

THE NOETHER INEQUALITIES FOR A FOLIATED SURFACE OF GENERAL TYPE

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ABSTRACT. Let (\mathcal{F}, S) be a foliated surface of general type with reduced singularities over the complex number. We establish the Noether type inequalities for (\mathcal{F}, S) . Namely, we prove that $\text{vol}(\mathcal{F}) \geq p_g(\mathcal{F}) - 2$, and that $\text{vol}(\mathcal{F}) \geq 2p_g(\mathcal{F}) - 4$ if moreover the surface S is also of general type. Examples show that both of the Noether type inequalities are sharp.

1. INTRODUCTION

The aim of this paper is to establish the Noether inequalities for a foliated surface of general type with reduced singularities over the complex number \mathbb{C} .

The classical Noether inequality [Noe70] asserts that

$$\text{vol}(S) \geq 2p_g(S) - 4, \quad (1.1)$$

for every complex smooth projective surface S of general type. Here we recall that the volume $\text{vol}(S)$ and the geometric genus $p_g(S)$ of S are defined as follows. Let K_S be the canonical divisor of S . Then

$$p_g(S) = \dim H^0(S, K_S);$$

$$\text{vol}(S) = \limsup_{n \rightarrow +\infty} \frac{\dim H^0(S, nK_S)}{n^2/2}.$$

These are two important birational invariants of S . If S is minimal, then the volume $\text{vol}(S)$ is equal to the intersection number K_S^2 , which is also the first Chern number of S . The Noether inequality (1.1) is one of the fundamental inequality in the surface theory. Minimal surfaces of general type with the equality are usually said on the Noether line, which has been systematically studied by Horikawa [Hor76].

In a series of nice works [Kob92, Che04, CCZ06, Che07, CC15, CH17, CCJ20b, CCJ20a], the following sharp Noether inequality has been established for every minimal 3-fold X of general type with $p_g(X) \leq 4$ or $p_g(X) \geq 11$.

$$\text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

It is also proved in [CJ17] that the Noether type inequality holds also in higher dimension: there exist positive numbers a_d and b_d , depending only on the dimension d of the variety X , such that

$$\text{vol}(X) \geq a_d p_g(X) - b_d.$$

Recently, the foliation theory has attracted more and more attention in algebraic geometry, especially in birational geometry. Miyaoka [Miy87, Miy88] introduced the use of the foliations to

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study the abundance conjecture. For instance, the foliations whose canonical divisor $K_{\mathcal{F}}$ is not pseudoeffective are characterised in [Miy87], see also [BM16]. In the beautiful paper [Cas21], Cascini proposed the minimal model program for foliated varieties. It stimulates a new project in birational geometry to generalize everything to foliated varieties.

Let (X, \mathcal{F}) be a foliated variety, and denote by $K_{\mathcal{F}}$ its canonical divisor. The canonical divisor $K_{\mathcal{F}}$ of a foliation plays a similar role in many aspects as the canonical divisor K_X of a variety. For instance, Miyaoka's theorem [Miy87] on foliated varieties with non-pseudoeffective canonical divisor, the minimal model theory on foliated surfaces due to Brunella and McQuillan etc. [Bru99, McQ08], and the deformation invariance of the pluri-genera by Cascini-Floris [CF18]. We are mainly interested in the following two birational invariants: the geometric genus $p_g(\mathcal{F})$ and volume $\text{vol}(\mathcal{F})$, which are defined similar to the case of algebraic varieties.

$$p_g(\mathcal{F}) = \dim H^0(X, K_{\mathcal{F}});$$

$$\text{vol}(\mathcal{F}) = \limsup_{n \rightarrow +\infty} \frac{\dim H^0(X, nK_{\mathcal{F}})}{n^d/d!}.$$

Here $d = \dim X$. By the asymptotic Riemann-Roch theorem, if (Y, \mathcal{F}_Y) is a minimal model of (X, \mathcal{F}) , then $\text{vol}(X, \mathcal{F}) = K_{\mathcal{F}_Y}^d$. The foliated variety (X, \mathcal{F}) is called of general type if $\text{vol}(\mathcal{F}) > 0$, i.e., the canonical divisor $K_{\mathcal{F}}$ is big. Inspired by the Noether type inequalities for algebraic varieties, it is natural to ask the following question.

Question 1.1. *Do there exist Noether type inequalities for a foliated variety of general type?*

In this paper, we try to settle the above question in dimension two.

Theorem 1.2. *Let (S, \mathcal{F}) be a foliated surface of general type with reduced singularities.*

(i). *The following Noether inequality holds*

$$\text{vol}(\mathcal{F}) \geq p_g(\mathcal{F}) - 2. \quad (1.2)$$

Moreover, if the equality in (1.2) holds, then the canonical map $\varphi_{|K_{\mathcal{F}}|}$ defines a birational map whose image is a surface of minimal degree (equal to $p_g(\mathcal{F}) - 2$) in $\mathbb{P}^{p_g(\mathcal{F})-1}$.

(ii). *Suppose furthermore that the surface S is of general type. Then*

$$\text{vol}(\mathcal{F}) \geq 2p_g(\mathcal{F}) - 4. \quad (1.3)$$

Moreover, if the equality in (1.3) holds, then the canonical map $\varphi_{|K_{\mathcal{F}}|}$ defines a two-to-one map whose image is a surface of minimal degree (equal to $p_g(\mathcal{F}) - 2$) in $\mathbb{P}^{p_g(\mathcal{F})-1}$.

Remark 1.3. (i). We will construct examples (cf. Example 6.1 and Example 6.4) reaching the equalities in both (1.2) and (1.3), which show that both of the Noether type inequalities are sharp.

(ii). For algebraic varieties of general type, there exists so-called "the second Noether inequality", at least in lower dimensions. When X is of dimension two, the volume is an integer, and hence $\text{vol}(X) \geq 2p_g(X) - 3$ if $\text{vol}(X) \neq 2p_g(X) - 4$; when X is a threefold, Hu-Zhang [HZ22] proved that $\text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{19}{6}$ if $\text{vol}(X) \neq \frac{4}{3}p_g(X) - \frac{10}{3}$. However, such a phenomenon disappears for foliated surfaces of general type. Indeed, we will construct in Example 6.2 a sequence of foliated surfaces (S_n, \mathcal{F}_n) of general type with reduced singularities such that $\text{vol}(\mathcal{F}_n) > p_g(\mathcal{F}_n) - 2$, but the difference $\text{vol}(\mathcal{F}_n) - (p_g(\mathcal{F}_n) - 2)$ can be arbitrarily close to zero.

(iii). The structure of a surface of minimal degree in \mathbb{P}^n is well-understood, see for instance [GH78, Chapter 4.3]. In a forthcoming paper, we will characterize the geometry of foliated surfaces of general type satisfying the Noether type equalities.

In the following, we briefly explain the strategy of our proof. As in the case proving the classical Noether inequality (1.1), the starting point is to analyze the canonical map $\varphi_{|K_{\mathcal{F}}|}$ defined by the complete linear system $|K_{\mathcal{F}}|$. However, there are two main difficulties compared to the case of $\varphi_{|K_S|}$:

(1). The canonical divisor $K_{\mathcal{F}}$ might not be nef for a foliation \mathcal{F} of general type with reduced singularities on a smooth surface S . It makes some trouble in estimating the lower bound of $\text{vol}(\mathcal{F})$. For instance, suppose that

$$|K_{\mathcal{F}}| = |M| + Z, \quad (1.4)$$

where Z is the fixed part of $|K_{\mathcal{F}}|$, and M is the moving part. Then it is no longer true that $\text{vol}(\mathcal{F}) \geq K_{\mathcal{F}} \cdot M$, cf. Remark 6.3. Of course, one can contract the support of the negative part of $K_{\mathcal{F}}$ to a normal surface S_0 , such that the induced foliation \mathcal{F}_0 on S_0 has the advantage that $K_{\mathcal{F}_0}$ is nef [Bru99, McQ08]. However, the surface S_0 would be singular with klt singularities and $K_{\mathcal{F}_0}$ is no longer a line bundle (but a \mathbb{Q} -bundle).

(2). The second difficulty occurring in the case when the canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration $f : S \rightarrow B$. In the case proving the classical Noether inequality (1.1), the general fiber F of f is of genus at least two, and hence $K_S \cdot F = 2g(F) - 2 \geq 2$. However, in our case it is only known that $K_{\mathcal{F}} \cdot F > 0$ since $K_{\mathcal{F}}$ is big. It can happen that $K_{\mathcal{F}} \cdot F = 1$, even when the surface S is also of general type (see Example 6.5 for such an example). Suppose that

$$K_{\mathcal{F}} = P + N, \quad (1.5)$$

is the Zariski decomposition of $K_{\mathcal{F}}$, where P is the nef part and N is the negative part of $K_{\mathcal{F}}$. It might happen that $N \cdot F > 0$, and hence $P \cdot F < K_{\mathcal{F}} \cdot F = 1$ (see Example 6.2 for such an example). This will cause trouble in estimating the lower bound of $\text{vol}(\mathcal{F}) = P^2$ along the usual way, cf. Remark 3.6.

To overcome the above two difficulties, we need to control the negative part in the Zariski decomposition of the canonical divisor $K_{\mathcal{F}}$, as well as the structure of $K_{\mathcal{F}}$ when the canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration. Let

$$\varphi = \varphi_{|K_{\mathcal{F}}|} : S \dashrightarrow \Sigma \subseteq \mathbb{P}^{p_g(\mathcal{F})-1},$$

be the rational map defined by $|K_{\mathcal{F}}|$. After some elementary reduction, we are reduced to the case when the image Σ is a curve and the canonical map $\varphi_{|K_{\mathcal{F}}|}$ has no base point. Hence the canonical map induces a fibration $f : S \rightarrow B$ by taking the normalization and Stein factorization:

$$\begin{array}{ccccc} & S & & & \\ f \swarrow & & \searrow \varphi & & \\ B & \xrightarrow{\quad} & \Sigma & \hookrightarrow & \mathbb{P}^{p_g(\mathcal{F})-1} \end{array}$$

Consider the two decompositions in (1.4) and (1.5). Since the moving part M is always nef, it follows that $M \leq P$, or equivalently $N \leq Z$. Let F be a general fiber of f . The moving part M consists of several fibers, whose cardinality is at least $p_g(\mathcal{F}) - 1$ by the Riemann-Roch theorem. Hence

$$\text{vol}(\mathcal{F}) = P^2 \geq P \cdot M \geq (p_g(\mathcal{F}) - 1)P \cdot F = (p_g(\mathcal{F}) - 1)(K_{\mathcal{F}} - N) \cdot F.$$

To obtain the Noether type inequalities, it suffices to get a lower bound on $(K_{\mathcal{F}} - N) \cdot F$, or equivalently, an upper bound on $N \cdot F$. We will show in Proposition 3.5 an upper bound on the coefficients a_C 's of $N = \sum a_C C$, from which we can prove the Noether type inequalities in most cases, except the extreme case where $K_{\mathcal{F}} \cdot F = 1$. In this extreme case, we will prove in Theorem 4.1 an explicit description on the canonical divisor $K_{\mathcal{F}}$, base on which the Noether type inequality follows.

The paper is organized as follows. In Section 2, we review some basic facts about foliations, especially the foliations on smooth surfaces. In Section 3, we recall the Zariski decomposition of the canonical divisor $K_{\mathcal{F}}$ for a relatively minimal foliation, and prove in Proposition 3.5 an upper bound on the coefficients a_C 's of the negative part $N = \sum a_C C$. In Section 4, we restrict ourselves to the special case when the canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration, and prove in Theorem 4.1 an explicit description on the canonical divisor $K_{\mathcal{F}}$ if $K_{\mathcal{F}} \cdot F = 1$ and S is of general type. The Noether type inequalities will be proved in Section 5. Finally, we will in Section 6 construct several examples to illustrate that the Noether type inequalities obtained in Theorem 1.2 are sharp.

