Hecke symmetries associated with twisted polynomial algebras in 3 indeterminates

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Abstract. We consider Hecke symmetries on a 3-dimensional vector space with the associated R-symmetric algebra isomorphic to the polynomial algebra $\Bbbk[x_1, x_2, x_3]$ twisted by an automorphism. The main result states that any such a Hecke symmetry is itself a twist of a Hecke symmetry with the associated R-symmetric algebra isomorphic to $\Bbbk[x_1, x_2, x_3]$. This allows us to describe equivalence classes of such Hecke symmetries.

Introduction

With each Hecke symmetry R on a vector space V one associates the R-symmetric algebra $\mathbb{S}(V, R)$ which is viewed as a noncommutative analog of the ordinary symmetric algebra $\mathbb{S}(V)$ [5]. In a preceding article [13] the second author gave a description of all Hecke symmetries on a 3-dimensional vector space V such that $\mathbb{S}(V, R) = \mathbb{S}(V)$. The new paper extends that work by dealing with the cases where $\mathbb{S}(V, R)$ is isomorphic to the algebra $\mathbb{S}(V)$ twisted by an automorphism. The main result stated in Theorem 1.1 says that any such a Hecke symmetry is itself a twist of some Hecke symmetry with the respective R-symmetric algebra equal to $\mathbb{S}(V)$, and thus all such Hecke symmetries can be easily determined.

The question we pursue is opposite in a sense to the common direction in the study of Nichols algebras where one attempts to understand the structure of algebras associated with some classes of braidings (see a survey by Andruskiewitsch [1]). For the Hecke symmetries corresponding to quantum analogs of GL_3 we know already that the respective algebras $\mathbb{S}(V, R)$ are Artin-Schelter regular of global dimension 3, and for each algebra in this class we want to find all braidings of Hecke type which yield the chosen algebra.

From the viewpoint of the classification of quantum GL_3 groups splitting up the set of all relevant Hecke symmetries according to the algebra $\mathbb{S}(V, R)$ appears quite natural as this algebra is a primary invariant of a Hecke symmetry. This association of Hecke symmetries with algebras is not immediately retrievable from a paper of Ewen and Ogievetsky [4] where the classification is done up to twisting, and so some information gets lost. The approach discussed in [4] involves solving large systems of algebraic equations which would be difficult to analyze by hand, taking into account quite a large number of cases considered separately.

Our proof of Theorem 1.1 is accomplished in a fully invariant manner for an arbitrary twisting operator $\zeta \in GL(V)$. Generalizing what has been done in [13] we associate with a Hecke symmetry R a collection of linear forms $\ell_{xy} \in V^*$ indexed by pairs of vectors $x, y \in V$, and then reformulate conditions imposed on R in terms of certain identities in the exterior algebra of the dual space V^*. However, the equations are now more complicated than those in [13], and we have to struggle to exclude the possibility of some other solutions. We apply geometric arguments to analyze identities satisfied by several polynomial maps arising in this context. The whole proof occupies sections 2 and 3 of the paper.

Explicit determination of Hecke symmetries with the prescribed R-symmetric algebra $\mathbb{S}(V,R)$ does depend on the twisting operator ζ. However, once it has been established that they are all twists, it requires only basic methods of linear algebra to select in the set of known Hecke symmetries with associated algebra $\mathbb{S}(V)$ those which commute with the linear operator $\zeta\otimes\zeta$. We discuss parametrization of such Hecke symmetries in section 4 of the paper.

Among all Artin-Schelter regular graded algebras of global dimension 3 the twisted polynomial algebras are characterized by the property that the scheme parametrizing their point modules is the whole projective plane $\mathbb{P}^2$ (see [2]). Thus the present paper gives a description of all Hecke symmetries associated with algebras of this type. Unfortunately, the remaining Artin-Schelter regular algebras for which the point scheme is a cubic divisor in $\mathbb{P}^2$ defy unified treatment, and one has to resort to the case-by-case analysis, as outlined in [4]. For some cases details have been supplied in [11], [12].

All our considerations are limited to the case $\dim V = 3$. The quantum Yang-Baxter equation is notoriously hard to solve in higher dimensions. One of natural questions is whether the conclusion of Theorem 1.1 remains true when $\dim V > 3$.

The twisted Hecke symmetries in our paper are related to some easily constructed 2-cocycles on the FRT bialgebras. Finding all 2-cocycles on the coordinate algebra of the group GL_n amounts to the determination of all quantum analogs of GL_n. In its full generality this problem is intractable. In our modest contribution to the problem twisting is obtained by a simple conjugation in the group $GL(V\otimes V)$, and we do not need to ever invoke the FRT bialgebras.

1. Twisting of graded algebras and Hecke symmetries

By the term "algebra" we understand a unital associative algebra over a field. Given a $\mathbb{Z}$-graded algebra $A=\bigoplus A_k$ and its automorphism σ preserving the grading, the *twist* of A by σ is the algebra A_σ which has the same elements as the algebra A, but a different multiplication $\cdot_\sigma$ defined by the rule

$$a\cdot_\sigma b = a\,\sigma^n(b) \qquad \text{for } a\in A_n,\ b\in A.$$

(see [3, section 8]).

If A is generated by its component $V=A_1$, then every automorphism of A is determined by its action on elements of degree 1. Let $A\cong\mathbb{T}(V)/I$ where $I=\bigoplus I_k$ is a graded ideal of the tensor algebra $\mathbb{T}(V)=\bigoplus_{k\ge 0}V^{\otimes k}$. An invertible linear map $\tau: V\to V$ extends to an automorphism of A if and only if $\tau^{\otimes k}(I_k)=I_k$ for all k. Let us denote by A_τ the twist of A by the automorphism whose restriction to A_1 is τ. This algebra is also generated by the degree 1 component. In fact, $A_\tau\cong\mathbb{T}(V)/I'$ where I' is the graded ideal of $\mathbb{T}(V)$ with homogeneous components

$$I'_k = (\mathrm{Id}_V\otimes\tau^{-1}\otimes\cdots\otimes\tau^{-(k-1)})(I_k), \qquad k\ge 2.$$

In particular, A_τ has quadratic defining relations whenever so does A, and in this case the ideal I' is generated by

$$I'_2 = (\mathrm{Id}_V\otimes\tau^{-1})(I_2) = (\tau\otimes\mathrm{Id}_V)(I_2).$$

Denote by $\mathbb{S}(V)$ the symmetric algebra of a vector space V. It is the factor algebra of $\mathbb{T}(V)$ by the ideal generated by $\{x\otimes y - y\otimes x \mid x,y\in V\}$. Every invertible linear operator $\tau : V\to V$ extends to an automorphism of $\mathbb{S}(V)$, and we can form the respective twisted algebra $\mathbb{S}(V)_\tau$.

There are more general twistings of graded algebras introduced by Zhang [14]. However, in the case of $\mathbb{S}(V)$ every generalized twist in the sense of Zhang is isomorphic as a graded algebra to the twist of $\mathbb{S}(V)$ by an automorphism (see [14, Prop. 5.13]). Twists by automorphisms can be realized as a very special case of cocycle twists of comodule algebras discussed, e.g., in [8, 10.3.2] and [10]. With respect to 2-cocycles on Manin's universally coacting Hopf algebra every Artin-Schelter regular graded algebra with the Hilbert series $1/(1-t)^n$ is a twist of the polynomial algebra in n indeterminates (see [7, Th. 5.1.1]).

Let V be a finite-dimensional vector space over a field $\Bbbk$. A *Hecke symmetry* with parameter $0\ne q\in\Bbbk$ on this vector space is a linear operator $R : V\otimes V\to V\otimes V$ satisfying the braid equation

$$(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R)(R\otimes \mathrm{Id}_V) = (\mathrm{Id}_V\otimes R)(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R) \tag{1.1}$$

and the quadratic Hecke relation

$$(R-q\cdot \mathrm{Id}_{V\otimes V})(R+\mathrm{Id}_{V\otimes V}) = 0. \tag{1.2}$$

This notion is due to Gurevich [5], while involutive symmetries corresponding to the parameter value $q=1$ were studied earlier by Lyubashenko [9].

The *R-symmetric algebra* $\mathbb{S}(V,R)$ associated with R is the factor algebra of $\mathbb{T}(V)$ by the graded ideal generated by the subspace

$$\mathrm{Im}\,(R-q\cdot \mathrm{Id}_{V\otimes V})\subset V^{\otimes 2}.$$

If $\zeta : V\to V$ is an invertible linear operator such that $\zeta\otimes\zeta$ commutes with R, then the linear operator

$$R_\zeta = (\zeta\otimes \mathrm{Id}_V)\circ R\circ(\zeta^{-1}\otimes \mathrm{Id}_V) = (\mathrm{Id}_V\otimes\zeta^{-1})\circ R\circ(\mathrm{Id}_V\otimes\zeta) \tag{1.3}$$

is also a Hecke symmetry called the *twist* of R by ζ. It satisfies the Hecke relation with the same parameter q, and the braid equation for R_ζ follows from the equalities

$$\begin{aligned}
&(R_\zeta\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R_\zeta)(R_\zeta\otimes \mathrm{Id}_V)\\
&\qquad = (\zeta\otimes \mathrm{Id}_V\otimes\zeta^{-1})(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R)(R\otimes \mathrm{Id}_V)(\zeta^{-1}\otimes \mathrm{Id}_V\otimes\zeta),\\
&(\mathrm{Id}_V\otimes R_\zeta)(R_\zeta\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R_\zeta)\\
&\qquad = (\zeta\otimes \mathrm{Id}_V\otimes\zeta^{-1})(\mathrm{Id}_V\otimes R)(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R)(\zeta^{-1}\otimes \mathrm{Id}_V\otimes\zeta).
\end{aligned}$$

Such twists were used by Ewen and Ogievetsky [4]. The condition that $\zeta\otimes\zeta$ commutes with R actually allows one to define a certain 2-cocycle on the FRT bialgebra associated with R. In coordinate form this condition is written out in [6, Lemma 3.1.4]. The FRT bialgebra associated with R_ζ is a cocycle twist of the initial bialgebra. However, we will never need this interpretation.

The ideal of the tensor algebra $\mathbb{T}(V)$ defining the R_ζ-symmetric algebra $\mathbb{S}(V,R_\zeta)$ is generated by the space

$$\mathrm{Im}(R_\zeta - q\cdot \mathrm{Id}_{V\otimes V}) = (\zeta\otimes \mathrm{Id}_V)\big(\mathrm{Im}(R-q\cdot \mathrm{Id}_{V\otimes V})\big).$$

Hence $\mathbb{S}(V,R_\zeta)\cong \mathbb{S}(V,R)_\zeta$.

The initial Hecke symmetry R is itself a twist of R_ζ. Indeed, $\zeta\otimes\zeta$ commutes with R_ζ, and $R=(R_\zeta)_{\zeta^{-1}}$. Our main result is

Theorem 1.1. *Let V be a vector space of dimension 3 over a field $\Bbbk$ of characteristic $\ne 2$, and let $\zeta \in GL(V)$. If R is a Hecke symmetry on V such that the identity transformation of V extends to an isomorphism of algebras $\mathbb{S}(V,R) \cong \mathbb{S}(V)_\zeta$, then R commutes with $\zeta \otimes \zeta$.*

As a consequence, such a Hecke symmetry is a twist of a Hecke symmetry with the associated R-symmetric algebra equal to $\mathbb{S}(V)$.

The final assertion of Theorem 1.1 is an obvious consequence of the first one. If R commutes with $\zeta \otimes \zeta$, then $R_{\zeta^{-1}}$ is a Hecke symmetry with the associated algebra $\mathbb{S}(V, R_{\zeta^{-1}}) = \mathbb{S}(V)$, and we have $R = (R_{\zeta^{-1}})_\zeta$.

In section 2 we will associate with the Hecke symmetry R a collection of linear forms $\ell_{xy} \in V^*$ indexed by pairs of vectors $x, y \in V$ and show that the braid equation for R implies that these linear forms satisfy a certain identity in the second exterior power $\bigwedge^2 V^*$ of the dual space V^*. Likewise the property that R commutes with $\zeta \otimes \zeta$ will be reformulated in terms of a certain identity in $\bigwedge^2 V^*$.

The proof of Theorem 1.1 will be completed in section 3 by analyzing all the identities found in section 2. The most difficult part presented in Proposition 3.2 excludes the possibility of one case which could lead to Hecke symmetries violating the conclusion of Theorem 1.1. The assumption $\operatorname{char} \Bbbk \ne 2$ will be used in the proof of Theorem 1.1 only once in Lemma 3.10.

The twisting transformation $R \mapsto R_\zeta$ is conjugation in the group $GL(V \otimes V)$ by means of the linear operator $\zeta \otimes \mathrm{Id}_V$. If R is any Hecke symmetry on V, then for any $\varphi \in GL(V)$ the linear operator

$$R' = (\varphi \otimes \varphi) \circ R \circ (\varphi \otimes \varphi)^{-1} \tag{1.4}$$

is again a Hecke symmetry on V, and φ extends to an isomorphism of graded algebras $\mathbb{S}(V,R) \to \mathbb{S}(V,R')$. Conjugation by the operators $\varphi \otimes \varphi$ defines an action of the group $GL(V)$ on the set of all Hecke symmetries on V which we call the *conjugating action*. We say that two Hecke symmetries on V are *equivalent* if they lie in the same orbit with respect to the conjugating action of $GL(V)$.

