STABILITY OF NAVIER-STOKES EQUATIONS WITH A FREE SURFACE

XING CHENG AND YUNRUI ZHENG

Abstract. We consider the viscous incompressible fluids in a three-dimensional horizontally periodic domain bounded below by a fixed smooth boundary and above by a free moving surface. The fluid dynamics are governed by the Navier-Stokes equations with the effect of gravity and surface tension on the free surface. We develop a global well-posedness theory by a nonlinear energy method in low regular Sobolev spaces. We use several techniques, including: the horizontal energy-dissipation estimates, a new tripled bootstrap argument inspired by Guo and Tice [Arch. Ration. Mech. Anal.(2018)]. Moreover, the solution decays asymptotically to the equilibrium in an exponential rate.

1. Introduction

1.1. **Formulation of the problem.** The viscous surface waves problem is presented for a viscous fluid of finite depth below the air. Concretely we consider an incompressible viscous fluid in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} | -1 < y_3 < \eta(y_1, y_2, t)\}.$$

The upper boundary of $\Omega(t)$ is free and assumed to be a graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. Here we assume $\Sigma = \mathbb{T}_{L_1} \times \mathbb{T}_{L_2}$, where $\mathbb{T}_{L_1}$ and $\mathbb{T}_{L_2}$ are the tori with periodicity lengths being L_1 and L_2.

For each $t \geq 0$, the fluid is described by its velocity and pressure functions $(u, p) : \Omega(t) \mapsto \mathbb{R}^3 \times \mathbb{R}$ which satisfy the following Navier-Stokes equations with boundary conditions:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = 0 & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ S(p, u)\nu = g\eta\nu - \kappa\mathcal{H}\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ u = 0 & \text{on } \{y_3 = -1\}. \end{cases} \tag{1.1}$$

In the system (1.1), $S(p, u)$ is the stress tensor which is defined by $S(p, u) = pI - \mu\mathbb{D}u$ for I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, and $\mu > 0$ the viscosity. Clearly, for the incompressible fluid, $\operatorname{div} S(p, u) = \nabla p - \mu\Delta u$. In the system (1.1), $\mathcal{N} = (-\partial_1\eta, -\partial_2\eta, 1)^\top$ is the outward-pointing normal on $\Sigma(t)$, and $\nu = \mathcal{N}/|\mathcal{N}|$ is the outward-unit-normal on $\Sigma(t)$. $g > 0$ is the strength of gravity, $\kappa > 0$ is the coefficient of surface tension, $\mathcal{H} = \nabla_* \cdot \left(\frac{\nabla_*\eta}{\sqrt{1+|\nabla_*\eta|^2}}\right)$ is the twice mean curvature of the free surface, where $\nabla_* = (\partial_1, \partial_2)$ is the horizontal gradient operator. We refer to [27] for the analysis of boundary conditions in (1.1). Note that in (1.1), we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure p_{atm} by $p = \bar{p} + gy_3 - p_{atm}$ for the actual pressure $\bar{p}$.

The problem (1.1) is studied when initial data (u_0, η_0) satisfy some certain compatibility conditions. We always assume that $\eta_0 + 1 > 0$, that avoids the contact lines generated by the intersection of free surface and bottom. Without loss of generality, we may assume that $\mu = g = \kappa = 1$ by scaling.

2020 Mathematics Subject Classification. 35A01,35Q30,35R35,76D05.

Key words and phrases. Free boundary problems, Navier-Stokes equations, global existence, stability.

The system (1.1) possesses a natural physical energy. For sufficiently regular (u,η), (1.1) possesses a relation between the change of physical energy and dissipation:

$$\begin{aligned}&\frac{1}{2}\int_{\Omega(t)}|u(t)|^2+\frac{1}{2}\int_{\Sigma}|\eta(t)|^2+2\left(\sqrt{1+|\nabla_*\eta|^2}-1\right)+\frac{1}{2}\int_0^t\int_{\Omega(s)}|\mathbb{D}u(s)|^2\\ =&\frac{1}{2}\int_{\Omega(0)}|u_0|^2+\frac{1}{2}\int_{\Sigma}|\eta_0|^2+2\left(\sqrt{1+|\nabla_*\eta_0|^2}-1\right).\end{aligned} \tag{1.2}$$

The first two integrals on the left-hand side represent the kinetic and potential energies, while the third one on the left-hand side represents the dissipation. This energy-dissipation relation (1.2) is our basis of the nonlinear energy method.

1.2. **Geometric reformulation.** The Navier-Stokes equations (1.1) admit a steady state $u_s=0,\eta_s=0$, that enables us to switch the problem (1.1) into a fixed domain in order to remove the difficulties created by the free surface $\Sigma(t)$ and deformable domain $\Omega(t)$. We reformulate (1.1) as in [12]. Now we consider the fixed equilibrium domain

$$\Omega:=\{x\in\Sigma\times\mathbb{R}|-1<x_3<0\}.$$

We will think of Σ as the upper boundary of Ω without confusion, and $\Sigma_b:=\{x_3=-1\}$ as the lower boundary. Then we define the harmonic extension of η:

$$\bar{\eta}=\sum_{n\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}e^{2\pi i n\cdot x'}e^{2\pi|n|x_3}\hat{\eta}(n),\quad x'\in\mathbb{T}_{L_1}\times\mathbb{T}_{L_2}, \tag{1.3}$$

where $\hat{\eta}(n)$ denotes the Fourier transform of $\eta(x')$ in discrete form.

Now we define the mapping $\Phi:\Omega\to\Omega(t)$ to be

$$\Omega\ni(x_1,x_2,x_3)\mapsto(x_1,x_2,x_3+\bar{\eta}\,(1+x_3))=(y_1,y_2,y_3)\in\Omega(t). \tag{1.4}$$

Clearly, $\Phi(\Sigma,t)=\{y_3=\eta(y_1,y_2,t)\}$ and $\Phi(\Sigma_b,t)=\{y_3=-1\}$; that is, Φ maps Σ and Σ_b to the free surface and lower boundary of $\Omega(t)$ respectively. The Jacobi matrix $\nabla\Phi$ and transform matrix $\mathcal{A}$ are

$$\nabla\Phi=\begin{pmatrix}1&0&0\\0&1&0\\A&B&J\end{pmatrix},\quad \mathcal{A}=(\nabla\Phi)^{-\top}=\begin{pmatrix}1&0&-AK\\0&1&-BK\\0&0&K\end{pmatrix}, \tag{1.5}$$

where

$$A=\partial_1\bar{\eta}W,\ B=\partial_2\bar{\eta}W,\ J=1+\bar{\eta}+\partial_3\bar{\eta}W,\ K=\frac{1}{J},\ W=1+x_3. \tag{1.6}$$

For any scalar-valued or vector valued function f and vector X, we define the transformed operators as follows:

$$\begin{aligned}(\nabla_{\mathcal{A}}f)_i=\mathcal{A}_{ij}\partial_jf,\ \nabla_{\mathcal{A}}\cdot X=\mathcal{A}_{ij}\partial_jX_i,\ \Delta_{\mathcal{A}}f=\nabla_{\mathcal{A}}\cdot\nabla_{\mathcal{A}}f,\\ S_{\mathcal{A}}(p,u)=pI-\mathbb{D}_{\mathcal{A}}u,\ (\mathbb{D}_{\mathcal{A}}u)_{ij}=\mathcal{A}_{ik}\partial_ku_j+\mathcal{A}_{jk}\partial_ku_i,\end{aligned} \tag{1.7}$$

where I is the 3×3 identity matrix and the summation is understood in the Einstein convection.

If η is sufficiently small (in an appropriate sense), the mapping Φ is a diffeomorphism. In the new coordinates, for each $t\geq0$, the original system (1.1) becomes

$$\begin{cases}\partial_tu-\partial_t\bar{\eta}WK\partial_3u+u\cdot\nabla_{\mathcal{A}}u+\nabla_{\mathcal{A}}p-\Delta_{\mathcal{A}}u=0 & \text{in }\Omega,\\ \operatorname{div}_{\mathcal{A}}u=0 & \text{in }\Omega,\\ S_{\mathcal{A}}(p,u)\mathcal{N}=\eta\mathcal{N}-\mathcal{H}\mathcal{N} & \text{on }\Sigma,\\ u=0 & \text{on }\Sigma_b,\\ \partial_t\eta=u\cdot\mathcal{N} & \text{on }\Sigma,\\ u(x,0)=u_0(x),\ \eta(x',0)=\eta_0(x'). \end{cases} \tag{1.8}$$

Here we still write $\mathcal{N}=(-\partial_1\eta,-\partial_2\eta,1)$ for the nonunit-normal to $\{y_3=\eta(y_1,y_2,t)\}$. If we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}}S_{\mathcal{A}}(p,u)=\nabla_{\mathcal{A}}p-\Delta_{\mathcal{A}}u$ for u satisfying $\operatorname{div}_{\mathcal{A}}u=0$. Clearly, all the spatial differential operators are related to the geometric structure of the free surface η.

1.3. **Main results.** To state our result, we need to explain the notation for spaces and norms. When we write $\left\|\partial_t^j u\right\|_{H^k}$, and $\left\|\partial_t^j p\right\|_{H^k}$, we always mean that the space is $H^k(\Omega)$, and when we write $\left\|\partial_t^j \eta\right\|_{H^s}$, we always mean that the space is $H^s(\Sigma)$, where $H^k(\Omega)$ and $H^s(\Sigma)$ are usual Sobolev spaces for $k, s \geq 0$.

When the coefficient of viscosity μ is zero, the system (1.1) is reduced to the free boundary problems of Euler equations with vorticity. There are many fruitful results for well-posedness and stability of (1.1) when $\mu = 0$, such as [6, 7, 21, 22, 26] and references therein. Many of these results are related to dispersive estimates.

