Periodic homogenisation for singular PDEs with generalised Besov spaces

Yilin Chen
Peking University

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Abstract

We consider periodic homogenisation problems for gPAM, Φ_3^4 and modified KPZ equations. Using Littlewood-Paley blocks, we establish para-products and commutator estimates for generalised Besov spaces associated with self-adjoint homgenisation elliptic operators. With this, we show that, the homogenisation and renormalisation procedures commute in aforementioned models under suitable assumptions.

Caution: There are some problems to be fixed and addressed in the current version. I am preparing a new version.

Contents

1 Introduction

We aim at studying homogenisation and renormalisation procedures in gPAM, Φ^4_3 and modified KPZ equations by the para-controlled distributions theory and generalised Besov spaces. Taking the periodic homogenisation of generalised parabolic Anderson model for example, gPAM in $\mathbf{T}^2$ is

$$(\partial_t - \mathcal{L}_\varepsilon)u^{(\delta)}_\varepsilon = g(u^{(\delta)}_\varepsilon)(\eta^{(\delta)} - C^{(\delta)}_\varepsilon g'(u^{(\delta)}_\varepsilon))$$

in which $\mathcal{L}_\varepsilon := \operatorname{div}(a_\varepsilon \nabla)$, $a_\varepsilon := a(\cdot/\varepsilon)$ and $\varepsilon = \varepsilon_n = \frac{1}{n}$. The coefficient matrix $a : \mathbf{T}^2 \to \mathbf{R}^{2\times 2}$ is symmetric, elliptic and Hölder continuous. η is a spatial white noise. $\eta^{(\delta)} := \eta * \rho_\delta$, in which $\rho \in C^\infty(\mathbf{T}^1)$ is a mollifier and $\rho_\delta = \rho(\cdot/\delta)$. The renormalisation coefficient $C^{(\delta)}_\varepsilon$ is given by

$$C^{(\delta)}_\varepsilon := \mathbf{E}(X^{(\delta)}_\varepsilon \circ_\varepsilon \eta^{(\delta)})$$

as that in [8]. X_ε and $\circ_\varepsilon$ are defined in (3.1) and (2.4). As in the classical renormalisation theory, the renormalisation coefficient is not unique. We choose the relatively nature one as in [8]. We fix this definition in our paper.

For a fixed $\delta > 0$, since $\eta^{(\delta)}$ is a smooth function, by the standard homogenisation theory, the solution $u^{(\delta)}_\varepsilon$ converges to $u^{(\delta)}_0$ that satisfies

$$\partial_t u^{(\delta)}_0 = \mathcal{L}_0 u^{(\delta)}_0 + g(u^{(\delta)}_0)(\eta^{(\delta)} - C^{(\delta)}_0 g'(u^{(\delta)}_0)),$$

where $\mathcal{L}_0 := \operatorname{div}(\bar{a}\nabla)$ and

$$\bar{a} := \int_{\mathbf{T}^2} a(\mathrm{id} + \nabla\chi)\mathrm{d}y \tag{1.1}$$

and the corrector is given by

$$\operatorname{div}(a(\mathrm{id} + \nabla\chi)) = 0$$

with $\int_{\mathbf{T}^2} \chi(y)\mathrm{d}y = 0$. Since $\bar{a}$ is constant, by para-controlled distributions established in [8], $u^{(\delta)}_0$ converges as δ tends to 0.

On the other hand, for a fixed $\varepsilon \in (0,1)$, since $a(\cdot/\varepsilon)$ is Hölder continuous, [15] proves the renormalisation limit converges. We denote the limit of $u^{(\delta)}_\varepsilon$ by u_ε. [4] proves u_ε converges as ε tends to 0 in the framework of classical Besov spaces. Obviously, classical Besov spaces are not compatible with $\mathcal{L}_\varepsilon$ and $\mathcal{I}_\varepsilon$. The authors of [4] manage to overcome the incompatibility by integration by part and completing products.

[1, 2] used heat semigroups and their integrals to build up para-products and commutator estimates for generalised Besov spaces associated with symmetric elliptic operators in divergence form. [1, Proposition 3.3] tells us that $\forall \alpha, \beta < 1$ and $\beta + \alpha > 0$, for the resonant terms,

$$\|\Pi^{(b)}(f,g)\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta}$$

where $\mathcal{C}^\alpha := \mathcal{B}^\alpha_{\infty,\infty}$ is given in [1, Definition 2.4]. In the setting of [2], the estimate for $\alpha, \beta < 3$ is valid under a strong assumption on heat semi-groups. This strong assumption in [2] seems not suitable for the periodic homogenisation setting.

The estimate for resonant terms is crucial in the proof of generalised parabolic Anderson model, if the fixed point scheme is built up as explained in [8]. Because the white noise η lives in $\mathcal{C}^{-1-}$ almost surely. We need to deal with this kind of para-products, $f \circ g$ where $f \in \mathcal{C}^{1+\kappa}$ and $g \in \mathcal{C}^{-1-\frac{\kappa}{2}}$ for $\kappa > 0$ sufficiently small.

To overcome the incompatiblity and obtain sufficient results about para-products, especially for resonant terms, using Littlewood-Paley blocks, we establish para-products and commutator estimates for generalised Besov spaces associated with symmetric elliptic operators in divergence form in Section 2. This theory can help us deal with periodic homogenisation problems for aformentioned models as in the classical theory of them. For example, in the classical theory, the fixed point mapping of gPAM can be given as

$$\begin{aligned}
u &= u^{\#} + g(u) \prec \mathcal{I}(\eta) + e^{t\Delta}u(0)\\
u^{\#} &= \mathcal{I}(g(u) \prec \eta) + [\mathcal{I}(u \prec \eta) - u \prec \mathcal{I}(\eta)]\\
&\quad + \mathcal{I}\big((g(u) - g'(u) \prec u) \circ \eta) + g'(u)(u^{\#} \circ \eta + e^{t\Delta}u(0) \circ \eta)\big)\\
&\quad + \mathcal{I}(\big(g(u) \prec \mathcal{I}(\eta)\big) \circ \eta - g(u)(\mathcal{I}(\eta) \circ \eta)) + g(u)[\mathcal{I}(\eta) \circ^{\diamond} \eta + J^{(\delta)}]
\end{aligned}$$

where $J^{(\delta)} := \lim\limits_{\delta\to 0^+}[C^{(\delta)} - \mathbf{E}\big((\mathcal{I}(\eta^{(\delta)})) \circ \eta^{(\delta)}\big)]$ and $\mathcal{I}_\varepsilon(f) := \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon} f(\tau)\mathrm{d}\tau$.

Similarly, we have the fixed point mapping below.

$$\begin{aligned}
u_\varepsilon &= u_\varepsilon^{\#} + g(u_\varepsilon) \prec_\varepsilon \mathcal{I}_\varepsilon(\eta) + e^{t\mathcal{L}_\varepsilon}u(0)\\
u_\varepsilon^{\#} &= \mathcal{I}_\varepsilon(g(u_\varepsilon) \prec_\varepsilon \eta) + [\mathcal{I}_\varepsilon(u_\varepsilon \prec_\varepsilon \eta) - u_\varepsilon \prec_\varepsilon \mathcal{I}_\varepsilon(\eta)]\\
&\quad + \mathcal{I}_\varepsilon\Big((g(u_\varepsilon) - g'(u_\varepsilon) \prec_\varepsilon u_\varepsilon) \circ_\varepsilon \eta) + g'(u)(u^{\#} \circ_\varepsilon \eta + e^{t\mathcal{L}_\varepsilon}u(0) \circ_\varepsilon \eta)\Big)\\
&\quad + \mathcal{I}_\varepsilon\Big(\big(g(u_\varepsilon) \prec_\varepsilon \mathcal{I}(\eta)\big) \circ_\varepsilon \eta - g(u_\varepsilon)(\mathcal{I}_\varepsilon(\eta) \circ_\varepsilon \eta)\Big) + g(u_\varepsilon)[(\mathcal{I}_\varepsilon(\eta) \circ_\varepsilon \eta)^{\diamond} + J_\varepsilon]
\end{aligned}$$

where

$$J_\varepsilon := \lim_{\delta\to 0^+}[C_\varepsilon^{(\delta)} - \mathbf{E}\big((\mathcal{I}_\varepsilon(\eta^{(\delta)})) \circ_\varepsilon \eta^{(\delta)}\big)]. \tag{1.2}$$

Analogously, we can establish uniform regularity estimates for solutions of aforementioned models in homogenisation cases in the classical way.

1.1 Organisation of this article

In Section 2, we build up para-products and commutator estimates for generalised Besov spaces. In Sections 3, 4 and 5, we apply the theory established in Section 2 to periodic homogenisation problems in gPAM, Φ^4_3 and modified KPZ equations. In Appendix A, we collect some necessary results from [12, 7, 6].

2 Paraproducts with generalised Besov spaces

For $d \in \mathbf{N}^+$, define $S(\lambda, \Lambda) := \{a; a : \mathbf{T}^d \to M_d(\mathbf{R}), a = a^\intercal, \lambda \le a \le \Lambda\}$ in which $M_d(\mathbf{R})$ is the set of all real $d\times d$ matrices. For a matrix a, $a \le M$ means $v^\intercal a v \le M|v|^2$, $\forall v \in \mathbf{R}^d$. $\forall a \in S(\lambda, \Lambda)$, define $\mathcal{L}_a := \operatorname{div}(a\nabla)$. Denote the Green's function of $(\partial_t - \mathcal{L}_a)|_{[0,T]\times\mathbf{T}^d}$ by Q_a. $S(M, \lambda, \Lambda) = \{a \in S(\lambda, \Lambda)$: Q_a satisfies Assumption 2.1$\}$.

Assumption 2.1. • *Small time Gaussian bound:*

$$|Q_a(t; x, y)| \le Mt^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Mt}}; \tag{2.1}$$

- *Hölder continuity:* $\forall |y-y'| \le \sqrt{t}$,

$$|Q_a(t;x,y) - Q_a(t;x,y')| \le M|y-y'|t^{\frac{d+1}{2}} e^{-\frac{|x-y|^2}{Mt}}; \tag{2.2}$$

- *Markov property:*

$$\int_{\mathbf{T}^d} Q_a(t;x,y)\mathrm{d}y = 1.$$

For any function F on the spectrum of $\sqrt{-\mathcal{L}_a}$, we define

$$F(\sqrt{-\mathcal{L}_a})f := \sum_{n=1}^{\infty} F(\lambda_{n,a}) < f, \psi_{n,a} > \psi_{n,a}$$

where $(\lambda_{n,a}, \psi_{n,a})$ is the nth non-negative eigen-pair of $\sqrt{-\mathcal{L}_a}$ on the torus $\mathbf{T}^d$. By the Weyl's formula [13], we have that the distribution of the eigenvalues of $-\mathcal{L}_a$ on the torus $\mathbf{T}^d$ is similar with that of $-\Delta$ on the torus $\mathbf{T}^d$. By [12, Theorem 3.1], we have the following lemma.

Lemma 2.2. *Let f be a smooth even function on $\mathbf{R}$, supported in $(-R,R)$ for some $R>1$. $\forall k \ge d+1$, $\forall a \in S(M,\lambda,\Lambda)$, $\forall \delta \in (0,1)$,*

$$|f(\delta\sqrt{-\mathcal{L}_a})(x,y)| \le c_k \delta^{-d}(1+\frac{|x-y|}{\delta})^{-k}$$

and

$$|\nabla_y f(\delta\sqrt{-\mathcal{L}_a})(x,y)| \le c_k \delta^{-d-1}(1+\frac{|x-y|}{\delta})^{-k}$$

in which

$$c_k := R^d\big((C_1 k)^k \|f\|_{L^\infty} + (C_2 R)^k \|f^{(k)}\|_{L^\infty}\big),$$

where C_1 and C_2 only depend on λ, Λ, M. Moreover,

$$\int_{\mathbf{T}^d} f(\sqrt{-\mathcal{L}_a})(x,y)\mathrm{d}y = f(0).$$

Remark 2.3. In this section, we use the $\mathbf{T}^d$ as the basic manifold for convenience. Essentially, we can obtain similar estimates for para-products and commutators with compact Riemannian manifolds with suitable conditions as basic manifolds. More general settings can be found in [12].

Let smooth functions $\tilde{\varphi}, \varphi_0 : \mathbf{R} \to [0,1]$ satisfy

$$\tilde{\varphi} + \sum_{j=0}^{\infty} \varphi_j = 1 \tag{2.3}$$

where $\varphi_j = \varphi_0(2^{-j}\cdot)$ and $\mathrm{supp}\varphi_0 \subset B(0,2)\backslash B(0,\frac{3}{4})$, $\mathrm{supp}\tilde{\varphi} \subset B(0,1)$. For convenience, we define $\varphi_{-1} := \tilde{\varphi}$. We define the generalised Besov space associated with $-\mathcal{L}_a$ on the torus $\mathbf{T}^d$.

$$\|f\|_{\mathcal{B}^{\alpha,-\mathcal{L}_a}_{p,q}(\mathbf{T}^d)} := \Big[\sum_{j=-1}^{\infty} \big(2^{\alpha j}\|\varphi_j(\sqrt{-\mathcal{L}_a})f\|_{L^p(\mathbf{T}^d)}\big)^q\Big]^{\frac{1}{q}}.$$

When $p = q = +\infty$, we define $\mathcal{C}^{\alpha,-\mathcal{L}_a}(\mathbf{T}^d)$ below

$$\|f\|_{\mathcal{C}^{\alpha,-\mathcal{L}_a}(\mathbf{T}^d)} := \sup_{j\geq -1}\Big(2^{\alpha j}\|\varphi_j(\sqrt{-\mathcal{L}_a})f\|_{L^\infty(\mathbf{T}^d)}\Big).$$

For convenience, we write $\|\cdot\|_{\mathcal{C}^{\alpha,-\mathcal{L}_a}}$ as $\|\cdot\|_{\alpha,a}$ and we write $\mathcal{C}^{\alpha,-\mathcal{L}_a}$ as $\mathcal{C}^{\alpha,a}$. The para-products associated with the generalised Besov space $\mathcal{B}^{\alpha,-\mathcal{L}_a}_{p,q}$ are defined as

$$f \prec_a g := \sum_{j=1}^{\infty} S_{j,a}(f)\Delta_{j,a}g$$

and

$$f \circ_a g := \sum_{j=-1}^{\infty}\sum_{|i-j|\leq 1}\Delta_{i,a}(f)\Delta_{j,a}g \tag{2.4}$$

where $S_{j,a}(f) := \sum_{l=-1}^{j-2}\Delta_{j,a}f$ and $\Delta_{j,a}f := \varphi_j(\sqrt{-\mathcal{L}_a})f$.

Denote the integral kernel of $\Delta_{k,a}$ by $\Delta_{k,a}(x,y)$. More precisely,

$$\Delta_{j,a}(x,y) := \sum_k \varphi_j(\lambda_{k,a})\psi_{k,a}(x)\psi_{k,a}(y), \quad \forall j \geq -1.$$

$\forall f \in C^\infty(\mathbf{T}^d)$,

$$(\Delta_{j,a}f)(x) := \int_{\mathbf{T}^d}\Delta_{j,a}(x,y)f(y)\mathrm{d}y.$$

2.1 Paraproduct estimates

Let $a \in S(M,\lambda,\Lambda)$. We denote the dimension of the basic manifold by d. The constants appearing in this subsection depend only on d, M, λ, Λ, if there is no other special statement. We denote the classical Besov spaces in $\mathbf{T}^d$ by $\mathcal{B}^\alpha_{p,q}(\mathbf{T}^d)$, for $\alpha \in \mathbb{R}$ and $p, q \in [1,\infty]$. $\mathcal{C}^\alpha(\mathbf{T}^d) := B^\alpha_{\infty,\infty}(\mathbf{T}^d)$ for $\alpha \in \mathbb{R}$ as in the classical Besov theory.

Lemma 2.4. *For $\alpha \in (0,1)$, we have*

$$\|f\|_{\mathcal{C}^\alpha(\mathbf{T}^d)} \lesssim_\alpha \|f\|_{\mathcal{C}^{\alpha,a}(\mathbf{T}^d)} \lesssim_\alpha \|f\|_{\mathcal{C}^\alpha(\mathbf{T}^d)}.$$

Proof. It follows from the point-wise estimate in Lemma 2.2 that

$$\|f\|_{\mathcal{C}^{\alpha,a}(\mathbf{T}^d)} \lesssim \|f\|_{\mathcal{C}^\alpha(\mathbf{T}^d)}.$$

On the other hand, when $|x-y| \geq 1$, and $x, y \in \mathbf{T}^d$,

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq 2\|f\|_{L^\infty(\mathbf{T}^d)} \lesssim \|f\|_{\alpha,a}.$$

When $|x-y| \leq 1$, there exists $N \in \mathbb{N}$ such that $2^N \leq |x-y|^{-1} < 2^{(N+1)}$. We divide the difference into two parts according to its frequency,

$$f(x) - f(y) = \sum_{j=-1}^{N} + \sum_{j=N+1}^{\infty}[(\Delta_{j,a}f)(x) - (\Delta_{j,a}f)(y)].$$

For the low-frequency part, via Lemma 2.2,

$$|\sum_{j=-1}^{N}[(\Delta_{j,a}f)(x)-(\Delta_{j,a}f)(y)]|\lesssim|x-y|2^{(1-\alpha)N}\|f\|_{\mathcal{C}^{\alpha,a}}.$$

For the high-frequency part, by definition of $\mathcal{C}^{\alpha,a}$,

$$|\sum_{j=N+1}^{\infty}[(\Delta_{j,a}f)(x)-(\Delta_{j,a}f)(y)]|\lesssim 2^{-\alpha N}\|f\|_{\mathcal{C}^{\alpha,a}}$$

Then $|f(x)-f(y)|\lesssim|x-y|^{\alpha}\|f\|_{\alpha,a}$. This completes the proof.

□

Lemma 2.5. $\forall\alpha\in(0,1)$, *we have*

$$\|f\|_{\mathcal{C}^{-\alpha}(\mathbf{T}^d)}\lesssim_{\alpha}\|f\|_{\mathcal{C}^{-\alpha,a}(\mathbf{T}^d)}\lesssim_{\alpha}\|f\|_{\mathcal{C}^{-\alpha}(\mathbf{T}^d)}.$$

Proof. For the first estimate, we divide $\Delta_j(f)$ according to frequency,

$$\Delta_j(f)=(\sum_{i=-1}^{j+1}+\sum_{i=j+2}^{\infty})\Delta_j\Big(\Delta_{i,a}(f)\Big).\tag{2.5}$$

For the first part of (2.5), by definition of $\mathcal{C}^{\alpha,a}$,

$$\|\Delta_j(\sum_{i=-1}^{j+1}\Delta_{i,a}(f))\|_{L^\infty}\lesssim 2^{\alpha j}\|f\|_{\alpha,a}.$$

For the second part of (2.5), $\forall i\geq 1$, since

$$\Delta_j(\Delta_{i,a}f)(x)=\Delta_j\Big((-\mathcal{L}_a)(-\mathcal{L}_a)^{-1}\Delta_{i,a}f\Big)(x),$$

then by integration by part,

$$\Delta_j(\Delta_{i,a}f)(x)=\int_{\mathbf{T}^d}\Big(\nabla_y\Delta_j(x,y)\Big)\cdot a(y)\Big(\nabla(-\mathcal{L}_a)^{-1}\Delta_{i,a}f\Big)(y)\mathrm{d}y.$$

Then by estimates in Lemma 2.2, $\|\Delta_j\big(\Delta_{i,a}f\big)\|_{L^\infty}\lesssim 2^{j+(\alpha-1)i}\|f\|_{\alpha,a}$, then

$$\|\sum_{i=j+2}^{\infty}\Delta_j\big(\Delta_{i,a}(f)\big)\|_{L^\infty}\lesssim 2^{\alpha j}.$$

This completes the proof of the first inequality.

