

BOUNDS ON THE DIMENSION OF LINEAL EXTENSIONS

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ABSTRACT. Let $E \subseteq \mathbb{R}^n$ be a union of line segments and $F \subseteq \mathbb{R}^n$ the set obtained from E by extending each line segment in E to a full line. Keleti's *line segment extension conjecture* posits that the Hausdorff dimension of F should equal that of E . Working in \mathbb{R}^2 , we use effective methods to prove a strong packing dimension variant of this conjecture, from which the generalized Kakeya conjecture for packing dimension immediately follows. This is followed by several doubling estimates in higher dimensions and connections to related problems.

1. INTRODUCTION AND MAIN RESULTS

Let $E = \bigcup \mathcal{I}$, where \mathcal{I} is a family of line segments in \mathbb{R}^n , $n \geq 2$. Throughout, \mathcal{I} is assumed to be *maximal* in the sense that, if I is a line segment and $I \subseteq \bigcup \mathcal{I}$, then $I \in \mathcal{I}$ —a hypothesis that results in no loss of generality in what follows. Denoting by $\mathcal{A}(n, 1)$ the affine Grassmannian of lines in \mathbb{R}^n , we define the **lineal extension of E** to be the set $\mathbf{L}(E)$ formed from E by extending each $I \in \mathcal{I}$ to the unique line $\ell_I \subset \mathbb{R}^n$ containing I :

$$\mathbf{L}(E) := \bigcup_{I \in \mathcal{I}} \ell_I = \bigcup \{ \ell \in \mathcal{A}(n, 1) : E \cap \ell \text{ contains a line segment} \}.$$

With this setup, Keleti [10] proposed the following conjecture. Let $\dim_{\mathbf{H}}$ denote Hausdorff dimension.

Conjecture 1 (Line segment extension conjecture). *Let $E \subseteq \mathbb{R}^n$ be a union of line segments and $\mathbf{L}(E)$ its lineal extension. Then $\dim_{\mathbf{H}} \mathbf{L}(E) = \dim_{\mathbf{H}} E$.*

Conjecture 1 is open in dimensions $n \geq 3$ but is known for $n = 2$.

Theorem 1.1 (Keleti [10]). *If $E \subseteq \mathbb{R}^2$ is a union of line segments, then $\dim_{\mathbf{H}} \mathbf{L}(E) = \dim_{\mathbf{H}} E$.*

This article concerns variants and extensions of this problem, emphasizing the planar case.

1.1. Line segment extension in \mathbb{R}^2 . We begin by introducing some notation. Let $\dim_{\mathbf{P}}$ denote packing dimension.

2020 *Mathematics Subject Classification.* Primary: 03D32, 28A80; Secondary: 68Q30.

Key words and phrases. Algorithmic complexity, Kakeya conjecture.

The second author was supported in part by NSF DMS-2037851 and NSF DMS-2246906.

Definition 1.1. For $s \in (0, 1]$ and $E \subseteq \mathbb{R}^n$ a union of subsets of lines with (packing or Hausdorff) dimension at least s , let

$$L_s^H(E) := \bigcup \{ \ell \in \mathcal{A}(n, 1) : \dim_H(\ell \cap E) \geq s \}$$

and

$$L_s^P(E) := \bigcup \{ \ell \in \mathcal{A}(n, 1) : \dim_P(\ell \cap E) \geq s \}$$

be, respectively, the *s-Hausdorff extension* and *s-packing extension of E*.¹ In the extreme case $s = 0$ we let

$$L_0(E) := \bigcup \{ \ell \in \mathcal{A}(n, 1) : \#(\ell \cap E) \geq 2 \},$$

which we call the *two-point extension of E*.

Our first result—a generalization of Theorem 1.1—pertains to the lineal extension of Furstenberg sets.

Proposition 1.2. *Let $s \in [0, 1]$ and let $E \subseteq \mathbb{R}^2$ be a union of (at least) s -Hausdorff-dimensional subsets of lines. Then $\dim_H(L_s^H(E)) \leq \dim_H E + 2 - 2s$. In particular, if E is a union of line segments and $\mathbf{L}(E)$ is its lineal extension, then $\dim_H \mathbf{L}(E) = \dim_H E$.*

Keleti’s proof of Theorem 1.1 combines Marstrand’s slicing theorem and the “Fubini inequality” for Hausdorff measures in a simple and elegant argument, whereas we prove Proposition 1.2 by effective methods that, in particular, hinges on a result [15] of N. Lutz and Stull (Theorem 3.3 below). We remark here that this is in fact implied by the Furstenberg set bound [16] of Molter and Rela; cf. §2 below.

One motivation for Proposition 1.2 is that the proof is morally similar to (but much simpler than) that of our main result.

Theorem 1.3. *If $E \subseteq \mathbb{R}^2$ is a union of 1-Hausdorff-dimensional subsets of lines, then $\dim_P(L_1^H(E)) = \dim_P E$. In particular, if E is a union of line segments and $\mathbf{L}(E)$ is its lineal extension, then $\dim_P \mathbf{L}(E) = \dim_P E$.*

This theorem follows from a different effective analogue that cannot be proved directly from the aforementioned Lutz–Stull result. The bulk of the proof in §4 is establishing this analogue, which involves a sort of multiscale application of the ideas underlying [15].

A strong “generalized Kakeya conjecture” for packing dimension (see §2.1) follows readily from the final step of the proof of Theorem 1.3. Let $\mathbb{P}^{n-1} := \mathbb{S}^{n-1}/\{\pm 1\}$ be the set of directions of lines in \mathbb{R}^n .

¹Alternatively, in the above definitions, we could drop the requirement that E be the union of certain subsets of lines and instead write $L_s^H(E) := \bigcup \{ \ell \in \mathcal{A}(n, 1) : \dim_H(E \cap \ell) \geq s \} \cup E$, and likewise for packing dimension. One could partition E into (1) the set of points covered by lines intersecting E in sets of Hausdorff dimension at least s and (2) the remainder of E ; calling the first of these E_{lines} , it is easy to see that $L_s^H(E) = L_s^H(E_{\text{lines}}) \cup E$. The same holds for packing dimension. Morally speaking, then, the actual definitions we use encompasses all of the interesting features of the problem.

Corollary 1.4. *Let $E \subseteq \mathbb{R}^2$ and let $D \subseteq \mathbb{P}^1$ be the set of directions of lines intersecting E in a set of Hausdorff dimension 1. If $D \neq \emptyset$, then*

$$\dim_{\mathbb{P}} D + 1 \leq \dim_{\mathbb{P}} E. \quad (1)$$

1.2. Line segment extension in \mathbb{R}^n and elementary Besicovitch set estimates. Partial results in higher dimensions, including some (non-)doubling bounds on the dimension of lineal extensions, follow from more rudimentary “two-part code” arguments in the spirit of [1] Theorem 1.2.

Proposition 1.5. *If $E \subseteq \mathbb{R}^n$ is a union of line segments, then*

$$\dim_{\mathbb{H}} \mathbf{L}(E) \leq \dim_{\mathbb{H}} E + \dim_{\mathbb{P}} E - 1 \quad \text{and} \quad \dim_{\mathbb{P}} \mathbf{L}(E) \leq 2 \dim_{\mathbb{P}} E - 1. \quad (2)$$

Such results are connected to the Kakeya conjecture via the following theorem. Call a subset of \mathbb{R}^n (not necessarily Borel) a **Besicovitch set** if it contains a unit line segment in every direction.

Theorem 1.6 (Keleti [10]).

- a. *If the line segment extension conjecture holds in \mathbb{R}^n for some $n \geq 2$, then every Besicovitch set in \mathbb{R}^n has Hausdorff dimension at least $n - 1$.*
- b. *If the line segment extension conjecture holds in \mathbb{R}^n for all $n \geq 2$, then, for every $n \geq 2$, every Besicovitch set in \mathbb{R}^n has packing dimension n .*

While Theorem 1.6 assumes the full strength of Conjecture 1, with a small modification of the final step in Keleti’s proof—to which the reader is referred—one obtains the following generalization.

Lemma 1.7. *Suppose there is a function $g: [0, n]^2 \rightarrow [0, n]$ such that the following holds:*

If E is a union of line segments in \mathbb{R}^n , then $g(\dim_{\mathbb{H}} E, \dim_{\mathbb{P}} E) \geq \dim_{\mathbb{H}} \mathbf{L}(E)$.

Then $g(\dim_{\mathbb{H}} K, \dim_{\mathbb{P}} K) \geq n - 1$ for every Besicovitch set $K \subseteq \mathbb{R}^n$.

As an immediate consequence of this lemma and Proposition 1.5, we obtain an elementary estimate on the dimension of Besicovitch sets in \mathbb{R}^n . (See also §2 for implications for the generalized Kakeya conjecture.)

Corollary 1.8. *If $K \subseteq \mathbb{R}^n$ is a Besicovitch set, then*

$$\dim_{\mathbb{H}} K + \dim_{\mathbb{P}} K \geq n.$$

This is of course far from state-of-the-art, but the implication of Kakeya inequalities from line segment extension inequalities has something of a practical importance that we describe below.

2. BACKGROUND ON THE LINE SEGMENT EXTENSION CONJECTURE AND ITS RELATIVES

2.1. History and context. Keleti [10] introduced the line segment extension problem as a natural follow-up to the constructions of Nikodym [17] and Larman [12] showing that a union of closed line segments in \mathbb{R}^n can have positive Lebesgue measure even when the union of the corresponding open line segments is Lebesgue-null.

In this same vein, Falconer and Mattila [5] introduced a “hyperplane extension problem,” which they treated as a slicing problem with a dual projection problem that is amenable to a Marstrand-type exceptional set estimate. The planar case of their Theorem 3.2 weakens the hypothesis in Theorem 1.1 that E contains many line segments to the hypothesis that it contains many positive-measure subsets of lines, which Proposition 1.2 further weakens to s -dimensional subsets of lines (possibly of \mathcal{H}^s -measure 0). Another consequence of [5] is an equation for the dimension of a family of hyperplanes in terms of that of its union. Héra, Keleti, and Máthé [8] pursued this direction in arbitrary dimension and codimension, bounding the dimension of families $\Lambda \subseteq \mathcal{A}(n, k)$ from above in terms of the dimension of any set giving large slices to $\bigcup \Lambda$.

Redirecting attention back to the connection between the line segment extension problem and the Kakeya problem, Keleti and Máthé [11] showed that Theorem 1.6 has a strong converse.

Theorem 2.1 (Keleti–Máthé [11]). *If the Kakeya conjecture is true in \mathbb{R}^n , then the line segment extension conjecture is true in \mathbb{R}^n .*

This they established as a corollary to the equivalence of the Kakeya conjecture with the generalized Kakeya conjecture.