2. PRELIMINARIES

In this section, we recall some basic facts about foliations, especially the foliations on smooth surfaces. For more details we refer to [Bru04, McQ08].

We work over the complex number \mathbb{C} . By a foliated variety (X, \mathcal{F}) we mean a foliation \mathcal{F} on the variety X .

Definition 2.1. A foliation \mathcal{F} of rank r on a normal variety X of dimension n is defined by a rank r coherent subsheaf $T_{\mathcal{F}} \subseteq T_X$ such that

- (i) $T_{\mathcal{F}}$ is saturated, i.e., $T_X/T_{\mathcal{F}}$ is torsion free;
- (ii) $T_{\mathcal{F}}$ is closed under the Lie bracket, i.e., $[\alpha, \beta] \in T_{\mathcal{F}}$ for any $\alpha, \beta \in T_{\mathcal{F}}$.

The singular locus $\text{sing}(\mathcal{F})$ of \mathcal{F} is the set of points $x \in X$ where either $x \in \text{sing}(X)$ or the quotient sheaf $N_{\mathcal{F}} := T_X/T_{\mathcal{F}}$ fails to be locally free at x . Note that the first condition above implies that the codimension of $\text{sing}(\mathcal{F})$ in X is at least two. The canonical divisor of \mathcal{F} is a divisor $K_{\mathcal{F}}$ on X such that $\mathcal{O}_X(K_{\mathcal{F}}) \cong \det(T_{\mathcal{F}})^*$. Locally in the Euclidean topology around a smooth point $x \in X \setminus \text{sing}(\mathcal{F})$, the foliation \mathcal{F} can be defined by a fibration; that is, there exists an analytic open neighborhood $x \in U \subseteq X$ and a morphism $f : U \rightarrow \mathbb{C}^{n-r}$ such that

$$T_{\mathcal{F}}|_U = \ker(df : T_U \rightarrow f^*T_{\mathbb{C}^{n-r}}).$$

Conversely, any morphism $f : X \rightarrow Y$ with connected fibers defines a foliation \mathcal{F} by taking the saturation of $\ker(df : T_X \rightarrow f^*T_Y)$ in T_X .

We next turn to the foliation on a smooth projective surface S . A foliation \mathcal{F} on S can be given by an exact sequence

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_S \longrightarrow I_{\Delta}(N_{\mathcal{F}}) \longrightarrow 0,$$

where $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$ are respectively the tangent bundle and normal bundle of \mathcal{F} , and I_{Δ} is an ideal sheaf supported on the singular locus of \mathcal{F} . Equivalently, a foliation on S is given by the data $\{(U_i, v_i)\}_{i \in I}$, where $\{U_i\}_{i \in I}$ is an open covering of S , v_i is a holomorphic vector field on U_i

with at most isolated zeros, and there exists $g_{ij} \in \mathcal{O}_S^*(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$ such that

$$v_i|_{U_i \cap U_j} = g_{ij} v_j|_{U_i \cap U_j}. \quad (2.1)$$

The cocycle $\{g_{ij}\}$ defines a line bundle which is nothing but the canonical divisor $K_{\mathcal{F}} = T_{\mathcal{F}}^*$.

Alternatively, one can also define \mathcal{F} using one-forms instead of vector fields. A foliation on S is given by a collection of one-forms $\omega_i \in \Omega_S^1(U_i)$ with at most isolated zeros and there exists $f_{ij} \in \mathcal{O}_S^*(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$ such that

$$\omega_i|_{U_i \cap U_j} = f_{ij} \omega_j|_{U_i \cap U_j}. \quad (2.2)$$

The cocycle $\{g_{ij}\}$ defines a line bundle which is the conormal bundle $N_{\mathcal{F}}^*$. One can also translate this into an exact sequence:

$$0 \longrightarrow N_{\mathcal{F}}^* \longrightarrow \Omega_S^1 \longrightarrow I_{\Delta}(K_{\mathcal{F}}) \longrightarrow 0.$$

For any surjective morphism $\Pi : \tilde{S} \rightarrow S$ from another smooth projective surface \tilde{S} , there is a natural foliation $\tilde{\mathcal{F}}$ induced by pulling-back \mathcal{F} on \tilde{S} . Suppose that $\{(U_i, \omega_i)\}_{i \in I}$ is a collection of local one-forms defining the foliation \mathcal{F} . At the first sight, one may think that $\tilde{\mathcal{F}}$ is simply given by $\{(\Pi^{-1}U_i, \Pi^*\omega_i)\}_{i \in I}$. But this is not the case in general, since $\Pi^*\omega_i$ may admit one-dimensional zeros. Alternatively, the foliation $\tilde{\mathcal{F}}$ is defined by the following data

$$\{(V_{ij}, \tilde{\omega}_{ij})\}, \quad i \in I, j \in J,$$

where V_{ij} is an open covering of \tilde{S} with $\Pi(V_{ij}) \subseteq U_i$, and $\tilde{\omega}_{ij} = \frac{\Pi^*(\omega_i)}{h_{ij}}$ with h_{ij} being some holomorphic function over V_{ij} satisfying $\text{div}(h_{ij}) = \text{div}(\Pi^*(\omega_i)|_{V_{ij}})$. In particular, for any blowing-up $\sigma : \tilde{S} \rightarrow S$ centered at $p \in S$, there is an induced foliation $\tilde{\mathcal{F}}$ on \tilde{S} , such that $\tilde{\mathcal{F}}|_{\tilde{S} \setminus E} \cong \mathcal{F}|_{S \setminus p}$ under the isomorphism $\tilde{S} \setminus E \cong S \setminus \{p\}$, where E is the exceptional curve.

Suppose that p is a singular point of \mathcal{F} , and v is a local vector field defining \mathcal{F} . The two eigenvalues λ_1, λ_2 of the linear part $(Dv)(p)$ are well-defined up to multiplication by a non-zero constant.

Definition 2.2. A singularity p of \mathcal{F} is called a *reduced singularity* if at least one of the two eigenvalues (say, λ_2) is not zero and the quotient $\lambda = \frac{\lambda_1}{\lambda_2}$ is not a positive rational number. The foliation \mathcal{F} is said to be reduced if any singularity of \mathcal{F} is reduced.

Remark that the quotient $\lambda = \frac{\lambda_1}{\lambda_2}$ is unchanged by multiplication of v by a nonvanishing holomorphic function. Of course, if $\lambda_1 \neq 0$, we could also consider the quotient $\lambda^{-1} = \frac{\lambda_2}{\lambda_1}$ instead of λ , but then $\lambda \notin \mathbb{Q}^+$ iff $\lambda^{-1} \notin \mathbb{Q}^+$. The complex number $\lambda = \frac{\lambda_1}{\lambda_2}$, with an inessential abuse due to the exchange $\lambda \leftrightarrow \lambda^{-1}$, is called the *eigenvalue* of \mathcal{F} at p following [Bru04]. A reduced singularity is called *non-degenerate* if both of the two eigenvalues λ_1 and λ_2 are non-zero; otherwise it is called a *saddle node*. Given any foliation, one can obtain a reduced one by a sequence of blowing-ups:

Theorem 2.3 (Seidenberg, [Bru04, Theorem 1.1]). *Given any foliated surface (S, \mathcal{F}) , there exists a sequence of blowing-ups $\pi : \tilde{S} \rightarrow S$, such that the foliation $\pi^*(\mathcal{F})$ is reduced.*

The birational geometry behaves well for foliated surfaces with reduced singularities, as showed in [Bru99, Bru04, Men00]. For instance, the pluri-genera $p_n(\mathcal{F})$'s keep invariant under birational maps between reduced foliated surfaces, where the pluri-genus $p_n(\mathcal{F})$ is defined

as

$$p_n(\mathcal{F}) = \dim H^0(S, nK_{\mathcal{F}}).$$

In particular, the geometric genus $p_g(\mathcal{F}) := p_1(\mathcal{F})$ as well as the volume $\text{vol}(\mathcal{F})$ are well-defined birational invariants for foliated surfaces with reduced singularities, where

$$\text{vol}(\mathcal{F}) = \limsup_{n \rightarrow +\infty} \frac{p_n(\mathcal{F})}{n^2/2} = \limsup_{n \rightarrow +\infty} \frac{\dim H^0(X, nK_{\mathcal{F}})}{n^2/2}.$$

An important way to produce foliations comes from fibration on smooth projective surfaces. Let $f : S \rightarrow B$ be a fibration of curves, i.e., f is a proper surjective morphism from S onto B with connected fibers. The fibration defines a foliation \mathcal{F} on S by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S . The canonical divisor $K_{\mathcal{F}}$ is simple:

$$K_{\mathcal{F}} = K_{S/B} \otimes \mathcal{O}_S \left(\sum (1 - a_i) C_i \right), \quad (2.3)$$

where $K_{S/B} = K_S - f^*(K_B)$ is the relative canonical divisor, the sum is taken over all components C_i 's in fibers of f , and a_i is the multiplicity of C_i in fibers of f . The foliation \mathcal{F} is reduced if and only if every possible singular fiber of f is normal crossing. In particular, if the fibration is semi-stable, i.e., any possible singular fiber of f is a reduced node curve, and any possible smooth rational component in such a singular fiber intersects other components at least two points, then \mathcal{F} is relatively minimal and $K_{\mathcal{F}} = K_{S/B}$ by (2.3).

An irreducible curve $C \subseteq S$ is said to be \mathcal{F} -invariant if the inclusion $T_{\mathcal{F}}|_C \hookrightarrow T_X|_C$ factors through T_C . By a curve $C \subseteq S$ we mean a reduced and compact algebraic curve. So it might be singular and reducible. Suppose that C is not \mathcal{F} -invariant, or more precisely every irreducible component of C is not \mathcal{F} -invariant. Then one defines the tangency of \mathcal{F} to C as follows. Let $p \in C$ be any point. Around p , let $\{f = 0\}$ be a local equation of C , and v be a local holomorphic vector field defining \mathcal{F} around p . Then the tangency of \mathcal{F} to C at p is defined to be

$$\text{tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,p}}{\langle f, v(f) \rangle}.$$

As C is not \mathcal{F} -invariant, $\text{tang}(\mathcal{F}, C, p) < +\infty$ and $\text{tang}(\mathcal{F}, C, p) = 0$ except for finitely many points. Hence one defines the tangency of \mathcal{F} to C .

$$\text{tang}(\mathcal{F}, C) = \sum_{p \in C} \text{tang}(\mathcal{F}, C, p).$$

Proposition 2.4 ([Bru04, Proposition 2.2]). *Let C be a curve on S which is not \mathcal{F} -invariant. Then*

$$\text{tang}(\mathcal{F}, C) = K_{\mathcal{F}}C + C^2.$$

In particular,

$$K_{\mathcal{F}}C + C^2 \geq 0.$$

We now suppose that C is \mathcal{F} -invariant, or more precisely every irreducible component of C is \mathcal{F} -invariant. Given any point $p \in C$, let $\{f = 0\}$ be a local equation of C , and ω be a local holomorphic one-form defining \mathcal{F} around p . Because C is \mathcal{F} -invariant, we may write

$$g\omega = hdf + f\eta,$$

for some holomorphic one-form η and holomorphic functions g, h around p , such that h and f are coprime. We define

$$Z(\mathcal{F}, C, p) = \text{vanishing order of } \frac{h}{g} \Big|_C \text{ at } p,$$

$$\text{CS}(\mathcal{F}, C, p) = \text{residue of } -\frac{\eta}{h} \Big|_C \text{ at } p.$$