For the second inequality, since

$$|\Delta_{j,a}(f)(x)|\lesssim\|\Delta_{i,a}(x,\cdot)\|_{L^1}^{1-\alpha}\|\nabla_y\Delta_{i,a}(x,\cdot)\|_{L^1}^{\alpha}\|f\|_{\mathcal{C}^{-\alpha}},$$

then by Lemma 2.2, we have $\forall j\geq-1$,

$$\|\Delta_{j,a}(f)\|_{L^\infty}\lesssim 2^{\alpha j}\|f\|_{\alpha,a}.$$

This completes the proof. □

Lemma 2.6. *$\forall i \geq 0$, $\forall m \in \mathbb{Z}$,*

$$\sup_{x\in\mathbf{T}^d} \|\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big)(x,\cdot)\|_{L^1(\mathbf{T}^d)} \leq C2^{2mi+\frac{5}{2}|m|}$$

where the constant C depends only on d, λ, Λ, M. Moreover,

$$\sup_{x\in\mathbf{T}^d} \|\nabla_x\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big)(x,\cdot)\|_{L^1(\mathbf{T}^d)} \leq C2^{2mi+\frac{5}{2}|m|+i}$$

where the constant C depends only on d, λ, Λ, M.

Proof. Define $F_m(x) = x^{2m}\varphi_0(x)$ where φ_0 is chosen and fixed in (2.3). Define

$$c_{m,k} := R^d\Big((C_1k)^k\|F_m\|_{L^\infty(\mathbf{T}^d)} + (C_2R)^k\|F_m^{(k)}\|_{L^\infty(\mathbf{T}^d)}\Big)$$

where we set $k = 2d$ and $R = 2$, since $\mathrm{supp}\varphi_0 \subset \overline{B(0,2)}$. Here, the constants C_1,C_2 are given in Lemma 2.2.

By explicit computations, we have

$$c_{m,k} \leq C2^{\frac{5|m|}{2}},$$

where the constant C depends only on d, λ, Λ, M.

For $F_m(2^{-i}\sqrt{-\mathcal{L}_a})$, by Lemma 2.2, we have

$$|F_m(2^{-i}\sqrt{-\mathcal{L}_a})(x,y)| \leq c_{m,k}2^{di}(1+\frac{|x-y|}{2^{-i}})^{-k}.$$

This completes proof of the first inequality with $(-\mathcal{L}_a)^m\Delta_{i,a} = 2^{2mi}F_m(2^{-i}\sqrt{-\mathcal{L}_a})$.

For the second one, we have

$$\nabla(-\mathcal{L}_a)^m\Delta_{i,a} = \Big(\nabla(\sum_{l=i-1}^{i+1}\Delta_{l,a})\Big)\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big).$$

More precisely,

$$\nabla\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big)(x,y) = \int_{\mathbf{T}^d}\Big(\nabla(\sum_{l=i-1}^{i+1}\Delta_{l,a})\Big)(x,z)\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big)(z,y)\mathrm{d}z.$$

By Lemma 2.2,

$$\sup_{x\in\mathbf{T}^d}\|\nabla_x(\sum_{l=i-1}^{i+1}\Delta_{l,a})(x,\cdot)\|_{L^1(\mathbf{T}^d)} \leq C2^i,$$

where the constant C depends only on d, λ, Λ, M.

By the first inequality,

$$\sup_{x\in\mathbf{T}^d} \|\Big((-\mathcal{L}_a)^m\Delta_{i,a}\Big)(x,\cdot)\|_{L^1(\mathbf{T}^d)} \leq C2^{2mi+\frac{5}{2}|m|},$$

where the constant C depends only on d, λ, Λ, M.

By the above estimates, we have

$$\sup_{x\in\mathbf{T}^d} \|\Big(\nabla_x(-\mathcal{L}_a)^m\Delta_{i,a}\Big)(x,\cdot)\|_{L^1(\mathbf{T}^d)} \leq C2^{2mi+\frac{5}{2}|m|+i},$$

where the constant C depends only on d, λ, Λ, M. This completes the proof of the second inequality. ∎

Lemma 2.7. *$\alpha, \beta \in \mathbf{R}$, $\exists C > 0$, $\forall j, k \le i-4$,*

$$\|\Delta_{i,a}(\Delta_{j,a}f\Delta_{k,a}g)\|_{L^\infty(\mathbf{T}^d)} \le C2^{-2i+j+k}2^{-(\alpha j+\beta k)}\|f\|_{\alpha,a}\|g\|_{\beta,a},$$

and $\exists C > 0$, $\forall i, j \le k-4$,

$$\|\Delta_{i,a}(\Delta_{j,a}f\Delta_{k,a}g)\|_{L^\infty(\mathbf{T}^d)} \le C2^{-2k+i+j}2^{-(\alpha j+\beta k)}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

Proof. For the first inequality, since $i \ge \max\{j,k\}+4$, then $\forall \lambda_{\theta,a} \in \operatorname{supp}(\varphi_i)$, $\lambda_{\Theta,a} \in \operatorname{supp}(\varphi_j)$, $\lambda_{\gamma,a} \in \operatorname{supp}(\varphi_k)$,

$$\lambda_{\theta,a}^2 > \lambda_{\Theta,a}^2 + \lambda_{\gamma,a}^2.$$

Then by Taylor's formula,

$$\frac{1}{\lambda_{\theta,a}^2 - \lambda_{\Theta,a}^2 - \lambda_{\gamma,a}^2} = \sum_{l=0}^{\infty}(\lambda_{\theta,a}^2)^{-(l+1)}\Big(\sum_{m=0}^{l} C_l^m (\lambda_{\Theta,a}^2)^m(\lambda_{\gamma,a}^2)^{l-m}\Big), \tag{2.6}$$

where we define $0! := 1$ and C_l^m is defined as

$$C_l^m := \frac{l!}{m!(l-m)!}, \quad \forall 0 \le m \le l, l \in \mathbb{N}.$$

By definition of $(\lambda_{n,a}, \psi_{n,a})$,

$$< \psi_{\theta,a}, \psi_{\Theta,a}\psi_{\gamma,a} >= -\frac{2 < \psi_{\theta,a}, \big(a\nabla\psi_{\Theta,a}\big)\cdot\nabla\psi_{\gamma,a} >}{\lambda_{\theta,a}^2 - \lambda_{\Theta,a}^2 - \lambda_{\gamma,a}^2}. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned}&\Delta_{i,a}(\Delta_{j,a}f\Delta_{k,a}g)(x)\\ =& -2\sum_{l=0}^{\infty}\sum_{m=0}^{l} C_l^m \int_{\mathbf{T}^d} P_{1,l,i}(x,y)\Big(a(y)\nabla P_{2,m,j}(y)\Big)\cdot\nabla P_{3,l-m,k}(y)\mathrm{d}y,\end{aligned}$$

where

$$\begin{aligned}P_{1,l,i}(x,y) &:= \sum_{\theta\in\mathbb{N}} \lambda_{\theta,a}^{-2(l+1)}\varphi_i(\lambda_{\theta,a})\psi_{\theta,a}(x)\psi_{\theta,a}(y);\\ P_{2,m,j}(y) &:= \sum_{\Theta\in\mathbb{N}} \lambda_{\Theta,a}^{2m}\varphi_j(\lambda_{\Theta,a}) < f, \psi_{\Theta,a} > \psi_{\Theta,a}(y);\\ P_{3,l-m,k}(y) &:= \sum_{\gamma\in\mathbb{N}} \lambda_{\gamma,a}^{2(l-m)}\varphi_k(\lambda_{\gamma,a}) < g, \psi_{\gamma,a} > \psi_{\gamma,a}(y).\end{aligned}$$

Here we have put one derivative on low frequency $\Delta_{j,a}f$ and $\Delta_{k,a}g$ each in order to gain high regularity of high frequency $\Delta_{i,a}$. Rewriting these formulas, we have

$$\begin{aligned}&P_{1,l,i}(x,y) := \big((-\mathcal{L}_a)^{-(l+1)}\Delta_{i,a}\big)(x,y);\\ &P_{2,m,j} := (-\mathcal{L}_a)^m\Delta_{j,a}f;\\ &P_{3,l-m,k} := (-\mathcal{L}_a)^{l-m}\Delta_{k,a}g.\end{aligned}$$

For $(-\mathcal{L}_a)^{-(l+1)}\Delta_{i,a}$, by Lemma 2.6 , we have

$$\sup_{x\in\mathbf{T}^d}\|\Big((-\mathcal{L}_a)^{-(l+1)}\Delta_{i,a}\Big)(x,\cdot)\|_{L^1}\leq C2^{-(l+1)(i-\frac{5}{4})}$$

where the constant C depends only on d,λ,Λ,M.

For $\nabla(-\mathcal{L}_a)^m\Delta_{j,a}$, by Lemma 2.6, we have

$$\|\nabla(-\mathcal{L}_a)^m\Delta_{j,a}f\|_{L^\infty}\leq C2^{j(1-\alpha)}\|f\|_{\alpha,a}2^{2m(j+\frac{5}{4})}$$

where the constant C depends only on d,λ,Λ,M.

For $\nabla(-\mathcal{L}_a)^{l-m}\Delta_{k,a}$, by Lemma 2.6, we have

$$\|\nabla(-\mathcal{L}_a)^{l-m}\Delta_{k,a}g\|_{L^\infty}\leq C2^{k(1-\beta)}\|g\|_{\beta,a}2^{2(l-m)(k+\frac{5}{4})}$$

where the constant C depends only on d,λ,Λ,M.

By these estimates, we have

$$\begin{aligned}\|\Delta_{i,a}(\Delta_{j,a}f\Delta_{k,a}g)\|_{L^\infty}\lesssim&\|f\|_{\alpha,a}\|g\|_{\beta,a}2^{j(1-\alpha)}2^{k(1-\beta)}\\&\sum_{l=0}^{\infty}\sum_{m=0}^{l}C_l^m2^{-(l+1)(i-\frac{5}{4})}2^{2m(j+\frac{5}{4})}2^{2(l-m)(k+\frac{5}{4})}.\end{aligned}$$

Since $i\geq\max\{j,k\}+4$, then

$$\sum_{l=0}^{\infty}2^{-2(l+1)(i-\frac{5}{4})}\sum_{m=0}^{l}C_l^m\big(2^{2(j+\frac{5}{4})}\big)^m\big(2^{2(k+\frac{5}{4})}\big)^{l-m}=\frac{1}{2^{2(i-\frac{5}{4})}-2^{2(j+\frac{5}{4})}-2^{2(k+\frac{5}{4})}}.$$

This completes the proof of the first one. The second one can be dealt with similarly. $\square$

Lemma 2.8.

$$\|g\prec_a f\|_{\alpha+\beta,a}\lesssim_{\alpha,\beta}\|f\|_{\alpha,a}\|g\|_{\beta,a},\quad\forall\alpha+\beta\in(-1,2),\beta<0;$$

and

$$\|f\circ_a g\|_{\alpha+\beta,a}\lesssim_{\alpha,\beta}\|f\|_{\alpha,a}\|g\|_{\beta,a},\quad\forall\alpha+\beta\in(0,2).$$

Moreover,

$$\|g\prec_a f\|_{\alpha,a}\lesssim_{\alpha}\|f\|_{\alpha,a}\|g\|_{L^\infty},\quad\forall\alpha\in(-1,2).$$

Proof. For the first, since $\alpha+\beta<2,\beta<1$, by the first estimate in Lemma 2.7 and definition of $\|\cdot\|_{\beta,a}$, we have

$$\sum_{j=-1}^{i-11}\|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty}\lesssim2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

Since $\beta<0$, then by definition,

$$\sum_{j=i-10}^{i+10}\|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty}\lesssim2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

Since $\alpha+\beta>-1, \beta<1$, by the second estimate in Lemma 2.7,

$$\sum_{j=i+11}^{\infty} \|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}$$

For the second, since $\alpha+\beta<2$, by the first estimate in Lemma 2.7,

$$\sum_{j=-1}^{i-11}\sum_{|j-k|\le 1} \|\Delta_{i,a}\Big(\Delta_{j,a}f\Delta_{k,a}g\Big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

By definition,

$$\sum_{j=i-10}^{i+10}\sum_{|j-k|\le 1} \|\Delta_{i,a}\Big(\Delta_{j,a}f\Delta_{k,a}g\Big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

Since $\alpha+\beta>0$, by definition,

$$\sum_{j=i+11}^{\infty}\sum_{|j-k|\le 1} \|\Delta_{i,a}\Big(\Delta_{j,a}f\Delta_{k,a}g\Big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the last, since $\alpha<2$, by the first estimate in Lemma 2.7 and definition of $\mathcal{C}^{\alpha,a}$,

$$\sum_{j=-1}^{i-11} \|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty} \lesssim 2^{-\alpha i}\|f\|_{\alpha,a}\|g\|_{L^\infty}.$$

By definition,

$$\sum_{j=i-10}^{i+10} \|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty} \lesssim 2^{-\alpha i}\|f\|_{\alpha,a}\|g\|_{L^\infty}.$$

Since $\alpha>-1$, by the second estimate in Lemma 2.7 and definition of $\mathcal{C}^{\alpha,a}$,

$$\sum_{j=i+11}^{\infty} \|\Delta_{i,a}\Big(\Delta_{j,a}f\big(\sum_{k=-1}^{j-2}\Delta_{k,a}g\big)\Big)\|_{L^\infty} \lesssim 2^{-\alpha i}\|f\|_{\alpha,a}\|g\|_{L^\infty}.$$

This completes the proof. $\square$

2.2 Commutator estimates

Lemma 2.9. *For $\alpha\in(0,1)$, we have*

$$||[\Delta_{j,a},f]g||_{L^\infty} \lesssim 2^{-\alpha j}\|f\|_{\alpha,a}\|g\|_{L^\infty}.$$

where $[\Delta_{j,a},f]g := \Delta_{j,a}(fg) - f\Delta_{j,a}g$.

Proof. By definition,

$$[\Delta_{j,a}, f]g = \Delta_{j,a}(fg) - f\Delta_{j,a}g = \int_{\mathbf{T}^2} \Delta_{j,a}(x,y)(f(y)-f(x))g(y)\mathrm{d}y$$

By Lemma 2.2, we have

$$|\int_{\mathbf{T}^2} \Delta_{j,a}(x,y)(f(y)-f(x))g(y)\mathrm{d}y| \lesssim \int_{\mathbf{T}^2} 2^j(1+2^j|x-y|)^{-10}|x-y|^{\alpha}\mathrm{d}y\|f\|_{\alpha,a}\|g\|_{L^\infty}.$$

This completes the proof. $\square$

Lemma 2.10. *For $\alpha \in (0,1)$ and $\alpha+\beta \in (0,2)$, we have*

$$||[\sqrt{-\mathcal{L}_a}, f \prec_a]g||_{\alpha+\beta-1,a} \lesssim \|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

where

$$[\sqrt{-\mathcal{L}_a}, f \prec_a]g := \sqrt{-\mathcal{L}_a}(f \prec_a g) - f \prec_a (\sqrt{-\mathcal{L}_a}g).$$

Proof. We divide the difference into three parts.

$$\Delta_{i,a}\big([\sqrt{-\mathcal{L}_a}, f \prec_a]g\big) = \Delta_{i,a}(\sum_{k=1}^{i-N-1} + \sum_{k=i-N}^{i+N} + \sum_{k=i+N+1}^{\infty})[\sqrt{-\mathcal{L}_a}, S_{k,a}f]\Delta_{k,a}g.$$

For the first part, it contains two terms. For the first term, by Lemma 2.7,

$$\|\sqrt{-\mathcal{L}_a}\Delta_{i,a}\big(\sum_{k=1}^{i-N-1}(S_{k,a}f\Delta_{k,a}g)\big)\|_{L^\infty} \lesssim 2^{(1-\alpha-\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

Similarly, for the other term,

$$\|\Delta_{i,a}\big(\sum_{k=1}^{i-N-1} S_{k,a}f\sqrt{-\mathcal{L}_a}\Delta_{k,a}g\big)\|_{L^\infty} \lesssim 2^{(1-\alpha-\beta)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the second part, we divide it into two terms.

$$\Delta_{i,a}\sum_{k=i-N}^{i+N}[\sqrt{-\mathcal{L}_a}, S_{k,a}f]\Delta_{k,a}g = \sum_{k=i-N}^{i+n}[\Delta_{i,a}\sqrt{-\mathcal{L}_a}, S_{k,a}f]\Delta_{k,a}g + [\Delta_{i,a}, S_{k,a}f]\big(\sqrt{-\mathcal{L}_a}\Delta_{k,a}g\big)$$

For the first term,

$$\sum_{k=i-N}^{i+N}[\Delta_{i,a}\sqrt{-\mathcal{L}_a}, S_{k,a}f]\Delta_{k,a}g = \sum_{k=i-N}^{i+n}\int_{\mathbf{T}^1}(\Delta_{i,a}\sqrt{-\mathcal{L}_a})(x,y)(S_{k,a}f(y) - S_{k,a}f(x))\Delta_{k,a}g(y)\mathrm{d}y.$$

By Lemma 2.2, then we have

$$\|\sum_{k=i-N}^{i+N}[\Delta_{i,a}\sqrt{-\mathcal{L}_a}, S_{k,a}f]\Delta_{k,a}g\|_{L^\infty} \lesssim 2^{-(\alpha+\beta-1)i}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the second term,

$$[\Delta_{i,a}, S_{k,a}f]\big(\sqrt{-\mathcal{L}_a}\Delta_{k,a}g\big) = \int_{\mathbf{T}^1} \Delta_{i,a}(x,y)(S_{k,a}f(y) - S_{k,a}f(x))\big(\sqrt{-\mathcal{L}_a}\Delta_{k,a}g\big)(y)\mathrm{d}y.$$

By Lemma 2.2, then we have

$$\|\sum_{k=i-N}^{i+N}[\Delta_{i,a}, S_{k,a}f]\big(\sqrt{-\mathcal{L}_a}\Delta_{k,a}g\big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta-1)i}.$$

For the third part, this part can be dealt with similarly as the first part. This completes the proof.