If R and R' are two Hecke symmetries such that $\mathbb{S}(V,R) = \mathbb{S}(V,R')$ in the sense that the two algebras have the same space of defining relations in $V^{\otimes 2}$, then equality (1.4) implies that φ extends to an automorphism of $\mathbb{S}(V,R)$. In other words, two Hecke symmetries with the same associated R-symmetric algebra A are equivalent if and only if they lie in the same orbit with respect to the conjugating action of the subgroup of $GL(V)$ consisting of all linear operators on V which extend to an automorphism of A.

In section 4 we will use Theorem 1.1 to describe equivalence classes of Hecke symmetries with the associated R-symmetric algebra isomorphic to $\mathbb{S}(V)_\zeta$.

2. Setup for the proof of the main result

Let V be a vector space of dimension 3 over a field $\Bbbk$. We fix a linear operator $\zeta \in GL(V)$. Let $A = \mathbb{T}(V)/I$ where I is the graded ideal of $\mathbb{T}(V)$ generated by the set $\{\zeta(x) \otimes y - \zeta(y) \otimes x \mid x, y \in V\}$. So $A \cong \mathbb{S}(V)_\zeta$.

Further on we will write elements of $\mathbb{T}(V)$ omitting the sign $\otimes$, i.e., $xy = x \otimes y$, $xyz = x \otimes y \otimes z$ for $x, y, z \in V$. Put

$$x \mathbin{\bar\wedge} y = \zeta(x)\, y - \zeta(y)\, x \in V^{\otimes 2} \qquad \text{and}$$
$$\begin{aligned} x \mathbin{\bar\wedge} y \mathbin{\bar\wedge} z = \zeta^2(x)\,\zeta(y)\,z + \zeta^2(y)\,\zeta(z)\,x + \zeta^2(z)\,\zeta(x)\,y \\ - \zeta^2(x)\,\zeta(z)\,y - \zeta^2(y)\,\zeta(x)\,z - \zeta^2(z)\,\zeta(y)\,x \in V^{\otimes 3}. \end{aligned}$$

Consider the graded subspace $\Upsilon = \bigoplus \Upsilon^{(k)}$ of $\mathbb{T}(V)$ where

$$\Upsilon^{(0)} = \Bbbk, \qquad \Upsilon^{(1)} = V, \qquad \Upsilon^{(2)} = I_2, \qquad \Upsilon^{(3)} = I_2 V \cap V I_2$$

and $\Upsilon^{(k)} = 0$ for $k > 3$. Its components $\Upsilon^{(2)}$ and $\Upsilon^{(3)}$ are spanned by the tensors, respectively, $x \mathbin{\bar\wedge} y$ and $x \mathbin{\bar\wedge} y \mathbin{\bar\wedge} z$. There are linear isomorphisms $\bigwedge^k V \cong \Upsilon^{(k)}$ given by the identity maps for $k = 0$ and $k = 1$, and such that

$$x \wedge y \mapsto x \mathbin{\bar\wedge} y, \qquad x \wedge y \wedge z \mapsto x \mathbin{\bar\wedge} y \mathbin{\bar\wedge} z.$$

for $k = 2$ and $k = 3$. By means of these isomorphisms we obtain an algebra structure on Υ, and $\bar\wedge$ can be understood as the respective multiplication. Thus $(\Upsilon, \bar\wedge)$ is a graded Frobenius algebra isomorphic to the exterior algebra of V.

Note that
$$\zeta^{\otimes 3}(x \mathbin{\bar\wedge} y \mathbin{\bar\wedge} z) = (\det \zeta)\, x \mathbin{\bar\wedge} y \mathbin{\bar\wedge} z \tag{2.1}$$
since this tensor is the image of $\zeta(x) \wedge \zeta(y) \wedge \zeta(z) \in \bigwedge^3 V$.

Suppose that R is a Hecke symmetry on V such that $\mathbb{S}(V, R) = A$. In other words, R satisfies the hypothesis of Theorem 1.1. Put

$$R' = (\zeta^{-1} \otimes \zeta^{-1}) \circ R \circ (\zeta \otimes \zeta). \tag{2.2}$$

Then $R' = R$ if and only if $\zeta \otimes \zeta$ commutes with R. In any case R' is a Hecke symmetry with the same associated algebra $\mathbb{S}(V, R') = A$ since $(\zeta \otimes \zeta)(I_2) = I_2$ (so ζ extends to an automorphism of A).

Let q be the parameter of the Hecke relation, and

$$Y = q \cdot \mathrm{Id}_{V \otimes V} - R \tag{2.3}$$

the *R-skewsymmetrizer*. Then $\operatorname{Im} Y = \Upsilon^{(2)}$ and $Y^2 = (q+1)Y$. Hence

$$Yw = (q+1)w \quad \text{for all } w \in \Upsilon^{(2)}. \tag{2.4}$$

The braid equation for R can be rewritten as the equation

$$\begin{aligned} (\mathrm{Id}_V \otimes Y)(Y \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes Y) - q \cdot (\mathrm{Id}_V \otimes Y) \\ = (Y \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes Y)(Y \otimes \mathrm{Id}_V) - q \cdot (Y \otimes \mathrm{Id}_V). \end{aligned} \tag{2.5}$$

Here the linear operators $\mathrm{Id} \otimes Y$ and $Y \otimes \mathrm{Id}$ acting on $V^{\otimes 3}$ have images, respectively, $V \otimes \Upsilon^{(2)}$ and $\Upsilon^{(2)} \otimes V$. Therefore the two equal operators in (2.5) have images in the 1-dimensional subspace

$$\Upsilon^{(3)} = (V \otimes \Upsilon^{(2)}) \cap (\Upsilon^{(2)} \otimes V) \subset V^{\otimes 3}.$$

So it follows that

$$(\mathrm{Id}_V \otimes Y)(Y \otimes \mathrm{Id}_V)w - qw \in \Upsilon^{(3)} \quad \text{for all } w \in V \otimes \Upsilon^{(2)}. \tag{2.6}$$

Fix a nonzero alternating trilinear form $\omega : V \times V \times V \to \Bbbk$. There is a linear bijection $\widetilde{\omega} : \Upsilon^{(3)} \to \Bbbk$ such that

$$\widetilde{\omega}(x \mathbin{\bar{\wedge}} y \mathbin{\bar{\wedge}} z) = \omega(x, y, z) \quad \text{for } x, y, z \in V. \tag{2.7}$$

Define linear forms $\ell_{xy}\,,\ \ell'_{xy} \in V^*$ by the rule

$$\begin{aligned} \ell_{xy}(z) &= \widetilde{\omega}\big(x \mathbin{\bar{\wedge}} Y\big(\zeta(y) z\big)\big), \\ \ell'_{xy}(z) &= \widetilde{\omega}\big(x \mathbin{\bar{\wedge}} Y'\big(\zeta(y) z\big)\big), \qquad x, y, z \in V, \end{aligned} \tag{2.8}$$

where $Y' = q \cdot \mathrm{Id} - R'$ is the R'-skewsymmetrizer. In this way we obtain two collections of linear forms on V indexed by pairs of vectors $x, y \in V$. Both ℓ_{xy} and ℓ'_{xy} depend linearly on x and on y. Note that

$$Y' = (\zeta^{-1} \otimes \zeta^{-1}) \circ Y \circ (\zeta \otimes \zeta) \tag{2.9}$$

and therefore

$$\ell'_{xy}(z) = (\det \zeta)^{-1}\, \ell_{\zeta(x)\zeta(y)}\big(\zeta(z)\big). \tag{2.10}$$

Containments (2.6) are not fully equivalent to the braid equation for R, but they are sufficient for our purposes. What is important, (2.6) can be interpreted in terms of the functions ℓ_{xy} and ℓ'_{xy}. We will not need the most general identity equivalent to (2.6) and will use only its special case stated below:

Lemma 2.1. *Containments* (2.6) *imply the following identity in* $\bigwedge^2 V^*$:

$$\ell_{xy} \wedge \ell'_{xy} = \ell_{xx} \wedge \ell'_{yy}\,, \qquad x, y \in V. \tag{2.11}$$

Proof. For $f, g \in V^*$ we identify $f \wedge g \in \bigwedge^2 V^*$ with an alternating bilinear form on V setting

$$(f \wedge g)(u, v) = f(u)\, g(v) - f(v)\, g(u), \qquad u, v \in V. \tag{2.12}$$

Let e_1, e_2, e_3 be any linear basis for the vector space V. In $V^{\otimes 2}$ and $V^{\otimes 3}$ we take the bases, respectively, $\{\zeta(e_i)\, e_j\}$ and $\{\zeta^2(e_i)\, \zeta(e_j)\, e_k\}$. They are better adapted to Υ than the natural bases $\{e_i e_j\}$ and $\{e_i e_j e_k\}$. Let Y_{ij}^{st}, $Y'^{st}_{ij} \in \Bbbk$ be the coefficients in the expressions

$$Y\big(\zeta(e_i)\, e_j\big) = \sum Y_{ij}^{st}\, \zeta(e_s)\, e_t, \qquad Y'\big(\zeta(e_i)\, e_j\big) = \sum Y'^{st}_{ij}\, \zeta(e_s)\, e_t.$$

Here and later it is assumed that the summation is over the indices repeated as subscripts and superscripts. Since Y and Y' have images in $\Upsilon^{(2)}$, we have

$$Y_{ij}^{st} = -Y_{ij}^{ts}, \quad Y_{ij}^{ss} = 0, \quad Y'^{st}_{ij} = -Y'^{ts}_{ij}, \quad Y'^{ss}_{ij} = 0.$$

Now

$$Y\big(\zeta^2(e_i)\, \zeta(e_j)\big) = (\zeta \otimes \zeta)\big(Y'\big(\zeta(e_i)\, e_j\big)\big) = \sum Y'^{rl}_{ij}\, \zeta^2(e_r)\, \zeta(e_l),$$

and for $w = \zeta^2(e_i)\big(\zeta(e_j)e_k - \zeta(e_k)e_j\big) \in V \otimes \Upsilon^{(2)}$ we find

$$(\mathrm{Id} \otimes Y)(Y \otimes \mathrm{Id})(w) = \sum \big(Y'^{rl}_{ij}\, Y^{st}_{lk} - Y'^{rl}_{ik}\, Y^{st}_{lj}\big)\, \zeta^2(e_r)\zeta(e_s)e_t.$$

If $s = r$, then the basis element $\zeta^2(e_r)\zeta(e_s)e_t$ has zero coefficient in the elements of $\Upsilon^{(3)}$, and if $i \ne r$, then $\zeta^2(e_r)\zeta(e_s)e_t$ has zero coefficient in w. Therefore (2.6) implies that

$$\sum \big(Y'^{rl}_{ij}\, Y^{rt}_{lk} - Y'^{rl}_{ik}\, Y^{rt}_{lj}\big) = 0 \quad \text{if } i \ne r \tag{2.13}$$

where the summation is over l. Now note that

$$Y^{23}_{ij} = \ell_{e_1 e_i}(e_j)\,\alpha, \qquad Y^{31}_{ij} = \ell_{e_2 e_i}(e_j)\,\alpha, \qquad Y^{12}_{ij} = \ell_{e_3 e_i}(e_j)\,\alpha,$$

$$Y'^{23}_{ij} = \ell'_{e_1 e_i}(e_j)\,\alpha, \qquad Y'^{31}_{ij} = \ell'_{e_2 e_i}(e_j)\,\alpha, \qquad Y'^{12}_{ij} = \ell'_{e_3 e_i}(e_j)\,\alpha$$

where $\alpha = \omega(e_1, e_2, e_3)^{-1}$. These formulas follow straightforwardly from (2.8). For example,

$$\ell_{e_1 e_i}(e_j) = \widetilde{\omega}\big(e_1 \barwedge Y\big(\zeta(e_i)e_j\big)\big) = Y^{23}_{ij}\, \omega(e_1, e_2, e_3) = Y^{23}_{ij}\, \alpha^{-1}$$

since $Y\big(\zeta(e_i)e_j\big) = Y^{23}_{ij}\, e_2 \barwedge e_3 + Y^{31}_{ij}\, e_3 \barwedge e_1 + Y^{12}_{ij}\, e_1 \barwedge e_2$.

If $r = t$ then each term in the left hand side of (2.13) vanishes. Suppose $r \ne t$, and let $p \in \{1, 2, 3\}$ be the remaining element $\ne r, t$. Making use of (2.12) we get

$$Y'^{rl}_{ij}\, Y^{rt}_{lk} - Y'^{rl}_{ik}\, Y^{rt}_{lj} = \begin{cases} 0 & \text{for } l = r \\ (\ell_{e_p e_t} \wedge \ell'_{e_p e_i})(e_k, e_j)\, \beta^2 & \text{for } l = t \\ (\ell_{e_p e_p} \wedge \ell'_{e_t e_i})(e_j, e_k)\, \beta^2 & \text{for } l = p \end{cases}$$

where $\beta = \omega(e_p, e_r, e_t)^{-1}$. Take $i = t$. So $i \ne r$, and (2.13) can be rewritten as

$$(\ell_{e_p e_t} \wedge \ell'_{e_p e_t} - \ell_{e_p e_p} \wedge \ell'_{e_t e_t})(e_k, e_j) = 0.$$

Since e_k and e_j are two arbitrary basis vectors, the above equality is exactly (2.11) with $x = e_p$ and $y = e_t$.

