The Continuous-Time Weighted-Median Opinion Dynamics*

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***Abstract*—Opinion dynamics models are important in understanding and predicting opinion formation processes within social groups. Although the weighted-averaging opinion-update mechanism is widely adopted as the micro-foundation of opinion dynamics, it bears a non-negligibly unrealistic implication: opinion attractiveness increases with opinion distance. Recently, the weighted-median mechanism has been proposed as a new microscopic mechanism of opinion exchange. Numerous advancements have been achieved regarding this new micro-foundation, from theoretical analysis to empirical validation, in a discrete-time asynchronous setup. However, the original discrete-time weighted-median model does not allow for "compromise behavior" in opinion exchanges, i.e., no intermediate opinions are created between disagreeing agents. To resolve this problem, this paper propose a novel continuous-time weighted-median opinion dynamics model, in which agents' opinions move towards the weighted-medians of their out-neighbors' opinions. It turns out that the proof methods for the original discrete-time asynchronous model are no longer applicable to the analysis of the continuous-time model. In this paper, we first establish the existence and uniqueness of the solution to the continuous-time weighted-median opinion dynamics by showing that the weighted-median mapping is contractive on any graph. We also characterize the set of all the equilibria. Then, by leveraging a new LaSalle invariance principle argument, we prove the convergence of the continuous-time weighted-median model for any initial condition and derive a necessary and sufficient condition for the convergence to consensus.**

***Index Terms*—Social Networks, Opinion Dynamics, Weighted Median, Consensus**

I. Introduction

A. Background and Motivations

Opinion dynamics focus on how opinions form, evolve, and stabilize within a group through interactions among individuals. Opinion dynamics also examine the roles of social structure, decision-making processes, and individual interactions, revealing complex patterns of how opinions evolve in these environments.

As the foundation of dynamic analysis, diverse mathematical models are proposed to understand how individual opinions lead to collective outcomes. The French-DeGroot model is a cornerstone in the study of opinion dynamic models and sets up a framework that incorporates a weighted-averaging mechanism into the opinion updating rules [1], [2]. Based on the weighted-averaging mechanism, numerous discrete- and continuous-time models have been developed and thoroughly examined [3]. However, the weighted-averaging mechanism is criticized for its unrealistic implication regarding the unreasonable attractiveness of distant opinions, and is challenged as a wildly-adopted foundation [4].

Recently, a new micro-foundation of opinion dynamics, i.e. the weighted-median mechanism, has been developed from theoretical formulations to empirical validations, offering much more realistic insights compared to traditional weighted-averaging methods [4]. The weighted-median mechanism updates individual opinions based on the weighted median of neighbours' opinions, and is supported by the psychological theory of cognitive dissonance [5]. This method has shown through empirical validation and numerical comparisons to more accurately predict opinion shifts and capture complex real-world opinion dynamics, such as the impact of group size and network structure on consensus formation. Recent research also analyses the dynamical behaviour of discrete-time weighted-median opinion dynamics rigorously including fully characterizing the equilibria set and establishing the almost-sure convergence for any initial condition [6]. Moreover, a necessary and sufficient condition for the almost-sure convergence to consensus is given, as well as a sufficient condition for almost-sure dissensus.

The original weighted-median opinion dynamics models are implemented in discrete-time settings, where each individual selects an opinion from either his or his neighbours' opinions at each step. This selection mechanism ensures that no new opinions are generated during the evolution of opinions, facilitating the analysis of opinion shifts within an unchanging set

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of opinion options. However, in the real world, individuals often compromise in opinions, i.e. individuals adjust their opinions towards opinions that don't fully align with any single one's initial opinion but reflect a blend. Thus, a new opinion dynamics model is motivated to achieve the compromise of opinions under the weighted-median mechanism.

B. Literature Review

The modelling of opinion behaviour has been a prominent research topic for approximately 70 years, attracting interest from various fields including sociology, economics, mathematics, physics and control theory. In this section, we will provide a brief review of classic opinion dynamics models as well as their variants. For more comprehensive and advanced surveys on opinion dynamic models, please refer to [3], [7] and [8].

By the domain of opinion value, opinion dynamics models can be classified into discrete opinions models, where opinions take only finitely many values like $\{0, 1\}$, and continuous opinion models, where opinions take values in a closed real interval like [0, 1] [9].

The threshold model, originally proposed by Granovetter [10] and Schelling [11], serves as a classic mechanism for understanding discrete opinion dynamics. It operates on a binary system where opinions are represented as 0 and 1, corresponding to "no" and "yes" in practical scenarios. The threshold model is based on the idea that individuals have a personal threshold for adopting a different opinion, which is influenced by the proportion of others in their neighbours who have already adopted that opinion. The ultimate or "equilibrium" number adopted by each opinion can be characterized by a frequency distribution of thresholds [10].

The voter model, another classic discrete opinion dynamics model, is proposed by Clifford and Sudbury to analyze species' competitions for territory [12] and was further analyzed by Holley and Liggett [13]. Clifford and Sudbury described a stochastic 'swapping process' in which two cells exchange their positions. Following the idea that individuals adopt the state of one of their neighbours, the voter model updates one's opinion in two steps: first, selecting a neighbour; and second, adopting the selected neighbour's opinion. Numerous variations of the voter model have been proposed, modifying the rules for neighbour selection and opinion switching: The q-voter model enhances the classic voter model by incorporating a parameter q, which represents the number of neighbours a voter must consult before potentially changing their opinion [14]; The noisy voter model introduces spontaneous state transitions, or "noise" to reflect a more realistic dynamic where changes in state are not solely due to peer influence [15]. Granovsky and Madras provide exact formulas for certain crucial time-dependent and equilibrium functionals by the use of duality. The impact of social networks on collective patterns has also been thoroughly explored. Sood and Redner offer significant insights into how the voter model behaves on heterogeneous graphs, and relate the mean time to reach consensus with the moment of the degree distribution [16]. For a more comprehensive tutorial on voter model dynamics, we refer to [17].

The threshold model and the voter model discussed above, however, do not take into account the degree of difference between opinions. That is, opinions that differ significantly exert the same influence on one's opinion switching as those that differ slightly, implying that opinion values are not ordered. By contrast, the weighted-median dynamics consider ordered discrete opinions, demonstrating that social influence depends on the weighted-median opinions of one's neighbour [4]. Mei et al. conduct a detailed analysis of the weighted-median opinion dynamics including characterizing the equilibria set and establishing the almost-sure convergence [6].

Although research into discrete opinion models is extensive, the inherent settings of these models prevent the generation of new opinions during the evolution, thereby limiting their ability to explain the phenomenon of compromise in opinions. On this point, continuous opinion dynamics models offer a more nuanced illustration of social influence.

The most fundamental mechanism to achieve compromise in opinions is the weighted-averaging mechanism originated from the French-DeGroot model. The French-DeGroot model proposes a process in which each individual repeatedly updates his/her opinion by averaging the opinions of his neighbours in a network [1], [2]. This model effectively describes how continuous interactions among group members lead to consensus, provided that the associated influence network satisfies certain requirements [18]. Its simplicity and explicit convergence endow it with border applications such as in coordination algorithms [19], [20]. However, as persistent disagreement is commonly observed in daily life, a great number of variants have been proposed and studied extensively to better model the diversity of opinions: The Friedkin-Johnsen (F-J) model introduces prejudice to explain the dissensus of opinions [21]; The Hegselmann-Krause (H-K) model assume that individuals only assign weights to opinions within certain distances from their own opinions [22], [23]; The Altafini model considered the presence of negative weights in the influence network [24].

Additionally, dynamics of other influence factors are incorporated into the weighted-averaging mechanism to suit sophisticated scenarios: [25] extends traditional opinion dynamics models to multidimensional settings, then the influence among different dimensions of opinions, which co-evolves as logic constraints, is discussed in [26]; For a general model, [27] considers four types of couplings of options and agents, then establish a nonlinear multioption opinion dynamics model covering a series of nonlinear consensus models. The behaviours of this general model include bifurcations of multistable agreement and disagreement, and is tractably analyzed by a few key parameters, offering both theoretical insights and practical methods for analyzing opinion dynamics within a unified framework. The weighted-averaging mechanism also advances social learning by averaging local Bayesian updates and beliefs of neighbours [28], [29]. Moreover, the impact of social networks structure on collect pattern is also deeply studied assuming the weighted-averaging mechanism [30]–

[32].

However, the weighted-averaging is criticized for having severely unrealistic implications. [4] exemplifies a simple but representative case where individual i is influenced by individuals j and k via the weighted-averaging mechanism, and rewrites the dynamics as

$$x_i(t+1) = x_i(t) + w_{ik}(x_k(t) - x_i(t)) + w_{ij}(x_j(t) - x_i(t)), \quad (1)$$

where $x_i(t)$ represents the opinion of i at time t and w_{ij} represents j's influence weight on i. Equation (1) indicates that a more divergent opinion unreasonably earns a higher "attractiveness" to drive one's opinion moving towards it, which results in a consensus under mild network connectivity conditions.

C. Contribution

As reviewed, the weighted-averaging mechanism is now considered inadequate as a foundation, while the original weighted-median dynamic discrete opinion model fails to account for the emergence of compromise in opinions. In this paper, we propose a continuous-time weighted-averaging model and conduct a rigorous analysis of its dynamics. The contributions of this paper are as follows:

First, we model the weighted-median mechanism within a continuous-time framework. Our proposed model extends the existing discrete opinion dynamics to a continuous opinion setting and captures the dynamics of compromise. Additionally, this dynamic is based on a more realistic foundation, preventing distant opinions from generating absurdly large attractiveness.

Second, we conduct a solid analysis on the solution to the proposed ordinary differential equation. We establish the existence and uniqueness of the solution to the dynamics by proving the weighted-median operator is non-expanding. We characterize the set of all the equilibria and prove their stability. Moreover, by refining the invariants sets, this paper obtains the convergence of the dynamics for any initial condition, and then derives a sufficient and necessary graph-theoretic condition for almost-sure convergence to consensus equilibria. Furthermore, a sufficient condition for almost-sure convergence to disagreement equilibria is provided.

D. Organization

The rest of the paper is organized as follows. Section II introduces some basic definitions and notions, as well as the formulation of the continuous-time weighted-median opinion dynamics. Section III contains all the theoretical analysis and proofs of the main results. Section IV concludes. Proofs of lemmas can be found in the appendices.

II. Basic Definitions And Model Setup

Let $\subseteq$ ($\subset$ resp.) be the symbols for subset (proper resp.). Denote by $\mathbb{N}$ the set of natural numbers, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{Z}$ ($\mathbb{Z}_+$ resp.) be the set of (positive resp.) integers. Let $\mathbb{1}_n$ ($\mathbb{0}_n$ resp.) be the n-dimension vector whose entries are all ones (zeros resp.).

