A Graphical Calculus for Stable Curvature Invariants

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Abstract

In this article we develop a graphical calculus for stable invariants of Riemannian manifolds akin to the graphical calculus for Rozansky–Witten invariants for hyperkähler manifolds; based on interpreting trivalent graphs with colored edges as stably invariant polynomials on the space of algebraic curvature tensors. In this graphical calculus we describe explicitly the Pfaffian polynomials $(\mathrm{pf}_n)_{n \in \mathbb{N}_0}$ central to the Theorem of Chern–Gauß–Bonnet and the normalized moment polynomials $(\Psi^\circ_n)_{n \in \mathbb{N}_0}$ calculating the moments of sectional curvature considered as a random variable on the Graßmannian of planes. Eventually we illustrate the power of this graphical calculus by deriving a curvature identity for compact Einstein manifolds of dimensions greater than 2 involving the Euler characteristic, the third moment of sectional curvature and the L^2–norm of the covariant derivative of the curvature tensor. A model implementation of this calculus for the computer algebra system Maxima is available [12].

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1 Introduction

Riemannian manifolds are studied extensively in Differential Geometry from different points of view, among which is the classical topic of classifying Riemannian manifolds up to homeomorphisms, diffeomorphisms or isometries. Riemannian invariants provide a direct method to distinguish non–isometric manifolds, such invariants can be constructed for example by integrating a scalar valued polynomial ψ in algebraic curvature tensors over the manifold in question. In order to have the integrand well–defined independent of the choice of coordinates ψ needs to be a polynomial on the space $\mathrm{Curv}^- T$ of algebraic curvature tensors invariant under the orthogonal group $\mathbf{O}(T, g)$. Tabulating the dimensions of the spaces of such invariant polynomials in dependence on their degree $n \in \mathbb{N}_0$ and the dimension $m \in \mathbb{N}$

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of the euclidean vector space with the help of the computer algebra system LiE we obtain:

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	$\underline{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\underline{1}$	$\underline{2}$	$\underline{4}$	3	3	3	3	3	3	3	3	3	3	3
3	$\underline{1}$	$\underline{3}$	$\underline{9}$	$\underline{7}$	$\underline{8}$	8	8	8	8	8	8	8	8	8
4	$\underline{1}$	$\underline{4}$	$\underline{19}$	$\underline{20}$	$\underline{24}$	$\underline{25}$	$\underline{28}$	26	26	26	26	26	26	26
5	$\underline{1}$	$\underline{5}$	$\underline{39}$	$\underline{51}$	$\underline{83}$	$\underline{84}$	$\underline{101}$	$\underline{89}$	$\underline{90}$	90	90	90	90	90
6	$\underline{1}$	$\underline{7}$	$\underline{82}$	$\underline{150}$	$\underline{361}$	$\underline{359}$	$\underline{509}$	$\underline{403}$	$\underline{409}$	$\underline{408}$	$\underline{412}$	409	409	409
7	$\underline{1}$	$\underline{8}$	$\underline{151}$	$\underline{431}$	$\underline{1697}$	$\underline{1761}$	$\underline{3125}$	$\underline{2194}$	$\underline{2407}$	$\underline{2240}$	$\underline{2281}$	$\underline{2245}$	$\underline{2246}$	2246 .

Certainly the most interesting aspect of these dimensions is that they stabilize for fixed degree $n \in \mathbb{N}_0$ provided the dimension m of the vector space T is sufficiently large, more precisely the dimension of the space of invariant polynomials of degree $n \in \mathbb{N}_0$ on $\mathrm{Curv}^- T$ is constant for $\dim T > 2n$ with a last drop from $\dim T = 2n$ to $\dim T = 2n+1$ for even n. This phenomenon gives rise to the concept of stable curvature invariants.

Stability is a somewhat vague concept, it can be made precise by introducing a suitably defined category **CM** of curvature models. Objects in this category are triples (T, g, R) formed by an algebraic curvature tensor $R \in \mathrm{Curv}^- T$ over a euclidean vector space (T, g), morphisms are the adjoints $F^* : T \longrightarrow \hat{T}$ of linear isometric maps $F : \hat{T} \longrightarrow T$ satisfying

$$\hat{R}(\,F^*X,\,F^*Y;\,F^*U,\,F^*V\,) \quad = \quad R(\,X,\,Y;\,U,\,V\,) \tag{1}$$

for all $X, Y, U, V \in T$. In turn a stable curvature invariant is a functor $\psi : \mathbf{CM} \longrightarrow \mathbb{R}$ from the curvature model category **CM** to the category $\mathbb{R}$ of real numbers with only the identities as morphisms. In other words a stable curvature invariant ψ associates a real number $\psi(R) \in \mathbb{R}$ to every curvature model (T, g, R) regardless of its dimension $m \in \mathbb{N}_0$ with equality $\psi(R) = \psi(\hat{R})$, whenever there exists a linear isometric map $F : \hat{T} \longrightarrow T$ whose adjoint $F^* : T \longrightarrow \hat{T}$ satisfies the constraint (1).

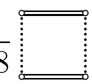
The category **CM** of curvature models captures the essence of the stabilization phenomenon observed above: The adjoint $F^* : T \longrightarrow \hat{T}$ of an isometric map $F : \hat{T} \longrightarrow T$ equals the composition $F^* = F^{-1} \circ \mathrm{pr}$ of the orthogonal projection to the regular subspace $\mathrm{im}\, F \subseteq T$ followed by the isometry $F^{-1} : \mathrm{im}\, F \longrightarrow \hat{T}$. In turn the existence of a morphism $F^* : (T, g, R) \longrightarrow (\hat{T}, \hat{g}, \hat{R})$ in the category **CM** tells us via equation (1) that the algebraic curvature tensor R is essentially the Cartesian product $\hat{R} \oplus 0$ of the algebraic curvature tensor $\hat{R}$ on $\mathrm{im}\, F \cong \hat{T}$ with the flat algebraic curvature tensor on $(\,\mathrm{im}\, F\,)^\perp$.

The purpose of this article is to develop a graphical calculus akin to the calculus of Rozansky–Witten invariants [8] to describe the algebra of stable curvature invariants $\psi : \mathbf{CM} \longrightarrow \mathbb{R}$ such that the induced map $\mathrm{Curv}^- T \longrightarrow \mathbb{R}$, $R \longmapsto \psi(\,R\,)$, is a polynomial on the vector space $\mathrm{Curv}^- T$ of algebraic curvature tensors for every euclidean vector space (T, g). Every such stable polynomial curvature invariant of degree n is necessarily an $2n$–fold iterated sum over an orthonormal basis $E_1, \ldots, E_m$ for T like the scalar curvature of degree $n = 1$

$$\kappa \;:=\; \sum_{\mu,\,\nu\,=\,1}^{m} R(\,E_\mu,\,E_\nu;\,E_\nu,\,E_\mu\,) \qquad \widehat{=} \qquad \frac{1}{4}$$

related to the Einstein–Hilbert functional or the norm square of the curvature tensor

$$g_{\Lambda^2 T^* \otimes \Lambda^2 T^*}(\,R,\,R\,) \;:=\; \frac{1}{4} \sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m} R(\,E_\mu,\,E_\nu;\,E_\alpha,\,E_\beta\,)^2 \qquad \widehat{=} \qquad \frac{1}{48}$$

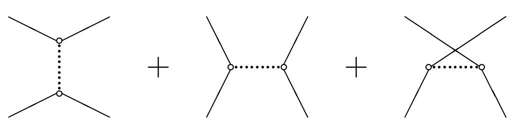


of degree $n = 2$. In order to encode the contraction pattern of these $2n$–fold iterated sums over orthonormal bases in graphs we consider trivalent graphs, possibly with loops or multiple edges between vertices, and with edges colored red and black, rendered in this article as dotted and solid lines respectively, such that every vertex is adjacent to exactly one red edge. Every black edge corresponds to one sum over an orthonormal basis, while every red edge connects two different vertices and corresponds to a copy of the sectional curvature tensor $\mathrm{Sec} \in \mathrm{Curv}^+ T$ associated to the algebraic curvature tensor $R \in \mathrm{Curv}^- T$. A concise formulation of this construction can be found in Definition 3.3.

Having outlined the general idea of how to convert a colored trivalent graph into a stable curvature invariant we define the graph algebra $\mathbb{A}^\bullet$ as the convolution algebra $\mathbb{R}\,\Gamma^\bullet$ of the graded monoid $\Gamma^\bullet$ of isomorphism classes of colored trivalent graphs under the disjoint union product. By construction this algebra comes along with an algebra homomorphisms

$$\mathrm{Inv}_{(\,T,\,g\,)} : \;\; \mathbb{A}^\bullet \;\longrightarrow\; [\,\mathrm{Sym}^\bullet(\,\mathrm{Curv}^- T\,)^*\,]^{\mathbf{O}(\,T,\,g\,)}, \qquad [\,\gamma\,] \;\longmapsto\; [\,\gamma\,]\;,$$

for every euclidean vector space T. It turns out that this algebra homomorphism factorizes over the quotient $\overline{\mathbb{A}}^\bullet$ of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs by the ideal of IHX–relations



on red edges, which arose historically in the study of invariants of knots; in the present context they reflect the first Bianchi identity. The reduced algebra $\overline{\mathbb{A}}^\bullet$ of colored trivalent graphs modulo the IHX–relations equals the algebra of stable curvature invariants:

Theorem 1.1 (Stable Algebra Isomorphism)
The algebra homomorphism from the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ to the algebra of invariant polynomials on the space of algebraic curvature tensors over a euclidean vector space T

$$\overline{\mathrm{Inv}}_{(\,T,\,g\,)} : \;\; \overline{\mathbb{A}}^\bullet \;\longrightarrow [\,\mathrm{Sym}^\bullet(\,\mathrm{Curv}^- T\,)^*\,]^{\mathbf{O}(\,T,\,g\,)}$$

induces isomorphisms $\overline{\mathbb{A}}^n \xrightarrow{\cong} [\,\mathrm{Sym}^n(\,\mathrm{Curv}^- T\,)^*\,]^{\mathbf{O}(\,T,\,g\,)}$ in all degrees $n < \frac{1}{2}\dim T$.

Unluckily we will not even outline the proof of this theorem in this article, mainly because it involves quite a lot of the representation theory of orthogonal and symplectic Lie algebras. However we intend to be more specific about Theorem 1.1 in a future publications, focussing for the time being on the description of the resulting graphical calculus. In case of doubt the reader may easily verify Theorem 1.1 directly in small degrees: Using the general relation between the Hilbert function of a free commutative associative graded algebra $\overline{\mathbb{A}}^\bullet$ and the numbers $m_d^{\overline{\mathbb{A}}} \in \mathbb{N}_0$ of generators of $\overline{\mathbb{A}}^\bullet$ necessary in degree $d \in \mathbb{N}$

$$\sum_{n>0}\Big(\sum_{d\,|\,n} d\,m_d^{\overline{\mathbb{A}}}\Big)\,t^n \;=\; t\frac{d}{dt}\,\ln\Big(\sum_{n\geq 0}(\dim\,\overline{\mathbb{A}}^n\,)\,t^n\Big) \tag{2}$$

we can deduce from the table of dimensions presented above that the algebra $\overline{\mathbb{A}}^\bullet$ of stable curvature invariants should be a free commutative graded algebra with 1, 2, 5, 15, 54, 270 and 1639 generators respectively of degrees 1, 2, 3, 4, 5, 6 and 7; numbers of generators which coincide up to degree 4 with the number of generators of the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ calculated in Section 3 as a direct consequence of Corollary 3.4.

Although characteristic numbers of compact manifolds like the Pontryagin numbers are polynomial curvature invariants, it turns out that they are not stable polynomial curvature invariants in the sense of this article, not to the least so, because the corresponding polynomial $\psi \in [\,\mathrm{Sym}^\bullet(\,\mathrm{Curv}^- T\,)^*\,]^{\mathbf{SO}(T,\,g)}$ is invariant under the special orthogonal subgroup $\mathbf{SO}(\,T,\,g\,)$, but not under the full orthogonal group $\mathbf{O}(\,T,\,g\,)$. The sole exception to this rule is the Euler characteristic: We can write the Theorem of Chern–Gauß–Bonnet for every compact, not necessarily oriented Riemannian manifold M of even dimension m in the form

$$\chi(\,M\,) \;=\; \frac{1}{(\,2\pi\,)^{\frac{m}{2}}}\int_M \mathrm{pf}_{\frac{m}{2}}(\,R\,)\,|\,\mathrm{vol}_g\,|$$

with a sequence $(\,\mathrm{pf}_n\,)_{n\in\mathbb{N}_0}$ of elements of degree n in algebra $\mathbb{A}^\bullet$ of colored trivalent graphs. This sequence of Pfaffian polynomials will be studied in more detail in Section 5 together with the sequence $(\,\Psi_n^\circ\,)_{n\in\mathbb{N}_0}$ of normalized moment polynomials, which calculate the moments of the sectional curvature considered as a random variable on the Graßmannian of planes. Both sequences have strikingly similar expansions in the power series completion

$$\sum_{n\geq 0}\mathrm{pf}_n \;=\; \exp\left(\sum_{[\,\gamma\,]\,\in\,\Gamma^\bullet_{\mathrm{conn}}}\frac{(-1)^{e(\gamma)}}{6^{n(\gamma)}}\,\frac{2^{g(\gamma)}}{\#\overline{\mathrm{Aut}}\,\gamma}\,[\,\gamma\,]\right) \tag{3}$$

$$\sum_{n\geq 0}\Psi_n^\circ \;=\; \exp\left(\sum_{\substack{[\,\gamma\,]\,\in\,\Gamma^\bullet_{\mathrm{conn}}\\ \gamma_{\mathrm{black}}\ \text{even cycles}}}(-1)^{n(\gamma)}\frac{2^{g(\gamma)}}{\#\overline{\mathrm{Aut}}\,\gamma}\,[\,\gamma\,]\right) \tag{4}$$

of the graph algebra $\mathbb{A}^\bullet$ derived in Lemmas 5.1 and 5.5 respectively, where $n(\gamma) := \frac{1}{2}\,\#\,\mathrm{Vert}\,\gamma$ is just the degree of the graph γ, whereas $e(\gamma)$ and $g(\gamma)$ denote the numbers of cycles of the bivalent black subgraph γ_{black} of even length and of length greater than 2 respectively.

In order to illustrate the power of the graphical calculus developed in this article we eventually consider in Section 6 the drastic algebraic simplifications in the reduced algebra $\overline{\mathbb{A}}^\bullet$ of colored trivalent graphs brought about by assuming the Riemannian manifold M to be an Einstein manifold and use these simplification to prove the following curvature identity:

Theorem 6.4 (Cubic Curvature Identity for Einstein Manifolds)
For every compact connected Einstein manifold M of dimension $m \geq 3$ with scalar curvature $\kappa \in \mathbb{R}$ the following identity of integrated stable curvature invariants of degree 3 holds true:

$$\int_M \mathrm{pf}_3(\,R\,)\,|\,\mathrm{vol}_g\,| \;-\; \frac{1}{40}\int_M \Psi_3^\circ(\,R\,)\,|\,\mathrm{vol}_g\,| \;+\; \frac{2}{15}\,\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*}$$
$$=\;\kappa^3\,\frac{m^2-18m+40}{60\,m^2}\,\mathrm{Vol}(\,M,\,g\,)\;+\;\kappa\,\frac{3m-104}{30\,m}\,\|\,R\,\|^2_{\Lambda^2T^*\otimes\Lambda^2T^*}$$

In Section 2 we provide a leisurely introduction to algebraic and sectional curvature tensors. Section 3 is certainly the central section of this article and details the construction of stable curvature invariants from graphs and the construction of the graph algebras $\mathbb{A}^\bullet$ and $\overline{\mathbb{A}}^\bullet$. Graphs are evaluated combinatorially on algebraic curvature tensors of constant sectional curvature in Section 4. In Section 5 we define the Pfaffian and normalized moment polynomials and present the combinatorial arguments behind the expansions (3) and (4) of their generating series. Last but not least we establish Theorem 6.4 in Section 6. A model implementation for the computer algebra system Maxima of the functionalities of the graphical calculus presented below can be found under the link [12].

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2 Algebraic and Sectional Curvature Tensors

Algebraic curvature tensors are discussed in detail in many textbooks on differential geometry, a good reference is for example [2]. In this introductory section we will focus on the less well–known symmetric counterparts of algebraic curvature tensors, the sectional curvature tensors, with the aim to establish the equivalence between both ways to describe curvature. Sectional curvature tensors are however easier to deal with, a statement illustrated by the derivation of the polarization formula for algebraic curvature tensors given in this section. In the development of a graphical calculus for stable curvature invariants this simplicity of sectional compared to algebraic curvature tensors will be a crucial advantage.

A euclidean vector space will be for the purpose of this article a finite dimensional vector space T over $\mathbb{R}$ endowed with a non–degenerate, not necessarily positive definite symmetric bilinear form $g : T \times T \longrightarrow \mathbb{R}$ called its scalar product. An isometric map between euclidean vector spaces T and $\hat{T}$ is a linear map $F : T \longrightarrow \hat{T}$ with the characteristic property that $\hat{g}(FX, FY) = g(X, Y)$ for all $X,\, Y \in T$, isometries are invertible isometric maps. Declaring for example the mutually inverse musical isomorphisms $\flat : T \longrightarrow T^*$, $X \longmapsto g(X, \cdot)$, and $\sharp := \flat^{-1}$ to be isometries defines a scalar product $g^{-1} : T^* \times T^* \longrightarrow \mathbb{R}$ such that $g^{-1}(\alpha, \beta) := \alpha(\beta^\sharp)$. The scalar products g and g^{-1} extend to symmetric and exterior powers of both T and T^* by using Gram's permanent or determinant respectively, for example:

$$g_{\Lambda^k T^*}(\, \alpha_1 \wedge \ldots \wedge \alpha_k,\ \beta_1 \wedge \ldots \wedge \beta_k \,) \;:=\; \det \Big\{\, g^{-1}(\, \alpha_\mu, \beta_\nu \,) \,\Big\}_{\mu,\,\nu\,=\,1,\,\ldots,\,k} .$$

Coming back to our main topic we define an algebraic curvature tensor on a euclidean vector space T to be a quadrilinear form $R : T \times T \times T \times T \longrightarrow \mathbb{R}$, which is skew symmetric in the first $R(X, Y; U, V) = -R(Y, X; U, V)$ and second pair $R(X, Y; U, V) = -R(X, Y; V, U)$ of its arguments $X,\, Y,\, Z,\, U,\, V \in T$ and satisfies the so called first Bianchi identity:

$$R(\, X, Y;\ Z, V \,) \;+\; R(\, Y, Z;\ X, V \,) \;+\; R(\, Z, X;\ Y, V \,) \;=\; 0 \,. \tag{5}$$

Due to the presence of the musical isomorphisms $\flat$ and $\sharp$ algebraic curvature tensors can be interpreted alternatively as trilinear products $R : T \times T \times T \longrightarrow T$, $(\, X,\, Y;\, U \,) \longmapsto R_{X,\,Y}U$ on the vector space T by setting $R_{X,\,Y}U := R(X, Y; U, \cdot\,)^\sharp$ or even as 2–forms with values in the Lie subalgebra $\mathfrak{so}(\, T,\, g \,) \subseteq \operatorname{End} T$ of skew symmetric endomorphisms of T:

$$R :\ T \times T \longrightarrow \mathfrak{so}(\, T,\, g \,), \qquad (\, X,\, Y \,) \longmapsto R_{X,\,Y} \,.$$

The latter interpretation is particularly interesting due to the vector space isomorphism

$$\Lambda^2 T \xrightarrow{\;\cong\;} \mathfrak{so}(\, T,\, g \,), \qquad X \wedge Y \longmapsto \Big(\, U \longmapsto g(\, X,\, U \,)\, Y \;-\; g(\, Y,\, U \,)\, X \,\Big)$$

characterized completely by the identity $g_{\Lambda^2 T}(\, \mathfrak{X},\, U \wedge V \,) = g(\, \mathfrak{X} U,\, V \,)$ for every bivector $\mathfrak{X} \in \Lambda^2 T$ and all $U,\, V \in T$. In accordance with this isomorphism an algebraic curvature tensor R can be interpreted as a bivector valued 2–form on T, namely the 2–form

$$R \;=\; \frac{1}{4} \sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m} R(\, E_\mu,\, E_\nu;\, E_\alpha,\, E_\beta \,)\, dE_\mu \wedge dE_\nu \,\otimes\, dE_\alpha^\sharp \wedge dE_\beta^\sharp \,, \tag{6}$$

where the sum is over an arbitrary pair of dual bases $\{\, E_\mu \,\}$ and $\{\, dE_\mu \,\}$ for the euclidean vector space T and its dual T^*. Of course the latter sum simplifies somewhat for an orthonormal basis due to the equations $E_\alpha^\flat = \pm dE_\alpha$ and $dE_\alpha^\sharp = \pm E_\alpha$ characterizing orthonormal bases in general. Last but not least we want to recall the definitions of the Ricci tensor

$$\operatorname{Ric}(\, X,\, Y \,) \;:=\; \operatorname{tr}\Big(\, U \longmapsto R_{U,\,X} Y \,\Big) \;\stackrel{!}{=}\; \sum_{\mu\,=\,1}^{m} R(\, E_\mu,\, X;\, Y,\, dE_\mu^\sharp \,) \tag{7}$$

and the scalar curvature $\kappa \in \mathbb{R}$ associated to an algebraic curvature tensor R:

$$\kappa \;:=\; \sum_{\nu\,=\,1}^{m} \operatorname{Ric}(\, E_\nu,\, dE_\nu^\sharp \,) \;=\; \sum_{\mu,\,\nu\,=\,1}^{m} R(\, E_\mu,\, E_\nu;\, dE_\nu^\sharp,\, dE_\mu^\sharp \,) \,. \tag{8}$$