$\square$

Lemma 2.11. *For $\alpha \in (0,1)$ and $\alpha+\beta \in (-1,2)$, we have*

$$\|R_{j,a}(f,g)\|_{\alpha+\beta,a} \lesssim \|f\|_{\alpha,a}\|g\|_{\beta,a},$$

where

$$\Delta_{j,a}(f \prec_a g) = f\Delta_{j,a}g + R_{j,a}(f,g).$$

Proof. First, we remove the high frequency part of f,

$$R_{j,a}(f,g) = R_{j,a}\big((\mathrm{id} - S_{j-1,a})f, g\big) + R_{j,a}(S_{j-1,a}f, g)$$

For the first part, since $\|(\mathrm{id} - S_{j-1,a})f\|_{L^\infty} \lesssim 2^{-j\alpha}\|f\|_{\alpha,a}$ and Lemma 2.7,

$$\|\Delta_{j,a}\big((\mathrm{id} - S_{j-1,a})f \prec_a g\big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}\|f\|_{\alpha,a}\|g\|_{\beta,a},$$

and

$$\|(\mathrm{id} - S_{j-1,a})f\Delta_{j,a}(g)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the second part,

$$R_{j,a}(S_{j-1,a}f, g) = [\Delta_{j,a}, S_{j-1,a}f]M_{i,a}g + \Delta_{j,a}(S_{j-1,a}f \prec_a S_{j,a}g) + \Delta_{j,a}(S_{j-1,a}f \prec_a U_{j,a}g)$$

where

$$M_{j,a}f := \sum_{|l-j|\lesssim 1}\Delta_{j,a}f\ , \quad U_{j,a}f := \sum_{l\geq j+2}\Delta_{j,a}f\ .$$

For the first term, by Lemma 2.9,

$$\|[\Delta_{j,a}, S_{j-1,a}f]M_{i,a}g\| \lesssim 2^{-(\alpha+\beta)j}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the last two terms, by Lemma 2.7 and definition of $\mathcal{C}^{\alpha,a}$,

$$\|\Delta_{j,a}(S_{j-1,a}f \prec_a S_{j,a}g)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}\|f\|_{\alpha,a}\|g\|_{\beta,a},$$

and

$$\|\Delta_{j,a}(S_{j-1,a}f \prec_a U_{j,a}g)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

This completes the proof. $\square$

Lemma 2.12. $\forall \alpha \in (0,1)$, $\beta < 0$,$\alpha+\beta \in (-1,1)$, $\forall j \geq -1$,

$$\|[\Delta_{j,a}, f]g - \Delta_{j,a}(f \circ_a g)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}.$$

Proof. By explicit computation,

$$[\Delta_{j,a}, f]g - \Delta_{j,a}(f \circ_a g) = \Delta_{j,a}(f \succ_a g) + R_{j,a}(f,g).$$

Then the assertion follows from Lemmas 2.8 and 2.11. □

Lemma 2.13. *For $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbf{R}$ such that $\alpha+\beta+\gamma \in (0,2)$, $\alpha+\beta \in (-1,2)$ and $\beta+\gamma < 0$, then we have*

$$\|C_a(f,g,h)\|_{\alpha+\beta+\gamma,a} \lesssim \|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}$$

where

$$C_a(f,g,h) := (f \prec_a g) \circ_a h - f(g \circ_a h).$$

Proof.

$$\Delta_{l,a} C_a(f,g,h) = \Delta_{l,a}\Big(\big(\sum_{j\leq l-11} + \sum_{j\geq l-10} \big) \big(\sum_{|i-j|\leq 1} R_{i,a}(f,g)\Delta_{j,a}h \big) \Big).$$

For the second term, by definition and Lemma 2.11,

$$\|R_{i,a}(f,g)\Delta_{j,a}h\|_{L^\infty} \lesssim 2^{-i(\alpha+\beta)-j\gamma}\|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}.$$

Then summing it up, with $\alpha+\beta+\gamma > 0$, we have

$$\| \sum_{j\geq l-10} \sum_{|i-j|\leq 1} R_{i,a}(f,g)\Delta_{j,a}h\|_{L^\infty} \lesssim 2^{-(\alpha+\beta+\gamma)l}\|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}.$$

For the first term, we divide it into two parts.

$$\begin{aligned}
&\Delta_{l,a}\Big(\sum_{j\leq l-11} \sum_{|i-j|\leq 1} R_{i,a}(f,g)\Delta_{j,a}h \Big) \\
=&\Delta_{l,a}\Big(\sum_{j\leq l-11} \sum_{|i-j|\leq 1} \Big(R_{i,a}(S_{l-10,a}f, g) + R_{i,a}((\mathrm{id} - S_{l-10,a})f, g)\Big)\Delta_{j,a}h \Big)
\end{aligned}$$

For the second part, since $\|(\mathrm{id} - S_{l-10,a})f\|_{L^\infty} \lesssim 2^{-\alpha l}$, then by definition,

$$\|(\mathrm{id} - S_{l-10,a})f\Delta_{i,a}g\Delta_{j,a}h\| \lesssim 2^{-(i(\alpha+\beta)+j\gamma)}\|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}.$$

Summing it up,

$$\| \sum_{j\leq l-11} \sum_{|i-j|\leq 1} (\mathrm{id} - S_{l-10,a})f\Delta_{i,a}g\Delta_{j,a}h\|_{L^\infty} \lesssim 2^{-(\alpha+\beta+\gamma)l}\|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}.$$

For the first part, with Lemma 2.7, we have

$$\|\Delta_{l,a}\Big(\sum_{j\leq l-11} \sum_{|i-j|\leq 1} R_{i,a}((\mathrm{id} - S_{l-10,a})f, g)\Delta_{j,a}h \Big)\| \lesssim 2^{-(\alpha+\beta+\gamma)l}\|f\|_{\alpha,a}\|g\|_{\beta,a}\|h\|_{\gamma,a}.$$

This completes the proof. □

Lemma 2.14. *For $0 < \beta \le \alpha < 1$,*

$$\|\mathcal{R}_{F,a}(u)\|_{\alpha+\beta,a} \lesssim \|F\|_{\mathcal{C}^{1+\frac{\beta}{\alpha}}}(1 + \|u\|_{\alpha,a}^{1+\frac{\beta}{\alpha}})$$

where

$$\mathcal{R}_{F,a}(u) := F(u) - F'(u) \prec_a u.$$

Moreover,

$$\|\mathcal{R}_{F,a}(u) - \mathcal{R}_{F,a}(v)\|_{\alpha+\beta,a} \lesssim \|F\|_{\mathcal{C}^{1+\frac{\beta}{\alpha}}}(1 + \|u\|_{\alpha,a}^{\frac{\beta}{\alpha}} + \|v\|_{\alpha,a}^{\frac{\beta}{\alpha}})\|u - v\|_{\alpha,a}.$$

Proof. By definition,

$$\Delta_{i,a}\mathcal{R}_{F,a}(u) = \big(\Delta_{i,a}(F(u)) - F'(u)\Delta_{i,a}u\big) - R_{j,a}(F'(u), u).$$

For the first term,

$$\begin{aligned}
&\Delta_{i,a}(F(u(x))) - F'(u(x))\Delta_{i,a}u(x) \\
&= \int_{\mathbf{T}^d} \Delta_{i,a}(x,y)\big(F(u(y)) - F(u(x)) - F'(u(x))(u(y) - u(x))\big)\mathrm{d}y
\end{aligned}$$

Since by definition,

$$|F(u(y)) - F(u(x)) - F'(u(x))(u(y) - u(x))| \lesssim \|F\|_{\mathcal{C}^{1+\frac{\beta}{\alpha}}}|u(y) - u(x)|^{1+\frac{\beta}{\alpha}}$$

then

$$\|\Delta_{i,a}(F(u)) - F'(u)\Delta_{i,a}u\|_{L^\infty} \lesssim 2^{-i(\alpha+\beta)}\|f\|_{\alpha,a}\|g\|_{\beta,a}.$$

For the second term, by Lemma 2.11, with $F'(u) \in \mathcal{C}^{\beta,a}$ and $u \in \mathcal{C}^{\alpha,a}$, then

$$\|R_{j,a}(F'(u), u)\| \lesssim 2^{-j(\alpha+\beta)}\|F'(u)\|_{\beta,a}\|u\|_{\alpha,a}.$$

This completes the proof for the first inequality.

The proof for the second one is given below. Now, we consider the second one. We divide the difference into two parts.

$$\begin{aligned}
\Delta_{j,a}(R_{F,a}(u) - R_{F,a}(v)) &= R_{j,a}(F'(u), u) - R_{j,a}(F'(v), v) \\
&\quad + \big[\Delta_{j,a}(F(u) - F(v)) - (F'(u)\Delta_{j,\varepsilon}u - F'(u)\Delta_{j,\varepsilon}u)\big].
\end{aligned}$$

For the first part, we divide it into two parts.

$$R_{j,a}(F'(u), u) - R_{j,a}(F'(v), v) = R_{j,a}(F'(u) - F'(v), u) + R_{j,a}(F'(v), u - v).$$

Then by Lemma 2.11, we have

$$\|R_{j,a}(F'(u), u) - R_{j,a}(F'(v), v)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)j}\|u - v\|_\alpha\|F\|_{2+\frac{\beta}{\alpha}}(1 + \|u\|_\alpha + \|v\|_\alpha)^{\frac{\beta}{\alpha}}.$$

For the second part, by definition,

$$\Delta_{j,a}(F(u) - F(v)) - F'(u)\Delta_{j,a}u + F'(v)\Delta_{j,a}v = \int_{\mathbf{T}^d} \Delta_{j,a}(x,y)\big(\Theta_u(x,y) - \Theta_v(x,y)\big)\mathrm{d}y$$

where
$$\Theta_u(x,y)=F(u(y))-F(u(x))-F'(u(x))(u(y)-u(x)).$$
By the estimates for Littlewood-Paley blocks derived from Lemma 2.2, it suffices to show
$$|\Theta_u(x,y)-\Theta_v(x,y)|\lesssim |x-y|^{\alpha+\beta}\|F\|_{\mathcal{C}^{1+\frac{\beta}{\alpha}}}(1+\|u\|_{\alpha,a}^{\frac{\beta}{\alpha}}+\|v\|_{\alpha,a}^{\frac{\beta}{\alpha}})\|u-v\|_{\alpha,a}.$$
Since
$$\Theta_u(x,y)=\big(u(y)-u(x)\big)\int_0^1\big[F'(z(u(y)-u(x))+u(x))-F(u(x))\big]\mathrm{d}z,$$
then
$$\begin{aligned}\Theta_u(x,y)-\Theta_v(x,y)=&\big((u-v)(y)-(u-v)(x)\big)\int_0^1\Psi_u(x,y;z)\mathrm{d}z\\&+\big(v(y)-v(x)\big)\int_0^1(\Psi_{F,u}(x,y;z)-\Psi_{F,v}(x,y;z))\mathrm{d}z=P_1+P_2\end{aligned}$$
where $\Psi_u(x,y;z)=F'(z(u(y)-u(x))+u(x))-F(u(x))$.

Since
$$|\Psi_u(x,y;z)|\lesssim |x-y|^\beta\|F'\|_{\frac{\beta}{\alpha}}\|u\|_\alpha^{\frac{\beta}{\alpha}}$$
then for the first part,
$$|P_1(x,y)|\lesssim |x-y|^{\alpha+\beta}\|u-v\|_\alpha\|F'\|_{\frac{\beta}{\alpha}}\|u\|_\alpha^{\frac{\beta}{\alpha}}.$$
For the second part, since
$$|\Psi_u(x,y;z)-\Psi_v(x,y;z)|\lesssim |x-y|^\beta\|u-v\|_\alpha^{\frac{\beta}{\alpha}}\|F\|_{2+\frac{\beta}{\alpha}},$$
then the estimate for P_2 is obtained. This completes the proof.

□

Lemma 2.15. *For $\alpha\in(0,2)$,*
$$\|e^{t\mathcal{L}_a}f\|_{L_T^\infty\mathcal{C}_x^{\alpha,a}\cap\mathcal{C}_T^{\frac{\alpha}{2}}L_x^\infty}\lesssim\|f\|_{\alpha,a}.$$
Moreover, $\forall\alpha\in\mathbf{R}$,
$$\|e^{t\mathcal{L}_a}f\|_{L_T^\infty\mathcal{C}_x^{\alpha,a}}\lesssim\|f\|_{\alpha,a}.$$
Proof. For the second estimate,
$$\Delta_{j,a}(e^{t\mathcal{L}_a}f)=\sum_{k\in\mathbb{N}^+}e^{-t\lambda_{k,a}^2}\varphi_j(\lambda_{k,a})<f,\psi_{k,a}>\psi_{k,a}.$$
Then by Lemma 2.2,
$$\|\Delta_{j,a}(e^{t\mathcal{L}_a}f)\|_{L^\infty}\lesssim\|f\|_{\alpha,a}.$$
Then by definition, $\|e^{t\mathcal{L}_a}f\|_{L_T^\infty\mathcal{C}_x^{\alpha,a}}\lesssim\|f\|_{\alpha,a}$.

For the time regularity, at first we consider the difference $e^{t\mathcal{L}_a} - Id$. $\forall t \in (0, \frac{1}{2})$, $\exists j \in \mathbf{N}$, s.t. $2^{-2j} \leq t < 2^{-2(j-1)}$. Then we divide the difference into two parts.

$$e^{t\mathcal{L}_a} f - f = (e^{t\mathcal{L}_a} - Id)(Id - S_{j,a})f + (e^{t\mathcal{L}_a} - Id)S_{j,a}f.$$

For the first part, by definition,

$$\|(e^{t\mathcal{L}_a} - Id)(Id - S_{j,a})f\|_{L^\infty} \lesssim 2^{-\alpha j}\|f\|_{\alpha,a} \lesssim t^{\frac{\alpha}{2}}\|f\|_{\alpha,a}.$$

For the second part, by Lemma 2.2,

$$\|(e^{t\mathcal{L}_a} - Id)S_{j,a}f\|_{L^\infty} \lesssim t2^{(2-\alpha)j}\|f\|_{\alpha,a} \lesssim t^{\frac{\alpha}{2}}\|f\|_{\alpha,a}.$$

Then $\forall s, t > 0$, and $|s - t| < \frac{1}{2}$,

$$\|(e^{t\mathcal{L}_a} - e^{s\mathcal{L}_a})f\|_{L^\infty} \lesssim (t - s)^{\frac{\alpha}{2}}\|f\|_{\alpha,a}.$$

This completes the proof. □

We define

$$\mathcal{I}_a(f) := \int_0^t e^{(t-\tau)\mathcal{L}_a}\big(f(\tau,\cdot)\big)\mathrm{d}\tau. \tag{2.8}$$

Lemma 2.16. *For $\alpha \in (0, 2)$,*

$$\|\mathcal{I}_a(f)\|_{\mathcal{C}_T^{\frac{\alpha}{2}} L_x^\infty \cap L_T^\infty \mathcal{C}_x^{\alpha,a}} \lesssim \|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}}.$$

Moreover for $\alpha \in \mathbf{R}$,

$$\|\mathcal{I}_a(f)\|_{L_T^\infty \mathcal{C}_x^{\alpha,a}} \lesssim \|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}}.$$

Proof. For space regularity of the first estimate, by definition,

$$\Delta_{j,a}(\mathcal{I}_a(f)) = \int_0^t \bigg(\sum_{k\in\mathbb{N}^+} e^{-t\lambda_{k,a}^2}\varphi_i(\lambda_{k,a}) < \sum_{|i-j|\leq 1} \Delta_{i,a}f, \psi_{k,a} > \psi_{k,a}\bigg)\mathrm{d}\tau.$$

then

$$\|\Delta_{j,a}(\mathcal{I}_a(f))\| \lesssim \int_0^t e^{-(t-\tau)2^{-2j+N}} \sum_{|l-j|\leq 1} \|\Delta_{l,a}f(\tau)\|_{L^\infty}\mathrm{d}\tau.$$

Since $\sum_{|l-j|\leq 1} \|\Delta_{l,a}f\|_{L^\infty} \lesssim 2^{-j(\alpha-2)}$, then

$$\|\Delta_{j,a}(\mathcal{I}_a(f))\|_{L^\infty} \lesssim 2^{-\alpha j}\|f\|_{\alpha-2,a}.$$

For the estimate about time regularity, $\forall 0 < s < t$,

$$\mathcal{I}_a(f)(t) - \mathcal{I}_a(f)(s) = (e^{(t-s)\mathcal{L}_a} - \mathrm{id})\Big(\mathcal{I}_a(s)\Big) + \int_s^t e^{(t-r)\mathcal{L}}\big(f(r)\big)\mathrm{d}r.$$

For the first part, by Lemma 2.15,

$$\|(e^{(t-s)\mathcal{L}_a} - \mathrm{id})\Big(\mathcal{I}_a(f)(s)\Big)\|_{L^\infty} \lesssim (t - s)^{\frac{\alpha}{2}}\|f\|_{L^\infty\mathcal{C}^{-\alpha,a}}.$$

For the second part, $\forall 0 < |t-s| \le \frac{1}{2}$, $\exists j \in \mathbf{R}$, s.t. $2^{-2j} \le t-s < 2^{-2(j-1)}$. Then we divide the second part into two terms,

$$\int_s^t e^{(t-r)\mathcal{L}_a}\big(f(r)\big)\mathrm{d}r = \int_s^t e^{(t-r)\mathcal{L}_a}\big((Id - S_{j,a})f(r)\big)\mathrm{d}r + \int_s^t e^{(t-r)\mathcal{L}_a}\big(S_{j,a}f(r)\big)\mathrm{d}r$$

For the first term, by definition and Lemma 2.2, we have

$$\|\int_s^t e^{(t-r)\mathcal{L}_a}\big((Id - S_{j,a})f(r)\big)\mathrm{d}r\|_{L^\infty} \lesssim 2^{-\alpha j}\|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}} \lesssim (t-s)^{\frac{\alpha}{2}}\|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}}.$$

For the second term,

$$\|\int_s^t e^{(t-r)\mathcal{L}_a}\big(S_{j,a}f(r)\big)\mathrm{d}r\|_{L^\infty} \lesssim \int_s^t (t-r)^{\frac{\alpha}{2}-1}\mathrm{d}r\|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}} \lesssim (t-s)^{\frac{\alpha}{2}}\|f\|_{L_T^\infty \mathcal{C}_x^{\alpha-2,a}}.$$

This completes the proof.

□

This estimate below is in spirit of the proof in [10, Lemma 18].

