Instanton's Insertions to arbitrary non flat Connections in $\mathbb{R}^4$

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Abstract

Given a connection A on a $SU(2)$-bundle P over $\mathbb{R}^4$ with finite Yang-Mills energy $YM(A)$ and nonzero curvature $F_A(0)$ at the origin, and given $\rho > 0$ small enough, we construct a new connection $\hat{A}$ on a bundle $\hat{P}$ of different Chern class ($|c_2(A) - c_2(\hat{A})| = 8\pi^2$), in such a way that $\hat{A}$ is gauge equivalent to A in $\mathbb{R}^4 \setminus B_\rho(0)$ and

$$YM(\hat{A}) \leq YM(A) + 8\pi^2 - \varepsilon_0 \rho^4 |F_A(0)|^2$$

for a universal constant $\varepsilon_0 > 0$.

I Introduction

Consider a connection $A \in \Omega^1(\mathbb{R}^4) \otimes su(2)$ on a $SU(2)$-bundle P. Up to modifying A outside a compact set, we can assume that A is the pull-back via stereographic projection of a smooth connection on a $SU(2)$-bundle over S^4. In particular, we can define its Chern class

$$c_2(A) := \int_{\mathbb{R}^4} \mathrm{tr}(F_A \wedge F_A) \in 8\pi^2 \mathbb{Z}\,, \tag{I.1}$$

and its Yang-Mills energy

$$YM(A) := \int_{\mathbb{R}^4} |F_A|^2 dx^4 < \infty\,. \tag{I.2}$$

The precise definition $\mathrm{tr}(F_A \wedge F_A)$ and $|F_A|$ will be given in the next section. The main result of this paper is the following theorem:

Theorem I.1. *Let $A \in \Omega^1(\mathbb{R}^4) \otimes su(2)$ be a smooth connection form on $\mathbb{R}^4$ with finite Yang-Mills energy such that $F_A(0) \neq 0$. Assume that*

$$|P_+ F_A(0)| \leq |P_- F_A(0)|\,, \tag{I.3}$$

where P_+ and P_- are respectively the projections of 2-forms onto self-dual or anti-self dual 2-forms in $\mathbb{R}^4$. Then, for ρ small enough, one can modify A into $\hat{A}$ such that A and $\hat{A}$ are gauge equivalent to each other on $\mathbb{R}^4 \setminus B_\rho(0)$,

$$\int_{\mathbb{R}^4} \mathrm{tr}(F_{\hat{A}} \wedge F_{\hat{A}}) = \int_{\mathbb{R}^4} \mathrm{tr}(F_A \wedge F_A) - 8\pi^2\ , \tag{I.4}$$

and

$$\int_{\mathbb{R}^4} |F_{\hat{A}}|^2\ dx^4 \leq \int_{\mathbb{R}^4} |F_A|^2\ dx^4 + 8\pi^2 - \varepsilon_0 \rho^4 |F_A(0)|^2\ , \tag{I.5}$$

for a positive constant ε_0 independent of A and ρ. □

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Remark I.1. *The corresponding result holds in case*

$$|P_+ F_A(0)| \geq |P_- F_A(0)| \ , \tag{I.6}$$

with (I.4) replaced by

$$\int_{\mathbb{R}^4} \mathrm{tr}(F_{\hat{A}} \wedge F_{\hat{A}}) = \int_{\mathbb{R}^4} \mathrm{tr}(F_A \wedge F_A) + 8\pi^2 \ . \tag{I.7}$$

□

The geometric meaning of (I.4) and (I.7) is that while A is a connection on a given $SU(2)$-bundle P, $\hat{A}$ is a connection on a different $SU(2)$-bundle $\hat{P}$ of Chern class $c_2(\hat{P}) = c_2(P) \pm 8\pi^2$, obtained by a gluing procedure, already described for instance in [4, 8, 14]. Roughly speaking, $\hat{P}$ is obtained gluing the trivial $SU(2)$-bundle over $B_\rho(0)$ to the trivial $SU(2)$-bundle over $\mathbb{R}^4 \setminus \{0\}$ via the non-trivial gauge changes $g(x) = \frac{x}{|x|}$, or $g(x) = \frac{\overline{x}}{|x|}$, where we identify $\mathbb{R}^4$ with the space of quaternions $\mathbb{H}$, see Section II.1.

Since such change of bundle/Chern class can be obtained by gluing a rescaled (concentrated) self-instanton SD_λ as given in (II.31), and since $YM(SD_\lambda) = 8\pi^2$, it is not difficult to construct $\hat{A}$ such that $YM(\hat{A}) \leq YM(A) + 8\pi^2 + \varepsilon$ with $\varepsilon > 0$ arbitrarily small. In fact, scaling arguments lead to the more quantitative

$$YM(\hat{A}) = YM(A) + 8\pi^2 + O(\rho^4), \quad \text{as } \rho \to 0 \ , \tag{I.8}$$

where ρ is the scale at which the gluing occurs. This can be made to hold true irrespective of whether condition (I.3) is satisfied or not. On the other hand, for our purposes it will be crucial to know that the term $O(\rho^4)$ in (I.8) is negative and of magnitude of order ρ^4 (and not $o(\rho^4)$). This is precisely the content of (I.5).

The importance of an estimate as in (I.5) was shown in the theory of harmonic maps. In the seminal work [1], Brezis and Coron showed that given a smooth map $u : \Omega \subset \mathbb{R}^2 \to S^2$ and $x_0 \in \Omega$ such that $\nabla u(x_0) \neq 0$, it is possible to change u in a ball $B_\rho(x_0)$ to a new smooth map $\hat{u}$ such that the Brouwer degree of $\hat{u}$ relative to u is ± 1[1]and

$$E(\hat{u}, \Omega) := \int_\Omega |\nabla \hat{u}|^2 \ dx^2 \leq \int_\Omega |\nabla u|^2 \ dx^2 + 8\pi - \varepsilon_0 \rho^2 \ , \tag{I.9}$$

for $\rho > 0$ sufficiently small. To fix the ideas, if $x_0 = 0$, $u(0) = (0, 0, 1) \in S^2$ and $\nabla u(0) \neq 0$, Brezis and Coron construct $\hat{u}$ such that $\hat{u} = u$ in $\Omega \setminus B_\rho(0)$, $\hat{u}$ is, *up to a rotation*, a rescaling v_λ of the inverse stereographic projection in $B_{\rho/2}(0)$, and $\hat{u}$ is an interpolation between u and v_λ in $B_\rho(0) \setminus B_{\rho/2}(0)$. The subtle part in [1] is proving that the interpolation energy $E(\hat{u}, B_\rho(0) \setminus B_{\rho/2}(0))$, which we will call "cost of gluing" is smaller than the "energy saving" $8\pi - E(v_\lambda, B_{\rho/2}(0)) + E(u, B_\rho(0))$, although *they are both of order* ρ^2. The difference between energy saving and cost of gluing will be called "energy gain".

The strict inequality of Brezis and Coron (I.9), was originally motivated by the open problem raised by Giaquinta and Hildebrant [5] of finding more than one harmonic map from the disk into S^2 for any non constant boundary datum. For this purpose, the weaker estimate

$$\int_\Omega |\nabla \hat{u}|^2 \ dx^2 < \int_\Omega |\nabla u|^2 \ dx^2 + 8\pi \ , \tag{I.10}$$

together with a variational argument to minimize in different homotopy classes and the fact that weakly converging sequence in $W^{1,2}$ can jump to a different homotopy class at a cost of at least 8π,

[1]depending on whether $\nabla u(x_0)$ preserves or reverses orientation, a condition that in complex coordinates can be expressed as $|\partial_{\bar{z}} u| \leq |\partial_z u|$, or $|\partial_z u| \leq |\partial_{\bar{z}} u|$, the counterparts of our conditions (I.3) and (I.6)

would have been sufficient. On the other hand the quantitative form (I.9), in which the energy gain is of the order ρ^2, turned out to be very relevant in the study of the relaxed Dirichlet energy for 3-dimensional maps into S^2. In the axially symmetric setting, Hardt, Lin and Poon [7] used (I.9) to construct 3-dimensional dipoles, i.e. maps from B^3 to S^2 with two singularities of opposite degree (in the spirit of Brezis, Coron and Lieb [2]) satisfying a strict inequality. This led them to a partial regularity theory for axially symmetric minimizers of the relaxed Dirichlet energy.

Here the term $-\varepsilon_0\rho^2$ in (I.9) cannot be replaced by a lower order term, such as $-\varepsilon_0\rho^4$, since in this case the 2-dimensional energy gain would be outspent by the cost of "closing the dipole" in the third direction, i.e. creating the singularities and gluing them to the original maps. This was made precise by the first author [9], working on n-axially symmetric maps for $n > 1$. In this case, $\nabla u = 0$ along the vertical axis, and the energy gain in (I.9) on disks transversal to the axis is only at most ρ^4, insufficiente to have a (symmetric) strict dipole. For this reason, the regularity theory for n-axially symmetric maps from B^3 into S^2 minimizing the relaxed Dirichlet energy is still open, and for the same reason a classification of tangent maps to minimizers of the Dirichlet relaxed energy is also open.

Still using the relaxed energy, together with a more general (non-symmetric) dipole construction satifying a strict inequality, the second author [11] proved the existence of weakly harmonic maps from B^3 into S^2 which are everywhere discontinuous.

Our Theorem I.1 is the starting point of an analogous regularity project for the Yang-Mills energy. In particular, for reasons similar to the ones explained above, in order to construct strict dipoles for the Yang-Mills functional (see [10]), it is fundamental that the energy gain in (I.5) is of order ρ^4. In fact, with a different construction, Taubes [14] was able to obtain

$$YM(\hat{A}) \leq YM(A) + 8\pi^2 - \varepsilon_0\rho^8, \quad \text{as } \rho \to 0 \ ,^2 \tag{I.11}$$

hence also obtaining a strict energy saving, but of much smaller entity. This sufficient for his purposes, and to find minimizers of the Yang-Mills energy in different Chern classes, as shown by Isobe and Marini [6], but not for the construction of dipoles, hence to address the regularity theory in dimension 5.

Regarding the main difficulty in proving (I.5), there is a strong analogy with (I.9), in that the energy saving due to the removal of the original connection in B_ρ and to the energy of the glued instanton (which is strictly less than $8\pi^2$), is of the same order ρ^4 of the cost of gluing the instanton to the original connection. As in [1], there seems to be no deep reason why the good term (energy saving) should be bigger than the bad term (the cost of gluing), giving a positive energy gain. In fact, by following in the steps of [1], i.e. replacing A in $B_{\rho/2}(0)$ with a self-instanton SD_λ, obtained by scaling by a suitable factor $\lambda > 0$, and gluing in the annulus $B_\rho(0) \setminus B_{\rho/2}(0)$ we would obtain that in the most degenerate (least symmetric) cases, the cost of gluing is bigger than the energy saving *for every choice of* $\lambda > 0$. It is only by inserting SD_λ in $B_{\tau\rho}(0)$ for some $\tau \in (0,1)$, then gluing in $B_\rho(0) \setminus B_{\tau\rho}(0)$ and fine-tuning both τ and λ that we can obtain a net energy gain (and actually quite small: the energy saving is only around 1% bigger than the gluing cost, in the worst possible case).

We conclude by mentioning that, similar to the above-mentioned work of Hardt, Lin a Poon [7], in symmetric situations the gluing is much easier, hence the estimates can be obtained with considerably less effort, as in Siebner, Siebner, Uhlenbeck [13].

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[2] actually in the estimate of Taubes the gain is $\varepsilon_0\rho^4$, but the gluing occurs on the bigger ball $B_{\sqrt{\rho}}(0)$, hence scaling to a ball a radius ρ we obtain (I.11).

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II Preliminaries and notations

II.1 Connection forms, curvatures and self-duality

Let $A \in \Omega^1(\mathbb{R}^4) \otimes su(2)$ be a smooth 1-form on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with values into $su(2)$, the Lie algebra of the special unitary group $SU(2)$. We identify $su(2)$ with the vector space $\Im m(\mathbb{H})$ of imaginary quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{II.12}$$

Observe that $\mathbf{i}\,\mathbf{j} = \mathbf{k}$, and in particular

$$[\mathbf{i}, \mathbf{j}] = 2\,\mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = 2\,\mathbf{i} \ , \quad [\mathbf{k}, \mathbf{i}] = 2\,\mathbf{j} \ . \tag{II.13}$$

Adding the identity matrix $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we can identify $\mathbb{R}^4$, $\mathbb{H}$ and the subspace $GL(2, \mathbb{C})$ spanned by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, namely we identify $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with

$$x = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \in \mathbb{H} \subset GL(2, \mathbb{C}), \tag{II.14}$$

and $SU(2)$ correspond to the set of unitary quaternions, i.e. x as in (II.14) with $x_1^2 + \cdots + x_4^2 = 1$. With this identification

$$|\mathbf{1}| = |\mathbf{i}|^2 = |\mathbf{j}|^2 = |\mathbf{k}|^2 = 2 \ . \tag{II.15}$$

Moreover, given $p = p_{\mathbf{i}}\,\mathbf{i} + p_{\mathbf{j}}\,\mathbf{j} + p_{\mathbf{k}}\,\mathbf{k} \in \Im m(\mathbb{H})$ we have

$$\begin{aligned} |p|^2 &= \mathrm{tr}(p\,\overline{p}) = -\mathrm{tr}\left((p_{\mathbf{i}}\,\mathbf{i} + p_{\mathbf{j}}\,\mathbf{j} + p_{\mathbf{k}}\,\mathbf{k})(p_{\mathbf{i}}\,\mathbf{i} + p_{\mathbf{j}}\,\mathbf{j} + p_{\mathbf{k}}\,\mathbf{k})\right) \\ &= -(p_{\mathbf{i}}^2 + p_{\mathbf{j}}^2 + p_{\mathbf{k}}^2)\mathrm{tr}(\mathbf{1}) = -2(p_{\mathbf{i}}^2 + p_{\mathbf{j}}^2 + p_{\mathbf{k}}^2) \\ &= -2\Re(p\,p) \end{aligned} \tag{II.16}$$

Using (II.12) we can write

$$A = \sum_{l=1}^4 A^l \ dx_l = \sum_{l=1}^4 \left(A^l_{\mathbf{i}}\ \mathbf{i} + A^l_{\mathbf{j}}\ \mathbf{j} + A^l_{\mathbf{k}}\ \mathbf{k}\right)\ dx_l \ , \tag{II.17}$$

with pointwise norm

$$|A(x)|^2 = \sum_{l=1}^4 \left(|A^l_{\mathbf{i}}(x)|^2|\mathbf{i}|^2 + |A^l_{\mathbf{j}}(x)|^2|\mathbf{j}|^2 + |A^l_{\mathbf{k}}(x)|^2|\mathbf{k}|^2\right) = 2\sum_{l=1}^4 \left(|A^l_{\mathbf{i}}(x)|^2 + |A^l_{\mathbf{j}}(x)|^2 + |A^l_{\mathbf{k}}(x)|^2\right) \ .$$

We also called such a A a *connection one form*[3] on $\mathbb{R}^4$.

Let us now define for $A, B \in \Omega^1(\mathbb{R}^4) \otimes su(2)$

$$[A, B](X, Y) = \frac{1}{2}\left([A(X), B(Y)] - [A(Y), B(X)]\right) \ . \tag{II.18}$$

In particular

$$[p\,dx_i, q\,dx_j] = \frac{1}{2}[p, q]dx_i \wedge dx_j, \quad \text{for } p, q \in su(2). \tag{II.19}$$

[3]We shall be exclusively considering $su(2)$ connections in this work.

Then, the curvature of A is the two form taking values into $su(2)$ given by[4]

$$\forall\ X,Y\in\mathbb{R}^4\quad F_A(X,Y):=dA(X,Y)+[A(X),A(Y)]$$

$$\begin{aligned}&=\sum_{l,m=1}^4\left(\partial_{x_l}A^m-\partial_{x_m}A^l\right)\ X_l\ Y_m+\left[\sum_{l=1}^4 A^l\,X_l,\sum_{m=1}^4 A^m\,Y_m\right]\\&=\sum_{l,m=1}^4\left(\partial_{x_l}A^m-\partial_{x_m}A^l+[A^l,A^m]\right)\ X_l\ Y_m\ ,\end{aligned}\tag{II.20}$$

and naturally we introduce the notation

$$F_A^{lm}:=\partial_{x_l}A^m-\partial_{x_m}A^l+[A^l,A^m]\ .\tag{II.21}$$

The norm of F_A is given by

$$|F_A|=\sum_{1\le i<j\le 4}\left|F_A(\partial_{x_i},\partial_{x_j})\right|^2\ ,$$

the norms on the RHS being given by (II.16).

