Joint calibration to SPX and VIX Derivative Markets with Composite Change of Time Models

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Abstract

The Chicago Board Options Exchange Volatility Index (VIX) is calculated from SPX options and derivatives of VIX are also traded in market, which leads to the so-called "consistent modeling" problem. This paper proposes a time-changed Lévy model for log price with a composite change of time structure to capture both features of the implied SPX volatility and the implied volatility of volatility. Consistent modeling is achieved naturally via flexible choices of jumps and leverage effects, as well as the composition of time changes. Many celebrated models are covered as special cases. From this model, we derive an explicit form of the characteristic function for the asset price (SPX) and the pricing formula for European options as well as VIX options. The empirical results indicate great competence of the proposed model in the problem of joint calibration of the SPX/VIX Markets.

Keywords: Time change; Lévy process; Option pricing; Consistent Modeling

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1 Motivation and Formulation

1.1 Consistent Modeling Problem

By the definition from CBOE, Volatility Index (VIX), as an indicator of implied volatility in the following 30 days, is given as

$$\mathrm{VIX}_t^2 = -\frac{2}{\tau} E^{\mathbb{Q}} \left[\ln \frac{e^{-r\tau} S_{t+\tau}}{S_t} \,\middle|\, \mathcal{F}_t \right],$$

where $\tau = 30/365$ and $\mathbb{Q}$ is the risk neutral measure of the equity market. If continuity of the asset price process is assumed, then equivalently

$$\begin{aligned} \mathrm{VIX}_t^2 &= \frac{1}{\tau} E^{\mathbb{Q}} \left[[\ln S]_{t+\tau} - [\ln S]_t \,\middle|\, \mathcal{F}_t \right] \\ &=: \frac{1}{\tau} E^{\mathbb{Q}} \left[\int_t^{t+\tau} v_s^{\mathrm{VIX}} ds \,\middle|\, \mathcal{F}_t \right], \end{aligned} \tag{1}$$

where v^{VIX} is the squared volatility of S. Even though the asset price is not continuous, formula (1) only leads to a third-order error $O\left(\left(\frac{dS_t}{S_{t-}} \right)^3 \right)$, as shown in Carr and Wu (2009). Hence the analysis below is still effective for general jump models to a large degree.

Likewise, we have the formula of VVIX, the volatility of volatility computed from VIX market:

$$\begin{aligned} \mathrm{VVIX}_t^2 &= -\frac{2}{\tau} E^{\mathbb{Q}} \left[\ln \frac{e^{-r\tau} \mathrm{VIX}_{t+\tau}}{\mathrm{VIX}_t} \,\middle|\, \mathcal{F}_t \right] \\ &= \frac{1}{\tau} E^{\mathbb{Q}} \left[[\ln \mathrm{VIX}]_{t+\tau} - [\ln \mathrm{VIX}]_t \,\middle|\, \mathcal{F}_t \right] \\ &=: \frac{1}{4\tau} E^{\mathbb{Q}} \left[\int_t^{t+\tau} v_s^{\mathrm{VVIX}} ds \,\middle|\, \mathcal{F}_t \right], \end{aligned} \tag{2}$$

where v^{VVIX} is the variance of $\ln \mathrm{VIX}^2$. It is important to note that the first equality holds if we believe that measure $\mathbb{Q}$ is risk-neutral in both SPX option market and the VIX market. That is, the two markets can be consistently modeled.

The problem of joint calibration for SPX market and VIX market is equivalent to (or at least incorporates) the calibration of current VIX and VVIX, which can be approximated by $\sqrt{v^{\mathrm{VIX}}}$ and $\sqrt{v^{\mathrm{VVIX}}}/2$, volatility and volatility of volatility respectively, if we consider a Markov setup and that τ is small. More generally, we have $\mathrm{VIX}_t^2 = g_1^M(v_t^{\mathrm{VIX}})$ and $\mathrm{VVIX}_t^2 = g_2^M(v_t^{\mathrm{VVIX}})$ by equation (1) and (2), where $g_1^M(\cdot), g_2^M(\cdot)$ are functions determined by model parameters. Therefore, it is crucial to study the relationship of v^{VIX} and v^{VVIX} in the problem of consistent modeling.

In the past literature on consistent modeling, there have been a lot of research on such volatility relationship. One line of work is aimed at reconstructing the widely used stochastic volatility models (SVMs) by allowing for more realistic and flexible vol-of-vol functionals. One such example is the 3/2 model in Drimus (2012) and in Baldeaux and Badran (2014). Fouque and Saporito (2018) proposed a Heston vol-of-vol model, where the vol-of-vol consists of additional stochastic factors. Additionally, following the work of Gatheral et al. (2018), authors including Bayer et al. (2016), Jacquier et al. (2018) and Gatheral et al. (2020) proposed rough volatility models in the joint calibration of stock and volatility smiles. And according to Lin and Chang (2010) and Kokholm and Stisen (2015), the role of jumps were studied and highlighted in the consistent modeling. Moreover, Papanicolaou (2022) studied the consistency condition of recovering SVMs from market models of the VIX futures term structure. Another category of models suggest a multi-factor specification of volatility. The first attempt was made by Gatheral (2008) and Bayer et al. (2013), who adopted a continuous diffusion model with double mean reverting structure. Multifactor affine specification was considered in Cont and Kokholm (2013), Cheng (2019) and Pacati et al. (2018). And Papanicolaou and Sircar (2014) considered a regime-switching Heston model, where sharp volatility regime shifts captures both volatility skews. Finally, there is some recent research that characterizes the volatility relationship using a non-parametric framework. Guo et al. (2022) introduced a time-continuous formulation of the joint calibration problem, followed by Guyon (2020), who built a non-parametric discrete-time model that achieved exact joint calibration.

While many models above achieves satisfactory consistent modeling results, our approach, apart from having good joint calibration performance, is capable of theoretically decoupling smiles from the two markets. Such nice interpretation of our model is achieved via time change technique by representing the vol-of-vol v^{VVIX} in a linear mixture form of v^{VIX} and another free factor. We present the following examples to show how restrictive or implicit the volatility relationship is in certain SVMs.

Example 1 (Heston Model)

$$\begin{cases} dS_t/S_t = rdt + \sqrt{v_t}dW_t, \\ dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dZ_t \end{cases}$$

with $E[W_tZ_t] = \rho t$. We may compute the volatility of volatility under the assumption that v_t approximates VIX_t:

$$v_t^{VVIX} \approx \frac{d[\ln v]_t}{dt} = \frac{\eta^2}{v_t} = \frac{\eta^2}{v_t^{VIX}}.$$

Such an inverse relationship is unrealistic for VIX and VVIX, and therefore explains the unfavorable calibration result of Heston model. In fact, empirical results show that Heston models generates downward-sloping volatility smiles in VIX market as opposed to the upward-sloping observed smile, as shown in Drimus (2012) and Baldeaux and Badran (2014). The model is poorly fitted under consistent modeling even if jump structures are incorporated, see Kokholm and Stisen (2015).

Example 2 (3/2 Model)

$$\begin{cases} dS_t/S_t = rdt + \sqrt{v_t}dW_t, \\ dv_t = \kappa v_t(\theta - v_t)dt + \eta v_t^{\frac{3}{2}} dZ_t \end{cases}$$

with $E[W_t Z_t] = \rho t$. By the same reasoning, we obtain

$$v_t^{VVIX} \approx \frac{d[\ln v]_t}{dt} = \eta^2 v_t = \eta^2 v_t^{VIX},$$

which is more close to empirical data, see figure 1.1. In fact, the correlation between VIX and VVIX is high in the past ten years (2013-2023), usually around 0.7.

Moreover, 3/2 model generates upward sloping volatility smiles and captures the behavior of VIX better than a variety of stochastic volatility models in Goard and Mazur (2013).

Despite its success in volatility market, the 3/2 model is restrictive in the situation of consistent modeling in the sense that v^{VVIX} is directly determined by v^{VIX}.

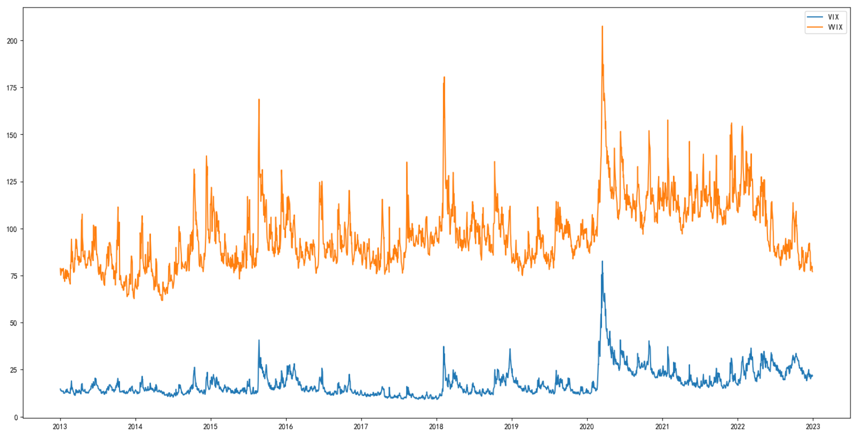

Figure 1: The time series of the VVIX and VIX between January 2013 and December 2022.

