decay rates of u for all $t > 0$ follows from (1.8).

4.1.2. **Decay rate of B.** When $\sigma_1 \leq \sigma_2$, $\|B(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}+\sigma_2}$ and $B \in L^\infty([0,\infty);\mathcal{Y}^{\sigma_2})$ from Lemma 4.1 and (4.4). So, we consider $\sigma_1 > \sigma_2$. Then, we have

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}+\sigma_1}.$$

By taking L^2 inner product of (1.7b) with B, we obtain

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = \int (B\cdot\nabla u)\cdot B \leq C^*\|u\|_{L^3}\|\nabla B\|_{L^2}^2,$$

and note that there exists $t_0 \geq 0$ such that

$$\|u(t)\|_{L^3} \leq \|u(t)\|_{L^2}^{1/2}\|\nabla u(t)\|_{L^2}^{1/2} < \frac{1}{2C^*} \quad \text{for all } t \geq t_0.$$

Hence, one has

$$\frac{d}{dt}\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 \leq 0 \quad \text{for all } t \geq t_0.$$

Since

$$\sup_{0\leq t<\infty}\sup_{|\xi|\leq 1}|\widehat{B}(t,\xi)| \leq C_0$$

from Lemma 4.1, Lemma 2.1 yields

$$\|B(t)\|_{L^2}^2 \leq C_0(1+t)^{-\frac{3}{2}}.$$

If $\sigma_2 \neq -1$, then $\sigma_1 - \sigma_2 < 2$. So. we obtain

$$\begin{aligned}|\xi|^{\sigma_2}|\widehat{B}(t,\xi)| &\leq \|B_0\|_{\mathcal{Y}^{\sigma_2}} + \int_0^t |\xi|^{1+\sigma_2}e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}\|B(\tau)\|_{L^2}\,d\tau + \int_0^t |\xi|^{2+\sigma_2}e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{L^2}^2\,d\tau \\ &\leq \|B_0\|_{\mathcal{Y}^{\sigma_2}} + C_0\int_0^t (t-\tau)^{-\frac{1+\sigma_2}{2}}(1+\tau)^{-\frac{3}{2}+\frac{\sigma_1}{2}}\,d\tau + C_0\int_0^t (t-\tau)^{-\frac{2+\sigma_2}{2}}(1+\tau)^{-\frac{3}{2}}\,d\tau \leq C_0,\end{aligned}$$

where we use

$$\int_0^t (t-\tau)^{-\frac{1+\sigma_2}{2}}(1+\tau)^{-\frac{3}{2}+\frac{\sigma_1}{2}}\,d\tau \leq \int_0^t (t-\tau)^{-\frac{1+\sigma_2}{2}}\tau^{-\frac{1}{2}+\frac{\sigma_2}{2}}\,d\tau = \mathcal{B}\left(\frac{1-\sigma_2}{2},\frac{1+\sigma_2}{2}\right).$$

Hence, by Lemma 2.1

$$\|B(t)\|_{L^2}^2 \le C_0(1+t)^{-\frac{3}{2}+\sigma_2}.$$

If $\sigma_2 = -1$, when $|\xi| \le 1$ we have

$$\begin{aligned}|\xi|^{-\frac{1}{2}}|\widehat{B}(t,\xi)| &\le \|B_0\|_{\mathcal{Y}^{-1}} + \int_0^t |\xi|^{\frac{1}{2}} e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}\|B(\tau)\|_{L^2}\,d\tau + \int_0^t |\xi|^{\frac{3}{2}} e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{L^2}^2\,d\tau \\ &\le \|B_0\|_{\mathcal{Y}^{-1}} + C_0\int_0^t (t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{3}{2}+\frac{\sigma_1}{2}}\,d\tau + C_0\int_0^t (t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}}\,d\tau \le C_0,\end{aligned}$$

which implies

$$\|B(t)\|_{L^2}^2 \le C_0(1+t)^{-2}.$$

Thus, we bound

$$\sup_{0\le t<\infty}\|B(t)\|_{\mathcal{Y}^{-1}} \le C_0$$

and obtain the desired decay rate of B by Lemma 2.1.

4.2. **Proof of Corollary 1.3.** We derive higher-order norms of (u, B) by applying Theorem 1.3 and Lemma 4.2. We note that $T_0 > 0$ can vary from expression to expression in this section.