We recall the definition of the Hodge operator $\star$ on the space of alternated 2-forms in $\mathbb{R}^4$

$$\forall\alpha\,,\,\beta\in\wedge^2\mathbb{R}^4\quad\alpha\wedge\star\beta=\langle\alpha,\beta\rangle\ \star 1\ .\tag{II.22}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\wedge^2\mathbb{R}^4$ and $\star 1$ is the standard volume form on $\mathbb{R}^4$ given by

$$\star 1=dx_1\wedge dx_2\wedge dx_3\wedge dx_4\ .\tag{II.23}$$

A form α is called *self-dual* (resp. *anti-self dual*) if $\star\alpha=\alpha$ (resp. $\star\alpha=-\alpha$). The 3 dimensional vector subspace of $\wedge^2\mathbb{R}^4$ made of self-dual (resp. anti-self dual) form is denoted $\wedge^2_+\mathbb{R}^4$ (resp. $\wedge^2_-\mathbb{R}^4$). An orthonormal basis of $(\wedge^2\mathbb{R}^4)^+$ (resp. $(\wedge^2\mathbb{R}^4)^-$) is given by

$$\left\{\begin{aligned}\omega_{\mathbf{i}}^{\pm}&=\frac{1}{\sqrt{2}}\ (dx_1\wedge dx_2\pm dx_3\wedge dx_4)\\\omega_{\mathbf{j}}^{\pm}&=\frac{1}{\sqrt{2}}\ (dx_1\wedge dx_3\pm dx_4\wedge dx_2)\\\omega_{\mathbf{k}}^{\pm}&=\frac{1}{\sqrt{2}}\ (dx_1\wedge dx_4\pm dx_2\wedge dx_3)\end{aligned}\right.\tag{II.24}$$

It is important for later purposes to observe

i) The sub-vector spaces $(\wedge^2\mathbb{R}^4)^+$ and $(\wedge^2\mathbb{R}^4)^-$ are orthogonal to each-other and

$$\wedge^2\mathbb{R}^4=(\wedge^2\mathbb{R}^4)^+\oplus(\wedge^2\mathbb{R}^4)^-\ .$$

ii) The family $(\omega_{\mathbf{i}}^+,\omega_{\mathbf{j}}^+,\omega_{\mathbf{k}}^+,\omega_{\mathbf{i}}^-,\omega_{\mathbf{j}}^-,\omega_{\mathbf{k}}^-)$ realizes an orthonormal basis of $\wedge^2\mathbb{R}^4$.

[4]We are taking the most spread notation according to which

$$dx_l\wedge dx_m(X,Y)=X_l\,Y_m-X_m\,Y_l\ .$$

We shall denote $P_\pm$ the orthogonal projections onto $(\wedge^2\mathbb{R}^4)^\pm$. We recall the well-known

$$\mathrm{tr}(F_A \wedge F_A) = -|P_+F_A|^2 + |P_-F_A|^2 \tag{II.25}$$

Also observe that

$$\begin{aligned} dx \wedge d\overline{x} &= (\mathbf{1}\, dx_1 + \mathbf{i}\, dx_2 + \mathbf{j}\, dx_3 + \mathbf{k}\, dx_4) \wedge (\mathbf{1}\, dx_1 - \mathbf{i}\, dx_2 - \mathbf{j}\, dx_3 - \mathbf{k}\, dx_4) \\ &= -2\sqrt{2}\left(\omega_{\mathbf{i}}^+\, \mathbf{i} + \omega_{\mathbf{j}}^+\, \mathbf{j} + \omega_{\mathbf{k}}^+\, \mathbf{k}\right) \end{aligned} \tag{II.26}$$

and

$$\begin{aligned} d\overline{x} \wedge dx &= (\mathbf{1}\, dx_1 - \mathbf{i}\, dx_2 - \mathbf{j}\, dx_3 - \mathbf{k}\, dx_4) \wedge (\mathbf{1}\, dx_1 + \mathbf{i}\, dx_2 + \mathbf{j}\, dx_3 + \mathbf{k}\, dx_4) \\ &= 2\sqrt{2}\left(\omega_{\mathbf{i}}^-\, \mathbf{i} + \omega_{\mathbf{j}}^-\, \mathbf{j} + \omega_{\mathbf{k}}^-\, \mathbf{k}\right) \end{aligned} \tag{II.27}$$

This implies in particular

$$\star(dx \wedge d\overline{x}) = dx \wedge d\overline{x}\ , \quad \star(d\overline{x} \wedge dx) = -d\overline{x} \wedge dx\ . \tag{II.28}$$

A special case of interest in this work is given by

$$A(x) := SD(x) = \Im m\left(\frac{x\, d\overline{x}}{1+|x|^2}\right) \tag{II.29}$$

where the $x \in \mathbb{R}^4$ is identified canonically with the quaternion as in (II.14) and $\overline{x} := x_1\mathbf{1} - x_2\,\mathbf{i} - x_3\,\mathbf{j} - x_4\,\mathbf{k}$ is its conjugate. The curvature of SD is given by the formula (see the appendix for details)

$$F_{SD} = \frac{dx \wedge d\overline{x}}{(1+|x|^2)^2}\ . \tag{II.30}$$

Pulling back via a dilation of factor $\lambda > 0$, gives the rescaled version of SD, namely

$$SD_\lambda(x) := \Im m\left(\frac{\lambda^2\, x\, d\overline{x}}{1+\lambda^2\,|x|^2}\right)\ , \tag{II.31}$$

with curvature

$$F_{SD_\lambda} = \frac{\lambda^2 dx \wedge d\overline{x}}{(1+\lambda^2|x|^2)^2}\ . \tag{II.32}$$

Thanks to (II.28), we observe

$$\star F_{SD_\lambda} = F_{SD_\lambda}\ , \tag{II.33}$$

that we can also rewrite

$$P_+(F_{SD_\lambda}) = F_{SD_\lambda} \quad \text{or, equivalently,} \quad P_-(F_{SD_\lambda}) = 0\ . \tag{II.34}$$

Denoting

$$ASD_\lambda(x) := \Im m\left(\frac{\lambda^2\, \overline{x}\, dx}{1+\lambda^2\,|x|^2}\right) = -\,\Im m\left(\frac{\lambda^2\, d\overline{x}\, x}{1+\lambda^2\,|x|^2}\right) \tag{II.35}$$

we obtain as before

$$F_{ASD_\lambda}(x) = \frac{\lambda^2\, d\overline{x} \wedge dx}{(1+\lambda^2\,|x|^2)^2} \tag{II.36}$$

Combining (II.27) and (II.36) is implying

$$P_-(F_{ASD_\lambda}) = F_{ASD_\lambda} \quad \Longleftrightarrow \quad P_+(F_{ASD_\lambda}) = 0 \quad \Longleftrightarrow \quad \star F_{ASD_\lambda} = -F_{ASD_\lambda}\ . \tag{II.37}$$

II.2 Gauge changes and the exponential/polar/radial gauge

Let A be a smooth $su(2)$ connection 1-form on $\mathbb{R}^4$ and g be a smooth map from $\mathbb{R}^4$ into $SU(2)$. Following [12] section III.3, we denote

$$A^g := g^{-1}\, A\, g + g^{-1}\, dg\ . \tag{II.38}$$

where we identify g with the corresponding unit quaternions. A direct computation (see [12] section III.3) gives

$$F_{A^g} = g^{-1}\, F_A\, g\ . \tag{II.39}$$

The passage from A to A^g corresponds on $\mathbb{R}^4$ to a change of trivialization (or *gauge change*) of the underlying principal trivial $SU(2)$ bundle $\mathbb{R}^4 \times SU(2)$ for which A is the one form representing a connection (see [12] section III). First of all we proceed to a first smooth gauge change ensuring $A(0) = 0$. This is possible by taking

$$g := \exp\left(-\sum_{l=1}^{4} x_l\ A^l(0)\right) \tag{II.40}$$

From now on we will only work with connection form satisfying $A(0) = 0$.

In [15] K.Uhlenbeck is considering a special gauge change g such that ultimately

$$\forall x \in \mathbb{R}^4 \qquad 0 = A^g(x) \llcorner \frac{\partial}{\partial r} := \sum_{l=1}^{4} (A^g)^l(x)\ \frac{x_l}{|x|} \tag{II.41}$$

where $\llcorner$ is the contraction operator between forms and vectors (more generally between contravariant and covariant tensors). Such a gauge is called in the literature *exponential* (or also *polar* or *radial*) gauge. We recall for the convenience of the reader the construction of g. The gauge g is obtained by "parallel transporting" with respect to A the identity at the origin along the rays given by the straight half lines emanating from 0. Precisely we solve for any $\sigma \in S^3$ and $r \in \mathbb{R}_+$

$$\left\{\begin{array}{l} \partial_r g(r,\sigma) = -A(r\,\sigma)\, g(r,\sigma) \\[3mm] g(0,\sigma) = \mathbf{1}\ , \end{array}\right. \tag{II.42}$$

where $\mathbf{1}$ is either the neutral element in the quaternions or the unit matrix in $SU(2)$. The existence and uniqueness $g \in C^1(\mathbb{R}_+, SU(2))$ for any $\sigma \in S^3$ follows using the smoothness of A by classical ODE theory. We first claim that $g \in Lip_{loc}(S^3 \times \mathbb{R}_+)$. Indeed for any pair σ and σ' in S^3 the quotient difference $u(r,\sigma,\sigma') := (g(r,\sigma) - g(r,\sigma'))/|\sigma - \sigma'|$ satisfies

$$\left\{\begin{array}{l} \partial_r u(r,\sigma,\sigma') = -A(r\,\sigma)\, u(r,\sigma,\sigma') - \dfrac{A(r\,\sigma) - A(r\,\sigma')}{|\sigma - \sigma'|}\, g(r,\sigma') \\[3mm] u(0,\sigma,\sigma') = 0 \end{array}\right. \tag{II.43}$$

Hence we have for $r < R$, that $|g(r,\sigma')| = \sqrt{2}$,

$$\partial_r |u| \le |\partial_r u| \le \|A\|_{L^\infty(B_R(0))}\ |u| + \sqrt{2} r\ \|\nabla A\|_{L^\infty(B_R(0))}\ . \tag{II.44}$$

This imples

$$\partial_r \left(e^{-r\,\|A\|_{L^\infty(B_R(0))}}\ |u| - \sqrt{2}^{-1}\ r^2\ \|\nabla A\|_{L^\infty(B_R(0))}\right) \le 0\ . \tag{II.45}$$

Since $|u|(0,\sigma,\sigma')=0$ we deduce

$$\forall\, r\le R\quad \forall\,\sigma,\sigma'\in S^3\qquad \frac{|g(r,\sigma)-g(r,\sigma')|}{|r\,\sigma-r\,\sigma'|}\le\ 2^{-1}\,e^{r\,\|A\|_{L^\infty(B_R(0))}}\,r\,\|\nabla A\|_{L^\infty(B_R(0))} \tag{II.46}$$

We have also from (II.42)

$$\forall\, r,r'\le R\quad \forall\,\sigma\in S^3\qquad \frac{|g(r,\sigma)-g(r',\sigma)|}{|r-r'|}\le\ \|A\|_{L^\infty(B_R(0))}\ . \tag{II.47}$$

Let $0<r<r'$ and σ,σ' in S^3 we have

$$\frac{|g(r,\sigma)-g(r',\sigma')|}{|r\,\sigma-r'\,\sigma'|}\le\frac{|g(r,\sigma)-g(r,\sigma')|}{|r\,\sigma-r'\,\sigma'|}+\frac{|g(r,\sigma')-g(r',\sigma')|}{|r\,\sigma-r'\,\sigma'|} \tag{II.48}$$

Because of the convexity of $B_r(0)$ in $\mathbb{R}^4$ the point $r\sigma'$ is the nearest point to $r'\sigma'$ in $B_r(0)$ hence

$$|r\,\sigma'-r'\,\sigma'|\le|r\,\sigma-r'\,\sigma'| \tag{II.49}$$

We have also

$$|r\sigma-r'\sigma'|^2=r^2+(r')^2-2\,r\,r'\,\sigma\cdot\sigma'\ge 2\,r\,r'-2\,r\,r'\,\sigma\cdot\sigma' \tag{II.50}$$

Cauchy-Schwartz inequality gives $|\sigma\cdot\sigma'|\le|\sigma|\,|\sigma'|=1$ hence

$$|r\sigma-r'\sigma'|^2\ge 2\,r\,r'\,(1-\sigma\cdot\sigma')\ge 2\,r^2\,(1-\sigma\cdot\sigma')=r^2+r^2-2\,r\,r\,\sigma\cdot\sigma'=|r\sigma-r\sigma'|^2 \tag{II.51}$$

Combining (II.48) with (II.49) and (II.51) gives then

$$\frac{|g(r,\sigma)-g(r',\sigma')|}{|r\,\sigma-r'\,\sigma'|}\le\frac{|g(r,\sigma)-g(r,\sigma')|}{|r\,\sigma-r\,\sigma'|}+\frac{|g(r,\sigma')-g(r',\sigma')|}{|r\,\sigma'-r'\,\sigma'|} \tag{II.52}$$

Combining now (II.52) with (II.46) and (II.47) is implying

$$\frac{|g(r,\sigma)-g(r',\sigma')|}{|r\,\sigma-r'\,\sigma'|}\le 2^{-1}\,e^{r\,\|A\|_{L^\infty(B_R(0))}}\,r\,\|\nabla A\|_{L^\infty(B_R(0))}+\|A\|_{L^\infty(B_R(0))}\ . \tag{II.53}$$

The map g obviously extend to a map on $\mathbb{R}^4$ and we have established the following lemma

Lemma II.1. *Let A be a smooth $su(2)$ connection one form on $\mathbb{R}^4$ such that $A(0)=0$. Let g be the map from $\mathbb{R}^4$ into $SU(2)$ solving (II.42). Then $g\in C^1(\mathbb{R}^4,SU(2))$ and for any $R>0$ the following estimate holds:*

$$\|\nabla g\|_{L^\infty(B_R(0))}\le 2^{-1}\,e^{r\,\|A\|_{L^\infty(B_R(0))}}\,r\,\|\nabla A\|_{L^\infty(B_R(0))}+\|A\|_{L^\infty(B_R(0))}\ . \tag{II.54}$$

moreover A^g is continuous at the origin and $A^g(0)=0$. □

Proof of lemma II.1 The fact that g is locally Lipschitz in $\mathbb{R}^4$ and that the estimate (II.54) holds is a direct consequence of (II.53). From the fact that $A(0)=0$ we obtain that

$$\lim_{R\to 0}\|\nabla g\|_{L^\infty(B_R(0))}=0 \tag{II.55}$$

This implies the lemma. □

We recall the standard trivialisation of TS^3 by the following orthonormal tangent frame

$$\left\{\begin{array}{l} e_1 := x_1\,\partial_{x_2} - x_2\,\partial_{x_1} + x_3\,\partial_{x_4} - x_4\,\partial_{x_3} \\[2mm] e_2 := x_1\,\partial_{x_3} - x_3\,\partial_{x_1} + x_4\,\partial_{x_2} - x_2\,\partial_{x_4} \\[2mm] e_3 := x_1\,\partial_{x_4} - x_4\,\partial_{x_1} + x_2\,\partial_{x_3} - x_3\,\partial_{x_2}\ , \end{array}\right. \tag{II.56}$$

To which we add the radial vector

$$\partial_r = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}\ .$$

We extend this frame by 0-homogeneity to $T(\mathbb{R}^4\setminus\{0\})$.