All considerations are valid with respect to any basis of V. If $x, y \in V$ are two linearly independent vectors, we can include them in some basis of V, and then (2.11) for these vectors x, y will follow by making use of the chosen basis. On the other hand, if both x and y are scalar multiples of some vector, then equality (2.11) is automatically true. $\square$

Remark. By analyzing equations more fully one can see that for the containments (2.6) to hold it is necessary and sufficient that the equality

$$(\ell_{xy} \wedge \ell'_{xz} - \ell_{xx} \wedge \ell'_{yz})(u, v) = q\, \omega(x, y, z)\, \omega(x, u, v)$$

be true for all $x, y, z, u, v \in V$. This generalizes the case $\zeta = \mathrm{Id}_V$ considered in [13, Lemma 2.2]. If $\zeta = \mathrm{Id}_V$, then $\ell'_{xy} = \ell_{xy}$, and (2.11) reduces to a much simpler identity $\ell_{xx} \wedge \ell_{yy} = 0$ which was used in [13] to determine all Hecke symmetries with the associated R-symmetric algebra $\mathbb{S}(V, R) = \mathbb{S}(V)$.

Equality (2.4) too can be interpreted in terms of the functions ℓ_{xy}. Indeed, it means that $\widetilde{\omega}(x \barwedge Yw) = (q + 1)\, \widetilde{\omega}(x \barwedge w)$ for all $x \in V$ and $w \in \Upsilon^{(2)}$. Taking $w = y \barwedge z = \zeta(y)\, z - \zeta(z)\, y$, we rewrite this as

$$\ell_{xy}(z) - \ell_{xz}(y) = (q + 1)\, \omega(x, y, z)\,, \qquad x, y, z \in V.$$

Since R' is a Hecke symmetry with the same associated algebra $\mathbb{S}(V, R') = \mathbb{S}(V, R)$, there is a similar identity with the functions ℓ_{xy} replaced by ℓ'_{xy}. Again, we will use only a special case of these identities obtained by taking $z = x$:

Lemma 2.2. *The condition that the linear operators Y and Y' act on the subspace $\Upsilon^{(2)} \subset V^{\otimes 2}$ as $(q+1)$ times the identity operator implies that*

$$\ell_{xy}(x) = \ell_{xx}(y), \qquad \ell'_{xy}(x) = \ell'_{xx}(y) \tag{2.14}$$

for all $x, y \in V$.

The braid equation implies other useful properties of the collection $\{\ell_{xy}\}$. They are immediate consequences of the following fact:

Lemma 2.3. *If some vector $a \in V$ has the property that either $Y(ax) = 0$ for all $x \in V$ or $Y(xa) = 0$ for all $x \in V$, then $a = 0$.*

Proof. The equality $Y(ax) = 0$ means that $R(ax) = qax$. If this holds for all $x \in V$ then, applying the operators in the braid equation (1.1) to the tensor $w = axy \in V^{\otimes 3}$, we get

$$q^2\,(\mathrm{Id}_V \otimes R)w = q\,(\mathrm{Id}_V \otimes R^2)w.$$

If $a \neq 0$, then it follows that $qR(xy) = R^2(xy)$ for all $x, y \in V$, whence $R = q \cdot \mathrm{Id}$ since R is an invertible operator. Thus $Y = 0$ by (2.3). However, this contradicts the assumption that $\operatorname{Im} Y = \Upsilon^{(2)}$ is a space of dimension 3.

In the other case when $Y(xa) = 0$ for all $x \in V$ we argue similarly, now working with the tensors of the form $w = xya$. $\square$

Lemma 2.4. *The set $\{\ell_{xy} \mid x, y \in V\}$ spans the whole space V^*. If $a \in V$ is such that either $\ell_{ax} = 0$ for all $x \in V$ or $\ell_{xa} = 0$ for all $x \in V$, then $a = 0$.*

Proof. For the first assertion we have to check that the only vector at which all linear forms ℓ_{xy} vanish is the zero vector. But $\ell_{xy}(a) = 0$ means that $x \barwedge Y\big(\zeta(y)a\big) = 0$, and if this equality holds for all $x, y \in V$, then $Y(va) = 0$ for all $v \in V$. So Lemma 2.3 applies.

If $\ell_{xa} = 0$ for all $x \in V$, then $Y\big(\zeta(a)v\big) = 0$ for all $v \in V$. If $\ell_{ax} = 0$ for all $x \in V$, then $a \barwedge t = 0$ for all $t \in \Upsilon^{(2)}$. The conclusion $a = 0$ follows in each case. $\square$

Our aim is to show that R commutes with $\zeta \otimes \zeta$, and we are going to reformulate this property in terms of the functions ℓ_{xy}, ℓ'_{xy}. This will be done in Lemma 2.6. First we mention one related result derived from an earlier work:

Lemma 2.5. *Any Hecke symmetry R satisfying the hypothesis of Theorem* 1.1 *commutes with the linear operator $(\zeta \otimes \zeta)^3$.*

Proof. The tensors t in the one-dimensional subspace $\Upsilon^{(3)} \subset V^{\otimes 3}$ satisfy the twisted cyclicity condition with the twisting operator $\psi = (\det \zeta)\,\zeta^{-3}$. This means that

$$t = (\mathrm{Id}_V \otimes \mathrm{Id}_V \otimes \psi)\, s_{123}(t)$$

where s_{123} is the permutation operator on $V^{\otimes 3}$ defined by the rule $s_{123}(xyz) = yzx$ for $x, y, z \in V$. Since $t = x \barwedge y \barwedge z$ for suitable x, y, z, the displayed equality is readily verified with the aid of (2.1). By [12, Theorem 3.8] R commutes with $\psi \otimes \psi$, and this gives the desired conclusion. $\square$

Under certain conditions on the eigenvalues of ζ the linear operators $\zeta \otimes \zeta$ and $(\zeta \otimes \zeta)^3$ have equal centralizers in the group $GL(V \otimes V)$. However, in general the conclusion of Lemma 2.5 is weaker than what we need, and we do not rely on it. Lemma 2.5 will be used only once to provide special treatment of the case $q = -1$ in Lemma 2.6. For $q \neq -1$ Lemma 2.5 will not be used.

Lemma 2.6. *The following conditions are equivalent:*

(i) *R commutes with $\zeta \otimes \zeta$,*

(ii) *$Y' = Y$,*

(iii) *$Y' = cY$ for some $c \in \Bbbk$,*

(iv) *there is $c \in \Bbbk$ such that $\ell'_{xy} = c\,\ell_{xy}$ for all $x, y \in V$.*

(v) *$\ell_{xy} \wedge \ell'_{xy} = 0$ for all $x, y \in V$.*

Proof. It is immediate from (2.2) that (i) holds if and only if $R' = R$, which in turn is equivalent to the equality $Y' = Y$ by the definition of these operators. We have to show that $c = 1$ whenever (iii) holds.

So suppose that $Y' = cY$. Recall that both Y and Y' act on the subspace $\Upsilon^{(2)}$ as $(q+1)$ times the identity operator. Hence $c(q+1) = q+1$. If $q \neq -1$, we do get $c = 1$.

If $q = -1$, we have to argue differently. Note that containments (2.6) are valid also for Y' since R' is a Hecke symmetry with the same associated algebra $S(V, R') = A$. However, the assumption $Y' = cY$ implies that

$$(\mathrm{Id}_V \otimes Y')(Y' \otimes \mathrm{Id}_V)w \equiv c^2 q w \pmod{\Upsilon^{(3)}} \qquad \text{for all } w \in V \otimes \Upsilon^{(2)}.$$

Hence $c^2 q = q$, i.e., $c^2 = 1$. We now make use of Lemma 2.5. Since the linear operator $(\zeta \otimes \zeta)^3$ commutes with R, it also commutes with Y. On the other hand, from (2.9) we deduce that

$$Y \circ (\zeta \otimes \zeta) = c\,(\zeta \otimes \zeta) \circ Y.$$

It follows that $c^3 = 1$. Together with $c^2 = 1$ this yields $c = 1$.

Recalling (2.8), we see that (iv) is expanded as

$$\widetilde{\omega}\big(x \barwedge Y'\big(\zeta(y)z\big)\big) = \widetilde{\omega}\big(x \barwedge c\,Y\big(\zeta(y)z\big)\big) \quad \text{for all } x, y, z \in V.$$

Since $\widetilde{\omega}$ induces a nondegenerate bilinear pairing $V \times \Upsilon^{(2)} \to \Bbbk$, this identity is equivalent to

$$Y'\big(\zeta(y)z\big) = c\,Y\big(\zeta(y)z\big) \quad \text{for all } y, z \in V,$$

which amounts to (iii). Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). It is also clear that (v) follows from (iv). The opposite implication is more complicated.

Suppose that (v) holds. If $x, y \in V$ are any two vectors such that $\ell_{xy} \neq 0$, then (v) implies that ℓ'_{xy} is a scalar multiple of ℓ_{xy}, and so $\ell'_{xy} = c(x,y)\,\ell_{xy}$ for some $c(x,y) \in \Bbbk$. We have to deal with the dependency of $c(x,y)$ on x and y.

For each $x \in V$ put $K_x = \{y \in V \mid \ell_{xy} = 0\}$. This is a vector subspace of V. By Lemma 2.4 we have $K_x \neq V$ unless $x = 0$. Put

$$X = \{x \in V \mid \dim K_x \leq 1\}.$$

Claim 1. *For each $x \in X$ there exists $c_\flat(x) \in \Bbbk$ such that $\ell'_{xy} = c_\flat(x)\,\ell_{xy}$ for all $y \in V$ simultaneously.*

Note that K_x is the kernel of the linear map $V \to V^*$ given by the assignment $y \mapsto \ell_{xy}$. It follows that ℓ_{xy_1} and ℓ_{xy_2} are linearly independent linear forms whenever $y_1, y_2 \in V$ are two vectors which are linearly independent modulo the one-dimensional subspace K_x. Then $\ell_{x(y_1+y_2)} = \ell_{xy_1} + \ell_{xy_2} \neq 0$ and

$$\ell'_{x(y_1+y_2)} = \ell'_{xy_1} + \ell'_{xy_2} = c(x, y_1)\,\ell_{xy_1} + c(x, y_2)\,\ell_{xy_2}.$$

On the other hand, the left hand side is equal to $c(x, y_1 + y_2)\,\ell_{x(y_1+y_2)}$. Hence

$$c(x, y_1) = c(x, y_1 + y_2) = c(x, y_2).$$

It follows that there exists $c_\flat(x) \in \Bbbk$ such that $c(x, y) = c_\flat(x)$ for all $y \in V$ such that $y \notin K_x$. Now the set

$$V_x = \{y \in V \mid \ell'_{xy} = c_\flat(x)\,\ell_{xy}\}$$

is a vector subspace of V which contains the set $V \smallsetminus K_x$. Since the latter spans the whole space V, we conclude that $V_x = V$.

Claim 2. *The set X is nonempty.*

Suppose that $X = \varnothing$. Then $\dim K_x > 1$ for each $x \in V$. If $x \neq 0$, then we must have $\dim K_x = 2$ since $K_x \neq V$, and therefore the set $T_x = \{\ell_{xy} \mid y \in V\}$ is a one-dimensional subspace of V^*. Pick some nonzero vectors $x_1 \in V$ and $a \in K_{x_1}$. The set

$$L = \{x \in V \mid \ell_{xa} = 0\}$$

is a vector subspace of V, and $L \neq V$ by Lemma 2.4. Hence $V \smallsetminus L$ spans the whole space V, and by Lemma 2.4 the linear span of the set $\{\ell_{xy} \mid x, y \in V,\ x \notin L\}$ must coincide with the whole V^* since it contains the functions ℓ_{xy} for all $x, y \in V$. We can find $x_2 \in V \smallsetminus L$ such that $T_{x_2} \neq T_{x_1}$.

Since $\ell_{x_2a} \neq 0$, we have $a \notin K_{x_2}$. Hence $K_{x_1} \neq K_{x_2}$, while these two spaces both have dimension 2. Take some vectors $y_1 \in K_{x_2} \smallsetminus K_{x_1}$ and $y_2 \in K_{x_1} \smallsetminus K_{x_2}$. Then $\ell_{x_1y_1}$ and $\ell_{x_2y_2}$ are nonzero, and moreover these two functions are linearly independent since they lie in two distinct one-dimensional subspaces T_{x_1} and T_{x_2}. On the other hand, $\ell_{x_1y_2} = \ell_{x_2y_1} = 0$. Hence

$$\ell_{(x_1+x_2)y_1} = \ell_{x_1y_1}\,, \qquad \ell_{(x_1+x_2)y_2} = \ell_{x_2y_2}\,,$$

and so $\dim T_{x_1+x_2} \geq 2$, in contradiction with the assumption $\dim K_{x_1+x_2} > 1$.

Claim 3. *The linear span $\Bbbk X$ of the set X coincides with V.*

Suppose that $\Bbbk X \neq V$. We have $\dim K_u = 2$ for all $u \in V$ such that $u \notin \Bbbk X$. If $u \notin \Bbbk X$ and $x \in X$, then $u + x \notin \Bbbk X$, whence $\dim K_{u+x} = 2$ as well. Furthermore, each element $a \in K_u \cap K_{u+x}$ is contained in K_x since $\ell_{xa} = \ell_{(u+x)a} - \ell_{ua} = 0$. This forces $K_u \neq K_{u+x}$ since $\dim K_x \leq 1$. Hence $K_u \cap K_{u+x}$ is a one-dimensional subspace of V, and we must have $K_x = K_u \cap K_{u+x}$.