4.2.1. **Decay rate of $\nabla^k u$.** When $\sigma_1 \ge \sigma_2$, decay rates of $\nabla^k u$ are straightforward from Lemma 4.2. So, we consider the case $\sigma_1 < \sigma_2 \le 0$.

By taking ∇^k to (1.7a) and by taking the inner product with $\nabla^k u$, and (2.2), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\nabla^k u\right\|_{L^2}^2 + \left\|\nabla^{k+1} u\right\|_{L^2}^2 &= -\int \nabla^k(u\cdot\nabla u)\cdot\nabla^k u + \int \nabla^k(B\cdot\nabla B)\cdot\nabla^k u \\ &= \int \nabla^k(u\otimes u) : \nabla^k\nabla u - \int \nabla^k(B\otimes B) : \nabla^k\nabla u \\ &\le C\|u\|_{L^3}\left\|\nabla^{k+1}u\right\|_{L^2}^2 + C\|B\|_{L^3}\left\|\nabla^{k+1}B\right\|_{L^2}\left\|\nabla^{k+1}u\right\|_{L^2} \\ &\le C\|u\|_{L^3}\left\|\nabla^{k+1}u\right\|_{L^2}^2 + C\|B\|_{L^3}^2\left\|\nabla^{k+1}B\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^{k+1}u\right\|_{L^2}^2\end{aligned}$$

from which we deduce from Theorem 1.3 and Corollary 1.3 that for $t > T_0$

$$\frac{d}{dt}\left\|\nabla^k u\right\|_{L^2}^2 + \left\|\nabla^{k+1}u\right\|_{L^2}^2 \le C\|B\|_{L^3}^2\left\|\nabla^{k+1}B\right\|_{L^2}^2 \le C\|B\|_{L^2}\|\nabla B\|_{L^2}\left\|\nabla^{k+1}B\right\|_{L^2}^2 \le C_k(1+t)^{-\frac{9}{2}+2\sigma_2-k}.$$

By modifying Lemma 2.1, we derive

$$\begin{aligned}\frac{d}{dt}\left[(1+t)^N\|\nabla^k u(t)\|_{L^2}^2\right] &\le N(1+t)^{N-1}\int_{\{|\xi|^2\le\frac{N}{1+t}\}}|\xi|^{2k}|\widehat{u}(t,\xi)|^2 + C_k(1+t)^{N-\frac{9}{2}+2\sigma_2-k} \\ &\le C_k(1+t)^{N-\frac{5}{2}+\sigma_1-k} + C_k(1+t)^{N-\frac{9}{2}+2\sigma_2-k} \le C_k(1+t)^{N-\frac{5}{2}+\sigma_1-k}.\end{aligned}$$

By integrating this in time, we obtain

$$\|\Lambda^k u(t)\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\sigma_1-k} \quad \text{for all } t > T_0.$$

4.2.2. **Decay rate of $\nabla^k B$.** We only consider the case $\sigma_1 > \sigma_2$. By taking ∇^k to (1.7b) and by taking the inner product with $\nabla^k B$, and using (2.2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k B\right\|_{L^2}^2 + \left\|\nabla^{k+1}B\right\|_{L^2}^2 = \int \nabla^k(u\times B)\cdot\nabla^k\nabla\times B - \int \nabla^k\operatorname{div}(B\otimes B)\cdot\nabla^k\nabla\times B \\ &\le C\|u\|_{L^3}\left\|\nabla^{k+1}B\right\|_{L^2}^2 + C\|B\|_{L^3}\left\|\nabla^{k+1}u\right\|_{L^2}\left\|\nabla^{k+1}B\right\|_{L^2} + C\|B\|_{L^\infty}\left\|\nabla^{k+1}B\right\|_{L^2}^2 \\ &\le C\|u\|_{L^3}\left\|\nabla^{k+1}B\right\|_{L^2}^2 + C\|B\|_{L^3}^2\left\|\nabla^{k+1}u\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^{k+1}B\right\|_{L^2}^2 + C\|B\|_{L^\infty}\left\|\nabla^{k+1}B\right\|_{L^2}^2,\end{aligned}$$

and so Theorem 1.3 and Lemma 4.2 imply that for $t > T_0$

$$\frac{d}{dt}\left\|\nabla^k B\right\|_{L^2}^2 + \left\|\nabla^{k+1}B\right\|_{L^2}^2 \le C\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^{k+1}u\|_{L^2}^2 \le C_k(1+t)^{-\frac{9}{2}+\frac{3\sigma_1+\sigma_2}{2}-k}.$$