The viscous surface waves have been studied intensively over a long period of time, since the significant progress made by Beale [3]. We first refer to the case of absence of surface tension. When $\Sigma = \mathbb{R}^2$ or $\mathbb{T}^2$, Beale [3] proved the local well-posedness in Sobolev-type spaces for any fixed bottom (not even flat), and Sylvester [23] proved the global well-posedness in the same framework of Beale. Recently, Gui [10] proved the global well-posedness using the Lagrangian structure. When the bottom is flat as in our setting, Hataya [16] proved the global well-posednss for $\mathbb{T}^2$, while Hataya and Kawashima [17] proved global well-posedss for $\mathbb{R}^2$. Both [16] and [17] assumed that $(u_0, \eta_0) \in H^r(\Omega) \times H^{r+1/2}(\Sigma)$ for $4 < r < 9/2$ were small and obtained the solutions with polynomial decay in time. The new breakthrough is due to Guo and Tice [12–14]. They developed a new two-tier energy method to prove the global well-posedness under the assumption that the initial data $(u_0, \eta_0) \in H^{4N}(\Omega) \times H^{4N+1/2}(\Sigma)$ with $N \geq 3$. They proved the solution decays to the equilibrium at a polynomial rate when $\Sigma = \mathbb{R}^2$ and at an almost exponential rate when $\Sigma = \mathbb{T}^2$. Recently, Ren, Xiang and Zhang [20] used a moving domain iteration introduced in [26] and obtained the local well-posedness without surface tension for large data $(u_0, \eta_0) \in H^2(\Omega) \times H^{5/2}(\Sigma)$.

For the viscous surface waves with surface tension, Beale [4] proved the global well-posedness in Sobolev-type spaces using the Fourier-Laplace method with the assumption that $(u_0, \eta_0) \in H^r(\Omega) \times H^{r+1/2}(\Sigma)$ with $5/2 < r < 3$ when $\Sigma = \mathbb{R}^2$. Then Beale and Nishida [5] showed the solution in [4] decays to the equilibrium in an optimal polynomial rate. Bae [2] proved the global well-posedness by an energy method with the same assumption as in [4] when $\Sigma = \mathbb{R}^2$. Nishida, Teramoto, and Yoshihara [19] proved the global well-posedness with the assumption that $(u_0, \eta_0) \in H^r(\Omega) \times H^{r+1/2}(\Sigma)$ with $2 < r < 5/2$ when $\Sigma = \mathbb{T}^2$ and proved that the solutions decay to the equilibrium in the exponential rate. Tan and Wang [24] considered the zero surface tension limit under the regularity assumption that the initial data $(u_0, \eta_0) \in H^{4N}(\Omega) \times H^{4N+1/2}(\Sigma)$ with $N \geq 3$.

We now consider the case $\Sigma = \mathbb{T}^2$ with surface tension. To present our results, we first define the energy

$$\mathcal{E} := \|u\|_{H^2(\Omega)}^2 + \|\partial_t u\|_{H^0(\Omega)}^2 + \|p\|_{H^1(\Omega)}^2 + \|\eta\|_{H^3(\Sigma)}^2 + \|\partial_t \eta\|_{H^{3/2}(\Sigma)}^2, \tag{1.9}$$

and dissipation

$$\mathcal{D} := \|u\|_{H^3(\Omega)}^2 + \|\partial_t u\|_{H^1(\Omega)}^2 + \|p\|_{H^2(\Omega)}^2 + \|\eta\|_{H^{7/2}(\Sigma)}^2 + \|\partial_t \eta\|_{H^{5/2}(\Sigma)}^2 + \left\|\partial_t^2 \eta\right\|_{H^{1/2}(\Sigma)}^2. \tag{1.10}$$

Then we denote $\mathcal{A}_0 = \mathcal{A}(t=0)$, $\mathcal{N}_0 = \mathcal{N}(t=0)$ and $\Pi_0(v)$ to be the projection of vector v on the surface $\{x_3 = \eta_0\}$. Our main result is stated as

Theorem 1.1. *Let $\eta_0 + 1 > \delta_0 > 0$ for some $\delta_0 > 0$ and $(u_0, \eta_0) \in H^2(\Omega) \times H^3(\Sigma)$ satisfy the compatibility condition*

$$\begin{cases} \operatorname{div}_{\mathcal{A}_0}(u_0) = 0 & \text{in } \Omega, \\ \Pi_0\left(\mathbb{D}_{\mathcal{A}_0} u_0 \mathcal{N}_0\right) = 0 & \text{on } \Sigma, \\ u_0 = 0 & \text{on } \Sigma_b, \end{cases} \tag{1.11}$$

and the zero average condition $\langle \eta_0 \rangle = \frac{1}{L_1 L_2} \int_\Sigma \eta_0 = 0$. There exists a small $\delta > 0$ such that if $\mathcal{E}(0) \leq \delta$, then (1.8) admits a unique global in time solution (u, p, η) achieving the initial data. Moreover, the solution satisfies

$$\sup_{t \geq 0} e^{\sigma t} \mathcal{E}(t) + \int_0^\infty \mathcal{D}(t)\, \mathrm{d}t \leq C \mathcal{E}(0)$$

for some universal constants $\sigma = \sigma(L_1, L_2) > 0$ and $C > 0$.

Remark 1.2. *From the divergence free condition, the zero average is conserved in all time:*

$$\frac{d}{dt}\int_\Sigma \eta(t) = \int_\Sigma u \cdot \mathcal{N} = \int_\Omega \operatorname{div}_{\mathcal{A}} uJ = 0.$$

The zero average condition is natural. If it fails, we need to change the domains and unknowns like

$$\Omega \mapsto \Omega - \{0, 0, \langle \eta_0 \rangle\}, \ \eta \mapsto \eta - \langle \eta_0 \rangle, \ \text{and } p \mapsto p - \langle \eta_0 \rangle.$$

The new unknowns still satisfy (1.8) in the new domain under the zero average condition.

Remark 1.3. *The initial data in $\mathcal{E}(0)$, except for u_0 and η_0, are constructed in the proceeding of local well-posedness. The smallness of η is to guarantee that the mapping Φ defined in (1.4) is a C^1 diffeomorphism, which enables us to switch the solutions in (1.8) into solutions in (1.1) so that (1.1) admits a unique global in time and decay solution. The proof of diffeomorphism is also contained in the proof of the local well-posedness.*

Remark 1.4. *Our main result yields the asymptotic stability of solutions in (1.8) around the equilibrium with exponentially decaying rate.*

The big difference between the progress in [19] and our main result is the regularity of initial data. The regularity of u_0 is weakened to $H^2(\Omega)$. We assume that $\eta_0 \in H^3(\Sigma)$, since it is naturally observed from the following variants of (1.2)

$$\int_{\Omega(t)} |u(t)|^2 + \int_\Sigma |\eta(t)|^2 + \frac{2\,|\nabla_*\eta|^2}{\sqrt{1+|\nabla_*\eta|^2}+1} + \int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 = \int_{\Omega(0)} |u_0|^2 + \int_\Sigma |\eta_0|^2 + \frac{2\,|\nabla_*\eta_0|^2}{\sqrt{1+|\nabla_*\eta_0|^2}+1},$$

that η formally gains one more derivative than u under the assumption of small data. Moreover, we still could prove the global well-posedness under our assumption. In addition, the assumption of initial data is consistent with our definition of energy defined in (1.9). In [19], the authors required much more regularity to prove the exponential decay rate for $\|u\|_2^2 + \|\eta\|_3^2$ due to their parabolic perturbation method. But in our result, the decay result is naturally included in the decay of energy. The idea of our method is borrowed partially from [12] and [30].

1.4. **Strategy of the proof.** Our proof by the nonlinear energy method is based on a higher-regular formulation of energy-dissipation equality (1.2) and a bootstrap argument of elliptic equations with variable coefficients.

The horizontal energy-dissipation estimates. The slab domain Ω determines we can only take horizontal derivatives ∂_t and ∇_* to preserve the form of energy-dissipation equality (1.2). Actually, we will take one temporal derivative and horizontal derivatives up to second order for (1.8) to estimate our energy, which is comparable with the fact that one temporal derivative behaves like two spatial derivatives. The choice of our energy and dissipation seems optimal in our framework, and could not be improved more.

The differential operators in (1.8) could not commute with the horizontal differential operators because of the variable coefficients. So there will be many nonlinear terms, which could be controlled by the horizontal energy, dissipation and energy, dissipation defined in (1.9) and (1.10). Then the following elliptic estimates will be combined to close our energy.

Elliptic estimates. Let us first explain the usual methods of treating elliptic estimates, such as [13, 24], etc., which is different from ours here. The method to deal with the elliptic estimates is usually employed by decoupling (u, p) with η. In particular, for any $\eta \in H^3$, we will apply the elliptic theory developed in [1] to

$$\begin{cases} -\Delta u + \nabla p = \partial_t u + \mathfrak{F}^1 & \text{in } \Omega, \\ \operatorname{div} u = \mathfrak{F}^2 & \text{in } \Omega, \\ S(p,u)e_3 = (\eta - \Delta_*\eta)e_3 + \mathfrak{F}^3 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b \end{cases} \tag{1.12}$$

and obtain the estimate for $\|u\|_{H^2}$ and $\|p\|_{H^1}$. Here $\mathfrak{F}^i$, $i=1,2,3$, are perturbation terms and $\Delta_*=\partial_1^2+\partial_2^2$. Then to enhance the regularity of (u,p,η), it is concerned with the system

$$\begin{cases} -\Delta u+\nabla p=\partial_t u+\mathfrak{F}^1 & \text{in } \Omega,\\ \operatorname{div} u=\mathfrak{F}^2 & \text{in } \Omega,\\ u=u & \Sigma,\\ u=0 & \text{on } \Sigma_b. \end{cases} \tag{1.13}$$

By applying the horizontal energy-dissipation estimates to obtain the bounds $\|u\|_{H^3}^2+\|p\|_{H^2}^2$. Then utilizing the equation $S(p,u)e_3=(\eta-\Delta_*\eta)e_3+\mathfrak{F}^3$ to bound $\|\eta\|_{7/2}$. Honestly, this method is effective for the higher regular Sobolev spaces, and also has no harm to our a priori estimates. Unfortunately, it is bad for the local well-posedness of our case, especially in the construction of approximate solutions, the energy estimates might not be closed due to the limitation of regularity of η. The main reason is probably that $\|u\|_{H^2}$ and $\|u\|_{H^3}$ can not be constructed independently, see [13], while the systems (1.12) and (1.13) work independently so that they may run into circular proof in the local theory.

One way to conquer this difficulty comes from our experience of research on contact lines [30], where we handle (u,p) coupling with η. This is natural since (u,p,η) are mutually dependent in the original system (1.8), which might probably not be handled separately in the much more critical spaces. So we construct the elliptic estimates for the triple (u,p,η) and obtain the bounds $\|u\|_{H^2}$ and $\|u\|_{H^3}$ in the same system, not in the different systems (1.12) and (1.13), respectively. This structure plays an important role in the a priori estimates and local well-posdness for large data. As far as we know, this view point has not been applied to consider the viscous surface waves previously.