Sectional curvature tensors on a euclidean vector space T are defined in complete analogy to algebraic curvature tensors as quadrilinear forms $S : T \times T \times T \times T \longrightarrow \mathbb{R}$, which are symmetric in the first and second pair $S(X,Y;U,V) = S(Y,X;U,V) = S(X,Y;V,U)$ of arguments and satisfy $S(X,X;X,V) = 0$ for all $X, V \in T$; the latter becomes

$$S(\,X,\,Y;\;Z,\,V\,) \;+\; S(\,Y,\,Z;\;X,\,V\,) \;+\; S(\,Z,\,X;\;Y,\,V\,) \;=\; 0 \tag{9}$$

upon polarization in $X, Y, Z \in T$. Algebraic curvature tensors are well–known to satisfy the symmetry in pairs $R(X,Y;U,V) = R(U,V;X,Y)$ identically in their arguments:

$$\begin{aligned}
2\,R(\,X,Y;U,V\,) &= +R(\,X,Y;U,V\,) - R(\,X,Y;V,U\,)\\
&= -R(\,Y,U;X,V\,) - R(\,U,X;Y,V\,) + R(\,Y,V;X,U\,) + R(\,V,X;Y,U\,)\\
&= +R(\,Y,U;V,X\,) + R(\,V,Y;U,X\,) - R(\,X,U;V,Y\,) - R(\,V,X;U,Y\,)\\
&= -R(\,U,V;Y,X\,) + R(\,U,V;X,Y\,)\\
&= \;2\,R(\,U,V;X,Y\,)\;.
\end{aligned}$$

Mutatis mutandis the reader may easily verify that sectional curvature tensors are symmetric in pairs $S(X,Y;U,V) = S(U,V;X,Y)$ as well. However it is more elegant to use the unpolarized first Bianchi identity $S(X,X;X,V) = 0$ directly to obtain the equality

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_0 S(\,X+tU,X+tU;\;X+tU,U\,) - \left.\frac{d}{dt}\right|_0 S(\,U+tX,U+tX;\;U+tX,X\,)\\
&= 2\,S(\,X,U;\;X,U\,) + S(\,X,X;\;U,U\,) - 2\,S(\,U,X;\;U,X\,) - S(\,U,U;\;X,X\,)\\
&= S(\,X,X;\;U,U\,) - S(\,U,U;\;X,X\,)
\end{aligned}$$

establishing the restricted symmetry $S(X,X;U,U) = S(U,U;X,X)$ for all $X, U \in T$. The general symmetry in pairs $S(X,Y;U,V) = S(U,V;X,Y)$ follows directly from its restricted version by means of the following polarization formula for sectional curvature tensors

$$\begin{aligned}
&16\,S(\,X,Y;\;U,V\,)\\
&\quad= +S(\,X+Y,X+Y;\;U+V,U+V\,) - S(\,X-Y,X-Y;\;U+V,U+V\,)\\
&\qquad - S(\,X+Y,X+Y;\;U-V,U-V\,) + S(\,X-Y,X-Y;\;U-V,U-V\,)
\end{aligned} \tag{10}$$

valid for all $X, Y, U, V \in T$, which is simply an iteration of the binomial polarization formula $4\,a(X,Y) = a(X+Y,X+Y) - a(X-Y,X-Y)$ for symmetric bilinear forms $a \in \mathrm{Sym}^2T^*$. By definition the sets of algebraic or sectional curvature tensors are natural subspaces Curv^-T or Curv^+T respectively of the vector space $\bigotimes^4 T^*$ of quadrilinear forms:

Lemma 2.1 (Equivalence of Algebraic and Sectional Curvature)
For every euclidean vector space T the vector spaces Curv^-T and Curv^+T of algebraic and sectional curvature tensors on T are naturally isomorphic via the mutually inverse isomorphisms $\Phi^+ : \mathrm{Curv}^-T \longrightarrow \mathrm{Curv}^+T$ and $\Phi^- : \mathrm{Curv}^+T \longrightarrow \mathrm{Curv}^-T$ defined by:

$$\begin{aligned}
(\,\Phi^+R\,)(\,X,\,Y;\,U,\,V\,) &:= -\,2\,\Big(\,R(\,X,\,U;\,Y,\,V\,) + R(\,X,\,V;\,Y,\,U\,)\,\Big)\\
(\,\Phi^-S\,)(\,X,\,Y;\,U,\,V\,) &:= -\,\frac{1}{6}\,\Big(\,S(\,X,\,U;\,Y,\,V\,) - S(\,X,\,V;\,Y,\,U\,)\,\Big)\;.
\end{aligned}$$

The scalar factors in the definitions of Φ^+ and Φ^- are a matter of taste up to their product being equal to $\frac{1}{3}$. With our choice the sectional curvature tensor $\mathrm{Sec} := \Phi^+ R$ associated to an algebraic curvature tensor $R \in \mathrm{Curv}^- T$ satisfies for all $X, U \in T$ the identity:

$$\frac{1}{4}\,\mathrm{Sec}(\,X,\,X;\,U,\,U\,) \;\;=\;\; R(\,X,\,U;\,U,\,X\,)\;.$$

In particular the inequalities $\mathrm{Sec}(\,X,X;\,U,U\,) \geq 0$ or $\mathrm{Sec}(\,X,X;\,U,U\,) > 0$ for all or for all linearly independent arguments $X,\,U \in T$ respectively characterize the cones of algebraic curvature tensors of non–negative or positive sectional curvature.

Proof: Leaving the straightforward verification of $\Phi^+ R \in \mathrm{Curv}^+ T$ and $\Phi^- S \in \mathrm{Curv}^- T$ for all $R \in \mathrm{Curv}^- T$ and $S \in \mathrm{Curv}^+ T$ to the reader we simply expand the definitions of the linear maps Φ^+ and Φ^- to find for example for every algebraic curvature tensor $R \in \mathrm{Curv}^- T$

$$\begin{aligned}
&(\,\Phi^-\Phi^+ R\,)(\,X,\,Y;\,U,\,V\,)\\
&\quad= \;-\frac{1}{6}\Big(\,(\,\Phi^+ R\,)(\,X,\,U;\,Y,\,V\,) \;-\; (\,\Phi^+ R\,)(\,X,\,V;\,Y,\,U\,)\,\Big)\\
&\quad= \;\;\frac{1}{3}\Big(\,R(\,X,Y;\,U,V\,) \;+\; R(\,X,V;\,U,Y\,) \;-\; R(\,X,Y;\,V,U\,) \;-\; R(\,X,U;\,V,Y\,)\,\Big)\\
&\quad= \;\;\frac{2}{3}\,R(\,X,\,Y;\,U,\,V\,) \;-\; \frac{1}{3}\Big(\,R(\,V,\,X;\,U,\,Y\,) \;+\; R(\,X,\,U;\,V,\,Y\,)\,\Big)\\
&\quad= \;\;\frac{2}{3}\,R(\,X,\,Y;\,U,\,V\,) \;+\; \frac{1}{3}\,R(\,U,\,V;\,X,\,Y\,) \;\;=\;\; R(\,X,\,Y;\,U,\,V\,)
\end{aligned}$$

using the first Bianchi identity (5) and the symmetry in pairs in the last line. The slightly simpler analogous argument establishing $\Phi^+\Phi^- S = S$ for every sectional curvature tensor $S \in \mathrm{Curv}^+ T$ is omitted to reduce redundancy. □

With Φ^+ and Φ^- being inverse isomorphisms we can write every algebraic curvature tensor in the form $R = \Phi^-\mathrm{Sec}$ for its associated sectional curvature tensor $\mathrm{Sec} := \Phi^+ R$ and so

$$24\,R(\,X,Y;\,U,V\,) \;\;=\;\; 4\,\mathrm{Sec}(\,X,V;\,Y,U\,) \;-\; 4\,\mathrm{Sec}(\,X,U;\,Y,V\,)$$

for all arguments $X,\,Y,\,U,\,V \in T$. In particular the polarization formula (10) for sectional curvature tensors entails a polarization formula for algebraic curvature tensors $R \in \mathrm{Curv}^- T$

$$\begin{aligned}
24R(X,Y;U,V) \;=\; &+R(X+V,Y+U;Y+U,X+V) - R(X+U,Y+V;Y+V,X+U)\\
&-R(X+V,Y-U;Y-U,X+V) + R(X+U,Y-V;Y-V,X+U)\\
&-R(X-V,Y+U;Y+U,X-V) + R(X-U,Y+V;Y+V,X-U)\\
&+R(X-V,Y-U;Y-U,X-V) - R(X-U,Y-V;Y-V,X-U)
\end{aligned}$$

for all $X,\,Y,\,U,\,V$. Likewise the definitions (7) and (8) of the Ricci tensor $\mathrm{Ric} \in \mathrm{Sym}^2 T^*$ and the scalar curvature $\kappa \in \mathbb{R}$ associated to an algebraic curvature tensor R can be rewritten

$$\mathrm{Ric}(\,X,\,Y\,) \;\;=\;\; \frac{1}{4}\sum_{\mu=1}^{m}\mathrm{Sec}(\,E_\mu, dE^\sharp_\mu;\,X,Y\,) \qquad\quad \kappa \;\;=\;\; \frac{1}{4}\sum_{\mu,\,\nu=1}^{m}\mathrm{Sec}(\,E_\mu, dE^\sharp_\mu;\,E_\nu, dE^\sharp_\nu\,)$$

in terms of the sectional curvature tensor $\mathrm{Sec} = \Phi^+ R$. In order to discuss a more important consequence of Lemma 2.1 we observe that the vector space $\mathrm{Sym}^2 T^* \otimes \mathrm{Sym}^2 T^*$ of quadrilinear forms symmetric in their first and second pair of arguments is spanned by the tensor products

$$(a \otimes b)(X, Y; U, V) \;\; := \;\; a(X, Y)\, b(U, V)$$

of symmetric bilinear forms $a, b \in \mathrm{Sym}^2 T^*$. In addition we can define the projection

$$\mathrm{pr} : \;\; \mathrm{Sym}^2 T^* \otimes \mathrm{Sym}^2 T^* \longrightarrow \mathrm{Curv}^+ T, \qquad S \longmapsto \mathrm{pr}\, S \; ,$$

from the vector space $\mathrm{Sym}^2 T^* \otimes \mathrm{Sym}^2 T^*$ to the vector space $\mathrm{Curv}^+ T$ by setting

$$\begin{aligned} 6\, (\mathrm{pr}\, S)(X, Y; U, V) \;\; := \;\; & + 2\, S(X, Y; U, V) + 2\, S(U, V; X, Y) \\ & - \;\; S(X, U; Y, V) - \;\; S(X, V; Y, U) \\ & - \;\; S(Y, U; X, V) - \;\; S(Y, V; X, U) \end{aligned} \tag{11}$$

for all arguments $X, Y, U, V \in T$; note that $\mathrm{pr}\, S = S$ for every sectional curvature tensor $S \in \mathrm{Curv}^+ T$ due to the first Bianchi identity (9), the symmetry in pairs and the compensation factor 6. On the other hand the image of every quadrilinear form $S \in \mathrm{Sym}^2 T^* \otimes \mathrm{Sym}^2 T^*$ symmetric in its first and second pair of arguments is a well–defined sectional curvature tensor $\mathrm{pr}\, S \in \mathrm{Curv}^+ T$, because the right hand of equation (11) is evidently symmetric under $X \leftrightarrow Y$ and $U \leftrightarrow V$ while vanishing for $X = Y = U$. With pr being a projection and thus surjective we conclude that the vector space $\mathrm{Curv}^- T$ of algebraic curvature tensors on a euclidean vector space T is spanned by the Nomizu–Kulkarni products

$$\begin{aligned} & (a \times b)(X, Y; U, V) \\ & \quad := \;\; a(X,U)\, b(Y,V) - a(X,V)\, b(Y,U) - a(Y,U)\, b(X,V) + a(Y,V)\, b(X,U) \end{aligned} \tag{12}$$

of symmetric bilinear forms $a, b \in \mathrm{Sym}^2 T^*$, because we find by expanding definition (11)

$$\begin{aligned} - 36\, (\Phi^- \circ \mathrm{pr})(a \otimes b)(X, Y; U, V) \\ = \;\; & + 6\, \mathrm{pr}(a \otimes b)(X, U; Y, V) - 6\, \mathrm{pr}(a \otimes b)(X, V; Y, U) \\ = \;\; & + 2\, a(X,U)\, b(Y,V) + 2\, a(Y,V)\, b(X,U) - 2\, a(X,V)\, b(Y,U) - 2\, a(Y,U)\, b(X,V) \\ & - \;\; a(X,Y)\, b(U,V) - \;\; a(X,V)\, b(U,Y) + \;\; a(X,Y)\, b(V,U) + \;\; a(X,U)\, b(V,Y) \\ & - \;\; a(U,Y)\, b(X,V) - \;\; a(U,V)\, b(X,Y) + \;\; a(V,Y)\, b(X,U) + \;\; a(V,U)\, b(X,Y) \\ = \;\; & + 3\, (a \times b)(X, Y; U, V) \end{aligned}$$

for all $X, Y, U, V \in T$. Put differently the Nomizu–Kulkarni product $a \times b \in \mathrm{Curv}^- T$ of two symmetric bilinear forms $a, b \in \mathrm{Sym}^2 T^*$ equals the algebraic curvature tensor corresponding to the sectional curvature tensor $\mathrm{pr}(a \otimes b) \in \mathrm{Curv}^+ T$ up to the scalar factor -12:

$$\Phi^+(a \times b) \;\; = \;\; - 12\, \mathrm{pr}(a \otimes b) \; . \tag{13}$$

Taking a closer look at definition (12) we see that the Nomizu–Kulkarni product is commutative $a \times b = b \times a$, hence we may polarize $a \times b = \frac{1}{4}(a+b) \times (a+b) - \frac{1}{4}(a-b) \times (a-b)$ to argue that the vector space $\mathrm{Curv}^- T$ is actually spanned by Nomizu–Kulkarni squares:

$$\mathrm{Curv}^- T \;\; = \;\; \mathrm{span}_{\mathbb{R}} \{ \; a \times a \; \mid \; a \in \mathrm{Sym}^2 T^* \text{ symmetric bilinear form } \} \;\; \subseteq \;\; \bigotimes{}^4 T^* \; . \tag{14}$$

Somewhat better every algebraic curvature tensor $R \in \mathrm{Curv}^- T$ can be written as a sum of Nomizu–Kulkarni squares of symmetric bilinear forms $a_1, \ldots, a_r \in \mathrm{Sym}^2 T^*$ of rank two:

$$R \;=\; a_1 \times a_1 \;+\; a_2 \times a_2 \;+\; \ldots \;+\; a_r \times a_r \;. \tag{15}$$

The argument relies on the following weak form of Sylvester's Theorem of Inertia: For every symmetric bilinear form $a \in \mathrm{Sym}^2 T^*$ on a finite dimensional vector space T over $\mathbb{R}$ there exists a unique tuple $(\,p,\,n\,) \in \mathbb{N}_0^2$ called the signature of a such that a can be written as a sum of signed symmetric squares of linearly independent forms $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_n \in T^*$

$$a \;=\; (\,\tfrac{1}{2}\alpha_1^2 \;+\; \ldots \;+\; \tfrac{1}{2}\alpha_p^2\,) \;-\; (\,\tfrac{1}{2}\beta_1^2 \;+\; \ldots \;+\; \tfrac{1}{2}\beta_n^2\,)\;;$$

by convention the square $\frac{1}{2}\alpha^2 \in \mathrm{Sym}^2 T^*$ of a linear form $\alpha \in T^*$ denotes the symmetric bilinear form $\alpha \otimes \alpha : T \times T \longrightarrow \mathbb{R}$, $(X, Y) \longmapsto \alpha(X)\alpha(Y)$. In particular a symmetric bilinear form $a \in \mathrm{Sym}^2 T^*$ is non–degenerate and hence defines a scalar product on T, if and only if its rank $p + n = m$ equals the dimension of T so that the $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_n$ become a basis of T^*, namely an orthonormal basis with respect to the dual scalar product a^{-1}.

According to equation (14) every algebraic curvature tensor $R \in \mathrm{Curv}^- T$ can be expanded into a finite sum of scaled Nomizu–Kulkarni squares $\lambda\, a \times a$ with $a \in \mathrm{Sym}^2 T^*$ and a scalar $\lambda \in \mathbb{R} \setminus \{0\}$. Replacing $a \rightsquigarrow \sqrt{|\lambda|}a$, $\lambda \rightsquigarrow \frac{\lambda}{|\lambda|}$ if necessary we may assume without loss of generality that all scalars occurring in this expansion of R equal $\lambda = \pm 1$ so that:

$$R \;=\; \pm\, a_1 \,\times\, a_1 \;\pm\; a_2 \,\times\, a_2 \;\pm\; \ldots \;\pm\; a_r \,\times\, a_r \;.$$

Sylvester's Theorem of Inertia allows us to expand each of the symmetric bilinear forms $a_1, \ldots, a_r \in \mathrm{Sym}^2 T^*$ further into a finite sum of signed symmetric squares. Using the distributivity law entailed by the bilinearity of the Nomizu–Kulkarni product $\times$ we thus arrive at an expansion of the algebraic curvature tensor R into a finite sum of terms

$$\pm\; \tfrac{1}{2}\alpha^2 \;\times\; \tfrac{1}{2}\beta^2$$

with linear forms $\alpha, \beta \in T^*$. Recalling the conventional equality $\frac{1}{2}\alpha^2 = \alpha \otimes \alpha$ we calculate

$$\begin{aligned}
(\,\tfrac{1}{2}\alpha^2 \times \tfrac{1}{2}\beta^2\,)(\,X, Y;\, U, V\,) \;&=\; +\,\alpha(X)\,\alpha(U)\,\beta(Y)\,\beta(V) \;-\; \alpha(X)\,\alpha(V)\,\beta(Y)\,\beta(U) \\
&\qquad -\,\alpha(Y)\,\alpha(U)\,\beta(X)\,\beta(V) \;+\; \alpha(Y)\,\alpha(V)\,\beta(X)\,\beta(U) \\
&=\; \Big(\alpha(X)\beta(Y) \;-\; \alpha(Y)\beta(X)\Big)\Big(\alpha(U)\beta(V) \;-\; \alpha(V)\beta(U)\Big) \\
&=\; (\,\alpha \wedge \beta \,\otimes\, \alpha \wedge \beta\,)(\,X,\,Y;\,U,\,V\,)
\end{aligned}$$

and conclude that $\frac{1}{2}\alpha^2 \times \frac{1}{2}\beta^2 = 0$ for linearly dependent forms $\alpha, \beta \in T^*$. In consequence

$$\pm\; \tfrac{1}{2}\alpha^2 \times \tfrac{1}{2}\beta^2 \;\;=\;\; \tfrac{1}{8}\,(\,\alpha^2 \;\pm\; \beta^2\,) \;\times\; (\,\alpha^2 \;\pm\; \beta^2\,)$$

due to the commutativity of the Nomizu–Kulkarni product and $\alpha^2 \times \alpha^2 = 0 = \beta^2 \times \beta^2$. Needless to say the symmetric bilinear forms $\frac{1}{\sqrt{8}}(\alpha^2 + \beta^2)$ and $\frac{1}{\sqrt{8}}(\alpha^2 - \beta^2)$ have rank 2 and signature $(2, 0)$ and $(1, 1)$ respectively unless α and β are linearly dependent forms.