Example 3 (Multi-factor Heston Model) *In the multi-factor Heston specification for volatility process, we have, by a reasoning similar to the Heston model,*

$$v_t^{VVIX} \approx \frac{\sum_{i=1}^n \eta_i^2 v_t^{(i)}}{(\sum_{i=1}^n v_t^{(i)})^2},$$

where $v^{(i)},\ i = 1, \ldots, n$ *are volatility factors and* $\eta_i,\ i = 1, \ldots, n$ *the corresponding coefficients. However, it is implicit how each factor* $v^{(i)}$ *acts upon the volatility of volatility.*

Example 4 (Heston vol-of-vol; Fouque and Saporito (2018))

$$\begin{cases} \mathrm{d}S_t/S_t = r\ \mathrm{d}t + \sqrt{v_t}S_t\ \mathrm{d}W_t, \\ \mathrm{d}v_t = \kappa\,(\theta - v_t)\,\mathrm{d}t + \eta_t\sqrt{v_t}\ \mathrm{d}B_t, \\ \mathrm{d}W_t\ \mathrm{d}B_t = \rho\ \mathrm{d}t \end{cases}$$

where $\eta_t = \eta\left(Y_t^{\varepsilon}, Z_t^{\delta}\right)$ *is a stochastic factor correlated with* W *and* B*. Then*

$$v_t^{VVIX} \approx \frac{d[\ln v]_t}{dt} = \frac{\eta_t^2}{v_t^{VIX}}.$$

Although the randomness of η *improves the calibration of* v^{VVIX}*,* η *directly depends on* v^{VIX} *and hence is not flexible enough.*

1.2 Decoupling Smiles via Composite Time Change Approach

To study the volatility relationship in composite time change (CTC) models . We first introduce some background knowledge on time change approaches. Time changes can be interpreted as the intensity of business activities that drives the variation in volatility of an asset. The original clock $\{t, t \geq 0\}$ is referred to as calendar time and time change process T is called business time. The log-price process of an asset is originally thought to be stationary and of independent increment, i.e. a Lévy process L. Every time a market event happens and drives the variation in volatility, the change is reflected in the time change, either via accelerating or slowing the business clock. And the real market price of the asset is updated under the business clock, namely L_T.

Originally, Clark (1973) and Geman et al. (2001) proposed a subordinated Brownian motion model for log price. The time change introduces jumps in volatility. Carr et al. (2003) and Carr and Wu (2004) introduced time-changed Lévy models, where the time change is absolutely continuous, through which stochastic volatility is introduced. Luciano and Schoutens (2006), Luciano and Semeraro

(2007) and Eberlein and Madan (2009) modeled the dependence of multi-assets via correlation of subordinated Brownian motions. Mendoza-Arriaga et al. (2010) extented the time-chanegd Lévy model by considering the combination of time change and a certain type of composite time change. And recently, Ballotta and Rayée (2022) established a unified TCLM structure that allows leverage via diffusion as well as jumps.

But like SVMs, TCMs with specifications shown in Carr and Wu (2004) are restricted by the relationship $v_t^{\mathrm{VVIX}} = f(v_t^{\mathrm{VIX}})$ for a model-determined function f. But when extended to composite time change, the model may generate the form of vol-of-vol as

$$v_t^{\mathrm{VVIX}} = a v_t^{\mathrm{VIX}} + b v_t^I, \tag{3}$$

where v^I is the idiosyncratic component of v^{VVIX} and a, b are determined by model parameters, which are typically considered steady in short periods. The linear combination satisfies both the need for the dependence of VVIX on VIX and the flexibility of VVIX. v^{VIX} is calibrated to the SPX market while a free factor v^I is calibrated to the VIX market.

Definition 1 *A composite time change has the form $T^{co} = T_V$, where $T = \{T_t, t \geq 0\}$ and $V = \{V_t, t \geq 0\}$ are time changes, i.e., $\{T_t\}, \{V_t\}$ are non-decreasing and right-continuous, and T_t, V_t are stopping times for every $t \geq 0$.*

Here we assume time changes to be absolutely continuous and the base Lévy process to be a standard Brownian motion. Specifically, $dT_t = u_t dt$ and $dV_t = v_t dt$, where u and v are two independent Itô processes. Then the instantaneous variance of the model is $v_t^{\mathrm{VIX}} = \frac{dT_{V_t}}{dt} = u_{V_t} v_t$. Then the product rule

$$dv_t^{\mathrm{VIX}} = d(u_{V_t} v_t) = u_{V_t} dv_t + v_t du_{V_t} + d[u_{V.}, v.]_t$$

gives

$$v_t^{\mathrm{VVIX}} \approx \frac{d[\ln v^{\mathrm{VIX}}]_t}{dt} = \frac{1}{v_t^2}\frac{d[v]_t}{dt} + \frac{1}{(u_{V_t})^2}\frac{d[u_V]_t}{dt}. \tag{4}$$

Equation (4) results in different linear mixture forms according to the specification of activity rate process. And the ideal form of equation (3) can be achieved by the composite 3/2 model proposed in section 3.2. In addition, the model parameters in time change T and V are naturally separated and linearly combined in form.

1.3 Organization of the Paper

The article is organized as follows. In section2, we summarize the theory of time-changed Lévy models developed by Carr and Wu (2004) and Ballotta and Rayée (2022) and we show how the technique of leverage neutral measure change helps form the characteristic function; section 3 develops the theory of composite time change models, where a general form is considered and characteristic fuctions derived. We also discusses some useful specifications of CTC models. In Section 4, we introduce the application of the model in derivative pricing, including European options and VIX options. We show that the European option pricing in CTC models can be conducted quite efficiently. In section 5, we perform real-market joint calibration and discusses the results. And the last section concludes.

2 Preliminaries: time-changed Lévy processes

2.1 Lévy processes

Definition 2 *A Lévy process, $L(t)$, on a filtered probability space $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ is a continuous-time process with independent and stationary increments with a characteristic function $\phi_L(m;t) = e^{t\Psi_L(m)}, m \in \mathbb{R}$ with characteristic exponent*

$$\Psi_L(m) = i\alpha m - \frac{m^2}{2}\sigma^2 + \int_{\mathbb{R}} \left(e^{imx} - 1 - imx1_{|x|\leq 1}\right)\nu(dx),$$

where $\alpha \in \mathbb{R}, \sigma \in \mathbb{R}^+$ and ν is a positive measure on $\mathbb{R}$ such that $v(\{0\}) = 0$, $\int_{\mathbb{R}} (|x|^2 \wedge 1)\, v(dx) < \infty$. The triplet $(\alpha, \sigma^2, v(dx))$ determines the Lévy process and is referred to as differential characteristics.

A subordinator $T = \{T_t, t \geq 0\}$ is a Lévy process such that $t \mapsto T_t$ is non-decreasing.

2.2 Time-changed Lévy Processes

Leverage neutral measure, first introduced in Carr and Wu (2004), is a complex-valued measure change technique that enables the explicit computation of the characteristic function (Ch.f.) of time-changed Lévy process L_T, especially when L and T are not independent.

Assumption 1 *The Lévy processes L considered in this paper are sufficiently integrable. That is, $\int_{|x|>1} e^{mx}\nu(dx) < \infty$ for all $m \in \mathbb{R}$.*

Assumption 2 *There are three versions of assumptions in the order of restrictiveness.*

1. *(Type 1) T satisfies sufficient regularity conditions. That is, the Laplace transforms considered in this paper always exist for every $t > 0$.*

2. *(Type 2) condition of type 1 and T is in synchronization with some adapted process X, i.e. X is constant a.s. on the interval $[T_{t-}, T_t]$ for each $t > 0$.*

3. *(Type 3) condition of type 1 and the time change is absolutely continuous with $T_t = \int_0^t v_s \mathrm{d}s$, $v_t > 0$ a.s. or an independent Lévy subordinator.*

Proposition 1 *Let the Lévy process L satisfy the assumption and time change T be of type 1. The log-price is $X_t = L_{T_t}$.*
1. If T is independent of L, then the Ch.f. of X_t is

$$\phi_X(m;t) := \mathrm{E}\left[\mathrm{e}^{\mathrm{i}mX_t}\right] = \phi_T(-i\Psi_L(m);t)$$

2. Otherwise,

$$\phi_X(m;t) := \mathrm{E}\left[\mathrm{e}^{\mathrm{i}mX_t}\right] = \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{T_t\Psi_L(m)}\right] = \phi_T^{\mathbb{Q}}\left(-i\Psi_L(m);t\right),$$

where $\mathrm{E}[\cdot]$ and $\mathrm{E}^{\mathbb{Q}}[\cdot]$ denote expectations under measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. The new class of complex-valued measures $\mathbb{Q}(m)$ is absolutely continuous with respect to $\mathbb{P}$ and is defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}(m)}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = M_t(m),$$

with

$$M_t(m) := \exp\left(\mathrm{i}mL_t + t\Psi_L(m)\right), \quad m \in \mathbb{R}.$$

Moreover, the Ch.e. of L under the leverage neutral measure is given by

$$\Psi_L^{\mathbb{Q}}(z) = \Psi_L(z+m) - \Psi_L(m), \quad m, z \in \mathbb{R}.$$

Remark 1 *Since the result above is a bit different from those proposed in Carr and Wu (2004) and Ballotta and Rayée (2022), we leave the proof in Appendix A.*

Extra conditions for T is required if we want to characterise the time-changed process X. Specifically, if T is of type 2, then semimartingale X has local characteristics

$$(\alpha dT_{t-}, \sigma^2 dT_{t-}, \nu(dx)dT_{t-}),$$

where $(\alpha, \sigma^2, \nu(dx))$ is the Lévy characteristic of L (see Küchler and Sorensen (2006)). Such an extra assumption makes the model identifiable, i.e., L and time change T are unique up to a constant multiplication.

3 Composite Time Change Models

In this section, we assume a composite time change model $X_t = L_{T_{V_t}}$, where both time changes T and V are absolutely continuous with activity rates u and v, respectively. Furthermore, we assume $\mathbb{P}$ to be the risk-neutral measure and $\mathcal{F}$ the original filtration under which the base Lévy process L is adapted and T, V are time changes. We denote by $\mathcal{F}^X$ with $\mathcal{F}_t^X := \sigma(\mathcal{F}_{T_{V_t}}) \vee \sigma(\mathcal{F}_{V_t})$ the filtration generated by X, T_V and V. Denote by $E_t[\cdot]$ the expectation taken under filtration $\mathcal{F}_t^X$.