Its dual basis is given for $x\in S^3$ by

$$\left\{\begin{array}{l} e_1^\ast := x_1\,dx_2 - x_2\,dx_1 + x_3\,dx_4 - x_4\,dx_3 \\[2mm] e_2^\ast := x_1\,dx_3 - x_3\,dx_1 + x_4\,dx_2 - x_2\,dx_4 \\[2mm] e_3^\ast := x_1\,dx_4 - x_4\,dx_1 + x_2\,dx_3 - x_3\,dx_2 \\[2mm] dr := x_1dx_1 + x_2dx_2 + x_3dx_3 + x_4dx_4\ , \end{array}\right. \tag{II.57}$$

again extended by 0-homogeneity to $T^\ast(\mathbb{R}^4\setminus\{0\})$. We recall also the three standard complex structure (I,J,K) on $\mathbb{R}^4$

$$\left\{\begin{array}{l} I\partial_{x_1} = \partial_{x_2} \quad \text{and} \quad I\partial_{x_3} = \partial_{x_4} \\[2mm] J\partial_{x_1} = \partial_{x_3} \quad \text{and} \quad J\partial_{x_4} = \partial_{x_2} \\[2mm] K\partial_{x_1} = \partial_{x_4} \quad \text{and} \quad K\partial_{x_2} = \partial_{x_3} \end{array}\right. \tag{II.58}$$

With these notations we have for instance

$$e_1 = I\partial_r \quad , \quad e_2 = J\partial_r \quad \text{and} \quad e_3 = K\partial_r\ , \tag{II.59}$$

and we have by duality

$$e_1^\ast = Idr \quad , \quad e_2^\ast = Jdr \quad \text{and} \quad e_3^\ast = Kdr\ , \tag{II.60}$$

We can then write for the exponential gauge change g given by the beginning of the subsection of a smooth connection A

$$A^g = \sum_{l=1}^3 A^g_{e_l}\ e_l^\ast \quad \text{where} \quad A^g_{e_l} = A^g\llcorner e_l \quad . \tag{II.61}$$

Using the fact that $A^g\llcorner\partial_r = 0$ we write for any $l=1,2,3$

$$F_{A^g}(\partial_r, e_l) = dA^g(\partial_r, e_l) + [A^g\llcorner\partial_r, A^g\llcorner e_l] = dA^g(\partial_r, e_l)\ . \tag{II.62}$$

Using the *Cartan Formula* for the exterior derivative of one forms we obtain

$$dA^g(\partial_r, e_l) = d(A^g\llcorner e_l)\llcorner\partial_r - d(A^g\llcorner\partial_r)\llcorner e_l - A^g\llcorner[\partial_r, e_l] = \partial_r\left(A^g_{e_l}\right) - A^g\llcorner[\partial_r, e_l] \tag{II.63}$$

where we used again $A^g\llcorner\partial_r = 0$. We have for any vector field Y in $\mathbb{R}^4$

$$[r\,\partial_r, Y] = \sum_{i,j}\left[x_j\,\partial_{x_j}Y^i - Y^j\,\delta_{ij}\right]\,\partial_{x_i} = \sum_i r\,\partial_r Y^i\,\partial_{x_i} - Y = r\,\partial_r Y - Y\ . \tag{II.64}$$

In particular, for $l = 1, 2, 3$

$$[r\,\partial_r, e_l] = -e_l \tag{II.65}$$

Using the formula

$$[f X, Y] = f\,[X, Y] - \nabla_Y f\, X \tag{II.66}$$

we obtain

$$[\partial_r, e_l] = -r^{-1}\, e_l \tag{II.67}$$

and

$$F_{A^g}(\partial_r, e_l) = dA^g(\partial_r, e_l) = \partial_r\left(A^g_{e_l}\right) + r^{-1}\, A^g \llcorner e_l = r^{-1}\,\partial_r\left(r\, A^g_{e_l}\right)\ . \tag{II.68}$$

Combining (II.61), (II.62) and (II.63) we obtain the following expression of the connection form in exponential gauge in terms of the curvature (which is maybe the main reason why this gauge is often used in the literature)

$$A^g(r\sigma) = r^{-1}\sum_{l=1}^{3}\int_0^r F_{A^g}(t\sigma)(\partial_r, e_l)\ t\,dt\ e_l^*\ . \tag{II.69}$$

where we have used from lemma II.1 that $A^g(0) = 0$.

From this expression we deduce the following lemma

Lemma II.2. *Let A be a smooth $su(2)$ connection one form on $\mathbb{R}^4$ such that $A(0) = 0$. Let g be the map from $\mathbb{R}^4$ into $SU(2)$ solving (II.42). Then*

$$\forall R > 0 \quad \forall x \in B_R(0) \qquad |A^g(x)| \le \|F_A\|_{L^\infty(B_R(0))}\ |x|\ , \tag{II.70}$$

and

$$A^g(x) = \frac{|x|}{2}\sum_{l=1}^{3} F_{A^g}(0)(\partial_r, e_l)e_l^* + O(|x|^2)\,, \quad \textit{as } |x| \to 0\ .$$

□

III Self-instanton insertion

Let A be the exponential gauge of a smooth one form connection which was originally chosen to be zero at the origin (unlike the previous section we do not mention the gauge change bringing to the exponential configuration anymore). To fix ideas we assume that A is a connection form issued from a smooth connection of a principal $SU(2)$ bundle E over S^4 via the stereographic projection in such a way that we have respectively

$$\int_{\mathbb{R}^4} |F_A|^2\ dx^4 < +\infty \tag{III.1}$$

and

$$c_2(E) = \int_{\mathbb{R}^4} \mathrm{tr}\,(F_A \wedge F_A) = \int_{\mathbb{R}^4}\left(-|P_+ F_A|^2 + |P_- F_A|^2\right)\, dx^4 = 8\pi^2\ k\ , \quad \text{where } k \in \mathbb{Z}\ . \tag{III.2}$$

Let $g_0 \in SU(2)$ (i.e. g_0 is constant on $\mathbb{R}^4$) to be fixed later. In a first step we consider the gauge transformed of A given by

$$A^{g_0} = g_0^{-1}\, A\, g_0 \tag{III.3}$$

This gauge is still exponential. Now, away from zero, we proceed to a gauge change of degree +1. Precisely we denote

$$\tilde{A}^{g_0} := \frac{x}{|x|}\, A^{g_0}\, \frac{\overline{x}}{|x|} + \frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) \ . \tag{III.4}$$

It is important at this stage to stress the fact that this new gauge (singular at the origin) is still exponential in the sense that

$$\tilde{A}^{g_0} \llcorner \partial_r = 0 \ . \tag{III.5}$$

Let η be a smooth increasing function from $\mathbb{R}_+$ into $[0,1]$ such that $\eta \equiv 0$ on $[0,\tau]$, for some $\tau \in (0,1)$ to be fixed later, and $\eta \equiv 1$ on $[1,+\infty)$. For $\rho > 0$ sufficiently small, we denote

$$\forall x \in \mathbb{R}^4 \qquad \eta_\rho(x) := \eta\left(\frac{|x|}{\rho}\right) \ . \tag{III.6}$$

Moreover, for a constant $c_0 > 0$ to be fixed independently of ρ we set

$$\lambda^2 = \frac{1}{c_0 \rho^4}, \quad \lambda \to \infty \text{ as } \rho \to 0^+ \ . \tag{III.7}$$

We introduce the following glueing between $\tilde{A}^{g_0}$ and SD_λ (compare to (II.31)). First we denote in $B_\rho(0) \setminus B_{\tau\rho}(0)$ for any 1-form A

$$\underline{A} := \phi_\rho^* A \tag{III.8}$$

where $\phi_\rho(x) := \rho\, x/|x|$. Then we define

$$\hat{A}(g_0,\rho,\lambda) := \eta_\rho\, \underline{\tilde{A}}^{g_0} + (1-\eta_\rho)\, SD_\lambda. \tag{III.9}$$

Using the identities

$$\frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) = \Im m\left(\frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right)\right) = \Im m\left(\frac{x}{|x|^2}\, d\overline{x}\right) = \frac{x}{|x|}\, \Im m\left(\frac{d\overline{x}\, x}{|x|^2}\right) \frac{\overline{x}}{|x|} \ , \tag{III.10}$$

and

$$\Im m\left(\frac{\overline{x}}{|x|}\, \frac{\lambda^2\, x\, d\overline{x}}{1+\lambda^2\, |x|^2}\, \frac{x}{|x|} - \frac{d\overline{x}\, x}{|x|^2}\right) = -\frac{1}{|x|^2}\, \Im m\left(\frac{d\overline{x}\, x}{1+\lambda^2\, |x|^2}\right) = -\frac{1}{1+\lambda^2\, |x|^2}\, d\left(\frac{\overline{x}}{|x|}\right) \frac{x}{|x|} \tag{III.11}$$

we obtain

$$\begin{aligned}
\hat{A}(g_0,\rho,\lambda) &= \frac{x}{|x|}\left(\eta_\rho\, g_0^{-1}\, \underline{A}\, g_0 + (1-\eta_\rho)\, \frac{\overline{x}}{|x|} \Im m\left(\frac{\lambda^2\, x\, d\overline{x}}{1+\lambda^2\, |x|^2}\right) \frac{x}{|x|}\right) \frac{\overline{x}}{|x|} + \eta_\rho\, \frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) \\
&= \frac{x}{|x|}\left(\eta_\rho\, g_0^{-1}\, \underline{A}\, g_0 + (1-\eta_\rho)\, \Im m\left(\frac{\overline{x}}{|x|}\, \frac{\lambda^2\, x\, d\overline{x}}{1+\lambda^2\, |x|^2}\, \frac{x}{|x|} - \frac{d\overline{x}\, x}{|x|^2}\right)\right) \frac{\overline{x}}{|x|} + \frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) \\
&= \frac{x}{|x|}\left(\eta_\rho\, g_0^{-1}\, \underline{A}\, g_0 - (1-\eta_\rho)\, \frac{1}{1+\lambda^2\, |x|^2}\, d\left(\frac{\overline{x}}{|x|}\right)\right) \frac{\overline{x}}{|x|} + \frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) \ ,
\end{aligned} \tag{III.12}$$

hence

$$\hat{A}(g_0,\rho,\lambda) = \frac{x}{|x|} \check{A}(g_0,\rho,\lambda) \frac{\overline{x}}{|x|} + \frac{x}{|x|}\, d\left(\frac{\overline{x}}{|x|}\right) \tag{III.13}$$

where

$$\check{A}(g_0,\rho,\lambda) = \eta_\rho\, g_0^{-1}\, \underline{A}\, g_0 - (1-\eta_\rho)\, \frac{1}{1+\lambda^2\, |x|^2}\, d\left(\frac{\overline{x}}{|x|}\right) \frac{x}{|x|} \ . \tag{III.14}$$

Observe that we have

$$\begin{cases} F_{\hat{A}} = \dfrac{x}{|x|}\, g_0^{-1}\, F_A\, g_0\, \dfrac{\overline{x}}{|x|} & \text{in } \mathbb{R}^4 \setminus B_\rho(0) \\ F_{\hat{A}} = F_{SD} \qquad \text{in } B_{\tau\rho}(0) \end{cases} \tag{III.15}$$

Lemma III.1. *We have*

$$\int_{\mathbb{R}^4} \mathrm{tr}\left(F_{\hat{A}} \wedge F_{\hat{A}}\right) = \int_{\mathbb{R}^4} \mathrm{tr}\left(F_A \wedge F_A\right) - 8\pi^2 \ .$$

Proof. Using the formula (see [4])

$$\mathrm{tr}\left(F_A \wedge F_A\right) = d\,\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge [A, A]\right) \ , \tag{III.16}$$

we get

$$\begin{aligned} \int_{B_R(0)} \mathrm{tr}\left(F_{\hat{A}} \wedge F_{\hat{A}}\right) &= \int_{\partial B_R(0)} \mathrm{tr}\left(\hat{A}^{g_0} \wedge d\hat{A}^{g_0} + \frac{1}{3}\hat{A}^{g_0} \wedge [\hat{A}^{g_0}, \hat{A}^{g_0}]\right) \\ &= \int_{\partial B_R(0)} \mathrm{tr}\left(\tilde{A}^{g_0} \wedge d\tilde{A}^{g_0} + \frac{1}{3}\tilde{A}^{g_0} \wedge [\tilde{A}^{g_0}, \tilde{A}^{g_0}]\right) \\ &= \int_{B_R(0)\setminus B_\varepsilon(0)} \mathrm{tr}\left(F_{\tilde{A}^{g_0}} \wedge F_{\tilde{A}^{g_0}}\right) + \int_{\partial B_\varepsilon(0)} \mathrm{tr}\left(\tilde{A}^{g_0} \wedge d\tilde{A}^{g_0} + \frac{1}{3}\tilde{A}^{g_0} \wedge [\tilde{A}^{g_0}, \tilde{A}^{g_0}]\right) \\ &= \int_{B_R(0)\setminus B_\varepsilon(0)} \mathrm{tr}\left(F_{A^{g_0}} \wedge F_{A^{g_0}}\right) + \int_{\partial B_\varepsilon(0)} \mathrm{tr}\left(A^{g_0} \wedge dA^{g_0} + \frac{1}{3}A^{g_0} \wedge [A^{g_0}, A^{g_0}]\right) \\ &\quad + \int_{\partial B_\varepsilon(0)} \mathrm{tr}(Q) \\ &= \int_{B_R(0)} \mathrm{tr}\left(F_{A^{g_0}} \wedge F_{A^{g_0}}\right) + \int_{\partial B_\varepsilon(0)} \mathrm{tr}(Q) \ , \end{aligned} \tag{III.17}$$

where Q denotes a 3-form with the expansion

$$Q = \frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right) \wedge d\left(\frac{\overline{x}}{|x|}\right) + \frac{2}{3}\frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right) \wedge \left[\frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right), \frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right)\right] + O(\varepsilon^{-2}),$$

as $\varepsilon \to 0$. We have

$$\begin{aligned} &\frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right) \wedge d\left(\frac{\overline{x}}{|x|}\right) = \frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right) \frac{\overline{x}}{|x|} \wedge \frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right) \\ &= -\frac{\mathbf{i}Idr + \mathbf{j}Jdr + \mathbf{k}Kdr}{|x|} \wedge \frac{\mathbf{i}Idr + \mathbf{j}Jdr + \mathbf{k}Kdr}{|x|} \wedge \left(-\frac{\mathbf{i}Idr + \mathbf{j}Jdr + \mathbf{k}Kdr}{|x|}\right) \\ &= \frac{6\,\mathbf{i}\,\mathbf{j}\,\mathbf{k}}{|x|^3} Idr \wedge Jdr \wedge Kdr = -\frac{6}{|x|^3} Idr \wedge Jdr \wedge Kdr. \end{aligned} \tag{III.18}$$

We have

$$\begin{aligned} \left[\frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right), \frac{x}{|x|} d\left(\frac{\overline{x}}{|x|}\right)\right] &= \left[-\frac{\mathbf{i}Idr + \mathbf{j}Jdr + \mathbf{k}Kdr}{|x|}, -\frac{\mathbf{i}Idr + \mathbf{j}Jdr + \mathbf{k}Kdr}{|x|}\right] \\ &= \frac{2}{|x|^2}\left(\mathbf{i}\, Jdr \wedge Kdr + \mathbf{j}\, Kdr \wedge Idr + \mathbf{k}\, Idr \wedge Jdr\right), \end{aligned} \tag{III.19}$$

hence

$$\begin{aligned}
&\frac{2}{3}\frac{x}{|x|}d\left(\frac{\overline{x}}{|x|}\right)\wedge\left[\frac{x}{|x|}d\left(\frac{\overline{x}}{|x|}\right),\frac{x}{|x|}d\left(\frac{\overline{x}}{|x|}\right)\right]\\
&=-\frac{4}{3|x|^3}(\mathbf{i}\,Idr+\mathbf{j}\,Jdr+\mathbf{k}\,Kdr)\wedge(\mathbf{i}\,Jdr\wedge Kdr+\mathbf{j}\,Kdr\wedge Idr+\mathbf{k}\,Idr\wedge Jdr)\\
&=\frac{4}{|x|^3}\mathbf{1}\,Idr\wedge Jdr\wedge Kdr.
\end{aligned}\tag{III.20}$$

Then

$$Q=-\frac{2}{|x|^3}\,\mathbf{1}\,Idr\wedge Jdr\wedge Kdr+O(\varepsilon^{-2}),$$

and, recalling that $\mathrm{tr}(\mathbf{1})=2$,

$$\int_{\partial B_\varepsilon(0)}\mathrm{tr}(Q)=-\frac{1}{\varepsilon^3}\int_{\partial B_\varepsilon(0)}4\,Idr\wedge Jdr\wedge Kdr+O(\varepsilon)=-8\pi^2+O(\varepsilon).$$