Corollary 2.2 (Description of Algebraic Curvature Tensors)
The vector space $\mathrm{Curv}^- T$ *of algebraic curvature tensors on a finite dimensional euclidean vector space* T *over* $\mathbb{R}$ *is spanned by the curvature tensors* $\frac{1}{2}\alpha^2 \times \frac{1}{2}\beta^2 = \alpha \wedge \beta \otimes \alpha \wedge \beta$:

$$\mathrm{Curv}^- T \;\;=\;\; \mathrm{span}_{\mathbb{R}}\{\;\alpha \,\wedge\, \beta \,\otimes\, \alpha \,\wedge\, \beta \;\mid\; \alpha, \beta \,\in\, T^* \textit{ linear forms }\;\} \;\;\subseteq\;\; \bigotimes{}^4 T^* \;.$$

3 Graph Algebras

Graphs are used in many different areas of mathematics as a means to encode information in a form easily accessible for humans. In mathematical physics for examples graphs or Feynman diagrams are used to encode specific analytic integrals by assigning Feynman rules to the different types of edges and vertices comprising a graph. In this section we will use trivalent graphs with edges colored red and black in a similar way to encode the contraction scheme corresponding to a stable curvature invariant. In due course we will construct the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs and its quotient $\overline{\mathbb{A}}^\bullet$ by the IHX–relations on red edges, which arose originally in knot theory, to obtain a graphical calculus for curvature polynomials.

Recall first of all that a labelling of a finite set S is a bijection $L : S \xrightarrow{\cong} \{1, \ldots, n\}$ with the set of the first $n = \#S$ natural numbers. In turn an orientation of S is an equivalence class $o = [\, L, \varepsilon\,]$ of a labelling L of S and a sign $\varepsilon \in \{\, +1, -1\, \}$ under the equivalence relation:

$$(L, \varepsilon\,) \;\sim\; (\hat{L}, \hat{\varepsilon}\,) \qquad \Longleftrightarrow \qquad \mathrm{sgn}(L \circ \hat{L}^{-1}) \;=\; \varepsilon\, \hat{\varepsilon}\ .$$

Because all one point sets as well as the empty set $\emptyset$ have unique labellings L_{can}, these sets have two distinguished orientations $o_\pm := [\, L_{\mathrm{can}}, \pm 1\,]$. In the same vein every labelling L of a finite set S represents the orientation $o_L := [\, L, +1\,]$. Finite sets with at least two elements have odd permutations and hence lack distinguished orientations, nevertheless we can still talk about the orientation $-o := [\, L, -\varepsilon\,]$ opposite to $+o = [\, L, +\varepsilon\,]$.

It is convenient for our purposes to define a finite graph as a quadruple $\gamma := (\, V, F; \theta, \mathbf{at}\,)$ consisting of finite sets V and F called the sets of vertices and flags of γ respectively, a fix point free involution $\theta : F \longrightarrow F$ called the flag involution and a map $\mathbf{at} : F \longrightarrow V$ called the attaching map. By assumption θ is fix point free with $f \neq \theta(f)$ for all $f \in F$, hence all orbits of θ in the set of flags have exactly two elements, namely the edges of the graph γ:

$$\mathrm{Edge}\,\gamma \;:=\; F/_{\langle\, \theta\, \rangle} \;=\; \{\ \{\, f, \theta(f)\, \} \ \mid\ f \in F\ \}\ .$$

The analogous back references $\mathrm{Vert}\,\gamma := V$ and $\mathrm{Flag}\,\gamma := F$ reduce the need to specify the vertex and flag sets of a finite graph γ by name, nevertheless θ and $\mathbf{at}$ will always refer to the flag involution and the attaching map of the finite graph in question. By definition every flag $f \in F$ is adjacent to the vertex $\mathbf{at}(f) \in V$, similarly an edge $e = \{f_1, f_2\}$ is adjacent to the not necessarily different vertices $\mathbf{at}(f_1)$ and $\mathbf{at}(f_2)$. The cardinality of the set $\mathrm{Flag}_v\gamma := \mathbf{at}^{-1}(\, v\,)$ of flags adjacent to a vertex $v \in \mathrm{Vert}\,\gamma$ is called its valence $\#\,\mathrm{Flag}_v\gamma$, in a bivalent and trivalent graphs respectively all vertices are required to have the same valence 2 or 3. According to the definitions above finite graphs may have multiple edges between vertices and loops, i.e. edges from a vertex to itself.

In order to talk about isomorphic graphs we consider finite graphs as the objects in a suitable category of graphs. Morphisms $\varphi : \gamma \longrightarrow \hat{\gamma}$ in this category are pairs of maps between the sets of vertices $\varphi_{\mathrm{Vert}} : \mathrm{Vert}\,\gamma \longrightarrow \mathrm{Vert}\,\hat{\gamma}$ and flags $\varphi_{\mathrm{Flag}} : \mathrm{Flag}\,\gamma \longrightarrow \mathrm{Flag}\,\hat{\gamma}$ respectively, which intertwine the flag involutions $\hat{\theta} \circ \varphi_{\mathrm{Flag}} = \varphi_{\mathrm{Flag}} \circ \theta$ and the attaching maps $\widehat{\mathbf{at}} \circ \varphi_{\mathrm{Flag}} = \varphi_{\mathrm{Vert}} \circ \mathbf{at}$. In consequence the automorphism group of a finite graph γ

comes along with a group homomorphism $\mathrm{Aut}\,\gamma \longrightarrow S_{\mathrm{Vert}\,\gamma}$, $\varphi \longmapsto \varphi_{\mathrm{Vert}}$, whose image and kernel are the groups of pure and trivial automorphisms of γ respectively:

$$\begin{array}{rcl} \overline{\mathrm{Aut}}\,\gamma & := & \{\ \varphi_{\mathrm{Vert}}\ \mid\ \varphi \text{ automorphism of } \gamma\ \} \\ \mathrm{Aut}_\circ\gamma & := & \{\ \varphi_{\mathrm{Flag}}\ \mid\ \varphi \text{ automorphism of } \gamma \text{ with } \varphi_{\mathrm{Vert}} = \mathrm{id}_{\mathrm{Vert}\,\gamma}\ \}\ . \end{array} \tag{16}$$

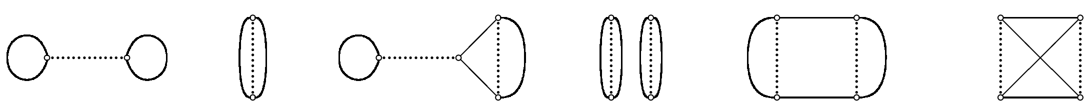
Remarkably the induced short exact sequence $\mathrm{Aut}_\circ\gamma \xrightarrow{\subset} \mathrm{Aut}\,\gamma \longrightarrow \overline{\mathrm{Aut}}\,\gamma$ splits on the right. The underlying argument is rather important for our calculations, because it ensures that the group $\overline{\mathrm{Aut}}\,\gamma$ of pure automorphisms is effectively computable as the subgroup of $S_{\mathrm{Vert}\,\gamma}$ preserving the adjacency numbers of the graph γ for all pairs $v,\,w \in \mathrm{Vert}\,\gamma$ of vertices

$$\overline{\mathrm{Aut}}\,\gamma\ =\ \{\ \sigma \in S_{\mathrm{Vert}\,\gamma}\ \mid\ \#\,\mathrm{Flag}_{\sigma(v),\,\sigma(w)}\gamma\ =\ \#\,\mathrm{Flag}_{v,\,w}\gamma \text{ for all } v,\,w \in \mathrm{Vert}\,\gamma\ \}\ ,$$

where $\mathrm{Flag}_{v,\,w}\gamma := \mathrm{Flag}_v\gamma \cap \theta(\,\mathrm{Flag}_w\gamma\,)$ is the set of flags adjacent to v on edges to w.

Definition 3.1 (Colored Trivalent Graphs)
A colored trivalent graph is a trivalent graph γ endowed with a coloring of its edges by colors red and black or equivalently by a θ–invariant coloring $\mathbf{c} : \mathrm{Flag}\,\gamma \longrightarrow \{\,\mathrm{red},\,\mathrm{black}\,\}$ of its flags in the sense $\mathbf{c} \circ \theta = \mathbf{c}$ such that every vertex of γ is adjacent to exactly one red flag.

The attaching map $\mathbf{at}$ of a colored trivalent graph γ restricts by definition to a bijection $\mathbf{at}_{\mathrm{red}} : \mathbf{c}^{-1}(\,\mathrm{red}\,) \longrightarrow \mathrm{Vert}\,\gamma$ between the set of red flags of γ and its set of vertices. In turn the flag involution θ induces a fix point free involution $\theta_{\mathrm{red}} := \mathbf{at}_{\mathrm{red}} \circ \theta \circ \mathbf{at}_{\mathrm{red}}^{-1}$ on the set of vertices of the graph γ. Evidently the red edges in a colored trivalent graph are completely determined by the fix point free involution θ_{red}, hence we may think of a colored trivalent graph γ as the bivalent graph γ_{black} we obtain by removing all red edges from γ endowed with the fix point free involution θ_{red} on its set of vertices. In diagrams we will depict the red and black edges of a colored graph by dotted and solid lines respectively, the six graphs



(17)

say are among the simplest colored trivalent graphs. Homomorphisms and isomorphisms of colored trivalent graphs $\varphi : \gamma \longrightarrow \hat{\gamma}$ are required to preserve the flag coloring $\mathbf{c} = \hat{\mathbf{c}} \circ \varphi_{\mathrm{Flag}}$ of course. With this definition in place we can consider the sets of isomorphism classes

$$\Gamma^n\ :=\ \{\ [\,\gamma\,]\ \mid\ \gamma \text{ colored trivalent graph with } 2n \text{ vertices }\ \}$$

of colored trivalent graphs with $2n$, $n \in \mathbb{N}_0$, vertices and their union $\Gamma^\bullet := \bigcup_{n \in \mathbb{N}_0} \Gamma^n$, which is a graded commutative monoid under the multiplication induced by the disjoint union $\dot{\cup}$

$$\mathrm{Vert}(\,\gamma\,\dot{\cup}\,\hat{\gamma}\,)\ :=\ \mathrm{Vert}\,\gamma\ \dot{\cup}\ \mathrm{Vert}\,\hat{\gamma} \qquad\qquad \mathrm{Flag}(\,\gamma\,\dot{\cup}\,\hat{\gamma}\,)\ :=\ \mathrm{Flag}\,\gamma\ \dot{\cup}\ \mathrm{Flag}\,\hat{\gamma}$$

of colored graphs with the obvious definitions for the flag involution θ, the attaching map $\mathbf{at}$ and the coloring $\mathbf{c}$. On the level of graphs the disjoint union may or may not be commutative and associative depending on the model of set theory in use, this minor nuisance however disappears on the level of isomorphism classes of colored trivalent graphs. Up to isomorphism for example the fourth graph in diagram (17) equals the disjoint union $\gamma \,\dot{\cup}\, \gamma$ of the second graph γ with itself. The unit element of the monoid $\Gamma^\bullet$ is represented by the unique empty colored trivalent graph $\gamma_{\mathrm{empty}} := (\,\emptyset,\,\emptyset;\,\theta,\,\mathbf{at},\,\mathbf{c}\,)$ without vertices or flags at all.

Definition 3.2 (Algebra of Colored Trivalent Graphs)
The algebra of colored trivalent graphs is the graded convolution algebra of the monoid $\Gamma^\bullet$ *over* $\mathbb{R}$. *Put differently the algebra of colored trivalent graphs is the free graded vector space*

$$\mathbb{A}^\bullet \;:=\; \mathbb{R}\,\Gamma^\bullet$$

over $\mathbb{R}$ *generated by the graded set* $\Gamma^\bullet = \bigcup \Gamma^n$ *with multiplication given by the* $\mathbb{R}$*–bilinear extension of the disjoint union multiplication of the commutative monoid* $\Gamma^\bullet$. *In passing we remark that the grading on* $\mathbb{A}^\bullet$ *is completely determined by the number operator derivation:*

$$N:\;\;\mathbb{A}^\bullet\;\longrightarrow\;\mathbb{A}^\bullet,\qquad [\,\gamma\,]\;\longmapsto\;\frac{\#\,\mathrm{Vert}\,\gamma}{2}\,[\,\gamma\,]\;.$$

From an algebraic point of view the algebra of colored trivalent graphs is not particularly interesting, because the underlying commutative monoid $\Gamma^\bullet$ is evidently the free commutative monoid generated by the subset $\Gamma^\bullet_{\mathrm{conn}} \subseteq \Gamma^\bullet$ of isomorphism classes of connected colored trivalent graphs. In consequence the inclusion $\mathbb{R}\Gamma^\bullet_{\mathrm{conn}} \subseteq \mathbb{A}^\bullet$ induces an algebra isomorphism

$$\mathrm{Sym}^\bullet(\,\mathbb{R}\,\Gamma^\bullet_{\mathrm{conn}}\,)\;\stackrel{\cong}{\longrightarrow}\;\mathbb{A}^\bullet$$

exhibiting $\mathbb{A}^\bullet$ as the free polynomial algebra generated by the graded vector space $\mathbb{R}\,\Gamma^\bullet_{\mathrm{conn}}$. From the differential geometric point of view it is much more interesting that every isomorphism class $[\,\gamma\,]$ of colored trivalent graphs with $2n$ vertices defines a homogeneous polynomial

$$[\gamma]:\;\;\mathrm{Curv}^-T\;\longrightarrow\;\mathbb{R},\qquad R\;\longmapsto\;[\gamma](\,R\,)\;,$$

of degree n on the space of algebraic curvature tensors on an arbitrary euclidean vector space:

Definition 3.3 (Stable Curvature Invariants)
Every isomorphism class of colored trivalent graphs γ *with* $2n$ *vertices gives rise to a homogeneous polynomial* $[\,\gamma\,]:\,\mathrm{Curv}^-T \longrightarrow \mathbb{R}$ *of degree* $n \in \mathbb{N}_0$ *for algebraic curvature tensors over an arbitrary euclidean vector space* T. *More precisely* γ *evaluates on an algebraic curvature tensor* $R \in \mathrm{Curv}^-T$ *to an iterated sum over an orthonormal basis* $E_1,\,\ldots,\,E_m$ *of* T

$$[\,\gamma\,](\,R\,)\;\;:=\sum_{\mu:\,\mathrm{Edge}\,\gamma_{\mathrm{black}}\longrightarrow\{1,\ldots,m\}}\;\;\prod_{\substack{\{\,v_+,\,v_-\,\}\subseteq\mathrm{Vert}\,\gamma\\ \theta_{\mathrm{red}}(v_+)\,=\,v_-}}\mathrm{Sec}(\,E_{\overline{\mu}(f_+^1)},\;E_{\overline{\mu}(f_+^2)};\;E_{\overline{\mu}(f_-^1)},\;E_{\overline{\mu}(f_-^2)}\,)\;,$$

where $\mathrm{Sec} := \Phi^+ R$ *denotes the corresponding sectional curvature tensor. Sum and product extend over all maps* $\mu:\,\mathrm{Edge}\,\gamma_{\mathrm{black}} \longrightarrow \{1,\ldots,m\}$ *and all orbits* $\{v_+, v_-\} \subseteq \mathrm{Vert}\,\gamma$ *of the fixed point free involution* θ_{red} *of* $\mathrm{Vert}\,\gamma$. *The flags* $f_+^1,\, f_+^2$ *and* $f_-^1,\, f_-^2$ *in this formula denote the pairs of black flags adjacent to* v_+ *and* v_-, *while* $\overline{\mu}:\,\mathrm{Flag}\,\gamma_{\mathrm{black}} \longrightarrow \{1,\ldots,m\}$ *refers to the composition of* μ *with the canonical projection* $\mathrm{Flag}\,\gamma_{\mathrm{black}} \longrightarrow \mathrm{Edge}\,\gamma_{\mathrm{black}}$.

In order to simplify the definition of the polynomial $[\,\gamma\,]$ we have omitted the usual sign factors $\pm$ appearing in iterated sums over orthonormal bases in cases the scalar product g of the euclidean vector space T is not positive definite. The polynomial $[\,\gamma\,]$ is of course well defined independent of the choices made in labelling the two vertices in an orbit of θ_{red} and in labelling the two black flags adjacent to each vertex. Extending the preceding construction linearly we may associate a polynomial on Curv^-T to every element of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs, more precisely we obtain a homomorphism of graded algebras

$$\mathrm{Inv}_{(\,T,\,g\,)}:\ \ \mathbb{A}^\bullet\ \longrightarrow\ \mathrm{Sym}^\bullet(\,\mathrm{Curv}^-T\,)^*,\qquad [\,\gamma\,]\ \longmapsto\ \Big(\ R\ \longmapsto\ [\,\gamma\,](\,R\,)\ \Big)\ ,$$

because both the sum and the product split over a disjoint union of graphs to provide for:

$$[\,\gamma\,\dot{\cup}\,\hat{\gamma}\,](\,R\,)\ \ =\ \ [\,\gamma\,](\,R\,)\ \cdot\ [\,\hat{\gamma}\,](\,R\,)\ .$$

A discussion of the dependence of the algebra homomorphism $\mathrm{Inv}_{(\,T,\,g\,)}$ on the euclidean vector space T necessarily involves the category **CM** of curvature models. Objects in this category are triples $(\,T,\,g,\,R\,)$ describing algebraic curvature tensors R on euclidean vector spaces $(\,T,\,g\,)$, morphisms are the adjoints $F^*:\,T\longrightarrow\hat{T}$ of isometric maps $F:\,\hat{T}\longrightarrow T$ between the euclidean vector spaces T and $\hat{T}$ satisfying either of the two equivalent constraints

$$\begin{array}{rcl}\widehat{R}(\,F^*X,\,F^*Y;\,F^*U,\,F^*V\,) & = & R(\,X,\,Y;\,U,\,V\,)\\ \widehat{\mathrm{Sec}}(\,F^*X,\,F^*Y;\,F^*U,\,F^*V\,) & = & \mathrm{Sec}(\,X,\,Y;\,U,\,V\,)\end{array}\tag{18}$$

for all $X,\,Y,\,U,\,V\,\in\,T$. For every such morphism $(\,T,\,g,\,R\,)\longrightarrow(\,\hat{T},\,\hat{g},\,\hat{R}\,)$ we have equality

$$[\,\gamma\,](\,R\,)\ \ =\ \ [\,\gamma\,](\,\hat{R}\,)\ ,\tag{19}$$

because the adjoint $F^*\,:\,T\,\longrightarrow\,\hat{T}$ of an isometric map $F\,:\,\hat{T}\,\longrightarrow\,T$ factors into the orthogonal projection to the regular subspace $\mathrm{im}\,F\,\subseteq\,T$ and the isometry $F^{-1}:\,\mathrm{im}\,F\longrightarrow\hat{T}$. In consequence we can choose an orthonormal basis $E_1,\,\ldots,\,E_m$ for the euclidean vector space T in such a way that $F^*E_1,\,\ldots,\,F^*E_{\hat{m}}$ is an orthonormal basis for $\hat{T}$, while $F^*E_\mu\,=\,0$ for all $\mu\,>\,\hat{m}$. Due to the constraint (18) imposed on morphisms in the category **CM** we find

$$\widehat{\mathrm{Sec}}(\,F^*E_{\overline{\mu}(f^1_+)},\ F^*E_{\overline{\mu}(f^2_+)};\ F^*E_{\overline{\mu}(f^1_-)},\ F^*E_{\overline{\mu}(f^2_-)}\,)\ \ =\ \ \mathrm{Sec}(\,E_{\overline{\mu}(f^1_+)},\ E_{\overline{\mu}(f^2_+)};\ E_{\overline{\mu}(f^1_-)},\ E_{\overline{\mu}(f^2_-)}\,)$$

for this particular choice of orthonormal bases and all $\mu:\,\mathrm{Edge}\,\gamma_{\mathrm{black}}\longrightarrow\{1,\ldots,\hat{m}\}$, while

$$\prod_{\substack{\{\,v_+,\,v_-\,\}\,\subseteq\,\mathrm{Vert}\,\gamma\\ \theta_{\mathrm{red}}(v_+)\,=\,v_-}}\mathrm{Sec}(\,E_{\overline{\mu}(f^1_+)},\ E_{\overline{\mu}(f^2_+)};\ E_{\overline{\mu}(f^1_-)},\ E_{\overline{\mu}(f^2_-)}\,)\ \ =\ \ 0$$

for all $\mu:\,\mathrm{Edge}\,\gamma_{\mathrm{black}}\longrightarrow\{1,\ldots,m\}$ with maximum larger than $\hat{m}$. A particular case of the invariance (19) of the polynomial $[\,\gamma\,]$ under morphisms in the category **CM** occurs for the adjoints $F^*\,=\,F^{-1}$ of self isometries $F\,:\,T\,\longrightarrow\,T$ of a given euclidean vector space T. In this case the invariance (19) reads $[\,\gamma\,](\,F\star\hat{R}\,)\,=\,[\,\gamma\,](\,\hat{R}\,)$ for all algebraic curvature tensors

$\hat{R} \in \mathrm{Curv}^{-}T$ in terms of the natural representation $\star$ of the orthogonal group $\mathbf{O}(T,\, g)$ of isometries of T on $\mathrm{Curv}^{-}T$; in other words the homomorphism of graded algebras

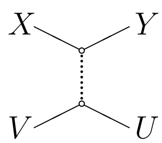
$$\mathrm{Inv}_{(T,g)} : \quad \mathbb{A}^{\bullet} \longrightarrow [\, \mathrm{Sym}^{\bullet}(\,\mathrm{Curv}^{-}T\,)^{*}\,]^{\mathbf{O}(T,g)}$$

take values in the subalgebra of $\mathbf{O}(T,\, g)$–invariant polynomials on $\mathrm{Curv}^{-}T$. A pretty similar argument to the one used to prove the invariance (19) implies that the polynomial associated to a linear combination $[\,\gamma\,] \in \mathbb{R}\Gamma^{\bullet}_{\mathrm{conn}}$ of connected colored trivalent graphs is additive

$$[\,\gamma\,](\, R \oplus \hat{R}\,) \;=\; [\,\gamma\,](\, R\,) \;+\; [\,\gamma\,](\,\hat{R}\,)$$

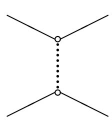
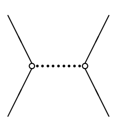
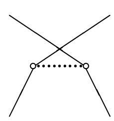
under Cartesian products of curvature models. For general elements of the algebra $\mathbb{A}^{\bullet}$ of colored trivalent graphs we can thus use the comultiplication $\Delta : \mathbb{A}^{\bullet} \longrightarrow \mathbb{A}^{\bullet} \otimes \mathbb{A}^{\bullet}$ induced by the algebra isomorphism $\mathbb{A}^{\bullet} \cong \mathrm{Sym}^{\bullet}(\,\mathbb{R}\Gamma^{\bullet}_{\mathrm{conn}}\,)$ to find $[\,\gamma\,](\,R \oplus \hat{R}\,) = [\,\Delta\gamma\,](\,R,\, \hat{R}\,)$.

