

Conserved currents in five-dimensional proposals for lattice chiral gauge theories

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We apply the Grabowska–Kaplan framework [1], originally proposed for lattice chiral gauge theories, to QCD. We show that the resulting theory contains a conserved and gauge invariant singlet axial current, both on the lattice and in the continuum limit. This must give rise to a difference with QCD, with the simplest possibility being a superfluous Nambu–Goldstone boson in the physical spectrum not present in QCD. We find a similar unwanted conserved current in the recent “disk” formalism [2, 3], this time limiting ourselves to the continuum formulation. A similar problem is expected when either of these formalisms is used for its original goal of constructing lattice chiral gauge theories. Finally we discuss a conjecture about the possible dynamics that might be associated with the unwanted conserved current, and the fate of ’t Hooft vertices.

I. INTRODUCTION

The original goal of Kaplan’s domain-wall fermions (DWFs) was to construct lattice chiral gauge theories [4]. Much later, a concrete realization of the original idea was proposed by Grabowska and Kaplan [1]. As in the standard lattice formulation of QCD using DWFs [5, 6], the basic geometry can be taken to be a five-dimensional “slab,” with Weyl fermions of opposite chiralities residing on the two four-dimensional boundaries.

In QCD, the four-dimensional gauge field is taken to be independent of the fifth coordinate. The Weyl fermions on both boundaries (or “walls”) couple to the gauge field with equal strength, thereby forming Dirac fermions: one Dirac fermion per each five-dimensional DWF field. By contrast, in the Grabowska–Kaplan framework the gauge field inside the five-dimensional (5D) slab is defined via gradient flow [7] in the fifth direction, with the dynamical four-dimensional (4D) gauge field on the near wall serving to start the flow. The goal is that, while preserving gauge invariance, the gauge field will die out well before reaching the Weyl fermion on the far wall, thereby decoupling it from the gauge field. If successful, then only the 4D Weyl fermion on the near wall of each 5D slab would remain coupled to the gauge field. All these Weyl fermions can be chosen to have the same handedness, and the construction could thus be used for a non-perturbative, gauge invariant definition of chiral gauge theories in four dimensions.

Recently, another proposal was put forward by Kaplan [2] and by Kaplan and Sen [3]. In the new proposal the slab geometry is replaced by a disk geometry, and the rim of the disk is identified with one of the four physical dimensions.¹ Remarkably, the rim supports a single Weyl fermion of one chirality only [2]. Moreover, this feature appears to survive lattice discretization [3], thereby circumventing the no-go theorems [8, 9]. By construction, the dynamical gauge field resides on the rim of the disk, and is extended into the whole disk via a radial gradient flow, once again maintaining gauge invariance.

In both the slab and disk geometries, it was argued that a necessary condition that the degrees of freedom in the extra dimension will fully decouple in the infrared is that the (chiral) fermion spectrum of the target 4D gauge theory satisfy the anomaly cancellation condition for the gauge symmetry. By contrast, if the fermion spectrum suffers from a gauge anomaly, then the effective low-energy 4D theory will remain non-local. Thus, both formalisms pass an important consistency test.

The question arises whether there are other possible stumbling blocks for the successful construction of lattice chiral gauge theories. In order to examine this question, here we turn our attention to the global flavor symmetries and their associated conserved currents. Our main finding is that, in both the slab and disk geometries, there is always one “superfluous” 4D current which is both gauge invariant and conserved, and which is not present in the target gauge theory. In order to highlight the persistence of this problem, we examine the application of the slab and disk frameworks for an alternative lattice definition of vector-like theories, taking one-flavor QCD as the target 4D theory for our main example. In the case of a vector-like theory, the unwanted conserved current is the singlet axial current, which should be anomalous. In the case that the target gauge theory is chiral, the additional conserved current is the total fermion number, which, once again, should be anomalous.

In Sec. II we consider the slab geometry, focusing mainly on one-flavor QCD to demonstrate the issue. We work on the lattice, and thus our conclusions apply both at finite

¹ The remaining three physical dimensions are cartesian. While we focus on $4 + 1$ dimensions in this paper, the discussion generalizes straightforwardly to $2 + 1$ dimension.

lattice spacing and in the continuum limit. In Sec. III we discuss the novel disk framework, limiting ourselves to the continuum formulation. Our findings are similar to those of the slab framework. In Sec. IV we conjecture on the dynamics that might be associated with the superfluous conserved axial current, focusing on the fate of the 't Hooft vertices of the target 4D theory. We conclude in Sec. V. In the appendix we discuss lattice gradient flows.

II. CONSERVED CURRENTS IN THE GRABOWSKA–KAPLAN FRAMEWORK

We discuss here the application of the Grabowska–Kaplan (GK) formalism [1] to QCD, using the one-flavor theory as an example. The main conclusion is that the GK framework gives rise to a singlet axial current which is simultaneously gauge invariant and conserved. The straightforward interpretation is that this, in turn, gives rise to a superfluous Nambu–Goldstone boson (NGB) in the physical spectrum.