It follows that $K_x \subset K_u$ for each $x \in X$ and each $u \in V \smallsetminus \Bbbk X$. Moreover, if we put $K = \bigcap_{u \notin \Bbbk X} K_u$, then $K_x = K$ for each $x \in X$ and $K \subset K_u$ for each $u \in V \smallsetminus \Bbbk X$. We have seen already that $K_x \neq 0$ for $x \in X$. Hence $K \neq 0$. If $a \in K$, then $\ell_{xa} = 0$ for all $x \in X$, as well as for all $x \in V \smallsetminus \Bbbk X$. This equality holds then also for all $x \in \Bbbk X$, and therefore for all $x \in V$. But this contradicts Lemma 2.4.

Claim 4. *The function $c_\flat : X \to \Bbbk$ is constant.*

We have to prove that $c_\flat(x_1) = c_\flat(x_2)$ whenever $x_1, x_2 \in X$. If x_2 is a scalar multiple of x_1, then this equality is obviously true. Suppose that x_1 and x_2 are linearly independent. Note that

$$c_\flat(x_1 + x_2)\,\ell_{(x_1+x_2)y} = \ell'_{(x_1+x_2)y} = \ell'_{x_1y} + \ell'_{x_2y} = c_\flat(x_1)\,\ell_{x_1y} + c_\flat(x_2)\,\ell_{x_2y}$$

for all $y \in V$. If ℓ_{x_1y} and ℓ_{x_2y} are linearly independent for some y, then, comparing the coefficients, we deduce that both $c_\flat(x_1)$ and $c_\flat(x_2)$ are equal to $c_\flat(x_1 + x_2)$, and we are done. We only have to be sure that such an element y exists.

Otherwise we would have $\ell_{x_1y} \wedge \ell_{x_2y} = 0$ for all $y \in V$. In this case ℓ_{x_2y} is a scalar multiple of ℓ_{x_1y} whenever $y \notin K_{x_1}$, and we can write $\ell_{x_2y} = \lambda(y)\ell_{x_1y}$ for some $\lambda(y) \in \Bbbk$. If $y_1, y_2 \in V$ are linearly independent modulo the one-dimensional subspace K_{x_1}, then $\ell_{x_1y_1}$ and $\ell_{x_1y_2}$ are linearly independent, and, expressing $\ell'_{x_2(y_1+y_2)}$ as their linear combination, we deduce that $\lambda(y_1) = \lambda(y_1 + y_2) = \lambda(y_2)$. It follows that λ is a constant function.

In other words, there is $\lambda \in \Bbbk$ such that $\ell_{x_2y} = \lambda\ell_{x_1y}$ for all $y \in V \smallsetminus K_{x_1}$. By linearity in y the last equality holds then for all $y \in V$, whence $\ell_{(x_2-\lambda x_1)y} = 0$ for all $y \in V$. However, this contradicts Lemma 2.4.

Thus Claim 4 is proved. It means that there is $c \in \Bbbk$ such that $\ell'_{xy} = c\,\ell_{xy}$ for all $x \in X$ and all $y \in V$. This equality holds then also for all $x \in \Bbbk X$, hence for all $x \in V$ by Claim 3. This shows that (v) $\Rightarrow$ (iv). $\square$

3. The proof

We will verify condition (v) of Lemma 2.6. This will be done by analyzing identities (2.11) and (2.14) involving the linear functions ℓ_{xy} and ℓ'_{xy}. Now one may safely forget the actual connection of these functions with Hecke symmetries. We will need only their properties stated in Lemma 2.4 and several simple consequences of (2.10).

Since the construction of R-symmetric algebras commutes with extensions of the base field, we may assume without loss of generality the field $\Bbbk$ to be algebraically closed. This will allow us to work comfortably with algebraic varieties and use some geometric arguments.

Denote by U and U' the subspaces of V^* spanned by the sets $\{\ell_{xx} \mid x \in V\}$ and $\{\ell'_{xx} \mid x \in V\}$ respectively. Note that $U' = \{f \circ \zeta \mid f \in U\}$ by (2.10). In particular, $\dim U' = \dim U$.

Proposition 3.1. *If $\dim U = 1$, then R commutes with $\zeta \otimes \zeta$.*

Proof. Since $\dim U' = 1$ too, we have $\dim(U + U') \le 2$. Therefore $\bigwedge^2(U + U')$ is at most 1-dimensional subspace of $\bigwedge^2 V^*$ which contains $\ell_{xx} \wedge \ell'_{yy}$ for any $x, y \in V$. Identity (2.11) yields

$$\ell_{xy} \wedge \ell'_{xy} \in \textstyle\bigwedge^2(U + U')$$

for all $x, y \in V$. The set $O = \{(x, y) \in V \times V \mid \ell_{xy} \notin U + U'\}$ is a Zariski open subset of $V \times V$. It is nonempty by Lemma 2.4. Note that for two linearly independent linear forms $\xi, \eta \in V^*$ the containment $\xi \wedge \eta \in \bigwedge^2(U + U')$ holds only when both ξ and η lie in $U + U'$. Therefore for $(x, y) \in O$ the displayed containment implies that $\ell_{xy} \wedge \ell'_{xy} = 0$.

On the other hand, $Z = \{(x,y) \in V \times V \mid \ell_{xy} \wedge \ell'_{xy} = 0\}$ is a Zariski closed subset of an irreducible algebraic variety $V \times V$. We conclude that $Z = V \times V$ since Z contains a nonempty Zariski open subset of $V \times V$. Thus $\ell_{xy} \wedge \ell'_{xy} = 0$ for all $x, y \in V$. Now Lemma 2.6 applies. □

Proposition 3.1 confirms the conclusion of Theorem 1.1 in the case when U has dimension 1. In fact this condition holds for all Hecke symmetries obtained by twisting from Hecke symmetries associated with the ordinary symmetric algebra $\mathbb{S}(V)$. It remains to consider the case $\dim U > 1$ which could possibly lead to another class of Hecke symmetries.

Proposition 3.2. *Assume that* $\operatorname{char} \Bbbk \neq 2$. *There does not exist any Hecke symmetry R such that the R-symmetric algebra $\mathbb{S}(V,R)$ is isomorphic to some twisted polynomial algebra in 3 indeterminates and for which* $\dim U > 1$.

The proof of Proposition 3.2 is much longer. It will be split into a series of lemmas which occupy the rest of this section. Let us assume that $\dim U > 1$. We will see eventually that this assumption leads to a contradiction.

The next lemma provides a tool to derive certain relations of linear dependence between the linear forms ℓ_{xy} and ℓ'_{xy}.

Lemma 3.3. *Let X be an irreducible affine algebraic variety over an algebraically closed field $\Bbbk$ with a factorial coordinate ring $\Bbbk[X]$. Suppose that W is a finite dimensional vector space over $\Bbbk$ and $\varphi_1, \ldots, \varphi_n, \psi : X \to W$ are morphisms in the category of algebraic varieties such that the set*

$$O = \{x \in X \mid \varphi_1(x), \ldots, \varphi_n(x) \text{ are linearly independent}\}$$

is nonempty, while for each $x \in X$ the $n+1$ vectors $\varphi_1(x), \ldots, \varphi_n(x), \psi(x) \in W$ are linearly dependent. Then there exist functions $p_1, \ldots, p_n, p_0 \in \Bbbk[X]$ such that

$$\gcd(p_1, \ldots, p_n, p_0) = 1 \tag{3.1}$$

and

$$p_0(x)\,\psi(x) = \sum_{i=1}^{n} p_i(x)\,\varphi_i(x) \quad \text{for all } x \in X. \tag{3.2}$$

Moreover, the equality $h_0(x)\,\psi(x) = \sum_{i=1}^{n} h_i(x)\,\varphi_i(x)$ holds identically on X for some collection of functions $h_1, \ldots, h_n, h_0 \in \Bbbk[X]$ if and only if there is $g \in \Bbbk[X]$ such that $h_i = gp_i$ for all i.

Proof. For each $x \in O$ the vector $\psi(x)$ is a linear combination of $\varphi_1(x), \ldots, \varphi_n(x)$ with uniquely determined coefficients. So

$$\psi(x) = \sum_{i=1}^{n} f_i(x)\,\varphi_i(x), \qquad x \in O,$$

for some functions $f_i : O \to \Bbbk$. It is easy to see that O is a Zariski open subset of X and $f_1, \ldots, f_n$ are regular on O, hence rational on X. We can find $0 \neq p_0 \in \Bbbk[X]$ such that $p_0 f_i = p_i \in \Bbbk[X]$ for each $i = 1, \ldots, n$. Equality (3.2) is satisfied on some nonempty Zariski open subset of X, but then it must hold everywhere. Factoring out the greatest common divisor of $p_1, \ldots, p_n, p_0$ we achieve (3.1).

In the second assertion we have

$$\sum_{i=1}^{n} \big(p_0(x)\, h_i(x) - h_0(x)\, p_i(x)\big)\, \varphi_i(x) = 0 \quad \text{for all } x \in X.$$

It follows from the definition of the set O that $p_0(x)h_i(x) = h_0(x)p_i(x)$ for all $x \in O$ and $i = 1, \ldots, n$. Any two regular functions $X \to \Bbbk$ must be equal provided that they agree on a nonempty Zariski open subset of X. Hence $p_0 h_i = h_0 p_i$ for each i. Factoriality of $\Bbbk[X]$ together with (3.1) ensures that p_0 divides h_0 in the ring $\Bbbk[X]$. So $g = h_0 p_0^{-1}$ will do. $\square$

We will apply Lemma 3.3 in the situation where X is either V or $V \times V$. So X is a vector space, and $\Bbbk[X] \cong \mathbb{S}(X^*)$ is the algebra of polynomial functions on X generated by the dual space X^* of linear functions on X.

Polynomial functions on $V \times V$ are functions of two arguments taken in V. We will encounter polynomial functions $f : V \times V \to \Bbbk$ which are homogeneous in each of the two arguments. We say that (m, n) is the *bidegree* of f if f is homogeneous of degree m in the first argument and homogeneous of degree n in the second. The *total degree* of f is the sum $m+n$ of its bidegree components. Note that any divisor of a bihomogeneous polynomial function is itself bihomogeneous.

Each of the expressions ℓ_{xy}, ℓ'_{xy}, ℓ_{xx}, ℓ'_{yy} regarded as a function of the pair (x, y) gives a quadratic polynomial map $V \times V \to V^*$. Since $\dim U > 1$, there exist $x, y \in V$ such that ℓ_{xx} and ℓ'_{yy} are linearly independent. On the other hand, the three elements ℓ_{xy}, ℓ_{xx}, ℓ'_{yy} of the vector space V^* are always linearly dependent, and so too are ℓ'_{xy}, ℓ_{xx}, ℓ'_{yy} since

$$\ell_{xy} \wedge \ell_{xx} \wedge \ell'_{yy} = 0 \qquad \text{and} \qquad \ell'_{xy} \wedge \ell_{xx} \wedge \ell'_{yy} = 0 \tag{3.3}$$

for all $x, y \in V$ by (2.11). Lemma 3.3 provides polynomial functions $p_i, p'_i \in \Bbbk[V \times V]$ for $i = 0, 1, 2$ such that

$$\gcd(p_0, p_1, p_2) = \gcd(p'_0, p'_1, p'_2) = 1, \tag{3.4}$$

and

$$\begin{aligned} p_0(x, y)\, \ell_{xy} &= p_1(x, y)\, \ell_{xx} + p_2(x, y)\, \ell'_{yy}\,, \\ p'_0(x, y)\, \ell'_{xy} &= p'_1(x, y)\, \ell_{xx} + p'_2(x, y)\, \ell'_{yy} \end{aligned} \tag{3.5}$$

for all $x, y \in V$. As seen from the final assertion of Lemma 3.3, those p_i and p'_i are polynomial functions of the smallest degree that can occur in such relations of linear dependence. Since every polynomial function on $V \times V$ is a sum of its bihomogeneous components, the functions p_i and p'_i are necessarily bihomogeneous.