By definition, both $Z(\mathcal{F}, C, p)$ and $\text{CS}(\mathcal{F}, C, p)$ are zero if p is not a singular point of \mathcal{F} . If \mathcal{F} is reduced, then $Z(\mathcal{F}, C, p) \geq 0$ for any $p \in C$ [Bru97]. Let

$$Z(\mathcal{F}, C) = \sum_{p \in C} Z(\mathcal{F}, C, p) = \sum_{p \in C \cap \text{Sing}(\mathcal{F})} Z(\mathcal{F}, C, p),$$

$$\text{CS}(\mathcal{F}, C) = \sum_{p \in C} \text{CS}(\mathcal{F}, C, p) = \sum_{p \in C \cap \text{Sing}(\mathcal{F})} \text{CS}(\mathcal{F}, C, p).$$

Proposition 2.5 ([Bru04, Proposition 2.2]). *Let C be a curve on S which is \mathcal{F} -invariant. Then*

$$Z(\mathcal{F}, C) = K_{\mathcal{F}}C + \chi(C), \quad \text{where } \chi(C) = -K_S C - C^2;$$

$$\text{CS}(\mathcal{F}, C) = C^2.$$

In particular, if \mathcal{F} is reduced and C is an \mathcal{F} -invariant curve, then

$$0 \leq K_{\mathcal{F}}C + \chi(C) = K_{\mathcal{F}}C - K_S C - C^2 = N_{\mathcal{F}}C - C^2.$$

3. THE ZARISKI DECOMPOSITION OF THE CANONICAL DIVISOR $K_{\mathcal{F}}$

In this section, we are concerned about the Zariski decomposition of the canonical divisor $K_{\mathcal{F}}$. We will first briefly recall the Zariski decomposition of the canonical divisor $K_{\mathcal{F}}$ for a relatively minimal foliation \mathcal{F} with $K_{\mathcal{F}}$ being pseudo-effective and refer to [McQ08] and [Bru04, Chapter 8] for more details. Then we give a careful analysis on the negative part, and prove a technical result about the coefficients appearing in the negative part, which will be helpful in proving the Noether inequalities.

Definition 3.1. Let \mathcal{F} be reduced foliation on a smooth projective S . An irreducible curve $C \subseteq S$ is \mathcal{F} -exceptional if

- (i). C is an exceptional curve of first kind on S , i.e., it is a smooth rational curve with $C^2 = -1$;
- (ii). the contraction of C to a point produces a new foliation (S_0, \mathcal{F}_0) which is still reduced.

Definition 3.2. A foliated surface (S, \mathcal{F}) is called relatively minimal if

- (i). the foliation \mathcal{F} is reduced;
- (ii). there is no \mathcal{F} -exceptional curve on S .

It is proved that any foliated surface (S, \mathcal{F}) has a relatively minimal model, cf. [Bru04, Proposition 5.1]. We assume in the following that \mathcal{F} is a relatively minimal foliation on a smooth projective surface S such that $K_{\mathcal{F}}$ is pseudo-effective. In fact, the canonical divisor $K_{\mathcal{F}}$ is pseudo-effective if and only if \mathcal{F} is not induced by a \mathbb{P}^1 -fibration, cf. [Miy87]. Denote the Zariski decomposition of $K_{\mathcal{F}}$ by

$$K_{\mathcal{F}} = P + N, \tag{3.1}$$

where P is the nef part and N is the negative one. By the Riemann-Roch theorem, one sees directly that

$$\text{vol}(\mathcal{F}) = P^2.$$

McQuillan proved that the support of the negative part N is a disjoint union of maximal \mathcal{F} -chains.

Definition 3.3. Let \mathcal{F} be a relatively minimal foliation on a smooth projective surface S . We say a curve $C \subseteq S$ is an \mathcal{F} -chain if

- (i). the curve C is a Hirzebruch-Jung string, i.e., $C = \cup_{j=1}^r C_j$, each C_j is a smooth rational curve with $C_j^2 \leq -2$, $C_j \cdot C_i = 1$ if $|i - j| = 1$ and 0 if $|i - j| \geq 2$;
- (ii). each irreducible component C_j is \mathcal{F} -invariant;
- (iii). $\text{Sing}(\mathcal{F}) \cap C$ are all reduced and non-degenerate;
- (iv). $Z(\mathcal{F}, C_1) = 1$, and $Z(\mathcal{F}, C_j) = 2$ for any $2 \leq j \leq r$.

Since each irreducible component C_j is a smooth rational curve, by Proposition 2.5 the last condition (iv) is also equivalent to

$$(iv)'. \quad K_{\mathcal{F}}C_1 = -1, \text{ and } K_{\mathcal{F}}C_j = 0 \text{ for any } 2 \leq j \leq r.$$

Theorem 3.4 ([Bru04, Theorem 8.1]). *Let \mathcal{F} be a relatively minimal foliation on a smooth projective surface S . Suppose that $K_{\mathcal{F}}$ is pseudo-effective with the Zariski decomposition as in (3.1). Then the support $\text{Supp}(N)$ is a disjoint union of maximal \mathcal{F} -chains, and $[N] = 0$.*

The above theorem shows that all the coefficients in N are less than 1. In fact, since the support $\text{Supp}(N)$ is a disjoint union of maximal \mathcal{F} -chains, which can be contracted to singularities of Hirzebruch-Jung type, these coefficients can be explicitly computed out using continued fractions [BHPV04, § III.5]. By contracting the support $\text{Supp}(N)$, one obtains a surface S_0 with finitely many singularities. Then the negative part N can be decomposed into

$$N = \sum_Q N_Q,$$

where the sum runs over all singularities Q on S_0 , and $\text{Supp}(N_Q)$ is supported on the inverse image of Q in S . Suppose that Q is a singularity of type $A_{n,q}$, and let $N_Q = \sum_{j=1}^r b_j C_j$ with $C = \cup_{j=1}^r C_j$ being a maximal \mathcal{F} -chain as above. Define $\lambda_{r+1} = 0$, $\lambda_r = 1$, and the rest λ_j 's by the following recursion formula:

$$\lambda_{j-1} - e_j \lambda_j + \lambda_{j+1} = 0, \tag{3.2}$$

where $e_j = -C_j^2 \geq 2$. Then it has been shown in [BHPV04, § III.5] that $\lambda_1 = q$, $\lambda_0 = n$. In fact, it holds that

$$\frac{n}{q} = e_1 - \frac{1}{e_2 - \frac{1}{\dots - \frac{1}{e_r}}}.$$

Moreover,

$$b_j = \frac{\lambda_j}{n}, \quad \forall 1 \leq j \leq r. \tag{3.3}$$

We provide a brief explanation of (3.3). According to the property of the Zariski decomposition

$$N_Q C_1 = K_{\mathcal{F}} C_1 = -1, \quad N_Q C_j = K_{\mathcal{F}} C_j = 0, \quad \forall 2 \leq j \leq r.$$

On the other hand, define $N' = \sum_{j=1}^r \frac{\lambda_j}{n} C_j$. Based on (3.2), one checks directly that

$$N' C_1 = -1 = N_Q C_1, \quad N' C_j = 0 = N_Q C_j, \quad \forall 2 \leq j \leq r.$$

Since the intersection matrix $(C_i C_j)$ is negatively definite, it follows that $N_Q = N'$, i.e., the equality (3.3) holds.

Proposition 3.5. *Let $N_Q = \sum_{j=1}^r b_j C_j$ with $C = \cup_{j=1}^r C_j$ being a maximal \mathcal{F} -chain as above.*

Then

$$b_j < \begin{cases} \frac{1}{e_1 - 1}, & \text{if } j = 1; \\ \frac{1}{e_j}, & \text{if } j \geq 2 \text{ and } e_j \geq 3, \end{cases} \quad (3.4)$$

where $e_j = -C_j^2 \geq 2$.

Proof. Let $\lambda_{r+1} = 0$, and $b_j = \frac{\lambda_j}{n}$ for $1 \leq j \leq r$. By the above arguments, $\lambda_r = 1$, and the rest λ_j 's can be computed by (3.2). As proved in [BHPV04, § III.5], $\lambda_1 = q$ and $\lambda_0 = n$. With the help of (3.2), one proves inductively that

$$\lambda_{j-1} > \lambda_j, \quad \forall 1 \leq j \leq r+1. \quad (3.5)$$

Hence

$$\lambda_{j-1} = e_j \lambda_j - \lambda_{j+1} > e_j \lambda_j - \lambda_j = (e_j - 1) \lambda_j.$$

In particular, $n = \lambda_0 > (e_1 - 1) \lambda_1$. Equivalently, $b_1 = \frac{\lambda_1}{n} < \frac{1}{e_1 - 1}$ as required.

Suppose now that $j \geq 2$. Then

$$n = \lambda_0 \geq \lambda_{j-2} = e_{j-1} \lambda_{j-1} - \lambda_j \geq 2 \lambda_{j-1} - \lambda_j > (2(e_j - 1) - 1) \lambda_j \geq e_j \lambda_j.$$

The last inequality follows from the assumption that $e_j \geq 3$. It follows that $b_j = \frac{\lambda_j}{n} < \frac{1}{e_j}$ if $j \geq 2$ and $e_j \geq 3$. This completes the proof of (3.4). \square

Remark 3.6. The above bounds on the coefficients of the negative part N_Q will be key to estimate the volume $\text{vol}(K_{\mathcal{F}})$. One can similarly define the volume $\text{vol}(L)$ for any big divisor L on a smooth projective surface S by

$$\text{vol}(L) = \limsup_{n \rightarrow +\infty} \frac{\dim H^0(X, nL)}{n^2/2}.$$

The naive Noether type inequalities do NOT hold for L . For instance, let $e > 0$ and $S = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ be the Hirzebruch surface admitting a unique section C_0 with $C_0^2 = -e < 0$. Let $f : S \rightarrow \mathbb{P}^1$ be the geometrical ruling on S , F be a general fiber of f , and $L = mF + C_0$. Suppose that $0 < m < e$. Then one checks easily that the Zariski decomposition of L is

$$L = \left(mF + \frac{m}{e} C_0\right) + \frac{e-m}{e} C_0,$$

and hence

$$\text{vol}(L) = \frac{m^2}{e}, \quad h^0(L) = m + 1.$$

Fixing m and letting $e \rightarrow +\infty$, one deduces that there can not exist positive constants a, b such that $\text{vol}(L) \geq ah^0(L) - b$.

Of course, in this concrete example, one shows easily that $L = K_{\mathcal{F}}$ for some reduced foliation \mathcal{F} on S only when $m = e - 1$ based on (3.2) and (3.3). If it is indeed the case, then

$$\text{vol}(\mathcal{F}) = \frac{m^2}{e} = \frac{(e-1)^2}{e} = e - 2 + \frac{1}{e} > e - 2 = p_g(\mathcal{F}) - 2.$$

We refer to Example 6.2 for a construction of such a foliation.

4. THE CANONICAL MAP INDUCES A FIBRATION

The main purpose of this section is to prove Theorem 4.1, which will be key to get the Noether inequalities when the canonical map of a foliated surface (S, \mathcal{F}) induces a fibration.