3.1 Model Theory

Under such assumptions, the model is specified as follows

$$\begin{cases} dL(t) = -\Psi(-i)dt + \sigma dW(t) + \eta dJ(t), \\ du(t) = \alpha_T(u(t))dt + \beta_T(u(t))dZ(T_t) + \gamma_T(u(t))dJ_T(T_t), \\ dv(t) = \alpha_V(v(t))dt + \beta_V(v(t))dB(V_t) + \gamma_V(v(t))dJ_V(V_t), \end{cases}$$

where W, Z, B are $(\mathbb{P}, \mathcal{F})$-Brownian motions, J, J_T, J_V are $(\mathbb{P}, \mathcal{F})$-subordinators and $\Psi(m) = -\frac{m^2}{2}\sigma^2 + \Psi_J(\eta m)$. In addition, $\alpha_i(\cdot), \beta_i(\cdot), \gamma_i(\cdot), i = T, V$ are unspecified functions. Finally, $\gamma_i(\cdot) \geq 0,\ i = T, V$ to guarantee the positivity of u and v.

To introduce leverage effect, we assume that $[W, Z]_t = \rho_T t$, $[W, B]_t = \rho_V t$ but Z and B are independent. We also assume that J and J_T have joint distribution F_T, J and J_V have joint distribution F_V. J_T and J_V are independent.

Note that if we let $\alpha_i(\cdot)$ be affine and $\beta_i(\cdot), \gamma_i(\cdot)$ be constant, then u, v both have an affine structure.

Next, we introduce the leverage neutral measure that will be useful for the computation with composite time change. The leverage neutral measure $\mathbb{Q}(m)$ is defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}(m)}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = M_t^m,$$

where

$$M_t^m = \exp\left(imL_t + t\Psi_L(m)\right).$$

is a $(\mathbb{P}, \mathcal{F})$-exponential martingale.

Then we have the following result of the Ch.f. of X.

Theorem 1 *The Ch.f. of the composite time-changed process X is given by*

$$\phi_X(m; t) := E\left[e^{imX_t}\right] = E^{\mathbb{Q}}\left[\phi_T^{\mathbb{Q}}(-i\Psi_L(m); V_t)\right] \tag{5}$$

where, under the leverage neutral measure $\mathbb{Q}(m)$, $u^{\mathbb{Q}}$ and $v^{\mathbb{Q}}$ are given by

$$du^{\mathbb{Q}}(t) = \left(\alpha_T(u^{\mathbb{Q}}(t)) + im\rho_T\sigma v^{\mathbb{Q}}(t)\beta_T(u^{\mathbb{Q}}(t))\right) dt + \beta_T(u^{\mathbb{Q}}(t))dZ^{\mathbb{Q}}(T_t) + \gamma_T(u^{\mathbb{Q}}(t))dJ_T^{\mathbb{Q}}(V_t)$$
$$dv^{\mathbb{Q}}(t) = \left(\alpha_V(v^{\mathbb{Q}}(t)) + im\rho_V\sigma v^{\mathbb{Q}}(t)\beta_V(v^{\mathbb{Q}}(t))\right) dt + \beta_V(v^{\mathbb{Q}}(t))dB^{\mathbb{Q}}(V_t) + \gamma_V(v^{\mathbb{Q}}(t))dJ_V^{\mathbb{Q}}(V_t),$$

where $B^{\mathbb{Q}}, Z^{\mathbb{Q}}$ are $(Q, \mathcal{F})$-Brownian motions. The Ch.e. of $J_i^{\mathbb{Q}},\ i = T, V$ are given by

$$\Psi_{J_i}^{\mathbb{Q}}(z) = \Psi_{J,J_i}(\eta m, z) - \Psi_J(\eta m), \tag{6}$$

where $\Psi_{J,J_i}(m, z)$ is the Ch.e. of $mJ + zJ_i$.

Remark 2 *Note that the time changes T and V are still real-valued and positive under the new measure. The dynamics of activity rates above is only effective when Ch.f. is computed.*

Proof By the independence of T and V,

$$\begin{aligned}\phi_X(m;t) &= E^{\mathbb{Q}}[e^{T_{V_t}\Psi_L(m)}]\\ &= E^{\mathbb{Q}}[E^{\mathbb{Q}}[e^{T_{V_t}\Psi_L(m)} \mid V_t]]\\ &= E^{\mathbb{Q}}[\phi_T^{\mathbb{Q}}(-i\Psi_L(m); V_t)].\end{aligned}$$

Next, we derive the dynamics of T and V under the risk-neutral measure $\mathbb{Q}$. By theorem 1, the Ch.e. of L under measure $\mathbb{Q}$ is given by

$$\Psi_L^{\mathbb{Q}}(z) = \Psi_L(z + m) - \Psi_L(m).$$

By the independence of Brownian motions and jump processes, the Ch.e. of W under measure $\mathbb{Q}$ is

$$\Psi_W^{\mathbb{Q}}(z) = \Psi_W(z + \sigma m) - \Psi_W(\sigma m) = -\frac{1}{2}z^2 - \sigma m z.$$

Therefore $W^{\mathbb{Q}}(t) := W(t) - im\sigma t$ is a $(\mathbb{Q}, \mathcal{F})$-Brownian motion. And by the correlation $[W, Z]_t = \rho_T t$, $Z^{\mathbb{Q}}(t) := W(t) - im\rho_T\sigma t$ is a $(\mathbb{Q}, \mathcal{F})$-Brownian motion. $B^{\mathbb{Q}}$ is defined likewise.

Next we show the Ch.f. of J_T and J_V. We note that the single joint distribution determines the joint distribution of $J(t)$ and $J_i(t),\ i = T, V$ at all time $t > 0$. Under the leverage neutral measure, the Ch.f. of J_V becomes

$$\begin{aligned}\phi_{J_V}^{\mathbb{Q}}(z;t) &= E\exp\left(izJ_V(t) + im\eta J(t) - t\Psi_J(\eta m)\right)\\ &= \exp\left(t(\Psi_{J,J_V}(\eta m, z) - \Psi_J(\eta m)\right),\end{aligned}$$

where $\Psi_{J,J^V}(m, z)$ is the Ch.e. of $mJ + zJ_V$. It follows that

$$\Psi_{J_V}^{\mathbb{Q}}(z) = \Psi_{J,J_V}(\eta m, z) - \Psi_J(\eta m).$$

□

Corollary 1 *When affine structure is imposed, that is, $\alpha_T(u(t)) = \kappa_T(\theta_T - u(t))$, $\alpha_V(v(t)) = \kappa_V(\theta_V - v(t))$ and $\beta_i(\cdot) \equiv \sigma_i$, $\gamma_i(\cdot) \equiv \eta_i$, $i = T, V$, then Ch.f. of X is explicit as follows:*

$$\phi_X(m;t) = \frac{1}{\pi}\int_0^\infty \int_0^\infty f(s;m)\,\mathrm{Re}\left[e^{-zs}\phi_V^{\mathbb{Q}}(-iz;t)\right] \mathrm{d}z_I \mathrm{d}s, \tag{7}$$

where $z = z_R + iz_I$ with $z_R > 0$ and

$$\phi_V^{\mathbb{Q}}(m;t) = e^{b_V(t)v(0)+c_V(t)},$$

with the affine exponents $b_V(t), c_V(t)$ solutions to the system of Riccati-type ODEs

$$\begin{aligned} b_V(t)' &= im - \kappa_V^{\mathbb{Q}} b_V(t) + \frac{\sigma_V^2}{2} b_V(t)^2 + \Psi_{J_V}^{\mathbb{Q}}(i\eta_V b_V(t)) \\ c_V(t)' &= \kappa_V \theta_V b_V(t). \end{aligned} \tag{8}$$

with $b_V(0) = c_V(0) = 0$, $\kappa_V^{\mathbb{Q}} = \kappa_V - im\rho_V\sigma_V\sigma$. The function f is given by

$$f(t;m) = e^{b_T(t)u(0)+c_T(t)} \tag{9}$$

with coefficients satisfying

$$\begin{aligned} b_T(t)' &= \Psi_L(m) - \kappa_T^{\mathbb{Q}} b_T(t) + \frac{\sigma_T^2}{2} b_T(t)^2 + \Psi_{J_T}^{\mathbb{Q}}(i\eta_T b_T(t)) \\ c_T(t)' &= \kappa_T \theta_T b_T(t). \end{aligned} \tag{10}$$

with $b_T(0) = c_T(0) = 0$, $\kappa_T^{\mathbb{Q}} = \kappa_T - im\rho_T\sigma_T\sigma$. The Ch.e. of J_i, $i = T, V$ are given in equation (6).

Proof By the inverse Laplace transform,

$$\begin{aligned} \phi_X(m;t) &= E^{\mathbb{Q}}[\phi_T^{\mathbb{Q}}(-i\Psi_L(m);V_t)] \\ &= \frac{1}{\pi}\int_0^\infty \int_0^\infty f(s;m)\,\mathrm{Re}\left[e^{-zs}\phi_V^{\mathbb{Q}}(-iz;t)\right] \mathrm{d}z_I \mathrm{d}s, \end{aligned}$$

where $z = z_R + iz_I$ with $z_R > 0$ and $f(t) = \phi_T^{\mathbb{Q}}(-i\Psi_L(m);t)$ (Here we denote $f(t;m)$ by $f(t)$ for simplicity). If an affine structure is imposed for T and V, then as is given in Filipović (2001), the Laplace transform of T_t is given by

$$f(t) = e^{b_T(t)u(0)+c_T(t)},$$

where coefficients b_T and c_T are given by the equation (10). Likewise, the Ch.f. of ϕ_V has coefficients given by the equation (8). $\square$

3.2 Specifications

Example 5 (Composite 3/2 Model)

$$\begin{cases} L(t) = -\frac{1}{2}dt + dW(t), \\ du_t = \kappa_T u_t(\theta_T - u_t)dt + \sigma_T u_t dZ_{T_t}, \\ dv_t = \kappa_V v_t(\theta_T - v_t)dt + \sigma_V v_t dB_{V_t} \end{cases}$$

with $E[W_t Z_t] = \rho_T t$, $E[W_t B_t] = \rho_V t$ *and* $E[Z_t B_t] = 0$.