We begin with a lattice setup containing two DWFs. In the standard DWF formulation of QCD [5, 6], this provides for the fermion content of the two-flavor theory. However, within the GK framework only the Weyl fermion on the near wall of each DWF is expected to couple to the gauge field. This will leave us with a total of two Weyl fermions (of opposite chiralities), in agreement with the fermion content of the one-flavor theory.

In more detail, the lattice theory contains two five-dimensional GK fields, both in the fundamental representation of $SU(3)$. The fifth coordinate takes values $s = 0, 1, \dots, N_5 - 1$. We assume that one of the GK fields, denoted $\Psi^{(R)}$, has a right-handed (RH) Weyl field on the near wall, while the other, denoted $\Psi^{(L)}$, has a left-handed (LH) Weyl field on the near wall. Taken together, we thus have a single Dirac fermion, the matter content of one-flavor QCD. The (bare) quark fields are identified with the 5D fields on the $s = 0$ layer, namely, $\psi_R(x) = \Psi^{(R)}(x, s = 0)$ and $\psi_L(x) = \Psi^{(L)}(x, s = 0)$. For simplicity, we assume that the quark mass is zero.

In practice, flipping the chirality of the Weyl fermion on the near wall is done by flipping the sign of γ_5 everywhere in the lattice action.² Therefore, there is no continuous symmetry that interchanges the two 5D fields $\Psi^{(R)}$ and $\Psi^{(L)}$. However, each 5D field is endowed with an exact $U(1)$ symmetry that acts on that field only. Following closely Refs. [6, 10], the corresponding Noether currents are

$$\mathcal{R}_\mu(x) = \frac{1}{2} \sum_{s=0}^{N_5-1} \left(\overline{\Psi}_{x,s}^{(R)} (1 + \gamma_\mu) U_{x,s,\mu} \Psi_{x+\hat{\mu},s}^{(R)} - \overline{\Psi}_{x+\hat{\mu},s}^{(R)} (1 - \gamma_\mu) U_{x,s,\mu}^\dagger \Psi_{x,s}^{(R)} \right), \quad (2.1a)$$

$$\mathcal{L}_\mu(x) = \frac{1}{2} \sum_{s=0}^{N_5-1} \left(\overline{\Psi}_{x,s}^{(L)} (1 + \gamma_\mu) U_{x,s,\mu} \Psi_{x+\hat{\mu},s}^{(L)} - \overline{\Psi}_{x+\hat{\mu},s}^{(L)} (1 - \gamma_\mu) U_{x,s,\mu}^\dagger \Psi_{x,s}^{(L)} \right). \quad (2.1b)$$

These currents are both gauge invariant and conserved. Notice that unlike in the standard DWF formulation of QCD, here the four-dimensional link variables depend on the fifth coordinate s via the gradient flow.

We may alternatively construct a vector and an axial current,

$$\mathcal{V}_\mu = \mathcal{R}_\mu + \mathcal{L}_\mu, \quad (2.2a)$$

$$\mathcal{A}_\mu = \mathcal{R}_\mu - \mathcal{L}_\mu. \quad (2.2b)$$

² In the standard DWF formulation of QCD, this operation can be undone by a reflection in the fifth dimension, because the gauge field is independent of the fifth coordinate.

These currents, too, are gauge invariant and conserved. The $U(1)$ symmetry associated with the vector current $\mathcal{V}_\mu$ rotates ψ_R and ψ_L with the same phase. This is baryon number symmetry, a good symmetry of (one-flavor) QCD. The other $U(1)$ symmetry, associated with the current $\mathcal{A}_\mu$, rotates ψ_R and ψ_L with opposite phases; this is the axial symmetry, which is anomalous in QCD. The problem is thus that in the GK framework the axial symmetry is an exact symmetry, too. As a result, also the (singlet) axial current $\mathcal{A}_\mu$ is both conserved and gauge invariant within the GK framework. This generates a conflict between the standard properties of (one-flavor) QCD, and the features of its GK formulation. If the GK framework for regulating QCD leads to a local and Lorentz invariant continuum limit, that continuum limit must be different from the one obtained from any of the standard lattice regularizations of QCD.

Let us elaborate on this conflict. In any standard formulation of QCD, the axial current A_μ is anomalous,

$$\partial_\mu A_\mu = X, \quad X = cg^2 \text{tr}(F\tilde{F}), \quad (2.3)$$

with c some non-vanishing numerical constant. This gives rise to anomalous Ward–Takahashi identities (WTIs). Consider for example the momentum-space WTI (in the continuum and chiral limits)

$$ip_\mu \langle A_\mu \eta \rangle(p) = \langle X \eta \rangle(p) + \Sigma, \quad (2.4)$$

where $\eta = \bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L$ is the pseudoscalar density, and $\Sigma = \langle \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \rangle$ is the fermion condensate. Thanks to the anomalous term, this WTI does not require the existence of any massless particle when $\Sigma \neq 0$, consistent with the large mass of the η' meson in QCD.

