# EXPONENTIAL DECAY FOR FRACTIONAL SCHRÖDINGER PARABOLIC PROBLEMS

JAN W. CHOLEWA[1] AND ANIBAL RODRIGUEZ-BERNAL[2]

[1]Institute of Mathematics
University of Silesia in Katowice
40-007 Katowice, Poland
E-mail: jan.cholewa@us.edu.pl

[2]Departamento de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
and
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM*, Spain
E-mail: arober@ucm.es

ABSTRACT. We discuss exponential decay in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$, of solutions of a fractional Schrödinger parabolic equation with a locally uniformly integrable potential. The exponential type of the semigroup of solutions is considered and its independence in of $1 \le p \le \infty$ is addressed. We characterise a large class of potentials for which solutions decay exponentially.

## 1. INTRODUCTION

In this paper we discuss exponential decay of solutions of fractional Schrödinger semigroups

$$\begin{cases} u_t + (-\Delta)^\mu u + V(x)u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with $0 < \mu \le 1$, $u_0 \in L^p(\mathbb{R}^N)$ with $1 \le p \le \infty$.

The nonnegative potential $V \ge 0$ belongs to the locally uniform space $L^{p_0}_U(\mathbb{R}^N)$, which, for $1 \le p_0 < \infty$, is composed of the functions $V \in L^{p_0}_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |V|^{p_0} \le C$$

endowed with the norm

$$\|V\|_{L^{p_0}_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|V\|_{L^{p_0}(B(x_0,1))}.$$

For $p_0 = \infty$ we define $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$.

For $\mu = 1$ and potentials $0 \le V \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ it was proved in [1] that the exponential decay in $L^p(\mathbb{R}^N)$ of solutions of (1.1) holds iff $V$ is sufficiently positive at infinity in the sense that

$$\int_G V(x)dx = \infty \tag{1.2}$$

for any open set $G \subset \mathbb{R}^N$ containing arbitrarily large balls, that is, such that for any $r > 0$ there exists $x_0 \in \mathbb{R}^N$ such that the ball of radius $r$ around $x_0$ is included in $G$.

*Date*: **April 26, 2024.**
Partially supported by Projects PID2019-103860GB-I00 and PID2022-137074NB-I00, MINECO, Spain.
Key words and phrases: fractional Schrödinger operator, linear evolution problem, stability.
Mathematical Subject Classification 2020: 35J10, 35R11, 47D06, 35B35.
*Partially supported by Severo Ochoa Grant CEX2019-000904-S funded by MCIN/AEI/10.13039/ 501100011033.

To extend this result to solutions of (1.1) with $0 < \mu \leq 1$ and $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ we first prove in Section 3 that if $p_0 > \max\{\frac{N}{2\mu}, 1\}$ then for any $1 \leq p \leq \infty$, (1.1) is well posed in $L^p(\mathbb{R}^N)$ and that it defines a contraction semigroup of solutions $u(t, u_0) = S_{\mu,V}(t)u_0$. This semigroup is order preserving, strongly continuous if $1 \leq p < \infty$ and analytic if $1 < p < \infty$, see Proposition 3.1. Moreover, for smooth initial data, solutions can be represented in terms of the fractional unperturbed semigroup (that is, when $V = 0$) and the variation of constants formula. Also, for nonnegative initial data, the larger the potential, the smaller the solution is, see Proposition 3.3. If $V \in L^\infty(\mathbb{R}^N)$ then the variation of constants formula applies to all initial data in $L^p(\mathbb{R}^N)$, see Proposition 3.4, and therefore it is natural to consider potentials $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ that can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded ones and to analyse the convergence of the corresponding semigroups, see Proposition 3.5 and Theorem 3.7.

With these tools, in Section 4 we address the exponential decay of solutions of (1.1) in $L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$ for potentials $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ satisfying (1.2). For this, we first prove that the exponential decay, actually, the exponential type of the semigroup $\{S_{\mu,V}(t)\}_{t\geq 0}$, does not depend on $1 \leq p \leq \infty$ and then characterise it in such a way that if solutions decay exponentially then (1.2) holds. Conversely, if the potential can be approximated by bounded ones, then (1.2) actually characterises the exponential decay, see Theorem 4.7.

Finally, in Section 2 we have included several known results for the unperturbed problem (1.1) with $V = 0$.

In this paper we denote by $c$ or $C$ generic constants that may change from line to line, whose value is not important for the results.

Also we will denote $A \sim B$ to denote quantities (norms or functions, for example) such that there exist positive constants $c_1, c_2$ such that $c_1 A \leq B \leq c_2 A$.

## 2. Basic results on fractional operators and semigroups

In this section we review several known results for the case of the unperturbed equation, that is, (1.1) with $V = 0$.

First, for the standard heat equation, $\mu = 1$, we have that for a wide class of initial data, including, $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$, the solution of (1.1) is given by

$$S(t)u_0(x) = \int_{\mathbb{R}^N} k(t,x,y)u_0(y)\, dy \quad t > 0,\ x \in \mathbb{R}^N, \tag{2.1}$$

for the selfsimilar convolution kernel

$$0 \leq k(t,x,y) = \frac{1}{t^{\frac{N}{2}}} k_0\left(\frac{x-y}{t^{\frac{1}{2}}}\right) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}}. \tag{2.2}$$

In particular, (2.1) defines an order preserving semigroup of contractions in $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$ and

$$\|S(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} = 1, \quad t > 0.$$

This semigroup is strongly continuous if $1 \leq p < \infty$ and analytic if $1 < p < \infty$. In particular, if $1 \leq p < \infty$ then for $u_0 \in L^p(\mathbb{R}^N)$ we have

$$S(t)u_0 \to u_0 \ \text{ as } \ t \to 0^+ \quad \text{in } L^p(\mathbb{R}^N) \text{ for any } 1 \leq p < \infty.$$

For all these classical results see e.g. [6].

If $p = \infty$ then for $u_0 \in L^\infty(\mathbb{R}^N)$

$$S(t)u_0 \to u_0 \ \text{ as } \ t \to 0^+ \quad \text{in } L^p_{loc}(\mathbb{R}^N) \text{ for any } 1 \leq p < \infty. \tag{2.3}$$

Finally, because of the regularity of the kernel, we have that for $1 \leq p \leq \infty$ and $u_0 \in L^p(\mathbb{R}^N)$, $u(t) = S(t)u_0$ in (2.1) is of class $C^\infty((0,\infty) \times \mathbb{R}^N)$ and satisfies the heat equation pointwise in $(0,\infty) \times \mathbb{R}^N$.

The semigroup of solutions possesses also several *smoothing* estimates among which we recall that for $1 \leq p \leq q \leq \infty$ there exists a constant $c_{p,q} > 0$ such that

$$\|S(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N), L^q(\mathbb{R}^N))} = \frac{c_{p,q}}{t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}}, \quad t > 0.$$

The fractional powers of $-\Delta$ can be defined in many equivalent ways, see [14]. From the heat semigroup (2.1) it can be defined as

$$(-\Delta)^{\mu}\phi(x) = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \left(S(t)\phi(x) - \phi(x)\right) \frac{dt}{t^{1+\mu}}$$

for $0 < \mu < 1$, which coincides with the nonlocal operator defined by

$$(-\Delta)^{\mu}\phi(x) = C_{N,\mu} \text{ P.V. } \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(z)}{|x-z|^{N+2\mu}} dz \tag{2.4}$$

with $C_{N,\mu} = \frac{2^{2\mu}\mu\Gamma(\frac{N}{2}+\mu))}{\pi^{N/2}\Gamma(1-\mu)}$, see [12]. When considered in $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ the domain of $(-\Delta)^{\mu}$ is the set of functions $\phi \in L^p(\mathbb{R}^N)$ such that $(-\Delta)^{\mu}\phi \in L^p(\mathbb{R}^N)$ with the natural norm

$$\|\phi\|_{L^p} + \|(-\Delta)^{\mu}\phi\|_{L^p},$$

see [7, Section A.2] when $1 \leq p < \infty$ and [8, Appendix A] when $p = \infty$. For $1 < p < \infty$ this domain coincides with the Bessel space $H_p^{2\mu}(\mathbb{R}^N)$. When $p = 2$ these spaces will be denoted by $H^{2\mu}(\mathbb{R}^N)$.