As we bound u above, we attain

$$\frac{d}{dt}\left[(1+t)^N\|\nabla^k B(t)\|_{L^2}^2\right] \le C_k(1+t)^{N-\frac{5}{2}+\sigma_2-k} + C_k(1+t)^{N-\frac{9}{2}+\frac{3\sigma_1+\sigma_2}{2}-k} \le C_k(1+t)^{N-\frac{5}{2}+\sigma_2-k}$$

since $3\sigma_1-\sigma_2\le 4$. Hence, we derive

$$\|\nabla^k B(t)\|_{L^2}^2 \le C_k(1+t)^{-\frac{3}{2}+\sigma_2-k} \quad \text{for all } t>T_0.$$

4.3. **Proof of Theorem 1.4.** From [20], we deduce that if $\|u_0\|_{\mathcal{X}^{-1}}+\|B_0\|_{\mathcal{X}^{-1}}+\|B_0\|_{\mathcal{X}^0}\le\epsilon$ is sufficiently small, $u\in C([0,\infty);\mathcal{X}^{-1})\cap L^1((0,\infty);\mathcal{X}^1)$, $B\in C([0,\infty);\mathcal{X}^{-1}\cap\mathcal{X}^0)\cap L^1((0,\infty);\mathcal{X}^1\cap\mathcal{X}^2)$, and

$$\|u(t)\|_{\mathcal{X}^{-1}}+\|B(t)\|_{\mathcal{X}^{-1}}+\|B(t)\|_{\mathcal{X}^0}+(1-C\epsilon)\int_0^t\left(\|u(\tau)\|_{\mathcal{X}^1}+\|B(\tau)\|_{\mathcal{X}^1}+\|B(\tau)\|_{\mathcal{X}^2}\right)d\tau\le C\epsilon \tag{4.5}$$

for all $t>0$. So, we only need to show $(u,B)\in L^\infty([0,\infty);\mathcal{Y}^\sigma)$ and derive the decay rate in $\mathcal{X}^{-1}$.

4.3.1. **Bounds in $\mathcal{Y}^\sigma$.** We split the estimates of (u,B) into 3 cases as in the proof of Theorem 1.3. For this, we repeatedly use (4.5) when we bound (u,B) in $\mathcal{Y}^\sigma$. However, since we can bound u as in the proof of Theorem 1.2, we only deal with B in details.

▶ (**Case 1**: $\sigma=1$) We first have

$$|\xi||\widehat{u}(t,\xi)|\le\|u_0\|_{\mathcal{Y}^1}+C\epsilon\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^1}+\|B(\tau)\|_{\mathcal{Y}^1}\right).$$

We now estimate B. In this case, we use

$$B(t)=e^{t\Delta}B_0+\int_0^t e^{(t-\tau)\Delta}\left(\nabla\times(u\times B)-\nabla\times((\nabla\times B)\times B)\right)(\tau)\,d\tau$$

instead of (4.1b). Then, using (3.7) with $\alpha=1$, we have

$$\begin{aligned}
|\xi||\widehat{B}(t,\xi)| &\le |\xi|e^{-t|\xi|^2}|\widehat{B}_0(\xi)|+\int_0^t|\xi|^2e^{-(t-\tau)|\xi|^2}\int_{\mathbb{R}^3}|\widehat{u}(\tau,\xi-\eta)||\widehat{B}(\tau,\eta)|\,d\eta d\tau\\
&\quad+\int_0^t|\xi|^2e^{-(t-\tau)|\xi|^2}\int_{\mathbb{R}^3}|\widehat{\nabla\times B}(\tau,\xi-\eta)||\widehat{B}(\tau,\eta)|\,d\eta d\tau\\
&\le\|B_0\|_{\mathcal{Y}^1}+\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^1}\|B(\tau)\|_{\mathcal{X}^{-1}}+\|B(\tau)\|_{\mathcal{Y}^1}\|u(\tau)\|_{\mathcal{X}^{-1}}\right)\int_0^t|\xi|^2e^{-(t-\tau)|\xi|^2}\,d\tau\\
&\quad+\sup_{0\le\tau\le t}\left(\|B(\tau)\|_{\mathcal{Y}^1}\|B(\tau)\|_{\mathcal{X}^0}\right)\int_0^t|\xi|^2e^{-(t-\tau)|\xi|^2}\,d\tau\\
&\le\|B_0\|_{\mathcal{Y}^1}+C\epsilon\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^1}+\|B(\tau)\|_{\mathcal{Y}^1}\right).
\end{aligned}$$