Let \mathcal{F} be a reduced foliation of general type on a smooth projective surface with canonical divisor $K_{\mathcal{F}}$. We restrict ourselves in this section to the case when the image $\varphi(S)$ is of one-dimension, where

$$\varphi = \varphi_{|K_{\mathcal{F}}|} : S \dashrightarrow \Sigma \subseteq \mathbb{P}^{p_g(\mathcal{F})-1},$$

is the rational map defined by $|K_{\mathcal{F}}|$. In this case, by the Stein factorization, we obtain a diagram as follows.

$$\begin{array}{ccccc} & S' & \xrightarrow{\sigma} & S & \\ & \downarrow \phi & & \downarrow \varphi & \\ B & \xrightarrow{\pi} & Y & \xrightarrow[\text{desingularization}]{\rho} & \Sigma \hookrightarrow \mathbb{P}^{p_g(\mathcal{F})-1} \\ & \nwarrow f & & & \end{array}$$

Here $\pi : B \rightarrow Y$ is finite, and $f : S' \rightarrow B$ is a fibration of curves with connected fibers. The foliation \mathcal{F} lifts to a foliation \mathcal{F}' on S' , which is also reduced. Since \mathcal{F} is reduced,

$$K_{\mathcal{F}'} = \sigma^* K_{\mathcal{F}} + \mathcal{E},$$

where \mathcal{E} is some effective divisor supported on the exceptional curves of the birational morphism σ . Moreover,

$$|K_{\mathcal{F}'}| = \sigma^* |K_{\mathcal{F}}| + \mathcal{E}.$$

It follows that the map $\varphi_{|K_{\mathcal{F}'}}|$ factors through $\varphi_{|K_{\mathcal{F}}|}$, and hence it induces the same fibration $f : S' \rightarrow B$. By replacing (S, \mathcal{F}) by (S', \mathcal{F}') , we may assume that the moving part of the linear system $|K_{\mathcal{F}}|$ is base-point-free, so that the canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration $f : S \rightarrow B$.

$$\begin{array}{ccccc} & S & & & \\ & \downarrow \phi & \searrow \varphi = \varphi_{|K_{\mathcal{F}}|} & & \\ B & \xrightarrow{\pi} & Y & \xrightarrow[\text{desingularization}]{\rho} & \Sigma \hookrightarrow \mathbb{P}^{p_g(\mathcal{F})-1} \\ & \nwarrow f & & & \end{array}$$

After such a replacement, the foliation \mathcal{F} is still reduced of general type, but not necessarily relatively minimal any more. Let F be a general fiber of f . As the foliation \mathcal{F} is assumed to be of general type, it follows that

$$K_{\mathcal{F}} \cdot F \geq 1.$$

The main purpose in this section is to prove the following.

Theorem 4.1. *Let \mathcal{F} be a reduced foliation of general type on a smooth projective surface S , such that its canonical map $\varphi = \varphi_{|K_{\mathcal{F}}|}$ induces a fibration $f : S \rightarrow B$ as above. Suppose that S is of general type and that $K_{\mathcal{F}} \cdot F = 1$. Then*

$$p_g(\mathcal{F}) = g(B). \quad (4.1)$$

More precisely, if $g(B) \geq 1$, then

$$|K_{\mathcal{F}}| = |f^*K_B| + C_0 + Z_v, \quad (4.2)$$

such that $|f^*K_B|$ (resp. $C_0 + Z_v$) is the moving part (resp. fixed part) of $|K_{\mathcal{F}}|$, where $C_0 \subseteq S$ is a section of f , and Z_v is effective whose support is contained in fibers of f . Here we understand that the moving part $|f^*K_B|$ is empty if $g(B) = 1$.

We divide the key points of the proof of Theorem 4.1 into several propositions as follows.

Proposition 4.2. *Let \mathcal{F} be a reduced foliation on S and $f : S \rightarrow B$ be a fibration on S . Suppose that $g(F) \geq 2$ and $K_{\mathcal{F}} \cdot F = 1$, where F is a general fiber of f . Then the foliation \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S .*

Proposition 4.3. *Let \mathcal{F} be a reduced foliation on S and $f : S \rightarrow B$ be a fibration on S . Suppose that \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S . Then the canonical divisor $K_{\mathcal{F}}$ has the following form*

$$K_{\mathcal{F}} = f^*K_B + Z, \quad (4.3)$$

where Z is effective.

Proposition 4.4. *Let \mathcal{F} be a reduced foliation on S and $f : S \rightarrow B$ be a fibration on S . Suppose that $g(F) \geq 2$ and $K_{\mathcal{F}} \cdot F = 1$, where F is a general fiber of f . Let Z be the effective divisor as in (4.3). Then*

$$Z = C_0 + \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} n_{ij} C_{ij}, \quad (4.4)$$

where $C_0 \subseteq S$ is a section of f , and C_{ij} 's are all contained in fibers of f . Moreover, the coefficients n_{ij} 's satisfy the following property: if $F_i = \sum_{j=1}^{r_i} m_{ij} C_{ij}$ is a fiber of f , then there exists at least one $1 \leq j \leq r_i$ such that $n_{ij} < m_{ij}$.

We postpone the proofs of the above three propositions, and first prove Theorem 4.1 based on them.

Proof of Theorem 4.1. Since S is of general type, the genus $g(F) \geq 2$ for a general fiber F of $f : S \rightarrow B$, cf. [BHPV04, § V]. Thus the foliation \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S by Proposition 4.2. According to Proposition 4.3 and Proposition 4.4,

$$K_{\mathcal{F}} = f^*K_B + C_0 + Z_v, \quad \text{with } Z_v = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} n_{ij} C_{ij}, \quad (4.5)$$

where $C_0 \subseteq S$ is a section of f , and C_{ij} 's are all contained in fibers of f . In particular,

$$p_g(\mathcal{F}) = h^0(S, K_{\mathcal{F}}) \geq h^0(B, K_B) = g(B).$$

As $g(F) \geq 2$ for a general fiber F of f , the section C_0 is certainly contained in the fixed part of $|K_{\mathcal{F}}|$. Hence

$$h^0(S, K_{\mathcal{F}}) = h^0(S, f^*K_B + Z_v).$$

According to Proposition 4.4, the coefficients n_{ij} 's satisfy the following property: if $F_i = \sum_{j=1}^{r_i} m_{ij} C_{ij}$ is a fiber of f , then there exists at least one $1 \leq j \leq r_i$ such that $n_{ij} < m_{ij}$. This means that $h^0(S, Z_v) = 1$, and that

$$f_* \mathcal{O}_S(Z_v) = \mathcal{O}_B.$$

Therefore,

$$h^0(S, K_{\mathcal{F}}) = h^0(S, f^* K_B + Z_v) = h^0(B, K_B \otimes f_* \mathcal{O}_S(Z_v)) = h^0(B, K_B) = g(B).$$

This proves (4.1), from which together with (4.5) the equality (4.2) follows. \square

We now come back to the proofs of Propositions 4.2-4.4. A key observation is the following.

Lemma 4.5. *Let \mathcal{F} be a reduced foliation on S and $f : S \rightarrow B$ be a fibration on S . Suppose that $g(F) \geq 2$ and $K_{\mathcal{F}} \cdot F = 1$, where F is a general fiber of f . Let F_0 be any fiber of $f : S \rightarrow B$. Then there is at least one component $C \subseteq F_0$ which is not \mathcal{F} -invariant. In particular, any smooth fiber of f is not \mathcal{F} -invariant.*

Proof. We prove by contradiction. Suppose that there exists a fiber $F_0 = \sum_{j=1}^r m_j C_j$, such that every component C_j is \mathcal{F} -invariant. Since \mathcal{F} is reduced, the fiber F_0 is normal crossing. According to Proposition 2.5, for any component $C_j \subseteq F_0$,

$$Z(\mathcal{F}, C_j) = K_{\mathcal{F}} C_j - K_S C_j - C_j^2. \quad (4.6)$$

By definition, if $p \in C_j$ is a singularity of \mathcal{F} , then $Z(\mathcal{F}, C_j, p) \geq 1$. Hence

$$Z(\mathcal{F}, C_j) \geq \sum_{i \neq j} C_i C_j.$$

Combining this with (4.6),

$$\sum_{i \neq j} C_i C_j \leq K_{\mathcal{F}} C_j - K_S C_j - C_j^2, \quad \text{for any fixed } 1 \leq j \leq r.$$

It follows that

$$\begin{aligned} \sum_{i=1}^r \sum_{j \neq i} m_j C_i C_j &= \sum_{j=1}^r \sum_{i \neq j} m_j C_i C_j \leq \sum_{j=1}^r (m_j K_{\mathcal{F}} C_j - m_j K_S C_j - m_j C_j^2) \\ &= (K_{\mathcal{F}} - K_S) \cdot \sum_{j=1}^r m_j C_j - \sum_{j=1}^r m_j C_j^2 \\ &= (K_{\mathcal{F}} - K_S) \cdot F_0 - \sum_{i=1}^r m_i C_i^2 \\ &= 1 - (2g(F) - 2) - \sum_{i=1}^r m_i C_i^2. \end{aligned}$$

Thus

$$1 - (2g(F) - 2) \geq \sum_{i=1}^r \left(\sum_{j \neq i} m_j C_i C_j + m_i C_i^2 \right) = \sum_{i=1}^r F_0 \cdot C_i = 0.$$

This gives a contradiction if $g(F) \geq 2$. \square

Proof of Proposition 4.2. This follows directly from Lemma 4.5. Indeed, if \mathcal{F} is the same as the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S , then every fiber of f is \mathcal{F} -invariant, which contradicts Lemma 4.5. \square

Proof of Proposition 4.3. Let ω_i be any local holomorphic one-form on the base curve B . Since \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S , the contraction of $f^*(\omega_i)$ with the local field v_i defining \mathcal{F} gives a non-zero local section of $K_{\mathcal{F}}$. Globally, it is just the following contraction map

$$\begin{aligned} H^0(S, T_S(T_{\mathcal{F}}^*)) \otimes f^*H^0(B, \Omega_B^1) &\longrightarrow H^0(S, K_{\mathcal{F}}) \\ (v, f^*\omega) &\mapsto (v, f^*\omega), \end{aligned}$$

where $v = (U_i, v_i)$ is a local vector field defining \mathcal{F} , which can be viewed as a section of $H^0(S, T_S(T_{\mathcal{F}}^*))$ by (2.1), and $\omega = (V_i, \omega_i) \in H^0(B, \Omega_B^1)$ with $f(U_i) \subseteq V_i$. Remark that the above contraction map also makes sense if ω is a rational one-form on B (in this case, the image $(v, f^*\omega)$ would be a rational section of $K_{\mathcal{F}}$ in general). To be concrete, let t_i be a local coordinate of B on V_i . Then (V_i, dt_i) defines a holomorphic section of $\omega \in H^0(B, \Omega_B^1(-K_B))$, i.e., ω is a twist one-form on B . Hence

$$\alpha := (v, f^*\omega) = \{(U_i, (v_i, f^*dt_i))\} \in H^0(S, K_{\mathcal{F}} \otimes f^*(-K_B)). \quad (4.7)$$

In other words,

$$K_{\mathcal{F}} - f^*K_B = \operatorname{div}(\alpha).$$

By construction, α is locally holomorphic, and hence $Z = \operatorname{div}(\alpha)$ is effective. This proves (4.3). \square

Proof of Proposition 4.4. Since $g(F) \geq 2$ and $K_{\mathcal{F}} \cdot F = 1$, by Proposition 4.2 the foliation \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_B)$ in T_S . According to the proof of Proposition 4.3 above, we have to determine the divisor $Z = \operatorname{div}(\alpha)$ in (4.7).