Using [7, Lemma 6.2] we have in particular, that for $1 < p < \infty$,

$$\|(-\Delta)^{\frac{1}{2}}\phi\|_{L^p(\mathbb{R}^N)} \sim \sum_{j=1}^{N} \|\partial_j \phi\|_{L^p(\mathbb{R}^N)} \quad \text{in } H_p^1(\mathbb{R}^N),$$

and for $\mu = \frac{1}{2} + \mu' \in (0,1)$

$$\|(-\Delta)^{\mu}\phi\|_{L^p(\mathbb{R}^N)} \sim \sum_{j=1}^{N} \|(-\Delta)^{\mu'}\partial_j \phi\|_{L^p(\mathbb{R}^N)}.$$

In particular, for $p = 2$ and for $\mu = \frac{1}{2} + \mu' \in (0,1)$

$$\|(-\Delta)^{\mu}\phi\|_{L^2(\mathbb{R}^N)} \sim \sum_{j=1}^{N} [\partial_j \phi]_{W^{2\mu',2}(\mathbb{R}^N)}$$

where in the right hand side we have the Gagliardo seminorms

$$[\phi]^p_{W^{\mu,p}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+\mu p}} \, dy dx.$$

The fractional semigroup, that is, the semigroup defined by the fractional Laplacian, can be constructed from the heat semigroup for $0 < \mu < 1$ and $1 \leq p < \infty$ as

$$S_{\mu}(t)\phi = \int_0^{\infty} f_{t,\mu}(s) S(s)\phi \, ds = \int_0^{\infty} f_{1,\mu}(s) S(st^{\frac{1}{\mu}})\phi \, ds, \quad t > 0 \tag{2.5}$$

and $S_{\mu}(0)\phi = \phi$ for $\phi \in L^p(\mathbb{R}^N)$, where $f_{t,\mu}$ is given by

$$0 \leq f_{t,\mu}(\lambda) = \begin{cases} \frac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^{\mu}} \, dz & \lambda \geq 0, \\ 0 & \lambda < 0, \end{cases}$$

$\sigma > 0$, and the branch for $z^{\mu}$ is chosen such that $Re(z^{\mu}) > 0$ if $Re(z) > 0$, and we have $\int_0^{\infty} f_{t,\mu}(s) \, ds = 1$, see [7, Appendix A.3], [19, p. 259] and [19, (20'), p. 264]. This semigroup is bounded, strongly continuous and analytic, see [7, Corollary 4.1] and [7, Appendix A.3]. For $p = \infty$ the same construction is possible, although the semigroup is bounded and analytic but not strongly continuous, see [8, Proposition B.1]. In particular, for $1 \leq p \leq \infty$ and $\mu \in (0,1)$, as the semigroup is analytic, see [15, Proposition 2.1.1], we have that $u(t) = S_\mu(t)u_0$ with $u_0 \in L^p(\mathbb{R}^N)$ solves

$$u_t + (-\Delta)^\mu u = 0, \quad x \in \mathbb{R}^N, \ t > 0, \qquad u(0) = u_0$$

and for $1 \leq p < \infty$ we have

$$S_\mu(t)u_0 \to u_0 \ \text{ as } \ t \to 0^+ \quad \text{in } L^p(\mathbb{R}^N).$$

When $p = \infty$ the following holds.

**Proposition 2.1.** *For $u_0 \in L^\infty(\mathbb{R}^N)$ and $0 < \mu < 1$*

$$S_\mu(t)u_0 \to u_0 \quad \textit{as} \quad t \to 0^+ \quad \textit{in } L^p_{loc}(\mathbb{R}^N) \textit{ for any } 1 \leq p < \infty.$$

**Proof.** Using the second expression in (2.5) and [19, formula (14), p. 262], if $u_0 \in L^\infty(\mathbb{R}^N)$ and we take a ball $B \subset \mathbb{R}^N$, using (2.3) and Lebesgue's dominated convergence theorem, we get

$$\|S_\mu(t)u_0 - u_0\|_{L^p(B)} \leq \int_0^\infty f_{1,\mu}(s)\|S_1(st^{\frac{1}{\mu}})u_0 - u_0\|_{L^p(B)} \to 0 \ \text{ as } \ t \to 0^+.$$

■

Also, from (2.5) and the properties of the heat semigroup, it is easy to obtain that the fractional semigroup is of contractions and order preserving in $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. Actually $\|S_\mu(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} = 1$ for all $t \geq 0$, see e.g. [7, Corollary 3.7]. Moreover it satisfies, among other, the smoothing estimates

$$\|S_\mu(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N), L^q(\mathbb{R}^N))} = \frac{c_{p,q,\mu}}{t^{\frac{N}{2\mu}(\frac{1}{p}-\frac{1}{q})}}, \quad t > 0, \tag{2.6}$$

for $1 \leq p \leq q \leq \infty$, see [5, Theorem 6.2] and [7, Proposition 6.5].

We also have for $0 \leq \gamma \leq \gamma' \leq 1$, $1 < p \leq q < \infty$

$$\|S_\mu(t)\|_{\mathcal{L}(H^{2\gamma}_p(\mathbb{R}^N), H^{2\gamma'}_q(\mathbb{R}^N))} \leq \frac{1}{t^{\frac{N}{2m\mu}(\frac{1}{p}-\frac{1}{q})}} \max\{c_{p,q,\mu}, \frac{c_{\gamma,\gamma',p,q,\mu}}{t^{\frac{\gamma'-\gamma}{\mu}}}\}, \quad t > 0. \tag{2.7}$$

The fractional semigroup has a $C^\infty$, positive convolution self similar kernel such that

$$S_\mu(t)u_0(x) = \int_{\mathbb{R}^N} k_\mu(t,x,y)u_0(y)\,dy \quad t > 0, \ x \in \mathbb{R}^N, \tag{2.8}$$

$$0 \leq k_\mu(t,x,y) = \frac{1}{t^{\frac{N}{2\mu}}} k_{0,\mu}\left(\frac{x-y}{t^{\frac{1}{2\mu}}}\right) \sim \frac{1}{t^{\frac{N}{2\mu}}} H_\mu\left(\frac{x-y}{t^{\frac{1}{2\mu}}}\right), \tag{2.9}$$

where

$$H_\mu(z) = \min\left\{1, \frac{1}{|z|^{N+2\mu}}\right\} \sim I_\mu(z) = \frac{1}{(1+|z|^2)^{\frac{N+2\mu}{2}}}, \quad z \in \mathbb{R}^N. \tag{2.10}$$

The profile $k_{0,\mu}$ is even and satisfies

$$(-\Delta)^\mu k_{0,\mu} = \frac{x}{2\mu}\nabla k_{0,\mu} + \frac{N}{2\mu}k_{0,\mu},$$

see [9, Section 6], [4, 5]. Since the kernel is $C^\infty$ so the solution of the fractional heat equation in (2.8) is $C^\infty((0,\infty) \times \mathbb{R}^N)$, see [5].

## 3. The fractional Schrödinger semigroup

In this section we study the perturbed equations

$$\begin{cases} u_t + (-\Delta)^\mu u + V(x)u = 0, & x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases}$$

for potentials in the uniform space $L^{p_0}_U(\mathbb{R}^N)$, wit $p_0 \geq 1$, that is, satisfying

$$\|V\|_{L^{p_0}_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|V\|_{L^{p_0}(B(x_0,1))} < \infty.$$

**Proposition 3.1.** *Assume $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$, for $p_0 > \frac{N}{2\mu}$.*

*Then $-(-\Delta)^\mu - V(x)$ defines an order preserving semigroup of contractions in $L^p(\mathbb{R}^N)$ for any $1 \leq p \leq \infty$ which is strongly continuous if $1 \leq p < \infty$ and analytic for $1 < p < \infty$ and that we denote $\{S_{\mu,V}(t)\}_{t \geq 0}$.*

*These semigroups are consistent in the sense that the semigroup in $L^p(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ give the same result for any $t > 0$ if $u_0 \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.*

*Also the semigroup in $L^{p'}(\mathbb{R}^N)$ is the adjoint of the semigroup in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

**Proof.** Consider the bilinear symmetric, nonnegative definite form

$$a(\phi, \psi) = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\mu}{2}} \phi (-\Delta)^{\frac{\mu}{2}} \psi + \int_{\mathbb{R}^N} V(x) \phi \psi.$$

We first prove that is is well defined in the Bessel space $H^\mu(\mathbb{R}^N)$. For this, denote by $\{Q_i\}$ the family of cubes centered at points of integer coordinates in $\mathbb{R}^N$ and with edges of length 1 parallel to the axes. Thus if $V \in L^{p_0}_U(\mathbb{R}^N)$, for any $\phi, \psi \in H^\mu(\mathbb{R}^N)$ we have, using Hölder's inequality,

$$\int_{\mathbb{R}^N} |V(x)\phi\psi| = \sum_i \int_{Q_i} |V(x)||\phi||\psi| \leq \sum_i \|V\|_{L^{p_0}(Q_i)} \|\phi\|_{L^r(Q_i)} \|\psi\|_{L^r(Q_i)}$$

with $\frac{1}{p_0} + \frac{2}{r} = 1$ and $H^\mu(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$, that is $\mu - \frac{N}{2} \geq -\frac{N}{r}$. This implies in turn that we must have $p_0 \geq \frac{N}{2\mu}$ (with strict inequality if $p_0 = 1$ which implies $r = \infty$). Hence, using embedding $H^\mu(Q_i) \subset L^r(Q_i)$, with constants independent of $i$, we get

$$\int_{\mathbb{R}^N} |V(x)||\phi||\psi| \leq C \|V\|_{L^{p_0}_U(\mathbb{R}^N)} \sum_i \|\phi\|_{L^r(Q_i)} \|\psi\|_{L^r(Q_i)}$$

$$\leq C \|V\|_{L^{p_0}_U(\mathbb{R}^N)} \Big( \sum_i \|\phi\|^2_{H^\mu(Q_i)} \Big)^{\frac{1}{2}} \Big( \sum_i \|\psi\|^2_{H^\mu(Q_i)} \Big)^{\frac{1}{2}} \leq C \|V\|_{L^{p_0}_U(\mathbb{R}^N)} \|\phi\|_{H^\mu(\mathbb{R}^N)} \|\psi\|_{H^\mu(\mathbb{R}^N)},$$

by Lemma 3.2 below.