1.5. **Notation and terminology.** Now, we mention some definitions, notation and conventions that we will use throughout this paper.

1. Constants. The constant $C>0$ will denote a universal constant that only depends on the parameters of the problem, N and Ω, but does not depend on the data, etc. They are allowed to change from line to line. We will write $C=C(z)$ to indicate that the constant C depends on z. And we will write $a\lesssim b$ to mean that $a\le Cb$ for a universal constant $C>0$.
2. Norms. Sometimes we will write $\|\cdot\|_k$ instead of $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_{H^k(\Sigma)}$. We assume that functions have natural spaces. For example, the functions u, p, and $\bar\eta$ live on Ω, while η lives on Σ. So we may write $\|\cdot\|_{H^k}$ for the norms of u, p, and $\bar\eta$ in Ω, and $\|\cdot\|_{H^s}$ for norms of η on Σ.

1.6. **Organization of the rest of this paper.** In section 2, we develop the machinery of basic linear energy-dissipation as our start point of the nonlinear energy method. In section 3, we give the elliptic analysis based on an bootstrap argument only depending on the surface function η in low regularity. In section 4, we construct two ingredients for our *a priori estimates*, that are horizontal energy-dissipation estimates and elliptic estimates. In section 5, we construct the *a priori estimates*, then develop the ideas of local well-posedness and complete the main result.

2. Weak formulation and energy

In this section, we develop the machinery of basic linear energy-dissipation, this is our start point of the nonlinear energy method.

Suppose that u and η are given, as well as $\mathcal{A}$, $\mathcal{N}$, etc., in terms of η. Then we consider the following system for triples (v,q,ζ):

$$\begin{cases} \partial_t v-\partial_t\bar\eta WK\partial_3 v+u\cdot\nabla_{\mathcal{A}}v+\operatorname{div}_{\mathcal{A}}S_{\mathcal{A}}(q,v)=F^1 & \text{in } \Omega,\\ \operatorname{div}_{\mathcal{A}}v=F^2 & \text{in } \Omega,\\ S_{\mathcal{A}}(q,v)\mathcal{N}=\zeta\mathcal{N}-\Delta_*\zeta\mathcal{N}+F^3 & \text{on } \Sigma,\\ \partial_t\zeta=v\cdot\mathcal{N}+F^4 & \text{on } \Sigma,\\ v=0 & \text{on } \Sigma_b. \end{cases} \tag{2.1}$$

Here (v, q, ζ) represents the one temporal derivative of (u, p, η) or the horizontal spatial derivatives of (u, p, η) up to second order. The following lemma is the start point of horizontal energy-dissipation estimates.

Lemma 2.1. *Suppose that u and η are given and (v, q, ζ) are sufficiently smooth satisfying (2.1). Then*

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|_{\mathcal{H}^0}^2 + \|\zeta\|_{H^1(\Sigma)}^2\right) + \frac{1}{2}\|v\|_{\mathcal{H}^1}^2 = \left(F^1, v\right)_{\mathcal{H}^0} + \left(q, F^2\right)_{\mathcal{H}^0} - \int_\Sigma v \cdot F^3 + \int_\Sigma \left(\zeta - \Delta_* \zeta\right) F^4, \tag{2.2}$$

where the space $\mathcal{H}^0$ and $\mathcal{H}^1$ are time-dependent spaces defined by

$$\mathcal{H}^0(t) = \left\{u : \Omega \to \mathbb{R}^3 | \sqrt{J} u \in L^2(\Omega)\right\} \text{ and } \mathcal{H}^1(t) = \left\{u : \int_\Omega \left|\mathbb{D}_{\mathcal{A}(t)} u\right|^2 J < \infty\right\},$$

respectively.

Proof. We take the L^2-inner product of the first equation in (2.1) with vJ over Ω and get

$$\int_\Omega \left(\partial_t v - \partial_t \bar{\eta} W K \partial_3 v + u \cdot \nabla_{\mathcal{A}} v\right) \cdot vJ + \int_\Omega \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q, v) \cdot vJ := I + II = \int_\Omega F^1 \cdot vJ. \tag{2.3}$$

A simple computation by changing of variables gives

$$I = \int_\Omega \left(\partial_t v - \partial_t \bar{\eta} W K \partial_3 v + u \cdot \nabla_{\mathcal{A}} v\right) \cdot vJ = \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 J. \tag{2.4}$$

Then by integration by parts,

$$\begin{aligned} II = \int_\Omega \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q, v) \cdot vJ &= \int_\Omega -q \operatorname{div}_{\mathcal{A}} vJ + \frac{1}{2}\left|\mathbb{D}_{\mathcal{A}} v\right|^2 J + \int_\Sigma S_{\mathcal{A}}(q, v)\mathcal{N} \cdot v \\ &= \int_\Omega -q \operatorname{div}_{\mathcal{A}} vJ + \frac{1}{2}\left|\mathbb{D}_{\mathcal{A}} v\right|^2 J + \int_\Sigma \left(\zeta - \Delta_* \zeta\right)(v \cdot \mathcal{N}) + v \cdot F^3. \end{aligned} \tag{2.5}$$

Then another integration by parts shows

$$\begin{aligned} \int_\Sigma \left(\zeta - \Delta_* \zeta\right)(v \cdot \mathcal{N}) = \int_\Sigma \left(\zeta - \Delta_* \zeta\right)\left(\partial_t \zeta - F^4\right) &= \int_\Sigma \zeta \partial_t \zeta + \nabla_* \zeta \partial_t \nabla_* \zeta - \left(\zeta - \Delta_* \zeta\right) F^4 \\ &= \frac{1}{2}\frac{d}{dt}\int_\Sigma |\zeta|^2 + |\nabla_* \zeta|^2 - \int_\Sigma \left(\zeta - \Delta_* \zeta\right) F^4. \end{aligned} \tag{2.6}$$

Consequently, (2.2) holds after combining (2.4)-(2.6). ☐

Remark 2.2. *The space $\mathcal{H}^0$ is endowed with the norm $\left(\int_\Omega |u|^2 J\right)^{1/2}$, that is comparable with the usual $L^2(\Omega)$ norm, and the space $\mathcal{H}^1$ is equivalent to the usual $H^1(\Omega)$. The equivalence is referred to [13, Lemma 2.1] for small data of η, and [28, Lemma 2.9] for large data of η. The $\mathcal{H}^0$ is introduced naturally, since after a change of variables, the $L^2(\Omega(t))$-norm is reduced to $\mathcal{H}^0$-norm.*

3. Elliptic estimates

In this section, we give the elliptic analysis based on the bootstrap argument in low regularity Sobolev spaces. The results are important in section 4 and section 5.

Suppose η, $\mathcal{A}$, $\mathcal{N}$, etc., are given. We consider the following elliptic systems

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q, v) = G^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} v = G^2 & \text{in } \Omega, \\ S_{\mathcal{A}}(q, v)\mathcal{N} = \xi \mathcal{N} - \Delta_* \xi \mathcal{N} + G^3 & \text{on } \Sigma, \\ v \cdot \mathcal{N} = G^4 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3.1}$$

where $\Delta_* = \partial_1^2 + \partial_2^2$ is the horizontal Laplace operator.

In order to solve (3.1), we first consider the case $\xi = \eta = 0$, so that (3.1) is reduced to the usual Stokes system with constant coefficient and we will handle (v, q) decoupling with ξ by the fixed point theory. We will treat (3.1) as a perturbation of the usual Stokes system and use the bootstrap argument to obtain the elliptic estimates. The key point of the bootstrap argument is the equation $\operatorname{div}_{\mathcal{A}} v = G^2$, since this ensures that the normal part of v could be represented by the horizontal components. If we suppose that $\|\eta\|_3^2 \le \delta$ for a sufficiently small δ, then by the Sobolev embedded theory for $\|\nabla_* \eta\|_{L^\infty(\Sigma)}^2 + \|\nabla \bar{\eta}\|_{L^\infty(\Omega)}^2 \lesssim \|\eta\|_3^2$, we have

$$\|\mathcal{N}\|_{L^\infty(\Sigma)}^2 + \|\mathcal{A}\|_{L^\infty(\Omega)}^2 \lesssim 1. \tag{3.2}$$

3.1. **Stokes system.** We first consider the following Stokes system of constant coefficients

$$\begin{cases} \operatorname{div} S(q,v) = R^1 & \text{in } \Omega, \\ \operatorname{div} v = R^2 & \text{in } \Omega, \\ \mathbb{D} v \nu \cdot \tau = R^3 & \text{on } \Sigma, \\ v \cdot \nu = R^4 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3.3}$$

where ν and τ are the unit normal and tangential of Σ, respectively. We now formulate a definition of weak solutions to (3.3).

Definition 3.1. *Suppose $R^1 \in (H^1)^*(\Omega)$, $R^2 \in H^0(\Omega)$, $R^3 \in H^{-1/2}(\Sigma)$ and $R^4 \in H^{1/2}(\Sigma)$ satisfying*

$$\int_\Omega R^2 = \int_\Sigma R^4. \tag{3.4}$$

A pair $(v,q) \in H^1(\Omega) \times H^0(\Omega)$ is called a weak solution of (3.3), provided that $\operatorname{div} v = R^2$, $v \cdot \nu = R^4$ on Σ, $v = 0$ on Σ_b, and

$$\frac{1}{2}(v,w)_1 - (q, \operatorname{div} w)_0 = \left\langle R^1, w \right\rangle_* + \left\langle R^3, w\cdot\tau \right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \tag{3.5}$$

for any $w \in \mathscr{W} = \left\{ w \in H^1(\Omega) | w \cdot \nu = 0 \text{ on } \Sigma,\ w = 0 \text{ on } \Sigma_b \right\}$.

Now we establish the existence and uniqueness of weak solutions for (3.1).