We will need very precise information about the functions p_i, p'_i. As a first step we deduce an identity which connects them. Making use of (3.5), we get

$$p_0(x, y)\, p'_0(x, y)\, \ell_{xy} \wedge \ell'_{xy} = \big(p_1(x, y)\, p'_2(x, y) - p_2(x, y)\, p'_1(x, y)\big)\, \ell_{xx} \wedge \ell'_{yy}.$$

The rule $(x, y) \mapsto \ell_{xx} \wedge \ell'_{yy}$ defines a nonzero polynomial map $V \times V \to \bigwedge^2 V^*$. Since $\ell_{xy} \wedge \ell'_{xy} = \ell_{xx} \wedge \ell'_{yy}$ and the ring $\Bbbk[V \times V]$ is a domain, it follows that

$$p_0 p'_0 = p_1 p'_2 - p_2 p'_1.$$

We will see later that p_0' can be taken equal to p_0. In this case the previous identity is written as

$$p_0^2 = p_1 p_2' - p_2 p_1'. \tag{3.6}$$

Take any basis ℓ_1, ℓ_2, ℓ_3 for the vector space V^*. We can write

$$\begin{aligned} \ell_{xx} &= \sum a_i(x)\,\ell_i, & \ell_{xy} &= \sum b_i(x,y)\,\ell_i, \\ \ell_{xx}' &= \sum a_i'(x)\,\ell_i, & \ell_{xy}' &= \sum b_i'(x,y)\,\ell_i \end{aligned} \tag{3.7}$$

for some quadratic forms a_i, a_i' and bilinear forms b_i, b_i' such that $a_i(x) = b_i(x,x)$ and $a_i'(x) = b_i'(x,x)$. Now

$$\ell_{xx} \wedge \ell_{yy}' = \Delta_1(x,y)\,\ell_2 \wedge \ell_3 + \Delta_2(x,y)\,\ell_1 \wedge \ell_3 + \Delta_3(x,y)\,\ell_1 \wedge \ell_2 \tag{3.8}$$

where $\Delta_i(x,y)$, $i = 1,2,3$, are the minors of order 2 of the matrix

$$\begin{pmatrix} a_1(x) & a_2(x) & a_3(x) \\ a_1'(y) & a_2'(y) & a_3'(y) \end{pmatrix}.$$

Lemma 3.4. *For each $i = 1,2,3$ the polynomial function Δ_i is divisible by p_0 and by p_0' in the ring $\Bbbk[V \times V]$.*

Proof. In terms of coordinate representation (3.7) the first equation in (3.3) means that the matrix

$$\begin{pmatrix} b_1(x,y) & b_2(x,y) & b_3(x,y) \\ a_1(x) & a_2(x) & a_3(x) \\ a_1'(y) & a_2'(y) & a_3'(y) \end{pmatrix}$$

has identically zero determinant. Hence there are 3 different relations of linear dependence between the rows of this matrix, the coefficients being the second order minors each time extracted from some pair of columns of the matrix. This gives 3 relations of linear dependence between ℓ_{xy}, ℓ_{xx}, ℓ_{yy}' in which the respective coefficient of ℓ_{xy} is $\Delta_i(x,y)$ for $i = 1,2,3$. For example,

$$\begin{vmatrix} a_1(x) & a_2(x) \\ a_1'(y) & a_2'(y) \end{vmatrix} \ell_{xy} - \begin{vmatrix} b_1(x,y) & b_2(x,y) \\ a_1'(y) & a_2'(y) \end{vmatrix} \ell_{xx} + \begin{vmatrix} b_1(x,y) & b_2(x,y) \\ a_1(x) & a_2(x) \end{vmatrix} \ell_{yy}' = 0.$$

The final assertion in Lemma 3.3 shows that p_0 divides Δ_i. Working with the second equation in (3.3) we deduce similarly that p_0' divides Δ_i. $\square$

Consider the greatest common divisor

$$d = \gcd(\Delta_1, \Delta_2, \Delta_3) \in \Bbbk[V \times V]. \tag{3.9}$$

Since $\Delta_1, \Delta_2, \Delta_3$ are bihomogeneous of bidegree $(2,2)$, the function d has to be bihomogeneous too, and its degree in each of the two arguments cannot exceed 2. The same can be said about p_0 and p_0' which are divisors of d by Lemma 3.4.

Lemma 3.5. *The three polynomial functions $\Delta_1, \Delta_2, \Delta_3$ are linearly independent. As a consequence,* $\deg d < 4$.

Proof. Suppose that $\Delta_1, \Delta_2, \Delta_3$ are linearly dependent. It is then seen from (3.8) that the set $\{\ell_{xx} \wedge \ell'_{yy} \mid x, y \in V\}$ spans a proper subspace of $\bigwedge^2 V^*$. However, the linear span of this set is a subspace containing $\xi \wedge \eta$ for all $\xi \in U$ and $\eta \in U'$, and we claim that it is the whole space $\bigwedge^2 V^*$.

If $U = V^*$, then $U' = V^*$ as well, and the claim is obvious. Suppose that $U \neq V^*$. Then $\dim U = \dim U' = 2$. If $U \neq U'$, then the claim is still true. Finally, if $U' = U$, then the argument in the proof of Proposition 3.1 shows that actually $\ell_{xy} \wedge \ell'_{xy} = 0$ for all $x, y \in V$. By (2.11) in this case $\ell_{xx} \wedge \ell'_{yy} = 0$ for all $x, y \in V$, but this contradicts the assumption $\dim U = 2$. □

Lemma 3.6. *The polynomial functions p_i, p'_i are all nonzero. As a consequence, p_0 and p'_0 are not functions of just one argument. In particular,* $\deg p_0 > 1$ *and* $\deg p'_0 > 1$.

Proof. Suppose that $p_2 = 0$, for example. By (3.5) and (3.7) then

$$p_0(x, y)\, b_i(x, y) = p_1(x, y)\, a_i(x)$$

for all $x, y \in V$ and each $i = 1, 2, 3$. Since the ring $\Bbbk[V \times V]$ is factorial and since $\gcd(p_0, p_1) = 1$ by (3.4), the function p_1 must divide each b_i in that ring.

The functions b_1, b_2, b_3 have bidegree $(1, 1)$. Hence each bidegree component of p_1 does not exceed 1. If p_1 has bidegree $(1, 1)$, then each b_i will be a scalar multiple of p_1, and so $\ell_{xy} = p_1(x, y)\, \ell_0$ for some linear form $\ell_0 \in V^*$ which depends neither on x nor on y. In this case all linear forms ℓ_{xy} are contained in the one-dimensional subspace of V^* spanned by ℓ_0, in contradiction with Lemma 2.4. Clearly p_1 must depend on y. Therefore the only possibility left is that p_1 has bidegree $(0, 1)$, i.e., $p_1(x, y) = \xi(y)$ for some $\xi \in V^*$. But then $\ell_{xy} = 0$ for all $x \in V$ and $y \in \operatorname{Ker} \xi$, again in contradiction with Lemma 2.4.

It is proved similarly that $p_1 \neq 0$, while $p_0 \neq 0$ is clear already from the assumption that $\ell_{xx} \wedge \ell'_{yy}$ does not vanish identically.

Let (m_i, n_i) be the bidegree of p_i. By the uniqueness of expressions (3.5) the basis linear forms ℓ_1, ℓ_2, ℓ_3 must occur with coefficients of equal bidegrees in each term of the first equation there. It follows that

$$m_0 = m_1 + 1 = m_2 - 1 \qquad \text{and} \qquad n_0 = n_1 - 1 = n_2 + 1. \tag{3.10}$$

Hence $m_0 > 0$ and $n_0 > 0$. The same argument applies to p'_0, p'_1, p'_2. □

Lemma 3.7. *The equality $\ell_{xx} \wedge \ell'_{xx} = 0$ holds for all $x \in V$. As a consequence, $U = U' = V^*$.*

Proof. Setting $y = x$ in (3.5), we get

$$\begin{aligned} \big(p_0(x, x) - p_1(x, x)\big)\, \ell_{xx} &= p_2(x, x)\, \ell'_{xx}\,, \\ \big(p'_0(x, x) - p'_2(x, x)\big)\, \ell'_{xx} &= p'_1(x, x)\, \ell_{xx}\,. \end{aligned} \tag{3.11}$$

If p_0 has degree 1 in the second argument, then by (3.10) p_2 must have degree 0 in the second argument, i.e., p_2 depends only on the first argument. In this case p_2 is not identically zero on the diagonal

$$D = \{(x,x) \mid x \in V\} \subset V \times V,$$

and for all $x \in V$ such that $p_2(x,x) \neq 0$ the linear function ℓ'_{xx} is a scalar multiple of ℓ_{xx} in view of (3.11). Similarly, if p'_0 has degree 1 in the first argument, then p'_1 depends only on the second argument. In this case p'_1 is not identically zero on D, and so ℓ_{xx} is a scalar multiple of ℓ'_{xx} for all $x \in V$ such that $p'_1(x,x) \neq 0$. In both cases the equality $\ell_{xx} \wedge \ell'_{xx} = 0$ holds for all x in a nonempty Zariski open subset of V, and therefore everywhere on V.

To confirm the first assertion of Lemma 3.7 it remains to note that one of the previous two conditions is satisfied in any event. Since p_0 and p'_0 are divisors of d, it follows from Lemma 3.6 that d has positive degree in each of the two arguments. But $\deg d < 4$ by Lemma 3.5. This leaves only 3 possibilities for the bidegree of d. It is $(1,1)$, or $(1,2)$, or $(2,1)$. If neither p_0 nor p'_0 have bidegree $(1,1)$, then both functions will be scalar multiples of d, and so they will have the same bidegree, either $(1,2)$ or $(2,1)$.

The equality $\ell_{xx} \wedge \ell'_{xx} = 0$ implies that ℓ'_{xx} is a scalar multiple of ℓ_{xx}, and so $\ell'_{xx} \in U$, whenever $\ell_{xx} \neq 0$. Thus $\ell'_{xx} \in U$ for all x in a nonempty Zariski open subset of V, and therefore for all $x \in V$. It follows that $U' \subset U$, and in fact $U' = U$ since these two spaces have the same dimension. Now

$$p_0(x,y)\,\ell_{xy} \in U + U' = U$$

for all $x, y \in V$ according to (3.5). Hence $\ell_{xy} \in U$ whenever $p_0(x,y) \neq 0$. In other words, $\ell_{xy} \in U$ for all pairs (x,y) in a nonempty Zariski open subset of $V \times V$, but then this containment holds everywhere on $V \times V$, and the equality $U = V^*$ follows from Lemma 2.4. □

Lemma 3.8. *The assumption that the three quadratic forms a_1, a_2, a_3 have a non-scalar common divisor in the ring $\Bbbk[V]$ leads to a contradiction. Hence we must have* $\gcd(a_1, a_2, a_3) = 1$. *Also,* $\gcd(a'_1, a'_2, a'_3) = 1$.

Proof. Suppose that $\xi \in \Bbbk[V]$ is a nonscalar polynomial function which divides each a_i. Clearly, ξ is homogeneous of degree ≤ 2. If $\deg \xi = 2$ then each a_i is a scalar multiple of ξ, and so $\ell_{xx} = \xi(x)\ell_0$ for some linear function $\ell_0 \in V^*$ which does not depend on x. This contradicts the assumption $\dim U > 1$, however.

Therefore $\deg \xi = 1$, i.e., $\xi \in V^*$. We have $\ell_{xx} = 0$ for all $x \in \operatorname{Ker} \xi$, whence $\ell'_{xx} = 0$ for all $x \in \zeta^{-1}(\operatorname{Ker} \xi)$ in view of (2.10). It follows that each a'_i is divisible in $\Bbbk[V]$ by the linear function $\eta = \xi \circ \zeta$. We can write

$$\ell_{xx} = \xi(x) \sum f_i(x)\ell_i\,, \qquad \ell'_{xx} = \eta(x) \sum g_i(x)\ell_i \tag{3.12}$$

for some linear forms $f_i, g_i \in V^*$, and we get $\Delta_i(x,y) = \xi(x)\,\eta(y)\,\delta_i(x,y)$ where $\delta_i(x,y)$ is the respective minor of order 2 of the matrix

$$\begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(y) & g_2(y) & g_3(y) \end{pmatrix}.$$

Each δ_i is a polynomial function on $V \times V$ of bidegree $(1,1)$, and $\delta_1, \delta_2, \delta_3$ are linearly independent by Lemma 3.5. Hence the degree of any common divisor of these three functions is less than 2.

Suppose that $\gcd(\delta_1, \delta_2, \delta_3) \neq 1$. Then there exists a linear form h on $V \times V$ of bidegree either $(1,0)$ or $(0,1)$ which divides each δ_i. If, for example, the bidegree of h is $(1,0)$, then $\operatorname{Ker} h = L \times V$ where L is a subspace of codimension 1 in V, and each δ_i must vanish on $L \times V$. This means that

$$f_j(x)\, g_k(y) = f_k(x)\, g_j(y) \quad \text{for all } \ j,k \in \{1,2,3\}, \ x \in L, \ y \in V.$$

Since $\dim U' = 3$ by Lemma 3.7, the functions g_1, g_2, g_3 must be linearly independent, and it follows from the displayed equalities above that each f_i vanishes on L. But then f_1, f_2, f_3 are scalar multiples of one function, and we see from (3.12) that all functions $\ell_{xx}, \ x \in V$, lie in a one-dimensional subspace of V^*. This contradicts the assumption $\dim U > 1$, however.

We conclude that $\gcd(\delta_1, \delta_2, \delta_3) = 1$, and therefore $\gcd(\Delta_1, \Delta_2, \Delta_3) = d$ where d is defined by the formula

$$d(x,y) = \xi(x)\, \eta(y), \qquad x, y \in V.$$

Since p_0 and p_0' are divisors of d of degree at least 2 by Lemma 3.6, both p_0 and p_0' must be scalar multiples of d. By scaling the functions we may assume without loss of generality that

$$p_0 = p_0' = d.$$

Here p_0 has bidegree $(1,1)$. This implies that p_1 and p_1' have bidegree $(0,2)$, while p_2 and p_2' have bidegree $(2,0)$.