Now if $\phi \in H^\mu(\mathbb{R}^N)$ then $|\phi| \in H^\mu(\mathbb{R}^N)$. To see this we will use the generalised Strook-Varopoulos inequality, see [11, Lemma 5.2], for $g(\phi), (-\Delta)^\mu \phi \in L^2(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} g(\phi)(-\Delta)^\mu \phi \geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} G(\phi)|^2 \geq 0 \tag{3.1}$$

with $g' = (G')^2$. In particular, with $g(s) = s$ and $G(s) = |s|$ we get for $\phi, (-\Delta)^\mu \phi \in L^2(\mathbb{R}^N)$, that is, for $\phi \in H^{2\mu}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \phi(-\Delta)^\mu \phi \geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} |\phi||^2 \geq 0$$

and we get, for $\phi \in H^{2\mu}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} |\phi||^2 \leq \int_{\mathbb{R}^N} \phi(-\Delta)^\mu \phi = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} \phi|^2.$$

For $\phi \in H^\mu(\mathbb{R}^N)$, by density,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} |\phi||^2 \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} \phi|^2.$$

Thus $|\phi| \in H^\mu(\mathbb{R}^N)$.

In particular, if $V \geq 0$ we have

$$a(|\phi|, |\phi|) \leq a(\phi, \phi), \qquad \phi \in H^\mu(\mathbb{R}^N). \tag{3.2}$$

On the other hand, if $0 \le \phi \in H^\mu(\mathbb{R}^N)$ then $v = \min\{\phi, 1\} \in H^\mu(\mathbb{R}^N)$. To see this observe that $v = g(\phi)$ with $g$ Lipschitz and $g'(s) = \mathcal{X}_{[0,1]}(s)$. Then in (3.1) we have $G(s) = g(s)$ and for smooth $\phi$

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} g(\phi)|^2 \le \int_{\mathbb{R}^N} g(\phi)(-\Delta)^\mu \phi = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\mu}{2}} g(\phi)(-\Delta)^{\frac{\mu}{2}} \phi$$

and Hölder's inequality gives

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} g(\phi)|^2 \le \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}} \phi|^2, \quad \phi \in H^{2\mu}(\mathbb{R}^{\mathbb{N}}).$$

Thus $g(\phi) \in H^\mu(\mathbb{R}^N)$. By density, if $\phi_n \in H^{2\mu}(\mathbb{R}^N)$ converges in $H^\mu(\mathbb{R}^N)$ to $\phi$, the inequality above and $g(\phi_n) \to g(\phi)$ in $L^2(\mathbb{R}^N)$ implies that $g(\phi_n) \to g(\phi)$ weakly in $H^\mu(\mathbb{R}^N)$ and we extend the inequality to $\phi \in H^\mu(\mathbb{R}^N)$.

In particular, if $V \ge 0$ we have

$$a(v, v) \le a(\phi, \phi), \qquad \phi \in H^\mu(\mathbb{R}^N). \tag{3.3}$$

Then from (3.2) and (3.3), [10, Theorem 1.3.2, pag 12] and [10, Theorem 1.3.3, pag 14], we have an order preserving semigroup of contractions in $L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$. Moreover, from [10, Theorem 1.4.1, pag 22] the semigroup in $L^p(\mathbb{R}^N)$ is the extension of the one in $L^2(\mathbb{R}^N)$, the semigroup in $L^p(\mathbb{R}^N)$ is the adjoint of the semigroup in $L^{p'}(\mathbb{R}^N)$ for $1 < p \le \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Also, the semigroup is analytic for $1 < p < \infty$ and strongly continuous for $1 \le p < \infty$. ■

The following result, used above, was proved in Lemma 2.4 in [2].

**Lemma 3.2.** *Let $\{Q_i\}$ be the family of cubes centered at points of integer coordinates in $\mathbb{R}^N$ and with edges of length 1 parallel to the axes.*

*Then for any $0 \le s \le 2$ and $1 < p < \infty$*

$$\sum_i \|\phi\|^p_{H^s_p(Q_i)} \le C \|\phi\|^p_{H^s_p(\mathbb{R}^N)} \qquad \text{for all} \quad \phi \in H^s_p(\mathbb{R}^N).$$

The next result shows in particular that, for suitable $p$, the semigroup $\{S_{\mu,V}(t)\}_{t \ge 0}$ above in $L^p(\mathbb{R}^N)$ can be represented in terms of the fractional semigroup $\{S_\mu(t)\}_{t \ge 0}$ and the variation of constants formula (a.k.a. Duhamel's principle), at least for smooth initial data.

**Proposition 3.3.** *For $0 \le V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$, the semigroup $u(t) = S_{\mu,V}(t)u_0$ satisfies the following.*

(i) *For $1 < p < \infty$ and $p \le p_0$, the operator $-(-\Delta)^\mu - V$ with domain $D((-\Delta)^\mu) = H^{2\mu}_p(\mathbb{R}^N)$ is a sectorial operator and it is the generator of the analytic semigroup $S_{\mu,V}(t)$. Moreover, for $u_0 \in H^{2\mu}_p(\mathbb{R}^N)$, we have $u \in C([0,\infty), H^{2\mu}_p(\mathbb{R}^N)) \cap C^1((0,\infty), L^p(\mathbb{R}^N))$ and*

$$u(t) = S_\mu(t)u_0 - \int_0^t S_\mu(t-s)Vu(s)\, ds, \quad t > 0.$$

(ii) *For $1 \le p \le \infty$ and $u_0 \in L^p(\mathbb{R}^N)$,*

$$|S_{\mu,V}(t)u_0| \le S_{\mu,V}(t)|u_0| \le S_\mu(t)|u_0|.$$

*In particular, the semigroup $S_{\mu,V}(t)$ satisfies the smoothing estimates (2.6).*

(iii) *For $1 \le p \le \infty$ and $0 \le u_0 \in L^p(\mathbb{R}^N)$, if $V_1 \ge V_2$ then for $u_0 \ge 0$ we have $0 \le S_{\mu,V_1}(t)u_0 \le S_{\mu,V_2}(t)u_0$.*

**Proof.** (i) We show that the domain of $(-\Delta)^\mu + V$ coincides with the domain $D((-\Delta)^\mu) = H^{2\mu}_p(\mathbb{R}^N)$ and is a sectorial operator. Actually, denote by $\{Q_i\}$ the family of cubes centered at points of integer coordinates in $\mathbb{R}^N$ and with edges of length 1 parallel to the axes. Thus if $V \in L^{p_0}_U(\mathbb{R}^N)$, for any $\phi \in H^{2\mu}_p(\mathbb{R}^N)$ we have, using Hölder's inequality,

$$\|V\phi\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |V|^p |\phi|^p = \sum_i \int_{Q_i} |V|^p |\phi|^p \le \sum_i \|V\|^p_{L^{p_0}(Q_i)} \|\phi\|^p_{L^r(Q_i)}$$

provided we chose $r$ such that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{r}$, which is possible since $p_0 \geq p$. Now we use the Sobolev embedding

$$\|\phi\|_{L^r(Q_i)} \leq C\|\phi\|_{H_p^{2\mu s}(Q_i)}$$

with constants independent of $i$, for $-\frac{N}{r} \leq 2\mu s - \frac{N}{p}$, which requires $\frac{N}{p_0} \leq 2\mu s$. Hence we get, using Lemma 3.2,

$$\|V\phi\|^p_{L^p(\mathbb{R}^N)} \leq C\|V\|^p_{L^{p_0}_U(\mathbb{R}^N)} \sum_i \|\phi\|^p_{H_p^{2\mu s}(Q_i)} \leq C\|V\|^p_{L^{p_0}_U(\mathbb{R}^N)} \|\phi\|^p_{H_p^{2\mu s}(\mathbb{R}^N)}. \tag{3.4}$$

Then, since $p_0 > \frac{N}{2\mu}$ we can take $0 < s < 1$ above and then by interpolation

$$\|V\phi\|_{L^p(\mathbb{R}^N)} \leq C\|V\|_{L^{p_0}_U(\mathbb{R}^N)} \|\phi\|_{H_p^{2\mu s}(\mathbb{R}^N)} \leq C\|V\|_{L^{p_0}_U(\mathbb{R}^N)} \|\phi\|^s_{H_p^{2\mu}(\mathbb{R}^N)} \|\phi\|^{1-s}_{L^p(\mathbb{R}^N)}$$

and then, for any $\varepsilon > 0$,

$$\|V\phi\|_{L^p(\mathbb{R}^N)} \leq \varepsilon\|(-\Delta)^\mu\phi\|_{L^p(\mathbb{R}^N)} + C_\varepsilon\|\phi\|_{L^p(\mathbb{R}^N)}, \qquad \phi \in H_p^{2\mu}(\mathbb{R}^N). \tag{3.5}$$

The sectoriality result follows from Theorem 2.1 in Chapter 3 in [16]. The regularity of $u(t) = S_{\mu,V}(t)u_0$ when $u_0 \in H_p^{2\mu}(\mathbb{R}^N)$ also follows from Chapter 1 in [16]. In this case $u$ satisfies

$$u_t = -(-\Delta)^\mu u - Vu, \quad t > 0$$

and we get the integral representation of $u$ as $Vu \in C([0,\infty), L^p(\mathbb{R}^N))$, see Chapter 4, Section 4.2 in [16].