By contrast, within the GK formulation of the one-flavor theory, the gauge invariant current $\mathcal{A}_\mu$ has no anomaly, and, after taking the continuum limit, we obtain the WTI³

$$ip_\mu \langle \mathcal{A}_\mu \eta \rangle(p) = \Sigma. \quad (2.5)$$

Note that now Σ is a true order parameter, since the axial symmetry is exact in the GK formulation. Assuming that chiral symmetry breaking takes place and $\Sigma \neq 0$, the (euclidean space) correlator $\langle \mathcal{A}_\mu \eta \rangle$ must now have a massless pole in order to saturate this WTI. After the analytic continuation to Minkowski space, if the theory is local and Lorentz invariant, so that its two-point correlation functions admit a spectral representation, then the massless pole is recognized as a NGB. This new NGB, being a singlet pseudoscalar meson, would signal a breakdown of universality. This is our main conclusion.

As discussed in Ref. [11], it is possible to split the current $\mathcal{A}_\mu$ into several pieces, one of which will behave as the—anomalous—axial current of QCD. Nonetheless, the existence within the GK framework of the axial current $\mathcal{A}_\mu$, which is both gauge invariant and conserved, means that there is no escape from the WTI (2.5), and its consequences for the physics of the theory.

The existence of a singlet pseudoscalar NGB in the physical spectrum, which is absent from the standard formulation of QCD, is a completely general phenomenon within the GK formulation. Generalizing the lattice setup to QCD with N_f flavors, it is easy to see that the global symmetry of the GK formulation will be $U(N_f)_L \times U(N_f)_R$, and not $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ as expected. Once again, we will have a singlet axial current which is both gauge invariant and conserved, and not anomalous. Again, assuming that the theory confines and breaks its chiral symmetry, and that it has a consistent continuum

³ At finite lattice spacing, p_μ is replaced by $(2/a) \sin(ap_\mu/2)$.

limit, then, in addition to the expected pions for $N_f \geq 2$, there will be a superfluous, singlet pseudoscalar NGB.

The problem persists when our goal is to construct a chiral gauge theory. A chiral gauge theory in four dimensions can be formulated in terms of LH fields only, and the total fermion number current is then always anomalous. But within the GK formulation, again there is a conserved and gauge invariant current associated with the total fermion number.

The violation of the global axial charge in vector-like theories, and of the total fermion number in chiral gauge theories, is believed to arise from 't Hooft vertices. In Sec. IV we discuss a conjecture about the fate of instantons and 't Hooft vertices within the GK formulation.

III. CONSERVED CURRENTS IN THE DISK FRAMEWORK

It is clear from the discussion of the previous section that the superfluous current, which is simultaneously conserved and gauge invariant, originates from the existence of a fermion number symmetry for each 5D field separately. This implies the existence of a conserved and gauge invariant 5D current for each GK field, from which one can construct a 4D conserved current. One linear combination of the 4D conserved currents will always be superfluous, as explained in the previous section.

In this section we turn to the disk framework, and demonstrate the existence of a similar, superfluous four-dimensional current. One can envisage various ways of discretizing the disk framework. For example, it is fairly obvious that there exist lattice discretizations that will preserve a discrete subgroup of the rotational symmetry of the disk. By contrast, in Ref. [3], a discretization based on a “trimmed” regular hypercubic lattice was preferred. In view of these rather different options for the lattice discretization, we will limit the discussion in this section to the continuum case only.

A. The conserved four-dimensional current

We first briefly introduce the disk framework. We will mostly disregard the gauge field, since it plays no role in the (classical) conservation equation. We stress, however, that in the presence of the gauge field, the current is gauge invariant. Moreover, since continuous symmetries do not have anomalies in odd spacetime dimensions, the conservation of the gauge invariant 5D current holds to all orders in perturbation theory. A similar conservation equation holds for the 5D lattice discretization of Ref. [3] as well.

In the (x, y) plane, the fermions are restricted to a disk of radius R . In addition, there are three cartesian coordinates, denoted z_i , $i = 1, 2, 3$. The 5D fermion field satisfies boundary conditions defined in terms of the radial projectors $P_{\pm}^r = \frac{1}{2}(1 \pm \gamma_r)$, where $\gamma_r = \gamma_x \cos \theta + \gamma_y \sin \theta$. As was shown in Refs. [2, 3], this construction gives rise to a single Weyl fermion on the rim of the disk. Hence, the rim of the disk is identified with the fourth ordinary dimension, which thus has a finite length $L = 2\pi R$ and periodic boundary conditions. It is described by the coordinate $R\theta$, with $0 \leq \theta < 2\pi$. The radial direction of the disk, with coordinate $0 \leq r \leq R$, corresponds to the fifth direction of the slab geometry.⁴

⁴ The limit $R \rightarrow \infty$ in the disk geometry corresponds for the slab geometry to simultaneously taking the limits $N_5 \rightarrow \infty$ together with the limit $L \rightarrow \infty$ for the cartesian direction identified with the rim of the disk.