Lemma 3.1. *Let $\left(R^1, R^2, R^3, R^4\right)$ be as in Definition 3.1, there exists a unique pair $(v,q) \in H^1(\Omega) \times H^0(\Omega)$ as a weak solution to (3.1). Furthermore,*

$$\|v\|_1^2 + \|q\|_0^2 \lesssim \left\|R^1\right\|_*^2 + \left\|R^2\right\|_0^2 + \left\|R^3\right\|_{-1/2}^2 + \left\|R^4\right\|_{1/2}^2. \tag{3.6}$$

Proof. We first consider

$$\begin{cases} \Delta \varphi = R^2 & \text{in } \Omega, \\ \nabla \varphi \cdot \nu = R^4 & \text{on } \Sigma, \\ \nabla \varphi \cdot \nu = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.7}$$

From the well-known theory of elliptic equations subjecting to Neumann boundary conditions, there exists $\varphi \in H^2(\Omega)$ satisfying

$$\|\nabla \varphi\|_1^2 \lesssim \left\|R^2\right\|_0^2 + \left\|R^4\right\|_{1/2}^2. \tag{3.8}$$

Then, as [3, Lemma 4.2], there exists a vector-valued function $\psi \in H^2(\Omega)$ satisfying

$$\begin{cases} \psi = 0,\ \nabla \psi \cdot \nu = \nu \times (-\partial_1 \varphi, -\partial_2 \varphi, 0)^T & \text{on } \Sigma_b, \\ \psi = 0 & \text{on } \Sigma \end{cases}$$

such that

$$\begin{cases} -\nabla \cdot (\nabla \times \psi) = 0 & \text{in } \Omega, \\ (\nabla \times \psi) \cdot \nu = 0 & \text{on } \Sigma, \\ \nabla \times \psi = -\left(\nabla_* \varphi, 0\right)^T & \text{on } \Sigma_b. \end{cases} \tag{3.9}$$

Combining the extension from the boundary $\partial\Omega$ to Ω (The similar method is also used in [18, page 25-27]), it satisfies

$$\|\nabla \times \psi\|_{H^1}^2 \lesssim \|\nabla \varphi\|_1^2 \lesssim \left\|R^2\right\|_0^2 + \left\|R^4\right\|_{1/2}^2.$$

So we can derive

$$\begin{cases} \operatorname{div}(v - \nabla\varphi - \nabla\times\psi) = 0 & \text{in } \Omega, \\ (v - \nabla\varphi - \nabla\times\psi)\cdot\nu = 0 & \text{on } \Sigma, \\ v - \nabla\varphi - \nabla\times\psi = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.10}$$

Then we switch the unknowns to $\bar{v} = v - \nabla\varphi - \nabla\times\psi$, and then restrict the test functions to $w \in {}^0_0H^1_\sigma(\Omega)$ so that $\operatorname{div}\bar{v} = 0$ and the pressureless weak formulation for v is

$$\frac{1}{2}(\bar{v}, w)_1 = -\frac{1}{2}(\nabla\varphi + \nabla\times\psi, w)_1 + \left\langle R^1, w\right\rangle_* + \left\langle R^3, w\cdot\tau\right\rangle_{H^{-1/2}(\Sigma)\times H^{1/2}(\Sigma)} \tag{3.11}$$

for any $w \in {}^0_0H^1_\sigma(\Omega)$. Here the space ${}^0_0H^1_\sigma(\Omega)$ is defined as

$${}^0_0H^1_\sigma(\Omega) = \left\{u \in H^1(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega,\ w\cdot\nu = 0 \text{ on } \Sigma,\ w = 0 \text{ on } \Sigma_b\right\}.$$

Then the Riesz representation theorem provides a unique $\bar{v} \in {}^0_0H^1_\sigma(\Omega)$ satisfying (3.11) and

$$\|\bar{v}\|_1^2 \lesssim \|\varphi\|_2^2 + \|\psi\|_2^2 + \left\|R^1\right\|_*^2 + \left\|R^3\right\|_{-1/2}^2 \lesssim \left\|R^1\right\|_*^2 + \left\|R^2\right\|_0^2 + \left\|R^3\right\|_{-1/2}^2 + \left\|R^4\right\|_{1/2}^2.$$

To introduce the pressure q, we define a linear functional on $(\mathscr{W})^*$ as the difference of left-hand side and the right-hand side of (3.11). Then treating pressure as the Lagrangian of Stokes operator(see for instance [25]), there exists a unique $q \in H^0(\Omega)$ (possibly up to a constant) satisfying

$$\|q\|_0^2 \lesssim \left\|R^1\right\|_*^2 + \left\|R^2\right\|_0^2 + \left\|R^3\right\|_{-1/2}^2 + \left\|R^4\right\|_{1/2}^2. \tag{3.12}$$

□

Now we develop the second-order regularity of (3.3).

Proposition 3.2. *Suppose $R^1 \in H^0(\Omega)$, $R^2 \in H^1(\Omega)$, $R^3 \in H^{1/2}(\Sigma)$ and $R^4 \in H^{3/2}(\Sigma_b)$ satisfying*

$$\int_\Omega R^2 = \int_\Sigma R^4. \tag{3.13}$$

Then the problem (3.3) admits a unique strong solution $(v, q) \in H^2(\Omega)\times H^1(\Omega)$ and

$$\|v\|_2^2 + \|q\|_1^2 \lesssim \left\|R^1\right\|_0^2 + \left\|R^2\right\|_1^2 + \left\|R^3\right\|_{1/2}^2 + \left\|R^4\right\|_{3/2}^2. \tag{3.14}$$

Proof. Since the Stokes system (3.3) is elliptic, following similar arguments in [25], we can get the result. Furthermore, [1, Theorem 10.5] provides (3.14). □

Rewriting the Stokes system (3.1) as the following perturbation form:

$$\begin{cases} \operatorname{div} S(q, v) = R^1 & \text{in } \Omega, \\ \operatorname{div} v = R^2 & \text{in } \Omega, \\ \mathbb{D}v\nu\cdot\tau = R^3 & \text{on } \Sigma, \\ v\cdot\nu = R^4 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3.15}$$

and

$$q - \frac{1}{|\mathcal{N}|^2}\mathbb{D}_{\mathcal{A}}v\mathcal{N}\cdot\mathcal{N} = \xi - \Delta_*\xi + \frac{1}{|\mathcal{N}|^2}G^3\cdot\mathcal{N} \quad \text{on } \Sigma, \tag{3.16}$$

where

$$R^1 = R^1(v, q) = G^1 + \operatorname{div}_{I-\mathcal{A}} S_{\mathcal{A}}(q, v) - \operatorname{div}\mathbb{D}_{I-\mathcal{A}}v, \qquad R^2 = R^2(v) = G^2 + \operatorname{div}_{I-\mathcal{A}} v,$$

$$R^3 = R^3(v) = -G^3\cdot\mathcal{T} + \mathbb{D}_{I-\mathcal{A}}v\mathcal{N}\cdot\mathcal{T} + \mathbb{D}v(\nu - \mathcal{N})\cdot\tau + \mathbb{D}v\mathcal{N}\cdot(\tau - \mathcal{T}), \quad R^4 = R^4(v) = G^4 + v\cdot(\nu - \mathcal{N}),$$

and $\mathcal{T}$ is the tangential of free surface.

Theorem 3.3. *Suppose $G^1 \in H^0(\Omega)$, $G^2 \in H^1(\Omega)$, $G^3 \in H^{1/2}(\Sigma)$, $G^4 \in H^{3/2}(\Sigma)$ and $\|\eta\|_{5/2} \leq \delta$ for δ sufficiently small. Then there exists a unique triple $(v, q, \xi) \in H^2(\Omega) \times H^1(\Omega) \times H^{5/2}(\Sigma)$ solving (3.1). Moreover,*

$$\|v\|_2^2 + \|q\|_1^2 + \|\xi\|_{5/2}^2 \lesssim \left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2. \tag{3.17}$$

Furthermore, if $G^1 \in H^1(\Omega)$, $G^2 \in H^2(\Omega)$, $G^3 \in H^{3/2}(\Sigma)$ and $G^4 \in H^{5/2}(\Sigma)$. Then

$$\|v\|_3^2 + \|q\|_2^2 + \|\xi\|_{7/2}^2 \lesssim \left\|G^1\right\|_1^2 + \left\|G^2\right\|_2^2 + \left\|G^3\right\|_{3/2}^2 + \left\|G^4\right\|_{5/2}^2 + \|\eta\|_{7/2}^2 \left(\left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2\right). \tag{3.18}$$

Proof. We will solve (3.15) and (3.16) by Banach's fixed point theorem. For any $(u, p) \in H^2(\Omega) \times H^1(\Omega)$, we have

$$\begin{aligned} &\left\|R^1(u,p)\right\|_0^2 + \left\|R^2(u)\right\|_1^2 + \left\|R^3(u)\right\|_{1/2}^2 + \left\|R^4(u)\right\|_{3/2}^2 \\ \lesssim &\left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2 + P\left(\|\eta\|_{5/2}^2\right)\left(\|u\|_2^2 + \|p\|_1^2\right), \end{aligned} \tag{3.19}$$

where $P(\cdot)$ is a polynomial satisfying $P(0) = 0$. Now we consider the linear problem:

$$\begin{cases} \operatorname{div} S(q, v) = R^1(u, p) & \text{in } \Omega, \\ \operatorname{div} v = R^2(u) & \text{in } \Omega, \\ \mathbb{D}v\nu \cdot \tau = R^3(u) & \text{on } \Sigma, \\ v \cdot \nu = R^4(u) & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.20}$$

From Proposition 3.2, there exists a unique pair $(v, q) \in H^2(\Omega) \times H^1(\Omega)$ which solves (3.20). Moreover, by (3.19), we have

$$\|v\|_2^2 + \|q\|_1^2 \lesssim \left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2 + P\left(\|\eta\|_{5/2}^2\right)\left(\|u\|_2^2 + \|p\|_1^2\right). \tag{3.21}$$

Similarly, suppose $(\tilde{u}, \tilde{p}) \in H^2(\Omega) \times H^1(\Omega)$ and $(\tilde{v}, \tilde{q})$ solves

$$\begin{cases} \operatorname{div} S(\tilde{q}, \tilde{v}) = R^1(\tilde{u}, \tilde{p}) & \text{in } \Omega, \\ \operatorname{div} \tilde{v} = R^2(\tilde{u}) & \text{in } \Omega, \\ \mathbb{D}\tilde{v}\nu \cdot \tau = R^3(\tilde{u}) & \text{on } \Sigma, \\ \tilde{v} \cdot \nu = R^4(\tilde{u}) & \text{on } \Sigma, \\ \tilde{v} = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.22}$$