Letting $\varepsilon\to 0$ in (III.17) we conclude. $\square$

Our goal now is to estimate

$$\int_{\mathbb{R}^4}|F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|F_A|^2\,dx^4\tag{III.21}$$

Thanks to Lemma III.1 and (II.25) we have

$$\begin{aligned}
&\int_{\mathbb{R}^4}|F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|F_A|^2\,dx^4\\
&=\int_{\mathbb{R}^4}|P_+F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|P_+F_A|^2\,dx^4+\int_{\mathbb{R}^4}|P_-F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|P_-F_A|^2\,dx^4\\
&=8\pi^2+2\int_{\mathbb{R}^4}|P_-F_{\hat{A}}|^2\,dx^4-2\int_{\mathbb{R}^4}|P_-F_A|^2\,dx^4
\end{aligned}\tag{III.22}$$

Combining (III.15) with the fact that F_{SD_λ} is self-dual gives

$$\int_{\mathbb{R}^4}|F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|F_A|^2\,dx^4=8\pi^2+2\int_{B_\rho(0)\setminus B_{\tau\rho}(0)}|P_-F_{\hat{A}}|^2\,dx^4-2\int_{B_\rho(0)}|P_-F_A|^2\,dx^4\tag{III.23}$$

Because of (III.13) we have on $B_\rho(0)\setminus B_{\tau\rho}(0)$

$$F_{\hat{A}}=\frac{\overline{x}}{|x|}\,F_{\check{A}}\,\frac{x}{|x|}\ ,\tag{III.24}$$

and in particular

$$P_-F_{\hat{A}}=\frac{\overline{x}}{|x|}\,P_-F_{\check{A}}\,\frac{x}{|x|}\ .\tag{III.25}$$

Hence finally

$$\begin{aligned}
&\int_{\mathbb{R}^4}|F_{\hat{A}}|^2\,dx^4-\int_{\mathbb{R}^4}|F_A|^2\,dx^4\\
&=8\pi^2+2\int_{B_\rho(0)\setminus B_{\tau\rho}(0)}|P_-F_{\check{A}}|^2\,dx^4-2\int_{B_\rho(0)}|P_-F_A|^2\,dx^4\ .
\end{aligned}\tag{III.26}$$

We are now estimating this difference in the limit $\rho \to 0$ by taking λ as in (III.7), where c_0 is going to be chosen later independent of ρ. This gives in particular

$$\left\|(1+\lambda^2\,|x|^2)^{-1}\right\|_{L^\infty(B_\rho(0)\setminus B_{\tau\rho}(0))} = O(\rho^2)\ . \tag{III.27}$$

Combining (II.70) in lemma II.2 with (III.25) is implying

$$\|\check{A}\|_{L^\infty(B_\rho(0)\setminus B_{\tau\rho}(0))} = O(\rho)\ . \tag{III.28}$$

We have also for any $x \in B_\rho(0) \setminus B_{\tau\rho}(0)$

$$\begin{aligned}
&|P_- d\check{A}| \le |d\check{A}| \le \|\eta'\|_\infty\ \rho^{-1}\left[|A(x)| + |x|^{-1}\,|(1+\lambda^2\,|x|^2)^{-1}|\right] + |dA|(x) + \lambda^2\,(1+\lambda^2\,|x|^2)^{-2}\\
&\le O(1) + |F_A|(x) + O(\rho^2) + O(\lambda^2\rho^4) = O(1)
\end{aligned} \tag{III.29}$$

Hence

$$\begin{aligned}
\left\||P_-F_{\check{A}}|^2 - |P_-d\check{A}|^2\right\|_{L^\infty(B_\rho(0)\setminus B_{\tau\rho}(0))} &= \left\|2\,\left\langle P_-d\check{A}, [\check{A},\check{A}]\right\rangle + \left|[\check{A},\check{A}]\right|^2\right\|_{L^\infty(B_\rho(0)\setminus B_{\tau\rho}(0))}\\
&= O(\rho^2)\ .
\end{aligned} \tag{III.30}$$

This gives

$$\begin{aligned}
&\int_{\mathbb{R}^4} |F_{\hat{A}}|^2\ dx^4 - \int_{\mathbb{R}^4} |F_A|^2\ dx^4\\
&= 8\pi^2 + 2\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_-d\check{A}|^2\ dx^4 - 2\int_{B_\rho(0)} |P_-F_A|^2\ dx^4 + O(\rho^6)\ .
\end{aligned} \tag{III.31}$$

This brings us naturally to the estimation of the square of the L^2 norm of $P_-d\check{A}$ on the intermediate annulus $B_\rho(0) \setminus B_{\tau\rho}(0)$.

Recall

$$\check{A}(g_0,\rho,\lambda) = \eta_\rho\, g_0^{-1}\,\underline{A}\,g_0 - (1-\eta_\rho)\ \frac{1}{1+\lambda^2\,|x|^2}\,d\left(\frac{\overline{x}}{|x|}\right)\ \frac{x}{|x|}\ . \tag{III.32}$$

We write

$$\begin{aligned}
d\check{A}\quad &= \eta'_\rho\, d|x| \wedge \left(g_0^{-1}\,\underline{A}\,g_0 + \frac{1}{1+\lambda^2\,|x|^2}\,d\left(\frac{\overline{x}}{|x|}\right)\ \frac{x}{|x|}\right)\\
&+\eta_\rho\ g_0^{-1}\,d\underline{A}\,g_0 + 2\,(1-\eta_\rho)\,\frac{\lambda^2\,|x|}{(1+\lambda^2\,|x|^2)^2}\,d|x| \wedge d\left(\frac{\overline{x}}{|x|}\right)\ \frac{x}{|x|}\\
&+(1-\eta_\rho)\ \frac{1}{1+\lambda^2\,|x|^2}\,d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right)\ .
\end{aligned} \tag{III.33}$$

Observe that

$$d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right) = \frac{d\overline{x} \wedge dx}{|x|^2} - \frac{d|x| \wedge \overline{x}\,dx}{|x|^3} - \frac{d\overline{x}\,x \wedge d|x|}{|x|^3}\ . \tag{III.34}$$

Hence, thanks to (II.28) we have

$$\begin{aligned}
P_-\left(d\left(\frac{\overline{x}}{|x|}\right) \wedge d\left(\frac{x}{|x|}\right)\right) &= \frac{d\overline{x} \wedge dx}{|x|^2} + P_-\left(\frac{d|x| \wedge (d\overline{x}\,x - \overline{x}\,dx)}{|x|^3}\right)\\
&= \frac{d\overline{x} \wedge dx}{|x|^2} + \frac{1}{2}\,P_-\left(\frac{d|x|^2 \wedge (d\overline{x}\,x - \overline{x}\,dx)}{|x|^4}\right)\ .
\end{aligned} \tag{III.35}$$

Observe trivially

$$0 = d|x|^2 \wedge (d\overline{x}\, x + \overline{x}\, dx)\ . \tag{III.36}$$

Combining (III.35) and (III.36) gives then

$$P_-\left(d\left(\frac{\overline{x}}{|x|}\right)\wedge d\left(\frac{x}{|x|}\right)\right) = \frac{d\overline{x}\wedge dx}{|x|^2} + \frac{2}{|x|}\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right) \tag{III.37}$$

We deduce

$$\begin{aligned}
&P_-\left(2\,\frac{\lambda^2\,|x|}{(1+\lambda^2\,|x|^2)^2}\,d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|} + \frac{1}{1+\lambda^2\,|x|^2}\,d\left(\frac{\overline{x}}{|x|}\right)\wedge d\left(\frac{x}{|x|}\right)\right)\\
&=\left(\frac{2\,\lambda^2\,|x|}{(1+\lambda^2\,|x|^2)^2}+\frac{2}{|x|\,(1+\lambda^2\,|x|^2)}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)+\frac{1}{|x|^2\,(1+\lambda^2\,|x|^2)}\,d\overline{x}\wedge dx\\
&=\frac{2+4\,\lambda^2\,|x|^2}{|x|\,(1+\lambda^2\,|x|^2)^2}\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)+\frac{1}{|x|^2\,(1+\lambda^2\,|x|^2)}\,d\overline{x}\wedge dx
\end{aligned} \tag{III.38}$$

Hence

$$\begin{aligned}
&P_- d\check{A} = \eta'_\rho\, g_0^{-1} P_-\,(d|x|\wedge \underline{A})\ g_0 + \eta_\rho\ g_0^{-1}\,P_- d\underline{A}\,g_0\\
&+\left(\eta'_\rho\frac{1}{1+\lambda^2\,|x|^2}+(1-\eta_\rho)\,\frac{2+4\,\lambda^2\,|x|^2}{|x|\,(1+\lambda^2\,|x|^2)^2}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)\\
&+\frac{1-\eta_\rho}{|x|^2\,(1+\lambda^2\,|x|^2)}\,d\overline{x}\wedge dx + O(\rho)
\end{aligned} \tag{III.39}$$

In $B_\rho(0)\setminus B_{\tau\rho}(0)$ we have respectively

$$\eta'_\rho\,\frac{1}{1+\lambda^2\,|x|^2} = \eta'_\rho\,\frac{1}{\lambda^2\,|x|^2}\,\frac{1}{1+O(\rho^2)} = \eta'_\rho\,\frac{1}{\lambda^2\,|x|^2} + O(\rho^3)\ , \tag{III.40}$$

and

$$(1-\eta_\rho)\,\frac{2+4\,\lambda^2\,|x|^2}{|x|\,(1+\lambda^2\,|x|^2)^2} = (1-\eta_\rho)\,\frac{4}{\lambda^2\,|x|^3}\ (1+O(\rho^2)) = (1-\eta_\rho)\,\frac{4}{\lambda^2\,|x|^3} + O(\rho^3)\ . \tag{III.41}$$

and

$$\frac{1-\eta_\rho}{(1+\lambda^2\,|x|^2)} = \frac{(1-\eta_\rho)}{\lambda^2\,|x|^2}\left(1+O(\rho^2)\right) = \frac{(1-\eta_\rho)}{\lambda^2\,|x|^2} + O(\rho^4)\ . \tag{III.42}$$

Thus in $B_\rho(0)\setminus B_{\tau\rho}(0)$ we have

$$\begin{aligned}
&\left(\eta'_\rho\frac{1}{1+\lambda^2\,|x|^2}+(1-\eta_\rho)\,\frac{2+4\,\lambda^2\,|x|^2}{|x|\,(1+\lambda^2\,|x|^2)^2}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)\\
&=\frac{1}{\lambda^2\,|x|^2}\ \left(\eta'_\rho + 4\,\frac{(1-\eta_\rho)}{|x|}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right) + O(\rho^2)
\end{aligned} \tag{III.43}$$

which gives

$$\begin{aligned}
&P_- d\check{A} = \rho^{-1}\,\eta'\left(\frac{x}{\rho}\right)\ g_0^{-1}P_-\,(d|x|\wedge\underline{A})\ g_0 + \eta_\rho\ g_0^{-1}\,P_- d\underline{A}\,g_0\\
&+\frac{1}{\lambda^2\,|x|^2}\ \left(\eta'_\rho + 4\,\frac{(1-\eta_\rho)}{|x|}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right) + \frac{1-\eta_\rho}{\lambda^2\,|x|^4}\,d\overline{x}\wedge dx + O(\rho)
\end{aligned} \tag{III.44}$$

We have

$$2\,|x|^3\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)=P_-\left(d|x|^2\wedge d\overline{x}\,x\right) \tag{III.45}$$

Moreover

$$\begin{aligned}
&P_-\left(d|x|^2\wedge d\overline{x}\,x\right)=P_-\left(d(\overline{x}\,x)\wedge d\overline{x}\,x\right)=\overline{x}\,P_-\left(\,dx\wedge d\overline{x}\right)\,x+P_-\left(d\overline{x}\,x\wedge d\overline{x}\,x\right)\\
&=P_-\left(d\overline{x}\,x\wedge d\overline{x}\,x\right)=P_-\left(d\overline{x}\,x\wedge d|x|^2\right)-P_-\left(d\overline{x}\,x\wedge\overline{x}\,dx\right)\\
&=-P_-\left(d|x|^2\wedge d\overline{x}\,x\right)-|x|^2\,d\overline{x}\wedge dx
\end{aligned} \tag{III.46}$$

This gives

$$P_-\left(d|x|^2\wedge d\overline{x}\,x\right)=-\tfrac{|x|^2}{2}\,d\overline{x}\wedge dx\ . \tag{III.47}$$

Hence combining (III.45) and (III.47) we obtain

$$P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)=-\frac{1}{4\,|x|}\,d\overline{x}\wedge dx\ . \tag{III.48}$$

This implies

$$\begin{aligned}
&\frac{1}{\lambda^2\,|x|^2}\,\left(\eta'_\rho+4\,\frac{(1-\eta_\rho)}{|x|}\right)\,P_-\left(d|x|\wedge d\left(\frac{\overline{x}}{|x|}\right)\,\frac{x}{|x|}\right)+\frac{1-\eta_\rho}{\lambda^2\,|x|^4}\,d\overline{x}\wedge dx\\
&=-\frac{1}{4\,\lambda^2\,|x|^3}\,\eta'_\rho\,d\overline{x}\wedge dx
\end{aligned} \tag{III.49}$$

Combining (III.44) and (III.49) we obtain

$$P_-d\check{A}=\rho^{-1}\,\eta'\left(\frac{x}{\rho}\right)\,g_0^{-1}P_-\left(d|x|\wedge\underline{A}\right)\,g_0+\eta_\rho\,g_0^{-1}\,P_-d\underline{A}\,g_0-\frac{1}{4\,\lambda^2\,|x|^3}\,\eta'_\rho\,d\overline{x}\wedge dx \tag{III.50}$$

We have, using (II.69),

$$P_-\left(d|x|\wedge\underline{A}\right)=\underline{A}\llcorner I\partial_r\;P_-\left(dr\wedge I dr\right)+\underline{A}\llcorner J\partial_r\;P_-\left(dr\wedge J dr\right)+\underline{A}\llcorner K\partial_r\;P_-\left(dr\wedge K dr\right) \tag{III.51}$$

Hence finally

$$P_-\left(d|x|\wedge\underline{A}\right)=\sqrt{2}^{-1}\,\left(\underline{A}\llcorner I\partial_r\;\omega^-_{\mathbf{i}}+\underline{A}\llcorner J\partial_r\;\omega^-_{\mathbf{j}}+\underline{A}\llcorner K\partial_r\;\omega^-_{\mathbf{k}}\right) \tag{III.52}$$

We have

$$\left\{\begin{aligned}
&\underline{A}\llcorner I\partial_r=(\phi_\rho^*A)\llcorner I\partial_r=A(\rho)\llcorner((\phi_\rho)_*I\partial_r)=\frac{\rho}{|x|}A(\rho)\llcorner(I\partial_r)=\frac{\rho^2}{2\,|x|}\,F_A(0)(\partial_r,I\partial_r)+O(\rho^2)\\
&\underline{A}\llcorner J\partial_r=(\phi_\rho^*A)\llcorner J\partial_r=A(\rho)\llcorner((\phi_\rho)_*J\partial_r)=\frac{\rho}{|x|}A(\rho)\llcorner(J\partial_r)=\frac{\rho^2}{2\,|x|}\,F_A(0)(\partial_r,J\partial_r)+O(\rho^2)\\
&\underline{A}\llcorner K\partial_r=(\phi_\rho^*A)\llcorner K\partial_r=A(\rho)\llcorner((\phi_\rho)_*K\partial_r)=\frac{\rho}{|x|}A(\rho)\llcorner(K\partial_r)=\frac{\rho^2}{2\,|x|}\,F_A(0)(\partial_r,K\partial_r)+O(\rho^2)
\end{aligned}\right. \tag{III.53}$$

Hence on $B_\rho(0)\setminus B_{\tau\rho}(0)$ there holds

$$
\begin{aligned}
P_-\,(d|x|\wedge \underline{A}) \quad &= \frac{\rho^2}{2\sqrt{2}|x|}\ \Big(F_A(0)(\partial_r, I\partial_r)\ \omega_{\mathbf{i}}^-(x)+F_A(0)(\partial_r, J\partial_r)\ \omega_{\mathbf{j}}^-(x)+F_A(0)(\partial_r, K\partial_r)\ \omega_{\mathbf{k}}^-(x)\Big)\\
&+O(|x|^2)
\end{aligned}
\tag{III.54}
$$

