# GLOBAL EXISTENCE OF A STRONG SOLUTION TO THE INITIAL VALUE PROBLEM FOR THE NERNST-PLANCK-NAVIER-STOKES SYSTEM IN $\mathbb{R}^N$

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ABSTRACT. We study the existence of a strong solution to the initial value problem for the Nernst-Planck-Navier-Stokes (NPNS) system in $\mathbb{R}^N, N \geq 3$. We obtain a global in-time strong solution without any smallness assumptions on the initial data.

## 1. INTRODUCTION

In this paper we investigate the existence of a strong solution to the initial value problem

$$\partial_t u + u\cdot\nabla u + \nabla p = \Delta u - \Psi\nabla\phi \ \text{ in } \mathbb{R}^N \times (0,T) \equiv Q_T, \tag{1.1}$$

$$\partial_t c_i + \nabla\cdot(c_i u) = \Delta c_i + \nabla\cdot(z_i c_i \nabla\phi) \ \text{ in } Q_T,\ i = 1,\cdots,I, \tag{1.2}$$

$$-\Delta\phi = \Psi \ \text{ in } Q_T, \tag{1.3}$$

$$\Psi = \sum_{i=1}^{I} z_i c_i, \tag{1.4}$$

$$\nabla\cdot u = 0 \ \text{ in } Q_T, \tag{1.5}$$

$$u(x,0) = u^{(0)}(x),\ \ c_i(x,0) = c_i^{(0)}(x) \ \text{ on } \mathbb{R}^N. \tag{1.6}$$

This problem can be used to describe the transport and diffusion of ions in electrolyte solutions. In this case, $c_i$ denote ionic concentrations, $z_i$ are the corresponding valences, $u \in \mathbb{R}^N$ is fluid velocity, $p$ is the pressure, $\phi$ is the electric potential. The system (1.1)-(1.5) is often called the Nernst-Planck-Navier-Stokes (NPNS) system, and it plays an important role in many physical and biological processes [1, 3], such as ion particles in the electrokinetic fluids [8, 12], and ion channels in cell membranes [2, 9]. An introduction to some of the basic physical, biological and mathematical issues can be found in [15].

Mathematical analysis of the NPNS system has attracted a lot of attentions recently. Most of the existing research deals with the case where the system is posed on a bounded domain with various types of boundary conditions. See [5, 6, 16] and the references therein. However, problems with unbounded domains present different mathematical challenges from those with bounded ones. The initial value problem such as ours was first considered in [11], where local existence of a smooth solution was established via analytic semi-group theory. The so-called energy dissipation equalities associated with the system were obtained in [13], from which a global-in-time weak solution was constructed. The objective of this paper is to generalize these results, More precisely, we have

**Theorem 1.1.** *Assume that*

$$|u^{(0)}| \in L^\infty(\mathbb{R}^N)\cap L^2(\mathbb{R}^N) \ \text{ and } \ \nabla\cdot u^{(0)} = 0, \tag{1.7}$$

$$c_i^{(0)} \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N) \ \text{ with } \ c_i^{(0)} \geq 0,\ i = 1,\cdots,I. \tag{1.8}$$

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*Then there exists a global strong solution to* (1.1)-(1.6) *with* $c_i \geq 0$ *for each* $i \in \{1, \cdots, I\}$*. Moreover, we have*

$$\|w\|_{\infty,Q_T} \equiv \left\|\sum_{i=1}^{I} c_i\right\|_{\infty,Q_T} \leq c, \tag{1.9}$$

$$\sup_{0\leq t\leq T} \|\phi(\cdot,t)\|_{W^{2,s}(\mathbb{R}^N)} \leq c \ \text{ for each } s > 1, \text{ and} \tag{1.10}$$

$$\|u\|_{\infty,Q_T} \leq c. \tag{1.11}$$

*Here and in what follows the constant* $c$ *depends only on* $N$, $T$, *and the initial data via the norms of the function spaces in* (1.7) *and* (1.8)*. The dependence of* $c$ *on* $T$ *is such that* $c$ *becomes unbounded only when* $T \to \infty$,

Obviously, under (1.11)-(1.10) higher regularity of the solution can be obtained via a bootstrap argument. In fact, a strong solution is understood to be a weak solution, as defined in [13], with the additional properties (1.11)-(1.10). A strong solution can be shown to satisfy system (1.1)-(1.3) in the a.e. sense [14]. We will not pursue the details here.

Note that if $I = 2$ then (1.9) was already obtained in [13]. However, as noted in the article, the method employed there cannot be extended to the case where $I > 2$. Also see [19]. By the classical Calderón-Zygmund estimate, (1.10) is an easy consequence of (1.3), (1.9), and (2.9) below.

Our approach is based upon an idea developed by the author in [18]. It combines scaling of the dependent variables with a De Giorgi iteration scheme. It seems to be very effective in dealing with the type of nonlinearity appearing in Navier-Stokes equations. Our development here further validates the approach in [18].

This work is organized as follows. In Section 2, we collect some relevant known results, while Section 3 is devoted to the proof of Theorem 1.1.

## 2. Preliminary results

In this section, we first collect some preliminary analysis on (1.1)-(1.6). Then we state a couple of relevant known results.

As shown in [13], a solution to (1.1)-(1.6) can be constructed as a limit of a sequence of smooth solutions to certain approximate problems. In view of the recent result in [18], we can design another approximation scheme by cutting off $\Psi$. That is, we approximate $\Psi$ by a sequence of bounded functions. This will lead to the existence of very "nice" approximate solutions. In the subsequent calculations we may assume that our solutions are these smooth approximating ones.

The following lemma is a consequence of the energy dissipation equalities in [13].

**Lemma 2.1.** *We have*

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(\int_{\mathbb{R}^N}|u|^2dx + \int_{\mathbb{R}^N}|\nabla\phi|^2dx\right) + \int_{Q_T}|\nabla u|^2dxdt + \int_{Q_T}\Psi^2dxdt \\ &+ \int_{Q_T}\sum_{i=1}^{I} z_i^2c_i|\nabla\phi|^2dxdt \leq c(N)\left(\|u^{(0)}\|_{2,\mathbb{R}^N}^2 + \left\|\sum_{i=1}^{I} z_i c_i^{(0)}\right\|_{\frac{2N}{N+2},\mathbb{R}^N}^2\right) \leq c. \end{aligned} \tag{2.1}$$

*Proof.* We easily verify that

$$u \cdot \nabla u \cdot u = \frac{1}{2} u \cdot \nabla |u|^2.$$

This together with (1.5) implies

$$\int_{\mathbb{R}^N} u \cdot \nabla u \cdot u \ dx = 0.$$

Similarly,

$$\int_{\mathbb{R}^N} u \cdot \nabla p \, dx = 0.$$

With these in mind, we use $u$ as a test function in (1.1) to deduce

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx = -\int_{\mathbb{R}^N} \Psi \nabla\phi \cdot u dx. \tag{2.2}$$

To estimate the term on the right-hand side, we differentiate (1.3) with respect to $t$ and use $\phi$ as a test function in the resulting equation to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla\phi|^2 dx = \int_{\mathbb{R}^N} \partial_t \Psi \phi dx. \tag{2.3}$$

Next, we use $z_i\phi$ as a test function in (1.2) to get

$$\begin{aligned} z_i\int_{\mathbb{R}^N} \phi \partial_t c_i dx &= z_i\int_{\mathbb{R}^N} c_i u \cdot \nabla\phi dx - z_i\int_{\mathbb{R}^N} \nabla c_i \cdot \nabla\phi dx - z_i^2\int_{\mathbb{R}^N} c_i|\nabla\phi|^2 dx \\ &= z_i\int_{\mathbb{R}^N} c_i u \cdot \nabla\phi dx - z_i\int_{\mathbb{R}^N} \Psi c_i dx - z_i^2\int_{\mathbb{R}^N} c_i|\nabla\phi|^2 dx. \end{aligned}$$

The last step is due to (1.3). Sum up the equations over $i$ to derive

$$\int_{\mathbb{R}^N} \partial_t \Psi \phi dx = \int_{\mathbb{R}^N} \Psi u \cdot \nabla\phi dx - \int_{\mathbb{R}^N} \Psi^2 dx - \int_{\mathbb{R}^N} \sum_{i=1}^I z_i^2 c_i |\nabla\phi|^2 dx.$$

Substitute this into (2.3) and add the resulting equation to (2.2) to deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla\phi|^2 dx \\ &+\int_{\mathbb{R}^N} \Psi^2 dx + \int_{\mathbb{R}^N} \sum_{i=1}^I z_i^2 c_i |\nabla\phi|^2 dx = 0. \end{aligned}$$