The conservation equation for the Noether current of the U(1) symmetry of a given 5D fermion field takes the form

$$0 = \sum_i \partial_i j_i + \partial_x j_x + \partial_y j_y = \sum_i \partial_i j_i + \frac{1}{r} \partial_r (r j_r) + \frac{1}{r} \partial_\theta j_\theta . \quad (3.1)$$

The middle expression is the divergence of the 5D current in cartesian coordinates, while in the rightmost expression we switched to radial coordinates for the (x, y) plane containing the disk. j_r (j_θ) is the component of the current in the radial (tangential) direction. Like the fermion field itself, the 5D current is restricted to $r < R$.

We define the 4D current by integrating along rays,

$$J_i(\theta; z_i) = \frac{1}{R} \int_0^R r dr j_i(r, \theta; z_i) , \quad i = 1, 2, 3 , \quad (3.2a)$$

$$J_\theta(\theta; z_i) = \int_0^R dr j_\theta(r, \theta; z_i) . \quad (3.2b)$$

Notice that the integration measure for the three transverse components is $(r/R)dr$, whereas the factor r/R is absent from the definition of J_θ , the component associated with the tangential direction along the rim of the disk. As we will now demonstrate, this is the right choice that leads to the conservation of the 4D current. Suppressing the coordinates $(\theta; z_i)$ one has

$$\begin{aligned} \sum_\mu \partial_\mu J_\mu &\equiv \sum_i \partial_i J_i + \frac{1}{R} \partial_\theta J_\theta \\ &= \frac{1}{R} \int_0^R r dr \left(\sum_i \partial_i j_i + \frac{1}{r} \partial_\theta j_\theta \right) \\ &= - \frac{1}{R} \int_0^R dr \partial_r (r j_r) \\ &= \frac{1}{R} (r j_r) \Big|_{r \rightarrow 0} - \frac{1}{R} (r j_r) \Big|_{r=R} = 0 . \end{aligned} \quad (3.3)$$

On the first line, the derivative of the tangential component is $(1/R)\partial/\partial\theta$, since $Rd\theta$ is the line element along the rim of the disk. On the second line we substituted the definitions (3.2), and on the next line we used the conservation of the 5D current, Eq. (3.1).

As might be expected, the divergence of the 4D current ends up being a surface term of the radial integral. The first term on the last line vanishes trivially, because of the $r \rightarrow 0$ limit.⁵ In addition, the surface term on the rim of the disk vanishes identically thanks to the boundary conditions. For definiteness let us assume that the boundary conditions at $r = R$ are $P_+^r \psi = 0$, $\bar{\psi} P_-^r = 0$.⁶ It follows that, on the rim, the radial component of the current is $j_r = \bar{\psi} \gamma_r \psi = \bar{\psi} P_+^r \gamma_r P_-^r \psi = 0$. This completes the proof.

⁵ Since the radial direction is ill-defined at the center, the lower end of the integration is defined as $r \rightarrow 0$, rather than $r = 0$.

⁶ These boundary conditions are consistent with the (Minkowski) relation $\bar{\psi} = \psi^\dagger \gamma_0$. We identify γ_0 as the Dirac matrix associated with one of the z_i directions, hence it anticommutes with γ_r .

B. Annulus geometry

It is interesting to explore the relation between the slab and disk geometries. The connection is provided by an annulus geometry. We start from the disk, and cut out a smaller disk of radius $R' < R$. For the boundary conditions we specified above on the outer rim $r = R$, the boundary conditions on the inner rim $r = R'$ will be $P_-^r \psi = 0$, $\bar{\psi} P_+^r = 0$.

The annulus geometry is topologically equivalent to the slab geometry. Starting from the slab geometry, let us take one of the four physical directions to be finite, and with periodic boundary conditions. The two-dimensional manifold consisting of this physical direction together with the (finite) fifth direction is then topologically equivalent to an annulus.

In the annulus geometry, the definition of the conserved 4D current remains the same as in Eq. (3.2), except that the lower end of the radial integration is now $r = R'$. The proof that this current is conserved works as in Eq. (3.3), with one notable change. The divergence of the 4D current is now the difference of two boundary terms

$$\sum_{\mu} \partial_{\mu} J_{\mu} = \frac{1}{R} (r j_r) \Big|_{r=R'} - \frac{1}{R} (r j_r) \Big|_{r=R} = 0, \quad (3.4)$$

and both terms vanish thanks to the boundary conditions imposed on the respective boundary.