Subtracting (3.22) from (3.20), we see $v - \tilde{v}$ satisfies

$$\begin{cases} \operatorname{div} S\left(q - \tilde{q}, v - \tilde{v}\right) = R^1(u, p) - R^1\left(\tilde{u}, \tilde{p}\right) & \text{in } \Omega, \\ \operatorname{div}\left(v - \tilde{v}\right) = R^2(u) - R^2(\tilde{u}) & \text{in } \Omega, \\ \mathbb{D}(v - \tilde{v})\nu \cdot \tau = R^3(u) - R^3\left(\tilde{u}\right) & \text{on } \Sigma, \\ (v - \tilde{v}) \cdot \nu = R^4(u) - R^4\left(\tilde{u}\right) & \text{on } \Sigma, \\ v - \tilde{v} = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.23}$$

Then by Proposition 3.2, we obtain

$$\|v - \tilde{v}\|_2^2 + \|q - \tilde{q}\|_1^2 \leq P\left(\|\eta\|_{5/2}^2\right)\left(\|u - \tilde{u}\|_2^2 + \|p - \tilde{p}\|_1^2\right). \tag{3.24}$$

We can take a small constant δ such that $P\left(\|\eta\|_{5/2}^2\right) < \frac{1}{2}$, provided $\|\eta\|_{5/2} \leq \delta$. Then by Banach's fixed point Theorem (see for instance [8]), the mapping $(u, p) \mapsto (v, q)$ has a unique fixed point. Therefore the Stokes problem (3.15) has a unique solution $(v, q) \in H^2(\Omega) \times H^1(\Omega)$ which satisfies

$$\|v\|_2^2 + \|q\|_1^2 \lesssim \left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2. \tag{3.25}$$

For any $\zeta \in H^1(\Sigma)$, by (3.16), we have

$$\int_\Sigma (\xi\zeta + \nabla_*\xi \cdot \nabla_*\zeta) = \int_\Sigma q\zeta - \frac{1}{|\mathcal{N}|^2}\mathbb{D}_\mathcal{A} v\mathcal{N}\cdot\mathcal{N}\zeta - \frac{1}{|\mathcal{N}|^2}G^3\cdot\mathcal{N}\zeta. \tag{3.26}$$

By the Hölder inequality and Sobolev inequality, we have

$$\int_\Sigma (\xi\zeta + \nabla_*\xi \cdot \nabla_*\zeta) \lesssim \|\xi\|_{L^2}\|\zeta\|_{L^2} + \|\nabla_*\xi\|_{L^2}\|\nabla_*\zeta\|_{L^2} \lesssim \|\xi\|_1^2 + \|\zeta\|_1^2. \tag{3.27}$$

Then the Lax-Milgram Theorem guarantees (3.16) has a unique weak solution $\xi \in H^1(\Sigma)$. By the regularity theory of elliptic equations and trace theory, we have

$$\begin{aligned} \|\xi\|_{5/2}^2 &\lesssim \|\xi\|_{1/2}^2 + \left\|\nabla_*^2\xi\right\|_{1/2}^2 \lesssim \|\xi\|_{1/2}^2 + \|q\|_{H^{1/2}(\Sigma)}^2 + \|v\|_{H^{3/2}(\Sigma)}^2 + \left\|G^3\right\|_{1/2}^2 \\ &\lesssim \left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2}^2 + \left\|G^4\right\|_{3/2}^2. \end{aligned} \tag{3.28}$$

We now use a bootstrap argument to enhance the regularity of solutions for (3.1). For $j = 1, 2$, we apply the horizontal derivative to (3.1) and get

$$\begin{cases} \operatorname{div}_\mathcal{A} S_\mathcal{A}(\partial_j q, \partial_j v) = G^{1,1} & \text{in } \Omega, \\ \operatorname{div}_\mathcal{A} \partial_j v = G^{2,1} & \text{in } \Omega, \\ S_\mathcal{A}(\partial_j q, \partial_j v)\mathcal{N} = \partial_j\xi\mathcal{N} - \Delta_*\partial_j\xi\mathcal{N} + G^{3,1} & \text{on } \Sigma, \\ \partial_j v\cdot\mathcal{N} = G^{4,1} & \text{on } \Sigma, \\ \partial_j v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3.29}$$

where

$$\begin{aligned} G^{1,1} &= \partial_j G^1 - \nabla_{\partial_j\mathcal{A}} q + \operatorname{div}_\mathcal{A}\mathbb{D}_{\partial_j\mathcal{A}} v + \operatorname{div}_{\partial_j\mathcal{A}}\mathbb{D}_\mathcal{A} v, \qquad & G^{2,1} &= \partial_j G^2 - \operatorname{div}_{\partial_j\mathcal{A}} v, \\ G^{3,1} &= \partial_j G^3 - S_\mathcal{A}(q,v)\partial_j\mathcal{N} + \mathbb{D}_{\partial_j\mathcal{A}} v\mathcal{N} + \xi\partial_j\mathcal{N} - \delta_*^2\xi\partial_j\mathcal{N}, \qquad & G^{4,1} &= \partial_j G^4 - v\partial_j\mathcal{N}. \end{aligned} \tag{3.30}$$

Then (3.17) guarantees

$$\|\partial_j v\|_2^2 + \|\partial_j q\|_1^2 + \|\partial_j\xi\|_{5/2}^2 \lesssim \left\|G^{1,1}\right\|_0^2 + \left\|G^{2,1}\right\|_1^2 + \left\|G^{3,1}\right\|_{1/2}^2 + \left\|G^{4,1}\right\|_{3/2}^2. \tag{3.31}$$

Direct calculation for the estimates of forcing terms $G^{k,1}$, $k = 1, 2, 3, 4$, by utilizing the Sobolev inequalities, the Hölder inequalities and interpolation estimate, are as follows. For ε sufficiently small and any $3/2 < s < 2$, we have

$$\begin{aligned} \left\|G^{1,1}\right\|_0^2 \lesssim{}& \left\|\partial_j G^1\right\|_0^2 + \|\partial_j\mathcal{A}\|_{L^4}^2\|\nabla q\|_{L^4}^2 + \|\mathcal{A}\|_{L^\infty}^2\left(\|\partial_j\mathcal{A}\|_{L^4}^2\left\|\nabla^2 v\right\|_{L^4}^2 + \|\nabla\partial_j\mathcal{A}\|_{L^2}^2\|\nabla v\|_{L^\infty}^2\right) \\ &+ \|\partial_j\mathcal{A}\|_{L^4}^2\left(\|\mathcal{A}\|_{L^\infty}^2\left\|\nabla^2 v\right\|_{L^2}^2 + \|\nabla\mathcal{A}\|_{L^4}^2\|\nabla v\|_{L^\infty}^2\right) \\ \lesssim{}& \left\|\partial_j G^1\right\|_0^2 + \|\eta\|_{5/2}^2\left(\|v\|_2^2 + \|\nabla v\|_s^2 + \|\nabla q\|_{L^2}^{1/2}\left\|\nabla^2 q\right\|_{L^2}^{3/2} + \left\|\nabla^2 v\right\|_{L^2}^{1/2}\left\|\nabla^3 v\right\|_{L^2}^{3/2}\right) \\ \lesssim{}& \left\|\partial_j G^1\right\|_0^2 + \|\eta\|_{5/2}^2\left(\|v\|_2^2 + \left\|\nabla^2 v\right\|_{L^2}^{2(2-s)}\left\|\nabla^3 v\right\|_{L^2}^{2(s-1)} + \|\nabla q\|_{L^2}^{1/2}\left\|\nabla^2 q\right\|_{L^2}^{3/2} + \left\|\nabla^2 v\right\|_{L^2}^{1/2}\left\|\nabla^3 v\right\|_{L^2}^{3/2}\right) \\ \lesssim{}& \left\|\partial_j G^1\right\|_0^2 + \|\eta\|_{5/2}^2\left(\|v\|_2^2 + \|q\|_1^2\right) + \varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2 + \left\|\nabla^2 q\right\|_{L^2}^2\right). \end{aligned} \tag{3.32}$$

Similarly, we have

$$\begin{aligned} \left\|G^{2,1}\right\|_1^2 &\lesssim \left\|\partial_j G^2\right\|_1^2 + \|\partial_j\mathcal{A}\|_{L^4}^2\left(\|\nabla v\|_{L^4}^2 + \left\|\nabla^2 v\right\|_{L^4}^2\right) + \|\nabla\partial_j\mathcal{A}\|_{L^2}^2\|\nabla v\|_{L^\infty}^2 \\ &\lesssim \left\|\partial_j G^2\right\|_1^2 + \|\eta\|_{5/2}^2\|v\|_2^2 + \varepsilon\left\|\nabla^3 v\right\|_{L^2}^2. \end{aligned} \tag{3.33}$$

Then by the trace embedding and extension theory in Sobolev spaces along with (3.2), we have