After an integration, we arrive at

$$\begin{aligned} &\sup_{0\le t\le T}\left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla\phi|^2 dx\right) + \int_{Q_T} |\nabla u|^2 dxdt + \int_{Q_T} \Psi^2 dxdt \\ &+ \int_{Q_T} \sum_{i=1}^I z_i^2 c_i |\nabla\phi|^2 dxdt \le 2\left(\int_{\mathbb{R}^N} |u^{(0)}(x)|^2 dx + \int_{\mathbb{R}^N} |\nabla\phi(x,0)|^2 dx\right). \end{aligned} \tag{2.4}$$

To bound the last term in the above inequality, we let $t = 0$ in (1.3) to get

$$-\Delta\phi(x,0) = \Psi(x,0) = \sum_{i=1}^I z_i c_i^{(0)}(x).$$

Use $\phi(x,0)$ as a test function to deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla\phi(x,0)|^2 dx &= \int_{\mathbb{R}^N} \sum_{i=1}^I z_i c_i^{(0)}(x)\phi(x,0) dx \\ &\le \left(\int_{\mathbb{R}^N} \left|\sum_{i=1}^I z_i c_i^{(0)}(x)\right|^{\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}} \left(\int_{\mathbb{R}^N} |\phi(x,0)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}}. \end{aligned} \tag{2.5}$$

Recall that the Sobolev inequality in the whole space asserts

$$\|f\|_{\frac{2N}{N-2},\mathbb{R}^N} \le c(N)\|\nabla f\|_{2,\mathbb{R}^N} \quad \text{for each } f \in H^1(\mathbb{R}^N). \tag{2.6}$$

This together with (2.5) implies

$$\|\nabla\phi(\cdot,0)\|_{2,\mathbb{R}^N} \le c\left\|\sum_{i=1}^{I} z_i c_i^{(0)}\right\|_{\frac{2N}{N+2},\mathbb{R}^N}.$$

Substitute this into (2.4) to complete the proof. □

Our next lemma enables us to avoid using the Carleman-type inequality as was done in [13], which would incur additional assumptions on the initial data.

**Lemma 2.2.** *There holds*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N} w dx \le c, \quad \int_{Q_T} w^{\frac{N+2}{N}} dxdt \le c, \tag{2.7}$$

*where $w$ is given as in* (1.9).

*Proof.* Integrate (1.2) over $\mathbb{R}^N$ to get

$$\frac{d}{dt}\int_{\mathbb{R}^N} c_i\ dx = 0 \ \text{ for } i =, \cdots, I, \tag{2.8}$$

from whence follows

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N} c_i dx \le \int_{\mathbb{R}^N} c_i^{(0)} dx. \tag{2.9}$$

This implies the first inequality in (2.7).

Before we continue, we must point out that $\ln c_i$ is not a legitimate test function for (1.2) because it is no longer a Sobolev function of the space variables. This point seems to have been overlooked in [13]. However, for each $\varepsilon >$ the function $\ln(c_i+\varepsilon)-\ln\varepsilon$ is. Upon using it, we derive

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N}\int_0^{c_i}[\ln(\mu+\varepsilon)-\ln\varepsilon]d\mu dx + \int_{\mathbb{R}^N}\frac{1}{c_i+\varepsilon}|\nabla c_i|^2 dx \\ &= \int_{\mathbb{R}^N}\frac{c_i}{c_i+\varepsilon}u\cdot\nabla c_i dx - z_i\int_{\mathbb{R}^N}\frac{c_i}{c_i+\varepsilon}\nabla\phi\cdot\nabla c_i dx. \end{aligned} \tag{2.10}$$

We can infer from (1.5) that

$$\int_{\mathbb{R}^N}\frac{c_i}{c_i+\varepsilon}u\cdot\nabla c_i dx = \int_{\mathbb{R}^N} u\cdot\nabla\int_0^{c_i}\frac{s}{s+\varepsilon}ds dx = 0.$$

The last term in (2.10) can be estimated as follows:

$$-z_i\int_{\mathbb{R}^N}\frac{c_i}{c_i+\varepsilon}\nabla\phi\cdot\nabla c_i dx \ \le\ \frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{c_i+\varepsilon}|\nabla c_i|^2 dx + \frac{z_i^2}{2}\int_{\mathbb{R}^N} c_i|\nabla\phi|^2 dx.$$

Use the preceding two results in (2.10) and integrate the resulting inequality with respect to $t$ to derive

$$\int_{Q_T}\frac{1}{c_i+\varepsilon}|\nabla c_i|^2 dxdt \le c\int_{\mathbb{R}^N}\int_0^{c_i^{(0)}}[\ln(\mu+\varepsilon)-\ln\varepsilon]d\mu dx + cz_i^2\int_{Q_T} c_i|\nabla\phi|^2 dxdt. \tag{2.11}$$

Here we have used the fact that

$$\int_0^{c_i}[\ln(\mu+\varepsilon)-\ln\varepsilon]d\mu \ge 0.$$

Note that

$$\ln(\mu+\varepsilon)-\ln\varepsilon = \ln\left(1+\frac{\mu}{\varepsilon}\right) \le \frac{\mu}{\varepsilon} \ \text{ for } \mu\ge 0\ .$$

This together with (2.1) and (1.8) implies that the right hand side of (2.11) is less that or equal to $c$ for each fixed $\varepsilon$. Unfortunately, the right-hand side blows up as $\varepsilon \to 0$. This will cause some complications. To address them, we easily see that

$$\sqrt{c_i+\varepsilon}-\sqrt{\varepsilon}=\frac{c_i}{\sqrt{c_i+\varepsilon}+\sqrt{\varepsilon}}\leq\sqrt{c_i}$$

With this, (2.6), (2.11), and (2.8) in mind, we calculate that

$$\begin{aligned}
&\int_{Q_T}\left(\sqrt{c_i+\varepsilon}-\sqrt{\varepsilon}\right)^{\frac{4}{N}+2}dxdt\\
&\leq \int_0^T\left(\int_{\mathbb{R}^N}\left(\sqrt{c_i+\varepsilon}-\sqrt{\varepsilon}\right)^2dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left(\sqrt{c_i+\varepsilon}-\sqrt{\varepsilon}\right)^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\
&\leq \left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}c_idx\right)^{\frac{2}{N}}\int_0^T\left(\int_{\mathbb{R}^N}\left(\sqrt{c_i+\varepsilon}-\sqrt{\varepsilon}\right)^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\
&\leq c\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}c_idx\right)^{\frac{2}{N}}\int_0^T\int_{\mathbb{R}^N}\frac{1}{c_i+\varepsilon}|\nabla c_i|^2dxdt\leq c. \qquad (2.12)
\end{aligned}$$

It is elementary to show that

$$c_i=\left(\sqrt{c_i+1}-1\right)^2+2\left(\sqrt{c_i+1}-1\right)\leq 2\left(\sqrt{c_i+1}-1\right)^2+1,$$

from whence follows that

$$\begin{aligned}
\int_{Q_T}c_i^{\frac{N+2}{N}}dxdt &= \int_{\{c_i\leq 1\}}c_i^{\frac{N+2}{N}}dxdt+\int_{\{c_i>1\}}c_i^{\frac{N+2}{N}}dxdt\\
&\leq \int_{Q_T}c_idxdt+c\int_{Q_T}\left(\sqrt{c_i+1}-1\right)^{\frac{2(N+2)}{N}}dxdt+c|\{c_i>1\}|\\
&\leq cT+c. \qquad (2.13)
\end{aligned}$$

The last step is due to (2.9) and (2.12). □

We would like to remark that estimate (2.13) is the only source where the constant $c$ depends on $T$. But this does not affect our global existence because the constant blows up only when $T\to\infty$.

The following lemma is the foundation of a De Giorgi iteration scheme, whose proof can be found in ([7], p.12).

**Lemma 2.3.** *Let $\{y_n\}, n=0,1,2,\cdots,$ be a sequence of positive numbers satisfying the recursive inequalities*

$$y_{n+1}\leq cb^ny_n^{1+\alpha}\quad\text{for some } b>1, c,\alpha\in(0,\infty).$$

*If*

$$y_0\leq c^{-\frac{1}{\alpha}}b^{-\frac{1}{\alpha^2}},$$

*then* $\lim_{n\to\infty}y_n=0$.

The following lemma from [18] is essentially a consequence of the interpolation inequality for $L^q$ norms in ([10], p.146). As we shall see, it provides a way to decrease certain exponents in our nonlinear terms.