Consider a group of n individuals labelled by $\mathcal{V} = \{1, \dots, n\}$. The interaction relationships among the group are described by a directed and weighted graph $\mathcal{G}(W)$, where $W = (w_{ij})_{n\times n}$ is the associated adjacency matrix. Here, $w_{ij} > 0$ means that there is a link from node i to node j with weight w_{ij}, or, equivalently, individual i assigns $w_{ij} > 0$ weight to individual j. Since nodes in $\mathcal{G}(W)$ represent individuals in opinion dynamics, we also call $\mathcal{G}(W)$ *influence matrix*. Denote by $\mathcal{N}_i$ the set of node i's out-neighbours, i.e., $\mathcal{N}_i = \{j \in \mathcal{V} | w_{ij} \neq 0\}$, which includes node i itself if $w_{ii} \neq 0$. Conventionally, an influence matrix W is assumed to be row-stochastic.

The formal definition of weighted median is given below.

Definition 1 (Weighted median): Given any n-tuple of real values $x = (x_1, \dots, x_n)$ and any n-tuple of non-negative weights $w = (w_1, \dots, w_n)$ with $\sum_{i=1}^n w_i = 1$, $x^* \in \{x_1, \dots, x_n\}$ is a weighted median of x associated with the weights w if x^* satisfies

$$\sum_{i:\, x_i < x^*} w_i \le 1/2, \quad \text{and} \quad \sum_{i:\, x_i > x^*} w_i \le 1/2.$$

For simplicity, we also say that x^* is a weighted median of x associated with w. The following lemma characterized some basic properties on the uniqueness of the weighted median [4].

Lemma 1 (Properties of weighted median): Given any n-tuple of real values $x = (x_1, \dots, x_n)$ and any n-tuple of non-negative weights $w = (w_1, \dots, w_n)$ with $\sum_{i=1}^n w_i = 1$,

1) the weighted median of x associated with w is unique if and only if there exists $x^* \in \{x_1, \dots, x_n\}$ such that
$$\sum_{i:\, x_i < x^*} w_i < \frac{1}{2}, \quad \sum_{i:\, x_i = x^*} w_i > 0, \quad \sum_{i:\, x_i > x^*} w_i < \frac{1}{2}.$$
Such an x^* is the unique weighted median;
2) the weighted medians of x associated with w are not unique if and only if there exists $z \in \{x_1, \dots, x_n\}$ such that $\sum_{i:\, x_i < z} w_i = \sum_{i:\, x_i \ge z} w_i = 1/2$. In this case, $\underline{x}^* \in \{x_1, \dots, x_n\}$ is the smallest weighted median if and only if
$$\sum_{i:\, x_i < \underline{x}^*} w_i < \frac{1}{2}, \quad \sum_{i:\, x_i = \underline{x}^*} w_i > 0, \quad \sum_{i:\, x_i > \underline{x}^*} w_i = \frac{1}{2},$$
and $\overline{x}^* \in \{x_1, \dots, x_n\}$ is the largest weighted median if and only if
$$\sum_{i:\, x_i < \overline{x}^*} w_i = \frac{1}{2}, \quad \sum_{i:\, x_i = \overline{x}^*} w_i > 0, \quad \sum_{i:\, x_i > \overline{x}^*} w_i < \frac{1}{2}.$$
Moreover, for any $\hat{x} \in \{x_1, \dots, x_n\}$ such that $\underline{x}^* < \hat{x} < \overline{x}^*$, $\hat{x}$ is also a weighted median and $\sum_{i: x_i = \hat{x}} w_i = 0$.

Compared to the straightforward picking of opinions at discrete time steps, the continuous-time variant aims to entail ongoing effects of the weighted-median mechanism. Consequently, the dynamics are constructed on the assumption that individuals' opinions gradually shifts towards the weighted median of their out-neighbours' opinions. The continuous-time weighted-median opinion dynamics is formally defined as follows:

Definition 2: Given any directed and weighted influence network $\mathcal{G}(W)$ with n individuals, denote by $x(t) \in \mathbb{R}^n$ the vector of the individuals' opinions at time t. Starting from any initial condition $x_0 \in \mathbb{R}^n$, the continuous-time weighted-median opinion dynamics is given as

$$\dot{x}(t) = f\big(x(t)\big) = \text{Med}\big(x(t); W\big) - x(t), \tag{2}$$

where $\text{Med}_i(x(t); W)$ is the weighted median of $x(t)$ associated with the weights given by the i-th row of W, i.e., $(w_{i1}, w_{i2}, \dots, w_{in})$. If the weighted-median is not unique, then let $\text{Med}_i\big(x(t); W\big)$ be the weighted median that is the closest to $x_i(t)$.

Note that compared with the discrete-time model, the continuous-time weighted-median model differs in two aspects:

- The updating of opinions has no time steps and is asynchronous, which means the convergence cannot be established by finding update sequences.
- New opinions can be generated during the dynamic.

III. Dynamical behavior of the continuous-time weighted-median opinion dynamics

In this section, we establish the existence and uniqueness of system (2)'s solution, characterize the set of all the equilibria of (2), and then prove its convergence. A necessary and sufficient graph theoretic condition for convergence to consensus, as well as a sufficient condition for convergence to disagreement equilibria, is also provided and proved.

A. Notations

Before proceeding to the rigorous analysis, we first introduce some notations frequently used in this section, see Table I.

TABLE I
NOTATIONS FREQUENTLY USED IN THIS PAPER

$\phi(t, x_0)$	the trajectory along the dynamics (2) starting from x_0 at time 0, i.e., $\phi(0, x_0) = x_0$.
∂B	the boundary of the set B.
$N_{\text{diff}}(x)$	the number of different values among the entries of any given vector x.
$\max^{(k)}(x)$	the k-th largest value among the $N_{\text{diff}}(x)$ different values of the entries of x ($k \in \{1, \dots, N_{\text{diff}}(x)\}$).
$\text{argmax}^{(k)}(x)$	the indices set $\{i \,\vert\, x_i = \max^{(k)}(x)\}$, for any given vector x, e.g., $\text{argmax}^{(1)}(x) = \text{argmax}_k\, x_k$ and $\text{argmax}^{(N_{\text{diff}}(x))}(x) = \text{argmin}_k\, x_k$. For convenience, let $\text{argmax}^{(0)}(x)$ be an empty set.
$\text{argmax}^{(\leq r)}(x)$	$\cup_{s \leq r} \text{argmax}^{(s)}(x)$. ($\text{argmax}^{(\leq 0)}(x)$ is empty.)
$\text{argmax}^{(\geq r)}(x)$	$\cup_{s \geq r} \text{argmax}^{(s)}(x)$. ($\text{argmax}^{(\geq N_{\text{diff}}(x)+1)}(x)$ is empty.)
$\text{argmax}^{(< r)}(x)$	$\cup_{s < r} \text{argmax}^{(s)}(x)$. ($\text{argmax}^{(<1)}(x)$ is empty.)
$\text{argmax}^{(> r)}(x)$	$\cup_{s > r} \text{argmax}^{(s)}(x)$. ($\text{argmax}^{(> N_{\text{diff}}(x))}(x)$ is empty.)

For any function $f: \mathbb{R} \to \mathbb{R}^n$, if f is right-differentiable, denote by $d^+ f(t)/dt$ the right derivative of f at t, i.e.,

$$\frac{d^+ f(t)}{dt} = \lim_{\Delta t \downarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

For any autonomous system $\dot{x} = f(x)$, denote the omega-limit set of any x by $\omega(x)$ and denote by $\phi(t, x)$ the solution at $t \geq 0$ starting from x, i.e., $\phi(0, x) = x$. Given a dynamical system $\dot{x} = f(x)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$, and a function $V: \mathbb{R}^n \to \mathbb{R}$, define $D^+ V(x)$ with respect to the dynamics $\dot{x} = f(x)$ as the lower Dini derivative:

$$D^+ V(x) = \liminf_{\Delta t \downarrow 0} \frac{V(x + \Delta t f(x)) - V(x)}{\Delta t}.$$

Note that $x + \Delta t f(x)$ is the first-order Euler approximation of $\phi(\Delta t, x)$, but the definition of $D^+ V(x)$ is NOT necessarily equivalent to the limit $\liminf_{\Delta t \downarrow 0} \Big(V\big(\phi(\Delta t, x)\big) - V(x) \Big) / \Delta t$.

When analysing the discrete-time weighted-median model, two network structures: the cohesive sets and the decisive links, are introduced to illustrate the dynamical behaviour of the weighted-median mechanism. These concepts are equally valuable in our analysis and we adopt the same definitions as presented in [6]. For a more detailed discussion and proofs of lemmas, we refer to [6].

Definition 3 (Cohesive set and maximal cohesive set): Given an influence network $\mathcal{G}(W)$ with node set V, a cohesive set $\mathcal{M} \subset \mathcal{V}$ is a subset of nodes that satisfies $\sum_{j \in \mathcal{M}} w_{ij} \geq 1/2$ for any $i \in \mathcal{M}$. A cohesive set $\mathcal{M}$ is a maximal cohesive set if there does not exist $i \in \mathcal{V} \setminus \mathcal{M}$ such that $\sum_{j \in \mathcal{M}} w_{ij} > 1/2$.

Given the concept of cohesive sets, we further introduce a process called *cohesive expansion*.

Definition 4 (Cohesive expansion): Given an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$ and a subset of nodes $\mathcal{M} \subset \mathcal{V}$, the cohesive expansion of $\mathcal{M}$, denoted by $\text{Expansion}(\mathcal{M})$, is the subset of $\mathcal{V}$ constructed via the following iteration algorithm:

1) Let $\mathcal{M}_0 = \mathcal{M}$;
2) For $k = 0, 1, 2, \dots$, if there exists $i \in \mathcal{V} \setminus \mathcal{M}_k$ such that $\sum_{j \in \mathcal{M}_k} w_{ij} > 1/2$, then let $\mathcal{M}_{k+1} = \mathcal{M}_k \cup \{i\}$;
3) Terminate the iteration at step k as long as there does not exist any $i \in \mathcal{V} \setminus \mathcal{M}_k$ such that $\sum_{j \in \mathcal{M}_k} w_{ij} > 1/2$, and let $\text{Expansion}(\mathcal{M}) = \mathcal{M}_k$.

The following lemma presents some important properties of cohesive sets and cohesive expansions.