$$\begin{aligned}\|S_{\mathcal{A}}(q,v)\partial_j\mathcal{N}\|_{H^{1/2}(\Sigma)}^2 &\lesssim \|S_{\mathcal{A}}(q,v)\partial_j\nabla_*\bar\eta\|_{H^1(\Omega)}^2\\ &\lesssim \left(\|q\|_s^2+\|\mathcal{A}\|_{L^\infty}^2\|\nabla v\|_s^2\right)\|\nabla\partial_j\nabla_*\bar\eta\|_{L^2}^2+\|\partial_j\nabla_*\bar\eta\|_{L^4}^2\left(\|q\|_{L^4}^2+\|\nabla q\|_{L^4}^2\right)\\ &\quad+\|\partial_j\nabla_*\bar\eta\|_{L^4}^2\|\mathcal{A}\|_{L^\infty}^2\left(\|\nabla v\|_{L^4}^2+\left\|\nabla^2 v\right\|_{L^4}^2\right)+\|\partial_j\nabla_*\bar\eta\|_{L^6}^2\|\nabla\mathcal{A}\|_{L^6}^2\|\nabla v\|_{L^6}^2\\ &\lesssim \|\eta\|_{5/2}^2\left(\|v\|_2^2+\|q\|_1^2\right)+\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2\right),\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}\left\|\mathbb{D}_{\partial_j\mathcal{A}}v\mathcal{N}\right\|_{H^{1/2}(\Sigma)}^2 &\lesssim \left\|\mathbb{D}_{\partial_j\mathcal{A}}v\right\|_1^2+\left\|\mathbb{D}_{\partial_j\mathcal{A}}v\nabla_*\eta\right\|_{H^{1/2}(\Sigma)}^2\\ &\lesssim \|\partial_j\mathcal{A}\|_{L^4}^2\left(\|\nabla v\|_{L^4}^2+\left\|\nabla^2 v\right\|_{L^4}^2\right)+\left(1+\|\nabla_*\bar\eta\|_{L^\infty}^2\right)\|\nabla\partial_j\mathcal{A}\|_{L^2}^2\|\nabla v\|_{L^\infty}^2\\ &\quad+\|\nabla_*\bar\eta\|_{L^\infty}^2\|\partial_j\mathcal{A}\|_{L^4}^2\left\|\nabla^2 v\right\|_{L^4}^2+\|\nabla\nabla_*\bar\eta\|_{L^6}^2\|\partial_j\mathcal{A}\|_{L^6}^2\|\nabla v\|_{L^6}^2\\ &\lesssim \|\eta\|_{5/2}^2\|v\|_2^2+\varepsilon\left\|\nabla^3 v\right\|_{L^2}^2.\end{aligned}\tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\begin{aligned}\left\|G^{3,1}\right\|_{1/2}^2 &\lesssim \left\|\partial_j G^3\right\|_{1/2}^2+\|\eta\|_{5/2}^2\left(\|v\|_2^2+\|q\|_1^2+\|\xi\|_{7/2}^2+\left\|G^5\right\|_{5/2}^2\right)+\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2\right)\\ &\lesssim \left\|\partial_j G^3\right\|_{1/2}^2+\|\eta\|_{5/2}^2\left(\|v\|_2^2+\|q\|_1^2+\|\eta\|_{5/2}^2+\left\|G^5\right\|_{5/2}^2\right)+\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.36}$$

For the term $G^{4,1}$, we have

$$\left\|G^{4,1}\right\|_{3/2}^2 \lesssim \left\|\partial_j G^4\right\|_{3/2}^2+\|v\|_2^2\|\eta\|_{7/2}^2.\tag{3.37}$$

Then (3.17), (3.31), (3.33), (3.36) and (3.37) imply for $j=1,2$,

$$\begin{aligned}&\|\partial_j v\|_2^2+\|\partial_j q\|_1^2+\|\partial_j\xi\|_{5/2}^2\\ \le& C\left(1+\|\eta\|_{7/2}^2\right)\left(\left\|G^1\right\|_0^2+\left\|G^2\right\|_1^2+\left\|G^3\right\|_{1/2}^2+\left\|G^4\right\|_{3/2}^2\right)\\ &+C\left(\left\|\partial_j G^1\right\|_0^2+\left\|\partial_j G^2\right\|_1^2+\left\|\partial_j G^3\right\|_{1/2}^2+\left\|\partial_j G^4\right\|_{3/2}^2\right)+C\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.38}$$

Finally, we turn to the estimate of $\partial_3 v$ and $\partial_3 q$. From the three components of $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q,v)=G^1$, we have

$$\begin{aligned}&-\left(\partial_{11}v_1+A\partial_{11}v_3\right)-\left(\partial_{22}v_1+A\partial_{22}v_3\right)-\left(1+A^2+B^2\right)K^2\left(\partial_{33}v_1+A\partial_{33}v_3\right)+2AK\left(\partial_{13}v_1+A\partial_{13}v_3\right)\\ &+2BK\left(\partial_{23}v_1+A\partial_{23}v_3\right)-\left(AK\partial_3(AK)+BK\partial_3(BK)-\partial_1(AK)-\partial_2(BK)+K\partial_3K\right)\left(\partial_3 v_1+A\partial_3 v_3\right)+\partial_1 q\\ =&G_1^1-\partial_1 G^2+AG_3^1,\end{aligned}$$

which contains the singular term $\left(1+A^2+B^2\right)K^2\left(\partial_{33}v_1+A\partial_{33}v_3\right)$. Applying ∂_3 on both sides of the above equation, the term $\left(1+A^2+B^2\right)K^2\left(\partial_3^3 v_1+A\partial_3^3 v_3\right)$ implies

$$\begin{aligned}\left\|\partial_3^3 v_1+A\partial_3^3 v_3\right\|_0^2 &\lesssim \left(1+\|\eta\|_{7/2}^2\right)\left(\left\|G^1\right\|_0^2+\left\|G^2\right\|_1^2+\left\|G^3\right\|_{1/2}^2+\left\|G^4\right\|_{3/2}^2\right)+\left\|\nabla G^1\right\|_0^2+\left\|\nabla G^2\right\|_1^2\\ &\quad+\left\|\nabla_* G^3\right\|_{1/2}^2+\left\|\nabla_* G^4\right\|_{3/2}^2+\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.39}$$

Similarly, we have

$$\begin{aligned}\left\|\partial_3^3 v_2+B\partial_3^3 v_3\right\|_0^2 &\lesssim \left(1+\|\eta\|_{7/2}^2\right)\left(\left\|G^1\right\|_0^2+\left\|G^2\right\|_1^2+\left\|G^3\right\|_{1/2}^2+\left\|G^4\right\|_{3/2}^2\right)+\left\|\nabla G^1\right\|_0^2+\left\|\nabla G^2\right\|_1^2\\ &\quad+\left\|\nabla_* G^3\right\|_{1/2}^2+\left\|\nabla_* G^4\right\|_{3/2}^2+\varepsilon\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^2 q\right\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.40}$$

Since the divergence equation $\operatorname{div}_{\mathcal{A}} v=G^2$ is equivalent to

$$K\left(1+A^2+B^2\right)\partial_3 v_3=G^2-\partial_1 v_1-\partial_2 v_2+AK\left(\partial_3 v_1+A\partial_3 v_3\right)+BK\left(\partial_3 v_2+B\partial_3 v_3\right),$$

we can take ∂_3^2 on both sides of the above equation and get

$$\begin{aligned}\|\partial_3^3 v_3\|_0^2 \lesssim &\left(1+\|\eta\|_{7/2}^2\right)\left(\|G^1\|_0^2+\|G^2\|_1^2+\|G^3\|_{1/2}^2+\|G^4\|_{3/2}^2\right)+\|\nabla G^1\|_0^2\\ &+\|\nabla G^2\|_1^2+\|\nabla_* G^3\|_{1/2}^2+\|\nabla_* G^4\|_{3/2}^2+\varepsilon\left(\|\nabla^3 v\|_{L^2}^2+\|\nabla^2 q\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.41}$$

Thus, (3.39)–(3.41) imply

$$\begin{aligned}\|\partial_3^3 v\|_0^2 \lesssim &\left(1+\|\eta\|_{7/2}^2\right)\left(\|G^1\|_0^2+\|G^2\|_1^2+\|G^3\|_{1/2}^2+\|G^4\|_{3/2}^2\right)+\|\nabla G^1\|_0^2\\ &+\|\nabla G^2\|_1^2+\|\nabla_* G^3\|_{1/2}^2+\|\nabla_* G^4\|_{3/2}^2+\varepsilon\left(\|\nabla^3 v\|_{L^2}^2+\|\nabla^2 q\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.42}$$

Then we take ∂_3 on both sides of the third component of $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q,v)=G^1$ and obtain

$$\begin{aligned}\|\partial_3^2 q\|_0^2 \lesssim &\left(1+\|\eta\|_{7/2}^2\right)\left(\|G^1\|_0^2+\|G^2\|_1^2+\|G^3\|_{1/2}^2+\|G^4\|_{3/2}^2\right)+\|\nabla G^1\|_0^2\\ &+\|\nabla G^2\|_1^2+\|\nabla_* G^3\|_{1/2}^2+\|\nabla_* G^4\|_{3/2}^2+\varepsilon\left(\|\nabla^3 v\|_{L^2}^2+\|\nabla^2 q\|_{L^2}^2+\|\xi\|_{7/2}^2\right).\end{aligned}\tag{3.43}$$

Hence

$$\begin{aligned}\|v\|_3^2+\|q\|_2^2 \lesssim &\left(1+\|\eta\|_{7/2}^2\right)\left(\|G^1\|_0^2+\|G^2\|_1^2+\|G^3\|_{1/2}^2+\|G^4\|_{1/2}^2\right)+\|\nabla G^1\|_0^2+\|\nabla G^2\|_1^2\\ &+\|\nabla_* G^3\|_{1/2}^2+\|\nabla_* G^4\|_{1/2}^2+\varepsilon\left(\|\nabla^3 v\|_{L^2}^2+\|\nabla^2 q\|_{L^2}^2+\|\xi\|_{7/2}^2\right),\end{aligned}$$

taking ε sufficiently small, we get (3.18). □

Remark 3.4. *Actually, the elliptic estimates (3.17) and (3.18) can be generalized under the assumption $\|\eta-\eta_0\|_{5/2}^2 \le \delta$ instead of $\|\eta\|_{5/2}^2 \le \delta$, which could be done by employing the argument that $\mathcal{A}$ is a perturbation of $\mathcal{A}(0)$. This is crucial to deal with the local well-posedness with large initial data.*

4. Energy and dissipation estimates

In this section, we first establish the horizontal energy-dissipation estimates, that is Theorem 4.1. We also give elliptic estimates in Theorem 4.6. The two estimates are two ingredients for our *a priori estimates* in the section 5.

4.1. **The horizontal energy–dissipation estimate.**

In order to close our *a priori estimates* for (2.1), we need to consider the unknowns up to twice order derivatives, thus we have to confirm the forcing terms on the right-hand side of (2.1) with respect to derivatives of order 0, 1. Actually, if we set the horizontal energy and dissipation as

$$\mathcal{E}_{\shortparallel}=\int_\Omega\left(|u|^2+|\partial_t u|^2+|\nabla_* u|^2+\left|\nabla_*^2 u\right|^2\right)J+\|\partial_t\eta\|_{1,\Sigma}^2+\|\eta\|_{3,\Sigma}^2,$$

and

$$\mathcal{D}_{\shortparallel}=\int_\Omega\left(|\mathbb{D}_{\mathcal{A}} u|^2+|\mathbb{D}_{\mathcal{A}}\partial_t u|^2+|\mathbb{D}_{\mathcal{A}}\nabla_* u|^2+\left|\mathbb{D}_{\mathcal{A}}\nabla_*^2 u\right|^2\right)J,$$

we can prove the following result.