**Lemma 2.4.** *Let $f\in L^\ell(Q_T)\cap L^\infty(Q_T)$ for some $\ell\geq 1$. Assume that there exist $q\in(\ell,\infty)$, $\delta>0$, and $c>0$ such that*

$$\|f\|_{\infty,Q_T}\leq c\|f\|_{q,Q_T}^{1+\delta}.$$

*If*

$$\delta < \frac{\ell}{q-\ell}, \tag{2.14}$$

*then*

$$\|f\|_{\infty,Q_T} \le c^{\frac{q}{\ell(1+\delta)-q\delta}} \|f\|_{\ell,Q_T}^{\frac{\ell(1+\delta)}{\ell(1+\delta)-q\delta}}. \tag{2.15}$$

*Proof.* We easily check

$$\begin{aligned} \|f\|_{\infty,Q_T} &\le c\left(\int_{Q_T} |f|^{q-\ell+\ell} dxdt\right)^{\frac{1+\delta}{q}} \\ &\le c\|f\|_{\infty,Q_T}^{\left(1-\frac{\ell}{q}\right)(1+\delta)} \|f\|_{\ell,Q_T}^{\frac{\ell}{q}(1+\delta)}. \end{aligned} \tag{2.16}$$

Condition (2.14) implies

$$\left(1-\frac{\ell}{q}\right)(1+\delta) < 1.$$

As a result, we can factor out $\|f\|_{\infty,Q_T}^{\left(1-\frac{\ell}{q}\right)(1+\delta)}$ from (2.16), thereby obtaining (2.15). □

Finally, the following two inequalities will be used without acknowledgment:

$$(|a|+|b|)^\gamma \le \begin{cases} 2^{\gamma-1}(|a|^\gamma + |b|^\gamma) & \text{if } \gamma \ge 1, \\ |a|^\gamma + |b|^\gamma & \text{if } \gamma \le 1. \end{cases}$$

.

## 3. Proof of Theorem 1.1

We begin with

*Proof of* (1.9) . We will adopt an idea from [18]. To do this, let $w$ be given as in (1.9). That is,

$$w = \sum_{i=1}^{I} c_i.$$

Define

$$\varphi = \frac{w}{\|w\|_{r,Q_T}}, \quad \psi = \frac{\Psi}{\|w\|_{r,Q_T}}, \tag{3.1}$$

where $r \in [1,\infty)$, whose precise value remains to be determined. Subsequently,

$$\|\varphi\|_{r,Q_T} = 1, \quad |\psi| \le \max_{1\le i\le I} |z_i|\varphi. \tag{3.2}$$

Sum (1.2) over $i$ and divide through the resulting equation by $\|w\|_{r,Q_T}$ to obtain

$$\partial_t\varphi - \Delta\varphi = -\nabla\cdot(\varphi u) + \nabla\cdot(\psi\nabla\phi) \text{ in } Q_T. \tag{3.3}$$

Obviously, $\varphi$ represents the scaling of $w$ by its $L^r$ norm. As we shall see, this will create another chance for us to use Lemma 2.4. Every time you are able to use the lemma, we are moving further away from "superlinear" and getting closer to "linear". This is the key idea behind our approach. Some of our subsequent proof is very similar to that in [18]. For completeness, we include them here. Our motivation is to reveal and expose the essence of the method developed in [18]. By applying the method to the current problem and achieving significantly better results for the problem than the existing ones, we once again demonstrate the power of the method.

To continue the proof, as in [18], we apply a De Giorgi-type iteration scheme to (3.3). For this purpose, select

$$k \ge 2\|\varphi(\cdot,0)\|_{\infty,\mathbb{R}^N} \tag{3.4}$$

as below. Define

$$k_n = k - \frac{k}{2^{n+1}} \quad \text{for } n = 0, 1, \cdots. \tag{3.5}$$

Fix

$$\beta > 1.$$

Then it is easy to check that the function

$$\left(\frac{1}{k_n^\beta} - \frac{1}{\varphi^\beta}\right)^+$$

is a legitimate test function for (3.3). Upon using it, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N}\int_{k_n}^{\varphi}\left(\frac{1}{k_n^\beta} - \frac{1}{\mu^\beta}\right)^+ d\mu dx + \beta\int_{\Omega_n(t)}\frac{1}{\varphi^{1+\beta}}\left|\nabla\varphi\right|^2 dx \\ &= \beta\int_{\Omega_n(t)}\frac{1}{\varphi^\beta}u\cdot\nabla\varphi dx - \beta\int_{\Omega_n(t)}\frac{\psi}{\varphi^{\beta+1}}\nabla\phi\cdot\nabla\varphi dx, \end{aligned} \tag{3.6}$$

where

$$\Omega_n(t) = \{x \in \mathbb{R}^N : \varphi(x,t) \geq k_n\}. \tag{3.7}$$

Evidently,

$$\beta\int_{\Omega_n(t)}\frac{1}{\varphi^{1+\beta}}\left|\nabla\varphi\right|^2 dx = \frac{4\beta}{(\beta-1)^2}\int_{\mathbb{R}^N}\left|\nabla\left(\frac{1}{k_n^{\frac{\beta-1}{2}}} - \frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right|^2 dx.$$

Note from (1.5) that

$$\beta\int_{\Omega_n(t)}\frac{1}{\varphi^\beta}u\cdot\nabla\varphi dx = \frac{\beta}{\beta-1}\int_{\mathbb{R}^N}u\cdot\nabla\left(\frac{1}{k_n^{\beta-1}} - \frac{1}{\varphi^{\beta-1}}\right)^+ dx = 0, \tag{3.8}$$

while (3.2) asserts that

$$\begin{aligned} -\beta\int_{\Omega_n(t)}\frac{\psi}{\varphi^{\beta+1}}\nabla\phi\cdot\nabla\varphi dx &\leq c\beta\int_{\Omega_n(t)}\frac{1}{\varphi^\beta}|\nabla\phi||\nabla\varphi| dx \\ &\leq \frac{\beta}{2}\int_{\Omega_n(t)}\frac{1}{\varphi^{\beta+1}}|\nabla\varphi|^2 dx + c\beta\int_{\Omega_n(t)}\frac{1}{\varphi^{\beta-1}}|\nabla\phi|^2 dx. \end{aligned} \tag{3.9}$$

We next claim

$$\int_{k_n}^{\varphi}\left(\frac{1}{k_n^\beta} - \frac{1}{\mu^\beta}\right)^+ d\mu \geq \frac{2\beta}{(1-\beta)^2}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}} - \frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2. \tag{3.10}$$

To see this, we compute

$$\begin{aligned} \left(\int_{k_{n+1}}^{\varphi}\left(\frac{1}{k_{n+1}^\beta} - \frac{1}{\mu^\beta}\right)^+ d\mu\right)'' &= \beta\varphi^{-1-\beta}, \\ \left(\frac{2\beta}{(1-\beta)^2}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}} - \frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2\right)'' &= \frac{2\beta}{\beta-1}\left(\beta - \frac{1+\beta}{2}\left(\frac{\varphi}{k_{n+1}}\right)^{\frac{\beta-1}{2}}\right)\varphi^{-1-\beta} \end{aligned}$$

We can easily verify that

$$\left(\int_{k_{n+1}}^{\varphi}\left(\frac{1}{k_{n+1}^\beta} - \frac{1}{\mu^\beta}\right)^+ d\mu\right)'' \geq \left(\frac{2\beta}{(1-\beta)^2}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}} - \frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2\right)'' \quad \text{for } \varphi \geq k_{n+1}.$$

Integrate this inequality over $[k_{n+1}, v]$ twice to obtain (3.10).