Lemma 2 (Properties of cohesive sets/expansions): Given an influence network $\mathcal{G}(W)$ with the node set $\mathcal{V}$, the following statements hold:

1) If $\mathcal{M}_1$, $\mathcal{M}_2 \subseteq \mathcal{V}$ are both cohesive sets, then $\mathcal{M}_1 \cup \mathcal{M}_2$ is also a cohesive set;
2) For any $\mathcal{M} \subset \mathcal{V}$, the cohesive expansion of $\mathcal{M}$ is unique, i.e., independent of the order of node additions;
3) For any $\mathcal{M}, \tilde{\mathcal{M}} \subset \mathcal{V}$, if $\mathcal{M} \subset \tilde{\mathcal{M}}$, then $\text{Expansion}(\mathcal{M}) \subset \text{Expansion}(\tilde{\mathcal{M}})$;
4) For any $\mathcal{M}, \tilde{\mathcal{M}} \subset \mathcal{V}$, $\text{Expansion}(\mathcal{M}) \cup \text{Expansion}(\tilde{\mathcal{M}}) \subset \text{Expansion}(\mathcal{M} \cup \tilde{\mathcal{M}})$; and
5) If $\mathcal{M}$ is a cohesive set, then $\text{Expansion}(\mathcal{M})$ is also cohesive and is the smallest maximal cohesive set that includes $\mathcal{M}$, that is, for any maximal cohesive set $\hat{\mathcal{M}}$ such that $\mathcal{M} \subset \hat{\mathcal{M}}$, we have $\text{Expansion}(\mathcal{M}) \subset \hat{\mathcal{M}}$.

Below we present another useful lemma on cohesive sets, which is a straightforward consequence of Definition 3.

Lemma 3 (Cohesive partition): Given an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$ and a cohesive set $\mathcal{M} \subset \mathcal{V}$, if $\mathcal{M}$ is maximally cohesive and $\mathcal{M} \subset \mathcal{V}$, then $\mathcal{V} \backslash \mathcal{M}$ is also maximally cohesive; If $\mathcal{M}$ is not maximally cohesive, then either of the following two statements holds:

1) Expansion$(\mathcal{M}) = \mathcal{V}$;
2) Expansion$(\mathcal{M})$ and $\mathcal{V} \setminus$ Expansion$(\mathcal{M})$ are both non-empty and maximally cohesive.

Another important concept involved in called *decisive link*.

Definition 5 (Decisive and indecisive out-links): Given an influence network $\mathcal{G}(W)$ with the node set $\mathcal{V}$, define the out-neighbor set of each node i as $\mathcal{N}_i = \{j \in \mathcal{V} \,|\, w_{ij} \neq 0\}$. A link (i, j) is a decisive out-link of node i, if there exists a subset $\theta \subset \mathcal{N}_i$ such that the following three conditions hold: (1) $j \in \theta$; (2) $\sum_{k\in\theta} w_{ik} > 1/2$; (3) $\sum_{k\in\theta\setminus\{j\}} w_{ik} < 1/2$. Otherwise, the link (i, j) is an indecisive out-link of node i.

B. Useful Lemmas

Before analyzing the dynamical behavior of the continuous-time weighted-median opinion dynamics (2), we present and prove some lemmas that will be used in the proof of our main theorems. The following lemma states that, if the trajectory of an ODE system converges to a set of stable equilibria, then it converges to a unique equilibrium. Its proof is given in Appendix A.

Lemma 4 (Convergence to unique equilibrium): Consider an ODE system $\dot{x} = f(x)$ with domain $D \subseteq \mathbb{R}^n$. Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ and let χ be a set of equilibria of the system, i.e., $f(x) = 0$ for any $x \in \chi$. Suppose that any equilibrium $x \in \chi$ is stable. Given any $x_0 \in D$, if its ω-limit set satisfies $\omega(x_0) \subseteq \chi$, then the solution trajectory $\phi(t, x_0)$ converges to a unique $x^* \in \chi$, i.e., $\omega(x_0) = \{x^*\}$.

The following lemma is a slightly more conservative version of the original LaSalle invariance principle proposed as Theorem 2 in [33].

Lemma 5 (LaSalle invariance principle): Consider an autonomous system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz on $\mathbb{R}^n$. If $G \subseteq \mathbb{R}^n$ is a compact and positively invariant set of the system; and there exists a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ such that $D^+V(x)$ exists and is non-positive for any $x \in G$, then for any $x \in G$, the solution $\phi(t, x)$ converges to M as $t \to \infty$, where M is the largest invariant set in $E = \{x \in G \,|\, D^+V(x) = 0\}$.

By applying LaSalle invariance principle we usually obtain the convergence of the solution to some subset of the dynamical systems' domain. The lemma below helps to further refine the set that the solution converges to. The proof is given in Appendix B.

Lemma 6 (Condition for not being an omega limit point): Consider an ODE system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz and $G \subseteq \mathbb{R}^n$ is compact and positively invariant with respect to the system dynamics. For any given $x \in G$, if there exists an open set $B \subseteq G$ such that

1) $x \notin B \cup \partial B$;
2) there exists $T > 0$ such that $\phi(t, x) \in B$ for any $t \geq T$;
3) $B \cup \partial B$ is positively invariant,

then $x \notin \omega(y)$ for any $y \in G$.

Lemma 7: For any $x \in \mathbb{R}^n$, the function $V : \mathbb{R}^n \to \mathbb{R}$ defined by $V(x) = \max\{x_1, x_2, \dots, x_n\}$ is globally Lipschitz.

Proof of Lemma 7: For any $x, y \in \mathbb{R}^n$, $|V(x) - V(y)| = |\max_i x_i - \max_j y_j|$. Let $i^* \in \operatorname{argmax}_i x_i$ and $j^* \in \operatorname{argmax}_j y_j$ and, without loss of generality, assume that $\max_i x_i \geq \max_j y_j$. We have

$$0 \leq |V(x) - V(y)| = |x_{i^*} - y_{j^*}| = x_{i^*} - y_{j^*} \leq x_{i^*} - y_{i^*} \leq \|x - y\|_\infty.$$

Therefore, $V(x)$ is globally Lipschitz in x. ■

C. Main results

We first establish the existence and uniqueness of the solution to the continuous-time weighted-median opinion dynamics (2) starting from any initial condition.

Theorem 8 (Existence and uniqueness of solution): Given the influence matrix W, Med$(x; W)$, as a map for x, is non-expanding in the sense that, for any $x, y \in \mathbb{R}^n$, $\|\text{Med}(x; W) - \text{Med}(y; W)\|_\infty \leq \|x - y\|_\infty$. Therefore, the function f in equation (2) is globally Lipschitz in $\mathbb{R}^n$. As a result, for any initial condition $x_0 \in \mathbb{R}^n$, the solution $\phi(t, x_0)$ exists and is unique.

Proof of Theorem 8: We prove by contradiction. Suppose

$$\|\text{Med}(x; W) - \text{Med}(y; W)\|_\infty > \|x - y\|_\infty,$$

i.e., there exists $i \in \mathcal{V}$ such that

$$\big|\text{Med}_i(x; W) - \text{Med}_i(y; W)\big| > |x_k - y_k|$$

for any $k \in \mathcal{V}$. Apparently this implies that

$$|\text{Med}_i(x; W) - \text{Med}_i(y; W)| > \|x - y\|_\infty \geq 0.$$

Without loss of generality, assume that $\text{Med}_i(x; W) > \text{Med}_i(y; W)$. Let

$$\mathcal{X}_i = \big\{j \in \mathcal{N}_i \,\big|\, x_j \geq \text{Med}_i(x; W)\big\}, \text{ and}$$
$$\mathcal{Y}_i = \big\{j \in \mathcal{N}_i \,\big|\, y_j \leq \text{Med}_i(y; W)\big\}.$$

According to Definition 1,

$$\sum_{j\in\mathcal{X}_i} w_{ij} \geq 1/2, \quad \text{and} \sum_{j\in\mathcal{Y}_i} w_{ij} \geq 1/2.$$

Now we show that these two inequalities lead to contradictions.

We first consider the case when the strict inequality holds in at least one of the two inequalities above. Since $\sum_{j\in\mathcal{N}_i} w_{ij} = 1$ and $w_{ij} \geq 0$ for any j, we have that $\mathcal{X}_i \cap \mathcal{Y}_i$ must be non-empty, otherwise we obtain

$$1 < \sum_{j\in\mathcal{X}_i\cup\mathcal{Y}_i} w_{ij} \leq \sum_{j=1}^{n} w_{ij},$$

which cannot be true. For any $j^* \in \mathcal{X}_i \cap \mathcal{Y}_i$, since $x_j \geq \text{Med}_i(x; W)$ and $y_j \leq \text{Med}_i(y; W)$, we have

$$x_{j^*} - y_{j^*} \geq \text{Med}_i(x; W) - \text{Med}_i(y; W) > 0,$$

which contradicts the assumption that $\big|\text{Med}_i(x;W) - \text{Med}_i(y;W)\big| > |x_k - y_k|$ for any $k \in \mathcal{V}$.

Now we consider the case when $\sum_{j\in \mathcal{X}_i} w_{ij} = 1/2$ and $\sum_{j\in \mathcal{Y}_i} w_{ij} = 1/2$. We aim to show that these equations lead to $x_i - y_i \geq \text{Med}_i(x;W) - \text{Med}_i(y;W)$, which contradicts the pre-assumption $\text{Med}_i(x;W) - \text{Med}_i(y;W) > \|x-y\|_\infty$. Let us first investigate the implication of the equation $\sum_{j\in \mathcal{X}_i} w_{ij} = 1/2$. Since

$$\sum_{j\in \mathcal{X}_i} w_{ij} = \sum_{j:\, x_j \geq \text{Med}_i(x;W)} w_{ij} = \frac{1}{2},$$

according to Lemma 1, the weighted median of x associated with the i-th row of W are not unique, and, among all such weighted medians, $\text{Med}_i(x;W)$ is not the smallest. As a result, $\text{Med}_i(x;W)$ is either the largest weighted median or a weighted median between the largest and the smallest. According to Definition 1 and Lemma 1, if $\text{Med}_i(x;W)$ is the largest weighted median, then $x_i \geq \text{Med}_i(x;W)$; if $\text{Med}_i(x;W)$ is between the largest and the smallest, then $x_i = \text{Med}_i(x;W)$ must hold. In either case, we have $x_i \geq \text{Med}_i(x;W)$. Following the same line of argument, $\sum_{j\in \mathcal{Y}_i} w_{ij} = 1/2$ implies that $\text{Med}_i(y;W)$ is either the smallest weighted median of y associated with the i-th row of W, or a weighted median between the smallest and the largest. As a consequence, we have $y_i \leq \text{Med}_i(y;W)$. Therefore, we obtain that

$$x_i - y_i \geq \text{Med}_i(x;W) - \text{Med}_i(y;W),$$

which contradicts the inequality

$$\text{Med}_i(x;W) - \text{Med}_i(y;W) > \|x-y\|_\infty.$$

This leads to the conclusion that the map $\text{Med}(s;W)$ is non-expanding. As straightforward results, the function $f(x) = \text{Med}(x;W) - x$ is globally Lipschitz in $\mathbb{R}^n$ and the solution to the dynamics (2) exists and is unique. ■

The following theorem characterizes the set of all the equilibria and their stability.