From these two bounds, we have

$$\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^1}+\|B(\tau)\|_{\mathcal{Y}^1}\right)\le\|u_0\|_{\mathcal{Y}^1}+\|B_0\|_{\mathcal{Y}^1}+C\epsilon\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^1}+\|B(\tau)\|_{\mathcal{Y}^1}\right).$$

▶ (**Case 2**: $\sigma\in(-1,1)$) In this case, we bound u as

$$\|u(t)\|_{\mathcal{Y}^\sigma}\le\|u_0\|_{\mathcal{Y}^\sigma}+C\epsilon\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^\sigma}+\|B(\tau)\|_{\mathcal{Y}^\sigma}\right).$$

For B, we have

$$\begin{aligned}
|\xi|^\sigma|\widehat{B}(t,\xi)| &\le|\xi|^\sigma e^{-t|\xi|^2}|\widehat{B}_0(\xi)|+\int_0^t|\xi|^{1+\sigma}e^{-(t-\tau)|\xi|^2}\int|\widehat{u}(\tau,\xi-\eta)||\widehat{B}(\tau,\eta)|\,d\eta d\tau\\
&\quad+\int_0^t|\xi|^{2+\sigma}e^{-(t-\tau)|\xi|^2}\int|\widehat{B}(\tau,\xi-\eta)||\widehat{B}(\tau,\eta)|\,d\eta d\tau\\
&\le\|B_0\|_{\mathcal{Y}^\sigma}+(\mathrm{I})+(\mathrm{II}).
\end{aligned}$$

Using (3.7) with $\alpha = \sigma$ and (2.3a), we first bound (I) as

$$
\begin{aligned}
\text{(I)} &\leq \int_0^t |\xi|^{1+\sigma} e^{-(t-\tau)|\xi|^2} \left(\|u(\tau)\|_{\mathcal{Y}^\sigma} \|B(\tau)\|_{\mathcal{X}^{-\sigma}} + \|B(\tau)\|_{\mathcal{Y}^\sigma} \|u(\tau)\|_{\mathcal{X}^{-\sigma}} \right) d\tau \\
&\leq \sup_{0\leq\tau\leq t} \|u(\tau)\|_{\mathcal{Y}^\sigma} \left(\int_0^t |\xi|^2 e^{-(t-\tau)\frac{2}{1+\sigma}|\xi|^2} \, d\tau \right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|B(\tau)\|_{\mathcal{X}^{-\sigma}}^{\frac{2}{1-\sigma}} \, d\tau \right)^{\frac{1-\sigma}{2}} \\
&\quad + \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma} \left(\int_0^t |\xi|^2 e^{-(t-\tau)\frac{2}{1+\sigma}|\xi|^2} \, d\tau \right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|u(\tau)\|_{\mathcal{X}^{-\sigma}}^{\frac{2}{1-\sigma}} \, d\tau \right)^{\frac{1-\sigma}{2}} \\
&\leq C \sup_{0\leq\tau\leq t} \|u(\tau)\|_{\mathcal{Y}^\sigma} \left(\sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{X}^{-1}} \right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|B(\tau)\|_{\mathcal{X}^1} \, d\tau \right)^{\frac{1-\sigma}{2}} \\
&\quad + C \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma} \left(\sup_{0\leq\tau\leq t} \|u(\tau)\|_{\mathcal{X}^{-1}} \right)^{\frac{1+\sigma}{2}} \left(\int_0^t \|u(\tau)\|_{\mathcal{X}^1} \, d\tau \right)^{\frac{1-\sigma}{2}} \\
&\leq C\epsilon \sup_{0\leq\tau\leq t} \left(\|u(\tau)\|_{\mathcal{Y}^\sigma} + \|B(\tau)\|_{\mathcal{Y}^\sigma} \right).
\end{aligned}
$$

When $\sigma \in [0,1)$, using

$$
|\xi|^\sigma \leq (|\xi - \eta| + |\eta|)^\sigma \leq C(|\xi - \eta|^\sigma + |\eta|^\sigma), \tag{4.6}
$$

we bound

$$
\begin{aligned}
\text{(II)} &\leq C \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \int |\xi - \eta|^\sigma |\widehat{B}(\tau, \xi - \eta)| |\widehat{B}(\tau, \eta)| \, d\eta d\tau \\
&\quad + C \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \int |\widehat{B}(\tau, \xi - \eta)| |\eta|^\sigma |\widehat{B}(\tau, \eta)| \, d\eta d\tau \\
&\leq C \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma} \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{X}^0} \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \, d\tau \leq C\epsilon \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma}.
\end{aligned}
$$