In this specification, we have by equation (4) that

$$\begin{aligned} v_t^{\mathrm{VVIX}} \approx \frac{d[\ln v^{\mathrm{VIX}}]_t}{dt} &= \frac{(\sigma_V u_{V_t})^2 v_t^3 + (\sigma_T v_t)^2 (u_{V_t}^3 v_t)}{(v_t^{\mathrm{VIX}})^2} \\ &= \sigma_V^2 v_t + \sigma_T^2 v_t^{\mathrm{VIX}}, \end{aligned}$$

a linear combination of factors as in equation (3).

We clearly see a separation of effects, including the effect of VIX and the idiosyncratic component. The SPX market calibrates v_t^{VIX} and implies a general relationship of model parameters σ_V and σ_T, while the VIX market calibrates v and determines the model parameters. In other specifications likewise, we also obtain a linear combination of factors.

To derive the Ch.f. under composite 3/2, we note that, according to Carr and Sun (2007), the Laplace transform of V has

$$\begin{aligned} &E^{\mathbb{Q}(m)}\left(\mathrm{e}^{-\lambda \int_t^\tau v_s \ \mathrm{ds}} \mid v_t\right) \\ &\quad = \frac{\Gamma(\gamma_V - \alpha_V)}{\Gamma(\gamma_V)} \left(\frac{2}{\sigma_V^2 y\,(t, v_t)}\right)^{\alpha_V} M\left(\alpha_V, \gamma_V, \frac{-2}{\sigma_V^2 y\,(t, v_t)}\right), \end{aligned}$$

where

$$\begin{aligned} y\,(t, v_t) &= v_t \frac{e^{\kappa_V \theta_V (\tau - t)} - 1}{\kappa_V \theta_V}, \\ \alpha_V &= -\left(\frac{1}{2} - \frac{p_V}{\sigma_V^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{p_V}{\sigma_V^2}\right)^2 + 2\frac{\lambda}{\sigma_V^2}} \\ \gamma_V &= 2\left(\alpha + 1 - \frac{p_V}{\sigma_V^2}\right), \\ p_V &= -\kappa_V^{\mathbb{Q}} := -\kappa_V + i\sigma_V \rho_V m, \end{aligned}$$

and $M(\alpha, \gamma, z)$ is the confluent hypergeometric function, defined as

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!},$$

and

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

And

$$\begin{aligned} f(t;m) &= \phi_T^{\mathbb{Q}}(-i\Psi_L(m);t) \\ &= \frac{\Gamma(\gamma_T-\alpha_T)}{\Gamma(\gamma_T)}\left(\frac{2}{\sigma_T^2 y_T\left(0,u_0\right)}\right)^{\alpha_T} M\left(\alpha_T,\gamma_T,\frac{-2}{\sigma_T^2 y_T\left(0,u_0\right)}\right) \end{aligned}$$

where

$$\begin{aligned} y_T\left(0,u_0\right) &= u_0\frac{e^{\kappa_T\theta_T t}-1}{\kappa_T\theta_T}, \\ \alpha_T &= -\left(\frac{1}{2}-\frac{p_T}{\sigma_T^2}\right)+\sqrt{\left(\frac{1}{2}-\frac{p_T}{\sigma_T^2}\right)^2+2\frac{q_T}{\sigma_T^2}} \\ \gamma_T &= 2\left(\alpha_T+1-\frac{p_T}{\sigma_T^2}\right), \\ p_T &= -\kappa_T^{\mathbb{Q}} := -\kappa_T + i\sigma_T\rho_T m, \\ q_T &= \frac{im}{2}+\frac{m^2}{2}. \end{aligned}$$

Despite its nice interpretation, the composite 3/2 can be time-consuming in pricing, particularly in VIX pricing. A modified version of composite 3/2 model is constructed:

$$\begin{cases} L(t) = -\frac{1}{2}dt + dW(t), \\ du_t = \kappa_T u_t(\theta_T - u_t)dt + \sigma_T u_t dZ_{T_t}, \\ dv_t = \kappa_V(\theta_T - v_t)dt + \sigma_V dB_{V_t}. \end{cases} \tag{11}$$

That is, the second time change V is substituted by a CIR process. It's shown in the section of VIX pricing that such formation is efficient in the method of exact simulation.

Example 6 (Composite 3/2 + Jump)

$$\begin{cases} L(t) = -\Psi(-i)dt + dW(t) + \eta dJ(t), \\ du_t = \kappa_T u_t(\theta_T - u_t)dt + \sigma_T u_t dZ_{T_t}, \\ dv_t = \kappa_V v_t(\theta_T - v_t)dt + \sigma_V dB_{V_t} \end{cases}$$

with $E\left[W_t Z_t\right] = \rho_T t$, $E\left[W_t B_t\right] = \rho_V t$, $E\left[Z_t B_t\right] = 0$ and J is a Lévy process. Under the specification, we only change q_T as

$$q_T = -\Psi_L(m) = \frac{im+m^2}{2} + im\Psi_J(-\eta i) - \Psi(\eta m).$$

Example 7 (Composite Heston Model)

$$\begin{cases} L(t) = -\frac{1}{2}dt + dW(t), \\ du(t) = \kappa_T(\theta_T - u(t))dt + \sigma_T dZ(T_t), \\ dv(t) = \kappa_V(\theta_V - v(t)dt + \sigma_V dB(V_t). \end{cases} \tag{12}$$

where $E[W_t Z_t] = \rho_T t, E[W_t B_t] = \rho_V t$ *and* $E[Z_t B_t] = 0$.

Likewise, we have

$$\begin{aligned} v_t^{\text{VVIX}} &\approx \frac{d[\ln v^{\text{VIX}}]_t}{dt} \\ &= \frac{(\sigma_V u_{V_t})^2 v_t + (\sigma_T v_t)^2 (u_{V_t} v_t)}{(v_t^{\text{VIX}})^2} \\ &= \frac{\sigma_V^2}{v_t} + \frac{\sigma_T^2}{v_t^{\text{VIX}}} \end{aligned}$$

The composite version of Heston inherits the inverse relationship between VIX and VVIX, but a linear mixture effect is incorporated.

Example 8 *Brought up in Mendoza-Arriaga et al. (2010), the composite time change* $T^{co} = T_V$ *is a pure-jump process.* T *and* V *are independent of* L *as well as of each other.*

$$\mathbb{E}\left[e^{iuX_t}\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(T_{V_t}\Phi_L(u)\right) \mid V_t\right]\right] = \mathbb{E}[\phi_T(-i\Psi_L(u); V_t)].$$

The model exhibits stochastic jump intensity even if L *has no jumps.*

However, the model is theoretically redundant in the sense that L_T *is in fact a Lévy process under the model assumption. Nevertheless, the composite version of Lévy process is convenient in exhibiting flexible moments of distribution, which is usually meaningful in practice.*

Example 9 (Leverage via Reflected Jumps)

$$\begin{cases} L_t = -\Psi(-i)dt + \eta dJ(t), \\ dv_t = \kappa(\theta - v_t)dt - \eta_J dJ^T(T_t), \end{cases}$$

where $J^T = J^-$ *is the negative part of the CGMY process* J.

The model has shown superior performance in Ballotta and Rayée (2022).

Example 10 (Compound Poisson With General Leverage)

$$\begin{cases} L(t) = -\Psi(-i)t + \eta \sum_{i=1}^{N_t} J_i, \\ du(t) = \kappa_T(\theta_T - u(t))dt + \eta_T d\left(\sum_{i=1}^{N_{T_t}} J_i^T\right), \\ dv(t) = \kappa_V(\theta_V - v(t))dt + \sigma_V dB(V_t) \end{cases}$$

where the jump sizes J and J^T are correlated with the joint distribution $F(x, y)$.

Under the leverage neutral measure, the Ch.e. of $Y_t := \sum_{i=1}^{N_t} J_i^T$ becomes

$$\Psi_Y^{\mathbb{Q}}(z) = \lambda(E \exp(izJ_1 + iuJ_1^T) - \phi_J(u)) = \lambda\phi_J(u)(\phi_{J^T}^u(z) - 1),$$

where

$$\phi_{J^T}^u(z) = \int e^{izy} \left(e^{iux - \Psi_J(u)} \mathrm{d}F(x, y)\right).$$

It can be interpreted as a new (complex-valued) compound Poisson process with jump intensity $\lambda\Psi_J(u)$ and jump size with a tilted distribution $\phi_{J^T}^u(z)$.

Example 11 (Composite Jump Heston)

$$\begin{cases} L(t) = -\Psi(-i)t + \eta J(t), \\ du(t) = \kappa_T(\theta_T - u(t))dt - \eta_T dJ(T_t)^-, \\ dv(t) = \kappa_V(\theta_V - v(t))dt + \eta_V dZ(V_t), \end{cases} \tag{13}$$

where $J(t)^-$ is the negative component of the CGMY processes $J(t)$, and Z is a Brownian motion. In this specification, leverage is introduced purely by simultaneous jumps of return and volatility. Ballotta and Rayée (2022) demonstrate a superior performance of a single time change JH model over other classic models, e.g. Heston, BNS.

Example 12 (Multifactor) *It has been shown in empirical study that risks reflected in diffusion and jumps are of different sources. It is therefore natural to consider a TCL of the form*

$$X = B_T + L_V.$$

4 Application In Derivatives Pricing

4.1 CTC-COS Method

Since we've obtained the expression for the Ch.f. of log price process X, the pricing of European options follows directly. For example, readers may consider acceleration methods such as FFT (Carr and Madan (1999)) or COS method (Fang and Oosterlee (2009)), to efficiently compute option prices.

In our CTC model, we show how the Ch.f. and option prices can be efficiently computed with a CTC-COS method as follows.

Theorem 2 (CTC-COS) *Given current time t and expiry date s, the price of a European call option with strike K is numerically approximated by*

$$C(K, \tau) \approx e^{-r\tau} \frac{2}{c} \int_0^c \left({\sum_{k=0}^{N-1}}' \operatorname{Re}\left\{ f(y; \frac{k\pi}{b-a}) A_k \right\} V_k \right) \left({\sum_{l=0}^{M-1}}' \operatorname{Re}\left\{ \phi_V^{\mathbb{Q}}(\frac{l\pi}{c}; s) \right\} \cos(\frac{l\pi y}{c}) \right) dy \tag{14}$$

where a, b, c are integration range and f is given in (9).