IV. DYNAMICAL CONSIDERATIONS: THE FATE OF 'T HOOFT VERTICES

In asymptotically free 4D gauge theories, it is widely believed that violation of the axial charge in vector-like theories, and of fermion number in chiral gauge theories, comes from instantons⁷ through the effective 't Hooft interactions they induce [12, 13].

Let us focus once again on the example of QCD with N_f flavors. Formulating the theory in terms of LH and RH fields in the fundamental representation, the Weyl fields are ψ_{Ri} and ψ_{Li} , where $i = 1, 2, \dots, N_f$. The 't Hooft interactions induced by the instantons violate the conservation of the fermion number of each Weyl field individually. In terms of global symmetries, the group $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ is respected by the 't Hooft interaction, while the global axial charge is not conserved.

Let us now turn to the GK or disk frameworks. As we have discussed in the previous sections, now the individual fermion numbers are conserved for each 5D field separately. The theory as a whole, and its 't Hooft interactions in particular, must therefore be invariant under the larger symmetry group $U(N_f)_L \times U(N_f)_R$. We emphasize that this behavior is completely general. In particular, it is true regardless of how the 4D gauge field is extended into 5D, as long as the construction preserves the 4D gauge invariance. We will argue that, since this symmetry is preserved on the lattice, it remains true for the effective 4D theory in the continuum limit of the lattice theory.

We will not attempt to discuss the 5D dynamics in complete generality, because the details can vary a lot, depending on how the 4D gauge field is extended into 5D. Instead, we consider in the appendix a family of lattice gradient flows suitable for both the slab and disk geometries. These flows are designed such that it is expected that all instantons will shrink in size under the flow, and eventually disappear. In particular, using such a lattice flow in

⁷ Other topologically non-trivial configurations may contribute as well.

the GK framework, we expect that in the limit of an infinite fifth dimension, $N_5 \rightarrow \infty$, the flowed gauge field on the far wall will always be a pure gauge with trivial topology, and hence that the far-wall Weyl fermions fully decouple.

Returning for simplicity to the example of the one-flavor theory discussed in Sec. II, in the field of an instanton we expect to have one zero mode for (say) the fermion field ψ_R , and another one for the antifermion field $\bar{\psi}_L$. The corresponding 't Hooft interaction is thus, schematically, $\bar{\psi}_L(x^0)\psi_R(x^0)$, where x_μ^0 are collective coordinates: the coordinates of the center of the instanton. These zero modes will arise from the Weyl fields that reside on the near walls in the GK framework, or on the rims of the disk in the alternative framework. The resulting 't Hooft operator is

$$O_{4D} \sim \bar{\psi}_L(x^0)\psi_R(x^0) . \quad (4.1)$$

The notation O_{4D} is to indicate that it accounts for the zero modes of the 4D fields of the target theory.

By itself, the operator O_{4D} violates the individual fermion numbers of both of the 5D fields $\Psi^{(R)}$ and $\Psi^{(L)}$. But the GK and disk frameworks preserve these individual fermion numbers, hence there must exist additional zero modes, to compensate for the 4D zero modes.

What is the dynamics responsible for the existence of such additional zero modes? We conjecture that one way for them to arise is as follows. As explained above, under the class of lattice flows we introduce in the appendix, the size of the instanton keeps shrinking with the fifth coordinate. After a long enough flow, the instanton's size will become comparable to the lattice spacing a , which enables it to eventually disappear altogether. The 5D point (x, s) inside the bulk where the instanton disappears must exhibit a *dislocation* of the flowed gauge field. We conjecture that a new zero mode may develop with support on this dislocation.⁸ In order to restore the conservation of the individual fermion numbers of the 5D fields, the bulk zero modes must have opposite U(1) charges from the 4D zero modes. Unlike the power-law decay of the familiar instanton zero modes, we expect the bulk zero modes to be exponentially localized.⁹

Let us illustrate the role of the novel bulk zero modes, again using the example of the one-flavor theory. Also, for simplicity, we will consider the GK framework, but a similar reasoning applies to the disk framework as well. For $N_f = 1$ we expect two bulk zero modes, which are represented by the operator

$$O_{\text{bulk}} \sim \bar{\Psi}^{(R)}(x, s)\Psi^{(L)}(x, s) . \quad (4.2)$$

The total 't Hooft vertex is the product of the terms coming from the near wall and from the bulk,

$$O_{\text{tot}} = O_{4D}O_{\text{bulk}} \sim \bar{\psi}_L(x^0)\psi_R(x^0)\bar{\Psi}^{(R)}(x, s)\Psi^{(L)}(x, s) . \quad (4.3)$$

Now the individual fermion numbers of the two 5D fields are preserved, as required by the $U(N_f)_L \times U(N_f)_R$ global symmetry.¹⁰

We will next argue that, in the continuum limit, O_{tot} vanishes as an operator acting on the states of the effective 4D theory. This would imply that the $U(N_f)_L \times U(N_f)_R$ symmetry of the underlying lattice theory is inherited by the effective 4D theory.