Theorem 9 (Equilibria set and stability): Consider the weighted-median opinion dynamics (2) with the influence matrix W. The set of all the equilibria is given by the following set:

$$X_{\text{eq}} = \Big\{ x\in\mathbb{R}^n \,\Big|\, \text{argmax}^{(\leq r)}(x) \text{ is a maximal cohesive set on } \mathcal{G}(W), \text{ for any } r\in\big\{1,\dots,N_{\text{diff}}(x)\big\}\Big\}$$

Moreover, each equilibrium $x \in X_{\text{eq}}$ is stable.

The proof of Theorem 9 is partly based on the following lemma and its proof is provided in Appendix C.

Lemma 10: Consider the weighted-median opinion dynamics (2) with the influence matrix W. If $\mathcal{C}$ is a cohesive set on the influence network $\mathcal{G}(W)$, then, for any $x \in \mathbb{R}^n$, $\max_{k\in\mathcal{C}} \phi_k(t,x)$ is non-increasing and $\min_{k\in\mathcal{C}} \phi_k(t,x)$ is non-decreasing, for any $t \geq 0$.

As a consequence of Lemma 10, if $\mathcal{C}$ is a cohesive set in $\mathcal{G}(W)$, then, for any $x \in \mathbb{R}^n$, $\phi(t,x)$, is bounded in the compact set $[\min_{k\in\mathcal{C}} x_k, \max_{k\in\mathcal{C}} x_k]^n$ for any $t\geq 0$. With Lemma 10, Theorem 9 is proved as follows.

Proof of Theorem 9: We first point out that an $x^*\in\mathbb{R}^n$ is an equilibrium of (2) if and only if $x_i^* = \text{Med}_i(x^*;W)$. That is, the continuous-time weighted-median opinion dynamics (2) and the original weighted-median dynamics given in [6] in fact have the same equilibria set. If $N_{\text{diff}}(x^*) = 1$, then x^* is a consensus state and it is in X_{eq} since $\text{argmax}^{(1)}(x^*) = \mathcal{V}$ is always maximally cohesive. Suppose that $N_{\text{diff}}(x^*) > 1$. For any $r \in \big\{1,\dots,N_{\text{diff}}(x^*)-1\big\}$, by applying *Theorem 6* from [6] for some $y \in [\max^{(r)}(x^*), \max^{(r+1)}(x^*))$, we immediately obtain that $x^* \in X_{\text{eq}}$ if and only if x^* satisfies the conditions in *Theorem 6* from [6]. Namely, X_{eq} in Theorem 9 is exactly the set of all the equilibria of (2).

Now we proceed to prove that each equilibrium is stable. For any $x^* \in X_{\text{eq}}$, either $x_i^* = x_j^*$ for any $i,\, j\in\mathcal{V}$, or $N_{\text{diff}}(x^*) > 1$ and $\text{argmax}^{(\leq r)}(x^*)$ is maximally cohesive for any $r \leq N_{\text{diff}}(x^*)$. we refer to these two cases as Case 1 and Case 2 respectively.

In Case 1, for any $\epsilon > 0$, let $\delta = \epsilon/2$. Then, for any x_0 such that $\|x_0 - x^*\|_\infty < \delta$, we have $\max_i x_{0,i} < x_i^* + \delta$ and $\min_i x_{0,i} > x_i^* - \delta$. According to Lemma 10, we have

$$\begin{aligned}
&\max_{i\in\mathcal{V}} \phi_i(t,x_0) \leq \max_{i\in\mathcal{V}} x_{0,i} < x_i^* + \delta, \quad \text{and}\\
&\min_{i\in\mathcal{V}} \phi_i(t,x_0) \geq \min_{i\in\mathcal{V}} x_{0,i} > x_i^* - \delta.
\end{aligned}$$

Therefore, $\|\phi(t,x_0) - x^*\|_\infty < \epsilon$ for any $t \geq 0$, i.e., x^* is stable.

In Case 2, for any $\epsilon > 0$, let

$$\delta = \min\Big\{\frac{\epsilon}{2}, \frac{\max^{(1)}(x^*) - \max^{(2)}(x^*)}{2}, \dots, \frac{\max^{(N_{\text{diff}}(x^*)-1)}(x^*) - \max^{(N_{\text{diff}}(x^*))}(x^*)}{2}\Big\} > 0.$$

For any x_0 such that $\|x_0 - x^*\|_\infty < \delta$, we have

$$\begin{aligned}
\max_{i\in\mathcal{V}} x_{0,i} &< \max_{i\in\mathcal{V}} x_i^* + \delta,\\
\min_{i\in\mathcal{V}} x_{0,i} &> \min_{i\in\mathcal{V}} x_i^* - \delta,
\end{aligned}$$

and, for any $r < N_{\text{diff}}$,

$$\begin{aligned}
\min_{i\in\text{argmax}^{(\leq r)}(x^*)} x_{0,i} &> \max_{i\in\text{argmax}^{(r)}(x^*)} -\delta\\
&> \max_{i\in\text{argmax}^{(r+1)}(x^*)} +\delta\\
&> \max_{i\in\text{argmax}^{(>r)}(x^*)} x_{0,i}.
\end{aligned}$$

Moreover, since $\text{argmax}^{(\leq r)}(x^*)$ is a maximal cohesive set for any $r \in \{1,\dots,N_{\text{diff}}\}$, according to Lemma 10, $\min_{i\in\text{argmax}^{(\leq r)}(x^*)} \phi_i(t,x_0)$ is non-decreasing and $\max_{i\in\text{argmax}^{(\leq r)}(x^*)} \phi_i(t,x_0)$ is non-increasing. Therefore, the inequalities above hold not only for x_0, but also for $\phi(t,x_0)$ with any $t\geq 0$. That is, $\|\phi(t,x_0) - x^*\|_\infty < \delta < \epsilon$ for any $t \geq 0$. Combining Case 1 and Case 2, we conclude that any $x^* \in X_{\text{eq}}$ is stable. ■

In what follows, we establish the convergence of the continuous-time weighted-median opinion dynamics, and then provide a sufficient and necessary graph-theoretic condition

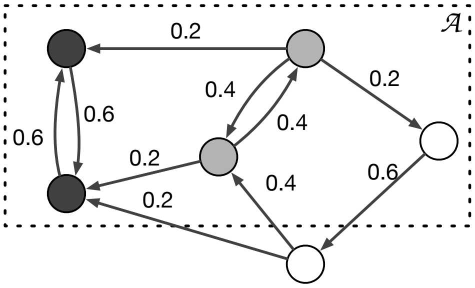

Fig. 1. Examples of maximal cohesive sets and the largest cohesive set contained in a nodes set $\mathcal{A}$ (the set of nodes in the dotted box). This figure visualized a weighted and directed influence network. For each node, the total weight of the out-going links is 1 and the self loops, whose weights can be thereby inferred, are omitted to avoid clutter. In this figure, the dark grey nodes form a maximal cohesive set but it is not the largest cohesive set contained in $\mathcal{A}$. The dark grey and light grey nodes together constitute the largest cohesive set in $\mathcal{A}$, which is however not a maximal cohesive set on the entire network.

for almost-sure convergence to consensus equilibria. A sufficient condition for almost-sure convergence to disagreement equilibria is also provided.

Theorem 11 (Convergence, consensus and disagreement): Consider the weighted-median opinion dynamics defined by equation (2) with the influence matrix W. The following statement hold:

1) For any $x_0 \in \mathbb{R}^n$, there exists $x^* \in X_{\text{eq}}$, depending on x_0, such that the solution $\phi(t,x_0)$ converges to x^* as $t \to \infty$;
2) If $\mathcal{V}$ is the only maximal cohesive set on $\mathcal{G}(W)$, then, for any $x_0 \in \mathbb{R}^n$, $\phi(t,x_0)$ converges to a consensus equilibrium.
3) If $\mathcal{V}$ is not the only maximal cohesive set on $\mathcal{G}(W)$, then there exists a subset $X_{0,\text{disagree}} \subseteq \mathbb{R}^n$ with non-zero measure, such that, for any $x_0 \in X_{0,\text{disagree}}$, $\phi(t,x_0)$ converges to a disagreement equilibrium;

In order to prove Theorem 11, we need to introduce the following definition and lemmas.

Lemma 12 (Convergence of a class of scalar ODEs): Consider a scalar ODE system $\dot{y}(t) = \beta(t) - y(t)$, where $\beta(t)$ is an integrable function. If there exists $y^* \in \mathbb{R}$ such that $\lim_{t\to+\infty} \beta(t) = y^*$, then $\lim_{t\to+\infty} y(t) = y^*$.
The proof of Lemma 12 is provided in Appendix D.

Definition 6 (Largest cohesive set): Given a set of nodes $\mathcal{A}$ on an influence network $\mathcal{G}(W)$, the largest cohesive set $\mathcal{C}$ contained in $\mathcal{A}$ is a cohesive subsets of $\mathcal{A}$ such that any subset $\tilde{\mathcal{C}}$ of $\mathcal{A}$ satisfying $\mathcal{C} \subset \tilde{\mathcal{C}}$ is not cohesive.

By Definition 6, the largest cohesive set contained in $\mathcal{A}$ is either empty (when none of the subsets of $\mathcal{A}$ is cohesive) or unique. The proof for uniqueness is straightforward as follows: Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ ($\mathcal{C}_1 \neq \mathcal{C}_2$) are both the largest cohesive sets contained in $\mathcal{A}$. Then, according to Lemma 2, $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{A}$ is also cohesive and therefore $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot be the largest cohesive set. The difference between "maximal cohesive set" and "the largest cohesive set" are illustrated in Fig. 1.

The largest cohesive set exhibits some important properties in the continuous-time weighted-median opinion dynamics (2), characterized by the following lemma. The proof is provided in Appendix E.