Conjecture 2 (Generalized Kakeya conjecture). *Let $E \subseteq \mathbb{R}^n$ and let $D \subseteq \mathbb{P}^{n-1}$ be the set of directions in which E contains a line segment. If $D \neq \emptyset$, then*

$$\dim_{\mathrm{H}} D + 1 \leq \dim_{\mathrm{H}} E.$$

In particular, this conjecture is true in \mathbb{R}^2 but open in all higher dimensions. Unlike Conjecture 2, the generalized Kakeya conjecture for packing dimension does not seem to readily imply the packing dimension analogue of Conjecture 1. On the Hausdorff side this implication follows from Marstrand’s slicing theorem, of which the packing dimension analogue is considerably weaker (cf. [4]). In fact, without additional hypotheses, replacing Hausdorff dimension with packing dimension in the proof of Conjecture 1 from Conjecture 2 only gives the trivial lower bound $\dim_{\mathrm{P}} E \geq 1$. It is furthermore not obvious to us that the argument used to establish the equivalence between Kakeya and generalized Kakeya in [11] easily adapts to packing dimension, and for these reasons it seems surprising that both Theorem 1.3 and Corollary 1.4 fall out of a single proof.

On a different note, it should also be remarked here that the implication in Theorem 2.1 is *not* quantitative, in the sense that absolute lower bounds on the size of Besicovitch sets

(or of unions of line segments more generally) do not translate into progress toward the line segment extension conjecture. This stands in contrast to Lemma 1.7, according to which partial results on the line segment conjecture *do* make headway on the Kakeya problem. In fact, the equivalence between the Kakeya and generalized Kakeya conjectures proved in [11] is also quantitative:

Theorem 2.2 (Keleti–Máthé [11]). *Let $E \subseteq \mathbb{R}^n$ and let $\emptyset \neq D \subseteq \mathbb{P}^{n-1}$ be the set of directions in which E contains a line segment. Then there exists a compact Besicovitch set $K \subset \mathbb{R}^n$ with*

$$\dim_{\mathrm{H}} K \leq n - 1 + \dim_{\mathrm{H}} E - \dim_{\mathrm{H}} D.$$

Our method gives such inequalities of generalized Kakeya type for packing and mixed Hausdorff-packing dimensions directly, without reference to a general result analogous to Theorem 2.2; see the remarks following the proof of Proposition 1.5 in §5.

2.2. Proposition 1.2 and the Furstenberg set conjecture. We take a moment to expound on the relationship between Proposition 1.2 and the Furstenberg set conjecture. Call a set $E \subseteq \mathbb{R}^2$ an **(s, t) -Furstenberg set** if there exists a t -Hausdorff-dimensional set $\Lambda \subseteq \mathcal{A}(2, 1)$ such that

$$E = \bigcup_{\ell \in \Lambda} (E \cap \ell), \quad \text{where} \quad \dim_{\mathrm{H}}(E \cap \ell) \geq s \quad \forall \ell \in \Lambda.$$

A special case of the aforementioned [5] Theorem 3.2 gives $\dim_{\mathrm{H}} L_s^{\mathrm{H}}(E) = t$, which in conjunction with Proposition 1.2 entails

$$t \leq \dim_{\mathrm{H}} E + 2 - 2s, \quad \text{i.e.,} \quad 2s + t - 2 \leq \dim_{\mathrm{H}} E.$$

When E is Borel, this is essentially a consequence of the Furstenberg set bound of Molter and Rela [16], and running the argument in reverse in turn yields Proposition 1.2 for Borel sets from the Furstenberg set estimate.²

More recently, Ren and Wang fully resolved the Furstenberg set conjecture in the plane.

Theorem 2.3 (Ren–Wang [21]). *If $E \subseteq \mathbb{R}^2$ is a Borel (s, t) -Furstenberg set, then*

$$\dim_{\mathrm{H}} E \geq \min \left\{ s + t, \frac{3s + t}{2}, s + 1 \right\}.$$

A corollary, then, is a sharpening of Proposition 1.2 for Borel sets.

Corollary 2.4. *Let $s \in [0, 1]$ and let $E \subseteq \mathbb{R}^2$ be a union of (at least) s -dimensional subsets of lines. If E is Borel, then*

$$\dim_{\mathrm{H}}(L_s^{\mathrm{H}}(E)) \leq \max \{ \dim_{\mathrm{H}} E + 1 - s, 2 \dim_{\mathrm{H}} E + 1 - 3s, 2 - s \}.$$

²The authors thank Tamás Keleti and Josh Zahl for sharing this observation.

The maximum here is attained according to the numerical relationship between $\dim_{\mathbb{H}}(L_s^{\mathbb{H}}(E))$ and s , and in fact the final step in the proof of Proposition 1.2 shows that

$$\dim_{\mathbb{H}}(L_s^{\mathbb{H}}(E)) \leq \dim_{\mathbb{H}} E + 1 - s \quad \text{when} \quad s \geq \dim_{\mathbb{H}}(L_s^{\mathbb{H}}(E)),$$

recovering the corresponding case of Corollary 2.4.

Regardless of its comparative strength in view of other Furstenberg set bounds, one reason for including Proposition 1.2 is that the setup for its proof mirrors that of Theorem 1.3.

3. PRELIMINARIES ON EFFECTIVE METHODS

3.1. Basic definitions. The main goal of this section is to collect in one place several tools which we use repeatedly in the remainder of the paper, especially for the benefit of readers less familiar with Kolmogorov complexity. We operate in the algorithmic framework laid out in [14], which we briefly recall here to establish terminology and notation. Let $\{0, 1\}^*$ be the collection of all finite strings over $\{0, 1\}$, including the empty string \emptyset . Fixing some prefix-free universal oracle Turing machine U , we define for each pair $\sigma, \tau \in \{0, 1\}^*$ the **Kolmogorov complexity of σ given τ** to be the minimal length of a program that, when given to U as an input with side information τ , returns σ as the output:

$$K(\sigma | \tau) := \min \{ |\pi| \in \mathbb{N} : \pi \in \{0, 1\}^*, U(\pi, \tau) = \sigma \}.$$

When $\tau = \emptyset$, we write $K(\sigma) := K(\sigma | \emptyset)$ and simply call this quantity the **Kolmogorov complexity of σ** .

Identifying the family of all rational vectors with $\{0, 1\}^*$ via some *effective encoding* $\bigcup_{n \in \mathbb{N}} \mathbb{Q}^n \hookrightarrow \{0, 1\}^*$, we may extend these definitions from strings to real vectors as follows. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, s \in \mathbb{N}$. We call

$$K_r(x) := \min_{p \in B(x, 2^{-r}) \cap \mathbb{Q}^n} K(p)$$

the **Kolmogorov complexity of x to precision r** and

$$K_{r,s}(x|y) := \max_{q \in B(y, 2^{-s}) \cap \mathbb{Q}^m} \left(\min_{p \in B(x, 2^{-r}) \cap \mathbb{Q}^n} K(p|q) \right)$$

the **Kolmogorov complexity of x to precision r given y to precision s** . When $s = r$ we simply write $K_r(x|y) := K_{r,r}(x|y)$, and when $y = x$ we write $K_{r,s}(x|x) := K_{r,s}(x)$.

The “universality” of U refers to the fact that, for every prefix-free oracle Turing machine M , there exists a program $\pi_M \in \{0, 1\}^*$ such that

$$U(\pi_M, \sigma) = M(\sigma) \quad \forall \sigma \in \{0, 1\}^*.$$

The length of the shortest such π_M is called the *machine constant of M* . When it is more awkward to work with $U(\pi_M, \cdot)$ than it is to work with M directly, we opt for the latter and then add the machine constant to the length of the shortest σ such that $M(\sigma) = \tau$ when computing the Kolmogorov complexity of τ .

By allowing a machine access to an *oracle* $A \subseteq \{0, 1\}^*$, we can *relativize* the above definitions to A , in which case we embellish the symbols U , K , and M with a superscript A . An oracle

represents extra information that an oracle Turing machine is allowed to use in computations. Access to an oracle can never make a computation meaningfully harder, as a machine can always “ignore” the oracle if its information is irrelevant. In particular, if A and B are oracles, then

$$K_r^{A,B}(x) \leq K_r^A(x) + O(\log r)$$

for all $x \in \mathbb{R}^n$.

Using some standard encoding, we can consider points in \mathbb{R}^n as oracles. Intuitively, conditional access to a point up to a certain precision should be no more useful than oracle access to *all* of the information in that point, and this is made precise by the inequality

$$K_r^{A,x}(y) \leq K_r^A(y|x) + O(\log r).$$

3.2. Some useful results. One key property of Kolmogorov complexity is *symmetry of information*. The following quantitative form will see repeated use in this paper.

Lemma 3.1 (Symmetry of information [15]). *For all $A \subseteq \{0, 1\}^*$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, s \in \mathbb{N}$ with $r \geq s$:*

- a. $|K_r^A(x|y) + K_r^A(y) - K_r^A(x, y)| \leq O(\log r) + O(\log \log \|y\|).$
- b. $|K_{r,s}^A(x) + K_s^A(x) - K_r^A(x)| \leq O(\log r) + O(\log \log \|x\|).$

In practice, the norms of the points we work with are fixed and independent of the level of precision, so we frequently use these facts in the (relativized) forms

$$K_r^A(x, y) \approx K_r^A(x|y) + K_r^A(y) \quad \text{and} \quad K_r^A(y) \approx K_{r,s}^A(y) + K_s^A(y),$$

where both equalities hold up to a logarithmic term in r . The latter of these is particularly useful as a tool to bound the complexity of y at a given precision: its repeated use allows us to *partition* the interval $[1, r]$ into smaller intervals on which the complexity function of y may have more desirable properties.

We add to this another result for understanding how complexity varies with precision. Case and J. Lutz [2] showed that, for any $A \subseteq \{0, 1\}^*$, $r, s \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

$$K_r^A(x) \leq K_{r+s}^A(x) \leq K_r^A(x) + ns + O(\log(s + r)).$$

This bound captures two essential features of the Kolmogorov complexity of points: it is non-decreasing, and its growth rate is essentially bounded by n on sufficiently long intervals.

Ultimately, we study the Kolmogorov complexity of points in $x \in \mathbb{R}^n$ to bound their asymptotic information density. Given an oracle $A \subseteq \{0, 1\}^*$, we call

$$\dim^A(x) := \liminf_{r \rightarrow \infty} \frac{K_r^A(x)}{r} \quad \text{and} \quad \text{Dim}^A(x) := \limsup_{r \rightarrow \infty} \frac{K_r^A(x)}{r}$$

the *effective Hausdorff dimension* and the *effective packing dimension of x relative to A* , respectively. The utility of effective dimensions in geometric measure theory stems from the following theorem of J. Lutz and N. Lutz.

Theorem 3.2 (Point-to-set principle [14]). *For every set $E \subseteq \mathbb{R}^n$,*

$$\dim_{\text{H}} E = \min_{A \subseteq \{0,1\}^*} \sup_{x \in E} \dim^A(x) \quad \text{and} \quad \dim_{\text{P}} E = \min_{A \subseteq \{0,1\}^*} \sup_{x \in E} \text{Dim}^A(x).$$

We frequently use the following immediate consequence of this theorem: given some $E \subseteq \mathbb{R}^n$, for any oracle A and $\varepsilon > 0$, there exists some $x \in E$ such that $\dim^A(x) > \dim_{\text{H}} E - \varepsilon$, and likewise for packing dimension.