When $\sigma \in (-1, 0)$, we use (2.3b) to obtain

$$
\begin{aligned}
\text{(II)} &\leq \int_0^t |\xi|^{2+\sigma} e^{-(t-\tau)|\xi|^2} \|B(\tau)\|_{\mathcal{Y}^\sigma} \|B(\tau)\|_{\mathcal{X}^{-\sigma}} \, d\tau \\
&\leq \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma} \left(\int_0^t |\xi|^2 e^{-(t-\tau)\frac{2}{2+\sigma}|\xi|^2} \, d\tau \right)^{\frac{2+\sigma}{2}} \left(\int_0^t \|B(\tau)\|_{\mathcal{X}^{-\sigma}}^{\frac{2}{-\sigma}} \, d\tau \right)^{\frac{-\sigma}{2}} \\
&\leq C \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma} \left(\sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{X}^0} \right)^{\frac{2+\sigma}{2}} \left(\int_0^t \|B(\tau)\|_{\mathcal{X}^2} \, d\tau \right)^{\frac{-\sigma}{2}} \leq C\epsilon \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^\sigma}.
\end{aligned}
$$

We combine the above estimates to yield

$$
\sup_{0\leq\tau\leq t} \left(\|u(\tau)\|_{\mathcal{Y}^\sigma} + \|B(\tau)\|_{\mathcal{Y}^\sigma} \right) \leq \|u_0\|_{\mathcal{Y}^\sigma} + \|B_0\|_{\mathcal{Y}^\sigma} + C\epsilon \sup_{0\leq\tau\leq t} \left(\|u(\tau)\|_{\mathcal{Y}^\sigma} + \|B(\tau)\|_{\mathcal{Y}^\sigma} \right).
$$

▶ (**Case 3**: $\sigma = -1$) In this case, we simply use (3.7) with $\alpha = 1$ to get

$$
|\xi|^{-1} |\widehat{u}(t, \xi)| \leq \|u_0\|_{\mathcal{Y}^{-1}} + \sup_{0\leq\tau\leq t} \|u(\tau)\|_{\mathcal{Y}^{-1}} \int_0^t \|u(\tau)\|_{\mathcal{X}^1} \, d\tau + \sup_{0\leq\tau\leq t} \|B(\tau)\|_{\mathcal{Y}^{-1}} \int_0^t \|B(\tau)\|_{\mathcal{X}^1} \, d\tau,
$$

and

$$
\begin{aligned}
|\xi|^{-1}|\widehat{B}(t,\xi)| &\le \|B_0\|_{\mathcal{Y}^{-1}} + \sup_{0\le\tau\le t}\|u(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|B(\tau)\|_{\mathcal{X}^1}\,d\tau + \sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\,d\tau \\
&\quad + \int_0^t |\xi| e^{-(t-\tau)|\xi|^2}\|B(\tau)\|_{\mathcal{Y}^{-1}}\|B(\tau)\|_{\mathcal{X}^1}d\tau \\
&\le \|B_0\|_{\mathcal{Y}^{-1}} + \sup_{0\le\tau\le t}\|u(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|B(\tau)\|_{\mathcal{X}^1}\,d\tau + \sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\,d\tau \\
&\quad + \sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{Y}^{-1}}\Big(\int_0^t |\xi|^2 e^{-2(t-\tau)|\xi|^2}\,d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|B(\tau)\|_{\mathcal{X}^1}^2 d\tau\Big)^{\frac{1}{2}} \\
&\le \|B_0\|_{\mathcal{Y}^{-1}} + \sup_{0\le\tau\le t}\|u(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|B(\tau)\|_{\mathcal{X}^1}\,d\tau + \sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{Y}^{-1}}\int_0^t \|u(\tau)\|_{\mathcal{X}^1}\,d\tau \\
&\quad + C\sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{Y}^{-1}}\left(\sup_{0\le\tau\le t}\|B(\tau)\|_{\mathcal{X}^0}\right)^{\frac{1}{2}}\left(\int_0^t \|B(\tau)\|_{\mathcal{X}^2}\,d\tau\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, we have

$$
\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^{-1}} + \|B(\tau)\|_{\mathcal{Y}^{-1}}\right) \le \|u_0\|_{\mathcal{Y}^{-1}} + \|B_0\|_{\mathcal{Y}^{-1}} + C\epsilon \sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\mathcal{Y}^{-1}} + \|B(\tau)\|_{\mathcal{Y}^{-1}}\right).
$$