Proof According to the cosine expansion method,

$$
\begin{aligned}
C(K,\tau) &\approx e^{-r\tau} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \phi_X\left(\frac{k\pi}{b-a}; s\right) A_k \right\} V_k \\
&= e^{-r\tau} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ E^{\mathbb{Q}}\left[f\left(V_s; \frac{k\pi}{b-a})\right)\right] A_k \right\} V_k \\
&= e^{-r\tau} \sum_{k=0}^{N-1}{}' \sum_{l=0}^{M-1}{}' \operatorname{Re}\left\{ \phi_V^{\mathbb{Q}}(\frac{l\pi}{c}; s) \right\} \operatorname{Re}\{U_{kl} A_k\} V_k \\
&\approx e^{-r\tau} \frac{2}{c} \int_0^c \left(\sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ f(y; \frac{k\pi}{b-a}) A_k \right\} V_k \right) \left(\sum_{l=0}^{M-1}{}' \operatorname{Re}\left\{ \phi_V^{\mathbb{Q}}(\frac{l\pi}{c}; s) \right\} \cos(\frac{l\pi y}{c}) \right) dy
\end{aligned}
$$

$$U_{kl} = \frac{2}{c} \int_0^c f(y; \frac{k\pi}{b-a}) \cos(\frac{l\pi y}{c}) dy,$$

$$A_k = \exp\left\{ -ik\pi \frac{a}{b-a} + \frac{ik\pi(\ln S_t/K + r\tau)}{b-a} \right\} \tag{15}$$

and

$$V_k = \frac{2}{b-a} \int_0^b K(e^y - 1) \cos\left(k\pi \frac{y-a}{b-a} \right) dy. \tag{16}$$

□

What we do in the transformation above is performing double cosine series expansions and then re-ordering the summation and integration. Since the double summations are separate, the overall complexity is $O(ND)$ under affine specifications, where D is the discretization degree of numerical integration. Such computation cost is only slightly higher than those of one-factor models with affine structure, e.g. Heston model.

In practice, when pricing the whole volatility surface, the number of strike prices does not add computational complexity since fourier-based methods like COS allows for a separation of strikes and the underlying asset. Meanwhile, the temporal discretization in the solving process of ODEs in function f and g, enabling the computation on all maturities at once. Therefore, we may price the whole volatility surface as efficently as in the case of a single option.

It's surprising that the CTC models have the same order of complexity as the single TCMs. Table 5.3 shows the speed of pricing the volatility surface (containing 4316 option quotes on Janurary, 9, 2023), where the composite Jump Heston and composition Heston models finish the pricing within 15s and 10s respectively.

4.2 VIX Options

To price VIX derivatives, it's common practice to first derive the relationship between the VIX and spot variance at maturity since the distribution of VIX is not directly available. It's known that VIX^2 is linearly dependent on the spot variance for the Heston model. However, for most other models, such relationship is implicit and requires simulation or a numerical procedure to derive.

Example 13 (Rough Bergomi) *As shown in Jacquier et al. (2018) for rough Bergomi models, the VIX_T^2 is expressed as an integral form and a computationally costly simulation is needed to obtain samples of VIX_T.*

Example 14 (3/2 Model) *The VIX-Spot relationship is also implicit for a 3/2 model. A numerical differentiation is needed and leads to additional computational cost in numerical pricing.*

Next, we will show that both single time change models and CTC models have simple VIX-Spot forms if the time changes have affine activity rates. Non-affine VIX-Spot relationship are also discussed.

4.2.1 Single Time Change Models

Proposition 2 *When a single time change $T = \int_0^\cdot v_s ds$ is considered with an affine process v,*

$$VIX_t^2 = av_t + b$$

with

$$a = 2(\Psi(-i) - \eta EJ(1))\phi(m)$$

$$b = 2(\Psi(-i) - \eta EJ(1))\frac{\kappa\theta}{m}(\phi(m) - 1)$$

and $\phi(x;\tau) = \frac{e^{x\tau}-1}{x\tau}$ and we denote $\phi(x)$ for short if there is no confusion.

Proof

$$\begin{aligned}
\text{VIX}_t^2 &= -\frac{2}{\tau}E_t\left[\ln\frac{e^{-r\tau}S_{t+\tau}}{S_t}\right] \\
&= -\frac{2}{\tau}E_t\left[X_{t+\tau} - X_t\right] \\
&= \frac{2}{\tau}\left(\Psi(-i)E_t\left[T_{t+\tau} - T_t\right] - E_t\left[\eta\left(J(T_{t+\tau}) - J(T_t)\right)\right]\right) \\
&= 2(\Psi(-i) - \eta EJ(1))g(v_t;\tau),
\end{aligned}$$

where $\tau = 30/365$ and $g(v_t;\tau) = \frac{1}{\tau}E_t\left[T_{t+\tau} - T_t\right]$. Since

$$Ev_t = v_0 + \kappa\theta t + m\int_0^t Ev_s ds,$$

where $m = \eta_J E J_1^T - \kappa < 0$. Differential form gives

$$g_t' = \kappa\theta + m g_t,$$

with initial condition $g_0 = 0$. The solution is

$$E[v_t|v_s] = v_s e^{m(t-s)} + \frac{\kappa\theta}{m}(e^{m(t-s)} - 1).$$

Then we have

$$g(v_t;\tau) = v_t\phi(m) + \frac{\kappa\theta}{m}(\phi(m) - 1).$$

□

Since the Ch.f. of v is easily obtained by solving an ODE system, the pricing formula of VIX options is given as:

Proposition 3 (Lian and Zhu (2013))

$$\begin{aligned} C^V(K,\tau) =& \frac{e^{-r\tau}}{2a\sqrt{\pi}} \\ & \times \int_0^\infty \mathrm{Re}\left[e^{\varphi b/a}\phi_v(\varphi;s)\frac{1-\mathrm{erf}(K\sqrt{\varphi/a})}{(\sqrt{\varphi/a})^3}\right]\mathrm{d}\varphi_I \end{aligned}$$

where $\varphi = \varphi_R + \varphi_I i$ is a complex variable with $\phi_R > 0$. $\phi_v(\varphi;s)$ is the characteristic function of v_s and a, b are given in Proposition 2.

4.2.2 CTC Models

For the case of affine CTC models, the VIX-spot relationship is still explicit and easy to obtain.

Proposition 4 *For a CTC model with affine activity rates,*

$$VIX_t^2 = A(v_t)u_{V_t} + Bv_t + C(v_t), \tag{17}$$

where coefficients

$$A(v_t) = M\frac{\phi_V(-im_T;t,t+\tau) - 1}{m_T\tau},$$

$$B = M\frac{\kappa_T\theta_T\phi(m_V)}{m_T}$$

and

$$C(v_t) = M\left(-\frac{\kappa_T\kappa_V\theta_T\theta_V(\phi(m_V)-1)}{m_Tm_V} + \frac{\kappa_T\theta_T}{m_T^2\tau}\left(\phi_V(-im_T;t,t+\tau)-1\right)\right)$$

with a common multiplier

$$M = 2(\Psi(-i) - \eta EJ(1)).$$

Proof

$$\begin{aligned}
\mathrm{VIX}_t^2 &= \frac{2(\Psi(-i)-\eta EJ(1))}{\tau}E_t\left[T_{V_{t+\tau}} - T_{V_t}\right] \\
&= \frac{2(\Psi(-i)-\eta EJ(1))}{\tau}E_t\left[E\left[T_{V_{t+\tau}} - T_{V_t} \mid \mathcal{F}_t, V_{t+\tau}\right]\right] \\
&= \frac{2(\Psi(-i)-\eta EJ(1))}{\tau}E_t\left[u_{V_t}\Delta_V(\tau)\phi(m_T;\Delta_V(\tau)) + \frac{\kappa_T\theta_T}{m_T}\Delta_V(\tau)\left(\phi(m_T;\Delta_V(\tau))-1\right)\right] \\
&= 2(\Psi(-i)-\eta EJ(1))\left\{\frac{(m_Tu_{V_t}+\kappa_T\theta_T)(E_te^{m_T\Delta_V(\tau)}-1)}{(m_T)^2\tau}\right. \\
&\qquad \left. -\frac{\kappa_T\theta_T}{m_T}\left(v_t\phi(m_V) + \frac{\kappa_V\theta_V}{m_V}(\phi(m_V)-1)\right)\right\} \\
&= 2(\Psi(-i)-\eta EJ(1))\left\{\frac{\phi_V(-im_T;t,t+\tau)-1}{m_T\tau}u_{V_t} - \frac{\kappa_T\theta_T\phi(m_V)}{m_T}v_t + \mathrm{Const}\right\},
\end{aligned} \tag{18}$$

where $\Delta_V(\tau) = V_{t+\tau} - V_t$ and constant

$$\mathrm{Const} = -\frac{\kappa_T\kappa_V\theta_T\theta_V(\phi(m_V)-1)}{m_Tm_V} + \frac{\kappa_T\theta_T}{m_T^2\tau}\left(\phi_V(-im_T;t,t+\tau)-1\right).$$

□

For other specification of time changes, there generally does not exist explicit expression for VIX_t. Still, we could recover it from the Laplace transform. That is,

$$\mathrm{VIX}_t = \frac{2(\Psi(-i)-\eta EJ(1))}{\tau}g(v_t, u_{V_t}, \tau), \tag{19}$$

where

$$g(v_t, u_{V_t}, \tau) := -\frac{\partial}{\partial l}E\left[\exp\left\{-l(T_{V_{t+\tau}} - T_{V_t})\right\} \mid u_{V_t}, v_t\right]\Bigg|_{l=0}$$

The Laplace transform can be explicitly computed via

$$
\begin{aligned}
& E\left[\exp\left\{-l(T_{V_{t+\tau}} - T_{V_t})\right\} \mid u_{V_t}, v_t\right] \\
\approx & \sum_{k=0}^{M-1} \mathrm{Re}\{\phi_V(\frac{k\pi}{c};\tau)\} \left(\frac{2}{c}\int_0^c \mathcal{L}^T(l;y)\cos(\frac{k\pi y}{c})dy\right) \\
= & \frac{2}{c}\int_0^c \mathcal{L}^T(l;y) \sum_{k=0}^{M-1} \left(\mathrm{Re}\{\phi_V(\frac{k\pi}{c};\tau)\}\cos(\frac{k\pi y}{c})\right) dy
\end{aligned}
$$

where ϕ_V and $\mathcal{L}^T$ are computed conditional on u_{V_t}, v_t.