Turning our attention to the effective dimension of points on lines, we note the following observation of N. Lutz and Stull: for any $A \subseteq \{0,1\}^*$ and any $x, a, b \in \mathbb{R}$,

$$K_r^A(x, ax + b) \leq K_r^A(x, a, b) + O_{x,a,b}(\log r). \quad (3)$$

This is because, for every large enough precision r , a Turing machine given x, a, b at precision r can perform a very accurate calculation of $ax + b$ at precision r . A main theme of [15] is how close this upper bound is to being a lower bound for the points on a line, the answer to which is expressed in the following theorem.

Theorem 3.3 (N. Lutz–Stull [15]). *For all $a, b, x \in \mathbb{R}$ and $A \subseteq \{0,1\}^*$,*

$$\dim^A(x|a, b) + \min \{ \dim^A(a, b), \dim^{a,b}(x) \} \leq \dim^A(x, ax + b).$$

This is the key ingredient in our proof of Proposition 1.2. However, as the effective packing dimension analogue of this statement is false,³ the proof of Theorem 1.3 will require a different strategy.

4. LINE SEGMENT EXTENSION IN THE PLANE

4.1. The Hausdorff dimension bound for s -Hausdorff extensions. We begin this section by using Theorem 3.3 of [15] to prove a bound on the Hausdorff dimension of line segment extensions in the plane. This proof takes Lutz and Stull’s result as a black box and illustrates the connection between effective and classical theorems in this setting, which we will need to prove Theorem 1.3.

Proposition 1.2, Restated. *Let $s \in [0, 1]$ and let $E \subseteq \mathbb{R}^2$ be a union of (at least) s -Hausdorff-dimensional subsets of lines. Then $\dim_{\text{H}}(L_s^{\text{H}}(E)) \leq \dim_{\text{H}} E + 2 - 2s$. In particular, if E is a union of line segments and $\mathbf{L}(E)$ is its lineal extension, then $\dim_{\text{H}} \mathbf{L}(E) = \dim_{\text{H}} E$.*

Proof. Write

$$E = \bigcup_{\ell \in \Lambda} E_{\ell},$$

where $\Lambda \subseteq \mathcal{A}(2, 1)$ is the family of lines ℓ such that $\dim_{\text{H}} E_{\ell} \geq s$ and $E_{\ell} := \ell \cap E$. By the point-to-set principle,

$$\dim_{\text{H}} E = \min_{A \subseteq \{0,1\}^*} \sup_{z \in E} \dim^A(z) = \min_{A \subseteq \{0,1\}^*} \sup_{\ell \in \Lambda} \sup_{z \in E_{\ell}} \dim^A(z)$$

³The authors appreciate Stull informing us of this in private communication.

and

$$\dim_{\text{H}} L_s^{\text{H}}(E) = \min_{A \subseteq \{0,1\}^*} \sup_{z \in L_s^{\text{H}}(E)} \dim^A(z) = \min_{A \subseteq \{0,1\}^*} \sup_{\ell \in \Lambda} \sup_{z \in \ell} \dim^A(z).$$

Comparing the right-hand sides of these equations, we see it suffices to show that

$$\sup_{z \in \ell} \dim^A(z) \leq \sup_{z \in E_\ell} \dim^A(z) + 2 - 2s \quad (4)$$

for every oracle $A \subseteq \{0,1\}^*$ and every line $\ell \in \Lambda$. Taking such an A and ℓ , we interchange the x - and y -coordinates if necessary so that ℓ is not vertical and we let (a, b) be the slope-intercept pair of ℓ . As was observed in the previous section, for each $x \in \mathbb{R}$ and each precision r ,

$$K_r^A(x, ax + b) \leq K_r^A(x, a, b) + o(r) \leq r + K_r^A(a, b) + o(r).$$

Hence $\dim^A(x, ax + b) \leq \min \{1 + \dim^A(a, b), 2\}$ and, consequently,

$$\sup_{z \in \ell} \dim^A(z) \leq \min \{1 + \dim^A(a, b), 2\}. \quad (5)$$

Now, let S be the projection of E_ℓ onto the x -axis. Then $\dim_{\text{H}}(S) \geq s$, so by the point-to-set principle, for every $\varepsilon > 0$, there exists $x_\varepsilon \in S$ such that $\dim^{A,a,b}(x_\varepsilon) \geq s - \varepsilon$. Applying Theorem 3.3, we have

$$\begin{aligned} \dim^A(x_\varepsilon, ax_\varepsilon + b) &\geq \dim^A(x_\varepsilon | a, b) + \min \{\dim^A(a, b), \dim^{a,b}(x_\varepsilon)\} \\ &\geq \dim^{A,a,b}(x_\varepsilon) + \min \{\dim^A(a, b), \dim^{A,a,b}(x_\varepsilon)\} \\ &\geq s - \varepsilon + \min \{\dim^A(a, b), s - \varepsilon\}. \end{aligned}$$

Letting ε go to zero gives

$$\sup_{z \in E_\ell} \dim^A(z) \geq \min \{s + \dim^A(a, b), 2s\}. \quad (6)$$

The difference in the minima is largest when $\dim^A(a, b) \geq 1$, which gives the desired bound. \square

4.2. The packing dimension bound for 1-Hausdorff extensions. Now we turn our attention to the packing dimension version of the line segment extension problem. The key issue is that we do not have an analogue of Lutz and Stull's bound for effective packing dimension. In fact, the statement obtained by replacing effective Hausdorff dimension with effective packing dimension in Theorem 3.3 is simply not true.

Essentially, the inequality fails because a high packing dimension for the pair (a, b) can be the result of $K_r^A(a, b)$ growing relatively slowly in r up to some level of precision, and then significantly more quickly up to a higher level of precision. At a key technical step in the proof, the complexity function of (a, b) needs to have certain properties, which can be guaranteed by reducing its complexity up to precision r via an oracle D . This “wastes” complexity growth of (x, a, b) that we would like to transfer to $(x, ax + b)$, but since effective Hausdorff dimension only reflects a *lower* bound on the asymptotic complexity growth, D does not reduce the complexity of $(x, ax + b)$ unacceptably in comparison to the effective Hausdorff dimension of (a, b) . Effective packing dimension, however, reflects an *upper* bound

on asymptotic complexity growth, which dashes any hope for the analogous packing inequality. We will proceed without proving an explicit lower bound on the packing dimension of arbitrary points on a line, but will still show that for x of essentially maximal complexity at certain precisions, $(x, ax + b)$ also has essentially maximal complexity. This will imply (a somewhat stronger version of) the line segment extension conjecture for packing dimension in the plane.

Theorem 1.3, Restated. *If $E \subseteq \mathbb{R}^2$ is a union of 1-Hausdorff-dimensional subsets of lines, then $\dim_{\mathbb{P}}(L_1^H(E)) = \dim_{\mathbb{P}} E$. In particular, if E is a union of line segments and $\mathbf{L}(E)$ is its lineal extension, then $\dim_{\mathbb{P}} \mathbf{L}(E) = \dim_{\mathbb{P}} E$.*

The proof will proceed in four main steps. We need to establish a variant of Theorem 3.3 that bounds the complexity growth of $(x, ax + b)$ on certain *intervals* of precision, as opposed to the entirety of $[1, r]$. We will need to consider two kinds of intervals, first *teal* intervals and then *yellow* intervals, named as in [22]. The third step is the construction of a certain partition of $[1, r]$ into yellow and teal intervals. Finally, we prove that for partitions of this form and certain choices of x , the lower bound arising from the partition essentially matches an upper bound at all sufficiently large precisions.

Before we begin, we will need several lemmas, including the following slight modification of Lemma 3.1 in [15].

Lemma 4.1. *Suppose $x, a, b \in \mathbb{R}$, $B \subseteq \{0, 1\}^*$, $r, t \in \mathbb{N}$, $\delta \in \mathbb{R}^+$, and $\varepsilon, \eta \in \mathbb{Q}^+$ satisfy $r > \log(2|a| + |x| + 5) + 1$, $t < r$, and the following:*

- (1) $K_{r,t}^B(a, b) \leq (\eta + \varepsilon)(r - t)$.
- (2) *For every $(u, v) \in B((a, b), 2^{-t})$ such that $ux + v = ax + b$,*

$$K_{r,t}^B(u, v) \geq (\eta - \varepsilon)(r - t) + \delta(r - s)$$

where $s = -\log |(a, b) - (u, v)| \in (t, r]$.

Then

$$K_{r,t}^B(x, ax + b) \geq K_{r,t}^B(x, a, b) - \frac{4\varepsilon}{\delta}(r - t) - K^B(\varepsilon) - K^B(\eta) - O(\log r).$$

We note that the implicit constant may depend on x, a , and b , but these will be fixed in each application. The alteration as compared to [15] is the introduction of the precision t , which enables us to apply this lemma not only on intervals $[1, r]$ but to the elements of a partition. This kind of lemma is often referred to as an “enumeration” lemma, as its proof depends on enumerating many short strings to find one that gives an output with the desired properties; enumeration lemmas are key technical elements of many proofs using effective methods because they give us conditions under which a desired lower bound holds. In the proof of our main theorem, showing that the two conditions are satisfied is a significant element in proving the desired bounds on yellow and teal intervals. We prove the lemma for completeness.

Proof. Let $x, a, b, B, r, t, \delta, \varepsilon$, and η be as above. We first prove that it suffices to show that

$$K_{r,t}^B(x, u, v | x, a, b) \leq K_{r,t}^B(x, ax + b | a, b, x) + K^B(\eta) + K^B(\varepsilon) + O(\log r). \quad (7)$$

whenever (u, v) satisfies $K_{r,t}^B(u, v) \leq (\eta + \varepsilon)(r - t)$ and condition (2) of the lemma. The second condition gives that (u, v) is distance not more than 2^{-t} from (a, b) , so

$$K_{r,t}^B(x, u, v | x, a, b) = K_{r,t}^B(x, u, v).$$

Because (x, a, b) contains at least as much information as $(x, ax + b)$ at precision t , we also have that

$$K_{r,t}^B(x, ax + b | a, b, x) \leq K_{r,t}^B(x, ax + b) + O(\log r).$$

Applying these two facts to (7) yields

$$K_{r,t}^B(x, u, v) \leq K_{r,t}^B(x, ax + b) + K^B(\eta) + K^B(\varepsilon) + O(\log r). \quad (8)$$

We see that (x, a, b) and (x, u, v) differ in two coordinates and completely agree up to precision t , so by the definition of s ,

$$K_{r,t}^B(x, u, v) \geq K_{r,t}^B(x, a, b) - 2(r - s) - O(\log r). \quad (9)$$

Our extra condition on (u, v) and the second condition of the lemma immediately imply

$$(\eta + \varepsilon)(r - t) \geq K_{r,t}^B(u, v) \geq (\eta - \varepsilon)(r - t) + \delta(r - s), \quad \text{so} \quad \frac{2\varepsilon}{\delta}(r - t) \geq (r - s).$$

This in conjunction with (9) entails

$$K_{r,t}^B(x, u, v) \geq K_{r,t}^B(x, a, b) - \frac{4\varepsilon}{\delta}(r - t) - O(\log r),$$

and combining this with (8) gives the desired bound.