(ii) Since $S_{\mu,V}(t)$ is order preserving, it is enough to prove the second inequality because for any order preserving semigroup in $L^p(\mathbb{R}^N)$, since $-|u_0| \leq u_0 \leq |u_0|$ we have

$$|S(t)u_0| \leq S(t)|u_0|.$$

First, for $1 < p \leq p_0$, if $0 \leq u_0 \in H_p^{2\mu}(\mathbb{R}^N)$ we know that $V \geq 0$, $u(t) = S_{\mu,V}(t)u_0 \geq 0$ and $S_\mu(t)$ is order preserving and then by (i) $\int_0^t S_\mu(t-s)Vu(s)\,ds \geq 0$ and

$$u(t) = S_\mu(t)u_0 - \int_0^t S_\mu(t-s)Vu(s)\,ds \leq S_\mu(t)u_0, \quad t > 0$$

and (ii) holds for such initial data. By density we get the result for $0 \leq u_0 \in L^p(\mathbb{R}^N)$. For initial data in $L^p(\mathbb{R}^N)$ with $p_0 < p < \infty$ the result follows again by density from the case above.

Finally, for $p = \infty$ and $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$, consider $0 \leq \varphi \in L^1(\mathbb{R}^N)$ and then

$$\langle S_{\mu,V}(t)u_0, \varphi\rangle = \langle u_0, S_{\mu,V}(t)\varphi\rangle \leq \langle u_0, S_\mu(t)\varphi\rangle = \langle S_\mu(t)u_0, \varphi\rangle$$

and we get the pointwise inequality.

Now it is clear that the estimates (2.6) apply to $S_{\mu,V}(t)$.

(iii) Again, first for $1 < p \leq p_0$, if $0 \leq u_0 \in H_p^{2\mu}(\mathbb{R}^N)$ and $u_i(t) = S_{\mu,V_i}(t)u_0 \geq 0$, by (i) $u_2(t)$ satisfies

$$(u_2)_t + (-\Delta)^\mu u_2 + V_1(x)u_2 + \big(V_2(x) - V_1(x)\big)u_2 = 0, \qquad x \in \mathbb{R}^N, \quad t > 0$$

and then, since $0 \leq \big(V_1 - V_2\big)u_2 \in C([0,\infty), L^p(\mathbb{R}^N))$,

$$u_2(t) = S_{\mu,V_1}(t)u_0 + \int_0^t S_{\mu,V_1}(t-s)(V_1 - V_2)u_2(s)\,ds \geq S_{\mu,V_1}(t)u_0, \quad t > 0.$$

By density, (iii) is proved for this range of $p$. For initial data in $L^p(\mathbb{R}^N)$ with $p_0 < p < \infty$ the result follows again by density from the case above.

Finally, for $p = \infty$ and $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$, consider $0 \leq \varphi \in L^1(\mathbb{R}^N)$ and then

$$\langle S_{\mu,V_1}(t)u_0, \varphi\rangle = \langle u_0, S_{\mu,V_1}(t)\varphi\rangle \leq \langle u_0, S_{\mu,V_2}(t)\varphi\rangle = \langle S_{\mu,V_2}(t)u_0, \varphi\rangle$$

and we get the pointwise inequality. ■

In the case of a bounded potential $0 \leq V \in L^\infty(\mathbb{R}^N)$, part (i) in Proposition 3.3 can be somehow improved and for all $1 \leq p \leq \infty$, the semigroup $\{S_{\mu,V}(t)\}_{t\geq 0}$ above in $L^p(\mathbb{R}^N)$ can be represented by the variation of constants formula for all initial data. Observe that actually the sign condition on $V$ is not needed in the proof below.

**Proposition 3.4.** *Assume $0 \leq V \in L^{\infty}(\mathbb{R}^N)$. Then for $1 \leq p < \infty$ the operator $-(-\Delta)^{\mu} - V$ with domain $D((-\Delta)^{\mu}) = H^{2\mu}_p(\mathbb{R}^N)$ is the generator of the semigroup $S_{\mu,V}(t)$.*

*For $1 \leq p \leq \infty$ and $u_0 \in L^p(\mathbb{R}^N)$, we have that $u(t) = S_{\mu,V}(t)u_0$ satisfies*

$$u(t) = S_{\mu}(t)u_0 - \int_0^t S_{\mu}(t-s)Vu(s)\,ds, \quad t > 0.$$

**Proof.** For $1 \leq p < \infty$ the semigroup $\{S_{\mu}(t)\}_{t\geq 0}$ is strongly continuous and the multiplication by $V$ is a bounded operator in $L^p(\mathbb{R}^N)$. Hence the result follows from Section 3.1 in [16].

Before dealing with the case $p = \infty$ observe first that for $1 \leq p \leq \infty$ and $T_0\|V\|_{L^{\infty}} < 1$ the mapping

$$\mathcal{F}_p(u)(t) = S_{\mu}(t)u_0 - \int_0^t S_{\mu}(t-s)Vu(s)\,ds$$

defines a contraction in $L^{\infty}((0,T_0), L^p(\mathbb{R}^N))$ as

$$\|\mathcal{F}_p(u)(t)\|_{L^p} \leq \|u_0\|_{L^p} + \int_0^t \|V\|_{L^{\infty}}\|u(s)\|_{L^p}\,ds$$

so $\mathcal{F}_p(u) \in L^{\infty}((0,T_0), L^p(\mathbb{R}^N))$ and

$$\|\mathcal{F}_p(u)(t) - \mathcal{F}_p(v)(t)\|_{L^p} \leq \int_0^t \|V\|_{L^{\infty}}\|u(s)-v(s)\|_{L^p}\,ds \leq T_0\|V\|_{L^{\infty}} \sup_{0\leq t\leq T_0} \|u(s)-v(s)\|_{L^p}.$$

Take $u_1(t)$ for $0 \leq t \leq T_0$ the unique fixed point in $L^{\infty}((0,T_0), L^p(\mathbb{R}^N))$ and consider for $t \geq T_0$,

$$\mathcal{H}_p(u)(t) = S_{\mu}(t-\tau)u_1(T_0) - \int_{T_0}^t S_{\mu}(t-s)Vu(s)\,ds.$$

With similar estimates as above it is immediate to get that this is also a contraction in $L^{\infty}((T_0,2T_0), L^p(\mathbb{R}^N))$ and has a unique fixed point $u_2(t)$ for $T_0 \leq t \leq 2T_0$. Then it is easy to get that the function

$$u(t) = \begin{cases} u_1(t), & 0 \leq t \leq T_0 \\ u_2(t), & T_0 \leq t \leq 2T_0 \end{cases}$$

is a fixed point of $\mathcal{F}_p$ in $L^{\infty}((0,2T_0), L^p(\mathbb{R}^N))$. Proceeding by induction we get a fixed point of $\mathcal{F}_p$ in $L^{\infty}((0,kT_0), L^p(\mathbb{R}^N))$ for any $k \in \mathbb{N}$. The uniqueness of the fixed point for $\mathcal{F}_p$ follows from Gronwall's lemma since for two such fixed points and $t > 0$

$$\|u(t) - v(t)\|_{L^p} \leq \int_0^t \|V\|_{L^{\infty}}\|u(s)-v(s)\|_{L^p}\,ds.$$

In particular, for $1 \leq p < \infty$ and $T > 0$, $u(t) = S_{\mu,V}(t)u_0$ for $0 \leq t \leq T$ is the unique fixed point of $\mathcal{F}_p$ in $L^{\infty}((0,T), L^p(\mathbb{R}^N))$ for all $T > 0$.

Hence, if $u_0 \in L^p(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $p > \frac{N}{2\mu}$ then the semigroup solution in $L^p(\mathbb{R}^N)$, $u(t) = S_{\mu,V}(t)u_0$ satisfies, using the integral expression and (2.6),

$$\|u(t)\|_{L^{\infty}} \leq \|u_0\|_{L^{\infty}} + \int_0^t \frac{1}{(t-s)^{\frac{N}{2p\mu}}}\|V\|_{L^{\infty}}\|u(s)\|_{L^p}\,ds$$

and therefore for any $T > 0$, $S_{\mu,V}(\cdot)u_0 \in L^{\infty}((0,T), L^{\infty}(\mathbb{R}^N))$ and so it is the fixed point of $\mathcal{F}_{\infty}$ in $[0,T]$ in $L^{\infty}((0,T), L^{\infty}(\mathbb{R}^N))$.

Now for $u_0 \in L^{\infty}(\mathbb{R}^N)$ take a sequence $u_0^n \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ with $p > \frac{N}{2\mu}$ such that $u_0^n \to u_0$ weak-* in $L^{\infty}(\mathbb{R}^N)$ (for example, take the truncation by zero of $u_0$ outside the ball $B(0,n) \subset \mathbb{R}^N$). Then for any $\varphi \in L^1(\mathbb{R}^N)$ and $t \geq 0$, as $n \to \infty$ we have, since $S_{\mu,V}(t)\varphi \in L^1(\mathbb{R}^N)$,

$$\langle S_{\mu,V}(t)u_0^n, \varphi\rangle = \langle u_0^n, S_{\mu,V}(t)\varphi\rangle \to \langle u_0, S_{\mu,V}(t)\varphi\rangle = \langle S_{\mu,V}(t)u_0, \varphi\rangle$$

i.e. $S_{\mu,V}(t)u_0^n \to S_{\mu,V}(t)u_0$ weak-* in $L^{\infty}(\mathbb{R}^N)$.