In practice, the formula above is computed based on a large amount of simuated v_t and u_{V_t} and the whole time cost is high. But in the special case where an affine v is considered, but not necessarily u, the Ch.f. ϕ_V is exponentially-affine with respect to v_t and, based on such explicit linear dependence, the computation is significantly faster.

4.2.3 Exact Simulation of Spot Variances

Despite its explicit VIX-Spot relationship, a simulation procedure for u_{V_s} and v_s is still needed. While Monte Carlo method is commonly slow in practice, we argue that there exists exact simulation methods for certain models. That is, we only need to simulation the distributions at maturity as opposed to the whole trajectories. As a result, there is no discretization error and the simulation procedure can be quite fast.

We assume that the Ch.f. of u is available and v follows a CIR process, our main concern is to simulate v_T and u_{V_T}. Then the sample of VIX is obtained from formula (17).

Step 1 Simulate v_T from a non-central chi-square distribution.

Step 2 Simulate the conditional distribution $(V_T \mid v_0, v_T)$ by the method of Glasserman and Kim (2011).

By the independence of u and v, we may simulate V_T first and then simulate u_{V_T} with a non-central chi-square distribution. Therefore, the conditional distribution of V_T, in the form of $\left(\int_0^T v_s ds \mid v_0, v_T\right)$, needs to be derived. This is the exactly same problem faced in the Heston simulation, as brought up in Broadie and Kaya (2006). We then apply the method of gamma expansion in Glasserman and Kim (2011), where the conditional distribution is efficiently simulated as a sum of independent variables.

Step 3 Simulate u_{V_T} by inverting the Ch.f. of u at V_T.

The Ch.f. of u is explicit for some models, e.g. Heston and 3/2 models given in Carr and Sun (2007). For composite JH model (13), we can ease the computation by precomputing the Ch.f. of u_t for a range of $t \in [0, \max V_T]$ along the numerical discretization of the corresponding ODE. The sample of u_{V_T} is then obtained by matching the precomputed values with V_T samples.

Step 4 Obtain VIX_T by formula (17) and compute $C^V(K,T)$ by taking the mean of the payoff.

To sum up, we price European options by

Algorithm 1 Calculate Call Option Prices

Input: Maturity T, strike K, discretization parameters N, M, Q, integration range a, b, c.
  for $y = \frac{c}{Q}, \ldots, c$ **do**
    for $k = 0, \cdots, N-1$ **do**
      Compute A_k, V_k according to (15) and (16)
      Compute $f(y; \frac{k\pi}{b-a}$ according to (9)
    end for
    for $l = 0, \cdots, M-1$ **do**
      Compute $\phi_V^{\mathbb{Q}}(\frac{l\pi}{c}; T)$ and $\cos(\frac{l\pi y}{c})$
    end for
    Compute the call price $C(K,T)$ by summing up according to (14)
  end for
Output: $C(K,T)$

Algorithm 2 Calculate VIX Option Prices

Input: Maturity T, strike K.
  Simulate v_T that follows a non-central chi-square distribution
  Simulate the conditional distribution $(V_T \mid v_0, v_T)$ by the method of Glasserman and Kim (2011)
  Given V_T, simulate u_{V_T} by inverting the Ch.f. of u
  Obtain the sample of VIX_T according to (17)
  Compute the call price $C^V(K,T)$ by taking the mean of the payoff function
Output: $C^V(K,T)$

5 Joint Calibration

5.1 Data

The data we consider spans two years, containing the implied volatility surfaces of the S&P 500 and the VIX from January 3, 2023, to February 28, 2023. The implied volatility is calculated using futures prices that are inferred from the put-call parity with ATM options.

Following the procedure of Bardgett et al. (2019), we clean the data by removing options with time to maturity less than 7 days or more than one year. Options quotes with negative bid-ask spreads, zero bid price, zero traded volume / open interest are also removed. For SPX options, we further exclude data with moneyness below 0.5 or above 1.4. We also remove all the ITM options and recover the corresponding futures prices using the ATM put-call parity. For VIX options, we exclude data with moneyness below 0.4 or above 3.3 and all the put options. If a VIX ITM call is illiquid, we use the put-call parity to infer the liquid price of the call from a more liquid VIX OTM put (Pacati et al. (2018)).

The final sample is made of 162,438 SPX options (daily average 4,165) and 10,254 VIX options (daily average 263).

We calibrate on a sample by solving the optimization problem below.

$$\widehat{\Theta} = \arg\min_{\Theta} \frac{1}{N_S} \sum_{i=1}^{N_S} \left(\sigma_S^i(\Theta) - \widehat{\sigma}_S^i\right)^2 + \frac{1}{N_V} \sum_{i=1}^{N_V} \left(\sigma_V^i(\Theta) - \widehat{\sigma}_V^i\right)^2,$$

where N_S and N_V are the number of SPX and VIX options in the sample.

5.2 Calibration Procedure

Calibrate the model based on the data on some specific dates. For example, we may choose the dates with remarkably high or low VIX / VVIX to test the robustness of the model in different market scenarios.

Since the calibration of VIX market is more challenging, we typically calibrate to VIX option data first, and then take the initial set of parameters in joint calibration based on the result in VIX calibration.

5.3 Calibration Results

The next table is the calibration result of SPX market. The RMSE in the table below is the root mean squared error of implied volatilities.

Models	Time Elapse (s)	RMSE
Heston	7.81	0.01722
Composite Heston	14.129	0.01272
JH	9.72	0.01052
Composite JH	47.808	0.00593
2 Factor	10.229	0.00862
3/2	10.859	0.02288
Composite 3/2	97.781	0.01438
3/2 + Jump	12.641	0.00791
Composite 3/2 + Jump	121.916	0.00599

Table 1: Calibration Results. Time elapse is the time needed for pricing the whole volatility surface (2600 option quotes)