Recall (3.5) and (3.4) to derive

$$\int_{k_n}^{\varphi}\left(\frac{1}{k_n^\beta}-\frac{1}{\mu^\beta}\right)^+ d\mu\Big|_{t=0}=0. \tag{3.11}$$

Use (3.9) and (3.8) in (3.6), integrate the resulting inequality with respect to $t$, and keep in mind (3.11) and (3.10) to deduce

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2 dx+\int_{Q_T}\left|\nabla\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right|^2 dxdt\le\frac{c}{k^{\beta-1}}\int_{Q_n}|\nabla\phi|^2dxdt,$$

where

$$Q_n=\{(x,t)\in Q_T:\varphi(x,t)\ge k_n\}. \tag{3.12}$$

Now set

$$y_n=|Q_n|. \tag{3.13}$$

We proceed to show that $\{y_n\}$ satisfies the condition in Lemma 2.3. By calculations similar to those in (2.12), we have

$$\begin{aligned}
&\int_{Q_T}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^{\frac{4}{N}+2}dxdt\\
\le\ &\int_0^T\left(\int_{\mathbb{R}^N}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\
\le\ &c\left(\sup_{0\le t\le T}\int_{\mathbb{R}^N}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^2dx\right)^{\frac{2}{N}}\int_{Q_T}\left|\nabla\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right|^2dxdt\\
\le\ &c\left(\frac{1}{k^{\beta-1}}\int_{Q_n}|\nabla\phi|^2dxdt\right)^{\frac{N+2}{N}}.
\end{aligned}\tag{3.14}$$

It is easy to verify that

$$\begin{aligned}
\int_{Q_T}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^{\frac{4}{N}+2}dxdt\ &\ge\ \int_{Q_{n+1}}\left[\left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{\varphi^{\frac{\beta-1}{2}}}\right)^+\right]^{\frac{4}{N}+2}dxdt\\
&\ge\ \left(\frac{1}{k_n^{\frac{\beta-1}{2}}}-\frac{1}{k_{n+1}^{\frac{\beta-1}{2}}}\right)^{\frac{4}{N}+2}|Q_{n+1}|\\
&=\ \left(\frac{\left(1-\frac{1}{2^{n+2}}\right)^{\frac{\beta-1}{2}}-\left(1-\frac{1}{2^{n+1}}\right)^{\frac{\beta-1}{2}}}{k^{\frac{\beta-1}{2}}\left(1-\frac{1}{2^{n+1}}\right)^{\frac{\beta-1}{2}}\left(1-\frac{1}{2^{n+2}}\right)^{\frac{\beta-1}{2}}}\right)^{\frac{4}{N}+2}|Q_{n+1}|\\
&\ge\ \frac{c|Q_{n+1}|}{2^{(\frac{4}{N}+2)n}k^{\frac{(\beta-1)(N+2)}{N}}}.
\end{aligned}$$

Combining this with (3.14) yields

$$y_{n+1} = |Q_{n+1}|^{\frac{N}{N+2}+\frac{2}{N+2}} \leq c4^n \int_{Q_n} |\nabla\phi|^2 dxdt |Q_{n+1}|^{\frac{2}{N+2}}. \tag{3.15}$$

In view of (1.3), we may apply the classical representation theorem ([10], p. 17) to obtain

$$\phi(x,t) = \int_{\mathbb{R}^N} \Gamma(y-x)\Psi(y,t)dy, \tag{3.16}$$

where $\Gamma(x)$ is the fundamental solution of the Laplace equation, i.e.,

$$\Gamma(x) = \frac{1}{N(N-2)\omega_N|x|^{N-2}}, \quad \omega_N = \text{the volume of the unit ball in } \mathbb{R}^N.$$

It immediately follows that

$$|\phi| \leq c\int_{\mathbb{R}^N} \frac{w}{|x-y|^{N-2}}dy, \quad |\nabla\phi| \leq c\int_{\mathbb{R}^N} \frac{w}{|x-y|^{N-1}}dy.$$

This together with Theorem 1 in ([17], p. 119) implies

$$\|\nabla\phi\|_{\frac{Ns}{N-s},\mathbb{R}^N} \leq c\|w\|_{s,\mathbb{R}^N} \quad \text{for } s \in (1,N). \tag{3.17}$$

Fix

$$q > N+2. \tag{3.18}$$

Then we can conclude from (3.17) that

$$\begin{aligned}
\int_{Q_n} |\nabla\phi|^2 dxdt &\leq \left(\int_{Q_T} |\nabla\phi|^q dxdt\right)^{\frac{2}{q}} |Q_n|^{1-\frac{2}{q}} \\
&\leq c\left(\int_0^T \left(\int_{\mathbb{R}^N} w^{\frac{Nq}{N+q}} dx\right)^{\frac{N+q}{N}} dt\right)^{\frac{2}{q}} y_n^{1-\frac{2}{q}} = c\|w\|^2_{\frac{Nq}{N+q},q,Q_T} y_n^{1-\frac{2}{q}},
\end{aligned}$$

where

$$\|w\|_{\frac{Nq}{N+q},q,Q_T} = \left(\int_0^T \|w(\cdot,t)\|^q_{\frac{Nq}{N+q},\mathbb{R}^N} dt\right)^{\frac{1}{q}}.$$

Use this in (3.15) to derive

$$y_{n+1} \leq c4^n\|w\|^2_{\frac{Nq}{N+q},q,Q_T} y_n^{1+\alpha}, \tag{3.19}$$

where

$$\alpha = -\frac{2}{q} + \frac{2}{N+2} > 0 \text{ due to (3.18)}. \tag{3.20}$$

Observe from (2.12) and (2.13) that

$$
\begin{aligned}
&\|w\|_{\frac{Nq}{N+q},q,Q_T}\\
\leq\ & c\|w\|_{\infty,Q_T}^{1-\frac{N+q+2}{Nq}}\left(\int_0^T\left(\int_{\mathbb{R}^N}\sum_{i=1}^I c_i^{\frac{N+q+2}{N+q}}dx\right)^{\frac{N+q}{N}}dt\right)^{\frac{1}{q}}\\
\leq\ & c\|w\|_{\infty,Q_T}^{1-\frac{N+q+2}{Nq}}\left(\int_0^T\left(\sum_{i=1}^I\left[\int_{\mathbb{R}^N}c_idx+\int_{\mathbb{R}^N}\left(\sqrt{c_i+1}-1\right)^{\frac{2(q+2)}{N+q}+\frac{2N}{N+q}}dx\right]\right)^{\frac{N+q}{N}}dt\right)^{\frac{1}{q}}\\
\leq\ & c\|w\|_{\infty,Q_T}^{1-\frac{N+q+2}{Nq}}\left(c+\int_0^T\sum_{i=1}^I\left(\int_{\mathbb{R}^N}c_idx\right)^{\frac{q+2}{N}}\left(\int_{\mathbb{R}^N}\left(\sqrt{c_i+1}-1\right)^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\right)^{\frac{1}{q}}\\
\leq\ & c\|w\|_{\infty,Q_T}^{1-\frac{N+q+2}{Nq}}\left[c+\sum_{i=1}^I\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}c_idx\right)^{\frac{q+2}{Nq}}\left(\int_{Q_T}|\nabla\left(\sqrt{c_i+1}-1\right)|^2dxdt\right)^{\frac{1}{q}}\right]\\
\leq\ & c\|w\|_{\infty,Q_T}^{1-\frac{N+q+2}{Nq}}. \qquad (3.21)
\end{aligned}
$$

Now we pick a number

$$\ell>r. \tag{3.22}$$

Choose $k$ so large that

$$\max\left\{L_1\|\varphi\|_{\ell,Q_T}^{\frac{\ell}{\ell-r}},\, L_2\|w\|_{r,Q_T}^{-1}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{Nq}{(N-1)q-N-2}}\right\}\leq k, \tag{3.23}$$

where $L_1$ and $L_2$ are two positive numbers to be determined. Note that the exponent of $\|\varphi\|_{\ell,Q_T}$ in the above inequality is the number $1+\delta$ in Lemma 2.4 as we can easily see from (2.16) that

$$
\begin{aligned}
\|\varphi\|_{\ell,Q_T}^{\frac{\ell}{\ell-r}} &= \|\varphi\|_{\ell,Q_T}^{1+\frac{r}{\ell-r}}\\
&\leq \|\varphi\|_{\infty,Q_T}^{\left(1-\frac{r}{\ell}\right)\left(1+\frac{r}{\ell-r}\right)}\|\varphi\|_{r,Q_T}^{\frac{r}{\ell-r}}=\|\varphi\|_{\infty,Q_T}. \qquad (3.24)
\end{aligned}
$$

The last step is due to (3.2). Inequality (3.21) indicates that the selection of the exponent of $\|w\|_{\frac{Nq}{N+q},q,Q_T}$ in (3.23) is based upon the same idea. For each

$$j>0$$

we have from (3.23) that

$$L_1^{j\alpha}\|\varphi\|_{\ell,Q_T}^{\frac{j\alpha\ell}{\ell-r}}\leq k^{j\alpha},\quad L_2^{j\alpha}\|w\|_{r,Q_T}^{-j\alpha}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{j\alpha Nq}{(N-1)q-N-2}}\leq k^{j\alpha}.$$

Use this in (3.19) to deduce

$$y_{n+1}\leq\frac{c4^n\|w\|_{r,Q_T}^{j\alpha}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{b}k^{2j\alpha}}{(L_1L_2)^{j\alpha}\|\varphi\|_{\ell,Q_T}^{\frac{j\alpha\ell}{\ell-r}}}y_n^{1+\alpha},$$

where

$$b=2-\frac{j\alpha Nq}{(N-1)q-N-2}. \tag{3.25}$$

The introduction of $j$ here is very crucial. As we shall see, by choosing $j$ suitably large, we can make certain exponents in our nonlinear terms negative. This will enable us to balance out large positive exponents.