On the other hand, setting $u^n(t) = S_{\mu,V}(t)u_0^n$ and $u(t) = S_{\mu,V}(t)u_0$, we have,

$$\langle u^n(t), \varphi\rangle = \langle u_0^n, S_\mu(t)\varphi\rangle - \int_0^t \langle u^n(s), VS_\mu(t-s)\varphi\rangle.$$

Therefore, for fixed $t > 0$ and for $0 < s < t$ we have, as $n \to \infty$,

$$\langle u^n(s), VS_\mu(t-s)\varphi\rangle \to \langle u(s), VS_\mu(t-s)\varphi\rangle$$

and for all $n \in \mathbb{N}$

$$|\langle u^n(s), VS_\mu(t-s)\varphi\rangle| \le \|VS_\mu(t-s)\varphi\|_{L^1} = g(s)$$

with $g \in L^1(0,t)$. Then Lebesgue's theorem implies that, as $n \to \infty$, we get

$$\langle u(t), \varphi\rangle = \langle u_0, S_\mu(t)\varphi\rangle - \int_0^t \langle u(s), VS_\mu(t-s)\varphi\rangle.$$

Hence, for all $T > 0$, $u(\cdot) = S_{\mu,V}(\cdot)u_0 \in L^\infty((0,T), L^\infty(\mathbb{R}^N))$ satisfies, for all $\varphi \in L^1(\mathbb{R}^N)$,

$$\langle u(t), \varphi\rangle = \langle S_\mu(t)u_0, \varphi\rangle - \int_0^t \langle S_\mu(t-s)Vu(s), \varphi\rangle, \quad 0 < t < T$$

and therefore

$$u(t) = S_\mu(t)u_0 - \int_0^t S_\mu(t-s)Vu(s)\, ds$$

and $u(\cdot) = S_{\mu,V}(\cdot)u_0$ is the unique fixed point of of $\mathcal{F}_\infty$ in $[0,T]$ in $L^\infty((0,T), L^\infty(\mathbb{R}^N))$. ■

In view of Propositions 3.3 and 3.4, given $1 \le p_0 < \infty$ we want to discuss the class of $V \in L^{p_0}_U(\mathbb{R}^N)$ that can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions and to analyse the convergence of the corresponding semigroups. Then we have the following result. Notice that below we use the class $\dot{L}^{p_0}_U(\mathbb{R}^N)$ of functions in $L^{p_0}_U(\mathbb{R}^N)$ such that the translations are continuous in the $L^{p_0}_U(\mathbb{R}^N)$ norm, that is, $V \in \dot{L}^{p_0}_U(\mathbb{R}^N)$ iff

$$\|\tau_y V - V\|_{L^{p_0}_U(\mathbb{R}^N)} \to 0, \quad \text{as } |y| \to 0$$

where $y \in \mathbb{R}^N$ and $\tau_y V(x) = V(x-y)$. This is a closed proper subspace of $L^{p_0}_U(\mathbb{R}^N)$, see [3].

**Proposition 3.5.** (i) *For $1 \le p_0 < \infty$ a function $V \in L^{p_0}_U(\mathbb{R}^N)$ can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions if and only if, defining*

$$V_M(x) = \begin{cases} M & \text{if } V(x) > M \\ V(x) & \text{if } -M \le V(x) \le M, \\ -M & \text{if } V(x) < -M \end{cases} \qquad M > 0,$$

*we have $V_M \xrightarrow[M\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$.*

(ii) *If $V \in \dot{L}^{p_0}_U(\mathbb{R}^N)$ then $V$ can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions.*

(iii) *Assume $V$ is not too large at infinity in the sense that there exists $M_0$ such that*

$$\lim_{|x|\to\infty} \int_{B(x,1)} |V(y) - V_{M_0}(y)|^{p_0}\, dy = 0.$$

*Then $V_M \xrightarrow[M\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$.*

(iv) *The class of functions in $L^{p_0}_U(\mathbb{R}^N)$ that can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions is a closed proper subspace of $L^{p_0}_U(\mathbb{R}^N)$.*

**Proof.** (i) Assume $V$ can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions. Then there exists $W_n \in L^\infty(\mathbb{R}^N)$ such that $W_n \xrightarrow[n\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$. Letting $M_n = \|W_n\|_{L^\infty}$, we have, for every $x \in \mathbb{R}^N$,

$$\int_{B(x,1)} |V(y) - V_{M_n}(y)|^{p_0}\, dy = \int_{B(x,1)\cap\{V\ge M_n\}} |V(y) - M_n|^{p_0}\, dy + \int_{B(x,1)\cap\{V\le -M_n\}} |V(y) + M_n|^{p_0}\, dy$$

and thus

$$\int_{B(x,1)} |V(y) - V_{M_n}(y)|^{p_0}\, dy \le \int_{B(x,1)} |V(y) - W_n(y)|^{p_0}\, dy.$$

Therefore $V_{M_n} \xrightarrow[n\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$. Since clearly $\|V - V_M\|_{L^{p_0}_U(\mathbb{R}^N)}$ decreases with $M$, then $V_M \xrightarrow[M\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$.

The converse is immediate.

(ii) If $V \in \dot{L}^{p_0}_U(\mathbb{R}^N)$ from the results in [3] we know that the solution of the heat equation satisfies $S(t)V \xrightarrow[t\to 0^+]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$ and $S(t)V \in L^\infty(\mathbb{R}^N)$.

(iii) Define the family of bounded continuous functions in $\mathbb{R}^N$, $H_M(x) = \int_{B(x,1)} |V(y) - V_M(y)|^{p_0}\, dy$ which are decreasing in $M$ and converge to 0 as $M \to \infty$ uniformly in bounded sets, by Lebesgue's theorem.

Then the assumption reads $\lim_{|x|\to\infty} H_{M_0}(x) = 0$. Therefore, given $\varepsilon > 0$ there exists $R > 0$ such that for $|x| > R$ and $M > M_0$ we have

$$0 \le H_M(x) \le H_{M_0}(x) \le \varepsilon.$$

This and the uniform convergence $H_M(x) \xrightarrow[M\to\infty]{} 0$ for $|x| \le R$ implies $H_M \xrightarrow[M\to\infty]{} 0$ uniformly in $\mathbb{R}^N$ and hence $V_M \xrightarrow[M\to\infty]{} V$ in $L^{p_0}_U(\mathbb{R}^N)$.

(iv) The closedness is immediate as this class is the closure of $L^\infty(\mathbb{R}^N)$ in $L^{p_0}_U(\mathbb{R}^N)$.

To show the subspace is proper, consider a sequence $|x_n| \to \infty$ with $|x_n - x_m| > 2$ for all $n, m \in \mathbb{N}$. Then define $V(x) = \sum_n n^{\frac{N}{p_0}} \mathcal{X}_{B(x_n, \frac{1}{n})}(x)$. Clearly $\|V\|_{L^{p_0}(B(x,1))} \le c$ for all $x \in \mathbb{R}^N$ since $B(x,1)$ contains at most one point of $\{x_n\}_n$, so $V \in L^{p_0}_U(\mathbb{R}^N)$. Also, for any $M > 0$ if $n > M$ we have

$$\|V - V_M\|^{p_0}_{L^{p_0}(B(x_n,1))} \ge c\frac{(n^{\frac{N}{p_0}} - M)^{p_0}}{n^N} \xrightarrow[n\to\infty]{} c > 0$$

and therefore $\|V - V_M\|^{p_0}_{L^{p_0}_U(\mathbb{R}^N)} \ge c > 0$ for all $M > 0$. ∎

**Remark 3.6.** (i) *Assume $V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > 1$ and can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions. Then $V$ can be approximated in $L^1_U(\mathbb{R}^N)$ by bounded functions. To see this just observe that uniform spaces are nested, that is, $L^p_U(\mathbb{R}^N) \subset L^q_U(\mathbb{R}^N)$, if $p > q$.*

(ii) *Conversely, if $V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > 1$ and can be approximated in $L^1_U(\mathbb{R}^N)$ by bounded functions, then for every $1 < q < p_0$, $V$ can be approximated in $L^q_U(\mathbb{R}^N)$ by bounded functions. To see this, just notice that like in bounded domains, we have the interpolation inequality*

$$\|f\|_{L^q_U} \le \|f\|^{1-\theta}_{L^{p_0}_U} \|f\|^{\theta}_{L^1_U}, \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{1}, \quad 0 < \theta < 1,$$

*for any $f \in L^{p_0}_U(\mathbb{R}^N)$.*

Now we analize the approximation of the corresponding semigroups. For this assume now $0 \le V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$ and consider the increasing sequence $0 \le V_M \le V$. Then, by Proposition 3.3, we have, for $1 \le p \le \infty$ and $0 \le u_0 \in L^p(\mathbb{R}^N)$,

$$0 \le S_{\mu,V}(t)u_0 \le S_{\mu,V_M}(t)u_0$$

and for $M_1 \le M_2$

$$0 \le S_{\mu,V_{M_2}}(t)u_0 \le S_{\mu,V_{M_1}}(t)u_0.$$

Hence, the monotonic decreasing limit

$$\tilde{S}_{\mu,V}(t)u_0 \stackrel{\text{def}}{:=} \lim_{M\to\infty} S_{\mu,V_M}(t)u_0$$

exists pointwise and in $L^p(\mathbb{R}^N)$ for all $t>0$ and satisfies $0 \le S_{\mu,V}(t)u_0 \le \tilde{S}_{\mu,V}(t)u_0$, $\tilde{S}_{\mu,V}(t+s)u_0 = \tilde{S}_{\mu,V}(t)\tilde{S}_{\mu,V}(s)u_0$ and $\lim_{t\to 0^+} \tilde{S}_{\mu,V}(t)u_0 = u_0$ in $L^p(\mathbb{R}^N)$. Using the positive and negative part of $u_0 \in L^p(\mathbb{R}^N)$ we can extend $\tilde{S}_{\mu,V}(t)$ to an order preserving $C^0$ semigroup of contractions in $L^p(\mathbb{R}^N)$.