Perhaps the best way to think about Definition 3.3 is as a set of Feynman rules for some unspecified field theory, which allow us to evaluate the isomorphism class $[\,\gamma\,]$ of a colored trivalent graph γ on the sectional curvature tensor $\mathrm{Sec} := \Phi^{+}R$ associated to an algebraic curvature tensor $R \in \mathrm{Curv}^{-}T$. These Feynman rules stipulate a summation over an orthonormal basis for every black edge and the multiplication of the red edge interactions:

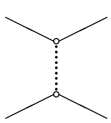
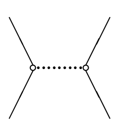
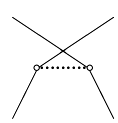


$$\widehat{=} \;\; \mathrm{Sec}(\, X,\; Y;\; U,\; V\,)\;. \tag{20}$$

In light of this Feynman rule description of Definition 3.3 we readily observe that the algebra homomorphism $\mathrm{Inv}_{(T,g)}$ can not be injective, because the Feynman interpretation of

 +  +  (21)

is exactly the left hand side of the first Bianchi identity (9) for sectional curvature tensors. Whenever colored trivalent graphs γ_I, γ_H and γ_X are isomorphic except for a pair of vertices connected by a red edge and four black flags attaching to these two vertices according to

γ_I :  γ_H :  γ_X :  , (22)

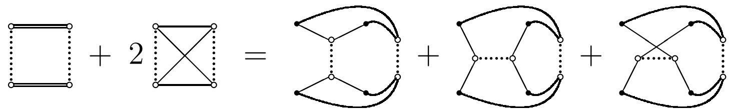
then $\mathrm{Inv}_{(T,g)}([\,\gamma_I\,] + [\,\gamma_H\,] + [\,\gamma_X\,]) = 0$. The subspace spanned by these IHX–relations

$$\langle\,\mathrm{IHX}\,\rangle \;:=\; \mathrm{span}_{\mathbb{R}}\{\; [\,\gamma_I\,] + [\,\gamma_H\,] + [\,\gamma_X\,] \;\mid\; \gamma_I,\, \gamma_H,\, \gamma_X \text{ isomorphic except for (22) } \} \tag{23}$$

is by construction a homogeneous ideal in the graph algebra $\mathbb{A}^{\bullet}$, because an IHX–relation for some red edge remains an IHX–relation at the same red edge after taking the disjoint union product with another graph. In consequence the algebra homomorphism $\mathrm{Inv}_{(T,g)}$ factorizes through the canonical projection to the reduced algebra of colored trivalent graphs

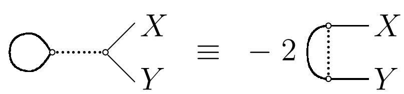
$$\overline{\mathbb{A}}^{\bullet} \;:=\; \mathbb{A}^{\bullet}/_{\langle\, \mathrm{IHX}\,\rangle} \tag{24}$$

and an algebra homomorphism $\overline{\mathrm{Inv}}_{(T,g)} : \overline{\mathbb{A}}^{\bullet} \longrightarrow [\,\mathrm{Sym}^{\bullet}(\,\mathrm{Curv}^{-}T\,)^{*}\,]^{\mathbf{O}(T,g)}$. One of the simplest examples of a congruence modulo the ideal $\langle\,\mathrm{IHX}\,\rangle$ is given by the IHX–relation:



$$\equiv \;\; 0\;. \tag{25}$$

Another useful family of congruences modulo $\langle\,\mathrm{IHX}\,\rangle$ is described by the Feynman rules:

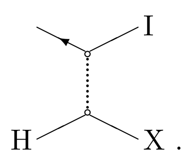


$$\widehat{=}\;\; 4\,\mathrm{Ric}(\,X,\,Y\,)\;. \tag{26}$$

Concerning the structure of the reduced graph algebra we remark

$$\mathbb{A}^{\bullet} \;\cong\; \mathrm{Sym}^{\bullet}(\,\mathbb{R}\Gamma^{\bullet}_{\mathrm{conn}}\,) \qquad \Longrightarrow \qquad \overline{\mathbb{A}}^{\bullet} \;\cong\; \mathrm{Sym}^{\bullet}\Big(\,\mathbb{R}\Gamma^{\bullet}_{\mathrm{conn}}/_{\mathbb{R}\Gamma^{\bullet}_{\mathrm{conn}}\,\cap\,\langle\,\mathrm{IHX}\,\rangle}\,\Big)$$

by decomposing a colored trivalent graph γ into its connected components as before. In order to establish a much stronger result we will make use of the bivalent black subgraph γ_{black} obtained from a colored trivalent graph γ by removing all its red edges. Consider for the moment an arbitrary red edge in a connected colored trivalent graph γ:



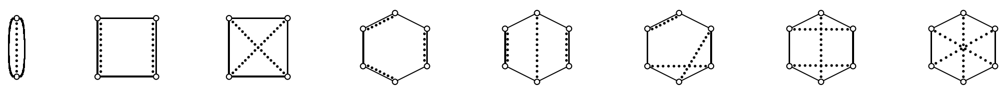
After leaving this red edge along the top left flag we will necessarily return to this red edge at some point or other along a different black flag, because every finite bivalent graph like γ_{black} equals the disjoint union of cycles of length ≥ 1. Depending on the black flag of first return exactly one of the three configurations in the IHX–relation corresponding to the chosen red edge will have one black cycle more than the other two configurations. Say the H configuration will have one cycle more than both the I and X configurations, if we return first along the black flag marked with H, analogous considerations apply for a first return along the black flag marked with I or X.

Given a connected graph γ such that γ_{black} has at least two cycles there necessarily exists a red edge connecting two different black cycles. In turn the IHX–relation for such a red edge becomes a congruence $[\,\gamma\,] \;\equiv\; -([\,\hat{\gamma}\,]+[\,\tilde{\gamma}\,])$ modulo the ideal $\langle\,\mathrm{IHX}\,\rangle$ with connected graphs $\hat{\gamma}$ and $\tilde{\gamma}$ such that $\hat{\gamma}_{\mathrm{black}}$ and $\tilde{\gamma}_{\mathrm{black}}$ both have one cycle less than γ_{black}. Repeating this process with $\hat{\gamma}$ and $\tilde{\gamma}$ we eventually end up with a congruence $[\,\gamma\,] \;\equiv\; \pm([\,\hat{\gamma}_1\,]+\ldots+[\,\hat{\gamma}_r\,])$ modulo the ideal $\langle\,\mathrm{IHX}\,\rangle$, in which all graphs $\hat{\gamma}_1,\,\ldots,\,\hat{\gamma}_r$ have connected black subgraphs:

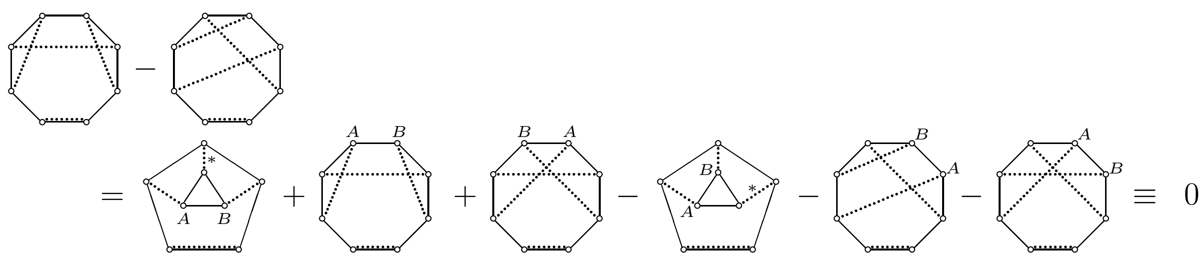
Corollary 3.4 (Generators for the Reduced Graph Algebra)
The reduced graph algebra $\overline{\mathbb{A}}^{\bullet}$*, the quotient of the algebra* $\mathbb{A}^{\bullet}$ *of colored trivalent graphs modulo the ideal* $\langle\,\mathrm{IHX}\,\rangle$ *of IHX–relations, is generated as an algebra, albeit not freely generated, by the classes* $[\,\gamma\,]$ *of colored trivalent graphs* γ *with connected bivalent black subgraph* γ_{black}*:*

$$\overline{\mathbb{A}}^{\bullet} \;:=\; \mathbb{A}^{\bullet}/_{\langle\,\mathrm{IHX}\,\rangle} \;=\; \langle\,\{\,[\,\gamma\,] + \langle\,\mathrm{IHX}\,\rangle \mid \gamma \textit{ colored trivalent graph, } \gamma_{\mathrm{black}} \textit{ connected}\,\}\,\rangle\;.$$

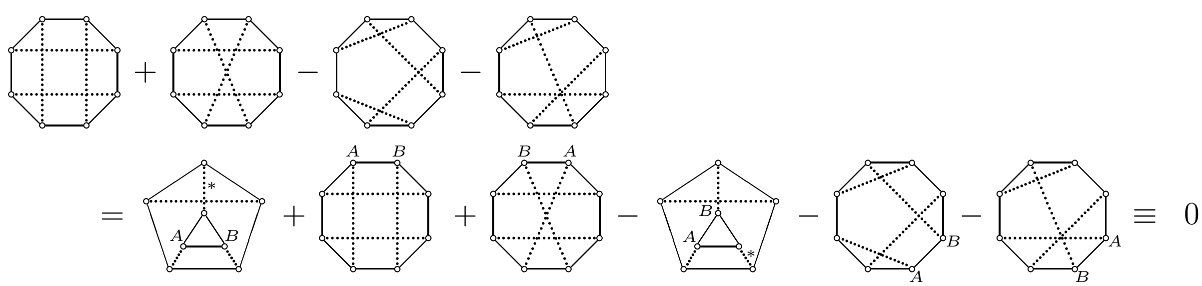
In consequence the list of generators of the reduced graph algebra $\overline{\mathbb{A}}^{\bullet}$ up to degree 3 reads:


. (27)

Interestingly there exist no further IHX–relations between these generators with connected black subgraphs up to degree 3. In degree 4 however there exist exactly two independent relations between the 17 isomorphism classes of colored trivalent graphs with 8 vertices and connected black subgraph in the reduced graph algebra $\overline{\mathbb{A}}^{\bullet}$ induced by the congruences



and

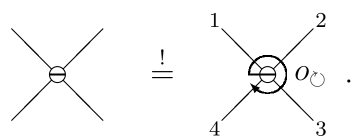


in the graph algebra $\mathbb{A}^{\bullet}$ modulo the ideal of IHX–relations. For the convenience of the reader we have marked the relevant red edge for these IHX–relations with $*$, in addition we have traced the images of a pair of interesting vertices marking them with the letters A and B.

Before closing this section let us briefly discuss a modification of the construction of the algebra $\mathbb{A}^{\bullet}$ of colored trivalent graphs and its quotient $\overline{\mathbb{A}}^{\bullet}$ by the IHX–relations, which allows us to use the curvature tensor R directly in the emerging graphical calculus for stable curvature invariants. For this purpose we need to allow tetravalent besides trivalent vertices, moreover all these tetravalent vertices need to come along with an orientation and an unordered partition of the set of adjacent flags into two pairs of flags:

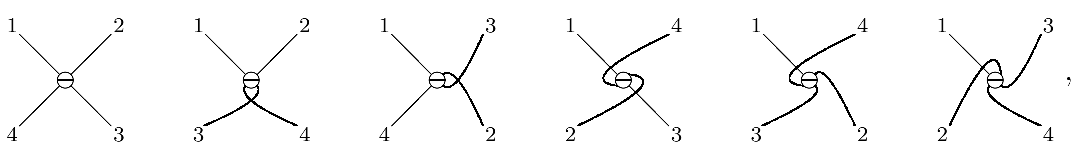
Definition 3.5 (Extended Colored Trivalent Graphs)
An extended colored trivalent graph is a graph γ with tri– and tetravalent vertices endowed with a coloring of its edges in red & black as before such that every trivalent vertex is adjacent to exactly one red flag and vice versa. Moreover the set $\mathrm{Flag}_v\gamma = \{ f_1, f_2, f_3, f_4 \}$ *of black flags adjacent to each tetravalent vertex* $v \in \mathrm{Vert}\,\gamma$ *is endowed with an orientation and an unordered partition* $\mathrm{Flag}_v\gamma = \{ f_1, f_2 \} \,\dot\cup\, \{ f_3, f_4 \}$ *into two pairs of flags.*

Concerning homomorphisms and isomorphisms we stipulate that every homomorphism between extended colored trivalent graphs $\varphi : \gamma \longrightarrow \hat\gamma$ needs to respect the θ–invariant coloring $\mathbf{c} : \mathrm{Flag}\,\gamma \longrightarrow \{\mathrm{red}, \mathrm{black}\}$ of the flags and the orientation as well as the partition of the sets of flags adjacent to tetravalent vertices. In diagrams we will depict tetravalent vertices by circles with horizontal or vertical chords indicating the partition of flags into two pairs:



Unless stated otherwise we will tacitly assume that the orientation on the set of flags adjacent to such a tetravalent vertex is the variant $o_\circlearrowleft$ of the so called blackboard orientation: Starting on one of the two end points of the chord and continuing counterclockwise around the vertex results in a labelling L of the adjacent flags representing $o_\circlearrowleft := [\, L, +1\,]$. Note that this orientation does not depend on which end of the chord we begin with.

In passing we remark that the joint stabilizer of the unordered partition $\{\{1, 2\}, \{3, 4\}\}$ into pairs and the tautological orientation of the set $\{1, 2, 3, 4\}$ in the symmetric group S_4 equals the normal Kleinian Four subgroup $K \subseteq S_4$. Instead of requiring both an orientation and a partition into pairs we may hence require alternatively that the set of flags adjacent to each tetravalent vertex v of the graph γ is decorated by an equivalence class of a labelling $L : \mathrm{Flag}_v\gamma \xrightarrow{\cong} \{ 1, 2, 3, 4 \}$ modulo postcomposition with elements of K. In this alternative formulation of Definition 3.5 the six different decorations on a tetravalent vertex read

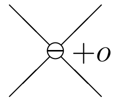


where the numbers indicate a representative labelling L under postcomposition with elements of $K \subseteq S_4$. Mimicking the construction of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs and its quotient $\overline{\mathbb{A}}^\bullet$ by the IHX–relations we consider the set of isomorphism classes

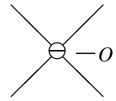
$$\Gamma^n_{\mathrm{ext}} \;:=\; \{\; [\,\gamma\,] \;|\; \gamma \text{ extended colored trivalent graph with } n \,=\, \tfrac{1}{2}\#\mathrm{Flag}\,\gamma - \#\mathrm{Vert}\,\gamma \;\}$$

of extended colored trivalent graphs with $2n$ vertices counting all tetravalent vertices twice. Unlike $\mathbb{A}^\bullet$ the algebra $\mathbb{A}^\bullet_{\mathrm{ext}}$ of extended colored trivalent graphs is not the convolution algebra $\mathbb{R}\,\Gamma^\bullet_{\mathrm{ext}}$ associated to the commutative monoid $\Gamma^\bullet_{\mathrm{ext}}$, but its quotient by the ideal $\langle\, \mathrm{O} \,\rangle \subseteq \mathbb{R}\,\Gamma^\bullet_{\mathrm{ext}}$

spanned by a change of orientation for the set of flags adjacent to some tetravalent vertex:



$+$

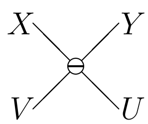


$. \qquad (28)$

Endowed with the $\mathbb{R}$–bilinear extension of the disjoint union the quotient $\mathbb{A}^\bullet_{\mathrm{ext}} := \mathbb{R}\,\Gamma^\bullet_{\mathrm{ext}}/\langle\, \mathrm{O}\,\rangle$ becomes a graded commutative algebra with its own version $\langle\,\mathrm{IHX}\,\rangle \subseteq \mathbb{A}^\bullet_{\mathrm{ext}}$ of the ideal of IHX–relations defined for all red edges by equations (22) and (23) as before. The quotient

$$\overline{\mathbb{A}}^\bullet_{\mathrm{ext}} \;=\; \mathbb{A}^\bullet_{\mathrm{ext}}/\langle\;\mathrm{IHX}\;\rangle$$

is the reduced algebra of extended colored trivalent graphs. In order to extend the definition of stable curvature invariants from $\mathbb{A}^\bullet$ to $\mathbb{A}^\bullet_{\mathrm{ext}}$ we specify the additional vertex interaction

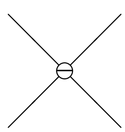


$$\widehat{=}\;\; R(\;X,\;Y;\;U,\;V\;) \tag{29}$$

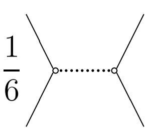
besides the red edge interaction (20). The entire point of this construction is that the extended graph algebra $\mathbb{A}^\bullet_{\mathrm{ext}}$ comes along with a surjective algebra homomorphism

$$\Phi^- : \;\; \mathbb{A}^\bullet_{\mathrm{ext}} \;\longrightarrow\; \mathbb{A}^\bullet\;,$$

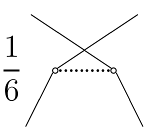
which descends to a surjective algebra homomorphism $\overline{\Phi}^- : \overline{\mathbb{A}}^\bullet_{\mathrm{ext}} \longrightarrow \overline{\mathbb{A}}^\bullet$ between the respective quotients by the ideal of IHX–relations. More precisely the algebra homomorphism Φ^- expands every tetravalent vertex into a pair of trivalent vertices connected by a red edge



$\stackrel{\Phi^-}{\longmapsto} \quad \frac{1}{6}$



$-\; \frac{1}{6}$



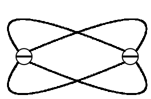
(30)

in accordance with the Feynman rules interpretation of the identity $R = \Phi^-\mathrm{Sec}$, namely:

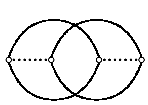
$$R(\;X,Y;\,U,V\;) \;\;=\;\; \frac{1}{6}\,\mathrm{Sec}(\;X,V;\;Y,U\;) \;-\; \frac{1}{6}\,\mathrm{Sec}(\;X,U;\;Y,V\;)\;.$$

An example of the usefulness of considering $\mathbb{A}^\bullet_{\mathrm{ext}}$ in addition to $\mathbb{A}^\bullet$ is the congruence

$\Phi^-\Big(\frac{1}{4}$



$\Big) \;=\; \frac{1}{144}\Big($



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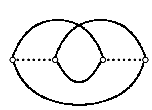

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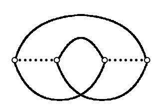

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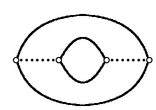
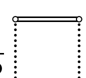
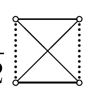


$\Big)$

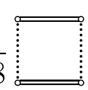
$= \frac{1}{72}$



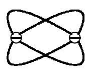
$- \frac{1}{72}$



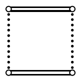
$\equiv \frac{1}{48}$



modulo the ideal $\langle\, \mathrm{IHX}\,\rangle$ based on the congruence (25). Using the Feynman rules (20) and (29) to work out the curvature invariants associated to the left and right hand sides we find



$$\left[\,\frac{1}{4}\quad\right](\,R\,) \;=\; \frac{1}{4}\sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m} R(\,E_\mu, E_\nu;\, E_\alpha, E_\beta\,)^2 \;=\; g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\;R,\;\;R\;)$$



$$\left[\,\frac{1}{4}\quad\right](\,R\,) \;=\; \frac{1}{4}\sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m} \mathrm{Sec}(\,E_\mu, E_\nu; E_\alpha, E_\beta\,)^2 \;=\; g_{\mathrm{Sym}^2T^*\otimes\mathrm{Sym}^2T^*}(\,\mathrm{Sec},\,\mathrm{Sec}\,)$$

and conclude that the isomorphisms Φ^+ and Φ^- of Section 2 are essentially isometries:

$$g_{\mathrm{Sym}^2T^*\otimes\mathrm{Sym}^2T^*}(\,\mathrm{Sec},\;\mathrm{Sec}\,) \;=\; 12\; g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\;R,\;R\;)\;. \tag{31}$$

4 Explicit Values of Curvature Invariants

In order to illustrate the rather abstract construction of stable curvature invariants from colored trivalent graphs discussed in Section 3 we want to present a combinatorial method to calculate explicitly the values of these stable curvature invariants on all space form algebraic curvature tensors, this is on all algebraic curvature tensors of constant sectional curvature. Central to this presentation is a family δ_m of homogeneous derivations of degree -1 of the graded graph algebras $\mathbb{A}^\bullet$ and $\overline{\mathbb{A}}^\bullet$ indexed by a formal dimension parameter m.