Lemma 13 (Dynamic properties of largest cohesive sets): Consider the continuous-time weighted-median opinion dynamics (2) on the influence network $\mathcal{G}(W)$. For any $x \in \mathbb{R}^n$, define $\mathcal{C}^{(1)}(x;W)$ as the largest cohesive set contained in $\text{argmax}^{(1)}(x)$. The following statements hold:

1) For any $j \notin \text{argmax}^{(1)}(x)$, $\phi_j(t,x) < \max^{(1)}(x)$ for any $t \geq 0$;
2) If $\mathcal{C}^{(1)}(x;W)$ is an empty set, then $\max^{(1)}\big(\phi(t,x)\big) < \max^{(1)}(x)$ for any $t > 0$;
3) If $\mathcal{C}^{(1)}(x;W)$ is non-empty, then, for any $i \in \mathcal{C}^{(1)}(x;W)$, $\phi_i(t,x) = \max^{(1)}(x)$ for any $t \geq 0$. Moreover, there exists $\Delta t > 0$ such that $\phi_j(\Delta t, x) < \max^{(1)}(x)$ for any $j \notin \mathcal{C}^{(1)}(x;W)$;
4) Denote by $\mathcal{E}^{(1)}(x;W)$ the cohesive expansion of $\mathcal{C}^{(1)}(x;W)$. If $\mathcal{C}^{(1)}(x;W)$ is non-empty, then $\lim_{t\to\infty} \phi_i(t,x) = \max^{(1)}(x)$ for any $i \in \mathcal{E}^{(1)}(x;W)$.

With all the preparations above, now we are ready to prove Theorem 11.

Proof of Theorem 11: The main part of this proof is the establishment of convergence. Regarding the convergence of system (2), the sketch of the proof is as follows: We first apply Lemma 5 and Lemma 6 to show that, for any given initial condition x_0, the solution $\phi(t,x_0)$ converges to the set of x's such that $\text{argmax}^{(1)}(x)$ forms a maximal cohesive set on the influence network $\mathcal{G}(W)$. Then we further refine the result by proving that, for any x in such a set but not in X_{eq}, x cannot be an omega limit point of x_0. That is, $\phi(t,x_0)$ converges to X_{eq}. Finally, according to the stability of any fixed point, which is already established in Theorem 9, we prove that, for any given $x_0 \in \mathbb{R}^n$, $\phi(t,x_0)$ must converge to a unique fixed point in X_{eq}.

First of all, in order to apply Lemma 5, we need to check that all the conditions in Lemma 5 are satisfied in system (2). According to Lemma 8, function $f(x) = \text{Med}(x;W) - x$ is globally Lipschitz. For any $x \in \mathbb{R}^n$, since

$$\begin{aligned}\text{Med}_i(x;W) &\leq x_i, \text{ for any } i \in \text{argmax}^{(1)}(x) \text{ and,}\\ \text{Med}_j(x;W) &\geq x_j, \text{ for any } j \in \text{argmax}^{(N_{\text{diff}}(x))}(x)\\ &\qquad\qquad\qquad = \text{argmin}_k\, x_k,\end{aligned}$$

we have that $\max_k \phi_k(t,x)$ is always non-increasing and $\min_k \phi_k(t,x)$ is always non-decreasing. Therefore, for any given $x_0 \in \mathbb{R}^n$, $G = \big[\min_k x_{0,k},\, \max_k x_{0,k}\big]^n$ is a compact and positively invariant set and $x_0 \in G$. Moreover, given any initial opinion $x_0 \in \mathbb{R}^n$, define

$$V(x) = \max{}^{(1)}(x) = \max_k x_k$$

for any $x \in G$. According to Lemma 7, $V(x)$ is globally

Lipschitz. Now we compute $D^+V(x)$. By definition,

$$\begin{aligned}
&D^+V(x)\\
&=\liminf_{\Delta t\downarrow 0}\frac{V(x+\Delta t f(x))-V(x)}{\Delta t}\\
&=\liminf_{\Delta t\downarrow 0}\frac{\max_k\Big(x_k+\Delta t\big(\mathrm{Med}_k(x;W)-x_k\big)\Big)-\max_k x_k}{\Delta t}.
\end{aligned}$$

For sufficiently small Δt, we have

$$\mathrm{argmax}_k\ x_k+\Delta t\big(\mathrm{Med}_k(x;W)-x_k\big)\ \subset\ \mathrm{argmax}^{(1)}(x)$$

Let

$$\begin{aligned}
\theta^{(1)}(x;W)=\Big\{i\in\mathrm{argmax}^{(1)}(x)\,\Big|\,&\mathrm{Med}_i(x;W)\geq\mathrm{Med}_j(x;W),\\
&\text{for any } j\in\mathrm{argmax}^{(1)}(x)\Big\}.
\end{aligned}$$

That is, $\theta^{(1)}(x;W)$ is a subset of $\mathrm{argmax}^{(1)}(x)$ such that the weighted-median opinion for any $i\in\theta^{(1)}(x;W)$ is no less than that of any other $j\in\mathrm{argmax}^{(1)}(x)\setminus\theta^{(1)}(x;W)$. Therefore,

$$\mathrm{argmax}_k\ x_k+\Delta t\big(\mathrm{Med}_k(x;W)-x_k\big)=\theta^{(1)}(x;W),$$

and, for any $i\in\theta^{(1)}(x;W)$,

$$\begin{aligned}
&D^+V(x)\\
&=\liminf_{\Delta t\downarrow 0}\frac{x_i+\Delta t\big(\mathrm{Med}_i(x;W)-x_i\big)-\max^{(1)}(x)}{\Delta t}\\
&=\liminf_{\Delta t\downarrow 0}\frac{\max^{(1)}(x)+\Delta t\big(\mathrm{Med}_i(x;W)-x_i\big)-\max^{(1)}(x)}{\Delta t}\\
&=\liminf_{\Delta t\downarrow 0}\frac{\Delta t\big(\mathrm{Med}_i(x;W)-x_i\big)}{\Delta t}\\
&=\lim_{\Delta t\downarrow 0}\frac{\Delta t\big(\mathrm{Med}_i(x;W)-x_i\big)}{\Delta t}\\
&=\mathrm{Med}_i(x;W)-x_i\leq 0.
\end{aligned}$$

Now, applying Theorem 5, we have that, as $t\to\infty$, $\phi(t,x_0)$ converges to the largest invariant set in

$$\begin{aligned}
E_1=\Big\{x\in G\,\Big|\,&\mathrm{Med}_i(x;W)-x_i=0,\\
&\text{for any } i\in\theta^{(1)}(x;W)\Big\}.
\end{aligned}$$

For this proof, we only need the result that $\phi(t,x_0)$ converges to E_1.

Now we refine this result on convergence by proving that $\phi(t,x_0)$ converges to a subset of E_1. We first prove that, for any $x\in E_1$, if $\mathrm{argmax}^{(1)}(x)$ does not contain any cohesive set, then x cannot be an ω-limit point of the given initial condition $x_0\in\mathbb{R}^n$. We prove this by using Lemma 6. In order to apply Lemma 6, we need to construct an open set satisfying all the conditions given in Lemma 6. Such an open set is constructed as follows: For any given $x\in E_1$, such that $\mathrm{argmax}^{(1)}(x)$ does not contain any cohesive set, according to the statement 2) of Lemma 13, there exists $\Delta t^{(1)}>0$ such that $\phi_i(\Delta t^{(1)},x)<\max^{(1)}(x)$ for any $i\in\mathcal{V}$. For this given $x\in E_1$, construct an open set

$$\begin{aligned}
B^{(1)}=\Big\{z\in G\,\Big|\,&\\
&\max^{(1)}(z)<\frac{1}{2}\Big(\max^{(1)}(x)+\max^{(1)}\big(\phi(\Delta t^{(1)},x)\big)\Big)\Big\}.
\end{aligned}$$

Obviously, if $x\in E_1$, then $x\notin B^{(1)}\cup\partial B^{(1)}$. Since $\max^{(1)}\big(\phi(t,z)\big)$ is non-increasing for any $z\in\mathbb{R}^n$, we have that $B^{(1)}\cup\partial B^{(1)}$ is positively invariant. Moreover, since

$$\begin{aligned}
\phi_i(\Delta t^{(1)},x)&\leq\max^{(1)}\big(\phi(\Delta t^{(1)},x)\big)\\
&<\big(\max^{(1)}(x)+\max^{(1)}(\phi(\Delta t^{(1)},x))\big)/2
\end{aligned}$$

for any $i\in\mathcal{V}$, we have $\phi(t,x)\in B^{(1)}$ for any $t\geq\Delta t^{(1)}$. According to Lemma 6, such an x cannot be an ω-limit point of x_0. Therefore, in order for $x\in E_1$ to be an ω-limit point of x_0, $\mathrm{argmax}^{(1)}(x)$ must contain a cohesive subset. That is, as $t\to\infty$, $\phi(t,x_0)$ converges to

$$\begin{aligned}
M_0=\big\{x\in G\,\big|\,&\mathrm{argmax}^{(1)}(x)\text{ contains}\\
&\text{at least one cohesive set.}\big\}\subset E_1.
\end{aligned}$$

Now we further refine the above result by proving that any $x\in M_0\setminus X_{\mathrm{eq}}$ is not an ω-limit point of x_0, i.e., $\omega(x_0)\subseteq X_{\mathrm{eq}}$. For any $r\in\mathbb{Z}_+$, define

$$\begin{aligned}
M_r=\Big\{x\in G\,\Big|\,&N_{\mathrm{diff}}(x)\geq r\text{ and }\mathrm{argmax}^{(\leq s)}(x)\\
&\text{is maximally cohesive }\forall\, s\leq r\Big\}.
\end{aligned}$$

Since $\mathcal{G}(W)$ only has a finite number of maximal cohesive sets, there exists $R(W)\in\mathbb{Z}_+$ such that $M_{R(W)}$ is non-empty and M_r is empty for any $r>R(W)$. In addition, we know that M_1 must be non-empty, since the node set $\mathcal{V}$ as an entirety must be maximally cohesive and any consensus state $x\in G$ satisfies $x\in M_1$. Namely, $R(W)\geq 1$ always holds. Moreover, according to the definition of M_r, we have

$$M_{R(W)}\subseteq M_{R(W)-1}\subseteq\cdots\subseteq M_1\subseteq M_0.$$

For any given $x\in M_0\setminus X_{\mathrm{eq}}$, according to Theorem 9, there exists $r\in\{0,1,\dots,R(W)-1\}$ such that $x\in M_r$ but $x\notin M_{r+1}$. For any such x, now we construct an open set B based on properties of the trajectory $\phi(t,x)$, and prove $x\notin\omega(x_0)$ by leveraging Lemma 6. For any $i\in\mathrm{argmax}^{(\leq r)}(x)$, since $x\in M_r$, we have $\mathrm{Med}_i(x;W)=x_i$ and thereby $\phi_i(t,x)=x_i>\max^{(r+1)}(x)$ for any $t\geq 0$. For any $i\in\mathcal{V}\setminus\mathrm{argmax}^{(\leq r)}(x)$, properties of the trajectory $\phi_i(t,x)$ are established via analyzing an auxiliary trajectory $\phi(t,y)$ starting from $y\in G$ defined as follows:

$$y_i=\begin{cases}\max^{(r+1)}(x) & \text{if } i\in\mathrm{argmax}^{(\leq r+1)}(x),\\ x_i & \text{if } i\in\mathcal{V}\setminus\mathrm{argmax}^{(\leq r+1)}(x).\end{cases}$$