Now we prove that $\tilde{S}_{\mu,V}(t) = S_{\mu,V}(t)$ for $t \ge 0$ if $V$ can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions.

**Theorem 3.7.** *Assume $0 \le V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$ can be approximated in $L^{p_0}_U(\mathbb{R}^N)$ by bounded functions.*

*Then $\tilde{S}_{\mu,V}(t) = S_{\mu,V}(t)$ for $t \ge 0$, as operators in $L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$.*

**Proof.** Take $0 \le u_0 \in C_c^\infty(\mathbb{R}^N)$ and $u_M(t) = S_{\mu,V_M}(t)u_0$ and $u(t) = S_{\mu,V}(t)u_0$ for $t \ge 0$. Since these semigroups are consistent in the Lebesgue spaces, it is enough to show that $u(t) = \lim_{M\to\infty} u_M(t)$ for $t>0$ in some of these spaces. Take $1 < p \le p_0$. As $u_0$ belongs to the domain of the generator, $H^{2\mu}_p(\mathbb{R}^N)$, we have, by Propositions 3.3 and 3.4,

$$u_M(t) = S_\mu(t)u_0 - \int_0^t S_\mu(t-s)V_M u_M(s)\,ds, \quad u(t) = S_\mu(t)u_0 - \int_0^t S_\mu(t-s)Vu(s)\,ds, \quad t>0.$$

Also by general properties of semigroups in [16], we have that $u_M(t), u(t) \in H^{2\mu}_p(\mathbb{R}^N)$ for $t>0$.

Now notice that (3.4), (3.5) and the closed mapping theorem also imply that the graph norm of $L = (-\Delta)^\mu + V$ in $H^{2\mu}_p(\mathbb{R}^N)$ is equivalent to the $H^{2\mu}_p(\mathbb{R}^N)$ norm, that is,

$$c_0\|\phi\|_{H^{2\mu}_p} \le \|L\phi\|_{L^p} + \|\phi\|_{L^p} \le c_1\|\phi\|_{H^{2\mu}_p}.$$

This implies, using that $Lu(t) = LS_{\mu,V}(t)u_0 = S_{\mu,V}(t)Lu_0$, that

$$\|u(t)\|_{H^{2\mu}_p} \le c\|u_0\|_{H^{2\mu}_p}, \quad t>0.$$

Now using the integral representations above, and adding and subtracting the term $\int_0^t S_\mu(t-s)V_M u(s)\,ds$, we get that $z_M(t) = u(t) - u_M(t)$ satisfies

$$z_M(t) = -\int_0^t S_\mu(t-s)\big(V - V_M\big)u(s)\,ds - \int_0^t S_\mu(t-s)V_M z_M(s)\,ds, \quad t>0.$$

Hence for $0<r<1$, using (2.7),

$$\|z_M(t)\|_{H^{2\mu r}_p} \le \int_0^t \frac{c}{(t-s)^r}\|(V-V_M)u(s)\|_{L^p}\,ds + \int_0^t \frac{c}{(t-s)^r}\|V_M z_M(s)\|_{L^p}\,ds$$

and (3.4) yields

$$\|z_M(t)\|_{H^{2\mu r}_p} \le \int_0^t \frac{c}{(t-s)^r}\|V-V_M\|_{L^{p_0}_U}\|u(s)\|_{H^{2\mu}_p}\,ds + \int_0^t \frac{c}{(t-s)^r}\|V_M\|_{L^{p_0}_U}\|z_M(s)\|_{H^{2\mu r}_p}\,ds$$

and then

$$\|z_M(t)\|_{H^{2\mu r}_p} \le Ct^{1-r}\|V-V_M\|_{L^{p_0}_U}\|u_0\|_{H^{2\mu}_p} + C\int_0^t \frac{1}{(t-s)^r}\|z_M(s)\|_{H^{2\mu r}_p}\,ds.$$

Then Henry's singular Gronwall Lemma 7.1.1 in [13] implies that for any $T>0$

$$\|z_M(t)\|_{H^{2\mu r}_p} \le C(T)\|V-V_M\|_{L^{p_0}_U} \xrightarrow[M\to\infty]{} 0, \quad 0<t<T.$$

Therefore $\tilde{S}_{\mu,V}(t)u_0 = S_{\mu,V}(t)u_0$ for $t \ge 0$ and $0 \le u_0 \in C_c^\infty(\mathbb{R}^N)$. By linearity and density we have the result. ■

## 4. Exponential decay

In this section we want to characterise certain classes of non negative potentials $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \frac{N}{2\mu}$ such that the fractional Schrödinger semigroup $\{S_{\mu,V}(t)\}_{t\geq 0}$ decays exponentially in $L^p(\mathbb{R}^N)$. Since the semigroup is of contractions in $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ then either $\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} = 1$ for all $t > 0$ or there exists $a > 0$ such that

$$\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} \leq e^{-at}, \quad t \geq 0.$$

Then we define the exponential type of the semigroup $\{S_{\mu,V}(t)\}_{t\geq 0}$ in $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, as

$$\omega_p \stackrel{\text{def}}{:=} \sup\{a \in \mathbb{R},\ \|S_{\mu,V}(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} \leq e^{-at}, \quad t \geq 0\},$$

and $\omega_p \geq 0$, since the semigroup is of contractions. The exponential type is related to the spectral bound of the generator $A_p = -(-\Delta)^\mu - V$, since these semigroups are order preserving, then for $1 \leq p < \infty$,

$$\omega_p = s(A_p) \stackrel{\text{def}}{:=} \sup\{Re(\lambda),\ \lambda \in \sigma(A_p)\},$$

see [17, 18].

Before proving our main results, we state the following lemma. Below we denote by $\|\cdot\|_{p\to q} = \|\cdot\|_{\mathcal{L}(L^p,L^q)}$.

**Lemma 4.1.** *For $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$, the semigroup $u(t) = S_{\mu,V}(t)u_0$ satisfies the following.*

(i) *For $1 \leq p < \infty$ and $t \geq 0$,*

$$\|S_{\mu,V}(t)\|_{2\to 2} \leq \|S_{\mu,V}(t)\|_{p\to p} = \|S_{\mu,V}(t)\|_{p'\to p'} \leq \|S_{\mu,V}(t)\|_{\infty\to\infty} = \|S_{\mu,V}(t)\|_{1\to 1}$$

(ii) *The exponential type $\omega_p$ is independent of $1 \leq p \leq \infty$.*

**Proof.** (i) Since for $1 \leq p < \infty$ the semigroup in $L^{p'}(\mathbb{R}^N)$ is the adjoint of the semigroup in $L^p(\mathbb{R}^N)$ we get the equalities in the statement.

Now, if $1 \leq p < 2 < p'$ then by the Riesz-Thorin interpolation we get $\|S_{\mu,V}(t)\|_{2\to 2} \leq \|S_{\mu,V}(t)\|_{p\to p} = \|S_{\mu,V}(t)\|_{p'\to p'}$.

(ii) From (i) we immediately get $\omega_2 \geq \omega_p = \omega_{p'} \geq \omega_\infty = \omega_1$.

Since the semigroup in $L^\infty(\mathbb{R}^N)$ is order preserving, then for $t > 0$, $\|S_{\mu,V}(t)\|_{\infty\to\infty} = \|S_{\mu,V}(t)\mathbf{1}\|_{L^\infty(\mathbb{R}^N)}$. Denote by $\mathcal{X}_R$ the characteristic function of the ball $B(0,R)$, then for $x \in \mathbb{R}^N$, using that the semigroup is of contractions in $L^\infty(\mathbb{R}^N)$ we get for $x \in \mathbb{R}^N$,

$$|S_{\mu,V}(t)\mathbf{1}(x)| \leq |S_{\mu,V}(t)\mathcal{X}_R(x)| + |S_{\mu,V}(t)(1-\mathcal{X}_R)(x)| \leq 1 + |S_{\mu,V}(t)\mathcal{X}_R(x)|.$$

Now since from part (ii) in Proposition 3.3 the semigroup $S_{\mu,V}(t)$ satisfies (2.6) we have, for $t \geq 1$,

$$S_{\mu,V}(t)\mathcal{X}_R(x) = S_{\mu,V}(1)S_{\mu,V}(t-1)\mathcal{X}_R(x)$$

and therefore

$$|S_{\mu,V}(t)\mathcal{X}_R(x)| \leq \|S_{\mu,V}(1)\|_{2\to\infty}\|S_{\mu,V}(t-1)\|_{2\to 2}\|\mathcal{X}_R\|_2 = C\|S_{\mu,V}(t-1)\|_{2\to 2}R^{\frac{N}{2}}.$$

Taking $R = t$ we finally get

$$\|S_{\mu,V}(t)\|_{\infty\to\infty} \leq c(1 + t^{\frac{N}{2}})\|S_{\mu,V}(t-1)\|_{2\to 2}$$

and therefore $\omega_\infty \geq \omega_2$ and the result is proved. ■

**Remark 4.2.** *Observe that if $\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} = 1$ for all $t > 0$ then there are solutions that converge to 0 arbitrarily slow.*