Multiplying the first equation in (3.5) by $p_2'(x,y)$, subtracting then the second equation multiplied by $p_2(x,y)$, and making use of (3.6), we get

$$\begin{aligned} p_0(x,y)\left(p_2'(x,y)\, \ell_{xy} - p_2(x,y)\, \ell_{xy}'\right) &= \left(p_2'(x,y)\, p_1(x,y) - p_2(x,y)\, p_1'(x,y)\right) \ell_{xx} \\ &= p_0(x,y)^2\, \ell_{xx} \end{aligned}$$

for all $x, y \in V$. Since the ring $\Bbbk[V \times V]$ is a domain, the common factor $p_0(x,y)$ in the left and right hand sides can be cancelled out, which results in the identity

$$p_2'(x,y)\, \ell_{xy} - p_2(x,y)\, \ell_{xy}' = p_0(x,y)\, \ell_{xx}\,.$$

Hence

$$p_2'(x,y)\, \ell_{xy}(x) - p_2(x,y)\, \ell_{xy}'(x) = p_0(x,y)\, \ell_{xx}(x) = \xi(x)\, \eta(y)\, \ell_{xx}(x)\,.$$

We have $p_2'(x,y) = p_2'(x,0)$ and $p_2(x,y) = p_2(x,0)$ since p_2 and p_2' depend only on the first argument. Now making use of (2.14), we rewrite the preceding equality as

$$p_2'(x,0)\, \ell_{xx}(y) - p_2(x,0)\, \ell_{xx}'(y) = \xi(x)\, \ell_{xx}(x)\, \eta(y)\,.$$

Since this holds for all $x, y \in V$, we get

$$p_2'(x,0)\, \ell_{xx} - p_2(x,0)\, \ell_{xx}' = \xi(x)\, \ell_{xx}(x)\, \eta \tag{3.13}$$

for all $x \in V$. The subset

$$O = \{x \in V \mid \ell_{xx} \text{ is not a scalar multiple of } \eta\}$$

is clearly Zariski open in V. It is also nonempty in view of Lemma 3.7. Suppose that $x \in O$. Then $\ell_{xx} \neq 0$, and the linear form in the left hand side of (3.13) is a scalar multiple of ℓ_{xx} since so is ℓ'_{xx} by Lemma 3.7. Since ℓ_{xx} and η are linearly independent, both sides of (3.13) must vanish. This yields $\xi(x)\,\ell_{xx}(x) = 0$.

We conclude that $\xi(x)\,\ell_{xx}(x) = 0$ for all $x \in V$ since this equality holds for all x in a nonempty Zariski open subset of V. Since $\xi \neq 0$, we get

$$\ell_{xx}(x) = 0 \quad \text{for all } x \in V.$$

With this at hand we go back to (3.12). Consider the bilinear form $\gamma : V \times V \to \Bbbk$ and linear forms $\gamma_x \in V^*$ defined by the rule

$$\gamma(x,y) = \sum f_i(x)\,\ell_i(y), \qquad \gamma_x(y) = \gamma(x,y)$$

for $x, y \in V$. Then $\ell_{xx} = \xi(x)\,\gamma_x$. Hence $\xi(x)\,\gamma(x,x) = \ell_{xx}(x) = 0$ for all $x \in V$, and therefore $\gamma(x,x) = 0$ for all $x \in V$. This shows that the bilinear form γ is alternating, which implies that its rank is even. Since $\dim V = 3$, any alternating bilinear form on V is degenerate. Hence there exists $0 \neq v \in V$ such that $\gamma_x(v) = 0$, and therefore also $\ell_{xx}(v) = 0$, for all $x \in V$. This contradicts the equality $U = V^*$ of Lemma 3.7.

At the beginning of the proof we have seen that the assumption $\gcd(a_1, a_2, a_3) \neq 1$ implies that $\gcd(a'_1, a'_2, a'_3) \neq 1$. The opposite implication is proved quite similarly. Therefore the assumption $\gcd(a'_1, a'_2, a'_3) \neq 1$ leads to a contradiction too. $\square$

Lemma 3.9. *The polynomial function $d = \gcd(\Delta_1, \Delta_2, \Delta_3)$ has bidegree $(1,1)$, i.e., $d : V \times V \to \Bbbk$ is a bilinear form on V. As a consequence, both p_0 and p'_0 are scalar multiples of d.*

Proof. For each $x \in V$ the linear forms ℓ_{xx} and ℓ'_{xx} are linearly dependent by Lemma 3.7. Applying Lemma 3.3 to the two polynomial maps $V \to V^*$ given by the assignments $x \mapsto \ell_{xx}$ and $x \mapsto \ell'_{xx}$, respectively, we deduce that there are polynomial functions $\xi, \eta \in \Bbbk[V]$ such that $\gcd(\xi, \eta) = 1$ and

$$\xi(x)\,\ell'_{xx} = \eta(x)\,\ell_{xx} \quad \text{for all } x \in V.$$

In the ring $\Bbbk[V]$ we have then $\xi a'_i = \eta a_i$ for each $i = 1, 2, 3$ (see (3.7)). Hence ξ is a common divisor of quadratic forms a_1, a_2, a_3, and so $\xi \in \Bbbk$ by Lemma 3.8. Clearly, $\deg \xi = \deg \eta$, whence $\eta \in \Bbbk$ as well. It follows that there is $0 \neq c \in \Bbbk$ such that $a'_i = c a_i$ for each i. But then each Δ_i is skewsymmetric, i.e.,

$$\Delta_i(y,x) = -\Delta_i(x,y) \quad \text{for all } x, y \in V,$$

and it follows that the degree of d in the second argument must be the same as its degree in the first argument. So the total degree of d is even and less than 4 by Lemma 3.5. Since p_0 and p'_0 are divisors of d, there is no possibility other than

$$\deg d = \deg p_0 = \deg p'_0 = 2,$$

and the bidegree of these functions is necessarily $(1,1)$. $\square$

Thus we may take $p_0 = p'_0 = d$ in (3.5).

Lemma 3.10. *The function p_0 is a bilinear form of rank* 1*, i.e., p_0 is the product of two linear forms of bidegrees* $(1, 0)$ *and* $(0, 1)$*.*

Proof. Identity (3.6) shows that p_0 is contained in the radical of the ideal of the ring $\Bbbk[V \times V]$ generated by p_1 and p_2. Hence p_0 vanishes on the set of common zeros of the two functions p_1 and p_2. Note that p_1 and p_1' have bidegree $(0, 2)$, while p_2 and p_2' have bidegree $(2, 0)$. Hence the zero sets of p_1 and p_2 are, respectively, $V \times Y$ and $X \times V$ where X and Y are some quadratic conical hypersurfaces in V. The common zero set of p_1 and p_2 is $X \times Y$. Since p_0 is bilinear, it vanishes on $\Bbbk X \times \Bbbk Y$ where $\Bbbk X$ and $\Bbbk Y$ are the linear spans of X and Y.

Since $p_0 \neq 0$, the equalities $\Bbbk X = V$ and $\Bbbk Y = V$ cannot hold simultaneously. Therefore either X or Y must be a linear subspace of codimension 1 in V. This means that at least one of the two functions p_1, p_2 is the square of a linear form on $V \times V$. If $\Bbbk X = V$ then Y is a linear subspace of V contained in the right kernel of p_0, whence $\operatorname{rank} p_0 = 1$. Similarly, $\operatorname{rank} p_0 = 1$ whenever $\Bbbk Y = V$.

Suppose that $\operatorname{rank} p_0 \neq 1$. Then both X and Y are linear subspaces of codimension 1 in V. Hence there exist $\xi, \eta \in V^*$ such that

$$p_1(x, y) = \eta(y)^2, \qquad p_2(x, y) = \xi(x)^2$$

for all $x, y \in V$. A similar argument shows that there are $\xi', \eta' \in V^*$ such that

$$p_1'(x, y) = \eta'(y)^2, \qquad p_2'(x, y) = \xi'(x)^2$$

for all $x, y \in V$. Identity (3.6) is now written as

$$p_0(x, y)^2 = \eta(y)^2\, \xi'(x)^2 - \xi(x)^2\, \eta'(y)^2.$$

Hence $p_0^2 = fg$ where $f, g \in \Bbbk[V \times V]$ are defined by the formulas

$$f(x, y) = \eta(y)\, \xi'(x) - \xi(x)\, \eta'(y), \qquad g(x, y) = \eta(y)\, \xi'(x) + \xi(x)\, \eta'(y).$$

Since p_0 is a bilinear form of rank > 1, it is an irreducible element of the factorial ring $\Bbbk[V \times V]$, and it follows from the factoriality that f and g are both scalar multiples of p_0. Then so too is $f + g$. But

$$(f + g)(x, y) = 2\, \eta(y)\, \xi'(x),$$

in contradiction with irreducibility of p_0. Note that here we do need the assumption that $\operatorname{char} \Bbbk \neq 2$. ☐

Lemma 3.11. *There are $\xi, \eta \in V^*$ such that*

$$p_0(x, y) = \xi(x)\, \eta(y) \tag{3.14}$$

and either $p_1(x, y) = \eta(y)^2$ or $p_2(x, y) = \xi(x)^2$ for all $x, y \in V$.

Proof. We continue the argument in the proof of Lemma 3.10. Since p_0 has rank 1, its left kernel $L = \{x \in V \mid p_0(x, V) = 0\}$ is a vector subspace of codimension 1 in V. If $X = L$, then, as we have seen in Lemma 3.10, p_2 is a square, i.e., $p_2 = \xi_1^2$ where ξ_1 is a linear form of bidegree $(1, 0)$. There is $\xi \in V^*$ such that $\xi_1(x, y) = \xi(x)$ for all $x, y \in V$. Since the function p_0 vanishes on $X \times V = \operatorname{Ker} \xi_1$, it is divisible by ξ_1 in the ring $\Bbbk[V \times V]$. This yields factorization (3.14).

Suppose that $X \neq L$. Then $X \not\subset L$ since X is a subvariety of codimension 1 in V, and it follows that $\Bbbk X + L = V$. Since p_0 vanishes on $(\Bbbk X + L) \times Y$, its right kernel $K = \{y \in V \mid p_0(V, y) = 0\}$ contains Y. Moreover, $K = Y$ since Y is a subvariety of codimension 1 in V. In particular, Y is a linear subspace. This implies that $p_1 = \eta_2^2$ where η_2 is a linear form of bidegree $(0,1)$. There is $\eta \in V^*$ such that $\eta_2(x, y) = \eta(y)$ for all $x, y \in V$. Since p_0 vanishes on $V \times Y = \operatorname{Ker} \eta_2$, it is divisible by η_2, whence formula (3.14) with a suitable ξ. □

The rest of the proof will go really fast. In the first equation of (3.5) we compare the coefficients of each basis linear form ℓ_i. This gives

$$p_0(x, y)\, b_i(x, y) = p_1(x, y)\, a_i(x) + p_2(x, y)\, a_i'(y)$$

for all $x, y \in V$. There are two possibilities described in Lemma 3.11. In one case both p_0 and p_2 are divisible by the linear form ξ_1 of bidegree $(1,0)$ corresponding to ξ. On the other hand, ξ_1 cannot divide the function p_1 of bidegree $(0,2)$, and it follows from the displayed equality above and factoriality that each a_i is divisible by ξ. In another case both p_0 and p_1 are divisible by the linear form η_2 of bidegree $(0,1)$ corresponding to η, but η_2 does not divide p_2. This implies that each a_i' is divisible by η. Now Lemma 3.8 eliminates both possibilities. □

This completes the proof of Proposition 3.2 and Theorem 1.1.

4. Explicit determination of Hecke symmetries

Given a graded factor algebra A of the tensor algebra $\mathbb{T}(V)$, we denote by

$$\operatorname{HeckeSym}(A)$$

the set of all Hecke symmetries on the vector space V such that $\mathbb{S}(V, R) = A$ where the exact equality means that the two algebras are factor algebras of $\mathbb{T}(V)$ by the same ideal. In the case when $A = \mathbb{T}(V)/I$ where I is the ideal of $\mathbb{T}(V)$ generated by $\{\zeta(x)y - \zeta(y)x \mid x, y \in V\} \subset V^{\otimes 2}$ for $\zeta \in GL(V)$ we write $\mathbb{S}(V)_\zeta$ instead of A, thus identifying the twisted algebra $\mathbb{S}(V)_\zeta$ with a factor algebra of $\mathbb{T}(V)$ by means of an isomorphism of graded algebras which acts as the identity operator on homogeneous elements of degree 1.

Assume that $\operatorname{char} \Bbbk \neq 2$. By Theorem 1.1 the set $\operatorname{HeckeSym}(\mathbb{S}(V)_\zeta)$ consists of the ζ-twists R_ζ of those Hecke symmetries R in the set $\operatorname{HeckeSym}(\mathbb{S}(V))$ which commute with $\zeta \otimes \zeta$. By [13, Theorem 5.1] each R in the latter set is given by the formula

$$R(xy) = \frac{q-1}{2} xy + \frac{q+1}{2} yx - g(x, y)\, a \barwedge b - x \barwedge Ty - y \barwedge Tx, \quad x, y \in V, \tag{4.1}$$

where $a, b \in V$ are two vectors, $g : V \times V \to \Bbbk$ a symmetric bilinear form satisfying

$$(q-1)^2 = 4\left(g(a, b)^2 - g(a, a)\, g(b, b)\right) \tag{4.2}$$

and $T : V \to V$ the linear operator defined by the rule

$$Tx = g(b,x)\,a - g(a,x)\,b, \qquad x \in V. \tag{4.3}$$

The product $\barwedge$ used in (4.1) is the one defined in section 1, however with respect to the identity operator, i.e., $x \barwedge y = xy - yx$ for $x, y \in V$.