The space of algebraic curvature tensors on a euclidean vector space T of dimension $m \geq 2$ contains a unique curvature tensor up to scale invariant under the isometry group $\mathbf{O}(\,T,\,g\,)$, namely the algebraic curvature tensor $R^{\mathbf{O}(\,T,\,g\,)} \in \mathrm{Curv}^-T$ of constant sectional curvature 1:

$$R^{\mathbf{O}(\,T,\,g\,)}(\;X,\;Y;\;U,\;V\;) \;:=\; -\,g_{\Lambda^2T}(\;X\wedge Y,\;U\wedge V\;)\;.$$

The corresponding sectional curvature tensor $\mathrm{Sec}^{\mathbf{O}(\,T,\,g\,)} \in \mathrm{Curv}^+T$ is given by:

$$\mathrm{Sec}^{\mathbf{O}(\,T,\,g\,)}(X,Y;U,V) \;=\; 4\,g(X,Y)g(U,V) \,-\, 2\,g(X,U)g(Y,V) \,-\, 2\,g(X,V)g(Y,U)\;. \tag{32}$$

The Ricci tensor of the algebraic curvature tensor $R^{\mathbf{O}(\,T,\,g\,)}$ of constant sectional curvature 1 is the simple multiple $\mathrm{Ric} \;=\; (m-1)\,g$ of the scalar product g and so its scalar curvature equals $\kappa \;=\; m(m-1)$. In passing we remark that $R^{\mathbf{O}(\,T,\,g\,)}$ can also be defined in terms of the Nomizu–Kulkarni product of equation (12), more precisely $R^{\mathbf{O}(\,T,\,g\,)} \;:=\; -\frac{1}{2}\,g \times g$.

Definition 4.1 (Curvature Derivation)
The curvature derivation in formal dimension $m \in \mathbb{N}_0$ is the homogeneous derivation

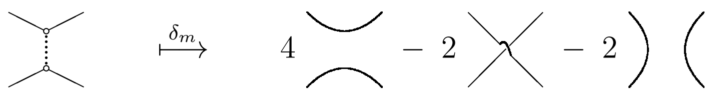
$$\delta_m:\;\; \mathbb{A}^\bullet \;\longrightarrow\; \mathbb{A}^{\bullet-1}, \qquad [\,\gamma\,] \;\longmapsto\; \delta_m[\,\gamma\,]\;,$$

of degree -1 of the graded algebra $\mathbb{A}^\bullet$ of colored trivalent graphs characterized by

$$\Big(\,\delta_m[\,\gamma\,]\,\Big)[\,R\,] \;=\; \frac{d}{dt}\bigg|_0\,[\,\gamma\,]\Big(\,R \,+\, tR^{\mathbf{O}(\,T,\,g\,)}\Big)$$

for every algebraic curvature tensor R on a euclidean vector space T of dimension m.

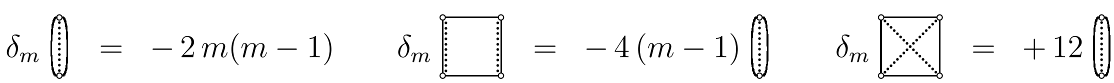
Due to the construction of stable curvature invariants in Definition 3.3 the value $[\,\gamma\,](\,R\,)$ of such an invariant on an algebraic curvature tensor R on a euclidean vector space T is an iterated sum over an orthonormal basis of T, whose parts are products over the red edges of the colored trivalent graph γ. Replacing R by $R + tR^{\mathbf{O}(T,g)}$ and taking the derivative $\frac{d}{dt}\big|_0$ we thus obtain a sum over the red edges of γ, in which the sectional curvature tensor factor $\mathrm{Sec} \in \mathrm{Curv}^+T$ associated to R has been replaced for this particular red edge by the factor $\mathrm{Sec}^{\mathbf{O}(T,g)}$. In consequence the curvature derivation δ_m is the homogeneous derivation of $\mathbb{A}^\bullet$ of degree -1, which expands each red edge in turn according to equation (32)



(33)

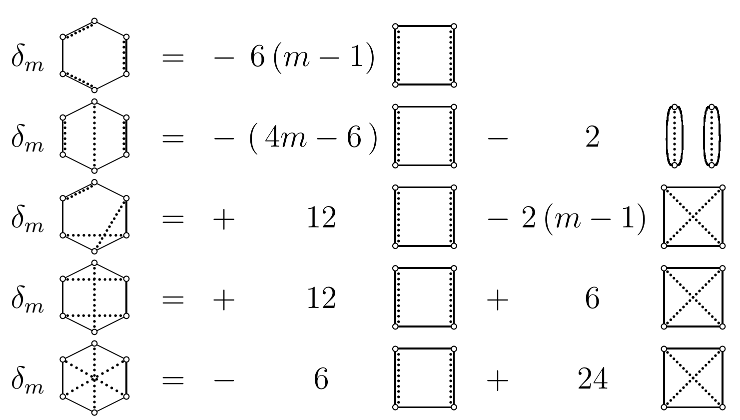
and sums all results together with the proviso that every black circle without any vertices at all occurred in the process needs to be interpreted as the scalar factor $m = \sum g(dE^\sharp_\mu, E_\mu)$. Needless to say this is the only way in which the dimension $m \in \mathbb{N}_0$ of the euclidean vector space T enters in the definition of the curvature derivation δ_m, in other words the derivation δ_m is at most a quadratic polynomial in the formal dimension m.

Evidently every IHX–relation of the form (21) vanishes under the expansion (33) of the curvature derivation, for this reason δ_m maps the ideal $\langle\,\mathrm{IHX}\,\rangle \subseteq \mathbb{A}^\bullet$ of all IHX–relations to itself. In turn the curvature derivation descends to a homogeneous derivation of degree -1 of the quotient algebra $\overline{\mathbb{A}}^\bullet$ of colored trivalent graphs modulo IHX–relations. Calculating δ_m for all the generators (27) of the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ up to degree 3 we find easily



(34)

for the generators of degree 1 and 2 as well as:



(35)

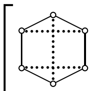
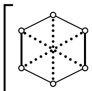
It should be pointed out that the curvature derivation allows us to calculate explicitly the values of stable curvature invariants on algebraic curvature tensors $R \in \mathrm{Curv}^-T$ of constant sectional curvature on every euclidean vector space T of dimension $m \geq 2$ using the formula

$$[\,\gamma\,](\,R\,) \;=\; \frac{1}{n!}\Big(\frac{\kappa}{m(m-1)}\Big)^n\,\delta^n_m[\,\gamma\,] \;\in\; \overline{\mathbb{A}}^0 \;=\; \mathbb{R} \tag{36}$$

valid for all isomorphism classes $[\,\gamma\,] \in \Gamma^n$ of colored trivalent graphs γ of degree $n \in \mathbb{N}_0$, where κ denotes the scalar curvature of R. In fact this formula is a direct consequence of the homogeneity of degree n of the polynomial $[\,\gamma\,]$ and the ensuing recursion relation:

$$\Big(\,\delta_m[\,\gamma\,]\,\Big)(\,R^{\mathbf{O}(T,g)}\,) \;=\; \frac{d}{dt}\Big|_0 (\,1+t\,)^n\,[\,\gamma\,](\,R^{\mathbf{O}(T,g)}\,) \;=\; n\,[\,\gamma\,](\,R^{\mathbf{O}(T,g)}\,)\;.$$

Based on equations (34) and (35) we find say for every space form curvature tensor R:

$\Big[$  $\Big](\,R\,) \;=\; +\,8\,\dfrac{(2m-5)\,\kappa^3}{m^2\,(m-1)^2}$ $\qquad$ $\Big[$  $\Big](\,R\,) \;=\; -\,8\,\dfrac{(m+11)\,\kappa^3}{m^2\,(m-1)^2}\;.$

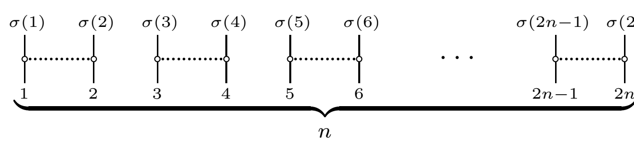
5 Pfaffian and Moment Polynomials

All characteristic numbers of a compact Riemannian manifold are integrated polynomial invariants of the curvature tensor, the corresponding polynomial however is not a stable curvature invariant in the sense of this article. The only stable characteristic number among the classical Euler and Pontryagin numbers turns out to be the Euler characteristic, in this section we will identity the corresponding elements in the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs, the Pfaffian polynomials $(\,\mathrm{pf}_n\,)_{n\in\mathbb{N}_0}$. Besides the Euler characteristic we will study the normalized moment polynomials $(\,\Psi^\circ_n\,)_{n\in\mathbb{N}_0}$ in this section, which calculate the moments $(\,\Psi_n\,)_{n\in\mathbb{N}_0}$ of the sectional curvature considered as a random variable on the Graßmannian of planes up to a normalization constant depending on the dimension.

The sequence of Pfaffian polynomials is a sequence $(\,\mathrm{pf}_n\,)_{n\in\mathbb{N}_0}$ of elements of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs defined for $n = 0$ by $\mathrm{pf}_0 = \mathbf{1}$ and in positive degrees $n > 0$ by:

$$\mathrm{pf}_n(\,R\,) \;:=\; \frac{1}{12^n\,n!}\sum_{\sigma\in S_{2n}}(\,\mathrm{sgn}\,\sigma\,)\sum_{\mu_1,\ldots,\mu_{2n}=1}^{m}\prod_{r=1}^{n}\mathrm{Sec}(\,E_{\mu_{2r-1}},\;dE^\sharp_{\mu_{\sigma(2r-1)}};\;E_{\mu_{2r}},\;dE^\sharp_{\mu_{\sigma(2r)}}\,)\;.$$

Right from this definition we can read off the corresponding element of the graph algebra $\mathbb{A}^\bullet$

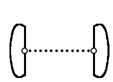
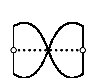
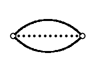
$$\mathrm{pf}_n \;=\; \frac{1}{12^n\,n!}\sum_{\sigma\in S_{2n}}(\,\mathrm{sgn}\,\sigma\,)$$ 

, (37)

where it is understood that a black edge runs between every pair of flags indexed by the same integer, one flag above and one below. According to the Theorem of Chern–Gauß–Bonnet the Euler characteristic of every compact Riemannian manifold M of even dimension $m \in 2\mathbb{N}_0$ with Riemannian metric g can be written as the integrated curvature invariant

$$\chi(\,M\,) \;=\; \frac{1}{(\,2\pi\,)^{\frac{m}{2}}}\int_M \mathrm{pf}_{\frac{m}{2}}(\,R\,)\,|\,\mathrm{vol}_g\,|\;. \tag{38}$$

associated to the sequence $(\,\mathrm{pf}_n\,)_{n\in\mathbb{N}_0}$. For Riemannian surfaces M for example we find

$\mathrm{pf}_1 \;=\; \dfrac{1}{12}\Big(\,+$  $-$  $\Big) \;\equiv\; -\,\dfrac{1}{4}$  $\;\widehat{=}\;\dfrac{\kappa}{2}$

using the basic IHX–relation (26) and conclude that $\mathrm{pf}_1(R)$ equals the Gaußian curvature of the surface M; keep in mind that κ denotes the scalar curvature throughout this article. In order to relate equation (38) to the Theorem of Chern–Gauß–Bonnet in higher dimensions we recall that it involves in its standard formulation the pointwise Berezin integral

$$[\ ]_{g,o}:\quad \Lambda^\bullet T^*M\otimes\Lambda^\circ TM\longrightarrow\Lambda^\bullet T^*M,\qquad \xi\otimes\mathfrak{X}\longmapsto\langle\,\mathrm{vol}_{g,o},\ \mathfrak{X}\,\rangle\,\xi\ ,$$

which depends on both the Riemannian metric g and an additional orientation o via the Riemannian volume form $\mathrm{vol}_{g,o}\in\Gamma(\Lambda^mT^*M)$. Considering the curvature as a bivector valued 2–form $R\in\Gamma(\Lambda^2T^*M\otimes\Lambda^2TM)$ like in equation (6) we may exponentiate it pointwise in the algebra bundle $\Lambda T^*M\otimes\Lambda TM$ and take the full Berezin integral of the result to obtain

$$\chi(M)\;=\;\int_{(M,o)}\Big[\ \exp\Big(-\frac{R}{2\pi}\Big)\Big]_{g,o}\tag{39}$$

according to Chern ([5],[3],[10]). One way to define the oriented integration of the Pfaffian differential form $\mathrm{Pf}(\frac{R}{2\pi}):=[\exp(-\frac{R}{2\pi})]_{g,o}$ over the oriented manifold M is to think of the orientation as the real line bundle isomorphism $o:\Lambda^mT^*M\longrightarrow\vartheta^1M$ determined by $o(\mathrm{vol}_{g,o})=|\,\mathrm{vol}_g\,|$ to turn the Pfaffian differential form into a multiple of the volume density $|\,\mathrm{vol}_g\,|\in\Gamma(\vartheta^1M)$. Combining the pointwise Berezin integral $[\ ]_{g,o}$ used to define $\mathrm{Pf}(\frac{R}{2\pi})$ with this orientation vector bundle isomorphism we obtain the vector bundle homomorphism

$$[\ ]_g:\ \Lambda^\bullet T^*M\otimes\Lambda^\circ TM\xrightarrow{[\ ]_{g,o}}\Lambda^\bullet T^*M\xrightarrow{o}\vartheta^1M,\qquad \xi\otimes\mathfrak{X}\longmapsto\langle\,\xi,\ \mathrm{pr}_{\Lambda^mTM}\mathfrak{X}\,\rangle\,|\,\mathrm{vol}_g\,|\ ,$$

where $\mathrm{pr}_{\Lambda^mTM}:\Lambda^\circ TM\longrightarrow\Lambda^mTM$ denotes the projection to the top dimensional term. This vector bundle homomorphism however is defined even for non–orientable manifolds M and so we can rewrite the Theorem of Chern–Gauß–Bonnet as an unoriented integral identity

$$\chi(M)\;=\;\frac{1}{(2\pi)^{\frac{m}{2}}}\int_M\Big[\ \frac{1}{(\frac{m}{2})!}\,\underbrace{(-R)\wedge\ldots\wedge(-R)}_{\frac{m}{2}}\ \Big]_g\ ,$$

in fact the projection of $\exp(-\frac{R}{2\pi})$ to the top dimensional term leaves us with the $\frac{m}{2}$–th power of R only. Replacing the algebraic curvature tensor $R=\Phi^-\mathrm{Sec}$ in its expansion (6) as a bivector valued 2–form by the corresponding sectional curvature tensor Sec we find

$$\begin{aligned}-\,R\\ &=\;+\,\frac{1}{24}\sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m}\Big(\mathrm{Sec}(E_\mu,E_\alpha;E_\nu,E_\beta)\,-\,\mathrm{Sec}(E_\mu,E_\beta;E_\nu,E_\alpha)\Big)dE_\mu\wedge dE_\nu\,\otimes\,dE^\sharp_\alpha\wedge dE^\sharp_\beta\\ &=\;+\,\frac{1}{12}\sum_{\substack{\mu_1,\,\mu_2\,=\,1\\ \alpha_1,\,\alpha_2\,=\,1}}^{m}\mathrm{Sec}(\,E_{\mu_1},\,E_{\alpha_1};\,E_{\mu_2},\,E_{\alpha_2}\,)\,dE_{\mu_1}\wedge dE_{\mu_2}\,\otimes\,dE^\sharp_{\alpha_1}\wedge dE^\sharp_{\alpha_2}\end{aligned}$$

and conclude that $\frac{1}{n!}(-R)^n\in\Gamma(\Lambda^{2n}T^*M\otimes\Lambda^{2n}TM)$ can be expanded for all $n\in\mathbb{N}_0$ into:

$$\frac{1}{n!}\underbrace{(-R)\wedge\ldots\wedge(-R)}_{n}\;=\;\frac{1}{12^n\,n!}\sum_{\substack{\mu_1,\,\ldots,\,\mu_{2n}\,=\,1\\ \alpha_1,\,\ldots,\,\alpha_{2n}\,=\,1}}^{m}\ \prod_{r\,=\,1}^{n}\mathrm{Sec}(E_{\mu_{2r-1}},E_{\alpha_{2r-1}};E_{\mu_{2r}},E_{\alpha_{2r}})\qquad dE_{\mu_1}\wedge\ldots\wedge dE_{\mu_{2n}}\otimes dE^\sharp_{\alpha_1}\wedge\ldots\wedge dE^\sharp_{\alpha_{2n}}\ .$$

In order to write the Theorem of Chern–Gauß–Bonnet in the form (38) it remains to observe

$$\langle\, dE_{\mu_1}\wedge\ldots\wedge dE_{\mu_{2n}},\, dE^\sharp_{\alpha_1}\wedge\ldots\wedge dE^\sharp_{\alpha_{2n}}\,\rangle \;=\; \sum_{\sigma\,\in\,S_{2n}} (\,\mathrm{sgn}\,\sigma\,)\, dE_{\mu_{\sigma(1)}}(\,dE^\sharp_{\alpha_1}\,)\,\ldots\, dE_{\mu_{\sigma(2n)}}(\,dE^\sharp_{\alpha_{2n}}\,)$$

and sum over $\alpha_1,\,\ldots,\,\alpha_{2n} = 1,\,\ldots,\,m$ using the equality $\xi^\sharp = \sum_\alpha \xi(\,dE^\sharp_\alpha\,)\,E_\alpha$. Although the argument presented above establishes the equivalence of the versions (38) and (39) of the Theorem of Chern–Gauß–Bonnet for all orientable compact manifolds, it does not per se prove the validity of the unoriented version (38) for all compact manifolds. In case of doubts the reader may simply apply equation (38) to the orientable 2–fold cover of a compact manifold M and divide both sides by 2. Better still would be to rework Flanders' proof [6] of the Theorem of Chern–Gauß–Bonnet with densities instead of differential forms.

Lemma 5.1 (Generating Series of Pfaffian Polynomials)
Every isomorphism class of colored trivalent graphs occurs with non–zero coefficient in the Pfaffian $\mathrm{pf}_n \in \mathbb{A}^n$ *of the appropriate degree* $n \in \mathbb{N}_0$*. In the power series completion of the graded algebra* $\mathbb{A}^\bullet$ *of colored trivalent graphs the total Pfaffian equals the exponential*

$$\sum_{n\,\geq\,0}\mathrm{pf}_n \;=\; \exp\left(\sum_{[\,\gamma\,]\,\in\,\Gamma^\bullet_{\mathrm{conn}}}\frac{(-1)^{e(\gamma)}}{6^{n(\gamma)}}\,\frac{2^{g(\gamma)}}{\#\overline{\mathrm{Aut}}\,\gamma}\,[\,\gamma\,]\right)\,,$$

where $n(\gamma) := \frac{1}{2}\#\mathrm{Vert}\,\gamma$ *equals the degree of* γ*, while* $e(\gamma)$ *and* $g(\gamma)$ *are the numbers of cycles of the black subgraph* γ_{black} *of* γ *of even and of length greater than* 2 *respectively.*

Proof: Consider to begin some $n \in \mathbb{N}$ and a non–void colored trivalent graph γ with $2n$ vertices. Choosing a direction of cyclically traversing each cycle in the associated bivalent black subgraph γ_{black} allows us to define a successor permutation $\sigma \in S_{\mathrm{Vert}\,\gamma}$ of the set of vertices of γ, which sends every vertex to the next vertex on the same cycle in the chosen direction. The signature $\mathrm{sgn}\,\sigma = (-1)^{e(\gamma)}$ of the successor permutation reflects the parity of the number $e(\gamma) \in \mathbb{N}_0$ of even length cycles as always with permutations. In addition we chose a labelling $L:\ \mathrm{Vert}\,\gamma \longrightarrow \{\,1,\ldots,2n\,\}$ of the vertices of γ such that red edges run exactly between the pairs $\{L^{-1}(1), L^{-1}(2)\}$, $\{L^{-1}(3), L^{-1}(4)\}$, $\ldots$, $\{L^{-1}(2n-1), L^{-1}(2n)\}$ of vertices. With such a choice of labelling the original colored trivalent graph γ turns out to be isomorphic to the summand graph in the sum (37) associated to the permutation $L\circ\sigma\circ L^{-1}$ of the index set $\{\,1,\ldots,2n\,\}$ of the same signature.