⁸ For related setups where a bulk zero mode appears, see Refs. [14, 15].

⁹ Such bulk zero modes would resemble the localized zero modes of the supercritical Wilson kernel of DWFs [16–18].

¹⁰ A similar result was anticipated in Ref. [1] for the case of a continuum flow. See also the appendix.

We start by examining the expectation value of O_{tot} itself, which satisfies

$$\langle O_{\text{tot}} \rangle \leq C e^{-2Ms} , \quad (4.4)$$

for some constant C , which in turn is independent of s . Here $M = O(a^{-1})$ is the mass of the bulk 5D fermions.¹¹ The bound arises because the propagator in the fifth direction falls off like $e^{-M|s-s'|}$, and the effective 4D fields are located on the boundary $s' = 0$. As long as the instanton's size is large compared to the lattice scale,¹² one expects that the product Ms diverges in the continuum limit. Hence, the expectation value of O_{tot} vanishes.

The above behavior generalizes to any correlation function involving O_{tot} together with any number of insertions of 4D fields residing on the $s' = 0$ boundary. In order to contract the fermion fields contained in O_{bulk} , for each 5D field we will need one propagator from the position (x, s) of the dislocation to the boundary. As we have just argued, this propagator is bounded from above by e^{-Ms} , again leading to a bound similar to Eq. (4.4) for the correlation function under consideration.

In the above argument we have used that the contraction $\langle \Psi^{(L)}(x, s) \bar{\Psi}^{(R)}(x, s) \rangle$ vanishes identically, because it does not preserve the individual fermion numbers of the 5D fields. The same result generalizes to the thermodynamical limit. In order to take the thermodynamical limit we introduce a small mass term $m_q(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$. Since the mass term couples the two 5D fields, now $\langle \Psi^{(L)}(x, s) \bar{\Psi}^{(R)}(x, s) \rangle$ is non-zero, and (quark-disconnected) terms that include this contraction must be considered as well. However, any propagation from one 5D field to the other must go through the mass insertion, which in turn lives on the boundary. Hence¹³

$$\left\langle \Psi^{(L)}(x, s) \bar{\Psi}^{(R)}(x, s) \right\rangle \lesssim m_q e^{-2Ms} . \quad (4.5)$$

It follows that a uniform upper bound by e^{-2Ms} still applies, and once again the correlation function vanishes in the continuum limit. The upshot is that the bulk part of the new 't Hooft vertex suppresses the original 4D 't Hooft vertex, thus providing a dynamical understanding of the exact, superfluous $U(1)_A$ symmetry.

In this section we have illustrated via a concrete scenario how the familiar 't Hooft vertices of QCD can get modified. Ultimately, in general, the key point is that $U(1)_A$ is an exact symmetry in a gauge invariant formalism. Regardless of the details of the 5D dynamics, in both the GK and disk frameworks, as well as in the “intermediate” annulus framework, the massless theory admits only 't Hooft vertices that preserve $U(1)_A$, just as they preserve $SU(N)_L \times SU(N)_R \times U(1)_B$. Under these circumstances, unless the low energy theory violates some fundamental properties, we expect that the existence of the singlet pseudoscalar NGB cannot be avoided when chiral symmetry breaking takes place, just like the existence of the familiar massless pions cannot be avoided.

V. DISCUSSION

In this paper we studied two proposals for the lattice construction of chiral gauge theories [1–3] having in common that the underlying fermion system is five-dimensional. We found that both formulations have a superfluous conserved current which is gauge invariant. If

¹¹ Also known as the domain-wall height.

¹² See the appendix for details.

¹³ The $\lesssim$ symbol indicates the presence of a proportionality factor, similarly to Eq. (4.4).

the target 4D theory is a chiral gauge theory, the superfluous conserved current is the fermion number current. If the target theory is vector-like, it is the singlet axial current. Correspondingly, the symmetry of the vector-like theory enlarges to $U(N_f)_L \times U(N_f)_R$, instead of $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ as would be expected. In all cases, the underlying reason is that the fermion number of each 5D field is separately conserved.

When the target 4D theory is QCD with a small enough N_f to enable chiral symmetry breaking, we concluded that the spectrum will contain a superfluous Nambu-Goldstone boson, which is a singlet pseudoscalar meson. Apart from chiral symmetry breaking, the further assumptions leading to this conclusions include Lorentz invariance and locality; all of them are well established properties of QCD.