Observe that $\underline{A}=A$ on $\partial B_\rho(0)$, while in $B_\rho(0)\setminus B_{\tau\rho}(0)$

$$
\begin{aligned}
&\phi_\rho^*(dr\wedge Idr)=\phi_\rho^*(dr\wedge Jdr)=\phi_\rho^*(dr\wedge Kdr)=0\,,\\
&\phi_\rho^*(Idr\wedge Jdr)=\frac{\rho^2}{|x|^2}Idr\wedge Jdr\,,\\
&\phi_\rho^*(Jdr\wedge Kdr)=\frac{\rho^2}{|x|^2}Jdr\wedge Kdr\,,\\
&\phi_\rho^*(Kdr\wedge Idr)=\frac{\rho^2}{|x|^2}Kdr\wedge Idr\,.
\end{aligned}
\tag{III.55}
$$

Then we have

$$
\begin{aligned}
&d\underline{A}=\phi_\rho^* dA=\phi_\rho^* F_A+O(\rho)=F_A(0)(I\partial_r,J\partial_r)\ \phi_\rho^*(Idr\wedge Jdr)+F_A(0)(J\partial_r,K\partial_r)\ \phi_\rho^*(Jdr\wedge Kdr)\\
&\quad +F_A(0)(K\partial_r,I\partial_r)\ \phi_\rho^*(Kdr\wedge Idr)+O(\rho)\,,
\end{aligned}
\tag{III.56}
$$

hence

$$
\begin{aligned}
d\underline{A}=\left(\frac{\rho}{|x|}\right)^2\quad &[F_A(0)(I\partial_r,J\partial_r)\ Idr\wedge Jdr+F_A(0)(J\partial_r,K\partial_r)\ Jdr\wedge Kdr\\
&+F_A(0)(K\partial_r,I\partial_r)\ Kdr\wedge Idr]+O(\rho)\ .
\end{aligned}
\tag{III.57}
$$

Observe that we have respectively

$$
\star(dr\wedge Idr)=Jdr\wedge Kdr\ ,\ \star(dr\wedge Jdr)=Kdr\wedge Idr\ ,\ \star(dr\wedge Kdr)=Idr\wedge Jdr
\tag{III.58}
$$

This gives in particular

$$
\left\{
\begin{aligned}
&P_-(Idr\wedge Jdr)=-\,P_-(dr\wedge Kdr)=-\sqrt{2}^{-1}\ \omega_{\mathbf{k}}^-(x)\\
&P_-(Jdr\wedge Kdr)=-\,P_-(dr\wedge Idr)=-\sqrt{2}^{-1}\ \omega_{\mathbf{i}}^-(x)\\
&P_-(Kdr\wedge Idr)=-\,P_-(dr\wedge Jdr)=-\sqrt{2}^{-1}\ \omega_{\mathbf{j}}^-(x)
\end{aligned}
\right.
\tag{III.59}
$$

Hence

$$
\begin{aligned}
P_-d\underline{A}(0)\quad &=-\sqrt{2}^{-1}\ \frac{\rho^2}{|x|^2}\ F_A(0)(J\partial_r,K\partial_r)\,\omega_{\mathbf{i}}^-(x)-\sqrt{2}^{-1}\ \frac{\rho^2}{|x|^2}\ F_A(0)(K\partial_r,I\partial_r)\,\omega_{\mathbf{j}}^-(x)\\
&-\sqrt{2}^{-1}\ \frac{\rho^2}{|x|^2}\ F_A(0)(I\partial_r,J\partial_r)\,\omega_{\mathbf{k}}^-(x)+O(\rho)
\end{aligned}
\tag{III.60}
$$

Combining (III.52) and (III.60) is giving

$$
\begin{aligned}
&\eta'_\rho\, g_0^{-1} P_-\,(d|x|\wedge \underline{A})\; g_0+\eta_\rho\; g_0^{-1}\, P_- d\underline{A}\, g_0\\
&=\frac{1}{2\sqrt{2}}\,\frac{\rho^2}{|x|}\,\eta'_\rho\, g_0^{-1}\,\Big(F_A(0)(\partial_r,I\partial_r)\;\omega_{\mathbf{i}}^-+F_A(0)(\partial_r,J\partial_r)\;\omega_{\mathbf{j}}^-+F_A(0)(\partial_r,K\partial_r)\;\omega_{\mathbf{k}}^-\Big)\; g_0\\
&-\frac{\eta_\rho}{\sqrt{2}}\,\frac{\rho^2}{|x|^2}\, g_0^{-1}\,\Big(F_A(0)(J\partial_r,K\partial_r)\,\omega_{\mathbf{i}}^-+F_A(0)(K\partial_r,I\partial_r)\,\omega_{\mathbf{j}}^-+F_A(0)(I\partial_r,J\partial_r)\,\omega_{\mathbf{k}}^-\Big)\; g_0+O(\rho)
\end{aligned}
\tag{III.61}
$$

Recall

$$
P_- d\check{A}=\eta'_\rho\, g_0^{-1} P_-\,(d|x|\wedge \underline{A})\; g_0+\eta_\rho\; g_0^{-1}\, P_- d\underline{A}\, g_0-\frac{1}{4\,\lambda^2\,|x|^3}\;\eta'_\rho\, d\overline{x}\wedge dx \tag{III.62}
$$

Combining (III.62) with (III.61) is giving in $B_\rho(0)\setminus B_{\tau\rho}(0)$

$$
\begin{aligned}
&g_0\, P_- d\check{A}\, g_0^{-1}\\
&=\frac{1}{2\sqrt{2}}\,\frac{\rho^2}{|x|}\,\eta'_\rho\;\Big(F_A(0)(\partial_r,I\partial_r)\;\omega_{\mathbf{i}}^-(x)+F_A(0)(\partial_r,J\partial_r)\;\omega_{\mathbf{j}}^-(x)+F_A(0)(\partial_r,K\partial_r)\;\omega_{\mathbf{k}}^-(x)\Big)\\
&-\frac{1}{\sqrt{2}}\,\frac{\rho^2}{|x|^2}\,\eta_\rho\;\Big(F_A(0)(J\partial_r,K\partial_r)\,\omega_{\mathbf{i}}^-(x)+F_A(0)(K\partial_r,I\partial_r)\,\omega_{\mathbf{j}}^-(x)+F_A(0)(I\partial_r,J\partial_r)\,\omega_{\mathbf{k}}^-(x)\Big)\\
&\quad-\frac{1}{4\,\lambda^2\,|x|^3}\;\eta'_\rho\, g_0\, d\overline{x}\wedge dx\, g_0^{-1}+O(\rho)
\end{aligned}
\tag{III.63}
$$

Thus

$$\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_- d\check{A}|^2 \, dx^4$$

$$= \frac{\rho^2}{8}\int_{\tau\rho}^{\rho} s\ \eta'\left(\frac{s}{\rho}\right)^2\ ds\ \int_{S^3}\left(|F_A(0)(\partial_r, I\partial_r)|^2 + |F_A(0)(\partial_r, J\partial_r)|^2 + |F_A(0)(\partial_r, K\partial_r)|^2\right) dvol_{S^3}$$

$$+\frac{\rho^4}{2}\int_{\tau\rho}^{\rho} \eta\left(\frac{s}{\rho}\right)^2 \frac{ds}{s}\ \int_{S^3}\left(|F_A(0)(I\partial_r, J\partial_r)|^2 + |F_A(0)(J\partial_r, K\partial_r)|^2 + |F_A(0)(K\partial_r, I\partial_r)|^2\right) dvol_{S^3}$$

$$+\frac{1}{16\,\lambda^4\,\rho^2}\int_{\tau\rho}^{\rho} \eta'\left(\frac{s}{\rho}\right)^2 \frac{ds}{s^3}\int_{S^3} |d\overline{x}\wedge dx|^2 dvol_{S^3}$$

$$-\frac{\rho^3}{2}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)\eta\left(\frac{s}{\rho}\right)\ ds\int_{S^3}\big[\,\langle F_A(0)(\partial_r, I\partial_r), F_A(0)(J\partial_r, K\partial_r)\rangle$$

$$+\,\langle F_A(0)(\partial_r, J\partial_r), F_A(0)(K\partial_r, I\partial_r)\rangle + \langle F_A(0)(\partial_r, K\partial_r), F_A(0)(I\partial_r, J\partial_r)\rangle\,\big]\ dvol_{S^3}$$

$$-\frac{1}{4\sqrt{2}\,\lambda^2}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)^2\frac{ds}{s}$$

$$\times\int_{S^3}\Big\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(\partial_r, I\partial_r)\ \omega_{\mathbf{i}}^-(x) + F_A(0)(\partial_r, J\partial_r)\ \omega_{\mathbf{j}}^-(x) + F_A(0)(\partial_r, K\partial_r)\ \omega_{\mathbf{k}}^-(x)\Big\rangle\, dvol_{S^3}$$

$$+\frac{\rho}{2\sqrt{2}\,\lambda^2}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)\eta\left(\frac{s}{\rho}\right)\ \frac{ds}{s^2}$$

$$\times\int_{S^3}\Big\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(J\partial_r, K\partial_r)\ \omega_{\mathbf{i}}^-(x) + F_A(0)(K\partial_r, I\partial_r)\ \omega_{\mathbf{j}}^-(x) + F_A(0)(I\partial_r, J\partial_r)\ \omega_{\mathbf{k}}^-(x)\Big\rangle\, dvol_{S^3}$$

$$+O(\rho^5) \tag{III.64}$$

Lemma III.2. *We have*

$$\begin{aligned}&\int_{S^3}\left(|F_A(0)(\partial_r, I\partial_r)|^2 + |F_A(0)(\partial_r, J\partial_r)|^2 + |F_A(0)(\partial_r, K\partial_r)|^2\right) dvol_{S^3}\\ &= \int_{S^3}\left(|F_A(0)(I\partial_r, J\partial_r)|^2 + |F_A(0)(J\partial_r, K\partial_r)|^2 + |F_A(0)(K\partial_r, I\partial_r)|^2\right) dvol_{S^3}\\ &= \pi^2 |F_A(0)|^2\ .\end{aligned} \tag{III.65}$$

Proof of Lemma III.2. We have

$$\begin{aligned}&\int_{S^3}\left(|F_A(0)(I\partial_r, J\partial_r)|^2 + |F_A(0)(J\partial_r, K\partial_r)|^2 + |F_A(0)(K\partial_r, I\partial_r)|^2\right) dvol_{S^3}\\ &= 2\,\pi^2\,|F_A(0)|^2 - \int_{S^3} |F_A(0)\llcorner\partial_r|^2\ dvol_{S^3}\ .\end{aligned} \tag{III.66}$$

Moreover

$$F_A(0) = \sum_{i<j} F_A^{ij}(0)\ dx_i\wedge dx_j = \frac{1}{2}\sum_{i,j=1}^{4} F_A^{ij}(0)\ dx_i\wedge dx_j\ , \tag{III.67}$$

and on S^3

$$F_A(0)\llcorner\partial_r=\frac{1}{2}\sum_{i,j=1}^4 F_A^{ij}(0)\ dx_i\,x_j-\frac{1}{2}\sum_{i,j=1}^4 F_A^{ij}(0)\ x_i\,dx_j=\sum_{i=1}^4\left(\sum_{j=1}^4 F_A^{ij}(0)\ x_j\right)\ dx_i\ . \tag{III.68}$$

Hence

$$\begin{aligned}
&\int_{S^3}|F_A(0)\llcorner\partial_r|^2\ dvol_{S^3}=\sum_{i=1}^4\int_{S^3}\left|\sum_{j=1}^4 F_A^{ij}(0)\ x_j\right|^2\ dvol_{S^3}\\
&=\sum_{i=1}^4\sum_{j,l=1}^4\left\langle F_A^{ij}(0),F_A^{il}(0)\right\rangle\ \int_{S^3}x_j\,x_l\ dvol_{S^3}\\
&=\sum_{i=1}^4\sum_{j=1}^4|F_A^{ij}(0)|^2\ \int_{S^3}x_j^2\ dvol_{S^3}=\frac{2\pi^2}{4}\sum_{i,j=1}^4|F_A^{ij}(0)|^2=\pi^2\sum_{i<j}|F_A^{ij}(0)|^2=\pi^2\,|F_A(0)|^2\ .
\end{aligned} \tag{III.69}$$

□

Lemma III.3. *We have*

$$\begin{aligned}
&\int_{S^3}\langle F_A(0)(\partial_r,I\partial_r),F_A(0)(J\partial_r,K\partial_r)\rangle+\langle F_A(0)(\partial_r,J\partial_r),F_A(0)(K\partial_r,I\partial_r)\rangle\ \ dvol_{S^3}\\
&+\int_{S^3}\langle F_A(0)(\partial_r,K\partial_r),F_A(0)(I\partial_r,J\partial_r)\rangle\ \ dvol_{S^3}=\frac{\pi^2}{\sqrt{2}}\ \left[|P_+F_A(0)|^2-|P_-F_A(0)|^2\right]
\end{aligned} \tag{III.70}$$

Proof of Lemma III.2. We have

$$\begin{aligned}
P_\pm F_A(0)\quad&=\sqrt{2}^{-1}\ \ (F_A(0)(\partial_r,I\partial_r)\pm F_A(0)(J\partial_r,K\partial_r))\ \omega_{\mathbf{i}}^\pm(x)\\
&+\sqrt{2}^{-1}\ \ (F_A(0)(\partial_r,J\partial_r)\pm F_A(0)(K\partial_r,I\partial_r))\ \omega_{\mathbf{j}}^\pm(x)\\
&+\sqrt{2}^{-1}\ \ (F_A(0)(\partial_r,K\partial_r)\pm F_A(0)(I\partial_r,J\partial_r))\ \omega_{\mathbf{k}}^\pm(x)\ .
\end{aligned} \tag{III.71}$$

Then

$$\begin{aligned}
&|P_+F_A(0)|^2-|P_-F_A(0)|^2=\frac{4}{\sqrt{2}}\ \langle F_A(0)(\partial_r,I\partial_r),F_A(0)(J\partial_r,K\partial_r)\rangle\\
&+\frac{4}{\sqrt{2}}\ \langle F_A(0)(\partial_r,J\partial_r),F_A(0)(K\partial_r,I\partial_r)\rangle+\frac{4}{\sqrt{2}}\ \langle F_A(0)(\partial_r,K\partial_r),F_A(0)(I\partial_r,J\partial_r)\rangle\ \ ,
\end{aligned} \tag{III.72}$$

and (III.70) follows. □

We denote

$$\left\{\begin{array}{l}
\mathbf{i}_{g_0}:=g_0\,\mathbf{i}\,g_0^{-1}\\
\mathbf{j}_{g_0}:=g_0\,\mathbf{j}\,g_0^{-1}\\
\mathbf{i}_{g_0}:=g_0\,\mathbf{k}\,g_0^{-1}
\end{array}\right. \tag{III.73}$$

Lemma III.4. *We have*

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, I\partial_r)\ \omega_{\mathbf{i}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, J\partial_r)\ \omega_{\mathbf{j}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, K\partial_r)\ \omega_{\mathbf{k}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ .
\end{aligned}
\tag{III.74}
$$

Proof of Lemma III.4. We have respectively

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, I\partial_r)\ \omega_{\mathbf{i}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}
\end{aligned}
\tag{III.75}
$$

We recall from (B.5)

$$
\left\{
\begin{aligned}
&\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{i}}^-(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2\\
&\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{i}}^-(x) = 2\ (x_2\, x_3 - x_1\, x_4)\\
&\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{i}}^-(x) = 2\,(x_1\, x_3 + x_2\, x_4)
\end{aligned}
\right.
\tag{III.76}
$$

Recall that on S^3 we have

$$
F_A(0)(\partial_r, I\partial_r) = F_A(0)(x_1\,\partial_{x_1} + x_2\,\partial_{x_2} + x_3\,\partial_{x_3} + x_4\,\partial_{x_4}, x_1\,\partial_{x_2} - x_2\,\partial_{x_1} + x_3\,\partial_{x_4} - x_4\,\partial_{x_3})
\tag{III.77}
$$