Each summand graph $\hat{\gamma}$ in the sum (37) on the other hand comes along with a tautological labelling $L:\ \mathrm{Vert}\,\hat{\gamma} \longrightarrow \{\,1,\ldots,2n\,\}$ of its vertices, moreover all its edges come along with a distinguished direction from the flag below to the flag above. These edge directions assemble together to a direction for traversing each cycle in $\hat{\gamma}_{\mathrm{black}}$ such that the associated successor permutation agrees with the permutation $\sigma \in S_{2n}$ indexing the summand graph $\hat{\gamma}$ in the first place. In consequence every colored trivalent graph γ with $2n$ vertices occurs at least once up to isomorphism in the sum (37) defining pf_n and all its occurrences in this sum share the same coefficient $\mathrm{sgn}\,\sigma = (-1)^{e(\gamma)}$ leaving no place for cancellations.

In order to prove the second statement we want to count the number of times a given colored trivalent graph γ is isomorphic to the summand graph $\hat{\gamma}$ indexed by a permutation $\sigma \in S_{2n}$. Fixing the labelling $L : \operatorname{Vert}\gamma \longrightarrow \{1, \ldots, 2n\}$ for the moment as above we observe that reversing the direction of a cycle of γ_{black} replaces the successor permutation σ by its inverse σ^{-1} on the vertices of this cycle. A cycle is different from its inverse however unless the cycle is of length less than or equal to 2, hence the different choices for the directions of the cycles with labelling L fixed account for exactly $2^{g(\gamma)}$ summands in the sum (37), where $g(\gamma) \in \mathbb{N}_0$ is the number of cycles of γ_{black} of length greater than 2.

Recall now that the labelling $L : \operatorname{Vert}\gamma \longrightarrow \{1, \ldots, 2n\}$ was required to be such that the pairs of vertices $\{L^{-1}(1), L^{-1}(2)\}$, $\{L^{-1}(3), L^{-1}(4)\}$, $\ldots$, $\{L^{-1}(2n-1), L^{-1}(2n)\}$ are connected by red edges. In general there are $2^n\, n!$ such labellings, however a different labelling $\hat{L}$ leads to different $2^{g(\gamma)}$ summand graphs $\hat{\gamma}$ in the sum (37) isomorphic to γ, if and only if $\hat{L}^{-1} \circ L \notin \overline{\operatorname{Aut}}\,\gamma$ fails to be a pure automorphism of γ. Hence the colored trivalent graph γ we consider occurs always with the same coefficient $(-1)^{e(\gamma)}$ in exactly

$$\frac{2^n\, n!}{\#\,\overline{\operatorname{Aut}}\,\gamma}\, 2^{g(\gamma)} \quad = \quad 12^n\, n!\, \Big(\, \frac{1}{6^n}\, \frac{2^{g(\gamma)}}{\#\,\overline{\operatorname{Aut}}\,\gamma} \,\Big)$$

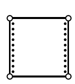
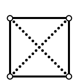
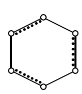
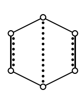
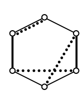
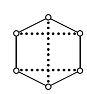
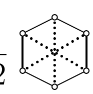
different summands of the sum (37) defining pf_n; put differently we obtain for all $n \in \mathbb{N}_0$:

$$\operatorname{pf}_n \quad = \quad \sum_{[\gamma] \in \Gamma^n} \frac{(-1)^{e(\gamma)}}{6^n}\, \frac{2^{g(\gamma)}}{\#\overline{\operatorname{Aut}}\,\gamma}\, [\,\gamma\,]\;.$$

Contemplating this formula a bit the reader may easily verify that the total Pfaffian is the stipulated exponential in the power series completion of the graph algebra $\mathbb{A}^\bullet$. $\square$

Needless to say the formula for the total Pfaffian given in Lemma 5.1 can still be simplified using the IHX–relations. After a little bit of computation we obtain for the power series expansion of the total Pfaffian up to degree 3 in the power series completion of $\overline{\mathbb{A}}^\bullet$:

$$\sum_{n \geq 0} \operatorname{pf}_n \quad = \quad \exp\Big(-\frac{1}{4}$$

$$-\frac{1}{8}$$

$$-\frac{1}{48}$$

$$-\frac{1}{24}$$

$$-\frac{1}{16}$$

$$-\frac{1}{24}$$

$$-\frac{5}{432}$$

$$-\frac{1}{432}$$

$$+ \ldots \Big)\;.$$

Remark 5.2 (Vanishing of Stable Curvature Invariants)
The Euler characteristic is of course multiplicative under taking Cartesian products of compact Riemannian manifolds, with the Theorem of Chern–Gauß–Bonnet in mind we anticipate that the Pfaffian of algebraic curvature tensors is multiplicative under taking direct sums

$$\operatorname{pf}_{\frac{m+\hat{m}}{2}}(\, R \oplus \hat{R}\,) \quad = \quad \operatorname{pf}_{\frac{m}{2}}(\, R\,)\, \operatorname{pf}_{\frac{\hat{m}}{2}}(\, \hat{R}\,)$$

of algebraic curvature tensors R and $\hat{R}$ on euclidean vector spaces of even dimensions m and $\hat{m}$ respectively. According to Lemma 5.1 the total Pfaffian is an exponential in the power series completion of $\mathbb{A}^\bullet$, hence the Pfaffian is in fact multiplicative as stipulated provided

$$\operatorname{pf}_n(\, R\,) \quad = \quad 0$$

for every algebraic curvature tensor R on a euclidean vector space T of dimension less than $2n$. Of course under this assumption already the n–th power $\frac{1}{n!}(-R)^n \in \Lambda^{2n}T^* \otimes \Lambda^{2n}T$ of the algebraic curvature tensor $R \in \mathrm{Curv}^- T$ considered as a bivector valued 2–form vanishes.

Let us now come to a different set of interesting polynomials in algebraic curvature tensors, namely the normalized moment polynomials Ψ_n° of degree $n \in \mathbb{N}_0$. Recall to begin with that every euclidean vector space T with scalar product g can be considered as a Riemannian manifold with translation invariant Riemannian metric g and associated Laplace–Beltrami operator. For the purposes of this article we will take this to be the positive Laplacian

$$\Delta_g \;:=\; -\Big(\,\frac{\partial^2}{\partial x_1^2} \;+\; \ldots \;+\; \frac{\partial^2}{\partial x_m^2}\,\Big)$$

in a system of linear coordinates on T provided by an orthonormal basis $x_1, \ldots, x_m$ of T^*:

Definition 5.3 (Normalized Moment Polynomials)
For every $n \in \mathbb{N}_0$ we define the normalized moment polynomial Ψ_n° as a homogeneous, stably invariant polynomial of degree n in algebraic curvature tensors via the generating series:

$$\sum_{n\,\geq\,0} \Psi_n^\circ(\,R\,) \;:=\; \exp\Big(\,-\,\Delta_g\,\Big)\Big|_0 \left(\; X \;\longmapsto\; \exp\Big(\,\sum_{r\,>\,0}\frac{1}{2r}\,\mathrm{tr}(\,R_{\,\cdot\,,\,X}X\,)^r\,\Big)\;\right)\;.$$

Depending on the signature of the scalar product g the positive Laplacian Δ_g may actually be the d'Alembert or wave operator $\Box$ of course. In order to justify calling $(\,\Psi_n^\circ\,)_{n\,\in\,\mathbb{N}_0}$ the normalized moment polynomials we consider a euclidean vector space T with positive definite scalar product g. Under this assumption the sectional curvature function $\sec_R : \mathrm{Gr}_2 T \longrightarrow \mathbb{R}$ associated to $R \in \mathrm{Curv}^- T$ is well–defined on the Graßmannian of planes in T by

$$\sec_R(\,E\,) \;:=\; \frac{R(\,X,\,Y;\,Y,\,X\,)}{g_{\Lambda^2 T}(\,X\wedge Y,\,X\wedge Y\,)} \;=\; \frac{1}{4}\,\frac{\mathrm{Sec}(\,X,\,X;\,Y,\,Y\,)}{g_{\Lambda^2 T}(\,X\wedge Y,\,X\wedge Y\,)}$$

for linearly independent vectors $X,\,Y$ spanning $E = \mathrm{span}_{\mathbb{R}}\{\,X,\,Y\,\} \in \mathrm{Gr}_2 T$. Endowing the Graßmannian with the Fubini–Study metric g_{FS} or measure $|\mathrm{vol}_{\mathrm{FS}}|$ we can think of the sectional curvature function as a random variable on $\mathrm{Gr}_2 T$, its moments are then given by:

Lemma 5.4 (Moments of Sectional Curvature [11])
In the positive definite case the normalized moment polynomials Ψ_n° calculate the moments Ψ_n, $n \in \mathbb{N}_0$, of the sectional curvature function $\sec_R : \mathrm{Gr}_2 T \longrightarrow \mathbb{R}$ considered as a random variable on the Graßmannian $\mathrm{Gr}_2 T$ of planes in T endowed with the Fubini–Study measure:

$$\Psi_n(\,R\,) \;:=\; \frac{1}{\mathrm{Vol}(\,\mathrm{Gr}_2 T,\;|\,\mathrm{vol}_{\mathrm{FS}}\,|\,)}\int_{\mathrm{Gr}_2 T}\sec_R^n\,|\,\mathrm{vol}_{\mathrm{FS}}\,| \;\stackrel{!}{=}\; \frac{n!}{[\,m+2n-2\,]_{2n}}\;\Psi_n^\circ(\,R\,)\;.$$

The normalization factor depends on the falling factorial $[\,x\,]_{2n} := x(x-1)\ldots(x-2n+1)$.

In passing we remark that it is possible to calculate the maximum and minimum of the sectional curvature function $\sec_R : \mathrm{Gr}_2 T \longrightarrow \mathbb{R}$ associated to an algebraic curvature tensor $R \in \mathrm{Curv}^- T$ in the positive definite case from its moments $\Psi_n(R)$, more precisely

$$\begin{aligned}\sec_{\max}(R) &:= \max_{E \in \mathrm{Gr}_2 T} \sec_R(E) \;=\; -\,\Lambda \;+\; \lim_{N \to \infty} \left(\sum_{n=0}^{N} \binom{N}{n} \Psi_n(R)\, \Lambda^{N-n} \right)^{\frac{1}{N}} \\ \sec_{\min}(R) &:= \min_{E \in \mathrm{Gr}_2 T} \sec_R(E) \;=\; -\,\Lambda \;+\; \lim_{N \to \infty} \left(\sum_{n=0}^{N} (-1)^n \binom{N}{n} \Psi_n(R)\, \Lambda^{N-n} \right)^{\frac{1}{N}}\end{aligned}$$

for all shift parameters $\Lambda \in \mathbb{R}$ satisfying $\Lambda \geq -\sec_{\min}(R)$ or $\Lambda \geq +\sec_{\max}(R)$ respectively. Needless to say both formulas are eventually formulas from probability theory for the essential supremum and infimum of an almost certainly bounded random variable.

In difference to the moment polynomials Ψ_n with their direct interpretation as moments the normalized moment polynomials Ψ_n° turn out to be stable curvature invariants. In order to identify the corresponding elements of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs we use the consequence $R(U, X; X, V) = \frac{1}{4} \mathrm{Sec}(X, X; U, V) = -\frac{1}{2} \mathrm{Sec}(X, U; X, V)$ of the first Bianchi identity (9) for Sec in order to expand the Jacobi operator $U \longmapsto R_{U,X} X$ into the sum

$$R_{\,\cdot\,,X} X \;=\; \sum_{\mu=1}^{m} R(\,\cdot\, , X;\, X,\, dE_\mu^\sharp)\, E_\mu \;=\; -\,\frac{1}{2} \sum_{\mu=1}^{m} \mathrm{Sec}(\, X,\, \cdot\,;\, X,\, dE_\mu^\sharp \,)\, E_\mu$$

over a dual pair of bases $\{ E_\mu \}$ and $\{ dE_\mu \}$ for the euclidean vector space T and its dual T^*. In turn the traces of the powers of the Jacobi operator become the iterated sums:

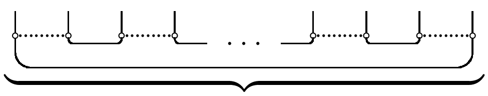
$$\begin{aligned}&\mathrm{tr}(R_{\,\cdot\,,X} X)^r \\ &= \Big(-\frac{1}{2} \Big)^r \sum_{\mu_1, \ldots, \mu_r = 1}^{m} \mathrm{Sec}(X, E_{\mu_1}; X, dE_{\mu_2}^\sharp)\, \mathrm{Sec}(X, E_{\mu_2}; X, dE_{\mu_3}^\sharp) \,\ldots\, \mathrm{Sec}(X, E_{\mu_r}; X, dE_{\mu_1}^\sharp) \,.\end{aligned}$$

Although we have used the exponential $\exp(-\Delta_g)\big|_0$ of the positive Laplacian on T in Definition 5.4, the rescaled exponential $\exp(-\frac{1}{2}\Delta_g)\big|_0$ has a rather compelling combinatorial description as a closure operation: Applying it to the diagonal polynomial $X \longmapsto a(X, \ldots, X)$ arising from a not necessarily symmetric form $a \in \bigotimes^{2n} T^*$ we obtain an iterated sum

$$\begin{aligned}\mathrm{closure}[\, a \,] &:= \exp(-\tfrac{1}{2}\Delta_g)\big|_0 \Big(X \longmapsto a(\underbrace{X, \ldots, X}_{2n}) \Big) \\ &= \sum_{\substack{\theta \in S_{2n} \text{ fix point free} \\ \text{involution of } \{ 1, \ldots, 2n \}}} \;\; \sum_{\substack{\mu : \{ 1, \ldots, 2n \} \longrightarrow \{ 1, \ldots, m \} \\ \mu = \mu \circ \theta \text{ invariant under } \theta}} a(E_{\mu(1)}, E_{\mu(2)}, \ldots, E_{\mu(2n)})\end{aligned}$$

over an orthonormal basis $E_1, \ldots, E_m$ of the euclidean vector space T, where the two sums extend over all fix point free involutions θ of the index set $\{ 1, \ldots, 2n \}$ and all θ–invariant maps $\mu : \{ 1, \ldots, 2n \} \longrightarrow \{ 1, \ldots, m \}$ in the sense $\mu = \mu \circ \theta$. For simplicity of the exposition we have pretended again that the scalar product g is positive definite to omit the sign factors necessary otherwise. In terms of colored trivalent graphs we can hence write the Definition 5.3 of the generating series of the normalized moment polynomials in the form

$$\sum_{n\geq 0}\Psi_n^\circ \;:=\; \mathrm{closure}\Bigg[\,\exp\Bigg(\,\sum_{r>0}\frac{(-1)^r}{2r}$$



r pairs of vertices

$$\Bigg)\,\Bigg]\,, \qquad (40)$$

where the closure operation now sums over all ways to join up the open flags of the argument in pairs to form a colored trivalent graph. In this formula the difference between the rescaled exponential $\exp(-\frac{1}{2}\Delta_g)\big|_0$ and the actual exponential $\exp(-\Delta_g)\big|_0$ of the Laplacian used in Definition 5.4 has been compensated by changing the original factor $(-\frac{1}{2})^r$ to $(-1)^r$.

Lemma 5.5 (Generating Series of Normalized Moment Polynomials)
In the power series completion of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs the generating series of the normalized moment polynomials $(\Psi_n^\circ)_{n\in\mathbb{N}_0}$ can be written as the exponential of a sum over all isomorphism classes of connected colored trivalent graphs γ with the additional property that all cycles of the associated black subgraph γ_{black} are cycles of even length:

$$\sum_{n\geq 0}\Psi_n^\circ \;=\; \exp\left(\sum_{\substack{[\gamma]\in\Gamma^\bullet_{\mathrm{conn}}\\ \gamma_{\mathrm{black}}\text{ even cycles}}}(-1)^{n(\gamma)}\frac{2^{g(\gamma)}}{\#\overline{\mathrm{Aut}}\,\gamma}\,[\gamma]\right).$$

As before $n(\gamma) := \frac{1}{2}\#\mathrm{Vert}\,\gamma$ and $g(\gamma)$ counts the cycles of γ_{black} of length greater than 2.

Proof: The most important observation by far to understand the proof is that the black edges in the expansion (40) of the generating series of the normalized moment polynomials $(\Psi_n^\circ)_{n\in\mathbb{N}_0}$ either are "old" edges present already in the argument or are "new" edges joined up during the closure operation. In order to formalize this idea we define a tricoloring for a given colored trivalent graph γ as an extension $\mathbf{c}_{\mathrm{ext}} : \mathrm{Edge}\,\gamma \longrightarrow \{\,\mathrm{red},\,\mathrm{black}_{\mathrm{old}},\,\mathrm{black}_{\mathrm{new}}\,\}$ of its coloring $\mathbf{c} : \mathrm{Edge}\,\gamma \longrightarrow \{\,\mathrm{red},\,\mathrm{black}\,\}$ in the sense $\mathbf{c} = \mathrm{pr}\circ\mathbf{c}_{\mathrm{ext}}$ for the projection pr implied by notation such that the edges adjacent to every vertex are all colored differently. By construction every summand in the expansion (40) of the generating series $\sum\Psi_n^\circ$ is naturally a colored trivalent graph with a distinguished tricoloring $\mathbf{c}_{\mathrm{ext}}$.

A necessary condition for the existence of a tricoloring for a colored trivalent graph γ is that all cycles of the black subgraph γ_{black} associated to γ are cycles of even length, because every tricoloring $\mathbf{c}_{\mathrm{ext}}$ necessarily colors the edges of γ_{black} alternately in $\mathrm{black}_{\mathrm{old}}$ and $\mathrm{black}_{\mathrm{new}}$ along each cycle. Provided this necessary condition is met we can on the other hand color the edges alternately along each of the $e(\gamma)\in\mathbb{N}$ cycles of γ_{black} starting with either $\mathrm{black}_{\mathrm{old}}$ or $\mathrm{black}_{\mathrm{new}}$ to obtain exactly $2^{e(\gamma)}$ different tricolorings for γ. In difference to the black subgraph γ_{black} associated to γ the colored bivalent subgraph γ_{old} obtained by removing all edges colored in $\mathrm{black}_{\mathrm{new}}$ certainly depends on the chosen tricoloring $\mathbf{c}_{\mathrm{ext}}$.

The set of all tricolorings of a colored trivalent graph γ comes along with a natural action of the automorphism group $\mathrm{Aut}\,\gamma$ of γ in such a way that the stabilizer subgroup of a tricoloring $\mathbf{c}_{\mathrm{ext}}$ agrees with the automorphism group $\mathrm{Aut}\,(\,\gamma,\mathbf{c}_{\mathrm{ext}}\,)$ of γ considered as a tricolored trivalent graph. Both the tricolored graph $(\,\gamma,\mathbf{c}_{\mathrm{ext}}\,)$ and the colored old subgraph γ_{old} associated to γ and a given tricoloring $\mathbf{c}_{\mathrm{ext}}$ have only pure automorphisms in the sense that all their automorphisms are completely determined by a permutation of their common set $\mathrm{Vert}\,\gamma$ of vertices, because the flags adjacent to each vertex are all colored differently:

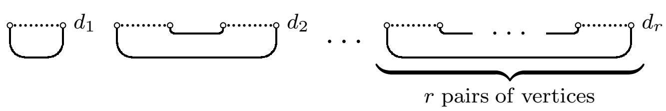
$$\mathrm{Aut}\,(\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,)\;\cong\;\overline{\mathrm{Aut}}\,(\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,)\qquad\qquad \mathrm{Aut}\,\gamma_{\mathrm{old}}\;\cong\;\overline{\mathrm{Aut}}\,\gamma_{\mathrm{old}}\;.$$

Interestingly the same conclusion does not hold for the original colored trivalent graph γ, it may have trivial automorphisms fixing all vertices, but not all flags. More precisely an automorphism of a colored trivalent graph γ fixing all its vertices can only swap the two flags or the two edges respectively in a short cycle of the black subgraph γ_{black} of length 1 or 2. Under the additional assumption that all cycles of γ_{black} have even length we thus obtain

$$\#\,\mathrm{Aut}_\circ\gamma \;=\; 2^{e(\gamma)\,-\,g(\gamma)}\;, \tag{41}$$

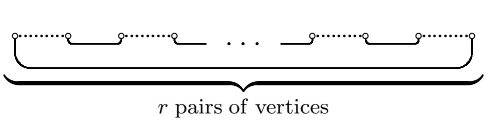
where $g(\gamma)$ equals the number of cycles of the black subgraph γ_{black} of length greater than 2.