Lemma 2.17. *Let $T > 0$. For $\alpha, \beta \in (0,1)$,*

$$\|[\mathcal{I}_a, f \prec_a]g\|_{L_T^\infty \mathcal{C}_x^{\alpha+\beta,a} \cap \mathcal{C}_T^{(\alpha\wedge\beta)/2} L_x^\infty} \lesssim \|f\|_{\mathcal{C}_s^\alpha([0,T]\times\mathbf{T}^2)}\|g\|_{L_T^\infty \mathcal{C}_x^{\beta-2,a}}.$$

where

$$[\mathcal{I}_a, f \prec_a]g := \mathcal{I}_a(f \prec_a g) - f\mathcal{I}_a(g).$$

Proof. For $\delta = (\delta_1, \delta_2)$ and δ_1, $\delta_2 > 0$, set

$$f_\delta = \rho_{\delta_2} * (\rho_{\delta_1} * f)$$

in which $\rho_{\delta_1}, \rho_{\delta_2}$ is the mollifier for time and space respectively. For $i \in \mathbb{N}^+$, we set $\delta_1 = 2^{-2i}$ and $\delta_2 = 2^{-i}$. We divide the commutator into two parts.

$$[\mathcal{I}_a, f \prec_a]g = [\mathcal{I}_a, (f - f_\delta) \prec_a]g + [\mathcal{I}_a, f_\delta \prec_a]g$$

For the first term, since $\|f - f_\delta\|_{L^\infty} \lesssim 2^{-\alpha i}$, then

$$\|\Delta_{i,a}\mathcal{I}_a((f - f_\delta) \prec_a g)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i},$$

and

$$\|\Delta_{i,a}\big((f - f_\delta) \prec_a \mathcal{I}_a(g)\big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}.$$

For the second term,

$$[\mathcal{I}_a, f_\delta \prec_a]g = \mathcal{I}_a\Big(-\big((\partial_t - \mathcal{L}_a)(f_\delta)\big) \prec_a \mathcal{I}_a(g) + 2\sum_{j=1}^{\infty} a_\varepsilon \nabla S_{j,a} f_\delta \cdot \nabla \Delta_{j,a}\mathcal{I}_a(g)\Big).$$

For the first part, since $\|\partial_t f_\delta\|_{L_{T,x}^\infty} \lesssim \delta_1^{\frac{\alpha}{2}-1}$, then we have

$$\|\mathcal{I}_a\big(\Delta_{i,a}(\partial_t f_\delta \prec_a \mathcal{I}_a(g))\big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}.$$

For the second part, since $\mathcal{L}_a(f_\delta) \in L_T^\infty \mathcal{C}_x^{\alpha-2,a}$, then

$$\|\Delta_{i,a}\mathcal{I}_a\Big(\mathcal{L}_a(f_\delta) \prec_a \mathcal{I}_a(g)\Big)\|_{L^\infty} \lesssim 2^{-(\alpha+\beta)i}.$$

For the third part, consider the localised frequency,

$$\Delta_{m,a}\bigg(\sum_{j=1}^{\infty} a\nabla\big(S_{j,a}f_\delta\big) \cdot \nabla\big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg) = \Delta_{m,a}\bigg(\big(\sum_{j=1}^{m} + \sum_{j=m+1}^{\infty}\big) a\nabla\big(S_{j,a}f_\delta\big) \cdot \nabla\big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg).$$

For the first term, by Lemma 2.2,

$$\|\Delta_{m,a}\bigg(\sum_{j=1}^{m} a\nabla\big(S_{j,a}f_\delta\big) \cdot \nabla\big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg)\|_{L^\infty} \lesssim \sum_{j=1}^{m} 2^{j(1-\alpha)}2^{j(1-\beta)} \lesssim 2^{(2-(\alpha+\beta))m}.$$

For the second term,

$$\begin{aligned}&\Delta_{m,a}\bigg(\sum_{j=m+1}^{\infty} a\nabla\big(S_{j,a}f_\delta\big) \cdot \nabla\big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg)\\ =&\Delta_{m,a}\mathrm{div}\bigg(\sum_{j=m+1}^{\infty} a\nabla\big(S_{j,a}f_\delta\big) \cdot \big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg) - \Delta_{m,a}\big(\mathcal{L}_a f_\delta \prec_a \mathcal{I}_a(g)\big).\end{aligned}$$

By Lemmas 2.8, since $\alpha+\beta \in (1,2)$ and $\alpha < 1$,

$$\bigg\|\Delta_{m,a}\mathrm{div}\bigg(\sum_{j=m+1}^{\infty} a\nabla\big(S_{j,a}f_\delta\big) \cdot \big(\Delta_{j,a}\mathcal{I}_a(g)\big)\bigg)\bigg\|_{L^\infty} \lesssim 2^m \sum_{j=m+1}^{\infty} 2^{(1-\alpha)j}2^{-\beta j} \lesssim 2^{(2-\alpha-\beta)m}.$$

By Lemma 2.2,

$$\|\Delta_{m,a}\big(\mathcal{L}_a f_\delta \prec_a \mathcal{I}_a(g)\big)\|_{L^\infty} \lesssim 2^{(2-\alpha-\beta)m}.$$

This completes the proof for space regularity. For the time regularity, since $(f-f_\delta) \prec_a \mathcal{I}_a(g) \in \mathcal{C}^{\alpha\wedge\beta/2}L^\infty$, then the time regularity of the commutator is controlled similarly as above by Lemma 2.16. $\square$

3 Periodic homogenisation for gPAM

Let a be a symmetric, elliptic and Hölder continuous matrix. Definie $a_\varepsilon := a(\cdot/\varepsilon)$. Then by [6], $\exists M, \lambda, \Lambda > 0$, s.t. $a_\varepsilon \in S(M, \lambda, \Lambda)$, $\forall \varepsilon \in (0,1)$. Define $\mathcal{L}_\varepsilon := \mathrm{div}(a_\varepsilon \nabla)$. We denote $\|\cdot\|_{\alpha,a_\varepsilon}, \Delta_{j,a_\varepsilon}$ by $\|\cdot\|_{\alpha,\varepsilon}, \Delta_{j,\varepsilon}$. Similarly, we denote $\prec_{a_\varepsilon}, \succ_{a_\varepsilon}, \circ_{a_\varepsilon}$ by $\prec_\varepsilon, \succ_\varepsilon, \circ_\varepsilon$ respectively, $\forall \varepsilon \in (0,1)$. We define

$$X_\varepsilon := \int_{\mathbf{T}^2} G_\varepsilon(x,y)\langle\eta\rangle(y)\mathrm{d}y \tag{3.1}$$

where $\langle\eta\rangle := \eta - \int_{\mathbf{T}^2}\eta(y)\mathrm{d}y$ and $G_\varepsilon(\cdot,\cdot)$ is the Green's function of $-\mathcal{L}_\varepsilon|_{\mathbf{T}^2}$.

3.1 Free Gaussian fields

For two Gaussian random fields f and g, we define

$$f \circ_\varepsilon^\diamond g := \sum_{|i-j|\le 1} \Delta_{i,\varepsilon} f \diamond \Delta_{j,\varepsilon} g$$

where $\Delta_{i,\varepsilon} f \diamond \Delta_{j,\varepsilon} g := \Delta_{i,\varepsilon} f \Delta_{j,\varepsilon} g - \mathbf{E}(\Delta_{i,\varepsilon} f \Delta_{j,\varepsilon} g)$.

Lemma 3.1. *$\forall j \ge -1$, $\forall \varepsilon \in (0,1)$,*

$$\mathbf{E}(\Delta_{j,\varepsilon} I_\varepsilon(\eta))^2 \lesssim 2^{-2j}.$$

Proof. For $-1 \le j \le 1$, by Green's function estimates, the desired result is obtained. By definition, for $j \ge 2$,

$$\Delta_{j,\varepsilon} I_\varepsilon(\eta) = \sum_n \varphi_j(\lambda_{n,\varepsilon}) \frac{1 - e^{-t\lambda_{n,\varepsilon}^2}}{\lambda_{n,\varepsilon}^2} < \eta, \psi_{n,\varepsilon} > \psi_{n,\varepsilon}.$$

Then

$$\mathbf{E}(\Delta_{j,\varepsilon} I_\varepsilon(\eta))^2 = \sum_n \varphi_j^2(\lambda_{n,\varepsilon}) (\frac{1 - e^{-t\lambda_{n,\varepsilon}^2}}{\lambda_{n,\varepsilon}^2})^2 \psi_{n,\varepsilon}^2.$$

By Lemma 2.2, the desired result is obtained. $\square$

Lemma 3.2. *$\forall j \ge -1$, $\forall \theta \in (0, \frac{1}{2})$,*

$$\mathbf{E}\Big[\Big(\Delta_{j,\varepsilon}\big[\mathcal{I}_\varepsilon(\eta) - X_\varepsilon\big]\Big)^2\Big] \lesssim t^{-\theta} 2^{-2j-2\theta}.$$

Proof. For $-1 \le j \le 1$, we can obtain the desired result by Green's function estimates. By definition,

$$\Delta_{j,\varepsilon}\big(X_\varepsilon - \mathcal{I}_\varepsilon(\eta)\big) = \sum_n \varphi_j(\lambda_{n,\varepsilon}) \frac{e^{-t\lambda_{n,\varepsilon}^2}}{\lambda_{n,\varepsilon}^2} < \eta, \psi_{n,\varepsilon} > \psi_{n,\varepsilon}.$$

Then

$$\mathbf{E}\Big[\Big(\Delta_{j,\varepsilon}\big[\mathcal{I}_\varepsilon(\eta) - X_\varepsilon\big]\Big)^2\Big] = \sum_n \varphi_j^2(\lambda_{n,\varepsilon}) \frac{e^{-2t\lambda_{n,\varepsilon}^2}}{\lambda_{n,\varepsilon}^4} \psi_{n,\varepsilon}^2.$$

This completes the proof. $\square$

Remark 3.3. Subsequent results of the above lemma and the standard homogenisation theory are $\forall \kappa > 0$, $\forall \theta \in (0, \frac{1}{2})$,

$$\|J_\varepsilon\|_{2\theta-\kappa} \lesssim t^{-\theta}, \quad \|J_\varepsilon - J_0\|_{2\theta-\kappa} \lesssim \varepsilon^{\frac{\kappa}{10}} t^{-\theta}. \tag{3.2}$$

where J_ε is defined in (1.2).

3.2 Enhanced stochastic terms

Lemma 3.4. $\forall p>1$, $\forall \kappa>0$,

$$\mathbf{E}\|(\mathcal{I}_\varepsilon(\eta)\circ_\varepsilon^\diamond \eta)\|_{-\kappa,\varepsilon}^p \lesssim 1.$$

Proof. By Lemma 2.12 and

$$\mathbf{E}\|\mathcal{I}_\varepsilon(\eta)\|_{1-\kappa,\varepsilon}^p \lesssim 1, \quad \mathbf{E}\|\eta\|_{-1-\kappa,\varepsilon}^p \lesssim 1, \forall \kappa>0, p>1$$

then it suffices to consider the behavior of $\big(\mathcal{I}_\varepsilon(\eta)(y)-\mathcal{I}_\varepsilon(\eta)(x)\big)\diamond\eta(y)$ for y near x. Since $\forall\kappa>0$,

$$\mathbf{E}(\int_{\mathbf{T}^2}\Delta_{j,\varepsilon}(x,y)(\big(\mathcal{I}_\varepsilon(\eta)(t,y)-\mathcal{I}_\varepsilon(\eta)(t,x)\big)\diamond\eta(y))\mathrm{d}y)^2\lesssim 2^{\kappa j}$$

then by the Gaussian hypercontractivity and embedding theorems, the assertion follows. $\square$

Lemma 3.5. $\forall p>1$, $\forall\kappa>0$, $\exists\theta>0$,

$$\left(\mathbf{E}\|(\mathcal{I}_\varepsilon(\eta)\circ_\varepsilon^\diamond\eta)-(\mathcal{I}_0(\eta)\circ_0^\diamond\eta)\|_{-\kappa,\varepsilon}^p\right)^{\frac{1}{p}}\lesssim\varepsilon^\theta 2^{\kappa j}.$$

Proof. By Lemma 2.12 it suffices to show the three terms

$$(\Delta_{j,\varepsilon}(fg)-f\Delta_{j,\varepsilon}g)^\diamond;\Delta_{j,\varepsilon}(f\succ_\varepsilon g)^\diamond;R_{j,\varepsilon}(f,g)^\diamond$$

converge in a suitable distribution space, in which $f=\mathcal{I}_\varepsilon(\eta)$ and $g=\eta$. Since

$$(\mathbf{E}\|\mathcal{I}_\varepsilon(\eta)-\mathcal{I}_0(\eta)\|_{1-\kappa,\varepsilon}^p)^{\frac{1}{p}}\lesssim\varepsilon^\theta,\quad \mathbf{E}\|\eta\|_{-1-\kappa,\varepsilon}^p\lesssim 1,\forall\kappa>0,p>1, \tag{3.3}$$

then for the first term, we have

$$\mathbf{E}(\int_{\mathbf{T}^2}\Delta_{j,\varepsilon}(x,y)[(\big(\mathcal{I}_\varepsilon(\eta)(y)-\mathcal{I}_\varepsilon(\eta)(x)\big)-(\big(\mathcal{I}_0(\eta)(y)-\mathcal{I}_0(\eta)(x)\big)]\diamond\eta(y))\mathrm{d}y)^2\lesssim\varepsilon^\theta 2^{\kappa j}.$$

For the second term, we divide it into two parts.

$$\mathcal{I}_\varepsilon(\eta)\succ_\varepsilon\eta-\mathcal{I}_0(\eta)\succ_0\eta=\big(\mathcal{I}_\varepsilon(\eta)-\mathcal{I}_0(\eta)\big)\succ_\varepsilon\eta+(\mathcal{I}_0(\eta)\succ_\varepsilon\eta-\mathcal{I}_0(\eta)\succ_0\eta)$$

Since (3.3), it suffices to consider the second part. Since the eigenfunctions of $\mathcal{L}_\varepsilon$ converge to $\mathcal{L}_0$ respectively, then the convergence of second part is controlled. Similarly, we divide the third term into two parts.

$$\begin{aligned}&\Delta_{j,\varepsilon}\big(\mathcal{I}_\varepsilon(\eta)\prec_\varepsilon\eta-\mathcal{I}_0(\eta)\prec_0\eta\big)-[\mathcal{I}_\varepsilon(\eta)-\mathcal{I}_0(\eta)]\Delta_{j,\varepsilon}(\eta)\\=&\Delta_{j,\varepsilon}(\mathcal{I}_0(\eta)\prec_\varepsilon\eta-\mathcal{I}_0(\eta)\prec_0\eta)+R_{j,\varepsilon}(\mathcal{I}_\varepsilon(\eta)-\mathcal{I}_0(\eta),\eta).\end{aligned}$$

Analogously, the convergence is obtained with the convergences of eigenfunctions of $\mathcal{L}_\varepsilon$ in [11]. $\square$

3.3 Convergence of the solution

$$\begin{cases} u_\varepsilon & = u_\varepsilon^\# + g(u_\varepsilon) \prec_\varepsilon \mathcal{I}_\varepsilon(\eta) \\ u_\varepsilon^\# & = \mathcal{I}_\varepsilon(\eta \prec_\varepsilon g(u_\varepsilon)) + [\mathcal{I}_\varepsilon, g(u_\varepsilon) \prec_\varepsilon](\eta) + \mathcal{I}_\varepsilon(\mathcal{R}_{F,\varepsilon}(u_\varepsilon) \circ_\varepsilon \eta) \\ & + \mathcal{I}_\varepsilon\big(g'(u_\varepsilon)(u_\varepsilon^\# \circ_\varepsilon \eta + C_\varepsilon(g(u_\varepsilon), \mathcal{I}_\varepsilon(\eta), \eta) + g(u_\varepsilon)(\mathcal{I}_\varepsilon(\eta) \circ_\varepsilon^\diamond \eta))\big) \end{cases}$$

Theorem 3.6. *$\alpha \in (\frac{2}{3}, 1)$, $\phi \in \mathcal{C}^\alpha$ and $\|g\|_{L^\infty(\mathbb{R})} + \|g'''\|_{L^\infty(\mathbb{R})} < \infty$. We define*

$$\begin{cases} \Gamma_1(u_\varepsilon, u_\varepsilon^\#) & = \Gamma_2(u_\varepsilon, u_\varepsilon^\#) + g(u_\varepsilon) \prec_\varepsilon \mathcal{I}_\varepsilon(\eta) + e^{t\mathcal{L}_\varepsilon}\phi \\ \Gamma_2(u_\varepsilon, u_\varepsilon^\#) & = \mathcal{I}_\varepsilon(\eta \prec_\varepsilon g(u_\varepsilon)) + [\mathcal{I}_\varepsilon, g(u_\varepsilon) \prec_\varepsilon](\eta) \\ & + \mathcal{I}_\varepsilon\big(\mathcal{R}_{F,\varepsilon}(u_\varepsilon) \circ_\varepsilon \eta\big) + \mathcal{I}_\varepsilon\big(C_\varepsilon(g(u_\varepsilon), \mathcal{I}_\varepsilon(\eta), \eta)\big) \\ & + \mathcal{I}_\varepsilon\Big(g'(u_\varepsilon)\big((u_\varepsilon^\# + e^{t\mathcal{L}_\varepsilon}\phi) \circ_\varepsilon \eta + g(u_\varepsilon)(\mathcal{I}_\varepsilon(\eta) \circ_\varepsilon^\diamond \eta)\big)\Big) \end{cases} \tag{3.4}$$

Then $\exists T > 0$, $\forall \varepsilon \in [0,1]$, $\exists (u_\varepsilon, u_\varepsilon^\#)$ is the unique fixed point of Γ_ε and

$$(u_\varepsilon, u_\varepsilon^\#) \in \big(\mathcal{C}_T^{\frac{\alpha}{2}} L_x^\infty \cap L_T^\infty \mathcal{C}_x^{\alpha,\varepsilon}\big) \times L_T^\infty \mathcal{C}_x^{2\alpha,\varepsilon}.$$

The uniform bounds of fixed points and existence time T_ε depend on the uniform bound of Υ_ε the vector of enhanced stochastic terms.

Proof. Using para-products and commutators in 2, we can obtain the desired results as the classical case. ☐

Theorem 3.7. *Let $(u_\varepsilon, u_\varepsilon^\#)$ be the fixed point of Γ_ε defined in Theorem 3.6. Let $(u_0, u_\varepsilon^\#)$ be the solution of*

$$\begin{cases} \Gamma_1(u_0, u_0^\#) & = \Gamma_2(u_0, u_0^\#) + g(u_0) \prec_0 \mathcal{I}_0(\eta) + e^{t\mathcal{L}_0}\phi \\ \Gamma_2(u_0, u_0^\#) & = \mathcal{I}_0(\eta \prec_0 g(u_0)) + [\mathcal{I}_0, g(u_0) \prec_0](\eta) \\ & + \mathcal{I}_0(\mathcal{R}_{F,0}(u_0) \circ_0 \eta) + \mathcal{I}_0\big(C_0(g(u_0), \mathcal{I}_0(\eta), \eta)\big) \\ & + \mathcal{I}_0\big(g'(u_0)((u_0^\# + e^{t\mathcal{L}_0}\phi) \circ_0 \eta + g(u_0)(\mathcal{I}_0(\eta) \circ_0^\diamond \eta))\big) \end{cases} \tag{3.5}$$

Then u_ε converges to u_0 in $\mathcal{C}_T^{\frac{\alpha}{2}-\kappa} L_x^\infty \cap L_T^\infty \mathcal{C}_x^{\alpha-\kappa}$ almost surely, $\kappa > 0$.