*More precisely, for any continuous function $g : [0,\infty) \to (0,1]$ such that $\lim_{t\to\infty} g(t) = 0$, there exists $u_0 \in L^p(\mathbb{R}^N)$ such that*

$$\limsup_{t\to\infty} \frac{\|S_{\mu,V}(t)u_0\|_{L^p}}{g(t)} = \infty.$$

*To see this, assume by contradiction that for all $u_0 \in L^p(\mathbb{R}^N)$ there exists $C = C(u_0)$ such that*

$$\frac{\|S_{\mu,V}(t)u_0\|_{L^p}}{g(t)} \leq C(u_0), \quad t \geq 0.$$

*Then, the uniform bounded principle implies that, for some $M > 0$,*

$$\frac{\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p)}}{g(t)} \leq M.$$

*But then for large enough $t$ we have $\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p)} \leq Mg(t) < 1$ which is a contradiction.*

Now we give conditions on the potential to have exponential decay. Following [1] we consider the following

**Definition 4.3.** *The class $\mathcal{G}$ consists of all open subsets of $\mathbb{R}^N$ containing arbitrarily large balls, that is, the sets such that for any $r > 0$ there exists $x_0 \in \mathbb{R}^N$ such that the ball of radius $r$ around $x_0$ is included in this set.*

**Theorem 4.4.** *Assume that $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$ and there exists $M > 0$ such that $0 \leq V_M(x) = \min\{V(x), M\}$ satisfies*

$$\int_G V_M(x)dx = \infty \quad \text{for all } \ G \in \mathcal{G}.$$

*Then there exists $a > 0$, independent of $1 \leq p \leq \infty$, such that*

$$\|S_{\mu,V}(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} \leq e^{-at}, \quad t \geq 0.$$

**Proof.** Since $0 \leq V_M(x) \leq V(x)$, by Proposition 3.3, it is enough to prove the decay for $S_{\mu,V_M}(t)$. Therefore we can assume $V$ is bounded.

From Lemma 4.1, to prove the exponential decay it is enough then to find $t > 0$ such that

$$\|S_{\mu,V}(t)\|_{\infty\to\infty} = \|S_{\mu,V}(t)\mathbf{1}\|_{L^\infty(\mathbb{R}^N)} < 1.$$

For this, notice that with $u_0 = \mathbf{1}$, we have in particular that, from Proposition 3.4, $u(t) = S_{\mu,V}(t)\mathbf{1}$ satisfies

$$u(t) = S_\mu(t)\mathbf{1} - \int_0^t S_\mu(t-s)Vu(s)\,ds = \mathbf{1} - \int_0^t S_\mu(t-s)Vu(s)\,ds.$$

Substituting the expresion above for $u(s)$ inside the integral term we get

$$u(t) = \mathbf{1} - \int_0^t S_\mu(t-s)V\,ds + \int_0^t S_\mu(t-s)V \int_0^s S_\mu(s-r)Vu(r)\,dr\,ds$$

We use that $u(s) \leq S_\mu(s)\mathbf{1} = \mathbf{1}$ and $0 \leq V \leq \|V\|_\infty \mathbf{1}$ and then

$$\int_0^t S_\mu(t-s)V \int_0^s S_\mu(s-r)Vu(r)\,dr\,ds \leq \|V\|_\infty \int_0^t S_\mu(t-s)V \int_0^s S_\mu(s-r)\mathbf{1}\,dr\,ds$$
$$= \|V\|_\infty \int_0^t sS_\mu(t-s)V\,ds = \|V\|_\infty \int_0^t (t-s)S_\mu(s)V\,ds \leq \|V\|_\infty t \int_0^t S_\mu(s)V\,ds.$$

Therefore

$$u(t) \leq \mathbf{1} - \int_0^t S_\mu(t-s)V\,ds + \|V\|_\infty t \int_0^t S_\mu(s)V\,ds = \mathbf{1} + (\|V\|_\infty t - 1)\int_0^t S_\mu(t-s)V\,ds.$$

From the results in Section 2

$$(S_\mu(s)V)(x) = \int_{\mathbb{R}^N} k_\mu(s,x,y)V(y)\,dy$$

where $k_\mu(s,x,y) = \frac{1}{t^{\frac{N}{2\mu}}} k_{0,\mu}\left(\frac{x-y}{t^{\frac{1}{2\mu}}}\right)$ is as in (2.2) and (2.9). Using [1, Proposition 1.4] we see in turn that there exist $c > 0$ and $r > 0$ such that

$$\int_{B(x,r)} V(y)dy \geq c \ \text{ for all } \ x \in \mathbb{R}^N.$$

For any $x \in \mathbb{R}^N$ and $s > 0$ we have

$$\begin{aligned}
S_\mu(s)V(x) &= \int_{\mathbb{R}^N} \frac{1}{s^{\frac{N}{2\mu}}} k_{0,\mu}\left(\frac{x-y}{s^{\frac{1}{2\mu}}}\right) V(y)\, dy \\
&\geq \int_{B(x,r)} \frac{1}{s^{\frac{N}{2\mu}}} k_{0,\mu}\left(\frac{x-y}{s^{\frac{1}{2\mu}}}\right) V(y)\, dy \\
&\geq \inf_{|z|\leq r} \frac{1}{s^{\frac{N}{2\mu}}} k_{0,\mu}\left(\frac{z}{s^{\frac{1}{2\mu}}}\right) \int_{B(x,r)} V(y)\, dy \\
&\geq c \left\{ \begin{array}{ll} (4\pi s)^{-\frac{N}{2}} e^{-\frac{r^2}{4s}}, & \mu = 1 \\ \frac{s}{(s^{\frac{1}{\mu}} + r^2)^{\frac{N+2\mu}{2}}}, & 0 < \mu < 1 \end{array} \right\} =: g_\mu(s)
\end{aligned}$$

where we have used the estimate of fractional heat equation kernel from (2.2) and (2.9)-(2.10).

Denoting $\mathscr{G}_\mu(t) \stackrel{\text{def}}{:=} \int_0^t g_\mu(s)\, ds$, we get, for $0 < t < \frac{1}{\|V\|_{L^\infty(\mathbb{R}^N)}}$ and $x \in \mathbb{R}^N$

$$(S_{\mu,V}(t)\mathbf{1})(x) \leq \mathbf{1} - (1 - t\|V\|_{L^\infty(\mathbb{R}^N)})\mathscr{G}_\mu(t) < 1.$$

■

In case $V$ can be approximated by bounded potentials, we have the following result.

**Proposition 4.5.** *Assume $V \in L^1_U(\mathbb{R}^N)$ can be approximated in $L^1_U(\mathbb{R}^N)$ by bounded functions. Then, they are equivalent*

(i) *$\int_G V_M = \infty$ for all $G \in \mathcal{G}$ with some $M$.*

(ii) *$\int_G V = \infty$ for all $G \in \mathcal{G}$.*

**Proof.** Since $0 \leq V_M \leq V$, clearly (i) implies (ii).

Conversely, from [1, Proposition 1.4], (ii) is equivalent to the existence of $c > 0$ and $r > 0$ such that

$$\inf_{x\in\mathbb{R}^N} \int_{B(x,r)} V \geq c.$$

By a simple covering argument, independent of $x$, we get that $\sup_{x\in\mathbb{R}^N} \|V_M - V\|_{L^1(B(x,r))} \leq C\|V_M - V\|_{L^1_U(\mathbb{R}^N)} \to 0$ as $M \to \infty$ and therefore, for some $M$ we have $\sup_{x\in\mathbb{R}^N} \|V_M - V\|_{L^1(B(x,r))} < \frac{c}{2}$. This implies that for all $x \in \mathbb{R}^N$

$$\int_{B(x,r)} V_M = \int_{B(x,r)} V + \int_{B(x,r)} (V_M - V) > c - \frac{c}{2} = \frac{c}{2}.$$

Again [1, Proposition 1.4] gives (i). ■

**Remark 4.6.** *Now we give an example that shows that in general in Proposition 4.5 (ii) does not imply (i). That is, we show that may have for all $M > 0$ and $r > 0$*

$$\inf_{x\in\mathbb{R}^N} \int_{B(x,r)} V_M = 0$$

*but for some $r > 0$,*

$$\inf_{x\in\mathbb{R}^N} \int_{B(x,r)} V > 0.$$

*Denote by $\{Q_i\}$ the family of cubes centered at points $i \in \mathbb{Z}^N$ of integer coordinates in $\mathbb{R}^N$ and with edges of length 1 parallel to the axes. Then notice that $B(i,1) \subset Q_i$ and define in $Q_i$ for $i \neq 0$,*

$$V(x) = c|i|^N \mathcal{X}_{B(i,\frac{1}{|i|})}$$

*and $V = \frac{1}{c}\mathcal{X}_{B(0,1)}$ in $Q_0$ where $c$ is the the measure of the unit ball.*

*Then*

(i) *For every $i$, $\int_{Q_i} V = 1$ so $V \in L^1_U(\mathbb{R}^N)$.*

(ii) *There exists an $r>0$ such that for every $x \in \mathbb{R}^N$, $B(x,r)$ contains at least one cube $Q_i$. Then*

$$\int_{B(x,r)} V \geq 1$$

(iii) *For every $M>0$ and every $r>0$ there exists $m(r) \in \mathbb{N}$ such that $B(x,r)$ is contained in $m(r)$ cubes and if $|x|$ is large then the centers of these cubes satisfy $|i| \geq |x|-2r>0$. For any such cubes*

$$\int_{Q_i} V_M \leq \frac{cM}{|i|^N}$$

*and then if we take $|x| \to \infty$ we see that*

$$\int_{B(x,r)} V_M \leq m(r) \frac{cM}{(|x|-2r)^N} \to 0.$$

Our next results characterises the exponential type of the fractional Schrödinger semigroup.