Coming back to the proof we consider in a first step the class of colored bivalent graphs with edges colored alternately with colors red and $\mathrm{black}_{\mathrm{old}}$. Due to the additional coloring the automorphism group of a connected colored bivalent graph acts simply transitively on its set of vertices, in turn the automorphism group of a colored bivalent graph with cycles up to length $2r$ and d_1 cycles of length 2, d_2 cycles of length 4 etc. is a finite group of order

$\#\mathrm{Aut}\Big($ 

$\Big) \;=\; d_1!\,2^{d_1}\,d_2!\,4^{d_2}\,\ldots\,d_r!\,(2r)^{d_r}\;,$

because we can permute all the cycles of length 2, all the cycles of length 4 etc. In consequence the generating power series of the set of isomorphism classes of colored bivalent graphs reads:

$\exp\Big(\,\sum_{r>0}\frac{(-1)^r}{2r}$ 

$\Big)\;=\;\sum_{[\,\gamma_{\mathrm{old}}\,]}\frac{(-1)^{n(\gamma_{\mathrm{old}})}}{\#\mathrm{Aut}\;\gamma_{\mathrm{old}}}[\,\gamma_{\mathrm{old}}\,]\;.$

By definition the closure operation in the expansion (40) of the generating power series of the normalized moment polynomials Ψ_n° is a sum over all fixed point free involutions σ of the vertex set $\mathrm{Vert}\,\gamma_{\mathrm{old}}$ of the colored bivalent graph argument γ_{old} connecting all orbit pairs $\{\,v,\,\sigma(v)\,\}$ by an edge colored $\mathrm{black}_{\mathrm{new}}$ to obtain a tricolored trivalent graph $(\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,)$.

The automorphism group $\mathrm{Aut}\,\gamma_{\mathrm{old}}\;\cong\;\overline{\mathrm{Aut}}\,\gamma_{\mathrm{old}}$ of the colored bivalent graph γ on the other hand acts on $\mathrm{Vert}\,\gamma_{\mathrm{old}}$ and in turn on the set of fixed point free involutions σ of $\mathrm{Vert}\,\gamma_{\mathrm{old}}$ by conjugation. The orbits of this action correspond bijectively to the isomorphism classes $[\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,]$ of tricolored trivalent graphs with old subgraph isomorphic to γ_{old}, while the stabilizer of a given fixed point free involution σ is essentially the automorphism group of the corresponding tricolored trivalent graph $(\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,)$. Counting the number of all fixed point free involutions $\tilde\sigma$ of $\mathrm{Vert}\,\gamma_{\mathrm{old}}$ resulting in a tricolored trivalent graph in the class $[\,\gamma,\,\mathbf{c}_{\mathrm{ext}}\,]$

thus amounts to calculating the length of the orbit of the fixed point free involution σ, in this way we obtain for the generating power series (40) of the normalized moment polynomials:

$$\sum_{n\geq 0} \Psi_n^\circ \;=\; \text{closure}\left[\,\sum_{[\,\gamma_{\text{old}}\,]} \frac{(-1)^{n(\gamma_{\text{old}})}}{\#\text{Aut}\;\gamma_{\text{old}}}[\,\gamma_{\text{old}}\,]\right] \;=\; \sum_{[\,\gamma,\,\mathbf{c}_{\text{ext}}\,]} \frac{(-1)^{n(\gamma)}}{\#\text{Aut}\;\gamma_{\text{old}}}\;\frac{\#\text{Aut}\;\gamma_{\text{old}}}{\#\text{Aut}\,(\,\gamma,\mathbf{c}_{\text{ext}}\,)}\,[\,\gamma\,]\;.$$

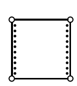
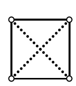
The problem with this expansion of the generating power series of the polynomials Ψ_n° is that the resulting sum is not effective, it sums multiples of the isomorphism class $[\,\gamma\,]$ of the colored trivalent graph underlying the tricolored isomorphism class $[\,\gamma,\,\mathbf{c}_{\text{ext}}\,]$. In order to obtain an effective formula for the generating power series (40) we thus need to sum for a given isomorphism class $[\,\gamma\,]$ of a colored trivalent graph γ over all possible isomorphism classes $[\,\gamma,\,\mathbf{c}_{\text{ext}}\,]$ of tricolored trivalent graphs extending γ.

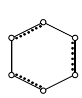
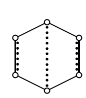
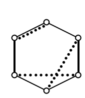
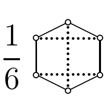
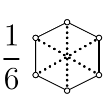
A necessary and sufficient condition for the existence of some isomorphism class extension $[\,\gamma,\,\mathbf{c}_{\text{ext}}\,]$ of a given colored trivalent graph γ is that all cycles of the black subgraph γ_{black} are cycles of even length; in this case the different isomorphism classes correspond to the orbits of the automorphism group $\text{Aut}\,\gamma$ of γ in the set of tricolorings $\mathbf{c}_{\text{ext}}$. Instead of summing over orbits of tricolorings it is more convenient to sum over tricolorings for γ weighted by the inverse length of their respective orbits, in this way we arrive eventually at the formula:

$$\begin{aligned}
\sum_{n\geq 0}\Psi_n^\circ \;&=\; \sum_{[\,\gamma,\,\mathbf{c}_{\text{ext}}\,]} \frac{(-1)^{n(\gamma)}}{\#\text{Aut}\,(\,\gamma,\mathbf{c}_{\text{ext}}\,)}\,[\,\gamma\,] \\
&=\; \sum_{\substack{[\,\gamma\,]\,\in\,\Gamma^\bullet\\ \gamma_{\text{black}}\text{ even cycles}}}\;\sum_{\mathbf{c}_{\text{ext}}\text{ tricoloring}} \left(\frac{\#\text{Aut}\,\gamma}{\#\text{Aut}(\,\gamma,\,\mathbf{c}_{\text{ext}}\,)}\right)^{-1} \frac{(-1)^{n(\gamma)}}{\#\text{Aut}\,(\,\gamma,\,\mathbf{c}_{\text{ext}}\,)}\,[\,\gamma\,] \\
&=\; \sum_{\substack{[\,\gamma\,]\,\in\,\Gamma^\bullet\\ \gamma_{\text{black}}\text{ even cycles}}} (-1)^{n(\gamma)}\;\frac{2^{e(\gamma)}}{\#\text{Aut}\;\gamma}\,[\,\gamma\,] \;=\; \sum_{\substack{[\,\gamma\,]\,\in\,\Gamma^\bullet\\ \gamma_{\text{black}}\text{ even cycles}}} (-1)^{n(\gamma)}\;\frac{2^{g(\gamma)}}{\#\overline{\text{Aut}}\;\gamma}\,[\,\gamma\,]\;.
\end{aligned}$$

For the last equality we have used equation (41) in the form $\#\text{Aut}\,\gamma \;=\; 2^{e(\gamma)-g(\gamma)}\,\#\overline{\text{Aut}}\,\gamma$. Similarly to the Pfaffian polynomials discussed above the mere form of the result tells us that the generating power series of the normalized moment polynomials Ψ_n° is the exponential of the sum on the right hand side taken over connected colored trivalent graphs only. $\square$

Of course Lemma 5.5 describes the expansion of the generating power series of the normalized moment polynomials Ψ_n° in the power series completion of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs. Simplifying the result using IHX–relations we find after some not too messy calculations in the power series completion of $\overline{\mathbb{A}}^\bullet$ the following expansion up to degree 3:

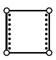
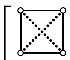
$\sum_{n\geq 0}\Psi_n^\circ \;=\; \exp\Big(\,-\,\frac{1}{2}$  $+\,\frac{1}{2}$  $-\,\frac{1}{4}$ 

$-\,\frac{1}{3}$  $-\,\frac{1}{2}$  $+$  $-\,\frac{1}{6}$  $-\,\frac{1}{6}$  $+\,\ldots\,\Big)\;.$

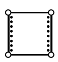
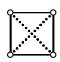
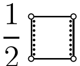
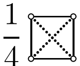
6 Curvature Identities for Einstein Manifolds

The graphical calculus for stable curvature invariants is well suited to derive curvature identities generalizing the Hitchin–Thorpe inequality in dimension $m = 4$. In a sense the Hitchin–Thorpe inequality deals with the expected value and the variance of sectional curvature, its third moment is related to the cubic polynomial $\Theta_3(\,R\,) := g_{\Lambda^2 T^* \otimes \Lambda^2 T^*}(\,q(R) \star R,\, R\,)$ of central importance to this section. Using the expansions of the Pfaffian and normalized moment polynomials we find an essentially unique linear relation between pf_3, Ψ_3° and Θ_3, whose integral over compact Einstein manifolds of dimension $m \geq 3$ results in Theorem 6.4. In contrast to the preceding, algebraic sections Riemannian manifolds are tacitly assumed to be endowed with positive definite metrics $g > 0$ throughout this section.

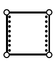
It is relatively easy to identify the generators (27) of the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ up to degree 2 using only the Ricci identity (26) and the argument leading to equation (31) together with the IHX–congruence (25). In this way we find [] $(\,R\,) = -2\kappa$ as well as:

[] $(\,R\,) \;=\; 8\, g_{\mathrm{Sym}^2 T^*}(\,\mathrm{Ric},\, \mathrm{Ric}\,)$ [] $(\,R\,) \;=\; -\,24\, g_{\Lambda^2 T^* \otimes \Lambda^2 T^*}(\,R,\, R\,)\,.$ (42)

Moreover the expansions of the total Pfaffian and the normalized moment polynomials given in Lemma 5.1 and Lemma 5.5 respectively duely simplified up to degree 3 in Section 5 imply:

$\mathrm{pf}_2 \;=\; \frac{1}{32}$  $^2 - \frac{1}{8}$  $- \frac{1}{48}$  $\qquad \Psi_2^\circ \;=\; \frac{1}{8}$  $^2 + \frac{1}{2}$  $- \frac{1}{4}$  .

Specializing to dimension $m = 4$ we observe that the actual second moment polynomial equals $\Psi_2 = \frac{1}{180}\Psi_2^\circ$ or equivalently $15\,\Psi_2 = \frac{1}{12}\Psi_2^\circ$, compare Lemma 5.4, moreover we can decompose the Ricci tensor into its trace free and scalar part $\mathrm{Ric} = \mathrm{Ric}_\circ + \frac{\kappa}{4} g$ to find:

$\Big[\,\mathrm{pf}_2 \,-\, 15\,\Psi_2\,\Big](\,R\,) \;=\; \Big[\,\frac{1}{48}$  $^2 \;-\; \frac{1}{6}$  $\Big](\,R\,) \;=\; -\,\frac{4}{3}\, g_{\mathrm{Sym}^2 T^*}(\,\mathrm{Ric}_\circ, \mathrm{Ric}_\circ\,) \;-\; \frac{\kappa^2}{12}\,.$

Integrating this identity over a compact 4–dimensional Riemannian manifold M we obtain

$$(\,2\pi\,)^2\,\chi(M) \;+\; \frac{4}{3}\,\|\,\mathrm{Ric}_\circ\|^2_{\mathrm{Sym}^2 T^*} \;=\; \int_M \frac{\kappa^2}{48}\,|\,\mathrm{vol}_g\,| \;+\; 15 \int_M \Big(\,\Psi_2(R) \;-\; \frac{\kappa^2}{144}\,\Big)\,|\,\mathrm{vol}_g\,| \qquad (43)$$

due to the formulation (38) of the Theorem of Chern–Gauß–Bonnet. For a compact Riemannian manifold M the right hand side is strictly positive unless M is flat, because the functions $\frac{\kappa}{12}$ and $\Psi_2(R) - (\frac{\kappa}{12})^2 \geq 0$ are the pointwise expected value and the pointwise variance respectively of the sectional curvature considered as a random variable. The resulting inequality $4\,\pi^2\,\chi(M) + \frac{4}{3}\,\|\,\mathrm{Ric}_\circ\,\|_{\mathrm{Sym}^2 T^*} \geq 0$ is a weak version of the Hitchin–Thorpe inequality for oriented manifolds [7], in particular every 4–dimensional Einstein manifold has strictly positive Euler characteristic unless it is flat [1]. In passing we recall that the signature is not a stable curvature invariant in the sense of this article.

In order to generalize the Hitchin–Thorpe inequality to Einstein manifolds of higher dimensions we will make use of the standard curvature term $q(R)$ defined in [9], albeit with a slightly different normalization. The letter q in our notation does not stand for quadratic, in contrast to the notation $Q(R)$ used in equation (46), but refers instead to the so–called quantization map from the symmetric to the universal enveloping algebra of a Lie algebra:

$$q: \ \operatorname{Sym}^{\leq\bullet}\mathfrak{g} \xrightarrow{\cong} \mathscr{U}^{\leq\bullet}\mathfrak{g}, \qquad \mathfrak{X}^r \longmapsto \mathfrak{X}^r \ .$$

Specializing to the orthogonal Lie algebra $\mathfrak{so}(T,\,g)$ of a euclidean vector space T we can convert every algebraic curvature tensor R over T via q linearly into various endomorphisms:

Definition 6.1 (Standard Curvature Term)
For the orthogonal Lie algebra $\mathfrak{so}(T,\,g)$ of skew symmetric endomorphisms of a euclidean vector space T the quantization map $q:\operatorname{Sym}^{\leq\bullet}\mathfrak{so}(T,\,g)\longrightarrow\mathscr{U}^{\leq\bullet}\mathfrak{so}(T,\,g)$ can be applied to every algebraic curvature tensor $R\in\operatorname{Curv}^-T$ over T considered in analogy to equation (6)

$$R \;=\; \frac{1}{2}\sum_{\mu,\,\nu=1}^{m}(\,dE^\sharp_\mu\wedge dE^\sharp_\nu\,)\otimes R_{E_\mu,\,E_\nu} \;\stackrel{!}{=}\; \frac{1}{4}\sum_{\mu,\,\nu=1}^{m}(\,dE^\sharp_\mu\wedge dE^\sharp_\nu\,)\cdot R_{E_\mu,\,E_\nu}$$

as an element of the symmetric square $\operatorname{Sym}^2\mathfrak{so}(T,\,g)$. Its image $q(\,R\,)\in\mathscr{U}^{\leq 2}\mathfrak{so}(T,\,g)$ becomes in turn an endomorphism in every representation $\star$ of the Lie algebra $\mathfrak{so}(T,\,g)$:

$$q(\,R\,)\star \;:=\; \frac{1}{4}\sum_{\mu,\,\nu=1}^{m}(\,dE^\sharp_\mu\wedge dE^\sharp_\nu\,)\;\star\;R_{E_\mu,\,E_\nu}\;\star\ .$$

In this definition of the standard curvature term the scalar factor $\frac{1}{4}$ is the proper choice, in difference to the rather unmotivated scalar factor $\frac{1}{2}$ used in [9]. For this reason the standard curvature term equals half the Ricci endomorphism, the symmetric endomorphism of T corresponding to the Ricci tensor, in the defining representation of the Lie algebra $\mathfrak{so}(T,\,g)$:

$$q(R)\star X \;=\; \frac{1}{2}\sum_{\mu,\,\nu=1}^{m}g(\,dE^\sharp_\mu,\,R_{E_\mu,\,E_\nu}X\,)\,dE^\sharp_\nu \;=\; \frac{1}{2}\sum_{\nu=1}^{m}\operatorname{Ric}(E_\nu,X)\,dE^\sharp_\nu \;=:\; \frac{1}{2}\operatorname{Ric}X\ .$$

On the bivector representations Λ^2T and Sym^2T the standard curvature term $q(R)$ acts essentially as the so called curvature operator and its symmetric counterpart:

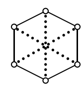
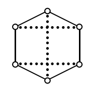
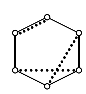
$$\begin{array}{rccccl} \mathscr{R}^{\mathrm{alt}}: & \Lambda^2T & \longrightarrow & \Lambda^2T\,, & X\wedge Y & \longmapsto\;-\,R_{X,Y} \\ \mathscr{R}^{\mathrm{sym}}: & \operatorname{Sym}^2T & \longrightarrow & \operatorname{Sym}^2T, & X\cdot Y & \longmapsto\;+\operatorname{Sec}_{X,Y}\ . \end{array}$$

In fact we find say for the adjoint or alternating bivector representation $\mathfrak{so}(T,\,g)\cong\Lambda^2T$

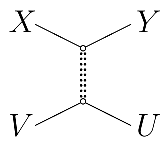
$$\begin{aligned} &q(\,R\,)\star(\,X\wedge Y\,) \\ &=\;\operatorname{Der}_{\frac{1}{2}\operatorname{Ric}}(\,X\wedge Y\,)+\frac{1}{2}\sum_{\mu,\,\nu=1}^{m}\Big(g(dE^\sharp_\mu,X)dE^\sharp_\nu\wedge R_{E_\mu,\,E_\nu}Y\;+\;R_{E_\mu,\,E_\nu}X\wedge g(dE^\sharp_\mu,Y)dE^\sharp_\nu\Big) \\ &=\;\operatorname{Der}_{\frac{1}{2}\operatorname{Ric}}(\,X\wedge Y\,)+\frac{1}{2}\sum_{\nu=1}^{m}dE^\sharp_\nu\wedge\Big(\,R_{X,\,E_\nu}Y\;-\;R_{Y,\,E_\nu}X\,\Big) \\ &=\;\frac{1}{2}\operatorname{Der}_{\operatorname{Ric}}(\,X\wedge Y\,)\;-\;\mathscr{R}^{\mathrm{alt}}(\,X\wedge Y\,)\ , \end{aligned}$$

where $\mathrm{Der}_{\frac{1}{2}\mathrm{Ric}}$ denotes the derivation extension of the Ricci endomorphism of T to an endomorphism of $\Lambda^2 T$ via $\mathrm{Der}_{\frac{1}{2}\mathrm{Ric}}(\,X \wedge Y\,) \;:=\; \frac{1}{2}\,\mathrm{Ric}\,X \wedge Y \;+\; \frac{1}{2}\,X \wedge \mathrm{Ric}\,Y$. Replacing the first Bianchi identity $R_{X,\,E_\nu}Y - R_{Y,\,E_\nu}X \;=\; R_{X,\,Y}E_\nu$ used in this argument by the variant $R_{X,\,E_\nu}Y + R_{Y,\,E_\nu}X \;=\; -\frac{1}{2}\,\mathrm{Sec}(X, Y;\, E_\nu,\,\cdot\,)^\sharp$ of the definition of the sectional curvature tensor $\mathrm{Sec} \;:=\; \Phi^+ R$ we find analogously $q(\,R\,)\star \;=\; \frac{1}{2}\,\mathrm{Der}_{\mathrm{Ric}} \;-\; \frac{1}{2}\,\mathscr{R}^{\mathrm{sym}}$ on the symmetric bivector representation $\mathrm{Sym}^2 T$. Let us now study the action of $q(\,R\,)$ on $R \,\in\, \mathrm{Curv}^- T$ itself:

Lemma 6.2 (Standard Curvature Term Polynomial)
Consider an algebraic curvature tensor $R \,\in\, \mathrm{Curv}^- T$ on a euclidean vector space T with scalar product g. The scalar product of $q(R) \star R$ with R equals the stable cubic polynomial:

$\Theta_3(\,R\,) \;:=\; g_{\Lambda^2 T^* \otimes \Lambda^2 T^*}(\,q(\,R\,) \star R,\; R\,) \;=\; \frac{1}{12}\Big[\frac{1}{6}$  $-\;\frac{2}{3}$  $+\;\frac{1}{2}$  $\Big](\,R\,)\,.$

Proof: Without discussing the details of the construction of such an algebra we consider an extended version of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs similar to the extended graph algebra $\mathbb{A}^\bullet_{\mathrm{ext}}$ discussed at the end of Section 3, in which a pair of vertices may be connected by a single or a double red edge with the associated additional Feynman rule:



$$\widehat{=}\;\;\Big(q(R) \,\star\, \mathrm{Sec}\Big)(\,X,\; Y;\; U,\; V\,)\;. \tag{44}$$

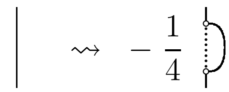
In order to expand elements of such an extended graph algebra into elements of the algebra $\mathbb{A}^\bullet$ of colored trivalent graphs we expand the additional quartic interaction into the sum

$$\begin{aligned}
&\Big(q(R) \,\star\, \mathrm{Sec}\Big)(\,X,\, Y;\, U,\, V\,)\\
&\quad= \sum_{\mu,\,\nu,\,\alpha,\,\beta\,=\,1}^{m} \mathrm{Sec}(E_\mu,\, E_\nu;\, E_\alpha,\, E_\beta)\, g\Big(q(R) \star (X \otimes Y \otimes U \otimes V),\; dE^\sharp_\mu \otimes dE^\sharp_\nu \otimes dE^\sharp_\alpha \otimes dE^\sharp_\beta\Big)
\end{aligned}$$