Hence we have successively using corollary B.2

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&=\ 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_2^2 - x_3^2 - x_4^2)\ (x_1^2 + x_2^2)\ dvol_{S^3}\ \left\langle \mathbf{i}_{g_0}, F_A^{12}(0)\right\rangle\\
&+ 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_2^2 - x_3^2 - x_4^2)\ (x_3^2 + x_4^2)\ dvol_{S^3}\ \left\langle \mathbf{i}_{g_0}, F_A^{34}(0)\right\rangle\\
&=\ 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_2^2 - x_3^2 - x_4^2)\ (x_1^2 + x_2^2)\ dvol_{S^3}\ \left\langle \mathbf{i}_{g_0}, F_A^{12}(0) - F_A^{34}(0)\right\rangle\\
&=\ \frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0)\right\rangle
\end{aligned}
\tag{III.78}
$$

we have

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega^-_{\mathbf{j}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&=\ 4\sqrt{2}\int_{S^3} x_2^2\,x_3^2\ \ dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{24}(0)+F_A^{13}(0)\right\rangle\\
&+4\sqrt{2}\int_{S^3} x_1^2\,x_4^2\ \ dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{24}(0)+F_A^{13}(0)\right\rangle\\
&=\ 8\sqrt{2}\int_{S^3} x_2^2\,x_3^2\ \ dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{24}(0)+F_A^{13}(0)\right\rangle\\
&=\ \frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0)+F_A^{24}(0)\right\rangle
\end{aligned}
\tag{III.79}
$$

and

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega^-_{\mathbf{k}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&= 4\sqrt{2}\int_{S^3} x_1^2\,x_3^2\ \ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle\\
&+4\sqrt{2}\int_{S^3} x_2^2\,x_4^2\ \ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle\\
&= 8\sqrt{2}\int_{S^3} x_1^2\,x_3^2\ \ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle\\
&=\ \frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle
\end{aligned}
\tag{III.80}
$$

We deduce

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(\partial_r, I\partial_r)\ \omega^-_{\mathbf{i}}(x)\right\rangle\ dvol_{S^3}\\
&=\frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{i}_{g_0}, F_A^{12}(0)+F_A^{43}(0)\right\rangle+\frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0)+F_A^{24}(0)\right\rangle\\
&+\frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle
\end{aligned}
\tag{III.81}
$$

We recall

$$
\begin{aligned}
P_-F_A(0)\quad &= 2^{-1}\ \left(F_A^{12}(0)+F_A^{43}(0)\right)(dx_1\wedge dx_2-dx_3\wedge dx_4)\\
&+2^{-1}\ \left(F_A^{13}(0)+F_A^{24}(0)\right)(dx_1\wedge dx_3-dx_4\wedge dx_2)\\
&+2^{-1}\ \left(F_A^{14}(0)+F_A^{32}(0)\right)(dx_1\wedge dx_4-dx_2\wedge dx_3)\\
&=\left[\frac{\left(F_A^{12}(0)+F_A^{43}(0)\right)}{\sqrt{2}}\ \omega^-_{\mathbf{i}}+\frac{\left(F_A^{13}(0)+F_A^{24}(0)\right)}{\sqrt{2}}\ \omega^-_{\mathbf{j}}+\frac{\left(F_A^{14}(0)+F_A^{32}(0)\right)}{\sqrt{2}}\ \omega^-_{\mathbf{k}}\right]
\end{aligned}
\tag{III.82}
$$

Thus

$$\left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, P_- F_A(0) \right\rangle = 2\,\sqrt{2}\, \left\langle \mathbf{i}_{g_0}\, \omega_{\mathbf{i}}^- + \mathbf{j}_{g_0}\, \omega_{\mathbf{j}}^- + \mathbf{k}_{g_0}\, \omega_{\mathbf{k}}^-, \right.$$

$$\left.\left[\frac{\left(F_A^{12}(0) + F_A^{43}(0)\right)}{\sqrt{2}}\; \omega_{\mathbf{i}}^- + \frac{\left(F_A^{13}(0) + F_A^{24}(0)\right)}{\sqrt{2}}\; \omega_{\mathbf{j}}^- + \frac{\left(F_A^{14}(0) + F_A^{32}(0)\right)}{\sqrt{2}}\; \omega_{\mathbf{k}}^- \right] \right\rangle \tag{III.83}$$

$$= 2\, \left\langle \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) \right\rangle + 2\, \left\langle \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) \right\rangle + 2\, \left\langle \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) \right\rangle$$

and finally

$$\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, I\partial_r)\; \omega_{\mathbf{i}}^-(x) \right\rangle\; dvol_{S^3} = \frac{\sqrt{2}}{3}\, \pi^2\, \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, P_- F_A(0) \right\rangle\; . \tag{III.84}$$

We have then respectively

$$\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, J\partial_r)\; \omega_{\mathbf{j}}^-(x) \right\rangle\; dvol_{S^3} \\
&= \int_{S^3} 2\,\sqrt{2}\; \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x)\; \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, J\partial_r) \rangle\; dvol_{S^3} \\
&+ \int_{S^3} 2\,\sqrt{2}\; \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x)\; \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, J\partial_r) \rangle\; dvol_{S^3} \\
&+ \int_{S^3} 2\,\sqrt{2}\; \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x)\; \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, J\partial_r) \rangle\; dvol_{S^3}
\end{aligned} \tag{III.85}$$

We recall from (B.5)

$$\left\{ \begin{array}{l}
\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x) = 2\,(x_1\, x_4 + x_2\, x_3) \\[2ex]
\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x) = x_1^2 + x_3^2 - x_2^2 - x_4^2 \\[2ex]
\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x) = 2\,(x_3\, x_4 - x_1\, x_2)
\end{array} \right. \tag{III.86}$$

Recall that on S^3 we have

$$F_A(0)(\partial_r, J\partial_r) = F_A(0)(x_1\, \partial_{x_1} + x_2\, \partial_{x_2} + x_3\, \partial_{x_3} + x_4\, \partial_{x_4}, x_1\, \partial_{x_3} - x_3\, \partial_{x_1} + x_4\, \partial_{x_2} - x_2\, \partial_{x_4}) \tag{III.87}$$

Hence we have successively

$$\begin{aligned}
&\int_{S^3} 2\,\sqrt{2}\; \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x)\; \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, J\partial_r) \rangle\; dvol_{S^3} \\
&= 4\,\sqrt{2}\; \int_{S^3} x_1^2\, x_4^2\, dvol_{S^3}\; < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\
&+ 4\,\sqrt{2}\; \int_{S^3} x_2^2\, x_3^2\; dvol_{S^3}\; < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\
&= 8\,\sqrt{2}\; \int_{S^3} x_1^2\, x_4^2\, dvol_{S^3}\; < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\
&= \frac{2\,\sqrt{2}}{3}\; \pi^2 \quad < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >
\end{aligned} \tag{III.88}$$

we have

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&=\ 2\sqrt{2}\ \int_{S^3} (x_1^2+x_3^2-x_2^2-x_4^2)(x_1^2+x_3^2)\ dvol_{S^3}\ \ <\mathbf{j}_{g_0}, F_A^{13}(0)>\\
&+2\sqrt{2}\ \int_{S^3} (x_1^2+x_3^2-x_2^2-x_4^2)(x_2^2+x_4^2)\ dvol_{S^3}\ \ <\mathbf{j}_{g_0}, F_A^{42}(0)>\\
&=\ 2\sqrt{2}\ \int_{S^3} (x_1^2+x_3^2-x_2^2-x_4^2)(x_1^2+x_3^2)\ dvol_{S^3}\ \ <\mathbf{j}_{g_0}, F_A^{13}(0)-F_A^{42}(0)>\\
&=\ \frac{2\sqrt{2}}{3}\ \pi^2\ \ \ <\mathbf{j}_{g_0}, F_A^{13}(0)+F_A^{24}(0)>
\end{aligned}
\tag{III.89}
$$

and

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&=4\sqrt{2}\ \int_{S^3} x_3^2\,x_4^2\ \ dvol_{S^3}\ \ <\mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)>\\
&+4\sqrt{2}\ \int_{S^3} x_1^2\,x_2^2\ \ dvol_{S^3}\ \ <\mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)>\\
&=\ \frac{2\sqrt{2}}{3}\ \pi^2\ \ \ <\mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)>
\end{aligned}
\tag{III.90}
$$

Hence we have also

$$
\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(\partial_r, J\partial_r)\ \omega_{\mathbf{j}}^-(x)\right\rangle\ dvol_{S^3} = \frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ . \tag{III.91}
$$

We have

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(\partial_r, K\partial_r)\ \omega_{\mathbf{k}}^-(x)\right\rangle\ dvol_{S^3}\\
&=\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&+\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&+\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3}
\end{aligned}
\tag{III.92}
$$

We recall from (B.5)

$$
\left\{
\begin{array}{l}
\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x) = 2\,(x_2\,x_4 - x_1\,x_3)\\[2ex]
\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x) = 2\ (x_1\,x_2 + x_3\,x_4)\\[2ex]
\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x) = x_1^2 + x_4^2 - x_2^2 - x_3^2
\end{array}
\right.
\tag{III.93}
$$

Recall that on S^3 we have

$$F_A(0)(\partial_r, K\partial_r) = F_A(0)(x_1\,\partial_{x_1} + x_2\,\partial_{x_2} + x_3\,\partial_{x_3} + x_4\,\partial_{x_4}, x_1\,\partial_{x_4} - x_4\,\partial_{x_1} + x_2\,\partial_{x_3} - x_3\,\partial_{x_2})\ . \quad \text{(III.94)}$$

Hence we have successively

$$\begin{aligned} &\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3} \\ &=\ 4\sqrt{2}\ \int_{S^3} x_2^2\, x_4^2\, dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\ &+ 4\sqrt{2}\ \int_{S^3} x_1^2\, x_3^2\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\ &=\ 8\sqrt{2}\ \int_{S^3} x_2^2\, x_4^2\, dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \\ &=\ \frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) > \end{aligned} \quad \text{(III.95)}$$

we have

$$\begin{aligned} &\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3} \\ &=\ 4\sqrt{2}\ \int_{S^3} x_1^2\, x_2^2\ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) > \\ &+4\sqrt{2}\ \int_{S^3} x_3^2\, x_4^2\ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) > \\ &=\ 8\sqrt{2}\ \int_{S^3} dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) > \\ &=\ \frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) > \end{aligned} \quad \text{(III.96)}$$

and

$$\begin{aligned} &\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(\partial_r, K\partial_r)\rangle\ dvol_{S^3} \\ &=\ 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_4^2 - x_2^2 - x_3^2)(x_1^2 + x_4^2)\ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{14}(0) > \\ &+2\sqrt{2}\ \int_{S^3} (x_1^2 + x_4^2 - x_2^2 - x_3^2)(x_2^2 + x_3^2)\ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{23}(0) > \\ &=\ 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_4^2 - x_2^2 - x_3^2)(x_1^2 + x_4^2)\ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{14}(0) - F_A^{23}(0) > \\ &=\ \frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) > \end{aligned} \quad \text{(III.97)}$$

Hence we have also

$$\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(\partial_r, K\partial_r)\ \omega_{\mathbf{k}}^-(x)\right\rangle\ dvol_{S^3} = \frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ . \quad \text{(III.98)}$$

$\square$

Lemma III.5. *We have*

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(J\partial_r, K\partial_r)\ \omega^-_{\mathbf{i}}(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(K\partial_r, I\partial_r)\ \omega^-_{\mathbf{j}}(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(I\partial_r, J\partial_r)\ \omega^-_{\mathbf{k}}(x)\right\rangle\ dvol_{S^3}\\
&= -\frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ .
\end{aligned}
\tag{III.99}
$$

Proof of Lemma III.5. We have

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(J\partial_r, K\partial_r)\ \omega^-_{\mathbf{i}}(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} 2\sqrt{2}\ \omega^-_{\mathbf{i}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega^-_{\mathbf{j}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega^-_{\mathbf{k}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}
\end{aligned}
\tag{III.100}
$$

We recall from (B.5)

$$
\left\{
\begin{array}{l}
\omega^-_{\mathbf{i}}\cdot\omega^-_{\mathbf{i}}(x) = x_1^2+x_2^2-x_3^2-x_4^2\\[2mm]
\omega^-_{\mathbf{j}}\cdot\omega^-_{\mathbf{i}}(x) = 2\ (x_2\,x_3 - x_1\,x_4)\\[2mm]
\omega^-_{\mathbf{k}}\cdot\omega^-_{\mathbf{i}}(x) = 2\,(x_1\,x_3 + x_2\,x_4)
\end{array}
\right.
\tag{III.101}
$$

Recall that on S^3 we have

$$
F_A(0)(J\partial_r, K\partial_r) = F_A(0)(x_1\,\partial_{x_3} - x_3\,\partial_{x_1} + x_4\,\partial_{x_2} - x_2\,\partial_{x_4}, x_1\,\partial_{x_4} - x_4\,\partial_{x_1} + x_2\,\partial_{x_3} - x_3\,\partial_{x_2})
\tag{III.102}
$$

Hence we have successively using corollary B.2

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega^-_{\mathbf{i}}\cdot\omega^-_{\mathbf{i}}(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&= 2\sqrt{2}\ \int_{S^3}(x_1^2+x_2^2-x_3^2-x_4^2)\ (x_1^2+x_2^2)\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{34}(0) >\\
&+ 2\sqrt{2}\ \int_{S^3}(x_1^2+x_2^2-x_3^2-x_4^2)\ (x_3^2+x_4^2)\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) >\\
&= 2\sqrt{2}\ \int_{S^3}(x_1^2+x_2^2-x_3^2-x_4^2)\ (x_1^2+x_2^2)\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, -F_A^{12}(0)+F_A^{34}(0) >\\
&= -\frac{2\sqrt{2}}{3}\,\pi^2\ \ < \mathbf{i}_{g_0}, F_A^{12}(0)+F_A^{43}(0) >
\end{aligned}
\tag{III.103}
$$

we have

$$\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&=\ 4\sqrt{2}\ \int_{S^3} x_2^2\, x_3^2\ \ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{42}(0) + F_A^{31}(0) >\\
&+4\sqrt{2}\ \int_{S^3} x_1^2\, x_4^2\ \ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{42}(0) + F_A^{31}(0) >\\
&= -\,8\sqrt{2}\ \int_{S^3} x_2^2\, x_3^2\ \ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{24}(0) + F_A^{13}(0) >\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) >
\end{aligned}\tag{III.104}$$

and

$$\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{i}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(J\partial_r, K\partial_r)\rangle\ dvol_{S^3}\\
&= 4\sqrt{2}\ \int_{S^3} x_1^2\, x_3^2\ \ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{41}(0) + F_A^{23}(0) >\\
&+4\sqrt{2}\ \int_{S^3} x_2^2\, x_4^2\ \ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{41}(0) + F_A^{23}(0) >\\
&= -\,8\sqrt{2}\ \int_{S^3} x_1^2\, x_3^2\ \ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) >\\
&=\ -\frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) >
\end{aligned}\tag{III.105}$$

We finally get

$$\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(J\partial_r, K\partial_r)\ \omega_{\mathbf{i}}^-(x)\right\rangle\ dvol_{S^3} = -\frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ .\tag{III.106}$$

We have

$$\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(K\partial_r, I\partial_r)\ \omega_{\mathbf{j}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}
\end{aligned}\tag{III.107}$$

We recall from (B.5)

$$\left\{\begin{array}{l}
\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x) = 2\,(x_1\, x_4 + x_2\, x_3)\\[2ex]
\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x) = x_1^2 + x_3^2 - x_2^2 - x_4^2\\[2ex]
\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x) = 2\,(x_3\, x_4 - x_1\, x_2)
\end{array}\right.\tag{III.108}$$

Recall that on S^3 we have

$$F_A(0)(K\partial_r, I\partial_r) = F_A(0)(x_1\,\partial_{x_4} - x_4\,\partial_{x_1} + x_2\,\partial_{x_3} - x_3\,\partial_{x_2}, x_1\,\partial_{x_2} - x_2\,\partial_{x_1} + x_3\,\partial_{x_4} - x_4\,\partial_{x_3}) \quad \text{(III.109)}$$