Since $K_{\mathcal{F}} \cdot F = 1$, the horizontal part of $\operatorname{div}(\alpha)$ consists of exactly one section of f , which we denote by C_0 . It remains to decide the vertical part of $\operatorname{div}(\alpha)$. Let $F_i = \sum_{j=1}^{r_i} m_{ij}C_{ij}$ be any fiber of f , and

$$\operatorname{div}(\alpha) = \sum_{j=1}^{r_i} n_{ij}C_{ij} + Z',$$

where the support of Z' does not contain any component of F_i . Then we have to show that

$$\text{there exists at least one } 1 \leq j \leq r_i \text{ such that } n_{ij} < m_{ij}. \quad (4.8)$$

Let $p_{ij} \in C_{ij}$ be a general point of C_{ij} . As $p_{ij} \in C_{ij}$ is general, there exist a local coordinate (x, y) around p_{ij} such that $C_{ij} = \{x = 0\}$, and a local coordinate t around $q = f(F_i)$ such that map f is defined by $t = x^{m_{ij}}$. Suppose that $v = h(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$ around p_{ij} . Then around the point p_{ij} , $f^*dt = m_{ij}x^{m_{ij}-1}$, and hence

$$\alpha = \left(h(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}, f^*dt \right) = m_{ij}h(x, y)x^{m_{ij}-1}.$$

Therefore,

$$n_{ij} = m_{ij} - 1 + \operatorname{order}_x(h(x, y)),$$

where

$$\text{order}_x(h(x, y)) = \max \{k \mid x^k \text{ divides } h(x, y)\}.$$

It follows that

$$n_{ij} \geq m_{ij}, \iff x \text{ divides } h(x, y), \iff C_{ij} \text{ is } \mathcal{F}\text{-invariant.}$$

Hence (4.8) follows immediately from Lemma 4.5. This completes the proof. \square

5. THE PROOF OF THE NOETHER TYPE INEQUALITIES

In this section, we prove the Noether type inequalities for a foliated surface of general type. As illustrated in Section 1, it relies on the canonical map $\varphi_{|K_{\mathcal{F}}|}$, where

$$\varphi_{|K_{\mathcal{F}}|} : S \dashrightarrow \Sigma \subseteq \mathbb{P}^{p_g(\mathcal{F})-1}, \quad (5.1)$$

is the rational map defined by $|K_{\mathcal{F}}|$. Theorem 1.2 will be proved in Proposition 5.1 if Σ is of dimension two, and in Proposition 5.4 if Σ is of dimension one.

Before going to the proof, let's do some preparations. As \mathcal{F} is of general type, $\text{vol}(\mathcal{F}) > 0$. Hence we will always assume in this section that

$$p_g(\mathcal{F}) = h^0(S, K_{\mathcal{F}}) \geq 3.$$

Given any foliated surface (S, \mathcal{F}) with reduced singularities, by [Bru04, Proposition 5.1], one can contract \mathcal{F} -exceptional curves to obtain a relatively minimal foliated surface (S', \mathcal{F}') such that $\text{vol}(\mathcal{F}') = \text{vol}(\mathcal{F})$ and $p_g(\mathcal{F}') = p_g(\mathcal{F})$. Therefore, we may assume that \mathcal{F} is relatively minimal as well. Let

$$|K_{\mathcal{F}}| = |M| + Z, \quad (5.2)$$

be the decomposition of the complete linear system $|K_{\mathcal{F}}|$, where Z is the fixed part. Let $\rho : Y \rightarrow \Sigma$ be the desingularization of Σ , where $\Sigma = \varphi_{|K_{\mathcal{F}}|}(S)$ is the image of S under the canonical map as in (5.1). By a sequence of blowing-ups $\sigma : \tilde{S} \rightarrow S$ centered on the base points of $|M|$, we obtain a well-defined morphism $\phi : \tilde{S} \rightarrow Y$ with the following diagram.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\sigma} & S \\ \downarrow \phi & & \downarrow \varphi = \varphi_{|K_{\mathcal{F}}|} \\ Y & \xrightarrow[\text{desingularization}]{\rho} & \Sigma \hookrightarrow \mathbb{P}^{p_g(\mathcal{F})-1} \end{array}$$

There are two possibilities of the dimension of the image Σ : $\dim \Sigma = 2$ or $\dim \Sigma = 1$.

Proposition 5.1. *Let (S, \mathcal{F}) be a foliated surface of general type with reduced singularities. Suppose that the image $\Sigma = \varphi_{|K_{\mathcal{F}}|}(S)$ is of dimension two. Then*

$$\begin{cases} \text{vol}(\mathcal{F}) \geq p_g(\mathcal{F}) - 2; \\ \text{vol}(\mathcal{F}) \geq 2p_g(\mathcal{F}) - 4, \quad \text{if moreover } S \text{ is of general type.} \end{cases} \quad (5.3)$$

Moreover, if the equality in any of the above two inequalities holds, then the image Σ is a surface of minimal degree (equal to $p_g(\mathcal{F}) - 2$) in $\mathbb{P}^{p_g(\mathcal{F})-1}$, and $\deg(\varphi) = 1$ (resp. $\deg(\varphi) = 2$) if the equality in the first (resp. second) inequality holds.

Proof. By construction, the map

$$\rho \circ \phi : \tilde{S} \longrightarrow \Sigma \hookrightarrow \mathbb{P}^N,$$

is defined by the complete linear system $|\widetilde{M}|$ with $\widetilde{M} = \sigma^*M - \sum a_j \mathcal{E}_j$. According to [Bea79, Lemme 1.2],

$$M^2 \geq \widetilde{M}^2 \geq \deg(\varphi) \cdot \deg(\Sigma).$$

On the other hand, let

$$K_{\mathcal{F}} = P + N,$$

be the Zariski decomposition of $K_{\mathcal{F}}$, where P is nef and N is negative. Then

$$\text{vol}(\mathcal{F}) = P^2 \geq M^2 \geq \widetilde{M}^2 \geq \deg(\varphi) \cdot \deg(\Sigma) \geq p_g(\mathcal{F}) - 2.$$

The last equality follows from [Bea79, Lemme 1.4]. Moreover, if the equality holds, then $\deg(\varphi) = 1$ and $\deg(\Sigma) = p_g(\mathcal{F}) - 2$ in $\mathbb{P}^{p_g(\mathcal{F})-1}$.

Suppose moreover that S is of general type. If $\deg(\varphi) = 1$, then Σ is a surface of general type birational to S , and hence by [Bea79, Lemme 1.4] and [Bea79, Remarque 1.5] one obtains that

$$\text{vol}(\mathcal{F}) \geq \deg(\varphi) \cdot \deg(\Sigma) = \deg(\Sigma) > 2(p_g(\mathcal{F}) - 2).$$

If $\deg(\varphi) \geq 2$, then

$$\text{vol}(\mathcal{F}) \geq \deg(\varphi) \cdot \deg(\Sigma) \geq 2(p_g(\mathcal{F}) - 2).$$

The equality holds implies that $\deg(\varphi) = 2$ and $\deg(\Sigma) = p_g(\mathcal{F}) - 2$. This completes the proof. \square

Remark 5.2. According to [Bea79, Lemme 1.4], for any non-ruled surface $\Sigma \subseteq \mathbb{P}^{p_g(\mathcal{F})-1}$, it holds

$$\deg(\Sigma) \geq 2p_g(\mathcal{F}) - 4.$$

Hence by the above proof, we have actually proved that

$$\text{vol}(\mathcal{F}) \geq 2p_g(\mathcal{F}) - 4,$$

for any foliation \mathcal{F} of general type with reduced singularities, if S is not a ruled surface and the image $\Sigma = \varphi|_{K_{\mathcal{F}}}(S)$ is of dimension two. Moreover, if the equality holds, then

- (i). either $\deg(\varphi) = 1$ and $\deg(\Sigma) = 2p_g(\mathcal{F}) - 4$;
- (ii). either $\deg(\varphi) = 2$ and $\deg(\Sigma) = p_g(\mathcal{F}) - 2$.

In the rest part of this section, we will assume that $\dim \Sigma = 1$. In this case, by the Stein factorization, we obtain a diagram as follows.

$$\begin{array}{ccccc} & \tilde{S} & \xrightarrow{\sigma} & S & \\ & \downarrow \phi & & \downarrow \varphi = \varphi|_{K_{\mathcal{F}}} & \\ B & \xleftarrow{f} & Y & \xrightarrow{\rho} & \Sigma \\ & \xrightarrow{\pi} & & \text{desingularization} & \end{array}$$

Here $\pi : B \rightarrow Y$ is finite, and $f : \tilde{S} \rightarrow B$ is a family of curves with connected fibers. By construction, the map

$$\rho \circ \phi : \tilde{S} \longrightarrow \Sigma \hookrightarrow \mathbb{P}^N,$$

is defined by the complete linear system $|\widetilde{M}|$, where $|\widetilde{M}|$ is obtained by blowing-up the base points of $|M|$. Since $|\widetilde{M}|$ is base-point-free and induces a fibration $f : \tilde{S} \rightarrow B$, it follows that

$$p_g(\mathcal{F}) = h^0(S, M) = h^0(\tilde{S}, \widetilde{M}) = h^0(B, f_*\mathcal{O}_{\tilde{S}}(\widetilde{M})).$$

According to the Riemann-Roch theorem,

$$\deg(L) \geq p_g(\mathcal{F}) - 1,$$

where $L = f_* \mathcal{O}_{\widetilde{S}}(\widetilde{M})$ is a line bundle on B . Note that $\widetilde{M} = f^*(L)$. Hence, we have numerically,

$$\widetilde{M} \equiv_{\text{num}} \deg(L) F, \quad \text{with } \deg(L) \geq p_g(\mathcal{F}) - 1, \quad (5.4)$$

where F is a general fiber of f .

Lemma 5.3. *If the linear system $|M|$ has a base point, then*

$$\text{vol}(\mathcal{F}) \geq (p_g(\mathcal{F}) - 1)^2 > 2p_g(\mathcal{F}) - 4.$$

Proof. Let $A = \sigma_*(F)$. If $|M|$ has a base point, then $A^2 \geq 1$. Hence

$$\text{vol}(\mathcal{F}) \geq P^2 \geq M^2 \geq ((p_g(\mathcal{F}) - 1)A)^2 \geq (p_g(\mathcal{F}) - 1)^2 \quad \square$$

We assume from now on that $|M|$ is base-point-free. In other words, $\varphi : S \rightarrow \Sigma$ is already a morphism. Hence one gets a commutative diagram as follows.

$$\begin{array}{ccccc} & & S & & \\ & f \swarrow & \downarrow \phi & \searrow \varphi = \varphi|_{K_{\mathcal{F}}} & \\ B & \xrightarrow{\pi} & Y & \xrightarrow[\text{desingularization}]{\rho} & \Sigma \hookrightarrow \mathbb{P}^{p_g(\mathcal{F})-1} \end{array}$$

Proposition 5.4. *Let (S, \mathcal{F}) be a foliated surface of general type with reduced singularities. Suppose that the canonical map $\varphi|_{K_{\mathcal{F}}}$ induces a fibration $f : S \rightarrow B$ as above. Let F be a general fiber of f . Then $K_{\mathcal{F}} \cdot F \geq 1$.*

(i). *The following inequalities holds.*

$$\text{vol}(\mathcal{F}) > \begin{cases} 2p_g(\mathcal{F}) - 4, & \text{if } K_{\mathcal{F}} \cdot F \geq 2; \\ p_g(\mathcal{F}) - 2, & \text{if } K_{\mathcal{F}} \cdot F = 1. \end{cases} \quad (5.5)$$

(ii). *If $K_{\mathcal{F}} \cdot F = 1$ and S is also of general type, then*

$$\text{vol}(\mathcal{F}) \geq 2p_g(\mathcal{F}) - 2. \quad (5.6)$$

Proof. Since \mathcal{F} is of general type, i.e., $K_{\mathcal{F}}$ is big, it follows that $K_{\mathcal{F}} \cdot F \geq 1$.