We now direct our attention to proving (7). Let M be an oracle Turing machine that does the following given access to B and inputs $(w_1, w_2, w_3) \in \mathbb{Q}^3$, $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 \in \{0, 1\}^*$ such that

$$U^B(\pi_2) = s_1 \in \mathbb{N}, \quad U^B(\pi_3) = s_2 \in \mathbb{N}, \quad U^B(\pi_4) = \zeta \in \mathbb{Q}, \quad U^B(\pi_5) = \iota \in \mathbb{Q}.$$

First, M^B computes $U^B(\pi_1, (w_1, w_2, w_3)) = (q_1, q_2)$. For every program $\sigma \in \{0, 1\}^*$ with length less than or equal to $(\zeta + \iota)(s_2 - s_1)$, it simulates $U^B(\sigma, (w_2, w_3)) = (p_1, p_2)$ in parallel. If one of the simulations halts with $(p_1, p_2) \in B((w_2, w_3), 2^{-s_1})$ such that $|p_1q_1 + p_2 - q_2| < 2^{-s_2}(|p_1| + |q_1| + 3)$, then M halts with output (q_1, p_1, p_2) . Let c_M be a constant for this machine.

Now let $\pi_1, \pi_2, \pi_3, \pi_4$, and π_5 testify to

$$K_{r,t}^B(x, ax + b | x, a, b), \quad K^B(t), \quad K^B(r), \quad K^B(\eta), \quad \text{and} \quad K^B(\varepsilon),$$

respectively, and let w_1, w_2, w_3 be the K -minimizing rationals at precision t for x, a, b . Then $U^B(\pi_1, (w_1, w_2, w_3))$ gives $(x, ax + b)$ to precision r . By our hypothesis that $K_{r,t}^B(a, b) \leq (\eta + \varepsilon)(r - t)$, there is a string σ of length at most $(\eta + \varepsilon)(r - t)$ such that $U^B(\sigma, (w_2, w_3))$ gives (a, b) to precision r . Thus, $M(\pi, (w_1, w_2, w_3))$ halts by the same geometric argument

as in [15], and the output (q_1, p_1, p_2) lies in $B((x, u, v), 2^{\gamma+1-r})$ for some γ depending only on $|x|$ and $|a|$. Absorbing γ into the error term gives

$$\begin{aligned} K_{r,t}^B(x, u, v | x, a, b) &\leq |\pi| + c_M + O(\log r) \\ &\leq K_{r,t}^B(x, ax + b | x, a, b) + K^B(t) + K^B(r) + K^B(\eta) + K^B(\varepsilon) + O(\log r) \\ &\leq K_{r,t}^B(x, ax + b | x, a, b) + K^B(\eta) + K^B(\varepsilon) + O(\log r). \end{aligned}$$

Finally observe that the string σ which computes (u, v) given access to its first t bits has length, by assumption, no greater than $(\eta + \varepsilon)(r - t)$. Hence $K_{r,t}^B(u, v) \leq (\eta + \varepsilon)(r - t)$, which in conjunction with the above inequality completes the proof. \square

We also make use of Lemmas 3.2 and 3.3 from [15], stated in the form we will need.⁴

Lemma 4.2. *Let $x, a, b \in \mathbb{R}$. For all $(u, v) \in B((a, b), 2^{-t})$ such that $ux + v = ax + b$ and for all $r \geq s := -\log |(a, b) - (u, v)|$,*

$$K_r^A(u, v) \geq K_s^A(a, b) + K_{r-s,r}^A(x | a, b) - O(\log r).$$

Lemma 4.3. *Let $A \subseteq \{0, 1\}^*$, $r \in \mathbb{N}$, $z \in \mathbb{R}^n$, and $\gamma \in \mathbb{Q}^+$. There is an oracle $D = D(A, n, r, z, \gamma)$ satisfying the following:*

- (1) *For every natural number $t \leq r$,*

$$K_t^{A,D}(z) = \min\{\gamma r, K_t^A(z)\} + O(\log r).$$

- (2) *For every $m, t \in \mathbb{N}$ and $y \in \mathbb{R}^m$,*

$$K_{t,r}^{A,D}(y | z) = K_{t,r}^A(y | z) + O(\log r) \quad \text{and} \quad K_t^{A,D,z}(y) = K_t^{A,z}(y) + O(\log r).$$

- (3) *If $B \subseteq \{0, 1\}^*$ satisfies $K_r^{A,B}(z) \geq K_r^A(z) - O(\log r)$*

$$K_r^{A,B,D}(z) \geq K_r^{A,D}(z) - O(\log r).$$

- (4) *For every $m, t \in \mathbb{N}$, $u \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$,*

$$K_r^A(u | w) \leq K_r^{A,D}(u | w) + K_r^A(z) - \gamma r.$$

Lemma 4.2 is the key geometric observation of Lutz and Stull, and it formalizes the statement that lines passing through the same point are either almost parallel (in which case they contain much of the same information), or they are transverse enough their approximations together determine the x -coordinate of the intersection to a high precision.

Lemma 4.3 is common in effective arguments. Although it is lengthy to state, the idea is rather simple: if you want to lower the complexity of a point z at some precision r , look back to find a precision $s < r$ at which the complexity of z is what you want it to be at precision r . Then, let D encode all of the new information in z from s to r . Property 1 says this oracle accomplishes the goal of lowering the complexity. By contrast, the remaining properties tell us that D is not *too* helpful, that is, D does not undesirably lower the complexity of other objects. Specifically D is not any more helpful in any calculation than knowing z up to

⁴Lemmas 3.2 and 3.3 are used in a relativized form in [15], so we state them in this way. The third and fourth properties in Lemma 4.3, which appear in the proof of [15] Lemma 3.3, can also be found in [22].

precision r (Property 2), it does not magically become more useful when combined with other unhelpful oracles for z (Property 3), and it does not reduce the complexity of any object more than it reduces the complexity of z at precision r (Property 4).

Proof of Theorem 1.3. By the same application of the point-to-set principle used in Proposition 1.2, it suffices to show that for any planar line ℓ with slope-intercept pair (a, b) and for any oracle $A \subseteq \{0, 1\}^*$, if $\dim^{A, a, b}(x_\varepsilon) \geq 1 - \frac{\varepsilon}{2}$ for a collection of x_ε , then

$$\lim_{\varepsilon \rightarrow 0} \text{Dim}^A(x_\varepsilon, ax_\varepsilon + b) = \sup_{z \in \ell} \text{Dim}^A(z). \quad (10)$$

With this aim in mind, let $a, b \in \mathbb{R}$ and $A \subseteq \{0, 1\}^*$ be given. Although $K_s^A(a, b)$ is a function defined on the natural numbers, for the purposes of our partitioning argument it is simpler to extend to all reals ≥ 1 by linear interpolation. For ease of notation, we continue to use $K_s^A(a, b)$ to denote this extended function. Let $\varepsilon \in (0, 1) \cap \mathbb{Q}$ and assume $x \in \mathbb{R}$ is such that $\dim^{A, a, b}(x) \geq 1 - \frac{\varepsilon}{2}$. In the following, we will always assume r is large enough that for $s > \log r$, $(1 - \varepsilon)s \leq K_s^{A, a, b}(x) \leq (1 + \varepsilon)s$.

Lower bound on teal intervals: Call an interval $[t, r]$ of precisions *teal* if

$$K_s^A(a, b) \geq K_r^A(a, b) - (r - s) \quad \forall s \in [t, r].$$

We want to prove a lower bound on the complexity growth of $(x, ax + b)$ on such intervals by applying Lemma 4.1. If possible,⁵ pick η to be an element of the finite set $\{\frac{i}{2^m} : m = 2 - \lceil \log \varepsilon \rceil \text{ and } 0 \leq i \leq 2^m\}$ such that

$$\frac{K_{r,t}^A(a, b)}{r - t} - 2\sqrt{\varepsilon} < \eta < \frac{K_{r,t}^A(a, b)}{r - t} - \sqrt{\varepsilon}.$$

Using Lemma 4.3, let $D = D(A, 2, r, (a, b), K_t^A(a, b) + \lfloor \eta(r - t) \rfloor)$. In particular, observe that D lowers the complexity of $K_r^A(a, b)$ by at most $2\sqrt{\varepsilon}(r - t) + O(\log r)$. Thus, with this choice of η , it is immediate that relative to (A, D) and for sufficiently large r , the first condition of Lemma 4.1 is satisfied.

Now we show the second is also satisfied. For $(u, v) \in B_{2^{-t}}(a, b)$, by symmetry of information, Lemma 4.2, and the second property of D , we have

$$\begin{aligned} K_{r,t}^{A,D}(u, v) &\geq K_r^{A,D}(u, v) - K_t^{A,D}(u, v) - O(\log r) \\ &\geq K_s^{A,D}(a, b) + K_{r-s,r}^{A,D}(x|a, b) - K_t^{A,D}(u, v) - O(\log r) \\ &= K_s^{A,D}(a, b) + K_{r-s,r}^{A,D}(x|a, b) - K_t^{A,D}(a, b) - O(\log r) \\ &\geq K_{s,t}^{A,D}(a, b) + K_{r-s,r}^{A,D}(x|a, b) - O(\log r) \\ &\geq K_{s,t}^{A,D}(a, b) + K_{r-s,r}^A(x|a, b) - O(\log r) \\ &\geq K_{s,t}^{A,D}(a, b) + K_{r-s}^{A, a, b}(x) - O(\log r). \end{aligned}$$

Because of how we choose D , our teal condition $K_s^A(a, b) \geq K_r^A(a, b) - (r - s)$ implies a slightly stronger teal condition for $K_s^{A,D}(a, b)$. Up to a log term, D lowers the complexity

⁵If the average growth of $K_{r,t}^A(a, b)$ is smaller than, say, $2\sqrt{\varepsilon}$, just set $\eta = 0$. We'll get the trivial bound of 0 for the growth on this interval with *this* choice of ε , but in practice, we will pick up any actual growth as we pass through with smaller and smaller ε .

of $K_r^{A,D}(a, b)$ by at least $\sqrt{\varepsilon}(r - t)$. Hence, instead of drawing a line of slope 1 intersecting the complexity function at r (the teal condition), we can draw a line of slope $(1 - \sqrt{\varepsilon})$ and $K_s^{A,D}(a, b)$ will still be above it on $[t, r]$. More concretely, $K_s^{A,D}(a, b) \geq K_r^{A,D}(a, b) - (1 - \sqrt{\varepsilon})(r - s) - O(\log r)$, since, up to a log term, D reduces the complexity of (a, b) (relative to A) more at precision r than any precision $s < r$. By two applications of symmetry of information, this in turn implies $K_{s,t}^{A,D}(a, b) \geq K_{r,t}^{A,D}(a, b) - (1 - \sqrt{\varepsilon})(r - s) - O(\log r)$. Hence,

$$K_{r,t}^{A,D}(u, v) \geq K_{r,t}^{A,D}(a, b) - (1 - \sqrt{\varepsilon})(r - s) + K_{r-s}^{A,a,b}(x) - O(\log r).$$