To apply Lemma 2.3, we first recall (3.2), (3.5), (3.12), and (3.13) to deduce

$$y_0 = |Q_0| \le \int_{Q_T}\left(\frac{2\varphi}{k}\right)^r dxdt = \frac{2^r}{k^r}.$$

Assume that

$$r > 2j. \tag{3.26}$$

Subsequently, we can pick $k$ so large that

$$\frac{2^r}{k^{r-2j}} \le \frac{(L_1L_2)^j\|\varphi\|_{\ell,Q_T}^{\frac{j\ell}{\ell-r}}}{c^{\frac{1}{\alpha}}4^{\frac{1}{\alpha^2}}\|w\|_{r,Q_T}^j\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{b}{\alpha}}}. \tag{3.27}$$

Lemma 2.3 asserts

$$\lim_{n\to\infty} y_n = |\{\varphi \ge k\}| = 0.$$

That is,

$$\sup_{Q_T}\varphi \le k. \tag{3.28}$$

According to (3.4), (3.23), and (3.27), it is enough for us to take

$$\begin{aligned} k &= 2\|\varphi(\cdot,0)\|_{\infty,\mathbb{R}^N} + L_1\|\varphi\|_{\ell,Q_T}^{\frac{\ell}{\ell-r}} + L_2\|w\|_{r,Q_T}^{-1}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{Nq}{(N-1)q-N-2}} \\ &\quad + c^{\frac{1}{\alpha(r-2j)}}4^{\frac{1}{\alpha^2(r-2j)}}(L_1L_2)^{-\frac{j}{r-2j}}\|\varphi\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{\frac{j}{r-2j}}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{b}{\alpha(r-2j)}}. \end{aligned} \tag{3.29}$$

Plug this into (3.28), take $L_1 = \frac{1}{2}$ in the resulting inequality, and make use of (3.24) to yield

$$\begin{aligned} \|\varphi\|_{\infty,Q_T} &\le 4\|\varphi(\cdot,0)\|_{\infty,\mathbb{R}^N} + 2L_2\|w\|_{r,Q_T}^{-1}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{Nq}{(N-1)q-N-2}} \\ &\quad + cL_2^{-\frac{j}{r-2j}}\|\varphi\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{\frac{j}{r-2j}}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{b}{\alpha(r-2j)}}. \end{aligned}$$

Recall (3.1) to deduce

$$\begin{aligned} \|w\|_{\infty,Q_T} &\le 4\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + 2L_2\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{Nq}{(N-1)q-N-2}} \\ &\quad + cL_2^{-\frac{j}{r-2j}}\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{1+\frac{j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{\frac{Nq}{N+q},q,Q_T}^{\frac{b}{\alpha(r-2j)}}. \end{aligned} \tag{3.30}$$

By the interpolation inequality for $L^q$ norms in ([10], p.146), we have

$$\|w\|_{\frac{Nq}{N+q},\mathbb{R}^N} \le \|w\|_{\frac{(N-1)q}{N},\mathbb{R}^N}^{1-\lambda}\|w\|_{1,\mathbb{R}^N}^{\lambda},$$

where

$$\lambda = \frac{\frac{N+q}{Nq} - \frac{N}{(N-1)q}}{1 - \frac{N}{(N-1)q}} = \frac{1}{N}.$$

Consequently,

$$\begin{aligned}\|w\|_{\frac{Nq}{N+q},q,Q_T} &\leq \left(\int_0^T \|w\|^{(1-\lambda)q}_{\frac{(N-1)q}{N},\mathbb{R}^N}\|w\|^{\lambda q}_{1,\mathbb{R}^N}dt\right)^{\frac{1}{q}}\\ &\leq \sup_{0\leq t\leq T}\|w\|^{\lambda}_{1,\mathbb{R}^N}\|w\|^{\frac{N-1}{N}}_{\frac{(N-1)q}{N},Q_T}\leq c\|w\|^{\frac{N-1}{N}}_{\frac{(N-1)q}{N},Q_T}.\end{aligned}$$

In view of (3.21), we may choose $L_2$ suitably small so that

$$2L_2\|w\|^{\frac{Nq}{(N-1)q-N-2}}_{\frac{Nq}{N+q},q,Q_T}\leq \frac{1}{2}\|w\|_{\infty,Q_T}.$$

Utilize the preceding two estimates in (3.30) to deduce

$$\begin{aligned}\|w\|_{\infty,Q_T} &\leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}\\ &\quad +c\|w\|^{-\frac{j\ell}{(r-2j)(\ell-r)}}_{\ell,Q_T}\|w\|^{1+\frac{j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}}_{r,Q_T}\|w\|^{\frac{(N-1)b}{N\alpha(r-2j)}}_{\frac{(N-1)q}{N},Q_T}.\end{aligned}\tag{3.31}$$

We proceed to show that we can extract enough information from this inequality by making suitable choice of the parameters

We first pick

$$r>\frac{(N-1)q}{N}.\tag{3.32}$$

Then we can invoke the interpolation inequality for $L^q$ norms again to deduce that

$$\|w\|_{r,Q_T}\leq\|w\|^{1-\lambda_1}_{\ell,Q_T}\|w\|^{\lambda_1}_{\frac{(N-1)q}{N},Q_T},\quad \lambda_1=\frac{(N-1)q(\ell-r)}{r[N\ell-(N-1)q]}.\tag{3.33}$$

Use this in (3.31) to derive

$$\begin{aligned}\|w\|_{\infty,Q_T} &\leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}\\ &\quad +c\|w\|^{-\frac{j\ell}{(r-2j)(\ell-r)}+(1-\lambda_1)\left(\frac{r-j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}\right)}_{\ell,Q_T}\|w\|^{\frac{(N-1)b}{N\alpha(r-2j)}+\lambda_1\left(\frac{r-j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}\right)}_{\frac{(N-1)q}{N},Q_T}.\end{aligned}\tag{3.34}$$

We further require that the last exponent in the above inequality be 0, i.e.,

$$\frac{(N-1)b}{N\alpha(r-2j)}+\lambda_1\left(\frac{r-j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}\right)=0.\tag{3.35}$$

Substitute $\lambda_1$ in (3.33) into this equation and then simplify to deduce

$$(b+\alpha q)\,\ell = \alpha qr-\alpha qj+\frac{b(N-1)q}{N}.\tag{3.36}$$

Later we will show how to pick our parameters so that this new condition can be made consistent with our old assumptions. Assume that this is the case for the moment. Substitute (3.35) and (3.36) into (3.34) to deduce

$$\begin{aligned}\|w\|_{\infty,Q_T} &\leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|^{1+\frac{N\alpha j+(N-1)b}{N\alpha(r-2j)}}_{\ell,Q_T}\\ &= 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|^{1+\frac{[N\alpha j+(N-1)b]q}{N(b+\alpha q)\ell-b(N-1)q-N\alpha jq}}_{\ell,Q_T}\\ &\equiv 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|^{1+\frac{M}{\ell-M}}_{\ell,Q_T}.\end{aligned}$$

That is, we have set

$$M=\frac{[N\alpha j+(N-1)b]q}{b+\alpha q}.\tag{3.37}$$

Plug $b$ in (3.25) into $M$ to obtain

$$M = \frac{(N+2)\left[\frac{2(N-1)[(N-1)q-N-2]}{(N+2)N\alpha} - j\right]}{\frac{(2+\alpha q)[(N-1)q-N-2]}{\alpha N q} - j}. \tag{3.38}$$

It is easy to verify that

$$\frac{2(N-1)[(N-1)q-N-2]}{(N+2)N\alpha} > \frac{(2+\alpha q)[(N-1)q-N-2]}{\alpha N q}.$$

We shall choose

$$j \in \left(0, \ \frac{(2+\alpha q)[(N-1)q-N-2]}{\alpha N q}\right). \tag{3.39}$$