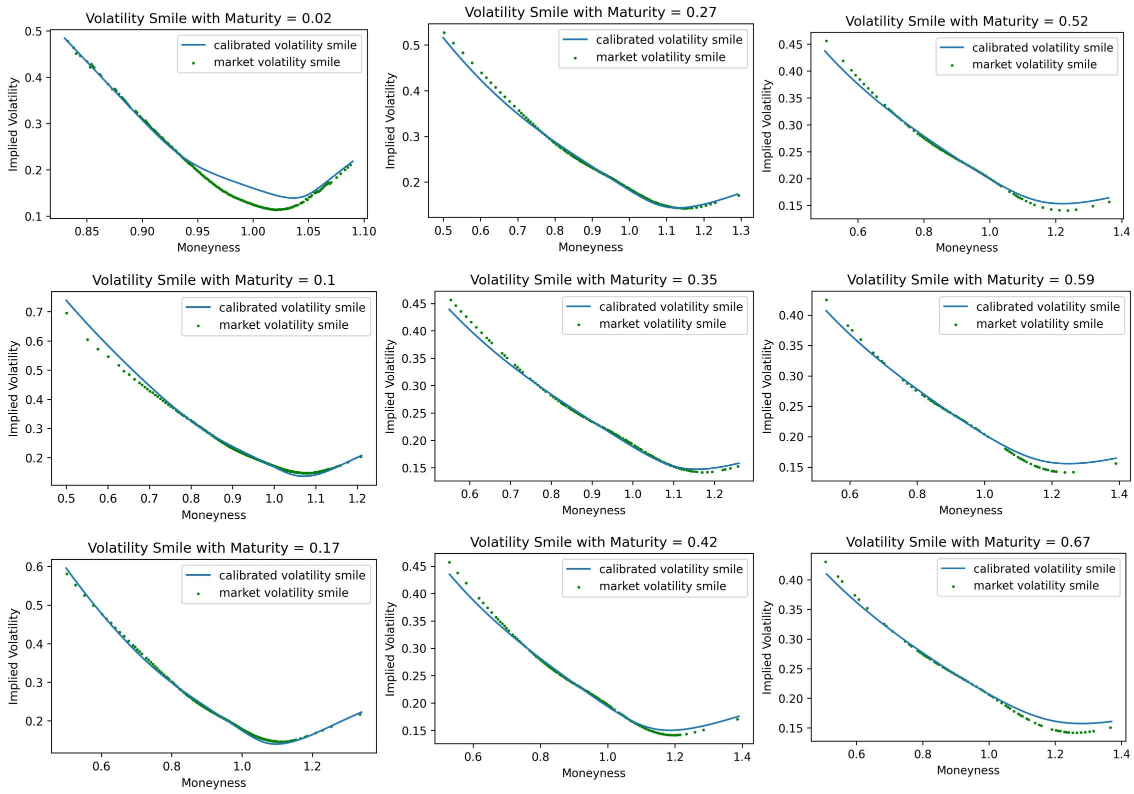

Figure 2: Calibration of 2 Factor Model

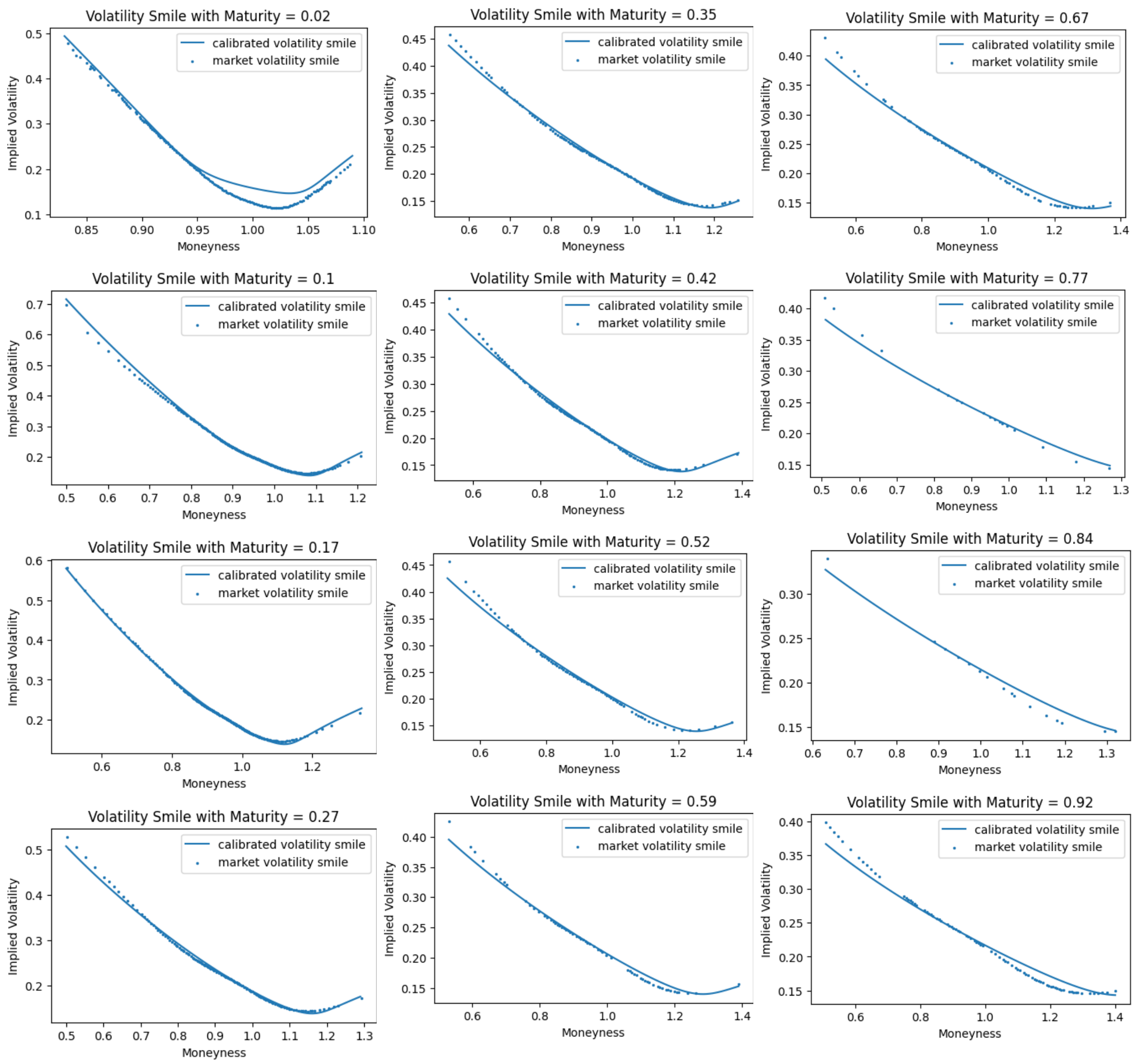

Figure 3: Calibration of 3/2 + Jump Model

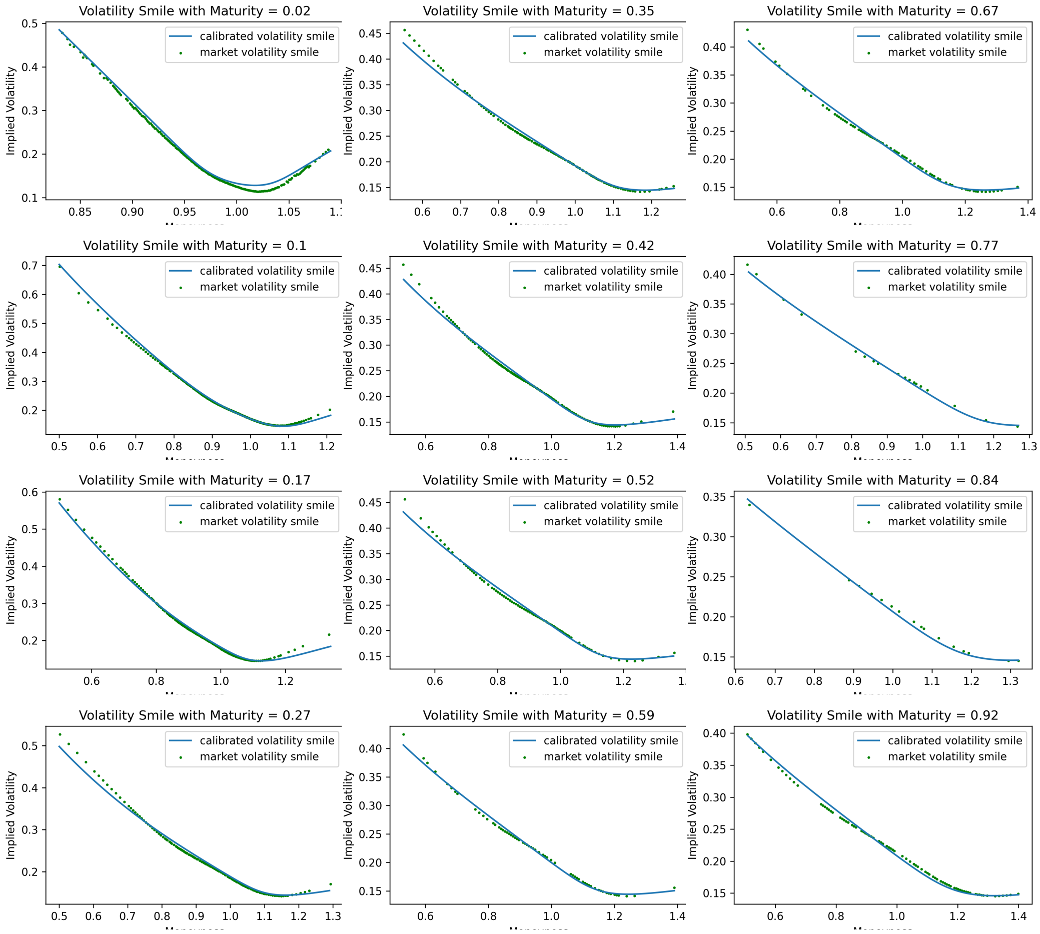

Figure 4: Calibration of Composite Jump Heston (Affine CGMY) Model

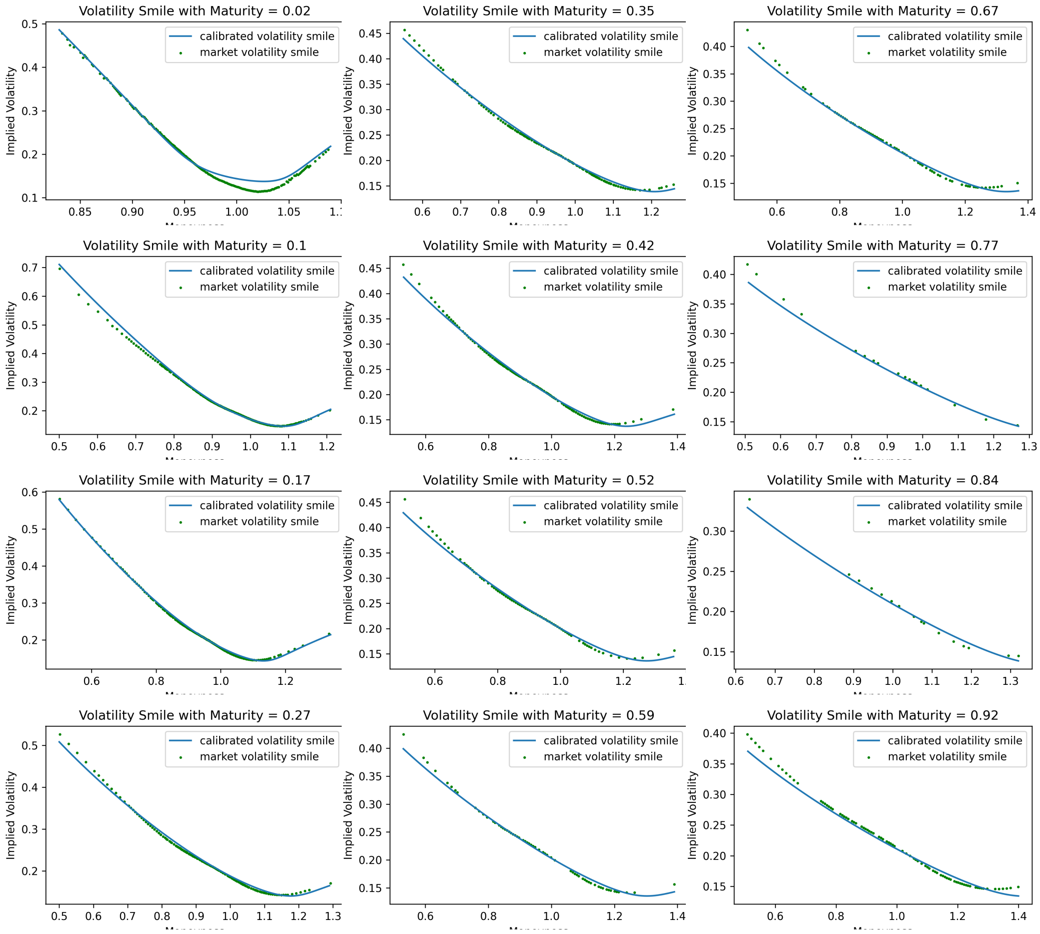

Figure 5: Calibration of Composite 3/2 + Jump Model

5.4 Joint Calibration

First, we test the performance of joint calibration on date 2023.01.13 (with VIX = 18.35) and date 2020.04.24 (with VIX = 35.93). We specifically consider the JH model, the composite JH model and the 2-factor JH model.