Proof. Define $A_n = (u_{\varepsilon_n}, u_{\varepsilon_n}^\#, R_{F,\varepsilon_n}(u_{\varepsilon_n}) \circ_{\varepsilon_n} \eta, u_{\varepsilon_n}^\# \circ_{\varepsilon_n} \eta)$, $\forall n \in \mathbb{N}^+$, where $\varepsilon_n = \frac{1}{n}$. For $\kappa \geq 0$ and $\tilde{\kappa} > 0$, define

$$\mathcal{S}_\kappa := (\mathcal{C}^{\frac{\alpha-\kappa}{2}} L^\infty \cap L^\infty \mathcal{C}^{\alpha-\kappa}) \times \mathcal{C}^{\theta-\kappa} L^\infty \cap L^\infty \mathcal{C}^{1-\kappa} \times \big[\mathcal{C}^{\theta-\kappa} L^\infty \cap L^\infty \mathcal{C}^{2\alpha-1-\kappa-\tilde{\kappa}}\big]^2.$$

Since Υ_ε the vector of enhanced stochastic terms is almost surely bounded in the corresponding space, $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$ almost surely. When $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$, then $\forall \kappa > 0$, $\{b_n\}_{n\in\mathbb{N}^+} \subset \mathbb{N}^+$, $\exists \{c_n\}_{n\in\mathbb{N}^+}$ a subsequence of $\{b_n\}_{n\in\mathbb{N}^+}$, $\exists A = (v, v^\#, w_1, w_2) \in \mathcal{S}_\kappa$, s.t.

$$A_{b_n} \to A \quad in \quad \mathcal{S}_\kappa.$$

Combining the above convergence and $u_{\varepsilon_{cn}}^\# = \Gamma_{\varepsilon_{cn},2}(u_{\varepsilon_{cn}}, u_{\varepsilon_{cn}}^\#)$, $\forall n \in \mathbb{N}^+$, we have

$$\begin{aligned} v^\# = & [\mathcal{I}_0, g(v) \prec_0](\eta) + \mathcal{I}_0\big(\eta \prec_0 g(v) + w_1 + C_0(g(v), \mathcal{I}_0(\eta), \eta)\big) \\ & + \mathcal{I}_0\big(g'(v)(w_2 + e^{t\mathcal{L}_0}\phi \circ_0 \eta + g(v)(\mathcal{I}_0(\eta) \circ_0^\diamond \eta))\big) \end{aligned} \tag{3.6}$$

This yields $v^\# \in L^\infty \mathcal{C}^{1+\kappa}$. Now, we are in the position to prove that

$$w_1 = \mathcal{R}_{F,0}(v) \circ_0 \eta \quad \textit{and} \quad w_2 = v \circ_0 \eta. \tag{3.7}$$

We show the second one. The first one can be dealt with similarly. We divide the difference into three parts.

$$w_2 - v \circ_0 \eta = w_2 - u^\#_{\varepsilon_{c_n}} \circ_{\varepsilon_{c_n}} \eta + \Big(\sum_{i=-1}^{N} + \sum_{i=N+1}^{\infty} \Big) \Big(\sum_{|i-j|\le 1} \Delta_{i,\varepsilon_{c_n}} u^\#_{\varepsilon_{c_n}} \Delta_{j,\varepsilon_{c_n}} \eta - \Delta_{i,0} v^\# \Delta_{j,0} \eta \Big)$$

For the third part, by the uniform bound of $u^\#_\varepsilon$, we have $\forall N \in \mathbb{N}^+$,

$$\| \sum_{i=N+1}^{\infty} \Big(\sum_{|i-j|\le 1} \Delta_{i,\varepsilon_{c_n}} u^\#_{\varepsilon_{c_n}} \Delta_{j,\varepsilon_{c_n}} \eta - \Delta_{i,0} v^\# \Delta_{j,0} \eta \Big) \|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then since A_{c_n} converges to A in the corresponding space, then $\forall N \in \mathbb{N}^+$,

$$\| w_2 - v \circ_0 \eta \|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then the second one of (3.7) is obtained.

With (3.7), (3.6) and uniqueness of solutions of classical two dimensional gPAM models in $\mathbf{T}^2$, we have

$$(v, v^\#) = (u_0, u_0^\#).$$

Then the limit is independent of the choices of sub-sequences. This completes the proof. □

Theorem 3.8. *Let $(u_\varepsilon, u^\#_\varepsilon)$ be the fixed point of Γ_ε defined in Theorem 3.6. Let $(u_0, u^\#_\varepsilon)$ be the solution of* (3.5)*. Then $t^{\frac{1-\alpha}{2}} a_\varepsilon \nabla u_\varepsilon$ converges to $t^{\frac{1-\alpha}{2}} \bar{a} \nabla u_0$ in $L^\infty_T \mathcal{C}^{-\kappa}_x$ almost surely, $\kappa > 0$.*

Proof. We divide the flux into four parts.

$$a_\varepsilon \nabla u_\varepsilon = a_\varepsilon \nabla e^{t\mathcal{L}_\varepsilon} \phi + g(u_\varepsilon) \big(a_\varepsilon \nabla \mathcal{I}_\varepsilon(\eta) \big) + a_\varepsilon \Lambda_\varepsilon(u_\varepsilon) + a_\varepsilon \nabla u^\#_\varepsilon,$$

where

$$\Lambda_\varepsilon(u_\varepsilon) := \nabla \big(g(u_\varepsilon) \prec_\varepsilon \mathcal{I}_\varepsilon(\eta) \big) - g(u_\varepsilon) \nabla \mathcal{I}_\varepsilon(\eta).$$

Then convergence of the first two parts follows from Green's function estimates. For the third part,

$$\begin{aligned} a_\varepsilon \Lambda_\varepsilon(u_\varepsilon) = & \sum_{j=1}^{\infty} \Big(a_\varepsilon \nabla S_{j,\varepsilon}(g(u_\varepsilon)) \Big) \Delta_{j,\varepsilon} \mathcal{I}_\varepsilon(\eta) - \sum_{j=1}^{\infty} \Delta_{j,\varepsilon} \big(g(u_\varepsilon) \big) \Big(a_\varepsilon \nabla S_{j,\varepsilon} \mathcal{I}_\varepsilon(\eta) \Big) \\ & - \sum_{j=-1}^{\infty} \sum_{|k-j|\le 1} \Delta_{k,\varepsilon} \big(g(u_\varepsilon) \big) \Big(a_\varepsilon \nabla \Delta_{j,\varepsilon} \mathcal{I}_\varepsilon(\eta) \Big). \end{aligned}$$

Since every term in the above three series converges to right limit respectively and the three series are absolutely convergent in $L^\infty(\mathbf{T}^2)$, then we have

$$\lim_{\varepsilon \to 0} \| a_\varepsilon \Lambda_\varepsilon(u_\varepsilon) - \bar{a} \Lambda_0(u_0) \|_{L^\infty(\mathbf{T}^2)} = 0,$$

where

$$\bar{a}\Lambda_0(u_0) := \sum_{j=1}^{\infty}\Big(\bar{a}\nabla S_{j,0}(g(u_0))\Big)\Delta_{j,0}\mathcal{I}_0(\eta) - \sum_{j=1}^{\infty}\Delta_{j,0}\big(g(u_0)\big)\Big(\bar{a}\nabla S_{j,0}\mathcal{I}_0(\eta)\Big)$$
$$- \sum_{j=-1}^{\infty}\sum_{|k-j|\le 1}\Delta_{k,0}\big(g(u_0)\big)\Big(\bar{a}\nabla\Delta_{j,0}\mathcal{I}_0(\eta)\Big).$$

Now, it suffices to deal with the last part. When $\{a_{\varepsilon_n}\nabla u^{\#}_{\varepsilon_n}\}_{n\in\mathbb{N}^+}$ is bounded in $L^\infty_{T,x}\cap\mathcal{C}^\theta_T\mathcal{C}^{-\kappa}$, then for any subsequence $\{a_{\varepsilon_{b_n}}\nabla u^{\#}_{\varepsilon_{b_n}}\}_{n\in\mathbb{N}^+}$, there is a sequence $\{c_n\}_{n\in\mathbb{N}^+}$ which is a subsequence of $\{b_n\}_{n\in\mathbb{N}^+}$, s.t.

$$a_{\varepsilon_{cn}}\nabla u^{\#}_{\varepsilon_{cn}} \rightharpoonup H \quad *-weakly \quad in \quad L^\infty_{T,x},$$
$$a_{\varepsilon_{cn}}\nabla u^{\#}_{\varepsilon_{cn}} \to H \quad strongly \quad in \quad L^\infty\mathcal{C}^{-\kappa}.$$

For any fixed $h\in C^\infty([0,T];\mathbf{T}^2)$, by integration by parts,

$$\int_0^T\langle a_\varepsilon\nabla u^{\#}_\varepsilon(r,\cdot),\Phi^*_\varepsilon(\cdot)h(r,\cdot)\rangle\mathrm{d}r = -\int_0^T\langle(u^{\#}_\varepsilon)(r,\cdot),a^{\mathsf{T}}_\varepsilon\Phi^*_\varepsilon(\cdot)\nabla h(r,\cdot)\rangle\mathrm{d}r$$

For the right hand side, by the convergence of $u^{\#}_\varepsilon$, as $\varepsilon\to 0$,

$$\int_0^T\langle u^{\#}_\varepsilon,a^{\mathsf{T}}_\varepsilon\Phi^*_\varepsilon\nabla h\rangle\mathrm{d}r \to \int_0^T\langle u^{\#}_0,\bar{a}^{\mathsf{T}}\nabla h\rangle\mathrm{d}r.$$

For the left hand side, since $\mathcal{L}_\varepsilon u^{\#}_\varepsilon\to\mathcal{L}_0 u^{\#}_0$ in $L^\infty_T\mathcal{C}^{2\alpha-2-\kappa}_x$, by the div-curl lemma in [14], we have

$$\int_0^T\langle a_\varepsilon\nabla u^{\#}_\varepsilon,\Phi^*_\varepsilon h\rangle\mathrm{d}r \to \int_0^T\langle H,h\rangle\mathrm{d}r.$$

Then $H=\bar{a}\nabla u^{\#}_0$ which is independent of the choice of sub-sequences. This completes the proof. □

4 Periodic homogenisation for Φ^4_3

4.1 Enhanced stochastic terms

Let G_ε be the Green's function of $(\partial_t-\mathcal{L}_\varepsilon+1)|_{\mathbf{T}^3}$. Define

$$X_\varepsilon := \int_{-\infty}^t\mathrm{d}\tau\int_{\mathbf{T}^3}G_\varepsilon(t-\tau;x,y)\xi(\tau,y)\mathrm{d}y$$

Define a random vector

$$\Upsilon_\varepsilon = \big(X_\varepsilon, X^{\diamond 2}_\varepsilon, \tilde{\mathcal{I}}_\varepsilon\big(X^{\diamond 3}_\varepsilon\big), \tilde{\mathcal{I}}_\varepsilon(X^{\diamond 3}_\varepsilon)\circ^\diamond_\varepsilon X_\varepsilon, \tilde{\mathcal{I}}_\varepsilon(X^{\diamond 2}_\varepsilon)\circ_\varepsilon X^{\diamond 2}_\varepsilon, \tilde{\mathcal{I}}_\varepsilon(X^{\diamond 3}_\varepsilon)\circ^\diamond_\varepsilon X^{\diamond 2}_\varepsilon\big).$$

That is

$$\Upsilon_\varepsilon = \big(X_\varepsilon, X^{\vee}_\varepsilon, X_\varepsilon, X_\varepsilon, X_\varepsilon, X_\varepsilon\big).$$

Its norm is given by

$$
\begin{aligned}
\|\Upsilon_\varepsilon\|_{\mathcal{G}_T} =& \|X_\varepsilon\|_{L_T^\infty \mathcal{C}^{-\frac{1}{2}-\kappa}} + \|X_\varepsilon^{\diamond 2}\|_{L_T^\infty \mathcal{C}^{-1-\kappa,\varepsilon}} + \|\mathcal{I}_\varepsilon\big(X_\varepsilon^{\diamond 3}\big)\|_{L_T^\infty \mathcal{C}^{\frac{1}{2}-\kappa}} \\
&+ \|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon^\diamond X_\varepsilon\|_{L_T^\infty \mathcal{C}^{-\kappa}} + \|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2}) \circ_\varepsilon^\diamond X_\varepsilon^{\diamond 2}\|_{L_T^\infty \mathcal{C}^{-\kappa}} + \|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon^\diamond X_\varepsilon^{\diamond 2}\|_{L_T^\infty \mathcal{C}^{-\frac{1}{2}-\kappa}}.
\end{aligned}
$$

And its renormalising version is

$$
\begin{aligned}
\Upsilon_\varepsilon^{(\delta)} = \big(& X_\varepsilon^{(\delta)}, (X_\varepsilon^{(\delta)})^{\diamond 2}, \mathcal{I}_\varepsilon\big((X_\varepsilon^{(\delta)})^{\diamond 3}\big), \mathcal{I}_\varepsilon((X_\varepsilon^{(\delta)})^{\diamond 3}) \circ_\varepsilon X_\varepsilon^{(\delta)}, \\
& \mathcal{I}_\varepsilon((X_\varepsilon^{(\delta)})^{\diamond 2}) \circ_\varepsilon (X_\varepsilon^{(\delta)})^{\diamond 2}, \mathcal{I}_\varepsilon((X_\varepsilon^{(\delta)})^{\diamond 3}) \circ_\varepsilon (X_\varepsilon^{(\delta)})^{\diamond 2}\big).
\end{aligned}
$$

That is

$$
\Upsilon_\varepsilon^{(\delta)} = \big(X_\varepsilon^{(\delta)}, (X_\varepsilon^{(\delta)})^{\vee}, (X_\varepsilon^{(\delta)}), ((X_\varepsilon^{(\delta)})), (X_\varepsilon^{(\delta)}), (X_\varepsilon^{(\delta)})\big).
$$

Lemma 4.1.

$$
|\mathbf{E}\big(X_\varepsilon(t,x)X_\varepsilon(\tau,y)\big)| \lesssim \frac{1}{\sqrt{|t-\tau|} + |x-y|}.
$$

Proof. Since by definition of Green function and the self-adjoint property of $\mathcal{L}_\varepsilon$

$$
\mathbf{E}\big(X_\varepsilon(t,x)X_\varepsilon(\tau,y)\big) = \int_{-\infty}^{t\wedge\tau} G_\varepsilon(2(t+\tau-2\tilde{\tau}); x, y)\mathrm{d}\tilde{\tau},
$$

then by Green's function estimates, we can obtain the desired result. □

Lemma 4.2. *$\forall \theta_2 > 0$, $\exists \theta_1 > 0$ s.t.*

$$
|\mathbf{E}\big[\big(X_\varepsilon(t,x) - X_0(t,x)\big)\big(X_\varepsilon(\tau,y)\big)\big]| \lesssim \frac{\varepsilon^{\theta_1}}{(\sqrt{|t-\tau|} + |x-y|)^{1+\theta_2}}.
$$

Proof. By definition,

$$
\begin{aligned}
&\mathbf{E}\big[\big(X_\varepsilon(t,x) - X_0(t,x)\big)\big(X_\varepsilon(\tau,y)\big)\big] \\
=& \int_{-\infty}^{t} \Big(\int_{\mathbf{T}^3} \big(G_\varepsilon(t-\tilde{\tau}; x, z) - G_0(t-\tilde{\tau}; x, z)\big) G_\varepsilon(\tau-\tilde{\tau}; y, z)\mathrm{d}z\Big)\mathrm{d}\tilde{\tau}
\end{aligned}
$$

with Green's function estimates, the desired result is obtained. □

Lemma 4.3.

$$
\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon X_\varepsilon\|^p_{L_T^\infty \mathcal{C}_x^{-\kappa}} \lesssim 1
$$

and

$$
\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon X_\varepsilon - \mathcal{I}_0(X_\varepsilon^{\diamond 3}) \circ_0 X_0\|^p_{L_T^\infty \mathcal{C}_x^{-\kappa}} \lesssim \varepsilon^{p\theta}
$$

Proof. With the same method in Lemmas 3.4 and 3.5, it suffices to consider the regularity of

$$
\int_{\mathbf{T}^3} \Delta_{j,\varepsilon}(x,y)\Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(x)]X_\varepsilon(y)\Big)^\diamond \mathrm{d}y
$$

$\forall \gamma \in (0, \frac{1}{2})$, by Green's function estimates,

$$
|\mathbf{E}\Big[\Phi_\varepsilon(t,x,y)\Phi_\varepsilon(t,x,\bar{y})\Big]| \lesssim |\bar{y}-y|^{-1}|x-y|^\gamma|x-\bar{y}|^\gamma \mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})\|^2_{\gamma,\varepsilon}
$$

in which

$$
\Phi_\varepsilon(t,y,x) := \Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(x)]X_\varepsilon(y)\Big)^\diamond.
$$

The desired estimate follows from the pointwise estimate for $\Delta_{j,\varepsilon}$. □

Using Lemmas 4.2 and 4.3, X_ε, $X_\varepsilon^{\diamond 2}$ and $\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})$ converge in $L^p_\omega\mathcal{C}^{-\frac{1}{2}-\kappa}$, $L^p_\omega\mathcal{C}^{-1-\kappa,\varepsilon}$ and $L^p_\omega\mathcal{C}^{\frac{1}{2}-\kappa}$ respectively $\forall\kappa > 0$.