(i). As explained at the beginning of this section, we may assume that the foliation \mathcal{F} is relatively minimal. By Lemma 5.3, we may assume that $|M|$ is base-point-free. The relation in (5.4) rephrases as

$$M \equiv_{\text{num}} \deg(L) F, \quad \text{with } \deg(L) \geq p_g(\mathcal{F}) - 1.$$

The moving part M is clear nef, from which it follows that $M \leq P$, or equivalently $N \leq Z$, where P is the nef part and N is the negative part of $K_{\mathcal{F}}$ in its Zariski decomposition, and Z is the fixed part of $|K_{\mathcal{F}}|$ as in (5.2). Hence

$$\text{vol}(\mathcal{F}) = P^2 \geq P \cdot M \geq (p_g(\mathcal{F}) - 1)P \cdot F = (p_g(\mathcal{F}) - 1)(K_{\mathcal{F}} - N) \cdot F. \quad (5.7)$$

Therefore, it suffices to prove a lower bound on $P \cdot F$, or equivalently an upper bound on $N \cdot F$.

The fibration $f : S \rightarrow B$ defines a natural foliation \mathcal{G} by taking the saturation of the kernel $\ker(df : T_S \rightarrow f^*T_B)$ in T_S . The given foliation \mathcal{F} may be equal to or different from \mathcal{G} .

Suppose that $\mathcal{F} = \mathcal{G}$. Then the genus $g(F) \geq 2$ since \mathcal{F} is assumed to be of general type. Moreover, the canonical divisor is easy to compute:

$$K_{\mathcal{F}} = K_{\mathcal{G}} = K_S \otimes f^* K_B^{-1} \otimes \mathcal{O}_S \left(\sum (1 - a_i) C_i \right),$$

where the sum is taken over all components in fibers of f , and a_i is the multiplicity of C_i in its fiber. Note that the support of the negative part N is a sum of \mathcal{F} -chains (cf. Theorem 3.4), and thus contained in fibers of f . Combining this with (5.7),

$$\begin{aligned} \text{vol}(\mathcal{F}) &\geq (p_g(\mathcal{F}) - 1) (K_{\mathcal{F}} - N) \cdot F \\ &= (p_g(\mathcal{F}) - 1) K_{\mathcal{F}} \cdot F \\ &= (2g(F) - 2)(p_g(\mathcal{F}) - 1) \geq 2(p_g(\mathcal{F}) - 1). \end{aligned}$$

In the rest part of the proof, we will always assume that \mathcal{F} is different from the foliation \mathcal{G} defined by taking the saturation of the kernel $\ker(df : T_S \rightarrow f^* T_B)$ in T_S . By (5.7), it suffices to prove that

$$P \cdot F > \begin{cases} \frac{2(p_g(\mathcal{F}) - 2)}{p_g(\mathcal{F}) - 1}, & \text{if } K_{\mathcal{F}} \cdot F \geq 2; \\ \frac{p_g(\mathcal{F}) - 2}{p_g(\mathcal{F}) - 1}, & \text{if } K_{\mathcal{F}} \cdot F = 1. \end{cases} \quad (5.8)$$

Let $Z = Z_h + Z_v$, where Z is the fixed part of $|K_{\mathcal{F}}|$ as in (5.2), and each component in Z_v is contained in fibers of f , while each component in Z_h maps surjectively to the base B . Similarly, we can decompose the negative as $N = N_h + N_v$. Let

$$\begin{aligned} Z_h &= \sum a_C C, & Z_v &= \sum a_D D, \\ N_h &= \sum b_C C, & N_v &= \sum b_D D. \end{aligned}$$

Then the coefficients $\{a_C, a_D\}$'s are positive integers; while $\{b_C, b_D\}$'s belong to $[0, 1)$ by Theorem 3.4, since the foliation \mathcal{F} is assumed to be relatively minimal.

Suppose first that there exists an irreducible component $C_0 \subseteq Z_h$ with $C_0^2 = -2$ and $b_{C_0} > 0$ (i.e., C_0 is contained in the support of N). Then

$$0 = P \cdot C_0 \geq ((a_{C_0} - b_{C_0})C_0 + (p_g(\mathcal{F}) - 1)F) \cdot C_0 = -2(a_{C_0} - b_{C_0}) + (p_g(\mathcal{F}) - 1)F \cdot C_0.$$

It follows that

$$a_{C_0} - b_{C_0} \geq \frac{(p_g(\mathcal{F}) - 1)F \cdot C_0}{2}.$$

Then

$$P \cdot F \geq a_{C_0} - b_{C_0} \geq \frac{p_g(\mathcal{F}) - 1}{2} \geq \frac{2(p_g(\mathcal{F}) - 2)}{p_g(\mathcal{F}) - 1}. \quad (5.9)$$

It remains to show that the above inequality is strict if $K_{\mathcal{F}} \cdot F \geq 2$. Indeed, if $P \cdot F = \frac{2(p_g(\mathcal{F}) - 2)}{p_g(\mathcal{F}) - 1}$, then $p_g(\mathcal{F}) = 3$ and

$$P \cdot F = a_{C_0} - b_{C_0} = F \cdot C_0 = 1.$$

Note that $b_{C_0} < 1$ by Theorem 3.4, and $b_{C_0} > 0$ by assumption. This gives a contradiction, since a_{C_0} is an integer. Hence the inequality (5.9) is strict as required.

We can now assume that for any component $C \subseteq D_h$, either C is not contained in the support of N , or $-C^2 \geq 3$. In other words, every possible irreducible curve C , which maps surjectively

onto the base C and occurs in the negative part N , satisfies that $C^2 \leq -3$. We will prove under the above assumption that

$$b_C < \frac{a_C}{p_g(\mathcal{F}) - 1}. \quad (5.10)$$

To prove (5.10), we may assume that $b_C > 0$, i.e., C is contained in the support of the negative part N , which is a union of maximal \mathcal{F} -chains. Suppose that C is contained in the maximal \mathcal{F} -chain $\sum_{j=1}^r C_j$. If C is not the first component C_1 in the maximal \mathcal{F} -chain, then

$$0 = K_{\mathcal{F}} \cdot C \geq a_C C^2 + (p_g(\mathcal{F}) - 1) F \cdot C \geq a_C C^2 + (p_g(\mathcal{F}) - 1).$$

Combining this with (3.4), one gets

$$b_C < \frac{1}{-C^2} \leq \frac{a_C}{p_g(\mathcal{F}) - 1}.$$

If C is the first component C_1 in the maximal \mathcal{F} -chain, then there are two possibilities: either the maximal \mathcal{F} -chain consists of exactly one component $C_1 = C$, or there is another curve, say C' , contained in the negative part N intersecting $C_1 = C$. If it is the first case, then

$$\begin{cases} -1 = N \cdot C = b_C C^2, \\ -1 = K_{\mathcal{F}} \cdot C \geq a_C C^2 + (p_g(\mathcal{F}) - 1) F \cdot C \geq a_C C^2 + (p_g(\mathcal{F}) - 1). \end{cases}$$

Hence

$$b_C = \frac{1}{-C^2} \leq \frac{a_C}{p_g(\mathcal{F})}.$$

If it is the second case, then

$$-1 = K_{\mathcal{F}} \cdot C \geq a_C C^2 + (p_g(\mathcal{F}) - 1) F \cdot C + a_{C'} \geq a_C C^2 + p_g(\mathcal{F}),$$

where $a_{C'}$ is the coefficient of C' in Z (as the component C' is in the support of N and $N \leq Z$, it follows that C' is contained in the support of Z). Combining this together with (3.4), one gets

$$b_C < \frac{1}{-C^2} \leq \frac{a_C}{p_g(\mathcal{F}) - 1}.$$

This completes the proof of (5.10).

Come back to the proof of (5.8). According to (5.10),

$$\begin{aligned} P \cdot F &= \sum (a_C - b_C) C \cdot F \geq \sum \left(a_C - \frac{a_C}{p_g(\mathcal{F}) - 1} \right) C \cdot F \\ &= \frac{p_g(\mathcal{F}) - 2}{p_g(\mathcal{F}) - 1} \cdot \sum a_C C \cdot F \\ &= \frac{p_g(\mathcal{F}) - 2}{p_g(\mathcal{F}) - 1} \cdot K_{\mathcal{F}} \cdot F. \end{aligned}$$

This proves (5.8), and hence completes the proof of (5.7).

(ii). Since it is assumed that $p_g(\mathcal{F}) \geq 3$, from Theorem 4.1 it follows that $g(B) = p_g(\mathcal{F}) \geq 3$ and the canonical linear system

$$|K_{\mathcal{F}}| = |f^* K_B| + C_0 + Z_v,$$

such that $|f^* K_B|$ (resp. $C_0 + Z_v$) is the moving part (resp. fixed part) of $|K_{\mathcal{F}}|$, where $C_0 \subseteq S$ is a section of f , and Z_v is effective whose support is contained in fibers of f . Since $g(C_0) = g(B) \geq 3$,

the curve C_0 is not contained in the support of negative part N in the Zariski decomposition $K_{\mathcal{F}} = P + N$. It follows that

$$\text{vol}(\mathcal{F}) = P^2 \geq P \cdot f^*K_B = (K_{\mathcal{F}} - N) \cdot f^*K_B = K_{\mathcal{F}} \cdot f^*K_B = 2g(B) - 2 = 2p_g(\mathcal{F}) - 2.$$

This proves (5.6). \square

Proof of Theorem 1.2. As \mathcal{F} is of general type, $\text{vol}(\mathcal{F}) > 0$, and hence we may assume that $p_g(\mathcal{F}) \geq 3$. Let $\varphi = \varphi_{|K_{\mathcal{F}}|} : S \dashrightarrow \Sigma \subseteq \mathbb{P}^{p_g(\mathcal{F})-1}$ be the rational map defined by $|K_{\mathcal{F}}|$. If the image Σ is of dimension two, it follows from Proposition 5.1; if Σ is of dimension one, it follows from Proposition 5.4. \square

6. EXAMPLES

In this section, we will construct several examples. Example 6.1 and Example 6.4 show that the two Noether type inequalities in Theorem 1.2 are both sharp. In Example 6.2 we construct a sequence of reduced foliated surfaces (S_n, \mathcal{F}_n) of general type, such that $\text{vol}(\mathcal{F}_n) > p_g(\mathcal{F}_n) - 2$ and that the difference $\text{vol}(\mathcal{F}_n) - (p_g(\mathcal{F}_n) - 2)$ tends to zero. This shows a phenomenon different from the case for algebraic varieties, where there exists so-called "the second Noether inequality", cf. [HZ22]. In Example 6.5, we will construct a sequence of reduced foliated surfaces (S, \mathcal{F}) of general type, whose canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration $f : S \rightarrow B$ with $g(B) \geq 2$, $g(F) \geq 2$ and $K_{\mathcal{F}} \cdot F = 1$, where F is a general fiber of f .