Models	N. Parameter	RMSE	MAE
JH	9	0.04663 (0.03471 + 0.05855)	0.03784
Composite JH	13	0.04113 (0.03391 + 0.04834)	0.03099
2 Factor TCM	13	0.03693 (0.01693 + 0.05693)	0.02822

Table 2: Results of Joint Calibration based on RMSE. Date 2023.1.13.

Models	N. Parameter	RMSE	MAE	MSRE
JH	9	0.0440 (0.0431 + 0.0448)	0.0352	0.0086
Composite JH	13	0.0375 (0.0344 + 0.0406)	0.0286	0.0036
2 Factor TCM	13	0.0419 (0.0384 + 0.0454)	0.0334	0.0085

Table 3: Results of joint calibration on date 2020.04.24.

To make comparison with other benchmark models, we compare the performances with the results in Kokholm and Stisen (2015) on 2008.10.22 and 2012.05.16.

Models	N. parameters	ARPE (%)	ARBAE (%)	Loss
SV*	5	12.0	5.6	-
SVJ*	8	9.25	2.9	-
SVJJ*	10	9.15	2.75	-
JH	9	8.93	2.52	0.01354
2 Factor TCM	13	8.95	2.40	0.01348
Composite JH	13	6.78	0.86	0.01026

Table 4: Results of joint calibration on 2008.10.22. VIX value **69.65**. Numbers of options: 440 + 78 + 5 (Original), 463 + 82 (Composite JH).

Models	N. parameters	ARPE (%)	ARBAE (%)	Loss
SV*	5	13.75	6.45	-
SVJ*	8	11.7	5.0	-
SVJJ*	10	10.4	3.7	-
JH	9	10.62	4.21	0.02384
2 Factor TCM	13	11.21	4.45	0.02440
Composite JH	13	9.56	3.67	0.02672

Table 5: Results of joint calibration on 2012.5.16. VIX value **22.27**. Calibration is based on optimizing RMSRE of option prices. Option numbers: 477 + 138 + 5 (Original), 622 + 146 (Composite JH)

It's shown in the results that, compared with 2-factor JH, composite JH models generally performs better. This demonstrates the effectiveness of time composition v.s. time combination. In particular, when the market volatility is large, composite JH models become even better.

6 Conclusion

Specifically, the contributions of our work are three-folded.

Firstly, we develop a generalized form of composite time-changed Lévy models. These models have the advantage of exhibiting various variance, skewness and kurtosis. Moreover, as we show in detail in section 4, CTC models typically have good tractability. It only requires a complexity of $O(N)$ if the affine structure of time changes is imposed.

Secondly, we theoretically demonstrate the effectiveness of our model in consistent modeling. Unlike historical works of consistent modeling, the composite time change models provide explicit interpretation in its decoupling mechanism of volatility and volatility of volatility.

And finally, we validates the superiority of our proposed model in the consistent modeling problem. We develop its option pricing theory and test its performance in the joint calibration problem of SPX and VIX option market. As shown in section 5, the composite time change models successfully calibrate the joint smiles in real market.

A Proof of Theorem 1

Lemma 1 *Consider a filtered space* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})$ *with probability measure* $\mathbb{P}$ *and a complex-valued measure* $\mathbb{Q}$*. Assume that* $\mathbb{Q}$ *is locally dominated by* $\mathbb{P}$ *with Radon-Nikodym derivative* M*, i.e.,*

$$M_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \quad t \geq 0$$

Then for any finite stopping time T*, we have* $\mathbb{Q}_T \ll \mathbb{P}_T$ *and*

$$M_T = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T}$$

The lemma is an extension of Jacod and Shiryaev (2013) (Theorem III.3.4.(ii)).

Proof Since $\mathcal{F}_{T\wedge n} \subseteq \mathcal{F}_n$, we see that $\mathbb{Q}_{T\wedge n} \ll \mathbb{P}_{T\wedge n}$, and as $\tau \wedge n$ is a bounded stopping time, it follows by the optional stopping theorem that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_{T\wedge n}} = E^{\mathbb{P}}[M_n \mid \mathcal{F}_{T\wedge n}] = M_{T\wedge n}.$$

To prove the theorem, it is enough to show $\mathbb{Q}(A) = \int_A M_T d\mathbb{P}$ for $A \in \mathcal{F}_T$. Then choose $A \in \mathcal{F}_T$, we have

$$\begin{aligned} E^{\mathbb{P}}\left(1_A M_T\right) &= \sum_{n\geq 1} E^{\mathbb{P}}\left(1_A 1_{\{n-1\leq T<n\}} E^{\mathbb{P}}[M_n \mid \mathcal{F}_T]\right) = \sum_{n\geq 1} E^{\mathbb{P}}\left(1_A 1_{\{n-1\leq T<n\}} M_n\right) \\ &= \sum_{n\geq 1} \mathbb{Q}(A \cap \{n-1 \leq T < n\}) = \mathbb{Q}(A). \end{aligned}$$

And the result follows. $\square$

Now we begin our proof of Theorem 1.

Proof When T is an independent time change, use iterated conditioning:

$$\mathrm{E}\left[\mathrm{e}^{\mathrm{i}uX_t}\right] = \mathrm{E}[\mathrm{E}\left[\mathrm{e}^{\mathrm{i}uL_{T_t}} \mid T_t\right]] = \mathrm{E}[e^{T_t\Psi_L(u)}] = \phi_T(-i\Psi_L(u); t).$$

When X is not independent of T. First we show that M is a $(\mathbb{P}, \mathcal{F}_T)$-martingale. Denote

$$N_t = \exp\left(iuL_t - t\Psi_L(u)\right).$$

which is a complex-valued martingale. As a result of lemma 1, $M = N_T$ is a $(\mathbb{P}, \mathcal{F}_T)$-martingale. And the Ch.f. of X_t follows from direct computation.

Since L is a Lévy process under filtration $\mathcal{F}$, we have that

$$\begin{aligned}
\phi_L^{\mathbb{Q}}(z; t) &= E\exp\left(izL_t\right)N_t \\
&= E\exp\left(i(u+z)L_t\right)/e^{t\Psi_L(u)} \\
&= \exp\left(t(\Psi_L(u+z) - \Psi_L(u)\right).
\end{aligned}$$

Thus, we have $\Psi_L^{\mathbb{Q}}(z) = \Psi_L(u+z) - \Psi_L(u)$. $\square$

B COS Method for VIX Options

According to COS method in Fang and Oosterlee (2009), the price at time t_0 of a European style option is

$$v\left(x, t_0\right) \approx e^{-r\Delta t}\sum_{k=0}^{N-1} \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}; x\right)e^{-ik\pi\frac{a}{b-a}}\right\}V_k,$$

where a, b are integration range and

$$V_k := \frac{2}{b-a}\int_a^b v(y, T)\cos\left(k\pi\frac{y-a}{b-a}\right)dy,$$

where $v(y, T)$ is the payoff function w.r.t. the underlying asset with value y. The analytically formula of V_k for call options is known if the log-asset price has analytical Ch.f.

$$V_k^{\text{call}} = \frac{2}{b-a}\int_0^b K(e^y - 1)\cos(k\pi\frac{y-a}{b-a})dy.$$

Since $S_t = S_0 e^{rt + L_{V_t} - \Psi_L(-i)V_t}$,

$$
\begin{aligned}
VIX_t &= -\frac{2}{\tau} E[\ln \frac{e^{-r\tau} S_{t+\tau}}{S_t} \mid \mathcal{F}_t] \\
&= \frac{2}{\tau} \left(\Psi_L(-i) E[V_{t+\tau} - V_t \mid \mathcal{F}_t] - E[L_{V_{t+\tau}} - L_{V_t} \mid \mathcal{F}_t] \right) \\
&= \frac{2}{\tau} E[V_{t+\tau} - V_t \mid \mathcal{F}_t](\Psi_L(-i) - EL_1) \\
&= \frac{2}{\tau} (\Psi_L(-i) - EL_1) f(v_t).
\end{aligned}
$$

In affine models, we can show that $VIX_t = mv_t + n$ for some constants m and m.

Let $a = \max(0, \frac{K^2 - n}{m})$, for VIX options, we have the analytical expression for the Ch.f. of VIX^2, then

$$
V_k^{\mathrm{VIX}} = \frac{2}{b} \int_a^b (\sqrt{my+n} - K) \cos(\frac{k\pi y}{b}) dy = \frac{2}{b} \int_a^b \sqrt{my+n} \cos(\frac{k\pi y}{b}) dy - \frac{2K}{b} \psi_k(a, b),
$$

the integral can be done numerically. Then the pricing formula becomes

$$
C^{\mathrm{VIX}}(v_{t_0}, t_0) \approx e^{-r\Delta t} \sum_{k=0}^{N-1} \mathrm{Re} \left\{ \phi_v \left(\frac{k\pi}{b}; v_{t_0} \right) \right\} V_k^{\mathrm{VIX}}.
$$

C Calibration Details

Spot index prices like SPX and VIX are not used in the calibration because they are not directly traded in the market, see also Lian and Zhu (2012). They are recovered from the market according to the put-call parity as discounted futures price. Meanwhile, such implied spot prices contain the term structure of future dividend expectations.

- SPX options with AM settlement are specially treated with 1 day less maturity
- The risk-free rate is quoted from daily U.S. treasury bond rates with Spline interpolation
- Implied volatility is a function of moneyness $k_t = \frac{K}{F_t}$ (and no additional rates) because

$$
\begin{aligned}
e^{-r\tau} E_t[S_T - K]_+ = C_t(K, T) &\equiv C^{BS}(k_t, K, \tau, IV_t) \\
&= K e^{-r\tau} \left[e^{k_t} \Phi(d_1) - \Phi(d_2) \right]
\end{aligned}
$$

yields

$$e^{k_t}\Phi\left(d_1\left(k_t, IV_t, \tau\right)\right) - \Phi\left(d_2\left(k_t, IV_t, \tau\right)\right) = E_t\left[e^{k_T} - 1\right]_+.$$

To price and compute IV, it's enough to obtain the moneyness. By put-call parity,

$$C - P = e^{-r\tau}(F - K),$$

if r is known, then

$$k_t = \frac{K}{K + (C_t - P_t)e^{r\tau}}.$$

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