Lemma 4.4.

$$\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2}) \circ_\varepsilon X_\varepsilon^{\diamond 2}\|^p_{L^\infty_T\mathcal{C}^{-\kappa}} \lesssim 1$$

and

$$\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2}) \circ_\varepsilon X_\varepsilon^{\diamond 2} - \mathcal{I}_0(X_0^{\diamond 2}) \circ_0 X_0^{\diamond 2}\|^p_{L^\infty_T\mathcal{C}^{-\kappa}} \lesssim \varepsilon^{p\theta}$$

Proof. With the same method in Lemmas 3.4 and 3.5, it suffices to consider the regularity of

$$\int_{\mathbf{T}^3} \Delta_{j,\varepsilon}(x,y)\Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})(x)]X_\varepsilon^{\diamond 2}(y)\Big)^\diamond \mathrm{d}y$$

$\forall\gamma \in (0,1)$, by Green's function estimates,

$$|\mathbf{E}\Big[\Phi_\varepsilon(t,x,y)\Phi_\varepsilon(t,x,\bar{y})\Big]| \lesssim |\bar{y}-y|^{-2}|x-y|^\gamma|x-\bar{y}|^\gamma\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})\|^2_{\gamma,\varepsilon}$$

in which

$$\Phi_\varepsilon(t,y,x) := \Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})(x)]X_\varepsilon^{\diamond 2}(y)\Big)^\diamond.$$

The desired estimate follows from the pointwise estimate for $\Delta_{j,\varepsilon}$. $\square$

Lemma 4.5.

$$\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon X_\varepsilon^{\diamond 2}\|^p_{L^\infty_T\mathcal{C}^{-\frac{1}{2}-\kappa}} \lesssim 1$$

and

$$\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3}) \circ_\varepsilon X_\varepsilon^{\diamond 2} - \mathcal{I}_0(X_0^{\diamond 3}) \circ_0 X_0^{\diamond 2}\|^p_{L^\infty_T\mathcal{C}^{-\frac{1}{2}-\kappa}} \lesssim \varepsilon^{p\theta}$$

Proof. With the same method in Lemmas 3.4 and 3.5, it suffices to consider the regularity of

$$\int_{\mathbf{T}^3} \Delta_{j,\varepsilon}(x,y)\Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 2})(x)]X_\varepsilon^{\diamond 2}(y)\Big)^\diamond \mathrm{d}y$$

$\forall\gamma \in (0,\frac{1}{2})$, by Green's function estimates,

$$|\mathbf{E}\Big[\Phi_\varepsilon(t,x,y)\Phi_\varepsilon(t,x,\bar{y})\Big]| \lesssim |\bar{y}-y|^{-2}|x-y|^\gamma|x-\bar{y}|^\gamma\mathbf{E}\|\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})\|^2_{\gamma,\varepsilon}$$

in which

$$\Phi_\varepsilon(t,y,x) := \Big([\mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(y) - \mathcal{I}_\varepsilon(X_\varepsilon^{\diamond 3})(x)]X_\varepsilon^{\diamond 2}(y)\Big)^\diamond.$$

The desired estimate follows from the pointwise estimate for $\Delta_{j,\varepsilon}$. $\square$

4.2 Convergence of solutions

Define

$$
\begin{aligned}
&\tilde X_\varepsilon := \mathcal{I}_\varepsilon(\xi);\\
&\tilde X_\varepsilon^{\vee} := X_\varepsilon^{\vee} + J_\varepsilon^{\vee};\\
&\tilde X_\varepsilon^{\mathsf{Y}} := \mathcal{I}_\varepsilon(\tilde X_\varepsilon^{\vee});\\
&\tilde X_\varepsilon^{\Psi} := \mathcal{I}_\varepsilon(\tilde X_\varepsilon^{\diamond 3}) + J_\varepsilon^{\Psi};\\
&\tilde X_\varepsilon^{\Psi\bullet} := \tilde X_\varepsilon^{\Psi} \diamond \tilde X_\varepsilon + J_\varepsilon^{\Psi\bullet};\\
&\tilde X_\varepsilon^{\Psi\vee} := \tilde X_\varepsilon^{\Psi} \diamond \tilde X_\varepsilon^{\vee} + J_\varepsilon^{\Psi\vee}\\
&\tilde X_\varepsilon^{\Psi\bullet\odot} := \tilde X_\varepsilon^{\Psi\bullet} \circ_\varepsilon^\diamond \tilde X_\varepsilon + J_\varepsilon^{\Psi\bullet\odot};\\
&\tilde X_\varepsilon^{\Psi\vee\odot} := \tilde X_\varepsilon^{\Psi\vee} \circ_\varepsilon^\diamond \tilde X_\varepsilon^{\vee} + J_\varepsilon^{\Psi\vee\odot};\\
&\tilde X_\varepsilon^{\mathsf{Y}\bullet\odot} := \tilde X_\varepsilon^{\mathsf{Y}\bullet} \circ_\varepsilon^\diamond \tilde X_\varepsilon^{\vee} + J_\varepsilon^{\mathsf{Y}\bullet\odot}.
\end{aligned}
$$

As explained in KPZ, these terms have similar uniform regularity and convergence as terms estimated in the last subsection.

Theorem 4.6. *$\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\phi \in \mathcal{C}^\alpha$, we define*

$$
\begin{aligned}
\Gamma_{1,\varepsilon} :=& \tilde X_\varepsilon + \tilde X_\varepsilon^{\Psi} + \Gamma_{2,\varepsilon};\\
\Gamma_{2,\varepsilon} :=& 3\Phi_\varepsilon \prec_\varepsilon \tilde X_\varepsilon^{\mathsf{Y}} + \Gamma_{3,\varepsilon} + e^{t\mathcal{L}_\varepsilon}\phi;\\
\Gamma_{3,\varepsilon} :=& [\mathcal{I}_\varepsilon, 3\Phi_\varepsilon \prec_\varepsilon](\tilde X_\varepsilon^{\vee}) + \mathcal{I}_\varepsilon(K_\varepsilon)
\end{aligned}
\tag{4.1}
$$

where $K_\varepsilon := K_{1,\varepsilon} + K_{2,\varepsilon}$. They are defined below.

$$
K_{1,\varepsilon} := (\Phi_\varepsilon + \tilde X_\varepsilon^{\Psi})^3 + 3(\Phi_\varepsilon^2 \tilde X_\varepsilon + 2\Phi_\varepsilon \tilde X^{\Psi\bullet} + \mathcal{I}_\varepsilon(\tilde X_\varepsilon^{\diamond 3})^2 \diamond \tilde X_\varepsilon) + 3\tilde X^{\Psi\vee},
$$

and

$$
K_{2,\varepsilon} := 3\Phi_\varepsilon \succ_\varepsilon \tilde X_\varepsilon^{\vee} + 3(\Phi_\varepsilon^{\#} + e^{t\mathcal{L}_\varepsilon}\phi) \circ_\varepsilon \tilde X_\varepsilon^{\vee} + C_\varepsilon(3\Phi_\varepsilon, \tilde X_\varepsilon^{\mathsf{Y}}, \tilde X_\varepsilon^{\vee}) + 3\Phi_\varepsilon \tilde X_\varepsilon^{\mathsf{Y}\bullet\odot}.
$$

Then $\exists T > 0$, $\forall \varepsilon \in [0,1]$, $\exists (u_\varepsilon, \Phi_\varepsilon, \Phi_\varepsilon^{\#})$ is the unique fixed point of Γ_ε and

$$
(u_\varepsilon, u_\varepsilon^{P_\varepsilon}, u_\varepsilon^{\#}) \in L_T^\infty \mathcal{C}_x^{-\alpha,\varepsilon} \times \left(\mathcal{C}_T^{\frac{\alpha}{2}} L_x^\infty \cap L_T^\infty \mathcal{C}_x^{\alpha,\varepsilon}\right) \times L_T^\infty \mathcal{C}_x^{2\alpha,\varepsilon}
$$

The uniform bounds of fixed points and existence time T_ε depend on the uniform bound of Υ_ε the vector of enhanced stochastic terms.

Theorem 4.7. *Let $(u_\varepsilon, u_\varepsilon^{P_\varepsilon}, u_\varepsilon^{\#})$ be the fixed point of Γ_ε defined in Theorem 5.7. Let $(u_0, u_0^{P_0}, u_0^{\#})$ be the solution of*

$$
\begin{aligned}
\Gamma_{1,0} :=& \tilde X_0 + I_0(\tilde X_0^{\diamond 3}) + \Gamma_{2,0};\\
\Gamma_{2,0} :=& 3\Phi_0 \prec_0 \mathcal{I}_0(\tilde X_0^{\diamond 2}) + \Gamma_{3,0} + e^{t\mathcal{L}_0}\phi;\\
\Gamma_{3,0} :=& [\mathcal{I}_0, 3\Phi_0 \prec_0]\mathcal{I}(\tilde X_0^{\diamond 2}) + \mathcal{I}_0(K_0)
\end{aligned}
\tag{4.2}
$$

where $K_0 := K_{1,0} + K_{2,0}$. They are defined below.

$K_{1,0} := (\Phi_0 + \mathcal{I}_0(\tilde{X}_0^{\diamond 3}))^3 + 3(\Phi_0^2 \tilde{X}_0 + 2\Phi_0(\mathcal{I}_0(\tilde{X}_0^{\diamond 3}) \diamond \tilde{X}_0) + \mathcal{I}_0(\tilde{X}_0^{\diamond 3})^2 \diamond \tilde{X}_0) + 3\mathcal{I}_0(\tilde{X}_0^{\diamond 3}) \diamond \tilde{X}_0^{\diamond 2},$

and

$K_{2,0} := 3\Phi_0 \succ_0 \mathcal{I}_0(\tilde{X}_0^{\diamond 2}) + 3(\Phi_0^{\#} + e^{t\mathcal{L}_\varepsilon}\phi) \circ_0 \tilde{X}_0^{\diamond 2} + C_0(3\Phi_0, \mathcal{I}_0(\tilde{X}_0^{\diamond 2}), \tilde{X}_0^{\diamond 2}) + 3\Phi_0(\mathcal{I}_0(\tilde{X}_0^{\diamond 2}) \circ_0 \tilde{X}_0^{\diamond 2}).$

Then u_ε converges to u_0 in $L_T^\infty \mathcal{C}_x^{-\alpha-\kappa}$ almost surely, $\kappa > 0$.

Proof. Define $A_n = (u_\varepsilon, \Phi_\varepsilon, \Phi_\varepsilon^{\#}, \Phi_{\varepsilon_n}^{\#} \circ_{\varepsilon_n} \tilde{\tilde{X}}_{\varepsilon_n}^{\diamond 2})$, $\forall n \in \mathbb{N}^+$, where $\varepsilon_n = \frac{1}{n}$. For $\kappa \geq 0$, $\tilde{\kappa} > 0$ and $\theta > 0$ sufficiently small, define

$\mathcal{S}_\kappa := (\mathcal{C}^\theta \mathcal{C}_x^{-\alpha-3\kappa-2\theta} \cap L_T^\infty \mathcal{C}_x^{-\alpha-\kappa}) \times (C^{\frac{\alpha-\kappa}{2}} L^\infty \cap L^\infty C^{\alpha-\kappa})^2 \times C_T^\theta \mathcal{C}_x^{2\alpha-1-2\kappa-2\theta,\varepsilon} \cap L_T^\infty \mathcal{C}_x^{2\alpha-1-\kappa,\varepsilon}$

Since Υ_ε the vector of enhanced stochastic terms is almost surely bounded in the corresponding space, $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$ almost surely. When $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$, then $\forall \kappa > 0$, $\{b_n\}_{n\in\mathbb{N}^+} \subset \mathbb{N}^+$, $\exists \{c_n\}_{n\in\mathbb{N}^+}$ a subsequence of $\{b_n\}_{n\in\mathbb{N}^+}$, $\exists A = (v, \Phi, \Phi^{\#}, w) \in \mathcal{S}_\kappa$, s.t.

$$A_{b_n} \to A \quad in \quad \mathcal{S}_\kappa.$$

Combining the above convergence and $u_{\varepsilon_{c_n}}^{\#} = \Gamma_{\varepsilon_{c_n},3}(u_{\varepsilon_{c_n}}, u_\varepsilon^{P_\varepsilon}, u_{\varepsilon_{c_n}}^{\#})$, $\forall n \in \mathbb{N}^+$, we have

$$\Phi^{\#} := [\mathcal{I}_0, 3\Phi \prec_0]\mathcal{I}(\tilde{X}_0^{\diamond 2}) + \mathcal{I}_0(\tilde{K}_0) \tag{4.3}$$

where $\tilde{K}_0 := \tilde{K}_{1,0} + \tilde{K}_{2,0}$. They are defined below.

$\tilde{K}_{1,0} := (\Phi_0 + \mathcal{I}_0(\tilde{X}_0^{\diamond 3}))^3 + 3(\Phi_0^2 \tilde{X}_0 + 2\Phi_0(\mathcal{I}_0(\tilde{X}_0^{\diamond 3}) \diamond \tilde{X}_0) + \mathcal{I}_0(\tilde{X}_0^{\diamond 3})^2 \diamond \tilde{X}_0) + 3\mathcal{I}_0(\tilde{X}_0^{\diamond 3}) \diamond \tilde{X}_0^{\diamond 2},$

and

$\tilde{K}_{2,0} := 3\Phi_0 \succ_0 \mathcal{I}_0(\tilde{X}_0^{\diamond 2}) + 3e^{t\mathcal{L}_\varepsilon}\phi \circ_0 \tilde{X}_0^{\diamond 2} + 3w + C_0(3\Phi_0, \mathcal{I}_0(\tilde{X}_0^{\diamond 2}), \tilde{X}_0^{\diamond 2}) + 3\Phi_0(\mathcal{I}_0(\tilde{X}_0^{\diamond 2}) \circ_0 \tilde{X}_0^{\diamond 2}).$

This implies $\Phi^{\#} \in L^\infty \mathcal{C}^{1+\kappa}$, for some $\kappa > 0$. Now, we are in the position to prove that

$$w = \Phi^{\#} \circ_0 \tilde{X}_0^{\diamond 2}. \tag{4.4}$$

We divide the difference into three parts.

$$\begin{aligned} w_2 - \Phi^{\#} \circ_0 \mathcal{D}_0 \tilde{X}_0 &= w_2 - \Phi_{\varepsilon_{c_n}}^{\#} \circ_{\varepsilon_{c_n}} \mathcal{D}_{\varepsilon_{c_n}} \tilde{X}_{\varepsilon_{c_n}} \\ +\Big(\sum_{i=-1}^{N} + \sum_{i=N+1}^{\infty}\Big)\Big(&\sum_{|i-j|\leq 1} \Delta_{i,\varepsilon_{c_n}} \Phi_{\varepsilon_{c_n}}^{\#} \Delta_{j,\varepsilon_{c_n}} \mathcal{D}_{\varepsilon_{c_n}} \tilde{X}_{\varepsilon_{c_n}} - \Delta_{i,0}\Phi^{\#}\Delta_{j,0}\mathcal{D}_0\tilde{X}_0\Big) \end{aligned}$$

For the third part, by the uniform bound of $\Phi_\varepsilon^{\#}$, we have $\forall N \in \mathbb{N}^+$,

$$\Big\| \sum_{i=N+1}^{\infty} \Big(\sum_{|i-j|\leq 1} \Delta_{i,\varepsilon_{c_n}} \Phi_{\varepsilon_{c_n}}^{\#} \Delta_{j,\varepsilon_{c_n}} \mathcal{D}_{\varepsilon_{c_n}} \tilde{X}_{\varepsilon_{c_n}} - \Delta_{i,0}\Phi^{\#}\Delta_{j,0}\mathcal{D}_0\tilde{X}_0\Big)\Big\|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then since A_{c_n} converges to A in the corresponding space, then $\forall N \in \mathbb{N}^+$,

$$\|w - \Phi^{\#} \circ_0 \eta\|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then (4.4) is obtained.

With (3.7), (3.6) and uniqueness of solutions of classical two dimensional gPAM models in $\mathbf{T}^2$, we have

$$(u, \Phi, \Phi^{\#}) = (u_0, \Phi_0, \Phi_0^{\#}).$$

Then the limit is independent of the choice of sub-sequences. This completes the proof. $\square$

The convergence for the flux $a_\varepsilon \nabla u_\varepsilon$ can be obtained by similar methods in KPZ and gPAM.

5 Periodic homogenisation for modified KPZ equations

In this section, we consider the periodic homogenisation problem for

$$\partial_t u_\varepsilon^{(\delta)} = \mathcal{L}_\varepsilon u_\varepsilon^{(\delta)} + (\sqrt{-\mathcal{L}_\varepsilon} u_\varepsilon)^2 - C_\varepsilon^{(\delta)} + \xi^{(\delta)}. \tag{5.1}$$

We choose this kind of modified KPZ equation because this kind of modification can preserve the relations between KPZ and stochastic Burgers' equation, illustrated in [9]. The corresponding stochastic Burgers' equation is

$$\partial_t v_\varepsilon^{(\delta)} = \mathcal{L}_\varepsilon v_\varepsilon^{(\delta)} + \sqrt{-\mathcal{L}_\varepsilon} (v_\varepsilon^{(\delta)})^2 + \sqrt{-\mathcal{L}_\varepsilon} \xi^{(\delta)}.$$

5.1 Enhanced stochastic terms

Let G_ε be the Green's function of $(\partial_t - \mathcal{L}_\varepsilon)|_{\mathbf{T}^1}$. Define

$$\tilde{\mathcal{I}}_\varepsilon f := \int_{-\infty}^t \mathrm{d}\tau \int_{\mathbf{T}^1} G_\varepsilon(t-\tau; x, y) \langle f \rangle (\tau, y) \mathrm{d}y.$$

where $\langle f \rangle = f - \int_{T^1} f(t,x)\mathrm{d}x$. We define

$$X_\varepsilon = \tilde{\mathcal{I}}_\varepsilon(\xi).$$

Let Υ_ε be a stochastic vector given below. For convenience, we denote $\sqrt{-\mathcal{L}_\varepsilon}$ by $\mathcal{D}_\varepsilon$.

$$X_\varepsilon, X_\varepsilon^{\vee}, X^{\vee\!\!\cdot}, X^{\vee\!\!\cdot\!\!\cdot}, X_\varepsilon^{\vee\vee}, \mathcal{D}_\varepsilon P_\varepsilon \circ_\varepsilon \mathcal{D}_\varepsilon X_\varepsilon.$$

in which $P_\varepsilon := \mathcal{D}_\varepsilon \tilde{\mathcal{I}}_\varepsilon (\mathcal{D}_\varepsilon X_\varepsilon)$.