As a result,

$$M > N+2.$$

Moreover, both the numerator and denominator of the fraction in (3.38) are positive. This implies that

$$N\alpha j + (N-1)b > 0, \quad b + \alpha q > 0.$$

However, $j$ can not be too close to 0 because (3.35) implies that

$$b < 0.$$

We easily see from (3.36) that

$$(b+\alpha q)(\ell - r) = -br - \alpha q j + \frac{(N-1)bq}{N} > 0.$$

Thus, we must have

$$r > \frac{\alpha q j - \frac{(N-1)bq}{N}}{-b}. \tag{3.40}$$

For simplicity, we take

$$b = -1.$$

Subsequently, by (3.25)

$$j = \frac{3[(N-1)q-N-2]}{\alpha N q}. \tag{3.41}$$

This number satisfies (3.39) if we further require

$$q > \frac{3(N+2)}{2}. \tag{3.42}$$

The right-hand side of (3.40) becomes

$$\frac{\alpha q j - \frac{(N-1)bq}{N}}{-b} = \frac{4(N-1)q - 3(N+2)}{N}.$$

Evaluate $M$ from (3.37) to get

$$M_0 = M|_{b=-1} = \frac{2(N-1)q - 3(N+2)}{-1+\alpha q} \to (N-1)(N+2) \ \text{ as } q \to \infty. \tag{3.43}$$

In summary, we have shown:

**Claim 3.1.** *Let $q$ be given as in (3.42). Define $j$ and $M_0$ as in (3.41) and (3.43), respectively. Then for each*

$$\ell > \max\left\{\frac{\alpha qj - \frac{(N-1)q}{N}}{-1+\alpha q}, \frac{6[(N-1)q-N-2]}{\alpha Nq}, \frac{4(N-1)q-3(N+2)}{N}\right\} \tag{3.44}$$

*there is a constant $c$ such that*

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1+\frac{M_0}{\ell-M_0}}. \tag{3.45}$$

*Proof.* Under the conditions of the claim, $b=-1$. Define $r$ via (3.36), i.e.,

$$r = \frac{(-1+\alpha q)\ell + \alpha qj + \frac{(N-1)q}{N}}{\alpha q} < \ell. \tag{3.46}$$

The last step is due to (3.44). It is easy to see from the proof of (3.40) that (3.32) holds. It remains to be seen that $r > 2j$. To this end, we derive from (3.46) that

$$\alpha q(r-2j) = (-1+\alpha q)\ell - \alpha qj + \frac{(N-1)q}{N} > 0.$$

That is, (3.26) holds. Thus, all the previous arguments are valid. The claim follows. □

To complete the proof of (1.9), we need to make use of the fact that $j$ can be large. That is, we must choose our parameters differently than what we did earlier. Let $q$ be given as (3.42). Fix

$$\varepsilon_0 > 0.$$

Then we pick $r, j$ so that

$$\alpha_1 \equiv \varepsilon_0 + \frac{j}{r-2j} = -\frac{(N-1)b}{N\alpha(r-2j)}, \tag{3.47}$$

where $\alpha$ is defined as in (3.20). Substitute (3.25) into this equation to yield

$$\alpha j = \frac{\varepsilon_0 N[(N-1)q-N-2]\alpha r + 2(N-1)[(N-1)q-N-2]}{2\varepsilon_0 N[(N-1)q-N-2] + N(N+2)}. \tag{3.48}$$

Subsequently,

$$\alpha(r-2j) = \frac{N(N+2)\alpha r - 4(N-1)[(N-1)q-N-2]}{2\varepsilon_0 N[(N-1)q-N-2] + N(N+2)}.$$

Hence, we need to assume

$$\begin{aligned} r &> \frac{4(N-1)[(N-1)q-N-2]}{\alpha N(N+2)} \\ &= \frac{2(N-1)[(N-1)q-N-2]q}{(q-N-2)N} \\ &> \frac{(N-1)q}{N}. \end{aligned} \tag{3.49}$$

As a result, (3.32) is guaranteed. We choose $\ell$ so that both (3.44) and (3.22) are satisfied.

Under our new choice of parameters, (3.31) reduces to

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}} \|w\|_{r,Q_T}^{1-\varepsilon_0+\alpha_1+\frac{j\ell}{(r-2j)(\ell-r)}} \|w\|_{\frac{(N-1)q}{N},Q_T}^{-\alpha_1}. \tag{3.50}$$

Note that the total sum of the last three exponents is $1-\varepsilon_0$. However, we must make the last three different norms into a single one. The first step in this direction is still (3.33). Upon using it in (3.50), we derive

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0+\beta_2}\|w\|_{\frac{(N-1)q}{N},Q_T}^{-\beta_2}, \tag{3.51}$$

where

$$\beta_2 = -\lambda_1\left(1-\varepsilon_0+\frac{j\ell}{(r-2j)(\ell-r)}\right)+(1-\lambda_1)\alpha_1.$$

We claim

$$\beta_2 > 0. \tag{3.52}$$

To see this, we calculate from (3.33) and (3.47) that

$$\begin{aligned}
\beta_2 &= -\lambda_1(1-\varepsilon_0) - \frac{\lambda_1 j\ell}{(r-2j)(\ell-r)} + (1-\lambda_1)\left(\varepsilon_0+\frac{j}{r-2j}\right)\\
&= \varepsilon_0 - \frac{(N-1)q(\ell-r)}{r[N\ell-(N-1)q]} - \frac{(N-1)qj\ell}{r[N\ell-(N-1)q](r-2j)} + \frac{\ell[Nr-(N-1)q]j}{r[N\ell-(N-1)q](r-2j)}\\
&= \frac{N\ell}{N\ell-(N-1)q}\left(\frac{[N\ell-(N-1)q]\varepsilon_0}{N\ell} + \frac{[Nr-2(N-1)q]\ell j-(N-1)q(\ell-r)(r-2j)}{Nr\ell(r-2j)}\right)\\
&= \frac{N\ell}{N\ell-(N-1)q}\left(\frac{[N\ell-(N-1)q]\varepsilon_0}{N\ell} + \frac{N\ell\left(j-\frac{(N-1)q}{N}\right)+(N-1)q(r-2j)}{N\ell(r-2j)}\right). \qquad (3.53)
\end{aligned}$$

It is easy to see that under (3.49) $\alpha j$ given in (3.48) is an increasing function of $\varepsilon_0$. That is, we have

$$j > \frac{2(N-1)[(N-1)q-N-2]}{N(N+2)\alpha} = \frac{(N-1)[(N-1)q-N-2]q}{N(q-N-2)} > \frac{(N-1)q}{N}.$$

This gives (3.52), from which follows that

$$\|w\|_{\ell,Q_T}^{\frac{N\ell\beta_2}{(N-1)q}} \leq \left(\|w\|_{\infty,Q_T}^{1-\frac{(N-1)q}{N\ell}}\|w\|_{\frac{(N-1)q}{N},Q_T}^{\frac{(N-1)q}{N\ell}}\right)^{\frac{N\ell\beta_2}{(N-1)q}}.$$

Combining this with (3.51) yields

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0}\left(\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}}\right)^{\frac{[N\ell-(N-1)q]\beta_2}{(N-1)q}}. \tag{3.54}$$

We would be tempted to take the limit $\ell\to\infty$ here. Unfortunately, the last exponent in the above inequality also goes to infinity with $\ell$. To address this issue, we appeal to (3.45). Since $\ell$ satisfies (3.44), (3.45) is indeed available to us. Divide through the inequality by $\|u\|_{\ell,Q_T}$ to obtain

$$\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}} \leq \frac{c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}}{\|w\|_{\ell,Q_T}} + c\|w\|_{\ell,Q_T}^{\frac{M_0}{\ell-M_0}}. \tag{3.55}$$

Without any loss of generality, we may assume

$$\text{the second term in (3.55)} \leq \text{ the third term there for all } \ell \text{ satisfying (3.44).}$$

Otherwise, we would have nothing more to prove. Equipped with this, we deduce

$$\left(\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}}\right)^{\frac{[N\ell-(N-1)q]\beta_2}{(N-1)q}} \leq c\|w\|_{\ell,Q_T}^{\frac{M_0[N\ell-(N-1)q]\beta_2}{(N-1)q(\ell-M_0)}},$$

Substitute this into (3.54) to get

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0+\frac{M_0[N\ell-(N-1)q]\beta_2}{(N-1)q(\ell-M_0)}}.$$

We must show that we can choose our parameters so that

$$\alpha_2 \equiv 1-\varepsilon_0+\frac{M_0[N\ell-(N-1)q]\beta_2}{(N-1)q(\ell-M_0)} = 1. \tag{3.56}$$

To this end, we evaluate from (3.53) that

$$\begin{aligned}
&\frac{M_0[N\ell-(N-1)q]\beta_2}{(N-1)q(\ell-M_0)}\\
&= \frac{M_0N\ell}{(N-1)q(\ell-M_0)}\left(\frac{[N\ell-(N-1)q]\varepsilon_0}{N\ell}+\frac{N\ell\left(j-\frac{(N-1)q}{N}\right)+(N-1)q(r-2j)}{N\ell(r-2j)}\right)\\
&= \frac{M_0[N\ell-(N-1)q]\varepsilon_0}{(N-1)q(\ell-M_0)}+\frac{M_0\left[N\ell\left(j-\frac{(N-1)q}{N}\right)+(N-1)q(r-2j)\right]}{(N-1)q(\ell-M_0)(r-2j)}.
\end{aligned}$$