Hence we have successively using corollary B.2

$$\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&= -4\sqrt{2}\ \int_{S^3} x_1^2\,x_4^2\,dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&-4\sqrt{2}\ \int_{S^3} x_2^2\,x_3^2\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&= -8\sqrt{2}\ \int_{S^3} x_1^2\,x_4^2\,dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >
\end{aligned} \quad \text{(III.110)}$$

we have

$$\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&= 2\sqrt{2}\ \int_{S^3} (x_1^2 + x_3^2 - x_2^2 - x_4^2)(x_1^2 + x_3^2)\ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{42}(0) >\\
&+2\sqrt{2}\ \int_{S^3} (x_1^2 + x_3^2 - x_2^2 - x_4^2)(x_2^2 + x_4^2)\ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{13}(0) >\\
&= -2\sqrt{2}\ \int_{S^3} (x_1^2 + x_3^2 - x_2^2 - x_4^2)(x_1^2 + x_3^2)\ dvol_{S^3}\ \ < \mathbf{j}_{g_0}, F_A^{13}(0) - F_A^{42}(0) >\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0) >
\end{aligned} \quad \text{(III.111)}$$

and

$$\begin{aligned}
&\int_{S^3} 2\sqrt{2}\,\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{j}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(K\partial_r, I\partial_r)\rangle\ dvol_{S^3}\\
&= -4\sqrt{2}\ \int_{S^3} x_3^2\,x_4^2\ \ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{41}(0) + F_A^{23}(0) >\\
&-4\sqrt{2}\ \int_{S^3} x_1^2\,x_2^2\ \ dvol_{S^3}\ \ < \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) >\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\quad < \mathbf{k}_{g_0}, F_A^{14}(0) + F_A^{32}(0) >\ \ .
\end{aligned} \quad \text{(III.112)}$$

We finally get

$$\int_{S^3} \left\langle g_0\,d\overline{x} \wedge dx\,g_0^{-1}, F_A(0)(K\partial_r, I\partial_r)\ \omega_{\mathbf{j}}^-(x)\right\rangle\ dvol_{S^3} = -\frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\,d\overline{x} \wedge dx\,g_0^{-1}, P_- F_A(0)\right\rangle\ . \quad \text{(III.113)}$$

We have

$$
\begin{aligned}
&\int_{S^3} \left\langle g_0\, d\overline{x} \wedge dx\, g_0^{-1}, F_A(0)(I\partial_r, J\partial_r)\ \omega_{\mathbf{k}}^-(x)\right\rangle\ dvol_{S^3}\\
&= \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&+ \int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}
\end{aligned}
\tag{III.114}
$$

We recall from (B.5)

$$
\left\{
\begin{array}{l}
\omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x) = 2\,(x_2\,x_4 - x_1\,x_3)\\[2mm]
\omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x) = 2\ (x_1\,x_2 + x_3\,x_4)\\[2mm]
\omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x) = x_1^2 + x_4^2 - x_2^2 - x_3^2
\end{array}
\right.
\tag{III.115}
$$

Recall that on S^3 we have

$$
F_A(0)(I\partial_r, J\partial_r) = F_A(0)(x_1\,\partial_{x_2} - x_2\,\partial_{x_1} + x_3\,\partial_{x_4} - x_4\,\partial_{x_3}), x_1\,\partial_{x_3} - x_3\,\partial_{x_1} + x_4\,\partial_{x_2} - x_2\,\partial_{x_4}). \tag{III.116}
$$

Hence we have successively

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{i}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{i}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&= -4\sqrt{2}\ \int_{S^3} x_2^2\,x_4^2\,dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&-4\sqrt{2}\ \int_{S^3} x_1^2\,x_3^2\ dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&= -8\sqrt{2}\ \int_{S^3} x_2^2\,x_4^2\,dvol_{S^3}\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\ \ < \mathbf{i}_{g_0}, F_A^{12}(0) + F_A^{43}(0) >
\end{aligned}
\tag{III.117}
$$

we have

$$
\begin{aligned}
&\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{j}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{j}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}\\
&= -4\sqrt{2}\ \int_{S^3} x_1^2\,x_2^2\ dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0)\right\rangle\\
&-4\sqrt{2}\ \int_{S^3} x_3^2\,x_4^2\ dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0)\right\rangle\\
&= -8\sqrt{2}\ \int_{S^3} dvol_{S^3}\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0)\right\rangle\\
&= -\frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{j}_{g_0}, F_A^{13}(0) + F_A^{24}(0)\right\rangle
\end{aligned}
\tag{III.118}
$$

and

$$\int_{S^3} 2\sqrt{2}\ \omega_{\mathbf{k}}^- \cdot \omega_{\mathbf{k}}^-(x)\ \langle \mathbf{k}_{g_0}, F_A(0)(I\partial_r, J\partial_r)\rangle\ dvol_{S^3}$$

$$= 2\sqrt{2}\ \int_{S^3} (x_1^2+x_4^2-x_2^2-x_3^2)(x_1^2+x_4^2)\ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, F_A^{23}(0)\right\rangle$$

$$+2\sqrt{2}\ \int_{S^3} (x_1^2+x_4^2-x_2^2-x_3^2)(x_2^2+x_3^2)\ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)\right\rangle \tag{III.119}$$

$$= 2\sqrt{2}\ \int_{S^3} (x_1^2+x_4^2-x_2^2-x_3^2)(x_1^2+x_4^2)\ dvol_{S^3}\ \left\langle \mathbf{k}_{g_0}, -F_A^{14}(0)+F_A^{23}(0)\right\rangle$$

$$= -\frac{2\sqrt{2}}{3}\ \pi^2\ \left\langle \mathbf{k}_{g_0}, F_A^{14}(0)+F_A^{32}(0)\right\rangle$$

We finally get

$$\int_{S^3} \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, F_A(0)(I\partial_r, J\partial_r)\ \omega_{\mathbf{k}}^-(x)\right\rangle\ dvol_{S^3} = -\frac{\sqrt{2}}{3}\,\pi^2\ \left\langle g_0\, d\overline{x}\wedge dx\, g_0^{-1}, P_- F_A(0)\right\rangle\ . \tag{III.120}$$

$\square$

Lemmas III.2, III.3, III.4 and III.5 imply

$$\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_- d\check{A}|^2\ dx^4 = \left(\frac{\rho^2}{8}\int_{\tau\rho}^{\rho} s\ \eta'\left(\frac{s}{\rho}\right)^2\ ds + \frac{\rho^4}{2}\int_{\tau\rho}^{\rho} \eta\left(\frac{s}{\rho}\right)^2 \frac{ds}{s}\right)\pi^2 |F_A(0)|^2$$
$$+\frac{6\pi^2}{\lambda^4\,\rho^2}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)^2\frac{ds}{s^3} - \frac{\pi^2\rho^3}{2\sqrt{2}}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)\eta\left(\frac{s}{\rho}\right)\ ds\ \left(|P_+F_A(0)|^2 - |P_-F_A(0)|^2\right)$$
$$-\frac{\pi^2}{\lambda^2}\left(\frac{1}{4}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)^2\frac{ds}{s} + \frac{\rho}{2}\int_{\tau\rho}^{\rho}\eta'\left(\frac{s}{\rho}\right)\eta\left(\frac{s}{\rho}\right)\ \frac{ds}{s^2}\right)\langle g_0 d\overline{x}\wedge dx g_0^{-1}, P_-F_A(0)\rangle + O(\rho^5)\ . \tag{III.121}$$

Then, with a change of variables and (III.7)

$$\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_- d\check{A}|^2\ dx^4 = \frac{\pi^2\rho^4}{2}\left(\frac{1}{4}\int_\tau^1 t\ \eta'(t)^2\ dt + \int_\tau^1 \eta(t)^2\frac{dt}{t}\right)|F_A(0)|^2$$
$$+12c_0^2\int_\tau^1\eta'(t)^2\frac{dt}{t^3} - \frac{1}{\sqrt{2}}\int_\tau^1\eta'(t)\,\eta(t)\ dt\ \left(|P_+F_A(0)|^2 - |P_-F_A(0)|^2\right)$$
$$-c_0\left(\frac{1}{2}\int_\tau^1\eta'(t)^2\frac{dt}{t} + \int_\tau^1\eta'(t)\,\eta(t)\ \frac{dt}{t^2}\right)\langle g_0 d\overline{x}\wedge dx g_0^{-1}, P_-F_A(0)\rangle + O(\rho^5)$$
$$= \frac{\pi^2\rho^4}{2}\Bigg[\left(\frac{1}{4}\int_\tau^1 t\ \eta'(t)^2\ dt + \int_\tau^1\eta(t)^2\frac{dt}{t} - \frac{1}{\sqrt{2}}\int_\tau^1\eta'(t)\,\eta(t)\ dt\right)|P_+F_A(0)|^2$$
$$+\left(\frac{1}{4}\int_\tau^1 t\ \eta'(t)^2\ dt + \int_\tau^1\eta(t)^2\frac{dt}{t} + \frac{1}{\sqrt{2}}\int_\tau^1\eta'(t)\,\eta(t)\ dt\right)|P_-F_A(0)|^2$$
$$-c_0\left(\frac{1}{2}\int_\tau^1\eta'(t)^2\frac{dt}{t} + \int_\tau^1\eta'(t)\,\eta(t)\ \frac{dt}{t^2}\right)\langle g_0 d\overline{x}\wedge dx g_0^{-1}, P_-F_A(0)\rangle + 12c_0^2\int_\tau^1\eta'(t)^2\frac{dt}{t^3}\Bigg] + O(\rho^5) \tag{III.122}$$

We write

$$\frac{P_-F_A(0)}{|P_-F_A(0)|} = \mathbf{a}\,\omega_{\mathbf{i}}^- + \mathbf{b}\,\omega_{\mathbf{j}}^- + \mathbf{c}\,\omega_{\mathbf{k}}^- \tag{III.123}$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ belong to $\Im m(\mathbb{H})$ and satisfy

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 = 1\ , \tag{III.124}$$

and we have

$$\left\langle g_0^{-1}\,\frac{P_-F_A(0)}{|P_-F_A(0)|}\,g_0, \mathbf{i}\,\omega_{\mathbf{i}}^- + \mathbf{j}\,\omega_{\mathbf{j}}^- + \mathbf{k}\,\omega_{\mathbf{k}}^-\right\rangle = \langle \mathbf{a}, \mathbf{i}_{g_0}\rangle + \langle \mathbf{b}, \mathbf{j}_{g_0}\rangle + \langle \mathbf{c}, \mathbf{k}_{g_0}\rangle \tag{III.125}$$

Recalling (II.16) and using Lemma B.1, we choose g_0 such that

$$\left\langle g_0^{-1}\,\frac{P_-F_A(0)}{|P_-F_A(0)|}\,g_0, \mathbf{i}\,\omega_{\mathbf{i}}^- + \mathbf{j}\,\omega_{\mathbf{j}}^- + \mathbf{k}\,\omega_{\mathbf{k}}^-\right\rangle \geq \sqrt{\frac{2}{3}}\ , \tag{III.126}$$

so that

$$\left\langle g_0 d\overline{x} \wedge dx g_0^{-1}, P_-F_A(0)\right\rangle \geq \frac{4}{\sqrt{3}}|P_-F_A(0)|\ . \tag{III.127}$$

Moreover, since

$$\frac{1}{4}t\eta'(t)^2 + \frac{\eta(t)^2}{t} - \frac{1}{\sqrt{2}}\eta'(t)\eta(t) \geq \left(\frac{\sqrt{t}}{2}\eta'(t) - \frac{\eta(t)}{\sqrt{t}}\right)^2 \geq 0$$

and $|P_+F_A(0)| \leq |P_-F_A(0)|$, we can further estimate from (III.122)

$$\begin{aligned}
&\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_-d\check{A}|^2\ dx^4 \leq \frac{\pi^2\rho^4}{2}\Bigg[\left(\frac{1}{2}\int_\tau^1 t\ \eta'(t)^2\ dt + 2\int_\tau^1 \eta(t)^2\frac{dt}{t}\right)|P_-F_A(0)|^2\\
&-\frac{2\,c_0}{\sqrt{3}}\left(\int_\tau^1 \eta'(t)^2\frac{dt}{t} + 2\int_\tau^1 \eta'(t)\,\eta(t)\ \frac{dt}{t^2}\right)|P_-F_A(0)| + 12c_0^2\int_\tau^1 \eta'(t)^2\frac{dt}{t^3}\Bigg] + O(\rho^5)
\end{aligned} \tag{III.128}$$

Finally:

$$\begin{aligned}
&\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_-d\check{A}|^2\ dx^4 - \int_{B_\rho(0)} |P_-F_A|^2 dx^4\\
&\leq \frac{\pi^2\rho^4}{2}\Bigg[\left(\frac{1}{2}\int_\tau^1 t\ \eta'(t)^2\ dt + 2\int_\tau^1 \eta(t)^2\frac{dt}{t} - 1\right)|P_-F_A(0)|^2\\
&-\frac{2c_0}{\sqrt{3}}\left(\int_\tau^1 \eta'(t)^2\frac{dt}{t} + 2\int_\tau^1 \eta'(t)\,\eta(t)\ \frac{dt}{t^2}\right)|P_-F_A(0)| + 12c_0^2\int_\tau^1 \eta'(t)^2\frac{dt}{t^3}\Bigg] + O(\rho^5)
\end{aligned} \tag{III.129}$$

Considering now that the minimum of a degree 2 polynomial ax^2+bx+c is $-\frac{b^2}{4a}+c$ and is attained at $x_0 = -\frac{b^2}{2a}$, we have that the minimum of the polynomial in c_0 inside the square brackets above is is

$$\begin{aligned}
m(\tau,\eta) = &\left[-\frac{\left(\int_\tau^1 \eta'(t)^2\frac{dt}{t} + 2\int_\tau^1 \eta'(t)\,\eta(t)\ \frac{dt}{t^2}\right)^2}{36\int_\tau^1 \eta'(t)^2\frac{dt}{t^3}} + \left(\frac{1}{2}\int_\tau^1 t\ \eta'(t)^2\ dt + 2\int_\tau^1 \eta(t)^2\frac{dt}{t} - 1\right)\right]|P_-F_A(0)|^2\\
&=: \varphi(\tau,\eta)|P_-F_A(0)|^2.
\end{aligned} \tag{III.130}$$

Choosing

$$\eta_0(t) := \frac{t-\tau}{1-\tau},$$

leads to

$$\begin{aligned} m(\tau,\eta_0) &= \left[-\frac{2\tau^2\left(\frac{3}{2(1-\tau)}\log\frac{1}{\tau}-1\right)^2}{9(1-\tau^2)} + 2\left(\frac{5-11\tau}{8(1-\tau)} + \frac{\tau^2}{(1-\tau)^2}\log\frac{1}{\tau} - \frac{1}{2}\right)\right] |P_- F_A(0)|^2 \\ &=: \varphi(\tau)|P_- F_A(0)|^2 \end{aligned} \tag{III.131}$$

and is attained for a constant $c_0 = c_0(\tau) > 0$. Now, considering that $\varphi(\tau,\eta_0) < 0$ for $\tau \in [0.3, 0.4]$, we can modify η_0 to a function $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 0$ for $t \le \tau$ and $\eta(t) = 1$ for $t \ge 1$, choose $\tau \in [0.3, 0.4]$, such that $\varphi(\eta,\tau) < 0$ and conclude

$$\int_{B_\rho(0)\setminus B_{\tau\rho}(0)} |P_- d\check{A}|^2 \, dx^4 - \int_{B_\rho(0)} |P_- F_A|^2 dx^4 \le \frac{\pi^2\rho^4}{2}\varphi(\tau,\eta)|P_- F_A(0)|^2 + O(\rho^5). \tag{III.132}$$

Appendix

A The self-instanton

Define

$$SD(x) = \Im m\left(\frac{x\, d\overline{x}}{1+|x|^2}\right) \tag{A.133}$$

where the $x \in \mathbb{R}^4$ is identified canonically with the quaternion $x := x_1 + x_2\,\mathbf{i} + x_3\,\mathbf{j} + x_4\,\mathbf{k}$ and $\overline{x} := x_1 - x_2\,\mathbf{i} - x_3\,\mathbf{j} - x_4\,\mathbf{k}$. As a consequence we have the more explicit formula

$$\begin{cases} SD^1(x) = \dfrac{x_2\,\mathbf{i} + x_3\,\mathbf{j} + x_4\,\mathbf{k}}{1+|x|^2} \\[2ex] SD^2(x) = \dfrac{-\,x_1\,\mathbf{i} - x_4\,\mathbf{j} + x_3\,\mathbf{k}}{1+|x|^2} \\[2ex] SD^3(x) = \dfrac{x_4\,\mathbf{i} - x_1\,\mathbf{j} - x_2\,\mathbf{k}}{1+|x|^2} \\[2ex] SD^4(x) = \dfrac{-\,x_3\,\mathbf{i} + x_2\,\mathbf{j} - x_1\,\mathbf{k}}{1+|x|^2} \end{cases} \tag{A.134}$$