More precisely,

$$
\begin{aligned}
&\Upsilon_{1,\varepsilon} := X_\varepsilon;\\
&X_\varepsilon^{\vee} := \tilde{\mathcal{I}}_\varepsilon\big[(\mathcal{D}_\varepsilon X_\varepsilon)^{\diamond 2}\big];\\
&X_\varepsilon^{\vee\!\!\vee} := \tilde{\mathcal{I}}_\varepsilon(\mathcal{D}_\varepsilon X_\varepsilon^{\vee}\mathcal{D}_\varepsilon X_\varepsilon);\\
&X^{\vee\!\!\vee\!\!\vee} := \tilde{\mathcal{I}}_\varepsilon\big[(\mathcal{D}_\varepsilon X_\varepsilon^{\vee\!\!\vee} \circ_\varepsilon^\diamond \mathcal{D}_\varepsilon X_\varepsilon)\big];\\
&X_\varepsilon^{\vee\vee\vee} := \tilde{\mathcal{I}}_\varepsilon\big[(\mathcal{D}_\varepsilon X_\varepsilon^{\vee})^{\diamond 2}\big];\\
&\Upsilon_{6,\varepsilon} := [\mathcal{D}_\varepsilon \tilde{\mathcal{I}}_\varepsilon(\sqrt{-\mathcal{L}_\varepsilon} X_\varepsilon) \circ_\varepsilon \mathcal{D}_\varepsilon X_\varepsilon]^\diamond.
\end{aligned}
$$

Lemma 5.1.

$$
|\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon)(t,x)(\mathcal{D}_\varepsilon X_\varepsilon)(\tau,y)\big)| \lesssim \frac{e^{-c|t-\tau|}}{\sqrt{|t-\tau|}+|x-y|}
$$

Proof. By definition,

$$
\mathbf{E}(\mathcal{D}_\varepsilon X_\varepsilon(t,x)\mathcal{D}_\varepsilon X_\varepsilon(\tau,y)) = \frac{1}{2}\sum_{n=1}^{\infty} e^{-|t-\tau|\lambda_{n,\varepsilon}^2}\psi_{n,\varepsilon}(x)\psi_{n,\varepsilon}(y)
$$

By suitable re-arrangement,

$$
\mathbf{E}(\mathcal{D}_\varepsilon X_\varepsilon(t,x)\mathcal{D}_\varepsilon X_\varepsilon(\tau,y)) = \frac{1}{2}(G_\varepsilon(t-\tau;x,y)-1).
$$

On the one hand, by Green's function estimates, we have

$$
|\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon)(t,x)(\mathcal{D}_\varepsilon X_\varepsilon)(\tau,y)\big)| \lesssim \frac{1}{\sqrt{|t-\tau|}+|x-y|}+1.
$$

One the other hand, by estimates for generalised Besov spaces in [12, Proposition 2.9], we have $\exists c>0$, s.t. $\forall |t-\tau|\geq 1$

$$
|\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon)(t,x)(\mathcal{D}_\varepsilon X_\varepsilon)(\tau,y)\big)| \lesssim e^{-c|t-\tau|}.
$$

This completes the proof. □

Lemma 5.2.

$$
\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(t,x)(\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(\tau,y)\big) \lesssim \log(2+\frac{1}{\sqrt{|t-\tau|}+|x-y|})
$$

Proof. By definition,

$$
\begin{aligned}
\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(t,x)(\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(\tau,y)\big) = 2\int_{(-\infty,t]\times(-\infty,\tau]\times T^2} &\mathcal{D}_\varepsilon G_\varepsilon(t-r;x,z)\\
&\mathcal{D}_\varepsilon G_\varepsilon(\tau-r';y,z')[\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon)(r,z)(\mathcal{D}_\varepsilon X_\varepsilon)(r',z')\big)]^2 \mathrm{d}r\mathrm{d}z\mathrm{d}r'\mathrm{d}z'
\end{aligned}
$$

By Green's function estimates,

$$
\begin{aligned}
&|\mathbf{E}\big((\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(t,x)(\mathcal{D}_\varepsilon X_\varepsilon^{\vee})(\tau,y)\big)|\\
\lesssim &\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{T^2}\frac{e^{-c(r+r')}}{(t-r+|x-z|^2)(\tau-r'+|y-z'|^2)(|r-r'|+|z-z'|^2)}\mathrm{d}r\mathrm{d}z\mathrm{d}r'\mathrm{d}z'
\end{aligned}
$$

This completes the proof. □

Lemma 5.3. $\forall \kappa > 0$,

$$\mathbf{E}\Big[\|(\Delta_{j,\varepsilon}X_\varepsilon^{\Psi}(t,\cdot)\|^2_{L^2(\mathbf{T}^1)}\Big] \lesssim 2^{-(3-\kappa)j}$$

Proof.

$$\|(\Delta_{j,\varepsilon}X_\varepsilon^{\Psi})(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} = \sum_k \Big[\int_{-\infty}^t e^{-(t-\tau)\lambda_{k,\varepsilon}^2}\varphi_j(\lambda_{k,\varepsilon}) < \mathcal{D}_\varepsilon X_\varepsilon^{\vee}(\tau)\mathcal{D}_\varepsilon X_\varepsilon(\tau), \psi_{k,\varepsilon} > \mathrm{d}\tau\Big]^2$$

Then

$$\mathbf{E}\|(\Delta_{j,\varepsilon}X_\varepsilon^{\Psi})(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} = \sum_k \int_{-\infty}^t\int_{-\infty}^t \Theta_\varepsilon^{(1)}(\tau,\tau';z,z')\Theta_\varepsilon^{(2)}(t,\tau,\tau';z,z')\mathrm{d}\tau\mathrm{d}\tau'$$

in which

$$\Theta_\varepsilon^{(1)}(\tau,\tau';z,z') = \mathbf{E}\Big(\mathcal{D}_\varepsilon X_\varepsilon^{\vee}(\tau,z)\mathcal{D}_\varepsilon X_\varepsilon(\tau,z)\mathcal{D}_\varepsilon X_\varepsilon^{\vee}(\tau',z')\mathcal{D}_\varepsilon X_\varepsilon(\tau',z')\Big)$$

and

$$\Theta_\varepsilon^{(2)}(t,\tau,\tau';z,z') = e^{-(2t-\tau-\tau')\lambda_{k,\varepsilon}^2}\varphi_j^2(\lambda_{k,\varepsilon})\psi_{k,\varepsilon}(z)\psi_{k,\varepsilon}(z').$$

Then by Lemma 5.1 and 5.2, we have

$$\mathbf{E}\|(\Delta_{j,\varepsilon}X_\varepsilon^{\Psi})(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} \lesssim \int_0^\infty\int_0^\infty \frac{\log(2+\frac{1}{\sqrt{|\tau-\tau'|}+|y-y'|})}{(\sqrt{|\tau-\tau'|}+|y-y'|)}|\Theta_\varepsilon^{(2)}(t,\tau,\tau';z,z')|\mathrm{d}z\mathrm{d}z'$$

When $\tau \geq 1$ or $\tau' \geq 1$, then $|\Theta_\varepsilon^{(2)}|$ can produce 2^{-jK}, $\forall K > 0$ and decay for time.When $\tau,\tau' \leq 1$, then it suffices to controll this integral.

$$\int_0^1 \mathrm{d}\tau\int_0^1\mathrm{d}\tau'\int_{\mathbf{T}^1}\int_{\mathbf{T}^1}\frac{\log(2+\frac{1}{\sqrt{|\tau-\tau'|}+|y-y'|})}{(\sqrt{|\tau-\tau'|}+|y-y'|)}\big(e^{-c|z-z'|2^n}2^n\big)2^{-(2-\kappa)n}\frac{1}{(\tau+\tau')^{1-\kappa}}\mathrm{d}z\mathrm{d}z'$$

Then by suitable scaling, we can obtain the desired result. $\square$

Lemma 5.4. $\forall \kappa > 0$,

$$\mathbf{E}(\|\Delta_{j,\varepsilon}(X^{\Psi\Psi})\|^2_{L^2(T^1)}) \lesssim 2^{-(4-\kappa)j}$$

Proof. By Lemmas 3.4 and 3.5, we consider

$$\mathbf{E}\Big(\int_{\mathbf{T}^2}\Delta_{j,\varepsilon}(x,y)\Phi_\varepsilon(t;x,y)\mathrm{d}y\Big)^2$$

in which

$$\Phi_\varepsilon(t;x,y) := (\mathcal{D}_\varepsilon X_\varepsilon^{\Psi}(t,y) - \mathcal{D}_\varepsilon X_\varepsilon^{\Psi}(t,x))\mathcal{D}_\varepsilon X_\varepsilon(t,y))^\diamond$$

By Lemma 5.1,

$$|\mathbf{E}\big(\Phi_\varepsilon(t;x,y)\Phi_\varepsilon(\tau;x,\bar y)\big)| \lesssim \frac{|x-y|^\gamma|x-\bar y|^\gamma}{\sqrt{|t-\tau|}+|y-\bar y|}E\big[\|X_\varepsilon^{\Psi}\|^2_{L_T^\infty\mathcal{C}_x^{\gamma+1,\varepsilon}}\big]$$

This completes the proof. $\square$

Lemma 5.5. $\forall \kappa > 0$,

$$\mathbf{E}\|\big(\Delta_{j,\varepsilon} X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} \lesssim 2^{-(4-\kappa)j}$$

Proof.

$$\Delta_{j,\varepsilon}(X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,x)) = \int_{-\infty}^t e^{-(t-\tau)\lambda^2_{k,\varepsilon}} \varphi(\lambda_{k,\varepsilon}) < (\partial_x X_\varepsilon^{\vee})^{\diamond 2}(\tau), \psi_{k,\varepsilon} > \psi_{k,\varepsilon}(x)\mathrm{d}\tau$$

Then by othorgonality of eigen-functions of $\sqrt{-\mathcal{L}_\varepsilon}$,

$$\|\Delta_{j,\varepsilon}(X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,\cdot))\|_{L^2} = \sum_{k=1}^\infty \Big[\int_{-\infty}^t e^{-(t-\tau)\lambda^2_{k,\varepsilon}} \varphi(\lambda_{k,\varepsilon}) < (\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}})^{\diamond 2}(\tau), \psi_{k,\varepsilon} > \mathrm{d}\tau\Big]^2$$

Then

$$\mathbf{E}\|\big(\Delta_{j,\varepsilon} X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} = \sum_{k,k'} \int_{-\infty}^t \int_{-\infty}^t \Big[e^{-(t-\tau)\lambda^2_{k,\varepsilon}} \varphi(\lambda_{k,\varepsilon}) e^{-(t-\tau)(\lambda^2_{k',\varepsilon}+1)} \varphi(\lambda_{k',\varepsilon})\Big]$$
$$\mathbf{E}\Big(< (\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}})^{\diamond 2}(\tau), \psi_{k,\varepsilon} >< (\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}})^{\diamond 2}(\tau'), \psi_{k',\varepsilon} > \Big)\mathrm{d}\tau\mathrm{d}\tau'$$

Then

$$\mathbf{E}\|\big(\Delta_{j,\varepsilon} X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} = \sum_{k,k'} \int_{-\infty}^t \int_{-\infty}^t \Big[e^{-(t-\tau)\lambda^2_{k,\varepsilon}} \varphi_j(\lambda_{k,\varepsilon}) e^{-(t-\tau)(\lambda^2_{k',\varepsilon}+1)} \varphi_j(\lambda_{k',\varepsilon})\Big]$$
$$\mathbf{E}\Big(< (\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}})^{\diamond 2}(\tau), \psi_{k,\varepsilon} >< (\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}})^{\diamond 2}(\tau'), \psi_{k',\varepsilon} > \Big)\mathrm{d}\tau\mathrm{d}\tau'$$

We re-write it as

$$\mathbf{E}\|\big(\Delta_{j,\varepsilon} X_\varepsilon^{\overset{\vee\vee}{\vee}}(t,\cdot)\|^2_{L^2(\mathbf{T}^1)} = 2\int_{-\infty}^t \int_{-\infty}^t \int_{\mathbf{T}^2} [\mathbf{E}\big(\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}}(\tau,z)\partial_x X_\varepsilon^{\overset{\vee\vee}{\vee}}(\tau',z')\big)]^2 \Theta_\varepsilon(z,z')\mathrm{d}z\mathrm{d}z'\mathrm{d}\tau\mathrm{d}\tau'$$

in which

$$\Theta_{j,\varepsilon}(t,\tau,z,z') := \sum_{k,k'} \Big[e^{-(t-\tau)\lambda^2_{k,\varepsilon}} \varphi_j(\lambda_{k,\varepsilon}) e^{-(t-\tau)\lambda^2_{k',\varepsilon}} \varphi_j(\lambda_{k',\varepsilon})\Big] \psi_{k,\varepsilon}(z)\psi_{k',\varepsilon}(z).$$

Then by estimates for blocks of generalised Besov spaces and Lemma 5.2, the assertion follows. □

Lemma 5.6. $\forall \kappa > 0$, $\forall j \geq -1$, $\forall x \in \mathbf{T}^1$,

$$\mathbf{E}(\int_{\mathbf{T}^1} \Delta_{j,\varepsilon}(x,y)\Phi_\varepsilon(t;x,y)\mathrm{d}y)^2 \lesssim 2^{\kappa j}$$

in which

$$\Phi_\varepsilon(t;x,y) := [\mathcal{D}_\varepsilon P_\varepsilon(t,y) - \mathcal{D}_\varepsilon P_\varepsilon(t,x)] \diamond \mathcal{D}_\varepsilon X_\varepsilon(t,y).$$

Proof. By definition,

$$\mathbf{E}(\mathcal{D}_\varepsilon X_\varepsilon(t,x)\mathcal{D}_\varepsilon X_\varepsilon(t,y)) = \sum_n \frac{\lambda_{n,\varepsilon}^2}{2(\lambda_{n,\varepsilon}^2+1)}\psi_{n,\varepsilon}(x)\psi_{n,\varepsilon}(y)$$

By suitable re-arrangement, we have

$$\mathbf{E}(\mathcal{D}_\varepsilon X_\varepsilon(t,x)\mathcal{D}_\varepsilon X_\varepsilon(t,y)) = \frac{1}{2}\delta_{x,y} - \sum_n \frac{1}{2(\lambda_{n,\varepsilon}^2+1)}\psi_{n,\varepsilon}(x)\psi_{n,\varepsilon}(y)$$

Since the second term of the above formula is $L^\infty_{x,y}$, then $\forall\gamma\in(0,\frac{1}{2})$, we have

$$\mathbf{E}(\int_{\mathbf{T}^1}\Delta_{j,\varepsilon}(x,y)\Phi_\varepsilon(t;x,y)\mathrm{d}y)^2 \lesssim \int_{\mathbf{T}^2}|\Delta_{j,\varepsilon}(x,y)|^2|x-y|^{2\gamma}\mathrm{d}y\mathbf{E}\|\mathcal{D}_\varepsilon P_\varepsilon\|^2_{L^\infty\mathcal{C}^{\gamma,\varepsilon}}.$$

This completes the proof. $\square$

5.2 Convergence of solutions

Let

$$\begin{aligned}
&\tilde X_\varepsilon := \mathcal{I}_\varepsilon(\xi);\\
&\tilde X_\varepsilon^{\vee} := \mathcal{I}_\varepsilon\big[(\mathcal{D}_\varepsilon\tilde X_\varepsilon)^{\diamond 2}\big] + J_\varepsilon^{\vee};\\
&\tilde X_\varepsilon^{\vee\!\!\cdot} := \mathcal{I}_\varepsilon(\mathcal{D}_\varepsilon\tilde X_\varepsilon^{\vee}\mathcal{D}_\varepsilon\tilde X_\varepsilon) + J_\varepsilon^{\vee\!\!\cdot};\\
&\tilde X^{\vee\!\!\cdot\!\!\cdot} := \mathcal{I}_\varepsilon\big[(\mathcal{D}_\varepsilon\tilde X_\varepsilon^{\vee\!\!\cdot}\circ_\varepsilon^\diamond\mathcal{D}_\varepsilon\tilde X_\varepsilon)\big] + J_\varepsilon^{\vee\!\!\cdot\!\!\cdot};\\
&\tilde X_\varepsilon^{\vee\vee} := \tilde{\mathcal{I}}_\varepsilon\big[(\mathcal{D}_\varepsilon\tilde X_\varepsilon^{\vee})^{\diamond 2}\big] + J_\varepsilon^{\vee\vee};\\
&\tilde P_\varepsilon := \tilde{\mathcal{I}}_\varepsilon(\sqrt{-\mathcal{L}_\varepsilon}\tilde X_\varepsilon).
\end{aligned}$$

The relation between X_ε and $\tilde X_\varepsilon$ is

$$\tilde X_\varepsilon := X - e^{t\mathcal{L}_\varepsilon}(X(0)) + \int_0^t \hat\xi(\tau,0)\mathrm{d}\tau.$$

Then $\tilde X_\varepsilon$ lives in the same spaces as X_ε.

$$J_\varepsilon^{\vee} = \mathbf{E}(\mathcal{I}_\varepsilon((\tilde X_\varepsilon + X_\varepsilon)(\int_0^t\hat\xi(\tau,0)\mathrm{d}\tau - e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))).$$

We analyse the second term. The first term is similar. We divide the second term into three parts.

$$\begin{aligned}
&\Delta_{j,\varepsilon}(J_\varepsilon^{\vee}) = \mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde X_\varepsilon + X_\varepsilon)\prec_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))\\
&+\mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde X_\varepsilon + X_\varepsilon)\succ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0))) + \mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde X_\varepsilon + X_\varepsilon)\circ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))
\end{aligned}$$

For the first part,

$$|\mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde X_\varepsilon + X_\varepsilon)\prec_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))| \lesssim 2^{-2j}2^{(1+\kappa)j}[\mathbf{E}\|\tilde X_\varepsilon + X_\varepsilon\|^2_{L^\infty\mathcal{C}^{-\frac{1+\kappa}{2}}}]^{\frac{1}{2}}[\mathbf{E}\|X_\varepsilon(0)\|^2_{\mathcal{C}^{-\frac{1+\kappa}{2}}}]^{\frac{1}{2}}$$

So this part is controlled by $2^{(1-\kappa)j}$. The second part can be dealt with similarly. For the third part,

$$|\mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde{X}_\varepsilon+X_\varepsilon)\circ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))|\lesssim\mathbf{E}[\sup_{t\in[0,T]}t^{\frac{1}{2}+\kappa}\|(\tilde{X}_\varepsilon+X_\varepsilon)\circ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)\|_{\mathcal{C}^\kappa}]2^{(1-\kappa)j}$$

By Lemma 2.8 and heat kernel properties,

$$\mathbf{E}\sup_{t\in[0,T]}t^{\frac{1}{2}+\kappa}\|(\tilde{X}_\varepsilon+X_\varepsilon)\circ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0))\|_{\mathcal{C}^\kappa}\lesssim\mathbf{E}(\|\tilde{X}_\varepsilon+X_\varepsilon\|_{L^\infty\mathcal{C}^{-\frac{1+\kappa}{2}}}\|X_\varepsilon(0)\|_{-\frac{1+\kappa}{2}})$$

This implies

$$|\mathbf{E}(\Delta_{j,\varepsilon}\mathcal{I}_\varepsilon((\tilde{X}_\varepsilon+X_\varepsilon)\circ_\varepsilon e^{t\mathcal{L}_\varepsilon}X_\varepsilon(0)))|\lesssim 2^{(1-\kappa)j}.$$

Then $J_\varepsilon^{\vee}$ lives in $L^\infty\mathcal{C}^{1-\kappa}$, $\forall\kappa>0$. Similarly, we can prove $J_\varepsilon^{\vee}$ converges in $L^\infty\mathcal{C}^{1-\kappa}$, $\forall\kappa>0$ as above. For other terms, we can obtain similar results.