Now by our assumption on x , either $r - s \leq \log(r)$ or $K_{r-s}^{A,a,b}(x) \geq (1 - \varepsilon)(r - s)$ holds. In both cases, we have

$$\begin{aligned} K_{r,t}^{A,D}(u, v) &\geq K_{r,t}^{A,D}(a, b) - (1 - \sqrt{\varepsilon})(r - s) + (1 - \varepsilon)(r - s) - O(\log r) \\ &= K_{r,t}^{A,D}(a, b) - (\varepsilon - \sqrt{\varepsilon})(r - s) - O(\log r) \\ &\geq \eta(r - t) - (\varepsilon - \sqrt{\varepsilon})(r - s) - O(\log r) \\ &= (\eta - \varepsilon + \varepsilon)(r - t) - (\varepsilon - \sqrt{\varepsilon})(r - s) - O(\log r) \\ &\geq (\eta - \varepsilon)(r - t) - (\varepsilon - \varepsilon - \sqrt{\varepsilon})(r - s) - O(\log r) \\ &\geq (\eta - \varepsilon)(r - t) + \sqrt{\varepsilon}(r - s) - O(\log r). \end{aligned}$$

Thus, for sufficiently large r , we have

$$K_{r,t}^{A,D}(u, v) \geq (\eta - \varepsilon)(r - t) + \frac{\sqrt{\varepsilon}}{2}(r - s).$$

This is precisely the second condition of Lemma 4.1 with $\delta = \frac{\sqrt{\varepsilon}}{2}$. Both conditions of Lemma 4.1 are satisfied, hence applying it without any additional oracle, we obtain

$$K_{r,t}^{A,D}(x, ax + b) \geq K_{r,t}^{A,D}(x, a, b) - 8\sqrt{\varepsilon}(r - t) - K(\varepsilon) - K(\eta) - O(\log r). \quad (11)$$

In practice, we will keep the same choice of ε throughout a partitioning argument even as r goes to infinity, and we chose η from a fixed set that depends only on ε . Thus, we can treat the complexity of these terms as constant in r . Furthermore, removing an oracle can only increase complexity, up to a log term, so

$$K_{r,t}^A(x, ax + b) \geq K_{r,t}^{A,D}(x, a, b) - 8\sqrt{\varepsilon}(r - t) - O(\log r). \quad (12)$$

Now, applying symmetry of information and the properties of D to $K_{r,t}^{A,D}(x, a, b)$, we obtain

$$\begin{aligned} K_{r,t}^{A,D}(x, a, b) &\geq K_r^{A,D}(x, a, b) - K_t^{A,D}(x, a, b) - O(\log r) \\ &\geq K_r^{A,D}(x|a, b) + K_r^{A,D}(a, b) - K_t^{A,D}(x|a, b) - K_t^{A,D}(a, b) - O(\log r) \\ &\geq K_r^{A,D,a,b}(x) + K_r^{A,D}(a, b) - K_t^{A,D,a,b}(x) - K_t^{A,D}(a, b) - O(\log r) \\ &\geq K_r^{A,a,b}(x) + K_r^{A,D}(a, b) - K_t^{A,a,b}(x) - K_t^{A,D}(a, b) - O(\log r) \\ &\geq K_r^{A,a,b}(x) + K_{r,t}^{A,D}(a, b) - K_t^{A,a,b}(x) - O(\log r). \end{aligned}$$

Recalling the definition of D , we have

$$K_{r,t}^{A,D}(x, a, b) \geq K_r^{A,a,b}(x) - K_t^{A,a,b}(x) + K_{r,t}^A(a, b) - 2\sqrt{\varepsilon}(r - t) - O(\log r).$$

Finally, we need a lower bound on the growth $K_r^{A,a,b}(x) - K_t^{A,a,b}(x)$. If $t < \log r$, then $K_r^{A,a,b}(x) - K_t^{A,a,b}(x) \geq (1 - \varepsilon)r - O(\log r)$, which is better than the bound we get below. Otherwise, recall the upper bound $K_t^{A,a,b}(x) \leq (1 + \varepsilon)t$. If t is too close to r , we cannot usefully combine this upper bound with the lower bound $K_t^{A,a,b}(x) \geq (1 - \varepsilon)t$. But, in this case, the interval $[t, r]$ will be too short to affect the sum of elements in the partition very much. So, we assume $t < (1 - \sqrt{\varepsilon})r$. Then

$$\begin{aligned} \frac{K_r^{A,a,b}(x) - K_t^{A,a,b}(x)}{r - t} &\geq \frac{(1 - \varepsilon)r - (1 + \varepsilon)t}{r - t} \\ &\geq \frac{(1 - \varepsilon)r - (1 + \varepsilon)(1 - \sqrt{\varepsilon})r}{r - (1 - \sqrt{\varepsilon})r} \\ &= 1 - 2\sqrt{\varepsilon} + \varepsilon \\ &\geq 1 - 2\sqrt{\varepsilon}. \end{aligned}$$

Hence

$$\begin{aligned} K_{r,t}^{A,D}(x, a, b) &\geq (1 - 2\sqrt{\varepsilon})(r - t) + K_{r,t}^A(a, b) - 2\sqrt{\varepsilon}(r - t) - O(\log r) \\ &\geq (1 - 4\sqrt{\varepsilon})(r - t) + K_{r,t}^A(a, b) - O(\log r). \end{aligned}$$

Combining this with inequality (12) when r is sufficiently large gives

$$K_{r,t}^A(x, ax + b) \geq (1 - 13\sqrt{\varepsilon})(r - t) + K_{r,t}^A(a, b) \quad (13)$$

or, in the case that $t \geq (1 - \sqrt{\varepsilon})r$,

$$K_{r,t}^A(x, ax + b) \geq K_{r,t}^A(a, b) - 13\sqrt{\varepsilon}(r - t). \quad (14)$$

Lower bound on yellow intervals: Having done the teal case, the yellow case is straightforward. We call an interval $[t, r]$ **yellow** if

$$K_s^A(a, b) \geq K_t^A(a, b) + (s - t) \quad \forall s \in [t, r].$$

In comparison with the previous case, the complexity function on teal intervals lies above the line of slope 1 passing through the right endpoint, while the complexity function on yellow intervals lies above the line of slope 1 passing through the left endpoint. It is an easy observation that, using an oracle D , we can reduce the complexity of $K_r^A(a, b)$ so that the teal property $K_s^{A,D}(a, b) \geq K_r^{A,D}(a, b) - (r - s) - O(\log r)$ holds on $[t, r]$, and the average growth rate of $K_s^{A,D}(a, b)$ on $[t, r]$ is only slightly smaller than 1. To be concrete, we pick D as follows.

Let η to be an element of the finite set $\{\frac{i}{2^m} : m = 2 - \lceil \log \varepsilon \rceil \text{ and } 0 \leq i \leq 2^m\}$ such that

$$1 - 2\sqrt{\varepsilon} < \eta < 1 - \sqrt{\varepsilon}.$$

Using Lemma 4.3, let $D = D(A, 2, r, (a, b), K_t^A(a, b) + \lfloor \eta(r - t) \rfloor)$. Thus, with this choice of η , it is immediate that relative to (A, D) and for sufficiently large r , the first condition of Lemma 4.1 is satisfied. If we defined $\tilde{D} = D(A, 2, r, (a, b), K_t^A(a, b) + 1(r - t))$, the teal property would hold relative to (A, \tilde{D}) . Comparatively, the oracle D further reduces $K_r^A(a, b)$ by at least $\sqrt{\varepsilon}(r - t)$, hence, as in the previous case, we have $K_{s,t}^{A,D}(a, b) \geq K_{r,t}^{A,D}(a, b) - (1 - \sqrt{\varepsilon})(r - s) - O(\log r)$. Since we have the $(1 - \sqrt{\varepsilon})$ -teal property relative

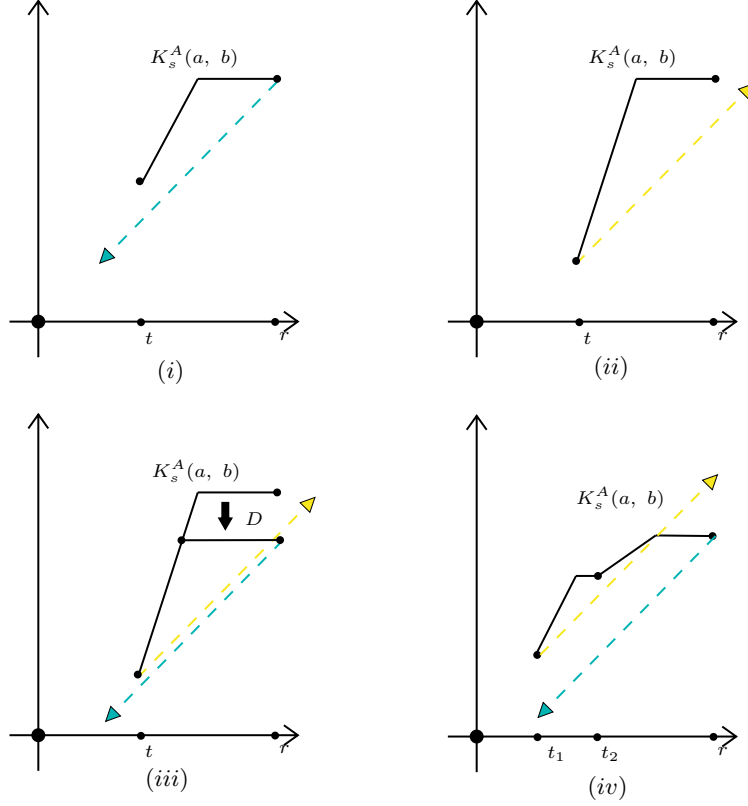


FIGURE 1. Several illustrations of the behavior of yellow and teal intervals. (i) A typical teal interval. (ii) A typical yellow interval. (iii) A yellow interval becomes a teal interval with average growth rate close to 1 upon the addition of a well-chosen oracle D . (iv) The union of a yellow interval (interval $[t_1, t_2]$) followed by a teal interval (interval $[t_2, r]$) is either a single yellow or a single teal interval, in this case teal.

to (A, D) , we omit the verification of the second condition, as it becomes identical to the teal case. Now, applying Lemma 4.1, we have

$$\begin{aligned}
 K_{r,t}^A(x, ax + b) &\geq K_{r,t}^{A,D}(x, ax + b) \\
 &\geq K_{r,t}^{A,D}(x, a, b) - 8\sqrt{\varepsilon}(r - t) - K(\varepsilon) - K(\eta) - O(\log r) \\
 &\geq K_{r,t}^{A,D}(x, a, b) - 8\sqrt{\varepsilon}(r - t) - O(\log r).
 \end{aligned}$$