► By collecting the bounds together, we arrive at

$$
\sup_{0\le t<\infty}\left(\|u(t)\|_{\mathcal{Y}^\sigma} + \|B(t)\|_{\mathcal{Y}^\sigma}\right) \le \|u_0\|_{\mathcal{Y}^\sigma} + \|B_0\|_{\mathcal{Y}^\sigma} + C\epsilon \sup_{0\le t<\infty}\left(\|u(t)\|_{\mathcal{Y}^\sigma} + \|B(t)\|_{\mathcal{Y}^\sigma}\right)
$$

for all $\sigma \in [-1,1]$. By restricting the size of ϵ as $2C\epsilon < 1$, we finally obtain

$$
\sup_{0\le t<\infty}\left(\|u(t)\|_{\mathcal{Y}^\sigma} + \|B(t)\|_{\mathcal{Y}^\sigma}\right) \le 2\left(\|u_0\|_{\mathcal{Y}^\sigma} + \|B_0\|_{\mathcal{Y}^\sigma}\right).
$$

4.3.2. **Decay rates.** We now investigate the temporal decay rate of (u,B) in $\mathcal{X}^{-1}$ using Lemma 2.2. Since the uniform bound of (u,B) in $\mathcal{Y}^\sigma$ implies (2.6), we only need to show that (u,B) satisfies (2.5) with $k=0$. In fact, by using (3.7) with $\alpha = 1$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_{\mathcal{X}^{-1}} + \|u(t)\|_{\mathcal{X}^1} &\le \int\!\!\int |\widehat{u}(t,\xi-\eta)||\widehat{u}(t,\eta)|\,d\eta d\xi + \int\!\!\int |\widehat{B}(t,\xi-\eta)||\widehat{B}(t,\eta)|\,d\eta d\xi \\
&\le \|u(t)\|_{\mathcal{X}^{-1}}\|u(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^{-1}}\|B(t)\|_{\mathcal{X}^1} \le C\epsilon\left(\|u(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\|B(t)\|_{\mathcal{X}^{-1}} + \|B(t)\|_{\mathcal{X}^1} &\le 2\int\!\!\int |\widehat{u}(t,\xi-\eta)||\widehat{B}(t,\eta)|\,d\eta d\xi + \int\!\!\int |\widehat{\nabla\times B}(t,\xi-\eta)||\hat{B}(t,\eta)|\,d\eta d\xi \\
&\le \|u(t)\|_{\mathcal{X}^{-1}}\|B(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^{-1}}\|u(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^0}\|B(t)\|_{\mathcal{X}^1} \\
&\le C\epsilon\left(\|u(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^1}\right).
\end{aligned}
$$

By combining these two estimates with ϵ restricted to satisfy $\theta = 1 - C\epsilon > 0$, we have

$$
\frac{d}{dt}\left(\|u(t)\|_{\mathcal{X}^{-1}} + \|B(t)\|_{\mathcal{X}^{-1}}\right) + \theta\left(\|u(t)\|_{\mathcal{X}^1} + \|B(t)\|_{\mathcal{X}^1}\right) \le 0
$$

and hence, the desired decay rate in $\mathcal{X}^{-1}$ follows from Lemma 2.2.

4.3.3. **Decay rates in $\mathcal{X}^{k-1}$.** Since we already have the uniform bound of (u,B) in $\mathcal{Y}^\sigma$, we only need to show that (u,B) satisfies (2.5). Using (4.6) and (2.3c),

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_{\mathcal{X}^{k-1}} + \|u(t)\|_{\mathcal{X}^{k+1}} &\le \int\!\!\int |\xi|^k|\widehat{u}(t,\xi-\eta)||\widehat{u}(t,\eta)|\,d\eta d\xi + \int\!\!\int |\xi|^k|\widehat{B}(t,\xi-\eta)||\widehat{B}(t,\eta)|\,d\eta d\xi \\
&\le C\|u(t)\|_{\mathcal{X}^0}\|u(t)\|_{\mathcal{X}^k} + C\|B(t)\|_{\mathcal{X}^0}\|B(t)\|_{\mathcal{X}^k} \\
&\le C\|u(t)\|_{\mathcal{X}^{-1}}\|u(t)\|_{\mathcal{X}^{k+1}} + C\|B(t)\|_{\mathcal{X}^{-1}}\|B(t)\|_{\mathcal{X}^{k+1}} \\
&\le C\epsilon\left(\|u(t)\|_{\mathcal{X}^{k+1}} + \|B(t)\|_{\mathcal{X}^{k+1}}\right)
\end{aligned}
$$