**Theorem 4.7.** *With the notations in Theorem 4.4, assume $0 \leq V \in L^{p_0}_U(\mathbb{R}^N)$ with $p_0 > \max\{\frac{N}{2\mu}, 1\}$. Then for all $1 \leq p \leq \infty$ the exponential type of the semigroup $\{S_{\mu,V}(t)\}_{t\geq 0}$ is given by*

$$\omega_p = a_* \stackrel{def}{:=} \inf\{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}\phi|^2 + \int_{\mathbb{R}^N} V|\phi|^2 \colon \phi \in C^\infty_c(\mathbb{R}^N),\ \|\phi\|_{L^2(\mathbb{R}^N)} = 1\} \geq 0.$$

*Moreover, if $a_*>0$ then*

$$\int_G V(x)dx = \infty \quad \text{for all } G \in \mathcal{G}. \tag{4.1}$$

*Conversely, if $V$ can be approximated in $L^1_U(\mathbb{R}^N)$ by bounded functions, then (4.1) implies $a_*>0$.*

**Proof.** From Lemma 4.1 it is enough to show that $a_* = \omega_2$. For this, for any $a<\omega_2$ we have that the semigroup $\{e^{at}S_{\mu,V}(t)\}_{t\geq 0}$ in $L^2(\mathbb{R}^N)$ is of contractions and from Section 3 it is strongly continuous, analytic and its generator is the operator $\mathscr{A} = -(-\Delta)^\mu - V + a$ in $L^2(\mathbb{R}^N)$ with domain $H^{2\mu}(\mathbb{R}^N)$. Then, by the Lummer-Phillips Theorem, see [16, Theorem 4.3], we have that $\mathscr{A}$ is accretive, that is,

$$\langle -\mathscr{A}\phi, \phi\rangle_{L^2(\mathbb{R}^N)} \geq 0 \quad \text{for all } \phi \in H^{2\mu}(\mathbb{R}^N).$$

In particular, taking $\phi \in C^\infty_c(\mathbb{R}^N)$ with $\|\phi\|_{L^2(\mathbb{R}^N)}=1$ then $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}\phi|^2 + \int_{\mathbb{R}^N} V|\phi|^2 \geq a > 0$. Hence $a_* \geq a$ and therefore $a_* \geq \omega_2$.

Conversely, if $a<a_*$ we take $f(t) = e^{2at}\|u(t)\|^2_{L^2(\mathbb{R}^N)}$ with $u(t) = S_{\mu,V}(t)\phi$ and then

$$f'(t) = 2ae^{2at}\int_{\mathbb{R}^N} u(t)^2 + 2e^{2at}\int_{\mathbb{R}^N} u(t)\frac{d}{dt}u(t) = 2ae^{2at}\int_{\mathbb{R}^N} u(t)^2 + 2e^{2at}\int_{\mathbb{R}^N} u(t)(-(-\Delta)^\mu u(t) - Vu(t))$$

and then

$$\frac{e^{-2at}}{2}f'(t) = -\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}u(t)|^2 + Vu(t)^2 + a\int_{\mathbb{R}^N} u(t)^2 \leq (a-a_*)\int_{\mathbb{R}^N} u(t)^2 \leq 0$$

which yields $f'(t) \leq 0$ and then $\|S_{\mu,V}(t)\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \leq e^{-at}$. Therefore $\omega_2 \geq a_*$.

Now, if $a_*>0$, assume that (4.1) fails, that is, there exists $G \in \mathcal{G}$ such that

$$\int_G V < \infty.$$

Then we choose a positive sequence satisfying $r_n > n^{\frac{1}{\mu}}$ and a sequence of points $\{x_n\} \subset \mathbb{R}^N$ such that $B(x_n, r_n) \subset G$ for each $n \in \mathbb{N}$.

Assume we had a sequence of functions such that $\{\phi_n\} \subset C^\infty_c(B(x_n,r_n))$, $\|\phi_n\|_{L^2(\mathbb{R}^N)} = 1$ such that

$$\lim_{n\to\infty} \|\phi_n\|_{L^\infty(\mathbb{R}^N)} = 0, \quad \lim_{n\to\infty} \|(-\Delta)^{\frac{\mu}{2}}\phi_n\|_{L^2(\mathbb{R}^N)} = 0.$$

Then we would have

$$0 < a_* \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}\phi_n|^2 + \int_{\mathbb{R}^N} V|\phi_n|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}\phi_n|^2 + \int_G V|\phi_n|^2$$
$$\leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\mu}{2}}\phi_n|^2 + \|\phi_n\|^2_{L^\infty(\mathbb{R}^N)} \int_G V \ \to 0 \ \text{ as } \ n \to \infty,$$

which is a contradiction and therefore (4.1) must be true. To construct a sequence as above we take $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |\psi|^2 = 1$ and defining

$$\psi_n(x) = n^{-\frac{N}{2\mu}}\psi(n^{-\frac{1}{\mu}}x), \quad x \in \mathbb{R}^N, \ n \in \mathbb{N}$$

we see that $\{\psi_n\} \subset C_c^\infty(B(0, r_n))$, $\|\psi_n\|_{L^2(\mathbb{R}^N)} = 1$, $\lim_{n\to\infty} \|\psi_n\|_{L^\infty(\mathbb{R}^N)} = 0$ and

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{\mu}{2}}\psi_n\|_{L^2(\mathbb{R}^N)} = \lim_{n\to\infty} n^{-1}\|(-\Delta)^{\frac{\mu}{2}}\psi\|_{L^2(\mathbb{R}^N)} = 0$$

since $(-\Delta)^{\frac{\mu}{2}}$ is an homogenous operator of degree $\mu$. Then $\phi_n(\cdot) = \psi_n(\cdot + x_n)$, $n \in \mathbb{N}$, satisfies the conditions above since $(-\Delta)^{\frac{\mu}{2}}$ is invariant under translations, see (2.4).

Conversely, if $V$ can be approximated in $L^1_U(\mathbb{R}^N)$ by bounded functions, then from Proposition 4.5 and Theorem 4.4 we get that (4.1) implies $a_* > 0$. ■

## References


[1] W. Arendt and C. J. K. Batty. Exponential stability of a diffusion equation with absorption. *Differential Integral Equations*, 6(5):1009–1024, 1993. Cited ↑ in page: 1, 13, 14

[2] J. M. Arrieta, J. W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Asymptotic behavior and attractors for reaction diffusion equations in unbounded domains. *Nonlinear Anal.*, 56(4):515–554, 2004. Cited ↑ in page: 6

[3] J. M. Arrieta, J. W. Cholewa, T. Dlotko, and A. Rodriguez-Bernal. Linear parabolic equations in locally uniform spaces. *Math. Models Methods Appl. Sci.*, 14(2):253–293, 2004. Cited ↑ in page: 9, 10

[4] K. Bogdan, T. Grzywny, T. Jakubowski, and D. Pilarczyk. Fractional Laplacian with Hardy potential. *Communications in Partial Differential Equations*, 44(1):20–50, 2019. Cited ↑ in page: 4

[5] M. Bonforte, Y. Sire, and J.L. Vázquez. Optimal existence and uniqueness theory for the fractional heat equation. *Nonlinear Analysis: Theory, Methods & Applications*, 153:142–168, 2017. Cited ↑ in page: 4

[6] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors. Cited ↑ in page: 2

[7] J. W. Cholewa and A. Rodriguez-Bernal. Sharp estimates for homogeneous semigroups in homogeneous spaces. Applications to PDEs and fractional diffusion in $\mathbb{R}^N$. *Commun. Contemp. Math.*, 24(1):Paper No. 2050070, 56, 2022. Cited ↑ in page: 3, 4

[8] J. W. Cholewa and A. Rodríguez-Bernal. On linear higher-order parabolic equations in Morrey spaces. *Anal. Appl. (Singap.)*, 21(6):1561–1608, 2023. Cited ↑ in page: 3, 4

[9] J. W. Cholewa and A. Rodriguez-Bernal. Self-similarity in homogeneous stationary and evolution problems. *J. Evol. Equ.*, 23(2):Paper No. 42, 39, 2023. Cited ↑ in page: 4

[10] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989. Cited ↑ in page: 6

[11] A. De Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, LXV:1242–1284, 2012. Cited ↑ in page: 5

[12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012. Cited ↑ in page: 3

[13] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981. Cited ↑ in page: 11

[14] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017. Cited ↑ in page: 3

[15] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995. Cited ↑ in page: 4

[16] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag New York, Inc., 1983. Cited ↑ in page: 7, 8, 11, 15

[17] L. Weis. The stability of positive semigroups on $L_p$ spaces. *Proc. Amer. Math. Soc.*, 123(10):3089–3094, 1995. Cited ↑ in page: 12

[18] L. Weis. A short proof for the stability theorem for positive semigroups on $L_p(\mu)$. *Proc. Amer. Math. Soc.*, 126(11):3253–3256, 1998. Cited ↑ in page: 12

[19] K. Yosida. *Functional analysis*. Springer-Verlag, Berlin-New York, fifth edition, 1978. Cited ↑ in page: 3, 4