As in the previous case,

$$K_{r,t}^{A,D}(x, a, b) \geq K_r^{A,a,b}(x) - K_t^{A,a,b}(x) + K_{r,t}^{A,D}(a, b) - O(\log r).$$

D was chosen such that the average growth rate on $[t, r]$ is at least $1 - 2\sqrt{\varepsilon}$, and the same bound on the complexity growth of x as in the previous case holds, hence, if $t < (1 - \sqrt{\varepsilon})r$,

then

$$\begin{aligned} K_{r,t}^{A,D}(x, a, b) &\geq K_r^{A,a,b}(x) - K_t^{A,a,b}(x) + (1 - 2\sqrt{\varepsilon})(r - t) - O(\log r) \\ &\geq (2 - 4\sqrt{\varepsilon})(r - t) - O(\log r). \end{aligned}$$

Combining this with our main inequality when r is sufficiently large gives

$$K_{r,t}^A(x, ax + b) \geq (2 - 13\sqrt{\varepsilon})(r - t); \quad (15)$$

or, in the case that $t \geq (1 - \sqrt{\varepsilon})r$,

$$K_{r,t}^A(x, ax + b) \geq (1 - 13\sqrt{\varepsilon})(r - t). \quad (16)$$

Partitioning $[1, r]$: Our aim is now to construct, given $A \subseteq \{0, 1\}^*$, $a, b \in \mathbb{R}$, and r sufficiently large, a partition \mathcal{P} of $[1, r]$ having (at least) one of the following forms:

- (1) \mathcal{P}_r consists of one element $[1, r]$, which is teal,
- (2) \mathcal{P}_r consists of one element $[1, r]$, which is yellow, or
- (3) \mathcal{P}_r consists of two elements, $[1, c_r]$ and $[c_r, r]$, where $[1, c_r]$ is teal and $[c_r, r]$ is yellow.

Note that the partition does not depend on x . We use a strategy similar to [22]. The difference is that we have no restriction on the maximum length of our intervals in our partition. Start with the partition $[1, 2], [2, 3], \dots, [r-1, r]$. The slopes are constant between consecutive integers, so every element is either teal or yellow. It is easy to observe that the union of yellow intervals is yellow, and the union of teal intervals is teal, so first, combine all adjacent yellow and teal intervals. Similarly, the union of a yellow interval followed by a teal interval is either yellow or teal, so next combine every yellow-teal pair. Repeat the first and second step in order until *neither* step changes the partition, then output the current partition as \mathcal{P}_r .

This process halts, since r is a finite number and each step can only increase the length of intervals. The halting condition implies \mathcal{P}_r contains no yellow-yellow, teal-teal, or yellow-teal pairs. Check that every three element partition contains one of these pairs; thus \mathcal{P}_r has at most two elements. But, the only valid pair is teal-yellow. So, \mathcal{P}_r is either a teal interval followed by a yellow interval, or consists of only one element, which validates the construction.

For convenience, we will always consider \mathcal{P}_r to be $\{[1, c_r], [c_r, r]\}$ by the convention that if \mathcal{P}_r is of form one then $c_r = r$, and if \mathcal{P}_r is of form two then $1 = c_r$.

Essentially tight bounds via \mathcal{P}_r : Now, we want to upper bound $K_r^A(x, ax + b)$ for any given $x \in \mathbb{R}$. We use two facts. The first is that on *any* interval, $K_{r,t}^A(x, ax + b)$ is essentially upper bounded by $2(r - t)$. The second, as we have used a few times, is that $K_r^A(x, ax + b)$ is essentially upper bounded by $K_r^A(x, a, b)$, since precision r approximations of x, a , and b are enough to compute $ax + b$ to a similar precision.⁶ We want to use the first bound on yellow intervals and the second on teal intervals.

⁶It is *not* true that $K_{r,t}^A(x, ax + b) \leq K_{r,t}^A(x, a, b)$, since x, a , and b could all be independently random on $[1, t]$ and then consist only of 0s on $[t, r]$; in this case, the complexity keeps growing for $(x, ax + b)$. Informally this is because the product of a up to precision t and x up to precision t can have length $2t$. This would

More formally, assume x is such that

$$\text{Dim}^A(x, ax + b) \geq \sup_{z \in \ell} \text{Dim}^A(z) - \varepsilon.$$

Clearly $(x, ax + b) \in \mathbb{R}^2$, so for sufficiently large r we have

$$K_{r,t}^A(x, ax + b) \leq 2(r - t) + o(r) \leq (2 + \varepsilon)(r - t). \quad (17)$$

At the same time, also for sufficiently large r ,

$$K_r^A(x, ax + b) \leq K_r^A(x, a, b) + o(r) \leq K_r^A(x, a, b) + \varepsilon r. \quad (18)$$

Now, assume r is large enough that the above bounds hold for any $s > \log r$. We use the partition \mathcal{P}_r , applying (18) on the interval $[1, c_r]$ and (17) on $[c_r, r]$. This gives

$$\begin{aligned} K_r^A(x, ax + b) &\leq K_{1,c_r}^A(x, ax + b) + K_{c_r,r}^A(x, ax + b) + O(\log r) \\ &\leq K_{c_r}^A(x, a, b) + \varepsilon c_r + 2(r - c_r) + \varepsilon(r - c_r) + O(\log r) \\ &\leq K_{c_r}^A(x, a, b) + 2(r - c_r) + 2\varepsilon r \\ &\leq c_r + K_{c_r}^A(a, b) + 2(r - c_r) + 3\varepsilon r. \end{aligned}$$

Now, we lower bound the complexity of x_ε , where x_ε satisfies the hypothesis of the yellow and teal results, namely that for sufficiently large r , $s > \log r$ implies $(1 - \varepsilon)s \leq K_s^{A,a,b}(x) \leq (1 + \varepsilon)s$. We use the fact that \mathcal{P}_r (and hence c_r) only depended on A, a, b , and r , and apply the yellow and teal interval lower bounds.

$$\begin{aligned} K_r^A(x_\varepsilon, ax_\varepsilon + b) &\geq K_{c_r}^A(x_\varepsilon, ax_\varepsilon + b) + K_{r,c_r}^A(x_\varepsilon, ax_\varepsilon + b) - O(\log r) \\ &\geq (1 - 13\sqrt{\varepsilon})c_r + K_{c_r}^A(a, b) + (2 - 13\sqrt{\varepsilon})(r - c_r) + 2\sqrt{\varepsilon}r - O(\log r) \\ &\geq c_r + K_{c_r}^A(a, b) + 2(r - c_r) - 28\sqrt{\varepsilon}r - O(\log r) \\ &\geq c_r + K_{c_r}^A(a, b) + 2(r - c_r) - 29\sqrt{\varepsilon}r, \end{aligned}$$

where the extra $2\sqrt{\varepsilon}r$ came from possibly discarding one of the intervals if it is too short to apply our yellow and teal bounds. Combining this lower bound with the previous upper bound gives that, for all sufficiently large r ,

$$K_r^A(x, ax + b) - K_r^A(x_\varepsilon, ax_\varepsilon + b) \leq 32\sqrt{\varepsilon}r.$$

Hence,

$$\text{Dim}^A(x, ax + b) - \text{Dim}^A(x_\varepsilon, ax_\varepsilon + b) \leq 32\sqrt{\varepsilon}.$$

Finally, by our choice of x , this gives

$$\text{Dim}^A(x_\varepsilon, ax_\varepsilon + b) \geq \sup_{z \in \ell} \text{Dim}^A(z) - 33\sqrt{\varepsilon}. \quad (19)$$

We could pick ε to be arbitrarily small, so this completes the proof. \square

Using one of the last inequalities of the above proof, we establish a slight strengthening of the generalized Kakeya conjecture for packing dimension in the plane. Specifically, we show

present a problem for the proof, if we did not have from the previous step that the partition cannot be a yellow and then teal interval.

Corollary 1.4, Restated. *Let $E \subseteq \mathbb{R}^2$ and let $D \subseteq \mathbb{P}^{n-1}$ be the set of directions of lines intersecting E in a set of Hausdorff dimension 1. If $D \neq \emptyset$, then*

$$\dim_{\mathbb{P}} D + 1 \leq \dim_{\mathbb{P}} E.$$

Proof. Let $\Lambda = \{\ell \in \mathcal{A}(2, 1) : \dim_{\mathbb{H}}(E \cap \ell) = 1\}$, let F_ℓ denote the orthogonal projection of $E \cap \ell$ onto the x -axis, and let A be a packing oracle for both E and D . Identifying D with the det of slopes of lines in Λ , we see by the point-to-set principle that the desired inequality (1) is equivalent to

$$\sup_{a \in D} \text{Dim}^A(a) + 1 \leq \sup_{\ell \in \Lambda} \sup_{x \in F_\ell} \text{Dim}^A(x, ax + b).$$

So it suffices to show that for every $a, b \in \mathbb{R}$, $A \subseteq \mathbb{N}$, and $S \subseteq \mathbb{R}$ of Hausdorff dimension 1,

$$\text{Dim}^A(a) + 1 \leq \sup_{x \in S} \text{Dim}^A(x, ax + b).$$

By the point-to-set principle, this follows if for any $x_\varepsilon \in \mathbb{R}$ such that $\dim^{A,a,b}(x_\varepsilon) > 1 - \frac{\varepsilon}{2}$,

$$\text{Dim}^A(a) + 1 \leq \lim_{\varepsilon \rightarrow 0} \text{Dim}^A(x_\varepsilon, ax_\varepsilon + b).$$

In the last section of the proof of the main theorem, we obtained the bound

$$K_r^A(x_\varepsilon, ax_\varepsilon + b) \geq c_r + K_{c_r}^A(a, b) + 2(r - c_r) - 29\sqrt{\varepsilon}r. \quad (20)$$

Assuming r is sufficiently large, this implies

$$\begin{aligned} K_r^A(x_\varepsilon, ax_\varepsilon + b) &\geq c_r + K_{c_r}^A(a, b) + 2(r - c_r) - 29\sqrt{\varepsilon}r \\ &\geq c_r + K_{c_r}^A(a) + 2(r - c_r) - 30\sqrt{\varepsilon}r \\ &\geq c_r + K_{c_r}^A(a) + (r - c_r) + K_{r,c_r}^A(a) - 31\sqrt{\varepsilon}r \\ &\geq c_r + K_r^A(a) + (r - c_r) - 32\sqrt{\varepsilon}r \\ &= r + K_r^A(a) - 32\sqrt{\varepsilon}r. \end{aligned}$$

This holds at every sufficiently large precision r , so dividing by r and taking the limit superior on both sides completes the proof. \square

4.3. Packing extensions. In comparison with 1-Hausdorff extensions, which do not increase either the Hausdorff or packing dimension of $E \subseteq \mathbb{R}^2$, the 1-packing extensions behave much more poorly. It is clear that the 1 packing extension can increase the Hausdorff dimension of set; just let E be a Hausdorff dimension 0, packing dimension 1 subset of a line. Similarly, the 1-packing extension can maximally increase the packing dimension of some sets E , illustrated in the following example.