As immediate results, we have

$$\begin{aligned}
&\max^{(s)}(y)=\max^{(r+s)}(x),\ \forall s\in\{1,\dots,N_{\mathrm{diff}}(x)-r\};\\
&\mathrm{argmax}^{(1)}(y)=\mathrm{argmax}^{(\leq r+1)}(x)\text{ and}\\
&\mathrm{argmax}^{(s)}(y)=\mathrm{argmax}^{(r+s)}(x),\ \forall s\in\{2,\dots,N_{\mathrm{diff}}(x)-r\}.
\end{aligned}$$

Now we establish the relations between $\phi(t,x)$ and $\phi(t,y)$. Since $\text{argmax}^{(\leq r)}(x)$ is a maximal cohesive set, according to Lemma 3, both $\text{argmax}^{(\leq r)}(x)$ and $\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$ are maximally cohesive. Therefore, according to Lemma 10, for any $t\geq 0$,

$$\begin{aligned}&\phi_i(t,x)>\max{}^{(r+1)}(x)\geq\phi_j(t,x),\\&\phi_i(t,y)=\max{}^{(r+1)}(x)\geq\phi_j(t,y),\end{aligned}\tag{3}$$

for any $i\in\text{argmax}^{(\leq r)}(x)$ and any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$. According to the definition of weighted median and the fact that $\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$ is maximally cohesive, as long as inequality (3) holds, the value of $\text{Med}_j\big(\phi(t,x);W\big)$ ($\text{Med}_j\big(\phi(t,y);W\big)$ resp.), for any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$ does not depend on the value of $\phi_i(t,x)$ ($\phi_i(t,y)$ resp.), for any $i\in\text{argmax}^{(\leq r)}(x)$. Moreover, since $x_j=y_j$ for any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$, we have

$$\phi_j(t,x)=\phi_j(t,y),$$

for any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$ and any $t\geq 0$. In the following, we establish some properties of the trajectory $\phi_j(t,x)$, for any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$, by analyzing the auxiliary trajectory $\phi(t,y)$.

According to how y is constructed, $\text{argmax}^{(1)}(y)$ contains at least one cohesive set, e.g., $\text{argmax}^{(\leq r)}(x)$. Denote by $\mathcal{LC}(y;W)$ the largest cohesive set contained in $\text{argmax}^{(1)}(y)$ and denote by $\mathcal{E}(y;W)$ the cohesive expansion of $\mathcal{LC}(y;W)$. Since $\text{argmax}^{(1)}(y)$ is not maximally cohesive, we have $\text{argmax}^{(1)}(y)\neq\mathcal{LC}(y;W)$, which in turn implies that either of the following two cases holds:

$$\begin{aligned}&\text{Case 1}: \quad \mathcal{LC}(y;W)\subseteq\text{argmax}^{(1)}(y)\subset\mathcal{E}(y;W);\\&\text{Case 2}: \quad \exists\, j\in\text{argmax}^{(1)}(y)\text{ such that } j\notin\mathcal{E}(y;W).\end{aligned}$$

In Case 1, since $\text{argmax}^{(1)}(y)\subset\mathcal{E}(y;W)$, it holds that $y_i<\max^{(1)}(y)$ for any $i\in\mathcal{E}(y;W)\setminus\text{argmax}^{(1)}(y)$. That is, $\min_{k\in\mathcal{E}(y;W)}y_k<\max^{(1)}(y)$. In addition, $\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)}y_k<\max^{(1)}(y)$ holds by definition. Let

$$\begin{aligned}b_1=\max\Big\{&\frac{\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)}y_k+\max^{(1)}(y)}{2},\\&\frac{\min_{k\in\mathcal{E}(y;W)}y_k+\max^{(1)}(y)}{2}\Big\},\end{aligned}$$

and construct the following open set:

$$\begin{aligned}B_1=\{z\in G\,|\,&z_i>b_1>z_j,\\&\forall\, i\in\mathcal{E}(y;W),\ \forall\, j\in\mathcal{V}\setminus\mathcal{E}(y;W)\}.\end{aligned}$$

Now we prove that x and B_1 satisfies the conditions in Lemma 6. Firstly, according to how y is constructed, we have

$$\min_{k\in\mathcal{E}(y;W)}x_k=\min_{k\in\mathcal{E}(y;W)}y_k<b_1,$$

which implies that $x\notin B_1\cup\partial B_1$. That is, condition 1) in Lemma 6 is satisfied. Secondly, since $\mathcal{E}(y;W)$ and $\mathcal{V}\setminus\mathcal{E}(y;W)$ are both cohesive, and

$$\min_{i\in\mathcal{E}(y;W)}z_i\geq b_1\geq\max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)}z_j,$$

for any $z\in B_1\cup\partial B_1$, according to Lemma 10, we have

$$\min_{i\in\mathcal{E}(y;W)}\phi_i(t,z)\geq b_1\geq\max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)}\phi_j(t,z),$$

which implies that $B_1\cup\partial B_1$ is positively invariant. That is, condition 3) in Lemma 6 is satisfied. Thirdly, since $\mathcal{V}\setminus\mathcal{E}(y;W)$ is cohesive, according to Lemma 10, we have

$$\max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)}\phi_j(t,y)\leq\max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)}y_j<b_1,$$

for any $t\geq 0$. In addition, since

$$\Big(\mathcal{V}\setminus\mathcal{E}(y;W)\Big)\subset\Big(\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)\Big)$$

and $\phi_j(t,x)=\phi_j(t,y)$ for any $j\in\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)$, we have

$$\max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)}\phi_j(t,x)<b_1,\quad t\geq 0.$$

Moreover, according to statement 4) of Lemma 13, $\lim_{t\to\infty}\phi_i(t,y)=\max^{(1)}(y)=\max^{(r+1)}(x)$ for any $i\in\mathcal{E}(y;W)$. As an immediate result, there exists $T_1>0$ such that $\phi_i(T_1,y)>b_1$ for any $i\in\mathcal{E}(y;W)$. In addition, since

$$\begin{aligned}&\phi_j(t,x)=\phi_j(t,y)\text{ and}\\&\phi_i(t,x)=\max{}^{(r+1)}(x)=\max{}^{(1)}(y)>b_1\end{aligned}$$

hold for any $t\geq 0$, any $j\in\Big(\mathcal{E}(y;W)\setminus\text{argmax}^{(\leq r)}(x)\Big)\subseteq\Big(\mathcal{V}\setminus\text{argmax}^{(\leq r)}(x)\Big)$, and any $i\in\text{argmax}^{(\leq r)}(x)$, we have

$$\min_{i\in\mathcal{E}(y;W)}\phi_i(T_1,x)>b_1.$$

Now we have proved that $\phi(T_1,x)\in B_1$, i.e., condition 2) in Lemma 6 is satisfied. Therefore, any given $x\in M_0\setminus X_{\text{eq}}$ such that the associated y is in Case 1 cannot be an ω-limit point of x_0.

In Case 2, according to statement 3) of Lemma 13 and the continuity of $\phi(t,y)$ in t, there exists sufficiently small Δt such that $\phi_j(\Delta t,y)<\max^{(1)}(y)$ for any $j\in\mathcal{V}\setminus\mathcal{LC}(y;W)$. Let

$$b_2=\frac{\max^{(1)}(y)+\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)}\phi_k(\Delta t,y)}{2}$$

and construct the open set

$$\begin{aligned}B_2=\{z\in G\,|\,&z_i>b_2>z_j,\\&\forall i\in\mathcal{E}(y;W),\ \forall j\in\mathcal{V}\setminus\mathcal{E}(y;W)\}.\end{aligned}$$

Now we prove that x and B_2 satisfies all the conditions in Lemma 6. Firstly, according to the assumptions of Case 2, there exists $j\in\text{argmax}^{(1)}(y)=\text{argmax}^{(\leq r+1)}(x)$ such that $j\notin\mathcal{E}(y;W)$. In addition, since $\text{argmax}^{(\leq r)}(x)\subset\mathcal{E}(y;W)$, such j can only be in $\text{argmax}^{(r+1)}(x)$. Therefore,

$$\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)}x_k=\max{}^{(r+1)}(x)=\max{}^{(1)}(y)>b_2,$$

i.e., $x\notin B_2\cup\partial B_2$. That is, condition 1) in Lemma 6 is satisfied. Secondly, since $\mathcal{E}(y;W)$ and $\mathcal{V}\setminus\mathcal{E}(y;W)$ are both cohesive and $\min_{i\in\mathcal{E}(y;W)} z_i > b_2 > \max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)} z_j$ for any $z \in B_2$, according to Lemma 10, we have

$$\min_{i\in\mathcal{E}(y;W)} \phi_i(t,z) \geq b_2 \geq \max_{j\in\mathcal{V}\setminus\mathcal{E}(y;W)} \phi_j(t,z),$$

for any $t \geq 0$. That is, the closure $B_2 \cup \partial B_2$ is positively invariant, i.e., condition 3) in Lemma 6 is satisfied. Thirdly, according to Lemma 10,

$$\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)} \phi_k(t,y) \leq \max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)} \phi_k(\Delta t,y) < b_2$$

for any $t \geq \Delta t$. Since $\big(\mathcal{V}\setminus\mathcal{E}(y;W)\big) \subseteq \big(\mathcal{V}\setminus \text{argmax}^{(\leq r)}(x)\big)$ and $\phi_j(t,x) = \phi_j(t,y)$ for any $j \in \mathcal{V}\setminus \text{argmax}^{(\leq r)}(x)$, we get

$$\max_{k\in\mathcal{V}\setminus\mathcal{E}(y;W)} \phi_k(t,x) < b_2, \text{ for any } t \geq \Delta t.$$

In addition, since

$$\begin{aligned}
\lim_{t\to\infty} \phi_i(t,y) &= \max{}^{(1)}(y) > b_2,\ \forall i \in \mathcal{E}(y;W),\\
\phi_i(t,x) &= \phi_i(t,y),\ \forall i \in \mathcal{E}(y;W)\setminus \text{argmax}^{(\leq r)}(x),\\
\text{and } \phi_i(t,x) &= x_i > \max{}^{(r+1)}(x) = \max{}^{(1)}(y) > b_2,\\
&\forall i \in \text{argmax}^{(\leq r)}(x),
\end{aligned}$$

there exists $T_2 \geq \Delta t > 0$ such that $\min_{i\in\mathcal{E}(y;W)} \phi_i(T_2,x) > b_2$. Therefore, $\phi(T_2,x) \in B_2$, i.e., condition 2) is satisfied. According to Lemma 6, any given $x \in M_0 \setminus X_{\text{eq}}$ such that the associated y is in Case 2 cannot be an ω-limit point of x_0.