and

$$\begin{aligned}\frac{d}{dt}\|B(t)\|_{\mathcal{X}^{k-1}}+\|B(t)\|_{\mathcal{X}^{k+1}}&\le 2\int\int|\xi|^k|\widehat{u}(t,\xi-\eta)||\widehat{B}(t,\eta)|\,d\eta d\xi+\int\int|\xi|^{k+1}|\widehat{B}(t,\xi-\eta)||\hat{B}(t,\eta)|\,d\eta d\xi\\&\le C\|u(t)\|_{\mathcal{X}^0}\|B(t)\|_{\mathcal{X}^k}+C\|B(t)\|_{\mathcal{X}^0}\|u(t)\|_{\mathcal{X}^k}+C\|B(t)\|_{\mathcal{X}^0}\|B(t)\|_{\mathcal{X}^{k+1}}\\&\le C\|u(t)\|_{\mathcal{X}^{-1}}\|u(t)\|_{\mathcal{X}^{k+1}}+C\|B(t)\|_{\mathcal{X}^{-1}}\|B(t)\|_{\mathcal{X}^{k+1}}+C\|B(t)\|_{\mathcal{X}^0}\|B(t)\|_{\mathcal{X}^{k+1}}\\&\le C\epsilon\left(\|u(t)\|_{\mathcal{X}^{k+1}}+\|B(t)\|_{\mathcal{X}^{k+1}}\right).\end{aligned}$$

By combining these two estimates with ϵ restricted to satisfy $\theta=1-C\epsilon>0$, we have

$$\frac{d}{dt}\left(\|u(t)\|_{\mathcal{X}^{k-1}}+\|B(t)\|_{\mathcal{X}^{k-1}}\right)+\theta\left(\|u(t)\|_{\mathcal{X}^{k+1}}+\|B(t)\|_{\mathcal{X}^{k+1}}\right)\le 0$$

and again, the decay rate follows from Lemma 2.2. This completes the proof.

Appendix A. L^1 bound of u

The decay rate in [23] implies that L^1 is an invariant space for all $t>0$. To show this, we first note that

$$\frac{d}{dt}\|u\|_{L^1}\le\|\mathbb{P}(u\cdot\nabla u)\|_{L^1},$$

where $\mathbb{P}$ is the Leray projection operator defined in (3.2). Since $u\cdot\nabla u\in\mathcal{H}$ (the Hardy space) with $\|u\cdot\nabla u\|_{\mathcal{H}}\le C\|u\|_{L^2}\|\nabla u\|_{L^2}$ [12] and $\mathbb{P}:\mathcal{H}\to L^1$ [27], we obtain

$$\frac{d}{dt}\|u\|_{L^1}\le C\|u\cdot\nabla u\|_{\mathcal{H}}\le C\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

Integrating this in time,

$$\begin{aligned}\|u(t)\|_{L^1}&\le\|u_0\|_{L^1}+C\int_0^t(1+\tau)^{-\frac{3}{4}}\|\nabla u(\tau)\|_{L^2}\,d\tau\\&\le\|u_0\|_{L^1}+C\left[\int_0^t\|\nabla u(\tau)\|_{L^2}^2\,d\tau\right]^{\frac12}\left[\int_0^t(1+\tau)^{-\frac32}d\tau\right]^{\frac12}\le\|u_0\|_{L^1}+C\|u_0\|_{L^2}\end{aligned}$$

for all $t>0$.

Appendix B. Alternative proof of Theorem 1.1

To prove Theorem 1.1, we take two steps. We first estimate $\left\|e^{t\Delta}u_0\right\|_{L^2}$ using $u_0\in L^2\cap\mathcal{Y}^\sigma$ from which we arrive at the decay rate of u in Theorem 1.1 by [30]. We then bound $u\in\mathcal{Y}^\sigma$ using the decay rate of u.