Theorem 4.1. *Suppose $(u(t),p(t),\eta(t))$ are solutions of (1.8) for $t\in[0,T]$ and $\sup_{t\in[0,T]}\mathcal{E}(t)\le\delta$ for the universal constant $\delta\in(0,1)$ given in Theorem 3.3. Then, for all $t\in[0,T]$,*

$$\frac{d}{dt}\left(\mathcal{E}_{\shortparallel}-\mathcal{P}\right)+\mathcal{D}_{\shortparallel}\lesssim\mathcal{E}^{1/2}\mathcal{D},$$

where

$$\mathcal{P}=\int_\Omega pF^2J.$$

Proof. Based on Lemma 2.1, we are supposed to take $(v,q,\xi)=(u,p,\eta)$, $(\partial_t u,\partial_t p,\partial_t\eta)$, $(\nabla_* u,\nabla_* p,\nabla_*\eta)$, and $(\nabla_*^2 u,\nabla_*^2 p,\nabla_*^2\eta)$, respectively, and estimate the corresponding forcing terms in the right-hand side of (2.2). The details of the proof is completely contained in Propositions 4.2–4.4. □

4.1.1. *The estimates for one temporal derivative.* When $(v,q,\xi)=(\partial_t u,\partial_t p,\partial_t\eta)$,

$$
\begin{aligned}
F^1 &= -\operatorname{div}_{\partial_t\mathcal{A}} S_{\mathcal{A}}(u,p) + \operatorname{div}_{\mathcal{A}} \mathbb{D}_{\partial_t\mathcal{A}} u - \partial_t u\cdot\nabla_{\mathcal{A}} u - u\cdot\nabla_{\partial_t\mathcal{A}} u + \partial_t^2\bar\eta KW\partial_3 u + \partial_t\bar\eta\partial_t KW\partial_3 u,\\
F^2 &= -\operatorname{div}_{\partial_t\mathcal{A}} u,\\
F^4 &= u\cdot\partial_t\mathcal{N},\\
F^3 &= \eta\partial_t\mathcal{N} - \Delta_*\eta\partial_t\mathcal{N} + \mathbb{D}_{\partial_t\mathcal{A}} u\mathcal{N} - S_{\mathcal{A}}(p,u)\partial_t\mathcal{N} - \frac{3\sum_{i,j=1}^2 \partial_i\eta\partial_j\eta\partial_{ij}\eta\left(\nabla_*\eta\cdot\nabla_*\partial_t\eta\right)}{\left(1+|\nabla_*\eta|^2\right)^{5/2}}\mathcal{N}\\
&\quad + \frac{|\nabla_*\eta|^2\Delta_*\eta}{\sqrt{1+|\nabla_*\eta|^2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)}\partial_t\mathcal{N} + \frac{\sum_{i,j=1}^2\partial_i\eta\partial_j\eta\partial_{ij}\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\partial_t\mathcal{N}\\
&\quad + \frac{|\nabla_*\eta|^2\Delta_*\eta\left(\nabla_*\eta\cdot\nabla_*\partial_t\eta\right)}{\left(1+|\nabla_*\eta|^2\right)^{3/2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)^2}\mathcal{N} + \frac{|\nabla_*\eta|^2\Delta_*\partial_t\eta + \left(\nabla_*\eta\cdot\nabla_*\partial_t\eta\right)\Delta_*\eta}{\sqrt{1+|\nabla_*\eta|^2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)}\mathcal{N}\\
&\quad + \frac{\sum_{i,j=1}^2\partial_i\eta\partial_j\eta\partial_{ij}\partial_t\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\mathcal{N} + \frac{2\sum_{i,j=1}^2\partial_i\partial_t\eta\partial_j\eta\partial_{ij}\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\mathcal{N}.
\end{aligned}
$$

We now give the estimates of the interaction between forcing terms and test functions in (2.2).

Proposition 4.2.

$$\left\|F^1 J\right\|_{H^0}^2 + \left\|F^3\right\|_{H^0}^2 + \left\|F^4\right\|_{H^0}^2 \lesssim \left(\mathcal{E}+\mathcal{E}^2\right)\mathcal{D}.$$

Proof. By the Hölder inequalities, the Sobolev inequality and trace theory along with (3.2), we have

$$
\begin{aligned}
\left\|F^1 J\right\|_{H^0}^2 &\lesssim \int_\Omega |\operatorname{div}_{\partial_t\mathcal{A}} S_{\mathcal{A}}(u,p)|^2 + |\operatorname{div}_{\mathcal{A}}\mathbb{D}_{\partial_t\mathcal{A}} u|^2 + |\partial_t u\cdot\nabla_{\mathcal{A}} u|^2 + |u\cdot\nabla_{\partial_t\mathcal{A}} u|^2\\
&\quad + \left|\partial_t^2\bar\eta KW\partial_3 u\right|^2 + |\partial_t\bar\eta\partial_t KW\partial_3 u|^2\\
&\lesssim \|\nabla\partial_t\bar\eta\|_{L^4}^2\left(\|\nabla p\|_{L^4}^2 + \left\|\nabla^2 u\right\|_{L^4}^2\right) + \|\nabla\partial_t\bar\eta\|_{L^6}^2\left\|\nabla^2\bar\eta\right\|_{L^6}^2\|\nabla u\|_{L^6}^2 + \left\|\nabla^2\partial_t\bar\eta\right\|_{L^4}^2\|\nabla u\|_{L^4}^2\\
&\quad + \|\partial_t u\|_{L^4}^2\|\nabla u\|_{L^4}^2 + \|u\|_{L^6}^2\|\nabla u\|_{L^6}^2\|\nabla\partial_t\bar\eta\|_{L^6}^2 + \left\|\partial_t^2\bar\eta\right\|_{L^4}^2\|\nabla u\|_{L^4}^2 + \|\partial_t\bar\eta\|_{L^6}^2\|\nabla\partial_t\bar\eta\|_{L^6}^2\|\nabla u\|_{L^6}^2\\
&\lesssim \|\partial_t\eta\|_{3/2}^2\left(\|p\|_2^2+\|u\|_3^2\right) + \left(\|\partial_t\eta\|_{5/2}^2 + \|\partial_t u\|_1^2 + \left\|\partial_t^2\eta\right\|_{1/2}^2\right)\|u\|_2^2 + \|u\|_2^4\|\partial_t\eta\|_{3/2}^2 + \|u\|_2^2\|\partial_t\eta\|_{3/2}^4\\
&\lesssim \left(\mathcal{E}+\mathcal{E}^2\right)\mathcal{D}.
\end{aligned}
$$

By (3.2) and the Sobolev inequalities, we have

$$
\begin{aligned}
\left\|F^3\right\|_0^2 &\lesssim \|(\eta-\Delta_*\eta+p)\nabla\partial_t\eta\|_0^2 + \|\nabla\partial_t\bar\eta\nabla u\|_{H^0(\Sigma)}^2 + \left\||\nabla_*\eta|^2\nabla_*\partial_t\eta\right\|_0^2\\
&\lesssim \|\eta\|_{7/2}^2\|\partial_t\eta\|_1^2 + \left(\|p\|_1^2+\|\eta\|_3^2\right)\|\partial_t\eta\|_{5/2}^2 + \|u\|_0^2\|\partial_t\eta\|_{3/2}^2 \lesssim \mathcal{E}\mathcal{D}.
\end{aligned}
$$

Finally, by the Sobolev inequalities, we have

$$\left\|F^4\right\|_0^2 \lesssim \|u\nabla\partial_t\eta\|_{H^0(\Sigma)}^2 \lesssim \|u\|_2^2\|\partial_t\eta\|_1^2 \lesssim \mathcal{E}\mathcal{D}.$$

□

We now consider the estimate of the pressure term. Since we do not have the estimate for $\partial_t p$, so we turn to the following argument.

Proposition 4.3.

$$\int_\Omega \partial_t p F^2 J - \frac{d}{dt}\int_\Omega p F^2 J \lesssim \left(\mathcal{E}^{1/2}+\mathcal{E}\right)\mathcal{D}.$$

Proof. We rewrite the integration as

$$\int_\Omega \partial_t p F^2 J = \frac{d}{dt}\int_\Omega pF^2 J - \int_\Omega p\partial_t F^2 J - \int_\Omega pF^2\partial_t J.$$

Direct calculation yields

$$\begin{aligned}
&\left|\int_\Omega p\partial_t F^2 J + \int_\Omega pF^2\partial_t J\right| \\
&\lesssim \int_\Omega |p|\left(\left|\nabla\partial_t^2\bar\eta\right|\left|\nabla u\right| + \left|\nabla\partial_t\bar\eta\right|\left|\nabla\partial_t u\right| + \left|\nabla\partial_t\bar\eta\right|^2\left|\nabla u\right|\right) \\
&\lesssim \|p\|_{L^4}\left(\left\|\nabla\partial_t^2\bar\eta\right\|_{L^2}\|\nabla u\|_{L^4} + \|\nabla\partial_t\bar\eta\|_{L^4}\|\nabla\partial_t u\|_{L^2}\right) + \|p\|_{L^4}\|\nabla\partial_t\bar\eta\|_{L^4}^2\|\nabla u\|_{L^4} \\
&\lesssim \|p\|_1^2\left(\left\|\partial_t^2\eta\right\|_{1/2}\|u\|_2 + \|\partial_t\eta\|_{3/2}\|\partial_t u\|_1 + \|\partial_t\eta\|_{3/2}^2\|u\|_2\right) \lesssim \left(\mathcal{E}^{1/2}+\mathcal{E}\right)\mathcal{D}.
\end{aligned}$$

□

Hence, we have

$$\frac{d}{dt}\int_\Omega\left(|\partial_t u|^2 J + p\left(\operatorname{div}_{\partial_t\mathcal{A}} u\right)J\right) + \frac{d}{dt}\|\partial_t\eta\|_{H^1(\Sigma)}^2\int_\Omega|\mathbb{D}_{\mathcal{A}}\partial_t u|^2 J \lesssim \left(\mathcal{E}^{1/2}+\mathcal{E}\right)\mathcal{D}.$$

Similarly, we also have

$$\frac{d}{dt}\int_\Omega |u|^2 J + \frac{d}{dt}\|\eta\|_{H^1(\Sigma)}^2 + \int_\Omega|\mathbb{D}_{\mathcal{A}} u|^2 J \lesssim \left(\mathcal{E}^{1/2}+\mathcal{E}\right)\mathcal{D}.$$