Combining the arguments for Case 1 and Case 2 together, we conclude that $\omega(x_0) \subseteq X_{\text{eq}} \cap G$. According to Lemma 4, there exists an equilibrium $x^* \in X_{\text{eq}} \cap G$ such that $\omega(x_0) = \{x^*\}$, i.e., $\lim_{t\to\infty} \phi(t,x_0) = x^*$. This concludes the proof of statement 1).

If $\mathcal{V}$ is the only maximal cohesive set of $\mathcal{G}(W)$, then M_r^* is empty for any $r > 1$, i.e., $X_{\text{eq}} = M_1^*$. Therefore, for any x_0, $\phi(t,x_0)$ converges to a point $x^* \in M_1^*$, which is a consensus equilibrium. This concludes the proof of statement 2).

If $\mathcal{V}_1 \subset \mathcal{V}$ is a maximal cohesive set, then $\mathcal{V}_2 = \mathcal{V}\setminus\mathcal{V}_2$ is also maximally cohesive. The set

$$X = \big\{x \in \mathbb{R}^n \,\big|\, x_i > x_j \text{ for any } i \in \mathcal{V}_1,\ j \in \mathcal{V}_2\big\}$$

is a subset of $\mathbb{R}^n$ with non-zero measure. For any such x, according to Lemma 10,

$$\min_{k\in\mathcal{V}_1} \phi_k(t,x) \geq \min_{k\in\mathcal{V}_1} x_k > \max_{k\in\mathcal{V}_2} x_k \geq \max_{k\in\mathcal{V}_2} \phi_k(t,x).$$

Therefore, $\phi(t,x)$ does not converges to any consensus equilibrium. This concludes the proof of statement 3). ■

IV. Conclusions

In conclusion, this paper has developed a continuous-time weighted-median opinion dynamics model that advances the illustration of opinion compromise based on the weighted-median mechanism. We have rigorously established the foundational mathematical properties of the model, including the existence and uniqueness of solutions and the characterization of equilibria. Moreover, we have derived both necessary and sufficient graph-theoretic conditions for consensus and provided sufficient conditions for dissensus. These insights are pivotal for designing strategies in social networks to foster agreement or understand persistent disagreements. Future work will focus on exploring the role of networks structure on these dynamics under this model under this model.

Appendix A
Proof of Lemma 4

Suppose x^*, y^* are both in $\omega(x_0) \subseteq \chi$, and $x^* \neq y^*$. There exists $\epsilon > 0$ such that

$$\inf_{z\in B(x^*,\epsilon)} \|y^* - z\| > \epsilon,$$

where $B(x^*,\epsilon) = \big\{z \in D \,\big|\, \|z - x^*\| < \epsilon\big\}$. Since x^* is stable, for the ϵ given above, there exists $\delta > 0$ such that, for any z satisfying $\|z - x^*\| < \delta$, $\phi(t,z) \in B(x^*,\epsilon)$ for any $t \geq 0$. Moreover, since x^* is an ω-limit point of x_0, there exists a time sequence $\{t_p\}_{p\in\mathbb{N}}$ such that $t_p \to \infty$ as $p \to \infty$, and $\lim_{p\to\infty} \phi(t_p,x_0) = x^*$. As a consequence, there exists P such that $\|\phi(t_P,x_0) - x^*\| < \delta$, which implies that $\phi(t,x_0) \in B(x^*,\epsilon)$ and thereby $\|\phi(t,x_0) - y^*\| > \epsilon$ for any $t \geq t_P$. Therefore, y^* cannot be an ω-limit point of x_0. This concludes the proof. ■

Appendix B
Proof of Lemma 6

Since $f(x)$ is globally Lipschitz, for any given $x \in G$, the solution $\phi(t,x)$ exists and is unique for any $t \geq 0$, and $\phi(t,x)$ is a continuous function of x.

Since B is an open set and there exists $T > 0$ satisfying $\phi(T,x) \in B$, there exists $\epsilon > 0$ such that, for any z satisfying $\|z - \phi(T,x)\| < \epsilon$, we have $z \in B$. Here $\|\cdot\|$ is some norm in $\mathbb{R}^n$. Moreover, since $B \cup \partial B$ is positively invariant, $\phi(t,z) \in B \cup \partial B$ for any $t \geq 0$. Additionally, since $\phi(t,x)$ is a continuous function of x, for the $\epsilon > 0$ given above, there exists $\xi > 0$ such that, for any $\hat{x}$ satisfying $\|\hat{x} - x\| < \xi$, $\|\phi(T,\hat{x}) - \phi(T,x)\| < \epsilon$. Therefore, $\phi(t+T,\hat{x}) \in B \cup \partial B$, for any $t \geq 0$.

Suppose there exists $y \in G$ such that $x \in \omega(y)$, then there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} \phi(t_k,y) = x$, which implies that there exists $K \in \mathbb{N}$ such that, for any $k \geq K$, $\|\phi(t_k,y) - x\| < \xi$. Therefore,

$$\|\phi\big(T,\phi(t_K,y)\big) - \phi(T,x)\| = \|\phi(T+t_K,y) - \phi(T,x)\| < \epsilon.$$

As a consequence, $\phi(t+T+t_K,y) \in B \cup \partial B$ for any $t \geq 0$, and thereby $\|\phi(t,y) - x\| \geq \text{dist}(x, B\cup\partial B) > 0$ for any $t \geq T + t_K$, which contradicts the assumption that $x \in \omega(y)$. This concludes the proof. ■

Appendix C
Proof of Lemma 10

Given any $x \in \mathbb{R}^n$ and any $t \geq 0$, let $\underline{c}_t = \min_{k\in\mathcal{C}} \phi_k(t,x)$ and $\overline{c}_t = \max_{k\in\mathcal{C}} \phi_k(t,x)$. For any $j \in \mathcal{C}$, since

$$\sum_{k:\,\phi_k(t,x)>\overline{c}_t} w_{jk} \leq \sum_{k\notin\mathcal{C}} w_{jk} < \frac{1}{2}$$

and

$$\sum_{k:\, \phi_k(t,x) < \underline{c}_t} w_{jk} \leq \sum_{k \notin \mathcal{C}} w_{jk} < \frac{1}{2},$$

according to Definition 1, we have that

$$\underline{c}_t \leq \mathrm{Med}_j\big(\phi(t,x); W\big) \leq \bar{c}_t$$

for any $j \in \mathcal{C}$. Therefore, for any $j \in \mathrm{argmax}_{k \in \mathcal{C}}\, \phi_k(t,x)$,

$$\frac{d\phi_j(t,x)}{dt} = \mathrm{Med}_j\big(\phi(t,x); W\big) - \bar{c}_t \leq 0,$$

and, for any $j \in \mathrm{argmin}_{k \in \mathcal{C}}\, \phi_k(t,x)$,

$$\frac{d\phi_j(t,x)}{dt} = \mathrm{Med}_j\big(\phi(t,x); W\big) - \underline{c}_t \geq 0.$$

This concludes the proof. ■

Appendix D
Proof of Lemma 12

From $\dot{y}(t) = \beta(t) - y(t)$ we immediately obtain

$$y(t) = \int_0^t \beta(\tau) e^{\tau} d\tau + y(0) e^{-t}.$$

Let $h(t) = \int_0^t \beta(\tau) e^{\tau} d\tau$. Since $\lim_{t\to+\infty} \beta(t) = y^*$, we have that, for any $\epsilon > 0$, there exists $T > 0$ such that $|\beta(t) - y^*| < \epsilon$ hold for any $\tau \geq T$. Therefore, for any $t \geq T$,

$$h(t) < \frac{\int_0^T \beta(\tau) e^{\tau} d\tau}{e^t} + \frac{(y^* + \epsilon)(e^t - e^{\tau})}{e^t}.$$

The right-hand side of the above inequality converges to $y^* + \epsilon$ as $t \to +\infty$. Therefore, we have $\limsup_{t\to+\infty} h(t) \leq y^* + \epsilon$. Similarly, we have

$$h(t) > \frac{\int_0^T \beta(\tau) e^{\tau} d\tau}{e^t} + \frac{(y^* - \epsilon)(e^t - e^{\tau})}{e^t},$$

of which the right-hand side converges to $y^* - \epsilon$ and thereby $\liminf_{t\to\infty} h(t) \geq y^* - \epsilon$. Now we have that

$$y^* - \epsilon \leq \liminf_{t\to+\infty} h(t) \leq \limsup_{t\to+\infty} h(t) \leq y^* + \epsilon$$

holds for any $\epsilon > 0$. Therefore, we obtain

$$\liminf_{t\to+\infty} h(t) = \limsup_{t\to+\infty} h(t) = y^*,$$

i.e., $h(t)$ converges to y^* as $t \to +\infty$. Moreover, since $y(t) = h(t) + y(0)e^{-t}$ and $\lim_{t\to+\infty} y(0)e^{-t} = 0$, we eventually have $\lim_{t\to+\infty} y(t) = y^*$. ■

Appendix E
Proof of Lemma 13

The proof of statement 1) is straightforward. According to Lemma 10, for any $x \in \mathbb{R}$, $\max^{(1)}\big(\phi(t,x)\big)$ is non-increasing with t. Therefore, for any $i \in \mathcal{V}$, we have

$$\begin{aligned}\frac{d^+\phi_i(t,x)}{dt} &= \mathrm{Med}_i\big(\phi(t,x); W\big) - \phi_i(t,x)\\ &\leq \max{}^{(1)}(x) - \phi_i(t,x),\end{aligned}$$

which leads to

$$\phi_i(t,x) \leq \max{}^{(1)}(x) - \big(\max{}^{(1)}(x) - x_i\big)e^{-t}.$$

Therefore, for any $i \in \mathrm{argmax}^{(1)}(x)$, $\phi_i(t,x) < \max^{(1)}(x)$ for any $t > 0$. This concludes the proof of statement 1).