In view of (3.43), we may choose $q$ large enough so that

$$\frac{M_0[N\ell-(N-1)q]}{(N-1)q(\ell-M_0)} < 1.$$

As a result, we can find a positive $\varepsilon_0$ to satisfy (3.56). That is, we have

$$\begin{aligned}
\|w\|_{\infty,Q_T} &\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|_{\ell,Q_T}\\
&\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|_{\infty,Q_T}^{1-\frac{(N+2)}{N\ell}}\|w\|_{\frac{N+2}{N},Q_T}^{\frac{(N+2)}{N\ell}}.
\end{aligned}$$

An application of Young's inequality ([10], p. 145) produces

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|_{\frac{N+2}{N},Q_T} \leq c.$$

□

*Proof of* (1.11). Set

$$A_r = \||u|^2\|_{r,Q_T} \quad \text{for } r > 1.$$

Subsequently, let

$$\Phi = \frac{|u|^2}{A_r}. \tag{3.57}$$

We proceed to derive an equation for $\Phi$ as in [18]. To this end, we take the dot product of both sides of (1.1) with $u$ to obtain

$$\partial_t u\cdot u + u\cdot\nabla u\cdot u + \nabla p\cdot u = \Delta u\cdot u - \Psi\nabla\phi\cdot u. \tag{3.58}$$

We calculate from (1.5) that

$$\begin{aligned}
u\cdot\partial_t u &= \frac{1}{2}\partial_t|u|^2,\\
u\cdot\nabla u\cdot u &= u_j\partial_{x_j}u_iu_i = \frac{1}{2}u\cdot\nabla|u|^2,\\
\nabla p\cdot u &= \nabla\cdot(pu),\\
\Delta u\cdot u &= \frac{1}{2}\Delta|u|^2-|\nabla u|^2.
\end{aligned}$$

Here we have employed the notation convention of summing over repeated indices. Substitute the preceding four equations into (3.58) and divide through the resulting equation by $\frac{A_r}{2}$ to derive

$$\partial_t\Phi + u\cdot\nabla\Phi - \Delta\Phi + 2A_r^{-1}|\nabla u|^2 = -2A_r^{-1}\nabla\cdot(pu) - 2A_r^{-1}\Psi\nabla\phi\cdot u. \tag{3.59}$$

We are in a position to employ the previous De Giorgi iteration scheme. Let

$$k \geq 2\|\Phi(\cdot,0)\|_{\infty,\mathbb{R}^N}.$$

Define $k_n$ as before. Use

$$(\ln\Phi - \ln k_n)^+$$

as a test function in (3.59) to derive

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\int_{k_n}^{\Phi}(\ln\mu - \ln k_n)^+\,d\mu dx + \int_{\{\Phi(\cdot,t)\geq k_n\}}\frac{1}{\Phi}|\nabla\Phi|^2dx\\
&\quad\leq\ -\int_{\mathbb{R}^N}u\cdot\nabla\Phi\,(\ln\Phi-\ln k_n)^+\,dx + 2A_r^{-1}\int_{\{\Phi(\cdot,t)\geq k_n\}}\frac{p}{\Phi}u\cdot\nabla\Phi dx\\
&\qquad -2A_r^{-1}\int_{\mathbb{R}^N}\Psi\nabla\phi\cdot u\,(\ln\Phi-\ln k_n)^+\,dx.
\end{aligned}\tag{3.60}$$

We proceed to analyze each term in the above inequality. First, note from (1.5) that

$$-\int_{\mathbb{R}^N}u\cdot\nabla\Phi\,(\ln\Phi-\ln k_n)^+\,dx = -\int_{\mathbb{R}^N}u\cdot\nabla\int_{k_n}^{\Phi}(\ln\mu-\ln k_n)^+\,d\mu = 0.$$

In view of (3.57), we have

$$\begin{aligned}
2A_r^{-1}\int_{\{\Phi(\cdot,t)\geq k_n\}}\frac{p}{\Phi}u\cdot\nabla\Phi dx\ &\leq\ 2A_r^{-\frac{1}{2}}\int_{\{\Phi(\cdot,t)\geq k_n\}}\frac{1}{\Phi^{\frac{1}{2}}}|p||\nabla\Phi|dx\\
&\leq\ \frac{1}{2}\int_{\{\Phi(\cdot,t)\geq k_n\}}\frac{1}{\Phi}|\nabla\Phi|^2dx + 2A_r^{-1}\int_{\{\Phi(\cdot,t)\geq k_n\}}|p|^2dx.
\end{aligned}$$

Remember from (1.4) that

$$|\Psi| \leq \max_{1\leq i\leq I}|z_i|w \leq c.$$

Moreover, it is easy to see from (3.17) that for each $s > \frac{N}{N-1}$ there holds

$$\|\nabla\phi\|_{s,\mathbb{R}^N} \leq c\|w\|_{\frac{Ns}{N+s},\mathbb{R}^N} \leq c\|w\|_{\infty,\mathbb{R}^N}^{1-\frac{N+s}{Ns}}\|w\|_{1,\mathbb{R}^N}^{\frac{N+s}{Ns}} \leq c. \tag{3.61}$$

The last term in (3.60) can be estimated as follows:

$$\begin{aligned}
&-2A_r^{-1}\int_{\mathbb{R}^N}\Psi\nabla\phi\cdot u\,(\ln\Phi-\ln k_n)^+\,dx\\
&\quad\leq\ cA_r^{-\frac{1}{2}}\int_{\mathbb{R}^N}|\nabla\phi|\sqrt{\Phi}\,(\ln\Phi-\ln k_n)^+\,dx\\
&\quad=\ cA_r^{-\frac{1}{2}}\int_{\mathbb{R}^N}|\nabla\phi|\left(\sqrt{\Phi}-\sqrt{k_n}\right)^+(\ln\Phi-\ln k_n)^+\,dx\\
&\qquad +cA_r^{-\frac{1}{2}}\sqrt{k_n}\int_{\mathbb{R}^N}|\nabla\phi|\,(\ln\Phi-\ln k_n)^+\,dx \equiv I_1 + I_2.
\end{aligned}$$

Fix

$$s > \frac{N}{2}.$$

Subsequently,

$$\frac{s}{s-1} < \frac{2N}{N-2}.$$

Also, it is easy to check from (3.57) that

$$\|\Phi\|_{r,Q_T} = 1.$$

Therefore,

$$|\{\Phi \geq k_n\}| \leq \int_{Q_T} \left(\frac{2\Phi}{k}\right)^r dxdt \leq 2^r k^{-r}.$$

With these in mind, we calculate from (2.6) and (3.61) that

$$\begin{aligned}
I_1 &\leq cA_r^{-\frac{1}{2}} \|\nabla\phi\|_{s,\mathbb{R}^N} \left\| \left(\sqrt{\Phi} - \sqrt{k_n}\right)^+ \right\|_{\frac{2N}{N-2},\mathbb{R}^N} \left\|(\ln\Phi - \ln k_n)^+\right\|_{\frac{2Ns}{(N+2)s-2N},\mathbb{R}^N} \\
&\leq cA_r^{-\frac{1}{2}} \left(\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx\right)^{\frac{1}{2}} \left\|(\ln\Phi - \ln k_n)^+\right\|_{\frac{2N}{N-2},\mathbb{R}^N} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{(N+2)s-2N}{2Ns} - \frac{N-2}{2N}} \\
&\leq cA_r^{-\frac{1}{2}} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{2s-N}{Ns}} \left(\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx\right)^{\frac{1}{2}} \left(\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi^2}|\nabla\Phi|^2 dx\right)^{\frac{1}{2}} \\
&\leq cA_r^{-\frac{1}{2}} k^{-\frac{1}{2}-\frac{r(2s-N)}{Ns}} \int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx. \qquad (3.62)
\end{aligned}$$

We choose $k$ so large that the coefficient of the last integral in (3.62) is less than $\frac{1}{8}$, i.e.,

$$\frac{1}{8} k^{\frac{1}{2}+\frac{r(2s-N)}{Ns}} \geq cA_r^{-\frac{1}{2}}.$$

Consequently,

$$I_1 \leq \frac{1}{8} \int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx.$$