The standard DWF formulation of QCD, which is also five-dimensional, is equivalent to a purely 4D formulation in which the effective Dirac operator satisfies the Ginsparg-Wilson relation [19]. This raises the question about a possible connection between the proposals of Refs. [1, 2], and Lüscher’s approach to the construction of lattice chiral gauge theories [20–22]. In the latter approach, the lattice theory is gauge invariant provided that the fermion integration measure can be properly defined, for which a necessary condition is that the fermion spectrum will have no gauge anomaly. In comparison with Refs. [1, 2], a key difference is that the total fermion number is violated in Lüscher’s approach. This behavior is a direct consequence of the way the fermion integration measure is defined in this approach [20].

Another approach to the construction of lattice chiral gauge theories is the gauge fixing approach [23–25]. In this approach, the lattice theory is not gauge invariant, and gauge invariance is restored only in the continuum limit. Like in the proposals of Refs. [1, 2], in the gauge-fixing approach there is also a $U(1)$ symmetry associated with the total fermion number. However, in this case the conserved current is not gauge invariant; conversely, the gauge invariant current is not conserved [23, 26]. The $U(1)$ fermion number symmetry is always spontaneously broken, but the associated Nambu-Goldstone boson does not belong to the gauge invariant physical spectrum [24].

We comment that the so-called “Symmetric Mass Generation” (SMG) approach¹⁴ to the construction of lattice chiral gauge theories does not have a similar $U(1)$ issue. While gauge invariance is always maintained, the $U(1)$ issue is avoided. The reason is that the multi-fermion and/or Yukawa interactions introduced into the lattice action are designed to break explicitly any symmetry not present in the target continuum theory, including the fermion number symmetry. Nevertheless this approach may fail for various dynamical reasons. Recently, we pointed out that the SMG approach has a potential issue with unwanted propagator zeros that could take the form of ghost states in the continuum theory. For details, as well as references to the original literature, see Ref. [28].

It is an open question whether or not the proposals of Refs. [1, 2] can lead to a consistent 4D quantum field theory in the continuum limit. If they do, this would signal a breakdown of universality, in the following sense. Considering once again the QCD example, the reason is that the new universality class would contain a singlet pseudoscalar Nambu-Goldstone boson in the physical spectrum if chiral symmetry breaking takes place, whereas the standard formulation of QCD does not have such a Nambu-Goldstone boson. If chiral symmetry breaking does not take place, and/or the 4D theory is not local (or Lorentz invariant), this would also mean that the theory is different from QCD as obtained in the continuum limit of one of the standard lattice regularizations. A similar breakdown of universality would

¹⁴ For a review of SMG, and its relation to lattice chiral gauge theories, see Ref. [27].

happen in the case of a chiral gauge theory, because in the proposals of Refs. [1, 2] there exists a conserved and gauge invariant fermion number current, while Lüscher's approach does not have such a current.

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Appendix A: Lattice gradient flows

In both the GK framework and the disk framework, a 4D gauge field must be extended into 5D while preserving the 4D gauge invariance. Naively, one may aim for a 5D extension of the gauge field that is smooth everywhere. In fact, whether or not this is possible, it is not desirable.

Let us start with the disk framework. In order to see why smoothness everywhere is not always possible, assume that the initial 4D gauge field is an instanton field. Now consider the resulting 5D gauge field in the vicinity of a cylinder defined by a circle of radius $0 < r < R$ inside the disk, together with the three transverse directions. If the 5D extension of the gauge field is smooth in the vicinity of every cylinder, then the 4D topological charge on every such cylinder will be $Q = 1$, same as for the initial instanton field. But this implies that at $r = 0$ we will necessarily generate a singularity!

In the case of the GK framework, it is possible in principle to use a continuum flow that preserves the topological charge, without creating a singularity anywhere in the 5D gauge field. However, this would imply that the Weyl field on the far wall will couple to a gauge field with the same topological charge as the initial gauge field [1]. The outcome will be that, instead of the (conjectured) bulk zero modes we discussed in Sec. IV, there will be fermion zero modes associated with the Weyl field on the far wall. These zero modes will replace of the bulk zero modes in the total 't Hooft vertex (see Eq. (4.3)). However, the far-wall zero modes represent a non-local modification of the target 4D theory, something we would like to avoid. Either way, the 't Hooft vertices will preserve the individual fermion numbers of each 5D field.

Following Refs. [1, 2], in this appendix we define the 5D gauge field via gradient flow, which automatically preserves the 4D gauge invariance. We will introduce a family of lattice gradient flows, under which the size of any instanton present in the initial gauge field is expected to decrease monotonically. Eventually, the instanton's size becomes $O(a)$, and the (now lattice-size) instanton disappears, leaving behind a *dislocation* in the 5D gauge field.