Theorem 5.7. *$\alpha\in(\frac{1}{3},\frac{1}{2})$, $\phi\in\mathcal{C}^\alpha$, we define*

$$\begin{aligned}\Gamma_{1,\varepsilon}:=&\tilde{X}_\varepsilon+\tilde{X}_\varepsilon^{\vee}+2\tilde{X}_\varepsilon^{\curlyvee}+\Gamma_{2,\varepsilon};\\ \Gamma_{2,\varepsilon}:=&u'_\varepsilon\prec_\varepsilon P_\varepsilon+\Gamma_{3,\varepsilon}+e^{t\mathcal{L}_\varepsilon}\phi;\\ \Gamma_{3,\varepsilon}:=&[\mathcal{I}_\varepsilon,u'_\varepsilon\prec_\varepsilon]\mathcal{D}_\varepsilon X_\varepsilon+\mathcal{I}_\varepsilon(K_\varepsilon)+X_\varepsilon^{\vee\vee}+\tilde{X}_\varepsilon^{\curlyvee\curlyvee}\end{aligned}\tag{5.2}$$

where $u'_\varepsilon:=2\mathcal{D}_\varepsilon u_\varepsilon+4\mathcal{D}_\varepsilon\tilde{X}_\varepsilon^{\curlyvee}$ and $K_\varepsilon:=K_{1,\varepsilon}+K_{2,\varepsilon}$. They are defined below.

$$K_{1,\varepsilon}:=(\mathcal{D}_\varepsilon u_\varepsilon^{P_\varepsilon})^2+2\mathcal{D}_\varepsilon u_\varepsilon\mathcal{D}_\varepsilon(\tilde{X}_\varepsilon^{\vee}+\tilde{X}_\varepsilon^{\curlyvee})+4X_\varepsilon^{\curlyvee}(\tilde{X}_\varepsilon^{\curlyvee}+\tilde{X}_\varepsilon^{\vee})+u'_\varepsilon\succ_\varepsilon\mathcal{D}_\varepsilon\tilde{X}_\varepsilon,$$

and

$$\begin{aligned}K_{2,\varepsilon}:=&(\mathcal{D}_\varepsilon u_\varepsilon^{\#}+[\mathcal{D}_\varepsilon,u'_\varepsilon\prec_\varepsilon]\mathcal{D}_\varepsilon P_\varepsilon+\mathcal{D}_\varepsilon e^{t\mathcal{L}_\varepsilon}\phi)\circ_\varepsilon\mathcal{D}_\varepsilon X_\varepsilon\\ &+C_\varepsilon(u'_\varepsilon,\mathcal{D}_\varepsilon P_\varepsilon,\mathcal{D}_\varepsilon X_\varepsilon)+u'_\varepsilon(\mathcal{D}_\varepsilon P_\varepsilon\circ_\varepsilon^{\diamond}\mathcal{D}_\varepsilon X_\varepsilon).\end{aligned}$$

Then $\exists T>0$, $\forall\varepsilon\in[0,1]$, $\exists(u_\varepsilon,u_\varepsilon^{P_\varepsilon},u_\varepsilon^{\#})$ is the unique fixed point of Γ_ε and

$$(u_\varepsilon,u_\varepsilon^{P_\varepsilon},u_\varepsilon^{\#})\in\left(\mathcal{C}_T^{\frac{\alpha}{2}}L_x^\infty\cap L_T^\infty\mathcal{C}_x^{\alpha,\varepsilon}\right)\times L_T^\infty\mathcal{C}_x^{\alpha+1,\varepsilon}\times L_T^\infty\mathcal{C}_x^{2\alpha+1,\varepsilon}.$$

The uniform bounds of fixed points and existence time T_ε depend on the uniform bound of Υ_ε the vector of enhanced stochastic terms.

Theorem 5.8. *Let $(u_\varepsilon,u_\varepsilon^{P_\varepsilon},u_\varepsilon^{\#})$ be the fixed point of Γ_ε defined in Theorem 5.7. Let $(u_0,u_0^{P_0},u_0^{\#})$ be the solution of*

$$\begin{aligned}\Gamma_{1,0}:=&\tilde{X}_0+\tilde{X}_0^{\vee}+2\tilde{X}_0^{\curlyvee}+\Gamma_{2,0};\\ \Gamma_{2,0}:=&u'_0\prec_0 P_0+\Gamma_{3,0}+e^{t\mathcal{L}_0}\phi;\\ \Gamma_{3,0}:=&[\mathcal{I}_0,u'_0\prec_0]\mathcal{D}_0\tilde{X}_0+\mathcal{I}_0(K_0)+\tilde{X}_0^{\vee\vee}+\tilde{X}_0^{\curlyvee\curlyvee}\end{aligned}\tag{5.3}$$

where $u'_0:=2\mathcal{D}_0u_0+4\mathcal{D}_0\tilde{X}_0^{\curlyvee}$ and $K_0:=K_{1,0}+K_{2,0}$. They are defined below.

$$K_{1,0}:=(\mathcal{D}_0u_0^{P_0})^2+2\mathcal{D}_0u_0\mathcal{D}_0(\tilde{X}_0^{\vee}+\tilde{X}_0^{\curlyvee})+4\tilde{X}_0^{\curlyvee}(\tilde{X}_0^{\curlyvee}+\tilde{X}_0^{\vee})+u'_0\succ_0\mathcal{D}_0\tilde{X}_0,$$

and

$$\begin{aligned}K_{2,0}:=&(\mathcal{D}_0u_0^{\#}+[\mathcal{D}_0,u'_0\prec_0]\mathcal{D}_0P_0+e^{t\mathcal{L}_\varepsilon}\phi)\circ_0\mathcal{D}_0\tilde{X}_0\\ &+C_0(u'_0,\mathcal{D}_0P_0,\mathcal{D}_0\tilde{X}_0)+u'_0(\mathcal{D}_0P_0\circ_0^{\diamond}\mathcal{D}_0\tilde{X}_0).\end{aligned}$$

Then u_ε converges to u_0 in $\mathcal{C}_T^{\frac{\alpha}{2}-\kappa}L_x^\infty\cap L_T^\infty\mathcal{C}_x^{\alpha-\kappa}$ almost surely, $\kappa>0$.

Proof. Define $A_n = (u_{\varepsilon_n}, u_{\varepsilon}^{P_\varepsilon}, u_{\varepsilon_n}^{\#}, u_{\varepsilon_n}^{\#} \circ_{\varepsilon_n} \mathcal{D}_{\varepsilon_n}\tilde{X}_{\varepsilon_n})$, $\forall n \in \mathbb{N}^+$, where $\varepsilon_n = \frac{1}{n}$. For $\kappa \geq 0$ and $\tilde{\kappa} > 0$, define

$$\mathcal{S}_\kappa := (C^{\frac{\alpha-\kappa}{2}}L^\infty \cap L^\infty C^{\alpha-\kappa}) \times (C^{\theta-\kappa}L^\infty \cap L^\infty C^{1-\kappa})^3.$$

Since Υ_ε the vector of enhanced stochastic terms is almost surely bounded in the corresponding space, $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$ almost surely. When $\{A_n\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $\mathcal{S}_0$, then $\forall\kappa > 0$, $\{b_n\}_{n\in\mathbb{N}^+} \subset \mathbb{N}^+$, $\exists\{c_n\}_{n\in\mathbb{N}^+}$ a subsequence of $\{b_n\}_{n\in\mathbb{N}^+}$, $\exists A = (v, v^P, v^{\#}, w) \in \mathcal{S}_\kappa$, s.t.

$$A_{b_n} \to A \quad in \quad \mathcal{S}_\kappa.$$

Combining the above convergence and $u_{\varepsilon_{cn}}^{\#} = \Gamma_{\varepsilon_{cn},3}(u_{\varepsilon_{cn}}, u_{\varepsilon}^{P_\varepsilon}, u_{\varepsilon_{cn}}^{\#})$, $\forall n \in \mathbb{N}^+$, we have

$$v^{\#} :=[\mathcal{I}_0, u_0' \prec_0]\mathcal{D}_0\tilde{X}_0 + \mathcal{I}_0(\tilde{K}_0) + \tilde{X}_0^{\vee\vee} + \tilde{X}_0^{\curlyvee\!\cdot} \tag{5.4}$$

where $u_0' := 2\mathcal{D}_0 u_0 + 4\mathcal{D}_0\tilde{X}_0^{\curlyvee}$ and $\tilde{K}_0 := K_{1,0} + K_{2,0}$. They are defined below.

$$\tilde{K}_{1,0} := (\mathcal{D}_0 u_0^{P_0})^2 + 2\mathcal{D}_0 u_0 \mathcal{D}_0(\tilde{X}_0^{\vee} + \tilde{X}_0^{\curlyvee}) + 4\tilde{X}_0^{\curlyvee}(\tilde{X}_0^{\curlyvee} + \tilde{X}_0^{\vee}) + u_0' \succ_0 \mathcal{D}_0\tilde{X}_0,$$

and

$$\begin{aligned}\tilde{K}_{2,0} :=&([\mathcal{D}_0, u_0' \prec_0]\mathcal{D}_0 P_0 + e^{t\mathcal{L}_\varepsilon}\phi) \circ_0 \mathcal{D}_0\tilde{X}_0 + w\\ &+C_0(u_0', \mathcal{D}_0 P_0, \mathcal{D}_0\tilde{X}_0) + u_0'(\mathcal{D}_0 P_0 \circ_0^\diamond \mathcal{D}_0\tilde{X}_0).\end{aligned}$$

Now, we are in the position to prove that

$$w = v^{\#} \circ_0 \mathcal{D}_0\tilde{X}_0. \tag{5.5}$$

We show the second one. The first one can be dealt with similarly. We divide the difference into three parts.

$$\begin{aligned}&w_2 - v^{\#} \circ_0 \mathcal{D}_0\tilde{X}_0 = w_2 - u_{\varepsilon_{cn}}^{\#} \circ_{\varepsilon_{cn}} \mathcal{D}_{\varepsilon_{cn}}\tilde{X}_{\varepsilon_{cn}}\\ &+(\sum_{i=-1}^{N} + \sum_{i=N+1}^{\infty})(\sum_{|i-j|\leq 1} \Delta_{i,\varepsilon_{cn}} u_{\varepsilon_{cn}}^{\#} \Delta_{j,\varepsilon_{cn}}\mathcal{D}_{\varepsilon_{cn}}\tilde{X}_{\varepsilon_{cn}} - \Delta_{i,0}v^{\#}\Delta_{j,0}\mathcal{D}_0\tilde{X}_0)\end{aligned}$$

For the third part, by the uniform bound of $u_\varepsilon^{\#}$, we have $\forall N \in \mathbb{N}^+$,

$$\|\sum_{i=N+1}^{\infty}(\sum_{|i-j|\leq 1} \Delta_{i,\varepsilon_{cn}} u_{\varepsilon_{cn}}^{\#} \Delta_{j,\varepsilon_{cn}}\mathcal{D}_{\varepsilon_{cn}}\tilde{X}_{\varepsilon_{cn}} - \Delta_{i,0}v^{\#}\Delta_{j,0}\mathcal{D}_0\tilde{X}_0)\|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then since A_{c_n} converges to A in the corresponding space, then $\forall N \in \mathbb{N}^+$,

$$\|w_2 - v \circ_0 \eta\|_{\mathcal{C}^{2\alpha-1-3\kappa}} \lesssim 2^{-\kappa N}$$

Then the second one of (3.7) is obtained.

With (3.7), (3.6) and uniqueness of solutions of classical two dimensional gPAM models in $\mathbf{T}^2$, we have

$$(v, v^P, v^{\#}) = (u_0, u_0^{P_0}, u_0^{\#}).$$

Then the limit is independent of the choice of sub-sequences. This completes the proof. $\square$

Theorem 5.9. *Let $(u_\varepsilon, u_\varepsilon^\#)$ be the fixed point of Γ_ε defined in Theorem 5.7. Let $(u_0, u_\varepsilon^\#)$ be the solution of (5.3). Then $t^{\frac{1-\alpha}{2}} a_\varepsilon \nabla u_\varepsilon$ converges to $t^{\frac{1-\alpha}{2}} \bar{a} \nabla u_0$ in $L_T^\infty \mathcal{C}_x^{-\kappa}$ almost surely, $\kappa > 0$.*

Proof. We divide the flux into two parts.

$$a_\varepsilon \nabla u_\varepsilon = a_\varepsilon \nabla (\tilde{X}_\varepsilon + \tilde{X}_\varepsilon^{\vee} + 2\tilde{X}_\varepsilon^{\vee\!\!\!\vee} + \Phi_\varepsilon) + a_\varepsilon \nabla u_\varepsilon^{P_\varepsilon}$$

Then convergence of the first part follows from Green's function estimates. It suffices to deal with the last term. When $\{a_{\varepsilon_n} \nabla u_{\varepsilon_n}^{P_\varepsilon}\}_{n\in\mathbb{N}^+}$ is bounded in $L_{T,x}^\infty \cap \mathcal{C}_T^\theta \mathcal{C}^{-\kappa}$, then for any subsequence $\{a_{\varepsilon_{b_n}} \nabla u_{\varepsilon_{b_n}}^{P_\varepsilon}\}_{n\in\mathbb{N}^+}$, there is a sequence $\{c_n\}_{n\in\mathbb{N}+}$ which is a subsequence of $\{b_n\}_{n\in\mathbb{N}^+}$, s.t.

$$\begin{aligned} a_{\varepsilon_{c_n}} \nabla u_{\varepsilon_{c_n}}^{P_\varepsilon} &\rightharpoonup H \quad *-weakly \quad in \quad L_{T,x}^\infty, \\ a_{\varepsilon_{c_n}} \nabla u_{\varepsilon_{c_n}}^{P_\varepsilon} &\to H \quad strongly \quad in \quad L^\infty \mathcal{C}^{-\kappa}. \end{aligned}$$

For any fixed $h \in C^\infty([0,T]; \mathbf{T}^2)$, by integration by parts,

$$\int_0^T \langle a_\varepsilon \nabla u_\varepsilon^{P_\varepsilon}(r,\cdot), \Phi_\varepsilon^*(\cdot) h(r,\cdot)\rangle \mathrm{d}r = -\int_0^T \langle (u_\varepsilon^{P_\varepsilon})(r,\cdot), a_\varepsilon^\intercal \Phi_\varepsilon^*(\cdot) \nabla h(r,\cdot)\rangle \mathrm{d}r$$

For the right hand side, by the convergence of $u_\varepsilon^{P_\varepsilon}$, as $\varepsilon \to 0$,

$$\int_0^T \langle u_\varepsilon^{P_\varepsilon}, a_\varepsilon^\intercal \Phi_\varepsilon^* \nabla h\rangle \mathrm{d}r \to \int_0^T \langle u_0^{P_0}, \bar{a}^\intercal \nabla h\rangle \mathrm{d}r.$$

For the left hand side, since $\mathcal{L}_\varepsilon u_\varepsilon^{P_\varepsilon} \to \mathcal{L}_0 u_0^{P_0}$ in $L_T^\infty \mathcal{C}_x^{2\alpha-1-\kappa}$, by the div-curl lemma in [14], we have

$$\int_0^T \langle a_\varepsilon \nabla u_\varepsilon^{P_\varepsilon}, \Phi_\varepsilon^* h\rangle \mathrm{d}r \to \int_0^T \langle H, h\rangle \mathrm{d}r.$$

Then $H = \bar{a} \nabla u_0^{P_0}$ which is independent of the choice of sub-sequences. This completes the proof. □

A Bounds on Green's functions

Let M_ε be the fundamental solution of $\partial_t - \mathcal{L}_\varepsilon$ in $\mathbf{R}^+ \times \mathbf{R}^d$. Let Q_ε be the Green's function of $\partial_t - \mathcal{L}_\varepsilon$ in $\mathbf{R}^+ \times \mathbf{T}^d$. Then

$$Q_\varepsilon(t;x,y) = \sum_{k\in\mathbb{Z}^d} M_\varepsilon(t;x,y+k) \tag{A.1}$$

By [6, (1-6)], essentially based on [3, 5, 7], we have

$$|M_\varepsilon(t;x,y)| \lesssim t^{-\frac{d}{2}} e^{-\frac{c|x-y|^2}{t}}.$$

The following lemma is the consequence of [6, (1-6)], [6, Theorem 2.7] and (A.1).

Proposition A.1. *For the Green's function Q_ε, we have $\forall x, y \in \mathbf{T}^d$, $\varepsilon \in [0,1]$,*

$$|Q_\varepsilon(t,x,y)| \lesssim t^{-\frac{d}{2}} e^{-\frac{c|x-y|^2}{t}} .$$

Moreover,

$$|\nabla_x Q_\varepsilon(t,x,y)| + |\nabla_y Q_\varepsilon(t;x,y)| \lesssim t^{-\frac{d+1}{2}} e^{-\frac{c|x-y|^2}{t}} .$$

And

$$|\nabla_y \nabla_x Q_\varepsilon(t;x,y)| \lesssim t^{-\frac{d+2}{2}} e^{-\frac{c|x-y|^2}{t}} .$$

Denote the Lebesgue measure of $S \subset \mathbf{R}^d$ by $|S|$. If $|S| < +\infty$, define

$$⨍_S f(x)\mathrm{d}x := \frac{1}{|S|}\int_S f(x)\mathrm{d}x.$$

Lemma A.2. $\exists C > 0$,$\forall \varepsilon \in [0,1]$, $\forall x, y \in \mathbf{T}^d$ *and* $x \neq y$,

$$|(\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon)(t;x,y)| \lesssim [\sqrt{|t|} + |x-y|]^{-d-1}.$$

where

$$\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon(t;x,y) := \sum_{n\in\mathbb{N}} \lambda_{n,\varepsilon} < Q_\varepsilon(t;\cdot,y), \psi_{n,\varepsilon} > \psi_{n,\varepsilon}(x).$$

Proof. Fix $(t_0,x_0),(t_1,x_1) \in \mathbf{R}^+ \times \mathbf{T}^d$ and $(t_0,x_0) \neq (t_1,x_1)$. define $r = \sqrt{|t_2 - t_1|} + |x_1 - x_2|$. Define $D_r(t_0,x_0) := (t_0 - r^2, B(x_0,r))$. For $f \in C_0^\infty\big(D_{\frac{r}{8}}((t_1,x_1))\big)$, define

$$u_{f,\varepsilon}(t,x) := \int_0^t \mathrm{d}r \int_{\mathbf{T}^d} (\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon)(t-\tau;x,y) f(\tau,y)\mathrm{d}y.$$

Since $\partial_t u_{f,\varepsilon} = \mathcal{L}_\varepsilon u_{f,\varepsilon}$ in $D_{\frac{r}{2}}(t_0,x_0)$, by [7, Theorem 1.3],

$$\|u_{f,\varepsilon}\|_{L^\infty(D_{\frac{r}{4}}(t_0,x_0))} \lesssim (⨍_{D_{\frac{r}{2}}(t_0,x_0)} |u_{f,\varepsilon}(t,x)|^2 \mathrm{d}x\mathrm{d}t)^{\frac{1}{2}}$$

Since $\|u_{f,\varepsilon}\|_{L^\infty L^2} \lesssim \|f\|_{L^2L^2}$, then

$$\|u_{f,\varepsilon}\|_{L^\infty(D_{\frac{r}{4}}(t_0,x_0))} \lesssim r^{-\frac{d}{2}} \|f\|_{L^2L^2}$$

Then

$$|⨍_{D_{\frac{r}{8}}(t_1,x_1)} (\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon)(t_0-\tau;x_0,y) f(\tau,y)\mathrm{d}y| \lesssim \frac{1}{r^{d+1}} (⨍_{D_{\frac{r}{8}}(t_1,x_1)} |f(\tau,y)|^2 \mathrm{d}y)^{\frac{1}{2}}$$

Then

$$(⨍_{D_{\frac{r}{8}}(t_1,x_1)} |(\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon)(t_0,\tau;x_0,y)|^2 \mathrm{d}y)^{\frac{1}{2}} \lesssim r^{-\frac{d+1}{2}}.$$

Using [7, Theorem 1.3],

$$|(\sqrt{-\mathcal{L}_\varepsilon} Q_\varepsilon)(t_0,t_1;x_0,x_1)| \lesssim r^{-\frac{d+1}{2}}.$$

This completes the proof. ∎

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