Similarly,

$$\begin{aligned}
I_2 &= cA_r^{-\frac{1}{2}} \sqrt{k_n} \int_{\mathbb{R}^N} |\nabla\phi| \left(\ln\Phi - \ln k_n\right)^+ dx \\
&\leq cA_r^{-\frac{1}{2}} \sqrt{k_n} \left(\int_{\mathbb{R}^N} |\nabla\phi|^s dx\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^N} \left[(\ln\Phi - \ln k_n)^+\right]^{\frac{s}{s-1}} dx\right)^{\frac{s-1}{s}} \\
&\leq cA_r^{-\frac{1}{2}} \sqrt{k_n} \left(\int_{\mathbb{R}^N} \left[(\ln\Phi - \ln k_n)^+\right]^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{s-1}{s} - \frac{N-2}{2N}} \\
&\leq cA_r^{-\frac{1}{2}} \sqrt{k_n} \left(\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi^2}|\nabla\Phi|^2 dx\right)^{\frac{1}{2}} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{(N+2)s-2N}{2Ns}} \\
&\leq cA_r^{-\frac{1}{2}} \left(\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx\right)^{\frac{1}{2}} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{(N+2)s-2N}{2Ns}} \\
&\leq \frac{1}{16} \int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi}|\nabla\Phi|^2 dx + cA_r^{-1} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{(N+2)s-2N}{Ns}}.
\end{aligned}$$

Collecting all the preceding results in (3.60) yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \int_{k_n}^{\Phi} (\ln\mu - \ln k_n)^+ d\mu dx + \frac{1}{16} \int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi} |\nabla\Phi|^2 dx \\
&\leq cA_r^{-1} \int_{\{\Phi(\cdot,t)\geq k_n\}} p^2 dx + cA_r^{-1} |\{\Phi(\cdot,t) \geq k_n\}|^{\frac{(N+2)s-2N}{Ns}}. \qquad (3.63)
\end{aligned}$$

We easily see that

$$\int_{\{\Phi(\cdot,t)\geq k_n\}} \frac{1}{\Phi} |\nabla\Phi|^2 \, dx = 4\int_{\mathbb{R}^N} \left| \nabla \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right|^2 dx.$$

We can also infer from the proof of (3.10) that

$$\int_{k_n}^{\Phi} (\ln\mu - \ln k_n)^+ \, d\mu \geq 2 \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^2.$$

Recall (3.5) and (3.4) to derive

$$\left. \int_{k_n}^{\Phi} (\ln\mu - \ln k_n)^+ \, d\mu \right|_{t=0} = 0.$$

Equipped with these estimates, we integrate (3.63) with respect to $t$ to deduce

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{\mathbb{R}^N} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^2 dx + \int_{Q_T} \left| \nabla \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right|^2 dxdt \\ &\leq \ cA_r^{-1} \int_{\{\Phi\geq k_n\}} p^2 dxdt + cA_r^{-1} \left| \{\Phi \geq k_n\} \right| \equiv I. \end{aligned}$$

Here we have taken $s = N$. Now set

$$y_n = |\{\Phi \geq k_n\}|.$$

We proceed to show that $\{y_n\}$ satisfies the condition in Lemma 2.3. By calculations similar to those in (2.12), we have

$$\begin{aligned} &\int_{Q_T} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^{\frac{4}{N}+2} dxdt \\ &\leq \ \int_0^T \left( \int_{\mathbb{R}^N} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^2 dx \right)^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} dt \\ &\leq \ c \left( \sup_{0\leq t\leq T} \int_{\mathbb{R}^N} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^2 dx \right)^{\frac{2}{N}} \int_{Q_T} \left| \nabla \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right|^2 dxdt \\ &\leq \ cI^{\frac{N+2}{N}}. \end{aligned} \tag{3.64}$$

It is easy to verify that

$$\begin{aligned} \int_{Q_T} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^{\frac{4}{N}+2} dxdt \ &\geq \ \int_{\{\Phi\geq k_{n+1}\}} \left[ \left( \sqrt{\Phi} - \sqrt{k_n} \right)^+ \right]^{\frac{4}{N}+2} dxdt \\ &\geq \ \left( \sqrt{k_{n+1}} - \sqrt{k_n} \right)^{\frac{4}{N}+2} |\{\Phi \geq k_{n+1}\}| \\ &\geq \ c \left( \frac{\sqrt{k}}{2^{n+2}} \right)^{\frac{4}{N}+2} |\{\Phi \geq k_{n+1}\}| \\ &\geq \ \frac{c|\{\Phi \geq k_{n+1}\}| k^{\frac{(N+2)}{N}}}{2^{(\frac{4}{N}+2)n}}. \end{aligned}$$

Combining this with (3.64) yields

$$\begin{aligned} y_{n+1} &= |\{\Phi \geq k_{n+1}\}|^{\frac{N}{N+2}+\frac{2}{N+2}} \\ &\leq c\frac{4^n I}{k}|\{\Phi \geq k_{n+1}\}|^{\frac{2}{N+2}} \\ &= \frac{c4^n}{A_r k}\left(\int_{\{\Phi\geq k_n\}} p^2 dxdt + |\{\Phi \geq k_n\}|\right)\{\Phi \geq k_{n+1}\}|^{\frac{2}{N+2}}. \end{aligned} \tag{3.65}$$

Now we turn our attention to $p$. Take the divergence of both sides of (1.1) to obtain

$$-\Delta p = \nabla\cdot(\Psi\nabla\phi) + \nabla\cdot(u\cdot\nabla u).$$

As in (3.16), we can also represent $p$ as

$$p(x,t) = \int_{\mathbb{R}^N}\Gamma(y-x)\left[\nabla\cdot(\Psi\nabla\phi) + \nabla\cdot(u\cdot\nabla u)\right]dy.$$

We observe from (1.5) that

$$\int_{\mathbb{R}^N}\Gamma(y-x)\nabla\cdot(u\cdot\nabla u)dy = \int_{\mathbb{R}^N}\Gamma_{y_iy_j}(y-x)u_iu_jdy.$$

It is a well known fact that $\partial^2_{y_iy_j}\Gamma(y)$ is a Calderón-Zygmund kernel. A result of [4] asserts that for each $\ell\in(1,\infty)$ there is a positive number $c_\ell$ determined by $N$ and $\ell$ such that

$$\left\|\int_{\mathbb{R}^N}\Gamma_{y_iy_j}(y-x)u_iu_jdy\right\|_{\ell,\mathbb{R}^N} \leq c_\ell\||u|^2\|_{\ell,\mathbb{R}^N}.$$

Combing this with Theorem 1 in ([17], p.119) yields

$$\|p\|_{\ell,\mathbb{R}^N} \leq c\|\Psi\nabla\phi\|_{\frac{N\ell}{N+\ell},\mathbb{R}^N} + c\||u|^2\|_{\ell,\mathbb{R}^N} \quad \text{for each } \ell > \tfrac{N}{N-1}. \tag{3.66}$$

Observe from (1.9) and (1.10) that

$$\|\Psi\nabla\phi\|_{\frac{N\ell}{N+\ell},\mathbb{R}^N} \leq c,$$

provided that

$$\ell > \frac{N}{N-2}.$$

As before, we pick

$$q > N+2.$$

Subsequently, by (3.66), we have

$$\int_{\{\Phi\geq k_n\}} p^2dxdt \leq \|p\|^2_{q,Q_T}|\{\Phi\geq k_n\}|^{1-\frac{2}{q}} \leq c(1+\|u\|^4_{2q,Q_T})|\{\Phi\geq k_n\}|^{1-\frac{2}{q}}.$$

Substitute this into (3.65) to get

$$y_{n+1} \leq \frac{c4^n}{A_r k}\left(1+\|u\|^4_{2q,Q_T} + |\{\Phi\geq k_n\}|^{\frac{2}{q}}\right)\{\Phi\geq k_{n+1}\}|^{1-\frac{2}{q}+\frac{2}{N+2}}. \tag{3.67}$$

Obviously, we may assume that

$$\|u\|^4_{2q,Q_T} \geq 1 + \left|\left\{\Phi \geq \frac{k}{2}\right\}\right|^{\frac{2}{q}}. \tag{3.68}$$

Consequently,

$$y_{n+1} \leq \frac{c4^n\|u\|^4_{2q,Q_T}}{A_r k}\{\Phi\geq k_{n+1}\}|^{1-\frac{2}{q}+\frac{2}{N+2}}.$$

We are in a position to repeat the argument in the proof of (1.9). Also see [18]. If (3.68) is not true, we can use the result in (3.67) to obtain the boundedness of $u$. The proof is rather standard. We shall omit here. The proof of Theorem 1.1 is now complete. □

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