Example 6.1. In this example, we construct a sequence of reduced foliated surfaces (S_n, \mathcal{F}_n) of general type, such that the volume $\text{vol}(\mathcal{F}_n)$ and the geometric genus $p_g(\mathcal{F}_n)$ tend to the infinity and that the following equality holds

$$\text{vol}(\mathcal{F}_n) = p_g(\mathcal{F}_n) - 2. \quad (6.1)$$

Let \mathcal{F}_0 be a foliation of degree two on \mathbb{P}^2 with reduced singularities. Such a foliation exists, cf. [LS20, Proposition 3.2]. In fact, any foliation of degree d on \mathbb{P}^2 can be generated by a vector of the form ([GO89, Bru04])

$$v = (P(x, y) + xR(x, y)) \frac{\partial}{\partial x} + (Q(x, y) + yR(x, y)) \frac{\partial}{\partial y},$$

where (x, y) is an affine coordinate of \mathbb{P}^2 , $P(x, y), Q(x, y)$ are polynomials of degree $\leq d$, and $R(x, y)$ is a homogeneous polynomial of degree d (plus some nondegeneracy conditions). Let

$$v_{\alpha, \beta} = x \left(-\alpha^2 + x^2 + y^2 \right) \frac{\partial}{\partial x} + y \left(2\alpha\beta + (2\alpha + \beta)y + x^2 + y^2 \right) \frac{\partial}{\partial y}, \quad (6.2)$$

where $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Q}$ are general. The foliation \mathcal{F}_0 defined by $v_{\alpha, \beta}$ admits 7 singularities:

$$\left\{ (0, 0), (0, -2\alpha), (0, -\beta), (\alpha, 0), (-\alpha, 0), \left(\frac{\alpha\sqrt{3(\alpha^2 - \beta^2)}}{2\alpha + \beta}, \frac{-\alpha(\alpha + 2\beta)}{2\alpha + \beta} \right), \left(\frac{-\alpha\sqrt{3(\alpha^2 - \beta^2)}}{2\alpha + \beta}, \frac{-\alpha(\alpha + 2\beta)}{2\alpha + \beta} \right) \right\}.$$

The eigenvalues at these 7 singularities are respectively equal to

$$\left\{ \lambda_1 = \frac{-\alpha}{2\beta}, \quad \lambda_2 = \frac{4\alpha - 2\beta}{3\alpha}, \quad \lambda_3 = \frac{\beta(\beta - 2\alpha)}{\beta^2 - \alpha^2}, \quad \lambda_4 = \lambda_5 = \frac{\alpha + 2\beta}{2\alpha}, \right. \\ \left. \lambda_6 = \lambda_7 = \frac{(\alpha - 2\beta)(2\alpha - \beta) - \sqrt{(\alpha - 2\beta)^2(2\alpha - \beta)^2 - 24\alpha(\alpha + 2\beta)(\beta^2 - \alpha^2)}}{(\alpha - 2\beta)(2\alpha - \beta) + \sqrt{(\alpha - 2\beta)^2(2\alpha - \beta)^2 - 24\alpha(\alpha + 2\beta)(\beta^2 - \alpha^2)}} \right\}.$$

If α, β are sufficiently general (for instance if $\{\alpha, \beta\} \in \mathbb{C} \setminus \mathbb{Q}$ are algebraic independent over \mathbb{Q}), then the eigenvalue $\lambda_i \in \mathbb{C} \setminus \mathbb{Q}$ for $1 \leq i \leq 7$, and hence these are all reduced singularities. Moreover, the two lines $L_0 := \{x = 0\}$ and $L_\infty := \{y = 0\}$ are both \mathcal{F}_0 -invariant in view of (6.2). Let $\sigma : S_1 \rightarrow \mathbb{P}^2$ be the blowing-up centered at $(0, 0)$, and \mathcal{F}_1 be the induced foliation on S_1 . Then \mathcal{F}_1 is still reduced, and

$$K_{\mathcal{F}_1} = \sigma^* K_{\mathcal{F}_0} = \sigma^* \mathcal{O}_{\mathbb{P}^2}(1) \sim C_0 + F,$$

where ' \sim ' stands for the linear equivalence, $C_0 = \mathcal{E} \subseteq S_1$ is the unique section (also the exceptional curve of σ) with $C_0^2 = -1$, and F_1 is a general fiber of f_1 . Hence

$$\text{vol}(\mathcal{F}_1) = \text{vol}(\mathcal{F}_0) = 1, \quad p_g(\mathcal{F}_1) = p_g(\mathcal{F}_0) = 3.$$

It follows that the foliations \mathcal{F}_0 and \mathcal{F}_1 (they are birational to each other) satisfy (6.1). By construction, S_1 is isomorphic to the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Let

$$f_1 : S_1 \rightarrow \mathbb{P}^1$$

be the ruling on S_1 , and F_0, F_∞ be the strict transforms of the two lines L_0, L_∞ respectively. As both L_0 and L_∞ are \mathcal{F}_0 -invariant, it follows that both F_0 and F_∞ are \mathcal{F}_1 -invariant. Let $\pi_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the cyclic cover of degree $n \geq 2$ branched over $f_1(F_0)$ and $f_1(F_\infty)$, and $S_n = S_1 \times_{\pi_n} \mathbb{P}^1$ be the fiber product as follows.

$$\begin{array}{ccc} S_n & \xrightarrow{\Pi_n} & S_1 \\ f_n \downarrow & & \downarrow f_1 \\ \mathbb{P}^1 & \xrightarrow{\pi_n} & \mathbb{P}^1 \end{array}$$

It is clear that $S_n \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Let \mathcal{F}_n be the induced foliation on S_n . Then \mathcal{F}_n is a reduced foliation since the eigenvalues $\lambda_i \in \mathbb{C} \setminus \mathbb{Q}$ at each of the 7 singularities of \mathcal{F}_0 . Since both F_0 and F_∞ are \mathcal{F}_1 -invariant, one obtains that (cf. [Bru04, § 2.3(4)]),

$$K_{\mathcal{F}_n} = \Pi_n^*(K_{\mathcal{F}_1}) = \Pi_n^*(C_0 + F_1) = \Pi_n^{-1}(C_0) + nF_n,$$

where F_n is a general fiber of f_n , and $\Pi_n^{-1}(C_0)$ is the strict transform of C_0 in S_n satisfying $\Pi_n^{-1}(C_0)^2 = -n$. Hence

$$\text{vol}(\mathcal{F}_n) = n, \quad p_g(\mathcal{F}_n) = n + 2.$$

Therefore, the equality (6.1) holds for the foliation \mathcal{F}_n . This completes the construction.

Example 6.2. In this example, we construct a sequence of reduced foliated surfaces (S_n, \mathcal{F}_n) of general type, such that the volume $\text{vol}(\mathcal{F}_n)$ and the geometric genus $p_g(\mathcal{F}_n)$ tend to the infinity and that the following equality holds

$$\text{vol}(\mathcal{F}_n) = p_g(\mathcal{F}_n) - 2 + \frac{1}{p_g(\mathcal{F}_n)}. \quad (6.3)$$

Let $n \geq 2$ and $S_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the Hirzebruch surface admitting a unique section C_0 with $C_0^2 = -n < 0$. Let $f : S_n \rightarrow \mathbb{P}^1$ be the geometrical ruling on S_n , which makes S_n as a \mathbb{P}^1 -bundle over the projective line \mathbb{P}^1 . Note that any \mathbb{P}^1 -bundle over an affine space is necessarily trivial. One can obtain the Hirzebruch surface S_n by gluing the two trivial \mathbb{P}^1 -bundles $\mathbb{C} \times \mathbb{P}^1$ by

$$\mathbb{C} \times \mathbb{P}^1 \longrightarrow \mathbb{C} \times \mathbb{P}^1,$$

$$(x, [Y_0, Y_1]) \mapsto \left(\frac{1}{x}, [Y_0, x^n Y_1] \right).$$

More explicitly, one may obtain S_n by gluing four affine spaces as follows. Let (x_i, y_i) be the affine coordinate on $U_i \cong \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ for $1 \leq i \leq 4$. The transition functions on their overlaps are given by

$$(x_1, y_1) = \left(x_2, \frac{1}{y_2} \right) = \left(\frac{1}{x_3}, x_3^n y_3 \right) = \left(\frac{1}{x_4}, \frac{x_4^n}{y_4} \right).$$

Moreover, $C_0 \cap U_1 = \{y_1 = 0\}$.

Let \mathcal{F}_n be the foliation on S_n defined (U_i, v_i) , where

$$\begin{cases} v_1 = h(x_1, y_1) \frac{\partial}{\partial x_1} + y_1^2 g(x_1, y_1) \frac{\partial}{\partial y_1}, \\ v_2 = y_2 h(x_2, 1/y_2) \frac{\partial}{\partial x_2} - y_2 g(x_2, 1/y_2) \frac{\partial}{\partial y_2}, \\ v_3 = -x_3 h(1/x_3, x_3^n y_3) \frac{\partial}{\partial x_3} + \left(n y_3 h(1/x_3, x_3^n y_3) + x_3^{n-1} y_3^2 g(1/x_3, x_3^n y_3) \right) \frac{\partial}{\partial y_3}, \\ v_4 = -x_4 y_4 h(1/x_4, x_4^n/y_4) \frac{\partial}{\partial x_4} - \left(n y_4^2 h(1/x_4, x_4^n/y_4) + x_4^{n-1} y_4 g(1/x_4, x_4^n/y_4) \right) \frac{\partial}{\partial y_4}, \end{cases}$$

where

$$h(x_1, y_1) = a_0 + y_1 \cdot \sum_{i=1}^n \tilde{a}_i x_1^i; \quad g(x_1, y_1) = \sum_{j=1}^{n-1} b_j x_1^j + y_1 \cdot \sum_{j=1}^{n-1} \tilde{b}_j x_1^j.$$

Here $a_i, \tilde{a}_i, b_j, \tilde{b}_j$ are some complex numbers satisfying certain non-degenerate conditions to insure these v_i 's contain no one-dimensional zeros. Then one checks easily by (2.1) that

$$K_{\mathcal{F}_n} = C_0 + (n-1)F,$$

where F is a general fiber of the ruling $f : S_n \rightarrow \mathbb{P}^1$. It follows that the Zariski decomposition of $K_{\mathcal{F}}$ is the following.

$$K_{\mathcal{F}_n} = P + N,$$

where $P = (n-1)F + \frac{n-1}{n}C_0$ is the nef part, and $N = \frac{1}{n}C_0$ is the negative part. Hence

$$\text{vol}(\mathcal{F}_n) = n - 2 + \frac{1}{n}, \quad p_g(\mathcal{F}_n) = n,$$

from which the equality (6.3) follows immediately. To complete the construction, we should insure that the foliation \mathcal{F}_n is reduced. A sufficient condition to ensure a singularity $p \in U_i$ of \mathcal{F}_n to be reduced is that both of the eigenvalues λ_1, λ_2 of $(Dv_i)(p)$ are non-zero and the quotient λ_1/λ_2 is not a positive rational number. This should be satisfied for a general choice of these complex numbers $\{a_i, \tilde{a}_i, b_j, \tilde{b}_j\}$, similar to the situation on \mathbb{P}^2 , [LS20, Proposition 3.2]. For instance, one can take

$$h(x_1, y_1) = 1 + y_1 x_1^n, \quad g(x_1, y_1) = \alpha x_1^{n-1} + y_1, \quad \text{where } \alpha \in \mathbb{C} \setminus \mathbb{Q}.$$

The number of singularities of \mathcal{F}_n is $\# \text{Sing}(\mathcal{F}) = 2n + 2$. There are $2n - 1$ singularities in U_1 :

$$\text{Sing}(\mathcal{F}) \cap U_1 = \{p_i = (\xi_i, -\alpha \xi_i^{n-1}), i = 1, \dots, 2n-1\}, \quad \text{where } \xi_i^{2n-1} = \frac{1}{\alpha}.$$

By direct computation, the two eigenvalues of $(Dv_1)(p_i)$ are

$$\frac{(\alpha - n) \pm \sqrt{(\alpha + n)^2 + 4(n - 1)\alpha^2\xi_i^{2n-2}}}{2\xi_i}.$$

Hence these p_i 's are reduced singularities of \mathcal{F}_n . The rest three singularities of \mathcal{F}_n are all on the fiber

$$F_\infty = \{x_3 = 0\} \cup \{x_4 = 0\}.$$

To be explicit, the rest three singularities are

$$\left\{(0, 0), \left(0, \frac{-n}{n + \alpha}\right)\right\} \subseteq U_3, \quad \text{and} \quad \{(0, 0)\} \subseteq U_4.$$

The eigenvalues of \mathcal{F}_n at these three singularities are respectively equal to $-n$, $\frac{n(n+\alpha)}{\alpha}$ and $n + \alpha$. Hence the foliation \mathcal{F}_n is reduced as required.