The linear operator given by (4.1) depends on the pair (t,g) where $t = a \wedge b$ is a bivector. A different choice of vectors a, b making the same bivector does not change the operator. The parameter q is determined by relation (4.2) in which the right hand side is a function of the pair (t,g). If either $t = 0$ or $g = 0$, then this operator is the flip R_0 which sends xy to yx for all $x, y \in V$. If $t \neq 0$ and $g \neq 0$, then another pair (t', g') produces the same operator if and only if $t' = ct$ and $g' = c^{-1}g$ for some $0 \neq c \in \Bbbk$.

This leads to the following parametrization of the set $\mathrm{HeckeSym}(\mathbb{S}(V))$. Define

$$\Delta(a \wedge b,\, g) = g(a,a)\,g(b,b) - g(a,b)^2 \tag{4.4}$$

and denote by P the set of all triples (t,g,q) where $t \in \bigwedge^2 V$ is a nonzero bivector, $g : V \times V \to \Bbbk$ a nonzero symmetric bilinear form, $q \in \Bbbk$ a nonzero scalar such that

$$(q-1)^2 = -4\,\Delta(t,g). \tag{4.5}$$

The multiplicative group $\Bbbk^\times$ of the field $\Bbbk$ acts on P according to the rule

$$c \cdot (t,\, g,\, q) = (ct,\, c^{-1}g,\, q), \qquad c \in \Bbbk^\times. \tag{4.6}$$

The elements of the set $\mathrm{HeckeSym}(\mathbb{S}(V)) \smallsetminus \{R_0\}$ are then in a bijective correspondence with the $\Bbbk^\times$-orbits in P. Furthermore, $\bigwedge^2 V$ and the space of symmetric bilinear forms on V are $GL(V)$-modules in a natural way. These module structures give rise to an action of $GL(V)$ on P under which q remains unaffected. Conjugation by the operators $\varphi \otimes \varphi$, $\varphi \in GL(V)$, is the corresponding action of $GL(V)$ on Hecke symmetries. Therefore two Hecke symmetries in the set $\mathrm{HeckeSym}(\mathbb{S}(V)) \smallsetminus \{R_0\}$ are equivalent if and only if they correspond to two triples in the set P lying in the same $(GL(V) \times \Bbbk^\times)$-orbit.

The flip R_0 commutes with $\zeta \otimes \zeta$ for any $\zeta \in GL(V)$. The corresponding twist $R_{0\zeta}$ is the linear operator on $V^{\otimes 2}$ such that

$$R_{0\zeta}(xy) = \zeta(y)\,\zeta^{-1}(x) \quad \text{for } x, y \in V. \tag{4.7}$$

It may be viewed as a distinguished element of the set $\mathrm{HeckeSym}(\mathbb{S}(V)_\zeta)$. The remaining Hecke symmetries in this set can be described in terms of triples in P:

Proposition 4.1. *Assume* $\operatorname{char} \Bbbk \neq 2$. *For* $\zeta \in GL(V)$ *put*

$$P(\zeta) = \{(t,g,q) \in P \mid \zeta \cdot (t,g,q) \in \Bbbk^\times \cdot (t,g,q)\},$$
$$G(\zeta) = \{\, \varphi \in GL(V) \mid \varphi\zeta\varphi^{-1} \textit{ is a scalar multiple of } \zeta\}.$$

There is a bijection between the set $\mathrm{HeckeSym}(\mathbb{S}(V)_\zeta) \smallsetminus \{R_{0\zeta}\}$ *and the set* $P(\zeta)/\Bbbk^\times$ *of* $\Bbbk^\times$*-orbits in* $P(\zeta)$*. Under this bijection the equivalence classes of Hecke symmetries correspond to the* $G(\zeta)$*-orbits in* $P(\zeta)/\Bbbk^\times$*.*

Proof. If $R \in \mathrm{HeckeSym}(\mathbb{S}(V))$ corresponds to a triple $(t,g,q) \in P$ then the Hecke symmetry $(\zeta \otimes \zeta) \circ R \circ (\zeta \otimes \zeta)^{-1}$ corresponds to $\zeta \cdot (t,g,q)$. Therefore R commutes with $\zeta \otimes \zeta$ if and only if $(t,g,q) \in P(\zeta)$. Such Hecke symmetries are in a bijective correspondence with the $\Bbbk^\times$-orbits in $P(\zeta)$. Composing this bijection with the twisting transformation $R \mapsto R_\zeta$ we get a bijection asserted in Proposition 4.1.

For any $\varphi \in GL(V)$ the space $\{\zeta(x)y - \zeta(y)x \mid x,y \in V\}$ of quadratic defining relations of the algebra $\mathbb{S}(V)_\zeta$ is stable under the linear operator $\varphi \otimes \varphi$ if and only if $\varphi\zeta\varphi^{-1} = c\zeta$ for some $c \in \Bbbk^\times$. In other words, φ extends to an automorphism of $\mathbb{S}(V)_\zeta$ if and only if $\varphi \in G(\zeta)$. This means that the equivalence classes of Hecke symmetries in the set $\mathrm{HeckeSym}(\mathbb{S}(V)_\zeta)$ are precisely the orbits with respect to the conjugating action of the group $G(\zeta)$ (see the discussion at the end of section 1).

If $\varphi \in G(\zeta)$, then $(\varphi \otimes \varphi)(\zeta \otimes \mathrm{Id}_V) = c(\zeta \otimes \mathrm{Id}_V)(\varphi \otimes \varphi)$ for some $c \in \Bbbk^\times$, and it follows that

$$(\varphi \otimes \varphi) \circ R_\zeta \circ (\varphi \otimes \varphi)^{-1} = R'_\zeta \quad \Leftrightarrow \quad (\varphi \otimes \varphi) \circ R \circ (\varphi \otimes \varphi)^{-1} = R'.$$

Hence the constructed bijection between $\mathrm{HeckeSym}(\mathbb{S}(V)_\zeta) \smallsetminus \{R_{0\zeta}\}$ and $P(\zeta)/\Bbbk^\times$ is $G(\zeta)$-equivariant. We thus get the second assertion of Proposition 4.1. $\square$

The set $P(\zeta)$ can be determined for each $\zeta \in GL(V)$, and this leads to the classification of the corresponding Hecke symmetries. In this paper we will investigate in detail the case of a diagonalizable twisting operator.

Equivalence classes in the set $\mathrm{HeckeSym}(\mathbb{S}(V))$ are of 8 types described in [13]. Types 1 and 2 include Hecke symmetries with parameter $q \neq 1$, while $q = 1$ in the other types. The action of a Hecke symmetry R of respective type is given with respect to a suitable basis x_1, x_2, x_3 of V by the following formulas:

Type 1.

$$\begin{array}{lll}
R(x_1^2) = qx_1^2 & R(x_1x_2) = (q-1)x_1x_2 + x_2x_1 & R(x_1x_3) = (q-1)x_1x_3 + x_3x_1 \\
R(x_2x_1) = qx_1x_2 & R(x_2^2) = qx_2^2 & R(x_2x_3) = qx_3x_2 \\
R(x_3x_1) = qx_1x_3 & R(x_3x_2) = (q-1)x_3x_2 + x_2x_3 & R(x_3^2) = qx_3^2 - x_1x_2 + x_2x_1
\end{array}$$

Type 2. The same formulas as in Type 1 with the exception that $R(x_3^2) = qx_3^2$.

Type 3.

$$\begin{array}{lll}
R(x_1^2) = x_1^2 + x_1x_2 - x_2x_1 & R(x_1x_2) = x_2x_1 & R(x_1x_3) = x_3x_1 - x_2x_3 + x_3x_2 \\
R(x_2x_1) = x_1x_2 & R(x_2^2) = x_2^2 & R(x_2x_3) = x_3x_2 \\
R(x_3x_1) = x_1x_3 - x_2x_3 + x_3x_2 & R(x_3x_2) = x_2x_3 & R(x_3^2) = x_3^2 + 2(x_1x_3 - x_3x_1)
\end{array}$$

Type 4. As in Type 3 with the exception that $R(x_3^2) = x_3^2 - x_1x_2 + x_2x_1$.

Type 5. As in Type 3 with the exception that $R(x_3^2) = x_3^2$.

Type 6.

$$\begin{array}{lll}
R(x_1^2) = x_1^2 & R(x_1x_2) = x_2x_1 & R(x_1x_3) = x_3x_1 \\
R(x_2x_1) = x_1x_2 & R(x_2^2) = x_2^2 & R(x_2x_3) = x_3x_2 \\
R(x_3x_1) = x_1x_3 & R(x_3x_2) = x_2x_3 & R(x_3^2) = x_3^2 + 2(x_1x_3 - x_3x_1)
\end{array}$$

Type 7. The same formulas as in Type 1, but with $q = 1$.

Type 8. R is the flip operator R_0 sending x_ix_j to x_jx_i.

Such an operator R corresponds to a pair (t, g) in which $t = x_1 \wedge x_2$, while the bilinear form g depends on the type to which R belongs (see [13]).

When forming the twist R_ζ, the condition that R should commute with $\zeta \otimes \zeta$ forces the basis vectors x_1, x_2, x_3 to be adapted to the twisting operator ζ in some way. Some types of Hecke symmetries may not be permitted by a particular twisting operator.

If ζ is a scalar operator, then $\mathbb{S}(V)_\zeta \cong \mathbb{S}(V)$, and twisting by ζ does not change Hecke symmetries. The case of nonscalar diagonalizable operators is described as follows:

Proposition 4.2. *Assume that* $\operatorname{char} \Bbbk \neq 2$ *and* $\zeta \in GL(V)$ *is a diagonalizable linear operator with at least two distinct eigenvalues. Any Hecke symmetry in the set* $\operatorname{HeckeSym}(\mathbb{S}(V)_\zeta)$ *is the* ζ*-twist of a Hecke symmetry* R *whose action is given by the formulas for one of Types 1–8 with respect to some basis of* V *consisting of eigenvectors* x_1, x_2, x_3 *of* ζ.

Moreover, with α_i *being the eigenvalue of* ζ *corresponding to the eigenvector* x_i,

Types 1 *and* 7 *occur only when* $\alpha_3^2 = \alpha_1\alpha_2$,
Type 3 *does not occur*,
Type 4 *occurs only when* $\alpha_1 = \alpha_2 = -\alpha_3$,
Type 5 *occurs only when* $\alpha_1 = \alpha_2$,
Type 6 *occurs only when* $\alpha_1 = \alpha_3$.

Proof. By Theorem 1.1 any Hecke symmetry in the set $\operatorname{HeckeSym}(\mathbb{S}(V)_\zeta)$ is R_ζ for some $R \in \operatorname{HeckeSym}(\mathbb{S}(V))$ such that R commutes with $\zeta \otimes \zeta$. If R is the flip R_0, then its matrix form does not depend on the choice of a basis for V. So we may assume that $R \neq R_0$. Let $(t, g, q) \in P(\zeta)$ be the corresponding triple.

There is a 2-dimensional subspace $V(t)$ of V spanned by any pair of vectors a, b such that $t = a \wedge b$. The condition that $\zeta \cdot t = ct$ for some $c \in \Bbbk^\times$ implies that $V(t)$ is invariant under ζ. Since ζ is diagonalizable, there exist its eigenvectors $x_1, x_2 \in V(t)$ such that $t = x_1 \wedge x_2$. Then $c = \alpha_1\alpha_2$ where α_1, α_2 are the respective eigenvalues. The condition that $\zeta \cdot g = c^{-1} g$ derived from (4.6) means that $g(u, v) = 0$ whenever u, v are two eigenvectors for ζ with eigenvalues λ, μ such that $\lambda\mu \neq \alpha_1\alpha_2$.

Let α_3 be the third eigenvalue of ζ corresponding to any eigenvector x_3 linearly independent from x_1 and x_2. There are several cases distinguished by some conditions on the triple of eigenvalues $\alpha_1, \alpha_2, \alpha_3$.

If $\alpha_1, \alpha_2, \alpha_3$ are pairwise distinct, then none of the products α_1^2, α_2^2, $\alpha_1\alpha_3$, $\alpha_2\alpha_3$ is equal to $\alpha_1\alpha_2$. In this case the matrix of the bilinear form g with respect to the basis x_1, x_2, x_3 is

$$\begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{4.8}$$

for some $\beta, \gamma \in \Bbbk$. In view of the previously mentioned condition on g we must have $\gamma = 0$ unless $\alpha_3^2 = \alpha_1\alpha_2$. By (4.4) and (4.5) we have $(q-1)^2 = 4\beta^2$. Hence β is either $(q-1)/2$ or $(1-q)/2$. Since $t = -x_2 \wedge x_1$, we can replace x_1, x_2 by the pair $-x_2$, x_1, and then β in the matrix of g will be changed to $-\beta$. Thus we can always find a basis of V consisting of eigenvectors of ζ with respect to which the matrix of g will have $\beta = (q-1)/2$. Note that $\beta\gamma \neq 0$ since $g \neq 0$. If $\gamma \neq 0$, then we can achieve $\gamma = 1$, replacing x_3 with its scalar multiple. Computing the action of R defined in (4.1) we obtain the formulas given for Types 1, 2, or 7.