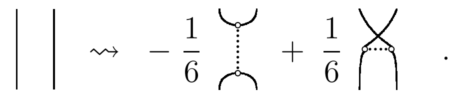
over an arbitrary pair of dual bases $\{\,E_\mu\,\}$ and $\{\,dE_\mu\,\}$. Representations of Lie algebras extend in general as derivations to the tensor algebra $\bigotimes T$ associated to T, hence the quadratic element $q(R) \,\in\, \mathscr{U}^{\leq 2}\mathfrak{so}(\,T,\, g\,)$ can be seen as a second order differential operator in the sense that its action on $\bigotimes T$ is determined by its action on the tensor square $\bigotimes^2 T$ with

$$\begin{aligned}
&g(\; q(R) \star (\,X \,\otimes\, Y\,),\; U \,\otimes\, V\,)\\
&\quad=\;\; g(\,\mathrm{Der}_{\frac{1}{2}\,\mathrm{Ric}}(\,X \,\otimes\, Y\,),\; U \,\otimes\, V\,) \;+\; \sum_{\mu,\,\nu\,=\,1}^{m} g(\,dE_\mu(X)\, dE^\sharp_\nu \,\otimes\, R_{E_\mu,\,E_\nu}Y,\; U \,\otimes\, V\,)\\
&\quad=\;\; g(\,\mathrm{Der}_{\frac{1}{2}\,\mathrm{Ric}}(\,X \,\otimes\, Y\,),\; U \,\otimes\, V\,) \;-\; \frac{1}{6}\,\mathrm{Sec}(\,X,\, Y;\, U,\, V\,) \;+\; \frac{1}{6}\,\mathrm{Sec}(\,X,\, V;\, Y,\, U\,)
\end{aligned}$$

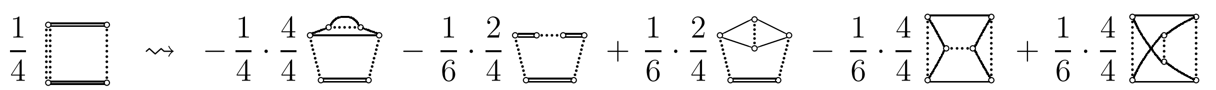
for all $X,\, Y,\, U,\, V \,\in\, T$ using $R \;=\; \Phi^-\mathrm{Sec}$. In consequence the full expansion of a double red edge (44) into ordinary colored trivalent graphs results in a sum of 16 terms grouped into two sums of 4 and 12 terms respectively. In the first of these two sums we replace the double by a single red edge and in addition each of the four adjacent black flags in turn by



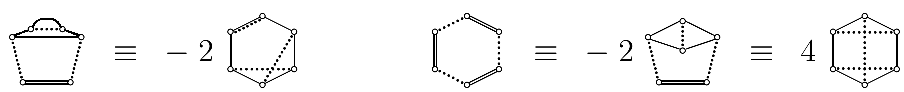
in order to implement $\mathrm{Der}_{\frac{1}{2}\mathrm{Ric}}$ via equation (26). In the second sum we replace similarly the double by a single red edge and moreover each of the six pairs of adjacent flags in turn by:



Implementing this expansion in the special case relevant to Lemma 6.2 we find without effort:

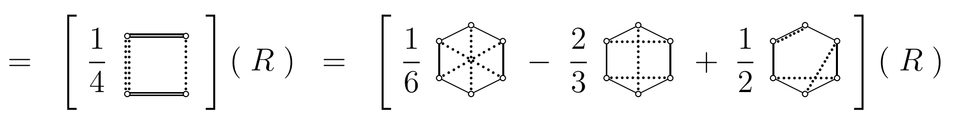


Interpreting the left hand side and simplifying the right hand side using the congruences



modulo the ideal of IHX–relations we eventually obtain the formula:

$$g_{\mathrm{Sym}^2T^*\otimes\mathrm{Sym}^2T^*}(\, q(\,R\,)\star\mathrm{Sec},\ \mathrm{Sec}\,)$$



In light of the isometry equation (31) this result is equivalent to the stipulated formula for the stable cubic polynomial $g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,q(R)\star R,\ R\,)$, after all the mutually inverse isomorphisms Φ^+ and Φ^- of Section 2 are equivariant under the action of the orthogonal group and thus commute $q(R)\star\Phi^-\mathrm{Sec} = \Phi^-(\,q(R)\star\mathrm{Sec}\,)$ with the standard curvature term. $\square$

The stable cubic curvature invariant Θ_3 of Lemma 6.2 arises naturally in the study of the curvature tensor $R\in\Gamma(\,\mathrm{Curv}^-TM\,)$ of Riemannian manifolds M with parallel Ricci curvature $\mathrm{Ric}\in\Gamma(\,\mathrm{Sym}^2T^*M\,)$, a class of Riemannian manifolds slightly larger than the class of Einstein manifolds of dimension $m\geq 3$ [2]. In order to relate Θ_3 to the Ricci curvature we compose the symmetrized second covariant derivative $\nabla^{[2]}_{X,Y} := \frac{1}{2}(\nabla^2_{X,Y}+\nabla^2_{Y,X})$ with the Nomizu–Kulkarni product $\times$ of equation (12) to obtain a second order differential operator:

$$\mathrm{Cross}:\ \Gamma(\,\mathrm{Sym}^2T^*M\,)\ \stackrel{\nabla^{[2]}}{\longrightarrow}\ \Gamma(\,\mathrm{Sym}^2T^*M\otimes\mathrm{Sym}^2T^*M\,)\ \stackrel{\times}{\longrightarrow}\ \Gamma(\,\mathrm{Curv}^-TM\,)\ .$$

This cross operator is a fundamental differential operator in Riemannian geometry, because it is in essence the linearization of the second order non–linear differential operator, which sends a Riemannian metric $g \in \Gamma(\,\mathrm{Sym}^2_{\mathrm{reg}}T^*M\,)$ to its curvature tensor $R \in \Gamma(\,\mathrm{Curv}^-TM\,)$, more precisely the curvature tensor varies under an arbitrary variation of the metric according to

$$\delta R \;=\; \frac{1}{2}\,\mathrm{Cross}\,\delta g \;+\; \frac{1}{4}\,\mathrm{Der}_{[\,\delta g\,]}R\;, \tag{45}$$

where $[\,\delta g\,]$ denotes the symmetric endomorphism field defined by $g([\,\delta g\,]X, Y) \;:=\; \delta g(X, Y)$.

Lemma 6.3 (Laplacian of Curvature Tensor [9])
The curvature tensor $R \in \Gamma(\,\mathrm{Curv}^-TM\,)$ of every Riemannian manifold M satisfies:

$$\nabla^*\nabla R \;+\; q(\,R\,) \star R \;\;=\;\; \mathrm{Cross}\;\mathrm{Ric}\;.$$

In particular the L^2–norm of the covariant derivative $\nabla R \in \Gamma(\,T^*M \otimes \mathrm{Curv}^-TM\,)$ of the curvature tensor of a compact Riemannian manifold M with parallel Ricci tensor equals:

$$\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*} \;\;=\;\; -\int_M g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,q(\,R\,) \star R,\; R\,)\,|\,\mathrm{vol}_g\,|\;.$$

Proof: Using the second Bianchi identity six times and keeping all the symmetries of algebraic curvature tensors self evident we obtain for every local basis $E_1, \ldots, E_m$:

$$\begin{aligned}
&(\,\nabla^*\nabla R\,)(\,X, Y; U, V\,)\\
&\quad=\; -\frac{1}{2}\sum_{\mu=1}^m\Big(\;+\;(\,\nabla^2_{E_\mu,\,dE^\sharp_\mu}R\,)(\,X, Y; U, V\,) \;+\; (\,\nabla^2_{E_\mu,\,dE^\sharp_\mu}R\,)(\,U, V; X, Y\,)\,\Big)\\
&\quad=\; +\frac{1}{2}\sum_{\mu=1}^m\Big(\;+\;(\,\nabla^2_{E_\mu,X}R\,)(\,Y, dE^\sharp_\mu; U, V\,) \;-\; (\,\nabla^2_{E_\mu,Y}R\,)(\,X, dE^\sharp_\mu; U, V\,)\\
&\qquad\qquad\qquad +\,(\,\nabla^2_{E_\mu,U}R\,)(\,V, dE^\sharp_\mu; X, Y\,) \;-\; (\,\nabla^2_{E_\mu,V}R\,)(\,U, dE^\sharp_\mu; X, Y\,)\,\Big)\\
&\quad=\; +\frac{1}{2}\sum_{\mu=1}^m\Big(\;+\;(\,R_{E_\mu,X}R\,)(\,Y, dE^\sharp_\mu; U, V\,) \;+\; (\,\nabla^2_{X,\,E_\mu}R\,)(\,U, V; Y, dE^\sharp_\mu\,)\\
&\qquad\qquad\qquad -\,(\,R_{E_\mu,Y}R\,)(\,X, dE^\sharp_\mu; U, V\,) \;-\; (\,\nabla^2_{Y,\,E_\mu}R\,)(\,U, V; X, dE^\sharp_\mu\,)\\
&\qquad\qquad\qquad +\,(\,R_{E_\mu,U}R\,)(\,V, dE^\sharp_\mu, X, Y\,) \;+\; (\,\nabla^2_{U,\,E_\mu}R\,)(\,X, Y; V, dE^\sharp_\mu\,)\\
&\qquad\qquad\qquad -\,(\,R_{E_\mu,V}R\,)(\,U, dE^\sharp_\mu, X, Y\,) \;+\; (\,\nabla^2_{V,\,E_\mu}R\,)(\,X, Y; U, dE^\sharp_\mu\,)\,\Big)\\
&\quad=\; +\frac{1}{2}\sum_{\mu=1}^m\Big(\;+\;(\,\nabla^2_{X,U}R\,)(\,V, E_\mu; dE^\sharp_\mu, Y\,) \;-\; (\,\nabla^2_{X,V}R\,)(\,U, E_\mu; dE^\sharp_\mu, Y\,)\\
&\qquad\qquad\qquad -\,(\,\nabla^2_{Y,U}R\,)(\,V, E_\mu; dE^\sharp_\mu, X\,) \;+\; (\,\nabla^2_{Y,V}R\,)(\,U, E_\mu; dE^\sharp_\mu, X\,)\\
&\qquad\qquad\qquad +\,(\,\nabla^2_{U,X}R\,)(\,Y, E_\mu; dE^\sharp_\mu, V\,) \;-\; (\,\nabla^2_{U,Y}R\,)(\,X, E_\mu; dE^\sharp_\mu, V\,)\\
&\qquad\qquad\qquad -\,(\,\nabla^2_{V,X}R\,)(\,Y, E_\mu; dE^\sharp_\mu, U\,) \;+\; (\,\nabla^2_{V,Y}R\,)(\,X, E_\mu; dE^\sharp_\mu, U\,)\,\Big)\\
&\qquad\quad -\frac{1}{2}\sum_{\mu=1}^m\Big(\;+\;(\,R_{E_\mu,X}R\,)(\,dE^\sharp_\mu, Y; U, V\,) \;+\; (\,R_{E_\mu,Y}R\,)(\,X, dE^\sharp_\mu; U, V\,)\\
&\qquad\qquad\qquad +\,(\,R_{E_\mu,U}R\,)(\,X, Y; dE^\sharp_\mu, V\,) \;+\; (\,R_{E_\mu,V}R\,)(\,X, Y; U, dE^\sharp_\mu\,)\,\Big)\;.
\end{aligned}$$

Evidently the first sum on the right hand side is just Cross Ric, note in particular that the symmetrized iterated covariant derivatives $\nabla^{[2]}$ appear naturally in this calculation. We leave it to the reader to verify that the second sum on the right hand side equals $-q(R) \star R$. $\square$

Combining Lemma 6.3 with the interpretation (45) of the cross operator as the linearization of the second order differential operator $\Gamma(\,\mathrm{Sym}^2_{\mathrm{reg}}T^*M\,) \longrightarrow \Gamma(\,\mathrm{Curv}^-TM\,)$, $g \longmapsto R$, we obtain the central formula of Hamilton's theory [4] for the Ricci Flow $\delta g := \pm 2\,\mathrm{Ric}$:

$$\delta R \;=\; \pm\Big(\,\mathrm{Cross\ Ric} \;+\; \frac{1}{2}\,\mathrm{Der}_{\mathrm{Ric}}R\,\Big) \;=\; \pm\Big(\,\nabla^*\nabla R \;+\; q(\,R\,)\star R \;+\; \frac{1}{2}\,\mathrm{Der}_{\mathrm{Ric}}R\,\Big)\;.$$

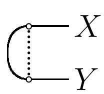
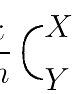
Comparing this formulation with the usual formulation of Hamilton's theory [4] we find

$$q(\,R\,)\,\star\,R \;=\; \frac{1}{2}\,\mathrm{Der}_{\mathrm{Ric}}R \;+\; Q(\,R\,) \tag{46}$$

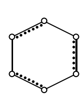
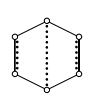
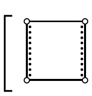
in terms of Hamilton's quadratic curvature expression $Q(\,R\,)$ defined by:

$$\begin{aligned} &Q(\,R\,)(\,X,\,Y;\,U,\,V\,)\\ &\quad:=\; 2\,g_{\Lambda^2T}(\,R_{X,Y},\,R_{U,V}\,) \;+\; 2\sum_{\mu=1}^{m}\Big(\,g(\,R_{E_\mu,X}U,\,R_{dE^\sharp_\mu,Y}V\,) \;-\; g(\,R_{E_\mu,X}V,\,R_{dE^\sharp_\mu,Y}U\,)\,\Big)\;.\end{aligned}$$

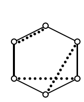
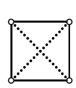
In order to illustrate the power of the graphical calculus developed in this article we want to discuss an interesting identity of stable curvature invariants of Einstein manifolds related to Lemma 6.3. Recall first of all that an algebraic curvature tensor of Einstein type is an algebraic curvature tensor $R \in \mathrm{Curv}^-T$ on a euclidean vector space T of dimension $m \in \mathbb{N}$ whose associated Ricci tensor $\mathrm{Ric} \;=\; \frac{\kappa}{m}\,g$ is a multiple of the metric g. On the subspace $\mathrm{Ein}(\,T,\,g\,) \subseteq \mathrm{Curv}^-T$ of curvature tensors of Einstein type the Ricci identity (26)


$$\;\widehat{=}\; -\,2\,\frac{\kappa}{m}\,g(\,X,\,Y\,) \;\widehat{=}\; -\,2\,\frac{\kappa}{m}$$


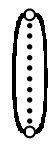
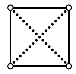
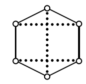
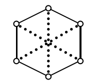
becomes an algebraic simplification on the level of the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ reducing the number of vertices at the expense of introducing κ as a new independent variable. Generators of the reduced graph algebra $\overline{\mathbb{A}}^\bullet$ with a pair of parallel red and black edges are thus redundant when restricted to $\mathrm{Ein}(\,T,\,g\,)$, for the first two of the generators (27) of degree 3 we find say

$\Big[$  $\Big](\,R\,) \;\stackrel{!}{=}\; \Big[$  $\Big](\,R\,) \;=\; -\,2\,\frac{\kappa}{m}\;\Big[$  $\Big](\,R\,) \;=\; -\,8\,\frac{\kappa^3}{m^2}$ (47)

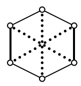
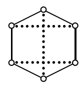
for every algebraic curvature tensor $R \in \mathrm{Ein}(\,T,\,g\,)$ of Einstein type in light of equation (42) in combination with $g_{\mathrm{Sym}^2T^*}(\frac{\kappa}{m}g,\,\frac{\kappa}{m}g) \;=\; \frac{\kappa^2}{2m}$. Similarly the third generator reduces to:

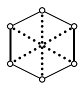
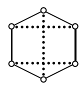
$\Big[$  $\Big](\,R\,) \;=\; -\,2\,\frac{\kappa}{m}\;\Big[$  $\Big](\,R\,) \;=\; +\,48\,\frac{\kappa}{m}\,g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,R,\,R\,)\;.$ (48)

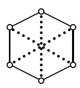
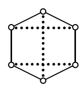
Effectively we are thus left with 4 instead of the 8 generators (27) of degree up to 3, namely

 $\widehat{=}\ -\ 2\,\kappa$  $\widehat{=}\ -\ 24\, g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,R,\,R\,)$   ,

when discussing stable curvature invariants of algebraic curvature tensors of Einstein type. In consequence there exists up to scale a unique linear combination of the Pfaffian pf_3, the normalized moment polynomial Ψ_3° and the cubic polynomial Θ_3 defined in Lemma 6.2

$\mathrm{pf}_3(R) \;=\; \Big[\,-\,\frac{1}{432}$  $-\,\frac{5}{432}$  $\Big](\,R\,)\;+\;\kappa^3\,\frac{m^2-12m+40}{48\,m^2}\;+\;\kappa\,\frac{m-8}{4\,m}\,|\,R\,|^2$

$\Psi_3^\circ(R) \;=\; \Big[\,-\,\frac{1}{6}$  $-\,\frac{1}{6}$  $\Big](\,R\,)\;+\;\kappa^3\,\frac{m^2+12m+40}{6\,m^2}\;+\;\kappa\,\frac{6\,(m+8)}{m}\,|\,R\,|^2$

$\Theta_3(R) \;=\; \Big[\,+\,\frac{1}{72}$  $-\,\frac{1}{18}$  $\Big](\,R\,)\;+\;\kappa\,\frac{2}{m}\,|\,R\,|^2$

in which both additional cubic generators are eliminated, the resulting identity reads:

$$\Big(\,\mathrm{pf}_3\;-\;\frac{1}{40}\,\Psi_3^\circ\;-\;\frac{2}{15}\,\Theta_3\,\Big)(\,R\,)\;\;=\;\;\kappa^3\,\frac{m^2-18m+40}{60\,m^2}\;+\;\kappa\,\frac{3m-104}{30\,m}\,g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,R,\,R\,)\;.$$

According to Lemma 6.3 the cubic curvature invariant $\Theta_3(\,R\,)\;:=\;-g_{\Lambda^2T^*\otimes\Lambda^2T^*}(\,q(R)\star R,\,R\,)$ integrates for a compact Einstein manifold M to the L^2–norm $\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*}$ of the covariant derivative of the curvature tensor, in this way we have proved:

Theorem 6.4 (Cubic Curvature Identity for Einstein Manifolds)
For every compact connected Einstein manifold M of dimension $m\,\geq\,3$ with scalar curvature $\kappa\,\in\,\mathbb{R}$ the following identity of integrated stable curvature invariants of degree 3 holds true:

$$\begin{aligned}\int_M\mathrm{pf}_3(\,R\,)\,|\,\mathrm{vol}_g\,|\;-\;\frac{1}{40}\int_M\Psi_3^\circ(\,R\,)\,|\,\mathrm{vol}_g\,|\;+\;\frac{2}{15}\,\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*}&\\ =\;\;\kappa^3\,\frac{m^2-18m+40}{60\,m^2}\,\mathrm{Vol}(\,M,\,g\,)\;+\;\kappa\,\frac{3m-104}{30\,m}\,\|\,R\,\|^2_{\Lambda^2T^*\otimes\Lambda^2T^*}&\end{aligned}$$

Specifically in dimension $m\,=\,5$ this identity reduces via $\Psi_3^\circ\,=\,10080\,\Psi_3$ to the identity

$$-\,252\int_M\Psi_3(R)\,|\,\mathrm{vol}_g\,|+\frac{2}{15}\,\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*}\;\;=\;\;-\,\frac{\kappa^3}{60}\,\mathrm{Vol}(\,M,\,g\,)-\frac{89\,\kappa}{150}\,\|\,R\,\|^2_{\Lambda^2T^*\otimes\Lambda^2T^*}\;,$$

because $\mathrm{pf}_3(\,R\,)\,=\,0$ *due to Remark 5.2, whereas is becomes for* $m\,=\,6$ *with* $\Psi_3^\circ\,=\,25200\,\Psi_3$*:*

$$\begin{aligned}(\,2\pi\,)^3\,\chi(\,M\,)\;-\;630\int_M\Psi_3(\,R\,)\,|\,\mathrm{vol}_g\,|\;+\;\frac{2}{15}\,\|\,\nabla R\,\|^2_{T^*\otimes\Lambda^2T^*\otimes\Lambda^2T^*}&\\ =\;\;-\,\frac{2\,\kappa^3}{135}\,\mathrm{Vol}(\,M,\,g\,)\;-\;\frac{43\,\kappa}{90}\,\|\,R\,\|^2_{\Lambda^2T^*\otimes\Lambda^2T^*}\;.&\end{aligned}$$

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