Now we prove statement 3). Statement 2) can be easily inferred from the proof of Statement 3). Suppose $\mathcal{C}^{(1)}(x) \subseteq \mathrm{argmax}^{(1)}(x)$ is non-empty. For any $i \in \mathcal{C}^{(1)}(x;W)$, since $\mathcal{C}^{(1)}(x;W)$ is cohesive, we have $\mathrm{Med}_i(x;W) = x_i = \max^{(1)}(x)$. Therefore, for any $i \in \mathcal{C}^{(1)}(x;W)$, $\phi_i(t,x) = x_i = \max^{(1)}(x)$ for any $t \geq 0$. Now we prove that, for any $j \notin \mathcal{C}^{(1)}(x;W)$, $\phi_j(\Delta t, x) < \max^{(1)}(x)$ for some $\Delta t > 0$. Suppose $\mathcal{C}^{(1)}(x;W) = \mathrm{argmax}^{(1)}(x)$, according to statement 1), $\phi_j(t,x) < \max^{(1)}(x)$ for any $j \notin \mathrm{argmax}^{(1)}(x) = \mathcal{C}^{(1)}(x;W)$ and any $t \geq 0$. Now suppose $\mathcal{C}^{(1)}(x;W) \subset \mathrm{argmax}^{(1)}(x;W)$. Define $\mathcal{V}_{-1} = \mathcal{V} \setminus \mathrm{argmax}^{(1)}(x)$. According to statement 1), for any $j \in \mathcal{V}_{-1}$, $\phi_j(t,x) < \max^{(1)}(x)$ for any $t \geq 0$. Moreover, since $\mathrm{argmax}^{(1)}(x)$ is not cohesive, there exists $j \in \mathrm{argmax}^{(1)}(x)$ such that $\sum_{k\in\mathcal{V}_{-1}} w_{jk} > 1/2$, which implies that $\mathrm{Med}_j(x;W) < \max^{(1)}(x)$. Define the set of all such j's as ϑ_1, i.e., the set of nodes in $\mathrm{argmax}^{(1)}(x)$ who assign more weights to the nodes outside $\mathrm{argmax}^{(1)}(x)$ than the nodes inside $\mathrm{argmax}^{(1)}(x)$. For any $j \in \vartheta_1$,

$$\begin{aligned}\frac{d^+\phi_j(t,x)}{dt}\Big|_{t=0} &= \mathrm{Med}_j(x;W) - x_j\\ &= \mathrm{Med}_j(x;W) - \max{}^{(1)}(x) < 0.\end{aligned}$$

Therefore, there exists $\Delta t > 0$ such that $\phi_j(t,x) < \phi_j(0,x) = \max^{(1)}(x)$ for any $t \in (0, \Delta t]$. We thereby have $\phi_j(\Delta t, x) < \max^{(1)}(x)$ for any $j \in \mathcal{V}_{-1} \cup \vartheta_1$.

With the nodes in ϑ_1 removed, the set $\mathrm{argmax}^{(1)}(x) \setminus \vartheta_1$ is still not necessarily cohesive, since there could exist $j \in \mathrm{argmax}^{(1)}(x) \setminus \vartheta_1$ such that j assigns more weights to $\vartheta_1 \cup \mathcal{V}_{-1}$ than $\mathrm{argmax}^{(1)}(x) \setminus \vartheta_1$. If such a j does exist, then $\mathcal{C}^{(1)}(x;W) \subset \mathrm{argmax}^{(1)}(x) \setminus \vartheta_1$ and $\mathrm{argmax}^{(1)}(x) \setminus \vartheta$ is still not cohesive, which implies that the set

$$\vartheta_2 = \Big\{ j \in \mathrm{argmax}^{(1)}(x) \setminus \vartheta_1 \,\Big|\, \sum_{k \in \mathcal{V}_{-1}\cup\vartheta_1} w_{jk} > 1/2 \Big\}$$

is non-empty. For any $j \in \vartheta_2$,

$$\begin{aligned}\frac{d^+\phi_j(t,x)}{dt} &= \mathrm{Med}_j\big(\phi(t,x); W\big) - \phi_j(t,x)\\ &\leq \max_{k\in\mathcal{V}_{-1}\cup\vartheta_1} \phi_k(t,x) - \phi_j(t,x).\end{aligned}$$

At any time instant $t \in (0, \Delta t]$, if

$$\phi_j(t,x) > \max_{k\in\mathcal{V}_{-1}\cup\vartheta_1} \phi_k(t,x),$$

then $d^+\phi_j(t,x)/dt < 0$; If

$$\phi_j(t,x) \leq \max_{k\in\mathcal{V}_{-1}\cup\vartheta_1} \phi_k(t,x),$$

then $\phi_j(t,x) < \max^{(1)}(x)$. Therefore, during the time period $(0, \Delta t]$, either $d^+\phi_j(t,x)/dt < 0$ or $\phi_j(t,x) \leq \max_{k\in\mathcal{V}_{-1}\cup\vartheta_1} \phi_k(t,x) < \max^{(1)}(x)$, which implies that $\phi_j(\Delta t, x) < \phi_j(0,x) = \max^{(1)}(x)$.

Following the same argument above, if $\text{argmax}^{(1)}(x)\setminus(\vartheta_1\cup\vartheta_2)$ is still not cohesive, then $\mathcal{C}^{(1)}\subset\text{argmax}^{(1)}(x)\setminus(\vartheta_1\cup\vartheta_2)$ and the set

$$\vartheta_3=\Big\{j\in\text{argmax}^{(1)}(x)\setminus(\vartheta_1\cup\vartheta_2)\,\Big|\sum_{k\in\mathcal{V}_{-1}\cup\vartheta_1\cup\vartheta_2}w_{jk}>1/2\Big\}$$

is non-empty. Moreover, for any $j\in\vartheta_3$, $\phi_j(\Delta t,x)<\max^{(1)}(x)$. Since the cardinality of $\text{argmax}^{(1)}(x)$ is finite, by repeating this argument, we eventually obtain a finite K such that $\mathcal{C}^{(1)}(x;W)=\text{argmax}^{(1)}(x)\setminus(\vartheta_1\cup\cdots\cup\vartheta_K)$, and for any $j\in\mathcal{V}\setminus\mathcal{C}^{(1)}(x;W)$, $\phi_j(\Delta t,W)<\max^{(1)}(x)$. Moreover, for any $i\in\mathcal{C}^{(1)}(x)$, since $\mathcal{C}^{(1)}(x)$ is cohesive, $\phi_i(t,x)=\max^{(1)}(x)$ for any $t\geq 0$. This concludes the proof of statement 3).

The argument above implies that, if $\mathcal{C}^{(1)}(x;W)$ is empty, then there exists a finite $\tilde{K}$ such that $\text{argmax}^{(1)}(x)=\vartheta_1\cup\cdots\cup\vartheta_{\tilde{K}}$ and thereby $\phi_i(\Delta t,x)<\max^{(1)}(x)$ for any $i\in\text{argmax}^{(1)}(x)$. This concludes the proof of statement 2).

Now we proceed to prove statement 4). Define

$$\mathcal{J}_1=\Big\{j\in\mathcal{E}^{(1)}(x;W)\setminus\mathcal{C}^{(1)}(x;W)\,\Big|\sum_{k\in\mathcal{C}^{(1)}(x;W)}w_{jk}>1/2.\Big\}.$$

If $\mathcal{J}_1$ is empty, then $\mathcal{C}^{(1)}(x;W)=\mathcal{E}^{(1)}(x;W)$ and we already show that, for any $i\in\mathcal{C}^{(1)}(x;W)$, $\phi_i(t,x)=\max^{(1)}(x)$ for any $t\geq 0$, which concludes the proof of statement 4). If $\mathcal{J}_1$ is non-empty, then, for any $j\in\mathcal{J}_1$, we have $\text{Med}_j(\phi(t,x),W)=\max^{(1)}(x)$ for any $t\geq 0$. Therefore,

$$\frac{d^+\phi_j(t,x)}{dt}=\max{}^{(1)}(x)-\phi_j(t,x),$$

which leads to $\lim_{t\to\infty}\phi_j(t,x)=\max^{(1)}(x)$. Define

$$\mathcal{J}_2=\Big\{j\in\mathcal{E}^{(1)}(x;W)\setminus\big(\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1\big)\,\Big|\sum_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1}w_{jk}>1/2.\Big\}.$$

If $\mathcal{J}_2$ is empty, then $\mathcal{E}^{(1)}(x;W)=\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1$ and we already prove that, for any $j\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1$, $\lim_{t\to\infty}\phi_j(t,x)=\max^{(1)}(x)$. If $\mathcal{J}_2$ is non-empty, then, for any $j\in\mathcal{J}_2$, since $\sum_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1}w_{jk}>1/2$, we have

$$\min_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1}\phi_k(t,x)\leq\text{Med}_j\big(\phi(t,x);W\big)\leq\max{}^{(1)}(x).$$

Since, for any $t\geq 0$, we have $\phi_j(t,x)\leq\max^{(1)}(x)$,

$$\begin{aligned}\min_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1}\phi_k(t,x)-\phi_j(t,x)&\leq\frac{d^+\phi_j(t,x)}{dt}\\&\leq\max{}^{(1)}(x)-\phi_j(t,x),\end{aligned}$$

and

$$\lim_{t\to\infty}\min_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1}\phi_k(t,x)=\max{}^{(1)}(x),$$

according to Lemma 12, we have $\lim_{t\to\infty}\phi_j(t,x)=\max^{(1)}(x)$. Define

$$\mathcal{J}_3=\Big\{j\in\mathcal{E}^{(1)}(x;W)\setminus\big(\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1\cup\mathcal{J}_2\big)\,\Big|\sum_{k\in\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1\cup\mathcal{J}_2}w_{jk}>1/2.\Big\}.$$

Following the same argument for $\mathcal{J}_2$, if $\mathcal{J}_3$ is empty, then $\mathcal{E}^{(1)}(x;W)=\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1\cup\mathcal{J}_2$. If $\mathcal{J}_3$ is non-empty, then, according to Lemma 12, $\lim_{t\to\infty}\phi_j(t,x)=\max^{(1)}(x)$ for any $j\in\mathcal{J}_3$. Repeating this argument, since $\mathcal{E}^{(1)}(x;W)$ is a finite set, there exists $K\in\mathbb{N}$ such that $\mathcal{E}^{(1)}(x;W)=\mathcal{C}^{(1)}(x;W)\cup\mathcal{J}_1\cup\cdots\cup\mathcal{J}_K$, and $\lim_{t\to\infty}\phi_j(t,x)=\max^{(1)}(x)$ for any $j\in\mathcal{E}^{(1)}(x;W)$. ■

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