Suppose now that $\alpha_1 = \alpha_2 \neq \alpha_3$. In this case $V(t)$ is a 2-dimensional eigenspace of ζ which is orthogonal to x_3 with respect to g because $\alpha_1\alpha_3 \neq \alpha_1^2$. If the restriction of g to $V(t)$ is nondegenerate, then there is a basis x_1, x_2 for $V(t)$ consisting of isotropic with respect to g vectors. With this choice the matrix of g will have the form (4.8), and we continue as in the previous case. So we do also when g vanishes on $V(t) \times V(t)$. If the restriction of g to $V(t)$ has rank 1, then we adjust the choice of x_1 and x_2 to obtain the following matrix of g:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{4.9}$$

If $\gamma \neq 0$, then we achieve $\gamma = 1$ by scaling. This is only possible when $\alpha_3^2 = \alpha_1^2$, i.e., $\alpha_3 = -\alpha_1$. The corresponding operator R acts by formulas of Type 4 or 5 depending on whether γ is 1 or 0.

Suppose finally that $\alpha_1 \neq \alpha_2$, but α_3 equals either α_1 or α_2. Since $\alpha_i^2 \neq \alpha_1\alpha_2$, we have $g(x_i, x_i) = 0$ for each i. Moreover, g has zero restriction to the 2-dimensional eigenspace U of ζ corresponding to the eigenvalue α_3. If $g(x_1, x_2) \neq 0$, then we can find $x_3 \in U$ orthogonal to both x_1 and x_2. In such a basis the matrix of g is of form (4.8) which has been considered already. If $g(x_1, x_2) = 0$, then we may assume $\alpha_3 = \alpha_1$, replacing x_1, x_2 with the pair $-x_2, x_1$ in the case when $\alpha_3 = \alpha_2$. With this assumption we will have $g(x_3, x_2) \neq 0$, and scaling the vectors brings the matrix of g to the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.10}$$

In this case the action of R is given by the formulas of Type 6. $\square$

The explicit formulas for the twisted operator R_ζ contain the same monomials as the formulas for R, but altered coefficients. For example, if R is of Type 1 or 2, then

$$R_\zeta(x_1x_2) = (q-1)x_1x_2 + \alpha_2\alpha_1^{-1}x_2x_1.$$

We do not write out such formulas in full because of space considerations.

It should be stressed that the eigenvectors x_1, x_2, x_3 in Proposition 4.2 are not fixed for the given operator ζ but each time are suited to a particular Hecke symmetry. This means that different Hecke symmetries in the description of that proposition require different numbering of eigenspaces and eigenvalues. For example, if ζ has an eigenvalue α_1 of multiplicity 1 and another eigenvalue α_2 of multiplicity 2, then the formulas for Type 5 should be used with respect to an ordered triple of eigenvectors with respective eigenvalues $\alpha_2, \alpha_2, \alpha_1$, while the formulas for Type 6 require a reordering, so that the respective eigenvalues should be $\alpha_2, \alpha_1, \alpha_2$.

The question concerning equivalence of twisted Hecke symmetries can be solved with the aid of Proposition 4.1. For this we need to know the group $G(\zeta)$.

Lemma 4.3. *The group $G(\zeta)$ either coincides with the centralizer of ζ in $GL(V)$ or contains this centralizer as a subgroup of index* 3. *In the latter case* $\operatorname{char} \Bbbk \neq 3$ *and the linear operator ζ has* 3 *distinct characteristic roots $\alpha_1, \alpha_2, \alpha_3$ in the algebraic closure of $\Bbbk$ such that*

$$\alpha_1\alpha_2^{-1} = \alpha_2\alpha_3^{-1} = \alpha_3\alpha_1^{-1} = \varepsilon \tag{4.11}$$

where ε is a primitive cube root of 1.

Proof. Extending the base field $\Bbbk$, we may assume $\Bbbk$ to be algebraically closed. If $\varphi \in G(\zeta)$, then $\varphi\zeta\varphi^{-1} = c\zeta$ for some $c \in \Bbbk^\times$. If $c \neq 1$, then ζ maps the generalized eigenspace of ζ corresponding to some eigenvalue α to another generalized eigenspace which corresponds to the eigenvalue $c\alpha$. It follows that α and $c\alpha$ have the same multiplicity as roots of the characteristic polynomial of ζ. Since $\dim V = 3$, there can occur at most one eigenvalue of multiplicity larger than 1. If α is such an eigenvalue, then $c\alpha = \alpha$, whence $c = 1$. In the other case ζ has 3 distinct eigenvalues which are permuted by the operator of scalar multiplication by c. Hence $c^3 = 1$, and so c is a primitive cube root of 1 whenever $c \neq 1$. □

Now define the type of a triple $(t, g, q) \in P$ as the type of the corresponding Hecke symmetry with the associated algebra $\mathbb{S}(V)$. Let $\zeta \in GL(V)$ be a diagonalizable linear operator. Note that its centralizer $C(\zeta)$ in $GL(V)$ consists of all invertible linear operators which leave stable each eigenspace of ζ.

In Proposition 4.2 we have seen that for each $(t, g, q) \in P(\zeta)$ there exist 3 linearly independent eigenvectors x_1, x_2, x_3 of ζ such that $t = x_1 \wedge x_2$ and the matrix of g with respect to x_1, x_2, x_3 has one of several possible forms depending on q and certain relations between the eigenvalues of ζ. If $q \neq 1$, then the condition that the matrix of g is (4.8) with $\beta = (q-1)/2$ determines such an ordered basis x_1, x_2, x_3 of V uniquely up to scaling of vectors.

Various choices of eigenvectors x_1, x_2, x_3 and permissible matrices of a symmetric bilinear form give in this way all triples in the set $P(\zeta)$.

Suppose first that the three eigenvalues of ζ are pairwise distinct. Then $P(\zeta)$ has precisely six $\Bbbk^\times$-orbits of Type 2 with any fixed $q \neq 1$, each corresponding to one of 6 possible orderings of the eigenvalues. Each of these orbits is invariant under the action of the centralizer $C(\zeta)$. This gives 6 elements of the set $P(\zeta)/\Bbbk^\times$ fixed by the action of $C(\zeta)$. If $\alpha_1, \alpha_2, \alpha_3$ satisfy (4.11), then the group $G(\zeta)$ contains transformations which permute cyclically the eigenspaces of ζ. In this case the 6 just mentioned elements form two $G(\zeta)$-orbits in $P(\zeta)/\Bbbk^\times$ with 3 elements in each. They correspond to two equivalence classes in the set $\operatorname{HeckeSym}(\mathbb{S}(V)_\zeta)$ with 3 different Hecke symmetries in each. If (4.11) does not hold, then $G(\zeta) = C(\zeta)$, whence we get 6 equivalence classes with only one Hecke symmetry in each.

If there is an eigenvalue λ of ζ such that λ^2 equals the product of the two other eigenvalues, then the set $P(\zeta)$ contains also triples of Type 1 and 7 with $\alpha_3 = \lambda$ for the respective choice of x_1, x_2, x_3. Since the action of $C(\zeta)$ allows arbitrary scaling of the value $g(x_3, x_3)$ while leaving x_1 and x_2 unchanged, each $C(\zeta)$-orbit of Type 1 or 7 in $P(\zeta)/\Bbbk^\times$ has infinitely many elements. If (4.11) does not hold, then there is only one eigenvalue λ with the property needed. In this case the set $P(\zeta)/\Bbbk^\times$ has two $C(\zeta)$-orbits of Type 1 with any fixed $q \neq 1$. However, for $q = 1$ there is only one $C(\zeta)$-orbit of Type 7 since g does not change when the triple (t, g, q) is formed with the basis x_1, x_2, x_3 replaced by $-x_2, x_1, x_3$. If the three eigenvalues of ζ satisfy (4.11), then each of them can be taken to obtain triples of Type 1 and 7. There are six $C(\zeta)$-orbits of Type 1 with any $q \neq 1$ and three $C(\zeta)$-orbits of Type 7, while extra transformations in the group $G(\zeta)$ permute these orbits in cycles of length 3. Thus, regardless of (4.11), the set $P(\zeta)/\Bbbk^\times$ always has two $G(\zeta)$-orbits of Type 1 with any $q \neq 1$ and one $G(\zeta)$-orbit of Type 7 with $q = 1$.

Suppose now that ζ has an eigenvalue α_1 of multiplicity 1 and an eigenvalue α_2 of multiplicity 2. In this case $G(\zeta) = C(\zeta) \cong \Bbbk^\times \times GL_2$. In correspondence with 3 possible orderings of the triple of eigenvalues $\alpha_1, \alpha_2, \alpha_2$, the set $P(\zeta)/\Bbbk^\times$ has three $G(\zeta)$-orbits of Type 2 with any fixed $q \neq 1$. If $\alpha_1 = -\alpha_2$, there is one $G(\zeta)$-orbit of Type 1 with any fixed $q \neq 1$. It consists of all elements of $P(\zeta)/\Bbbk^\times$ with representatives $(t, g, q) \in P(\zeta)$ such that the bivector t corresponds to the 2-dimensional eigenspace of ζ and the bilinear form g is nondegenerate. Considering all triples in $P(\zeta)$ corresponding to the Hecke symmetries of Type either 5 or 6, as discussed in the proof of Proposition 4.2, we see that they form one $G(\zeta)$-orbit in $P(\zeta)/\Bbbk^\times$, and so do the triples of Type 4 and Type 7 in the case when $\alpha_1 = -\alpha_2$. Each of these orbits of Type 1, 2, 4, 5, 6, and 7 has infinitely many elements.

In Corollary 4.4 we will summarize the preceding conclusions. Note that for a diagonalizable linear operator ζ with eigenvalues $\alpha_1, \alpha_2, \alpha_3$ the algebra $\mathbb{S}(V)_\zeta$ has defining relations

$$x_3x_2 = p_1x_2x_3, \qquad x_1x_3 = p_2x_3x_1, \qquad x_2x_1 = p_3x_1x_2 \tag{4.12}$$

where

$$p_1 = \alpha_2\alpha_3^{-1}, \qquad p_2 = \alpha_3\alpha_1^{-1}, \qquad p_3 = \alpha_1\alpha_2^{-1}. \tag{4.13}$$

Conversely, the skew polynomial algebra with defining relations (4.12) is isomorphic to the algebra $\mathbb{S}(V)_\zeta$ for a suitable ζ provided that $p_1p_2p_3 = 1$.

Any cyclic permutation of parameters p_1, p_2, p_3 results in an isomorphic algebra. For this reason we do not mention cyclically permuted triples of parameters in the list of conditions given in the next corollary.

Corollary 4.4. *Assume that* $\operatorname{char}\Bbbk \neq 2$*. Let A be the graded algebra with generators x_1, x_2, x_3 and defining relations* (4.12) *where $p_1p_2p_3 = 1$. For each nonzero $q \in \Bbbk$ the set* $\operatorname{HeckeSym}(A)$ *contains finitely many equivalence classes of Hecke symmetries with the chosen parameter q. The number of equivalence classes depends on q and p_1, p_2, p_3 as shown in the table below:*

	$q \neq 1$	$q = 1$
p_1, p_2, p_3 are pairwise distinct and $p_i \neq 1$ for each i	6	1
$p_1 = p_2 \neq p_3$ and $p_i \neq 1$ for each i	8	2
$p_1 = p_2 = p_3 = \varepsilon$ where ε is a primitive cube root of 1	4	2
$p_1 \neq p_2, \quad p_3 = 1$	3	3
$p_1 = p_2 = -1, \quad p_3 = 1$	4	5
$p_1 = p_2 = p_3 = 1$	2	6

Only $R_{0\zeta}$ and the twists of Hecke symmetries of Type 2 *in the case when $p_i \neq 1$ for each i form equivalence classes of finite cardinality. In particular,* $\operatorname{HeckeSym}(A)$ *contains finitely many Hecke symmetries with some fixed value of q only when p_1, p_2, p_3 are pairwise distinct and $p_i \neq 1$ for each i.*

Proof. If i, j, k are three distinct elements of the set $\{1, 2, 3\}$, then it follows from (4.13) that $p_i = 1$ if and only if $\alpha_j = \alpha_k$, and also $p_j = p_k$ if and only if $\alpha_i^2 = \alpha_j\alpha_k$. By Proposition 4.1 the equivalence classes in the set $\mathrm{HeckeSym}(A)$ are in a bijective correspondence with the $G(\zeta)$-orbits in the set $P(\zeta)/\Bbbk^\times$. So we can refer to the counting of orbits in the discussion preceding Corollary 4.4. The first three lines of the table correspond to the case of pairwise distinct eigenvalues $\alpha_1, \alpha_2, \alpha_3$. The last line records the 8 types of Hecke symmetries with the associated algebra $\mathbb{S}(V)$. $\square$

Motivated by Corollary 4.4 we are led to ask

Question 4.5. *Let A be an arbitrary graded Artin-Schelter regular algebra of global dimension 3 with quadratic defining relations. Is it always true that for each nonzero $q \in \Bbbk$ there are only finitely many equivalence classes of Hecke symmetries with the chosen parameter q and the property that $\mathbb{S}(V, R) \cong A$?*

It is very unlikely that finiteness of this kind can be always satisfied in dimensions larger than 3.

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