Example 4.1 ($L_1^{\mathbb{P}}(E)$ can increase the packing dimension). Let r_i be a rapidly increasing sequence. Define

$$\begin{aligned} X &= \{x = 0.x_1x_2x_3 \cdots \in [0, 1] : j \in [r_{2i-1}, r_{2i}) \text{ for some } i \in \mathbb{N} \Rightarrow x_j = 0\} \quad \text{and} \\ S &= \{a = 0.a_1a_2a_3 \cdots \in [0, 1] : j \in [r_{2i}, r_{2i+1}) \text{ for some } i \in \mathbb{N} \Rightarrow a_j = 0\}. \end{aligned}$$

Note that X and S both have packing dimension 1. Further define

$$E = \{(x, ax) \in \mathbb{R}^2 : x \in X, a \in S\}.$$

Clearly E has packing dimension at least 1. We get equality using the point to set principle. For any $A \subseteq \{0, 1\}^*$, $x \in X$, and $a \in S$,

$$\begin{aligned} K_r^A(x, ax) &\leq K_r(x, ax) + O(\log r) \\ &\leq K_r(x, a) + O(\log r) \\ &\leq K_r(x + a) + O(\log r) \\ &\leq r + O(\log r), \end{aligned}$$

where the second-to-last inequality holds because x and a can only have nonzero digits on disjoint sets of precisions, so their sum is enough to compute both of them. Taking the limit superior, the effective packing dimension of any point is no more than 1, so $\dim_{\text{P}} E = 1$.

Since E was defined to be a collection of packing dimension 1 subsets of lines with slope a ,

$$\{(x, ax) \in \mathbb{R}^2 : x \in \mathbb{R}, a \in S\} \subseteq L_1^{\text{P}}(E).$$

Let A be a packing oracle for $L_1^{\text{P}}(E)$. Let $a \in S$ be such that, removing the intervals $[r_{2i}, r_{2i+1} - 1]$ from its binary expansion and letting \tilde{a} denote the remaining string, \tilde{a} is random relative to A . Finally, let x be random relative to (a, A) . Let $\varepsilon > 0$ be given. By the condition on a , for sufficiently large i , adding an oracle D makes $[r_{2i}, r_{2i+1}]$ a teal interval with average loss less than ε , hence

$$\begin{aligned} K_{r_{2i+1}}^A(x, ax) &\geq K_{r_{2i+1}, r_{2i}}^A(x, ax) - O(\log r_{2i+1}) \\ &\geq K_{r_{2i+1}, r_{2i}}^{A, D}(x, ax) - O(\log r_{2i+1}) \\ &\geq K_{r_{2i+1}, r_{2i}}^A(x, a) - \varepsilon(r_{2i+1} - r_{2i}) - O(\log r_{2i+1}) \\ &\geq K_{r_{2i+1}, r_{2i}}^A(x, a) - \varepsilon r_{2i+1} - o(r_{2i+1}) \\ &\geq K_{r_{2i+1}}^A(x, a) - \varepsilon r_{2i+1} - o(r_{2i+1}) \\ &\geq K_{r_{2i+1}}^A(x) + K_{r_{2i+1}}^A(a) - \varepsilon r_{2i+1} - o(r_{2i+1}) \\ &\geq r_{2i+1} + r_{2i+1} - \varepsilon r_{2i+1} - o(r_{2i+1}) \\ &\geq 2r_{2i+1} - 2\varepsilon r_{2i+1}. \end{aligned}$$

Taking the limit superior and letting ε go to zero shows that $\dim_{\text{P}}(L_1^{\text{P}}(E)) = 2$.

These examples illustrate the lack of nontrivial bounds on the increase of Hausdorff and packing dimension for 1-packing extensions in the plane. More generally, for $s > 0$ and $E \subseteq \mathbb{R}^2$ we could ask for upper bounds on

$$\dim_H(L_s^{\text{P}}(E)) - \dim_H(E) \quad \text{and} \quad \dim_{\text{P}}(L_s^{\text{P}}(E)) - \dim_{\text{P}}(E).$$

Slightly modifying the above examples shows that the upper bound is at least 1 in both cases, but would not be surprised by better examples.⁷ We note a related problem, dimension estimates for the exceptional sets of orthogonal directions. For Hausdorff dimension in the plane, this was solved in [21], and is indeed closely connected to the Furstenberg set problem

⁷We consider the extreme case of “two-point” extensions in the next section.

and thus to line segment extension. Analogous to the worse behavior of s -packing extensions, attempting to bound the packing dimension of exceptional sets or modifying the definition of exceptional sets using packing dimension results in a rather different problem; see [9], [4] and [18].

5. EXTENSION IN HIGHER DIMENSIONS AND RELATED PROBLEMS

Given a point z on a line, there are two degrees of freedom in choosing collinear points x, y such that $x + t(y - x) = z$ for some $t \in \mathbb{R}$. To prove Proposition 1.5, we choose x and y such that (along with another condition) x 's first coordinate encodes the largest possible quantity of information about y . We justify the possibility of such an encoding as a lemma.

Lemma 5.1. *For all $y \in \mathbb{R}^n$, $A \subseteq \{0, 1\}^*$, and $\varepsilon > 0$, there exists a dense set of points $x \in \mathbb{R}$ such that, for all sufficiently large r (depending on x),*

$$K_r^A(y|x) \leq \max \{K_r^A(y) - (1 - \varepsilon)r, \varepsilon r\}. \quad (21)$$

The idea of the proof is simple; build the point $x \in \mathbb{R}$ such that successive segments of its binary expansion are strings that aid in the computation of successive segments of y .

Proof. Given a rational $0 < \delta < 1$, we build a point $x_\delta \in \mathbb{R}$ as follows. For each $i \in \mathbb{N}$, let $[r_i, r_{i+1}]$ be an interval of length $\lceil (1 + \delta)^i \rceil$, where $r_0 = 1$. Let σ_i denote a string testifying to $K_{r_i, r_{i+1}}^A(y)$ and let $x \in [0, 1]$ be the real number with binary expansion $0.\sigma_0\sigma_1\dots$

Let $\pi = \pi_1\pi_2\pi_3$, where $U^A(\pi_2)$ is a finite list of positive integer lengths l_0, \dots, l_m . Given a side information string τ , define an oracle Turing machine M that computes $M^A(\pi|\tau)$ as follows. The machine M first calculates $U^A(\tau) = q \in \mathbb{Q}$ and determines the successive strings q_0, \dots, q_j of lengths l_0, \dots, l_j formed from the binary digits of q . It then iteratively computes $U^A(q_1) = p_1$, $U^A(q_2, p_1) = p_2$, \dots , $U^A(q_j, p_{j-1}) = p_j$ and returns $M^A(\pi|\tau) = U^A(\pi_1, p_j)$ as the output. Let c_M be a constant for this machine.

Now let $r \in \mathbb{N}$ be sufficiently large and let $t = r_k$ be the lesser of (1) the largest precision r_i as defined above such that $K_{r_i}^A(y) \leq r$ and (2) the smallest precision r_i such that $r_i \geq r$. (This r can be assumed to be large enough that $K_{r_1}^A(y) \leq r$.) If there is no largest such r_i in (1), we consider this term to be ∞ and default to the r_i given by (2). As $\{r_i\}_{i \in \mathbb{N}}$ is an increasing sequence tending to infinity, such an r_i clearly exists.

Let π_1 testify to $K_{r, t}^A(y)$, let π_2 testify to $K^A(K_{r_0}^A(y), K_{r_1, r_0}^A(y), \dots, K_{r_k, r_{k-1}}^A(y))$, and let τ testify to $K_r^A(x_\delta)$. It is easy to check that, on these inputs, M outputs a precision r estimate of y ; from the definition of x , the q_i are precisely the strings that give the additional information in y from precision r_i to precision r_{i+1} . Hence

$$\begin{aligned} K_r^A(y|x_\delta) &\leq K_{r, t}^A(y) + K^A(K_{r_0}^A(y), K_{r_1, r_0}^A(y), \dots, K_{r_k, r_{k-1}}^A(y)) + c_M \\ &\leq K_{r, t}^A(y) + O(k \log r), \end{aligned}$$

where the second inequality follows from the fact that each complexity in the list is some integer between 0 and r . Since $k \leq O(\log r)$ by the definition of t ,

$$K_r^A(y|x_\delta) \leq K_{r,t}^A(y) + O((\log r)^2).$$

Now consider the two cases for the choice of t . If $t > r$, then $K_{r,t}^A(y) \leq O(\log r)$ and for sufficiently large r

$$K_r^A(y|x_\delta) \leq O((\log r)^2) < 3n\delta r. \quad (22)$$

For the case that $t < r$, let s be the smallest integer precision such that $K_s^A(y) \geq r$. Then $t \leq s$, so

$$\begin{aligned} K_r^A(y|x_\delta) &\leq K_{r,s}^A(y) + K_{s,t}^A(y) + O((\log r)^2) \\ &\leq K_r^A(y) - K_s^A(y) + K_{s,t}^A(y) + O((\log r)^2) \\ &\leq K_r^A(y) - r + K_{s,t}^A(y) + O((\log r)^2) \\ &\leq K_r^A(y) - r + n(s - t) + O((\log r)^2) \end{aligned}$$

by the choice of t and s , $s < r_{k+1}$. Hence,

$$\begin{aligned} K_r^A(y|x_\delta) &\leq K_r^A(y) - r + n(r_{k+1} - r_k) + O((\log r)^2) \\ &\leq K_r^A(y) - r + 2n\delta r_{k+1} + O((\log r)^2) \\ &\leq K_r^A(y) - r + 2n\delta r + O((\log r)^2). \end{aligned}$$

So, for all r sufficiently large that $n\delta r$ dominates the $O((\log r)^2)$ term, we have

$$K_r^A(y|x_\delta) \leq K_r^A(y) - r + 3n\delta r. \quad (23)$$

Picking $\delta < \frac{\varepsilon}{3n}$ and combining (22) and (23) gives the existence of one x satisfying (21). Appending the digits of x to any dyadic number (which are dense) gives the same property for sufficiently large r , completing the proof. \square

For simplicity and considering how it is used below, we stated the lemma for $x \in \mathbb{R}$, but an almost identical proof allows one to encode information about $y \in \mathbb{R}^n$ within $x \in \mathbb{R}^m$.