We first briefly introduce gradient flow [7]. The general form of the flow equation in the continuum is

$$\frac{\partial B_\mu}{\partial t} = -g^2 \frac{\delta S}{\delta B_\mu} . \quad (\text{A1})$$

Here $t \geq 0$ is the flow time parameter, and B_μ is the flowed gauge field, which is subject to the boundary condition $B_\mu = A_\mu$ at $t = 0$, where A_μ is the dynamical gauge field. In the slab geometry, t is identified with the (cartesian) fifth coordinate s , while in the disk geometry it is identified with the radial coordinate $R - r$.

The right-hand side of the flow equation is the functional derivative of some continuum action S with respect to the (flowed) gauge field. Provided that the action S that generates the flow is gauge invariant, the flow equation is gauge covariant. If we use the standard continuum gauge action, then the right-hand side of the flow equation is proportional to the Yang-Mills equation of motion, and thus all classical solutions are unmodified by the flow. This includes in particular topologically non-trivial solutions, such as instantons.

On the lattice, the action that generates the flow¹⁵ can be expanded as a power series in the (squared) lattice spacing a^2 , much like in the Symanzik improvement program [29–33]. Here we will limit ourselves to the two simplest lattice gauge actions. The first is the single plaquette (Wilson) action

$$S_W = \frac{1}{g_0^2} \sum_{x,\mu\nu} \text{Re tr} (1 - P_{x,\mu\nu}) , \quad (\text{A2})$$

where $P_{x,\mu\nu}$ is the oriented product of link variables around the plaquette, and g_0 is the bare lattice gauge coupling. The second is the rectangle action

$$S_r = \frac{2}{g_0^2} \sum_{x,\mu\nu} \text{Re tr} (1 - P_{x,\mu\nu}^{\text{rect}}) , \quad (\text{A3})$$

where $P_{x,\mu\nu}^{\text{rect}}$ is the oriented product of links around the rectangle, with μ (ν) corresponding to the short (long) side of the rectangle.¹⁶ Expanding to $O(a^2)$, these actions are given by [30, 31]

$$S_W = \frac{1}{g_0^2} \int d^4x \left(\frac{1}{2} \text{tr} G^2 - \frac{1}{12} a^2 S_3 + \dots \right) , \quad (\text{A4a})$$

$$S_r/8 = \frac{1}{g_0^2} \int d^4x \left(\frac{1}{2} \text{tr} G^2 - \frac{5}{24} a^2 S_3 + \dots \right) . \quad (\text{A4b})$$

Here $G^2 = \sum_{\mu\nu} G_{\mu\nu}^2$ is the (squared) field strength. The first term on the right-hand sides of both equations is recognized as the familiar continuum gauge action. S_3 is the dimension-6 operator

$$S_3 = \sum_{\mu\nu} \text{tr} \left((\mathcal{D}_\mu G_{\mu\nu})^2 \right) , \quad (\text{A5})$$

where $\mathcal{D}_\mu$ is the covariant derivative in the adjoint representation. Notice that S_3 is invariant under hypercubic rotations only. With S_W and S_r at hand, the total lattice action that generates the flow will be assumed to be

$$S = c_p S_W + c_r S_r . \quad (\text{A6})$$

Requiring that the leading continuum term in the Symanzik expansion, $\text{tr} G^2$, will have its standard normalization implies the constraint¹⁷

$$c_p + 8c_r = 1 . \quad (\text{A7})$$

¹⁵ For the lattice version of the flow equation, see Ref. [7].

¹⁶ Equation (A3) uses the normalization of the rectangle term most common in the literature.

¹⁷ The so-called Symanzik action has $c_p = 5/3$ and $c_r = -1/12$, which removes the S_3 term from the action in the tree approximation.

Next, if we substitute into Eqs. (A4) an instanton field with size collective coordinate ρ , we obtain [34]

$$S_W = \frac{8\pi^2}{g_0^2} \left(1 - \frac{1}{5}(a/\rho)^2 + \dots \right), \quad (\text{A8a})$$

$$S_r/8 = \frac{8\pi^2}{g_0^2} \left(1 - \frac{1}{2}(a/\rho)^2 + \dots \right), \quad (\text{A8b})$$

where $8\pi^2/g_0^2$ is the classical instanton action. We see that for both S_W and S_r , the $O(a^2)$ term lowers the action, and the effect becomes stronger if we decrease the instanton size ρ . Related, a general feature of the flow, both in the continuum and on the lattice, is that the action S that generates the flow is always a monotonically decreasing function of the flow it generates [7].

This motivates us to propose the following strategy: *Use a lattice flow driven by the plaquette and rectangle terms, where both c_p and c_r are positive.* The advocated range corresponds to $0 < c_p < 1$, while c_r is determined via Eq. (A7). For any flow with these features, both S_W and S_r will decrease in absolute value under the flow. Moreover, in view of Eqs. (A8), we may expect that the size of every instanton present in the initial gauge field will shrink monotonically under the flow, until eventually the instanton's size will become $O(a)$, and the instanton will disappear at a dislocation. For a recent related numerical study, see Ref. [35].

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