Remark 6.3. In the above example, $|K_{\mathcal{F}_n}| = |(n - 1)F| + C_0$, where $|M| := |(n - 1)F|$ is the moving part and C_0 is the fixed part of $|K_{\mathcal{F}}|$. It follows that

$$\text{vol}(\mathcal{F}) = n - 1 + \frac{1}{n} = P \cdot M < K_{\mathcal{F}} \cdot M.$$

Example 6.4. In this example, we construct a sequence of reduced foliated surfaces (S, \mathcal{F}) of general type with S being a surface of general type and

$$\text{vol}(\mathcal{F}) = 2p_g(\mathcal{F}) - 4. \tag{6.4}$$

Let $f : S \rightarrow \mathbb{P}^1$ be a semi-stable fibration of curves of genus $g = 2$, whose slope

$$\lambda_f := \frac{K_{S/\mathbb{P}^1}^2}{\deg f_* \mathcal{O}_S(K_{S/\mathbb{P}^1})} = 2, \quad \text{where } K_{S/\mathbb{P}^1} = K_S - f^*K_{\mathbb{P}^1}.$$

Such a semi-stable fibration $f : S \rightarrow \mathbb{P}^1$ exists; indeed, one can construct such a fibration as follows. Let $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the Hirzebruch surface with the ruling $h : Y \rightarrow \mathbb{P}^1$, and $L_m = 6C_0 + 2m\Gamma$, where C is the section with $C_0^2 = -n$ and Γ is a general fiber of h . Then L_m is very ample if m is sufficiently large (in fact $m > 3n$ is enough, cf. [Har77, Cor V.2.18]). By the Bertini theorem [Har77, Thm II.8.18], a general element $R \in |L_m|$ satisfies that

- (i). the divisor R is smooth;
- (ii). the restricted map $h|_R : R \rightarrow \mathbb{P}^1$ has only simply ramified points, i.e., the ramification indices are all equal to 2.

Let $\pi : S \rightarrow Y$ be the double cover branched over such a general divisor R , and $f = h \circ \pi : S \rightarrow \mathbb{P}^1$ the induced fibration. Then f is a semi-stable fibration of curves of genus 2. Moreover, one checks easily that

$$\begin{aligned} K_{S/\mathbb{P}^1}^2 &= 2(K_{Y/\mathbb{P}^1} + R/2)^2 = 4m - 6n, \\ \deg f_* \mathcal{O}_S(K_{S/\mathbb{P}^1}) &= \frac{1}{4}R \cdot (K_{Y/\mathbb{P}^1} + R/2) = 2m - 3n. \end{aligned}$$

Let \mathcal{F} be the foliation on S defined by taking the saturation of $\ker(df : T_S \rightarrow f^*T_{\mathbb{P}^1})$ in T_S . Then \mathcal{F} is reduced, relatively minimal, and $K_{\mathcal{F}} = K_{S/\mathbb{P}^1}$. Hence

$$\text{vol}(\mathcal{F}) = K_{\mathcal{F}}^2 = 4m - 6n.$$

Moreover, since the Hodge bundle $\deg f_* \mathcal{O}_S(K_{S/\mathbb{P}^1})$ is of rank two and semi-positive, it follows that

$$p_g(\mathcal{F}) = h^0(S, K_{\mathcal{F}}) = h^0(\mathbb{P}^1, f_* \mathcal{O}_S(K_{S/\mathbb{P}^1})) = 2 + \deg f_* \mathcal{O}_S(K_{S/\mathbb{P}^1}) = 2m - 3n + 2.$$

Therefore, the Noether equality (6.4) holds for (S, \mathcal{F}) . Moreover, if $m > \frac{3n+4}{2}$, then $K_S^2 = K_{S/\mathbb{P}^1}^2 - 8 > 0$, and hence S would be a surface of general type.

Example 6.5. In this example, we construct a sequence of reduced foliated surface (S, \mathcal{F}) of general type, whose canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces a fibration $f : S \rightarrow B$ with $g(B) \geq 2$, $g(F) = g \geq 2$, and (where F is a general fiber of f)

$$\begin{cases} K_{\mathcal{F}} \cdot F = 1, \\ \text{vol}(\mathcal{F}) = 4p_g(\mathcal{F}) - 4 = 4g(B) - 4. \end{cases} \quad (6.5)$$

Let B be a curve of genus $g(B) \geq 2$, and $\psi : B \rightarrow \mathbb{P}^1$ is a finite cover of degree $2m$ with only simply ramified points, i.e., the ramification indices are all equal to 2. Let $\hat{Y} = \mathbb{P}^1 \times B$, with two projections \hat{h}_1, \hat{h}_2 . The curve B can be embedded into \hat{Y} by mapping $x \in B$ to $(\psi(x), x) \in \hat{Y}$ as a section of \hat{h}_2 :

$$\begin{array}{ccccc} B & \xhookrightarrow{i} & \hat{Y} & \xrightarrow{\hat{h}_2} & B \\ & \searrow \psi & \downarrow \hat{h}_1 & & \\ & & \mathbb{P}^1 & & \end{array}$$

Denote by $\hat{D} = i(B) \subseteq \hat{Y}$, and let $\hat{\Gamma}_i = \hat{h}_1^{-1}(p_i)$'s ($1 \leq i \leq 2g+1$) be $2g+1$ general fibers of \hat{h}_1 , where $g \geq 0$. Then one sees that $\hat{R} = \hat{D} + \sum_{i=1}^{2g+1} \hat{\Gamma}_i$ is an even divisor, and hence one can construct a double cover $\hat{\pi} : \hat{S} \rightarrow \hat{Y}$ branched exactly over \hat{R} . The surface \hat{S} is singular. In fact, each $\hat{\Gamma}_i$ intersects \hat{D} transversely at $2m$ points, since $\hat{\Gamma}_i$ is general. One can perform a canonical resolution to resolve the singularities on \hat{S} , cf. [BHPV04, § V.22]. Let $\sigma : Y \rightarrow \hat{Y}$ be the birational map by blowing up these intersection points $\hat{D} \cap \sum_{i=1}^{2g+1} \hat{\Gamma}_i$, and let $\pi : S \rightarrow Y$ be the induced double cover. Then S would be a smooth surface with following diagram.

$$\begin{array}{ccccc} & & f=f_2 & & \\ & & \curvearrowright & & \\ S & \xrightarrow{\pi} & Y & \xrightarrow{h_2} & B \\ \downarrow \rho & & \downarrow \sigma & \nearrow \hat{h}_2 & \\ \hat{S} & \xrightarrow{\hat{\pi}} & \hat{Y} & & \\ \downarrow h_1 & & \downarrow \hat{h}_1 & & \\ & & \mathbb{P}^1 & & \end{array} \quad \begin{array}{l} \curvearrowleft f_1 \\ \curvearrowleft \end{array}$$

The fibration $f_1 : S \rightarrow \mathbb{P}^1$ defines a foliation \mathcal{F} on S by taking the saturation of $\ker(df_1 : T_S \rightarrow f_1^* T_{\mathbb{P}^1})$ in T_S . We want to check that the foliation \mathcal{F} satisfies our requirements.

Let D (resp. C) be the strict transform of \hat{D} in Y (resp. S), and Γ_i (resp. Δ_i) be the strict transform of $\hat{\Gamma}_i$ in Y (resp. S). Let \mathcal{E}_i be the union of exceptional curves intersecting Γ_i (\mathcal{E}_i

consists of $2m$ components). Then

$$K_Y = \sigma^*(K_{\hat{Y}}) + \sum_{i=1}^{2g+1} \mathcal{E}_i, \quad \Gamma_i = \sigma^*(\hat{\Gamma}_i) - \mathcal{E}_i, \quad D = \sigma^*(\hat{D}) - \sum_{i=1}^{2g+1} \mathcal{E}_i.$$

Hence

$$K_S = \pi^*(K_Y) + C + \sum_{i=1}^{2g+1} \Delta_i = (\sigma \circ \pi)^* \left(K_{\hat{Y}} + \frac{1}{2} (\hat{D} + \sum_{i=1}^{2g+1} \hat{\Gamma}_i) \right). \quad (6.6)$$

By construction, any singular fiber of f_1 is normal crossing. Hence the foliation \mathcal{F} defined by taking the saturation of $\ker(df_1 : T_S \rightarrow f_1^* T_{\mathbb{P}^1})$ in T_S is reduced. Moreover, the curves Δ_i 's are of multiplicity equal to two in fibers of f_1 , and all the other components in fibers of f_1 are of multiplicity one. It follows that

$$\begin{aligned} K_{\mathcal{F}} &= K_{S/\mathbb{P}^1} - \sum_{i=1}^{2g+1} \Delta_i = \pi^*(K_{Y/\mathbb{P}^1}) + C = (\sigma \circ \pi)^*(K_{\hat{Y}/\mathbb{P}^1}) + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C \\ &= (\hat{h}_2 \circ \sigma \circ \pi)^*(K_B) + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C \\ &= f^* K_B + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C, \end{aligned}$$

where $\bar{\mathcal{E}}_i \subseteq S$ is the strict transform of \mathcal{E}_i . Let F be a general fiber $f = f_2 : S \rightarrow B$. Then $\pi(F) \cong \mathbb{P}^1$ and F is double cover of $\pi(F)$ branched over $2g+2$ points (the branched divisor is $(D + \sum_{i=1}^{2g+1} \Gamma_i) \cap \pi(F)$). By the Hurwitz formula, one obtains

$$g(F) = g.$$

Moreover,

$$K_{\mathcal{F}} \cdot F = \left(f^* K_B + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C \right) \cdot F = C \cdot F = 1.$$

Since $C \cdot F = 1$ and $g(F) = g \geq 2$, the curve C must be contained in the fixed part of the linear system $|K_{\mathcal{F}}|$. It follows that

$$p_g(\mathcal{F}) = h^0(S, K_{\mathcal{F}}) = h^0\left(S, f^* K_B + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i\right) = h^0(B, K_B) = g(B).$$

In fact, one proves moreover that

$$|K_{\mathcal{F}}| = |f^* K_B| + Z,$$

where $Z = \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C$ is the fixed part of $|K_{\mathcal{F}}|$. In particular, the canonical map $\varphi_{|K_{\mathcal{F}}|}$ is nothing but the same as the map $\varphi_{|f^* K_B|}$ defined by the linear system $|f^* K_B|$. Hence the canonical map $\varphi_{|K_{\mathcal{F}}|}$ induces the fibration $f : S \rightarrow B$ as required. It remains to check the equality

$$\text{vol}(\mathcal{F}) = 4g(B) - 4. \quad (6.7)$$

To this purpose, we first compute the Zariski decomposition of $K_{\mathcal{F}}$. By construction, every irreducible component in $\sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i$ is a smooth rational curve with self-intersection -2 . Moreover,

$$(\sigma \circ \pi)^*(\hat{D}) = \pi^*\left(D + \sum_{i=1}^{2g+1} \mathcal{E}_i\right) = 2C + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i.$$

Hence

$$K_{\mathcal{F}} = f^*K_B + \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i + C = P + N,$$

where

$$P = (\sigma \circ \pi)^*\left(\hat{h}_2^*(K_B) + \frac{\hat{D}}{2}\right), \quad N = \frac{1}{2} \sum_{i=1}^{2g+1} \bar{\mathcal{E}}_i.$$

One checks easily that this is the Zariski decomposition of $K_{\mathcal{F}}$. Hence

$$\text{vol}(\mathcal{F}) = P^2 = 2\left(\hat{h}_2^*(K_B) + \frac{\hat{D}}{2}\right)^2 = 2(2g(B) - 2).$$

This proves (6.7). Finally, By (6.6), one computes that

$$K_S^2 = 4(g-1)(2g(B) - 2 + m) > 0.$$

In particular, S is of general type.

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