The curvature of this connection form is given by

$$F_{SD}(X,Y) := d\,\Im m\left(\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) + \left[\Im m\left(\frac{x\,\overline{X}}{1+|x|^2}\right), \Im m\left(\frac{x\,\overline{Y}}{1+|x|^2}\right)\right] \tag{A.135}$$

Recall that for any pairs of quaternions (p,q) one has respectively

$$\begin{aligned} [p,q] &= p\,q - q\,p \\ &= (\Re(p) + \Im m(p))\,(\Re(q) + \Im m(q)) - (\Re(q) + \Im m(q))\,(\Re(p) + \Im m(p)) \\ &= [\Im m(p), \Im m(q)]\ . \end{aligned} \tag{A.136}$$

Hence in particular

$$F_{SD}(X,Y) = d\,\Im m\left(\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) + \left[\frac{x\,\overline{X}}{1+|x|^2}, \frac{x\,\overline{Y}}{1+|x|^2}\right] \tag{A.137}$$

We have first

$$d\,\Im m\left(\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) = \Im m\left(\frac{dx\wedge d\overline{x}}{1+|x|^2}\right)(X,Y) - \left(\frac{d|x|^2}{1+|x|^2}\wedge A\right)(X,Y)\ . \tag{A.138}$$

Observe that

$$\begin{aligned}
\overline{dx\wedge d\overline{x}} &= 2^{-1}\sum_{l,m=1}^{4} \overline{\partial_{x_l}x\,\partial_{x_m}\overline{x} - \partial_{x_m}x\,\partial_{x_l}\overline{x}}\ dx_l\wedge dx_m\\
&= 2^{-1}\sum_{l,m=1}^{4} \partial_{x_m}x\,\partial_{x_l}\overline{x} - \partial_{x_l}x\,\partial_{x_m}\overline{x}\ dx_l\wedge dx_m\\
&= 2^{-1}\sum_{l,m=1}^{4} \partial_{x_l}x\,\partial_{x_m}\overline{x} - \partial_{x_m}x\,\partial_{x_l}\overline{x}\ dx_m\wedge dx_l\\
&= -\,dx\wedge d\overline{x}\ .
\end{aligned} \tag{A.139}$$

Hence

$$d\,\Im m\left(\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) = \frac{dx\wedge d\overline{x}}{1+|x|^2}(X,Y) - \left(\frac{d|x|^2}{1+|x|^2}\wedge SD\right)(X,Y)\ . \tag{A.140}$$

Observe that

$$\Re e\left(\frac{x\,d\overline{x}}{1+|x|^2}\right) = \frac{1}{2}\,\frac{x\,d\overline{x}+dx\,\overline{x}}{1+|x|^2} = \frac{1}{2}\,\frac{d|x|^2}{1+|x|^2}\ , \tag{A.141}$$

where we have used the identity $x\overline{x} = |x|^2$. Combining (A.140) and (A.141) gives then

$$\begin{aligned}
d\,\Im m\left(\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) &= \frac{dx\wedge d\overline{x}}{1+|x|^2}(X,Y) - \left(\frac{d|x|^2}{1+|x|^2}\wedge\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y)\\
= \frac{dx\wedge d\overline{x}}{1+|x|^2}(X,Y) &- \left(\frac{x\,d\overline{x}}{1+|x|^2}\wedge\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y) - \left(\frac{dx\,\overline{x}}{1+|x|^2}\wedge\frac{x\,d\overline{x}}{1+|x|^2}\right)(X,Y)\\
= \frac{dx\wedge d\overline{x}}{1+|x|^2}(X,Y) &- \frac{x\,\overline{X}\,x\,\overline{Y} - x\,\overline{Y}\,x\,\overline{X}}{(1+|x|^2)^2} - |x|^2\,\frac{dx\wedge d\overline{x}}{(1+|x|^2)^2}(X,Y)\ .
\end{aligned} \tag{A.142}$$

Combining (A.137) and (A.142) is implying

$$F_{SD}(X,Y) = \frac{dx\wedge d\overline{x}}{(1+|x|^2)^2}(X,Y) \tag{A.143}$$

B Some lemma

Lemma B.1. *For any triple of 3 vectors in $\mathbb{R}^3$, $(\vec{a},\vec{b},\vec{c})$ there exists a positive orthonormal basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ such that*

$$\vec{a}\cdot\vec{e}_1 + \vec{b}\cdot\vec{e}_2 + \vec{c}\cdot\vec{e}_3 \geq \frac{1}{\sqrt{3}}\,\sqrt{|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2} \tag{B.1}$$

□

Proof of lemma B.1. By linearity we can assume $\sqrt{|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2}=1$. There exists a vector of length at least $1/\sqrt{3}$. Assume this is $\vec{a}$. We choose $\vec{e}_1 := \vec{a}/|\vec{a}|$. We choose $\vec{e}_2$ and $\vec{e}_3$ arbitrary such that $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ is forming a positive orthonormal basis.. If

$$\vec{e}_2\cdot\vec{b}+\vec{e}_3\cdot\vec{c}<0$$

we change $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ into $(\vec{e}_1,-\vec{e}_2,-\vec{e}_3)$ and we get

$$\vec{a}\cdot\vec{e}_1+\vec{b}\cdot\vec{e}_2+\vec{c}\cdot\vec{e}_3\geq\frac{1}{\sqrt{3}}\ .$$

The result is optimal by taking $(\vec{a},\vec{a},\vec{a})$ where $|\vec{a}|^2=3^{-1}$. Hence the lemma is proved. □

Lemma B.2. *We have*

$$\begin{aligned}\int_{S^3}x_i^2\,d\sigma&=\frac{\pi^2}{2}\quad\textit{for } 1\leq i\leq 4\,,\\ \int_{S^3}x_i^4\,d\sigma&=\frac{\pi^2}{4}\quad\textit{for } 1\leq i\leq 4\,,\\ \int_{S^3}x_i^2x_j^2\,d\sigma&=\frac{\pi^2}{12}\quad\textit{for } 1\leq i<j\leq 4\,.\end{aligned}$$

Proof of Lemma B.2. We recall the following well-known formula, see e.g. [3]: Given a monomial $p(x_1,x_2,x_3,x_4)=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}$ with $\alpha_1,\ldots,\alpha_4\in 2\mathbb{N}$, we have

$$\int_{S^3}p\,d\sigma=\frac{2\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\beta_4)}{\Gamma(\beta_1+\beta_2+\beta_3+\beta_4)},\tag{B.2}$$

where $\beta_l=\frac{1}{2}(\alpha_l+1)$ for $l=1,\ldots,4$, and Γ denotes the usual Γ function.

The lemma follows (B.2), using that $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2})=\frac{3}{4}\sqrt{\pi}$, and $\Gamma(k)=(k-1)!$ for k positive integer. □

Lemma B.3. *The set of* 2*-forms*

$$\begin{cases}\omega_{\mathbf{i}}^-(x)=\sqrt{2}\,P_-(dr\wedge I dr)\\[2ex] \omega_{\mathbf{j}}^-(x)=\sqrt{2}\,P_-(dr\wedge J dr)\\[2ex] \omega_{\mathbf{k}}^-(x)=\sqrt{2}\,P_-(dr\wedge K dr)\end{cases}\tag{B.3}$$

realizes an orthonormal basis of $(\Lambda^2\mathbb{R}^4)^-$.

The compatibility with the previous notations is given by

$$\begin{cases}\omega_{\mathbf{i}}^-=\omega_{\mathbf{i}}^-(1,0,0,0)\\[2ex] \omega_{\mathbf{j}}^-=\omega_{\mathbf{j}}^-(1,0,0,0)\\[2ex] \omega_{\mathbf{k}}^-=\omega_{\mathbf{k}}^-(1,0,0,0)\end{cases}\tag{B.4}$$

Proof of Lemma B.3. We compute

$$\begin{aligned}
r^2\, dr \wedge I dr &= (x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3 + x_4\, dx_4) \wedge (x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_4 - x_4\, dx_3) \\
&= (x_1^2 + x_2^2)\, dx_1 \wedge dx_2 + (x_3^2 + x_4^2)\, dx_3 \wedge dx_4 + x_1\, x_3\, (dx_1 \wedge dx_4 + dx_3 \wedge dx_2) \\
&+ x_1\, x_4\, (dx_3 \wedge dx_1 + dx_4 \wedge dx_2) + x_2\, x_3\, (dx_2 \wedge dx_4 + dx_1 \wedge dx_3) \\
&+ x_2\, x_4\, (dx_3 \wedge dx_2 + dx_1 \wedge dx_4) = (x_1^2 + x_2^2)\, dx_1 \wedge dx_2 + (x_3^2 + x_4^2)\, dx_3 \wedge dx_4 \\
&+ (x_1\, x_3 + x_2\, x_4)\, \sqrt{2}\, \omega_{\mathbf{k}}^- + (x_2\, x_3 - x_1\, x_4)\, \sqrt{2}\, \omega_{\mathbf{j}}^- \\
&= (x_1^2 + x_2^2)\, \sqrt{2}^{-1} (\omega_{\mathbf{i}}^+ + \omega_{\mathbf{i}}^-) + (x_3^2 + x_4^2)\, \sqrt{2}^{-1} (\omega_{\mathbf{i}}^+ - \omega_{\mathbf{i}}^-) \\
&+ (x_1\, x_3 + x_2\, x_4)\, \sqrt{2}\, \omega_{\mathbf{k}}^- + (x_2\, x_3 - x_1\, x_4)\, \sqrt{2}\, \omega_{\mathbf{j}}^- \\
&= \frac{r^2}{\sqrt{2}}\, \omega_{\mathbf{i}}^+ + \frac{(x_1^2 + x_2^2 - x_3^2 - x_4^2)}{\sqrt{2}}\, \omega_{\mathbf{i}}^- + (x_1\, x_3 + x_2\, x_4)\, \sqrt{2}\, \omega_{\mathbf{k}}^- + (x_2\, x_3 - x_1\, x_4)\, \sqrt{2}\, \omega_{\mathbf{j}}^- \ .
\end{aligned} \tag{B.5}$$

Hence

$$P_+(r^2\, dr \wedge I dr) = 2^{-1}\, |x|^2\, (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = \frac{r^2}{\sqrt{2}}\, \omega_{\mathbf{i}}^+ \ . \tag{B.6}$$

This implies

$$r^2\, dr \wedge I dr = \sqrt{2}^{-1}\, r^2\, \omega_{\mathbf{i}}^+ + P_-(r^2\, dr \wedge I dr) \ . \tag{B.7}$$

Similarly

$$\begin{aligned}
r^2\, dr \wedge J dr &= (x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3 + x_4\, dx_4) \wedge (x_1\, dx_3 - x_3\, dx_1 + x_4\, dx_2 - x_2\, dx_4) \\
&= (x_1^2 + x_3^2)\, dx_1 \wedge dx_3 + (x_2^2 + x_4^2)\, dx_4 \wedge dx_2 + x_1\, x_4\, (dx_1 \wedge dx_2 + dx_4 \wedge dx_3) \\
&+ x_1\, x_2\, (dx_4 \wedge dx_1 + dx_2 \wedge dx_3) + x_2\, x_3\, (dx_1 \wedge dx_2 + dx_4 \wedge dx_3) \\
&+ x_3\, x_4\, (dx_3 \wedge dx_2 + dx_1 \wedge dx_4) = (x_1^2 + x_3^2)\, dx_1 \wedge dx_3 + (x_2^2 + x_4^2)\, dx_4 \wedge dx_2 \\
&+ (x_1\, x_4 + x_2\, x_3)\, \sqrt{2}\, \omega_{\mathbf{i}}^- + (x_3\, x_4 - x_1\, x_2)\, \sqrt{2}\, \omega_{\mathbf{k}}^- \\
&= \frac{r^2}{\sqrt{2}}\, \omega_{\mathbf{j}}^+ + \frac{(x_1^2 + x_3^2 - x_4^2 - x_2^2)}{\sqrt{2}}\, \omega_{\mathbf{j}}^- + (x_1\, x_4 + x_2\, x_3)\, \sqrt{2}\, \omega_{\mathbf{i}}^- + (x_3\, x_4 - x_1\, x_2)\, \sqrt{2}\, \omega_{\mathbf{k}}^- \ .
\end{aligned} \tag{B.8}$$

Hence

$$P_+(r^2\, dr \wedge J dr) = 2^{-1}\, |x|^2\, (dx_1 \wedge dx_3 + dx_4 \wedge dx_2) = \frac{r^2}{\sqrt{2}}\, \omega_{\mathbf{j}}^+ \ . \tag{B.9}$$

This implies

$$r^2\, dr \wedge J dr = \sqrt{2}^{-1}\, r^2\, \omega_{\mathbf{j}}^+ + P_-(r^2\, dr \wedge J dr) \ . \tag{B.10}$$

We have also

$$r^2\,dr\wedge Kdr = (x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3 + x_4\,dx_4)\wedge(x_1\,dx_4 - x_4\,dx_1 + x_2\,dx_3 - x_3\,dx_2)$$

$$= (x_1^2+x_4^2)\,dx_1\wedge dx_4 + (x_2^2+x_3^2)\,dx_2\wedge dx_3 + x_1\,x_3(dx_2\wedge dx_1 + dx_3\wedge dx_4)$$

$$+x_1\,x_2\,(dx_1\wedge dx_3 + dx_2\wedge dx_4) + x_2\,x_4\,(dx_1\wedge dx_2 + dx_4\wedge dx_3)$$

$$+x_3\,x_4\,(dx_1\wedge dx_3 + dx_2\wedge dx_4) = (x_1^2+x_4^2)\,dx_1\wedge dx_4 + (x_2^2+x_3^2)\,dx_2\wedge dx_3$$

$$+(x_2\,x_4 - x_1\,x_3)\sqrt{2}\,\omega^-_{\mathbf{i}} + (x_1\,x_2 + x_3\,x_4)\,\sqrt{2}\,\omega^-_{\mathbf{j}}$$

$$= \frac{r^2}{\sqrt{2}}\,\omega^+_{\mathbf{k}} + \frac{(x_1^2+x_4^2-x_2^2-x_3^2)}{\sqrt{2}}\,\omega^-_{\mathbf{k}} + (x_2\,x_4 - x_1\,x_3)\,\sqrt{2}\,\omega^-_{\mathbf{i}} + (x_1\,x_2 + x_3\,x_4)\,\sqrt{2}\,\omega^-_{\mathbf{j}}\ . \tag{B.11}$$

Hence

$$P_+(r^2\,dr\wedge Kdr) = 2^{-1}\,|x|^2(dx_1\wedge dx_4 + dx_2\wedge dx_3) = \sqrt{2}^{-1}\,r^2\,\omega^+_{\mathbf{k}}\ . \tag{B.12}$$

This implies

$$r^2\,dr\wedge Kdr = \sqrt{2}^{-1}\,r^2\,\omega^+_{\mathbf{k}} + P_-(r^2\,dr\wedge Kdr)\ . \tag{B.13}$$

From (B.7), (B.10) and (B.13) we deduce that

$$\begin{aligned} 1 &= |dr\wedge Idr|^2 = 2^{-1} + |P_-(dr\wedge Idr)|^2\ ,\\ 1 &= |dr\wedge Jdr|^2 = 2^{-1} + |P_-(dr\wedge Jdr)|^2\ ,\\ 1 &= |dr\wedge Kdr|^2 = 2^{-1} + |P_-(dr\wedge Kdr)|^2\ . \end{aligned} \tag{B.14}$$

Moreover

$$\begin{aligned} 0 &= \langle dr\wedge Idr, dr\wedge Jdr\rangle = \langle P_-(dr\wedge Idr), P_-(dr\wedge Jdr)\rangle\ ,\\ 0 &= \langle dr\wedge Idr, dr\wedge Kdr\rangle = \langle P_-(dr\wedge Idr), P_-(dr\wedge Kdr)\rangle\ ,\\ 0 &= \langle dr\wedge Jdr, dr\wedge Kd\rangle >= \langle P_-(dr\wedge Jdr), P_-(dr\wedge Kdr)\rangle\ . \end{aligned} \tag{B.15}$$

This implies that $\{\omega^-_{\mathbf{i}}(x),\,\omega^-_{\mathbf{j}}(x),\,\omega^-_{\mathbf{k}}(x)\}$ is a orthonomal basis. □

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