Proposition 1.5, Restated. *If $E \subseteq \mathbb{R}^n$ is a union of line segments, then*

$$\dim_{\mathbb{H}} \mathbf{L}(E) \leq \dim_{\mathbb{H}} E + \dim_{\mathbb{P}} E - 1 \quad \text{and} \quad \dim_{\mathbb{P}} \mathbf{L}(E) \leq 2 \dim_{\mathbb{P}} E - 1.$$

Proof. Let $A \subseteq \{0, 1\}^*$ be both a Hausdorff oracle and a packing oracle for E and let $\varepsilon > 0$. Given a point $z \in \mathbb{R}^n$, we construct a machine M that operates as follows. Take any $y \in \mathbb{R}^n$ such that $[2^c y_1] = [2^c z_1]$ for some $c \in \mathbb{N}$, where $[\cdot]$ denotes the fractional part of a real number; that is, the first coordinates of y and z agree from the c th binary digit onwards. Next, let $x \in \mathbb{R}^n$ with $x_1 \neq y_1$ and let

$$t = \frac{z_1 - x_1}{y_1 - x_1}.$$

We form a program $\pi = \pi_1 \pi_2 \pi_3 \pi_4$, where $U^A(\pi_1) = p$ is a precision $r + s$ approximation of x , $U^A(\pi_2|\pi_1) = q$ is a precision $r + s$ approximation of y , and $U^A(\pi_3|\pi_1, \pi_2, \pi_4) = u$ is a precision $r + s$ approximation of t , where $U^A(\pi_4) = [2^c z_1]$ and $s \in \mathbb{N}$ is large enough that $p + u(q - p) \in B(z, 2^{-r})$. (Note that s depends only on $|z_1 - x_1|$ and $|z_1 - y_1|$. In particular,

it does not depend on r .) With this data, $M^A(\pi)$ computes $p + u(q - p)$ by computing p , q , and u as just described.

Now let $z \in \ell_I \subseteq F$ and let $y \in I$ be a point such that, up to a permutation of the axes, $[2^c y_1] = [2^c z_1]$ for some $c \in \mathbb{N}$. Next, let $x \in I$ be a point with $x_1 \neq y_1$ such that x_1 assists in the computation of y as in Lemma 5.1, i.e., such that

$$K_r^A(y | x_1) \leq \max \{K_r^A(y) - (1 - \varepsilon)r, \varepsilon r\}$$

for all sufficiently large r . Up to a loss of $O(\log r)$, the same inequality holds with x in place of x_1 , the other $n - 1$ coordinates of x being ignored in the computation of y . Finally, let t be as before, and let π_1, π_2, π_3 , and π_4 witness $K_{r+s}^A(x)$, $K_{r+s}^A(y|x)$, $K_{r+s}^A(t|x, y, \lceil 2^c z_1 \rceil)$, and $K^A(\lceil 2^c z_1 \rceil)$, respectively. Since t is computable from x , y , and $\lceil 2^c z_1 \rceil$, we have

$$K_{r+s}^A(t|x, y, \lceil 2^c z_1 \rceil) = o(r + s) = o(r).$$

Hence, by the design of M , the choice of x , and symmetry of information, the following holds for all large $r \in \mathbb{N}$:

$$\begin{aligned} K_r^A(z) &\leq |\pi_1 \pi_2 \pi_3 \pi_4| + c_M \\ &\leq K_{r+s}^A(x) + K_{r+s}^A(y|x) + K_{r+s}^A(t|x, y, \lceil 2^c z_1 \rceil) + K^A(\lceil 2^c z_1 \rceil) + c_M \\ &\leq K_{r+s}^A(x) + K_{r+s}^A(y|x) + o(r) \\ &\leq K_r^A(x) + K_r^A(y|x) + 2sn + o(r) \\ &\leq K_r^A(x) + \max \{K_r^A(y) - (1 - \varepsilon)r, \varepsilon r\} + o(r). \end{aligned}$$

Dividing through by r and taking the limit inferior as $r \rightarrow \infty$ gives

$$\begin{aligned} \dim^A(z) &\leq \dim^A(x) + \max \{ \text{Dim}^A(y) - (1 - \varepsilon), \varepsilon \} \\ &\leq \dim_H E + \dim_P E - (1 - \varepsilon), \end{aligned}$$

where we have chosen the first alternative in the maximum because $\dim_P E \geq 1$. Taking the supremum over all $z \in F$ gives the first inequality in (2) modulo an ε , which we let decrease to 0. To obtain the second inequality in (2), we simply take the limit superior instead of the limit inferior. \square

On some level, the argument is morally similar to that of [6] Theorem 6, which leverages the dimension inequalities for product sets. Their Kakeya set estimate

$$\dim_H K + \dim_P K \geq n + 1$$

improves on Corollary 1.8 by +1, but this is to be expected, as they prove their estimate directly rather than by way of line segment extension.

By taking the oracle A in the proof of Proposition 1.5 to encode the set $D \subseteq \mathbb{P}^{n-1}$ of directions of the segments in \mathcal{I} and then computing z using x , $\frac{x-y}{|x-y|}$ and t , one can modify the conclusions of Proposition 1.5 to

$$\begin{aligned} \dim_H \mathbf{L}(E) &\leq \min \{ \dim_H E + \dim_P D, \dim_H D + \dim_P E \} \quad \text{and} \\ \dim_P \mathbf{L}(E) &\leq \dim_P E + \dim_P D. \end{aligned}$$

As suggested, instead of bounding the complexity of z in terms of the complexities of x , y , and t , one bounds this in terms of the complexities of x , $\frac{x-y}{|x-y|}$ and t , although this renders Lemma 5.1 inapplicable as stated—hence the disappearance of the -1 terms. Bounding the dimension of D in terms of the dimension of E is the generalized Keakeya problem, so in practice these inequalities are no more useful than those in (2). Playing with the choices of x and y in the proof likewise allows one to derive similar inequalities that, in the absence of specific information about E , do not lead to more profound information about $\mathbf{L}(E)$.

In this vein, at the cost of control over the base point x and the scalar t , one can turn the inequality for line segment extensions into an inequality for “two-point extensions” (recall Definition 1.1).

Proposition 5.2. *If $E \subseteq \mathbb{R}^n$, then*

$$\dim_{\mathbf{H}} L_0(E) \leq \dim_{\mathbf{H}} E + \dim_{\mathbf{P}} E + 1 \quad \text{and} \quad \dim_{\mathbf{P}} L_0(E) \leq 2 \dim_{\mathbf{P}} E + 1. \quad (24)$$

Proof. The proof is the same in spirit as that of Proposition 1.5, but forgoing the encodings greatly simplifies the matter. Let $z \in L_0(E)$, so that $z = x + ty$ for some $x, y \in E$ and some $t \in \mathbb{R}$. Taking the limit inferior of both sides of

$$K_r^A(z) \leq K_r^A(x) + K_r^A(y) + K_r^A(t) + o(r) \leq K_r^A(x) + K_r^A(y) + r + o(r)$$

gives the first inequality and taking the limit superior gives the second. \square

Example 5.1 (Sharpness of Proposition 5.2). The inequalities in (24) are sharp in \mathbb{R}^2 . Let $C_\alpha \subset [0, 1]$ be the middle- α Cantor set, $\alpha \in (\frac{1}{2}, 1)$, so that

$$s := \dim_{\mathbf{H}} C_\alpha = \dim_{\mathbf{P}} C_\alpha = \frac{\log 2}{\log \frac{1}{2}(1 - 2\alpha)} \in (0, \frac{1}{2}).$$

Then $\mathcal{H}^{2s}(C_\alpha \times C_\alpha) > 0$, so by Marstrand’s projection theorem, $\dim_{\mathbf{H}}(tC_\alpha - C_\alpha) = 2s$ for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$. In fact, by Proposition 1.3 of [19], it is also the case that $\dim_{\mathbf{P}}(tC_\alpha - C_\alpha) = 2s$ for a.e. $t \in \mathbb{R}$, so we fix a t satisfying both these equations and let

$$E := (\{0\} \times C_\alpha) \cup (\{1\} \times tC_\alpha).$$

Then the set of slopes of lines in $L_0(E)$ is simply $(tC_\alpha - C_\alpha) \cup \{\infty\}$, and in particular the set $D \subset \mathbb{P}^1$ of directions in which $L_0(E)$ contains a line has both Hausdorff and packing dimension $2s$. By the generalized Keakeya conjecture in the plane (cf. Conjecture 2 and Theorem 2.2 above),

$$\begin{aligned} \dim_{\mathbf{P}} L_0(E) &\geq \dim_{\mathbf{H}} L_0(E) \geq \dim_{\mathbf{H}} D + 1 = 2s + 1 \\ &= \dim_{\mathbf{H}} E + \dim_{\mathbf{P}} E + 1 = 2 \dim_{\mathbf{P}} E + 1. \end{aligned}$$

Hence, both inequalities in (24) hold with equality. Taking an intersection of the exceptional sets in [19] Proposition 1.3 shows that our argument similarly works when $\alpha = \frac{1}{2}$ (although $tC_\alpha - C_\alpha$ may have zero 1-dimensional packing *measure*), yielding the extreme case $\dim_{\mathbf{H}} E = \dim_{\mathbf{P}} E = \frac{1}{2}$. In the other extreme, any two-point set poses a sharp example for $\dim_{\mathbf{H}} E = \dim_{\mathbf{P}} E = 0$.

Interestingly, $\dim_p E$ cannot be replaced with $\dim_H E$ in the first inequality of Proposition 5.2. In fact, no nontrivial inequality bounding the Hausdorff dimension of the two-point extension of a set is possible solely in terms of the Hausdorff dimension of the original set.

Example 5.2 (Failure of $\dim_H L_0(E) \leq 2 \dim_H E + 1$). Through a simple argument (in the spirit of [3] Example 7.8, [7], or the construction in [1] Theorem 1.4) we observe that the analogous bound for the Hausdorff dimension of two-point extensions severely fails. Let $E = \{x \in \mathbb{R}^n : \dim(x) = 0\}$. It is immediate by the point-to-set principle that this set has Hausdorff dimension 0 (although this also follows from a simpler counting argument; see [13] Theorem 3.3.1). Its two-point extension is

$$\begin{aligned} L_0(E) &= \{x + t(y - x) \in \mathbb{R}^n : \dim(x) = 0, \dim(y) = 0, t \in \mathbb{R}\} \\ &\supseteq \{2y - x \in \mathbb{R}^n : \dim(x) = 0, \dim(y) = 0\}, \end{aligned}$$

and since scaling a vector by a nonzero rational does not change its pointwise dimension, it follows that

$$L_0(E) \supseteq \{x + y \in \mathbb{R}^n : \dim(x) = 0, \dim(y) = 0\}.$$

Now observe that any $z \in \mathbb{R}^n$ can be written as the sum of two Hausdorff dimension-0 points. We illustrate for $z \in [0, 1]$ with binary representation $0.z_1z_2z_3\dots$. Let $x = 0.x_1x_2x_3\dots$, where $x_i = z_i$ when there exists even $j \in \mathbb{N}$ such that $j! \leq i < (j+1)!$ and $x_i = 0$ otherwise. If $y = z - x$, then x and y both consist of alternating blocks of zeros which rapidly increase in length; hence, they both have effective Hausdorff dimension 0. Repeating the same construction in each coordinate gives the result in \mathbb{R}^n . Consequently, $L_0(E) = \mathbb{R}^n$, so

$$\dim_H L_0(E) = n \quad \text{but} \quad \dim_H E = 0.$$

This also poses a counterexample to any sort of “reverse continuum Beck’s theorem”; see [20] Corollary 1.5.

ACKNOWLEDGEMENTS

The authors thank Tamás Keleti, Don Stull, Raymond Tana, and Josh Zahl for helpful comments on an earlier version of this paper.

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