► We recall the argument in [30]: if $\left\|e^{t\Delta}u_0\right\|_{L^2}^2\le C(1+t)^{-\alpha_0}$, then a weak solution of (1.1) decays in time:

$$\|u(t)\|_{L^2}^2\le C_0(1+t)^{-\min\left\{\alpha_0,\frac52\right\}}.$$

We now bound $\left\|e^{t\Delta}u_0\right\|_{L^2}$ with $u_0\in L^2\cap\mathcal{Y}^\sigma$:

$$\|e^{t\Delta}u_0\|_{L^2}^2=\int e^{-2t|\xi|^2}|\widehat{u}_0(\xi)|^2\,d\xi=\int_{|\xi|\le1}e^{-2t|\xi|^2}|\widehat{u}_0(\xi)|^2\,d\xi+\int_{|\xi|\ge1}e^{-2t|\xi|^2}|\widehat{u}_0(\xi)|^2\,d\xi=\text{(I)}+\text{(II)}.$$

We first deal with (I):

$$\begin{aligned}\text{(I)}&=(2t)^{-\frac32+\sigma}\int_{|\xi|\le1}(2t|\xi|^2)^{-\sigma}e^{-2t|\xi|^2}\left(|\xi|^\sigma|\widehat{u}_0(\xi)|\right)^2(2t)^{\frac32}\,d\xi\\&\le Ct^{-\frac32+\sigma}\|u_0\|_{\mathcal{Y}^\sigma}^2\int_{|\eta|\le\sqrt{2t}}|\eta|^{-2\sigma}e^{-|\eta|^2}d\eta\le Ct^{-\frac32+\sigma}\|u_0\|_{\mathcal{Y}^\sigma}^2,\end{aligned}$$

where we use the change of variables $\eta=\sqrt{2t}\xi$ and

$$\int|\eta|^{-2\sigma}e^{-|\eta|^2}\,d\eta=C\int_0^\infty r^{2-2\sigma}e^{-r^2}\,dr\le C\int_0^1 r^{2-2\sigma}dr+C\int_1^\infty e^{-r^2}dr\le C$$

when $\sigma<\frac32$. Using (2.1) and the condition $\sigma<\frac32$, we now bound (II):

$$\text{(II)}=(2t)^{-\frac32+\sigma}\int_{|\xi|\ge1}(2t|\xi|^2)^{\frac32-\sigma}e^{-2t|\xi|^2}|\xi|^{2\sigma-3}|\widehat{u}_0(\xi)|^2\,d\xi\le Ct^{-\frac32+\sigma}\int|\widehat{u}_0(\xi)|^2\,d\xi\le Ct^{-\frac32+\sigma}\|u_0\|_{L^2}^2.$$

Therefore, these two bounds lead to

$$\|e^{t\Delta}u_0\|_{L^2}^2 \le Ct^{-\frac{3}{2}+\sigma}\left(\|u_0\|_{\mathcal{Y}^\sigma}^2 + \|u_0\|_{L^2}^2\right).$$

Since $\alpha_0 = \frac{3}{2} - \sigma \le \frac{5}{2}$ when $\sigma \ge -1$, we arrive at the desired decay rate of u in Theorem 1.1.

► We here show that $u \in \mathcal{Y}^\sigma$ using (1.4). When $\sigma = 1$, we first have

$$\begin{aligned}
|\xi||\widehat{u}(t,\xi)| &\le |\xi|e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \\
&\le \|u_0\|_{\mathcal{Y}^1} + \|u_0\|_{L^2}^2 \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}\,d\tau \le C_0
\end{aligned}$$

from which we deduce that

$$\sup_{0\le t<\infty} \|u(t)\|_{\mathcal{Y}^1} \le C_0.$$

When $\sigma \in [-1, 1)$, we bound u as (3.5):

$$\begin{aligned}
|\xi|^\sigma|\widehat{u}(t,\xi)| &\le |\xi|^\sigma e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t |\xi|^{1+\sigma} e^{-(t-\tau)|\xi|^2}\|u(\tau)\|_{L^2}^2\,d\tau \\
&\le \|u_0\|_{\mathcal{Y}^\sigma} + C_0\int_0^t (t-\tau)^{-\frac{1+\sigma}{2}}(1+\tau)^{-\frac{3}{2}+\sigma}\,d\tau \le C_0.
\end{aligned}$$

So, we have

$$\sup_{0\le t<\infty} \|u(t)\|_{\mathcal{Y}^\sigma} \le C_0.$$

Acknowledgments

H. B. was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) (grant No.2022R1A4A1032094).

J. J. was supported by NRF grant (No.RS-2022-00165600).

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