4.1.2. *The estimates for spatial derivatives.* We now give the explicit forms of forcing terms in (2.2) when we take spatial derivatives on (u,p,η). We will use Einstein summation for the sake of simplicity. When $(v,q,\zeta) = (\partial_j u, \partial_j p, \partial_j\eta)$, $j=1,2$, we have

$$\begin{aligned}
F^1 &= \partial_j\partial_t\bar\eta KW\partial_3 u + \partial_t\bar\eta\partial_j KW\partial_3 u - u\cdot\nabla_{\partial_j\mathcal{A}}u - \partial_j u\cdot\nabla_{\mathcal{A}}u - \nabla_{\partial_j\mathcal{A}}p + \operatorname{div}_{\partial_j\mathcal{A}}\mathbb{D}_{\mathcal{A}}u + \operatorname{div}_{\mathcal{A}}\mathbb{D}_{\partial_j\mathcal{A}}u, \\
F^2 &= -\operatorname{div}_{\partial_j\mathcal{A}} u, \\
F^4 &= -u\cdot\partial_j\mathcal{N}, \\
F^3 &= (\eta-\Delta_*\eta)\,\partial_j\mathcal{N} + \mathbb{D}_{\partial_j\mathcal{A}}u\mathcal{N} - S_{\mathcal{A}}(p,u)\partial_j\mathcal{N} + \frac{|\nabla_*\eta|^2\Delta_*\eta}{\sqrt{1+|\nabla_*\eta|^2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)}\partial_j\mathcal{N} \\
&\quad + \frac{\sum_{k,\ell=1}^2\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\partial_j\mathcal{N} + \frac{|\nabla_*\eta|^2\Delta_*\partial_j\eta + (\nabla_*\eta\cdot\nabla_*\partial_j\eta)\,\Delta_*\eta}{\sqrt{1+|\nabla_*\eta|^2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)}\mathcal{N} + \frac{2\sum_{k,\ell=1}^2\partial_k\partial_j\eta\partial_\ell\eta\partial_{k\ell}\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\mathcal{N} \\
&\quad + \frac{|\nabla_*\eta|^2\Delta_*\eta(\nabla_*\eta\cdot\nabla_*\partial_j\eta)}{\left(1+|\nabla_*\eta|^2\right)^{3/2}\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)}\mathcal{N} + \frac{\sum_{k,\ell=1}^2\partial_k\eta\partial_\ell\eta\partial_{k\ell}\partial_j\eta}{\left(1+|\nabla_*\eta|^2\right)^{3/2}}\mathcal{N} \\
&\quad + \frac{|\nabla_*\eta|^2\Delta_*\eta\,(\nabla_*\eta\cdot\nabla_*\partial_j\eta)}{\left(1+|\nabla_*\eta|^2\right)\left(1+\sqrt{1+|\nabla_*\eta|^2}\right)^2}\mathcal{N} - 3\frac{\sum_{k,\ell=1}^2\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta\,(\nabla_*\eta\cdot\nabla_*\partial_j\eta)}{\left(1+|\nabla_*\eta|^2\right)^{5/2}}\mathcal{N}.
\end{aligned}$$

We also need to take horizontal derivatives on (u,p,η) twice in our definitions of horizontal energy and dissipation. Taking $(v,q,\zeta)=(\partial_{ij}u,\partial_{ij}p,\partial_{ij}\eta)$, $i,j=1,2$, we have

$$
\begin{aligned}
F^1 &= \partial_{ij}\partial_t\bar{\eta}KW\partial_3 u+\partial_j\partial_t\bar{\eta}\partial_i KW\partial_3 u+\partial_j\partial_t\bar{\eta}KW\partial_{i3}u+\partial_i\partial_t\bar{\eta}\partial_j KW\partial_3 u+\partial_t\bar{\eta}\partial_{ij}KW\partial_3 u+\partial_t\bar{\eta}\partial_j KW\partial_{3i}u\\
&\quad-\partial_i u\cdot\nabla_{\partial_j\mathcal{A}}u-u\cdot\nabla_{\partial_{ij}\mathcal{A}}u-u\cdot\nabla_{\partial_j\mathcal{A}}\partial_i u-\partial_{ij}u\cdot\nabla_{\mathcal{A}}u-\partial_j u\cdot\nabla_{\partial_i\mathcal{A}}u-\partial_j u\cdot\nabla_{\mathcal{A}}\partial_i u+\operatorname{div}_{\mathcal{A}}\mathbb{D}_{\partial_{ij}\mathcal{A}}u\\
&\quad-\nabla_{\partial_j\mathcal{A}}\partial_i p+\operatorname{div}_{\partial_{ij}\mathcal{A}}\mathbb{D}_{\mathcal{A}}u+\operatorname{div}_{\partial_j\mathcal{A}}\mathbb{D}_{\partial_i\mathcal{A}}u+\operatorname{div}_{\partial_j\mathcal{A}}\mathbb{D}_{\mathcal{A}}\partial_i u+\operatorname{div}_{\partial_i\mathcal{A}}\mathbb{D}_{\partial_j\mathcal{A}}u+\operatorname{div}_{\mathcal{A}}\mathbb{D}_{\partial_j\mathcal{A}}\partial_i u-\nabla_{\partial_{ij}\mathcal{A}}p\\
&\quad+\partial_t\bar{\eta}\partial_i KW\partial_{3j}u+\partial_i\partial_t\bar{\eta}KW\partial_{3j}u-\partial_i u\cdot\nabla_{\mathcal{A}}\partial_j u-u\cdot\nabla_{\partial_i\mathcal{A}}\partial_j u-\operatorname{div}_{\partial_i\mathcal{A}}S_{\mathcal{A}}(\partial_j p,\partial_j u)+\operatorname{div}_{\mathcal{A}}\mathbb{D}_{\partial_i\mathcal{A}}\partial_j u,\\
F^2 &= -\operatorname{div}_{\partial_{ij}\mathcal{A}}u-\operatorname{div}_{\partial_j\mathcal{A}}\partial_i u-\operatorname{div}_{\partial_i\mathcal{A}}\partial_j u,\\
F^4 &= -\partial_i u\cdot\partial_j\mathcal{N}-u\cdot\partial_{ij}\mathcal{N}-\partial_j u\cdot\partial_i\mathcal{N},
\end{aligned}
$$

$$
\begin{aligned}
F^3 &= \left(\partial_i\eta - \Delta_*\partial_i\eta\right)\partial_j\mathcal{N} + \left(\eta - \Delta_*\eta\right)\partial_{ij}\mathcal{N} + \mathbb{D}_{\partial_{ij}\mathcal{A}}u\mathcal{N} + \mathbb{D}_{\partial_j\mathcal{A}}\partial_i u\mathcal{N} + \mathbb{D}_{\partial_j\mathcal{A}}u\partial_i\mathcal{N} - S_{\mathcal{A}}\left(\partial_i p, \partial_i u\right)\partial_j\mathcal{N} \\
&+ 2\mathbb{D}_{\partial_i\mathcal{A}}\partial_j u\mathcal{N} - S_{\mathcal{A}}\left(\partial_j p, \partial_j u\right)\partial_i\mathcal{N} - S_{\mathcal{A}}(p,u)\partial_{ji}\mathcal{N} + \frac{2\left|\nabla_*\eta\right|^2\Delta_*\partial_i\eta}{\sqrt{1+\left|\nabla_*\eta\right|^2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\partial_j\mathcal{N} \\
&+ \frac{3\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)\Delta_*\eta}{\sqrt{1+\left|\nabla_*\eta\right|^2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\partial_j\mathcal{N} + \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_i\eta}{\sqrt{1+\left|\nabla_*\eta\right|^2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\partial_{ij}\mathcal{N} \\
&+ \frac{\sum_{k,\ell=1}^2\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\partial_{ij}\mathcal{N} + \frac{2\partial_k\eta\partial_\ell\eta\partial_{ik\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\partial_j\mathcal{N} - \frac{6\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{5/2}}\partial_j\mathcal{N} \\
&+ 3\frac{\partial_i\left(\partial_k\eta\partial_\ell\eta\right)\partial_{k\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\partial_j\mathcal{N} - \frac{3\partial_k\eta\partial_\ell\eta\partial_{jk\ell}\eta\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{5/2}}\mathcal{N} + \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_{ij}\eta + \left(\nabla_*\eta\cdot\nabla_*\partial_{ij}\eta\right)\Delta_*\eta}{\sqrt{1+\left|\nabla_*\eta\right|^2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} \\
&+ \frac{3\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)\Delta_*\partial_j\eta + \left(\nabla_*\partial_i\eta\cdot\nabla_*\partial_j\eta\right)\Delta_*\eta}{\sqrt{1+\left|\nabla_*\eta\right|^2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} - \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_j\eta\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} \\
&- \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_j\eta\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)^2}\mathcal{N} + \frac{15\left(\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta\right)\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{7/2}}\mathcal{N} \\
&+ \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_i\eta\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} + \frac{\left|\nabla_*\eta\right|^2\Delta_*\partial_i\eta\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)^2}\mathcal{N} \\
&+ \frac{\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\partial_i\eta\cdot\nabla_*\partial_j\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} + \frac{\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\eta\cdot\nabla_*\partial_{ij}\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} \\
&- \frac{3\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{5/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)}\mathcal{N} + \frac{\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\partial_i\eta\cdot\nabla_*\partial_j\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)^2}\mathcal{N} \\
&+ \frac{\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\eta\cdot\nabla_*\partial_{ij}\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)^2}\mathcal{N} - \frac{2\left|\nabla_*\eta\right|^2\Delta_*\eta\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)\left(\nabla_*\eta\cdot\nabla_*\partial_i\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}\left(1+\sqrt{1+\left|\nabla_*\eta\right|^2}\right)^4}\mathcal{N} \\
&+ \frac{\partial_{ki}\eta\partial_\ell\eta\partial_{jk\ell}\eta + \partial_k\eta\partial_{i\ell}\eta\partial_{jk\ell}\eta + \partial_k\eta\partial_\ell\eta\partial_{ijk\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\mathcal{N} + \frac{2\partial_{kj}\eta\partial_\ell\eta\partial_{ik\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\mathcal{N} + \frac{2\partial_{kj}\eta\partial_{i\ell}\eta\partial_{k\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\mathcal{N} \\
&- \frac{3\partial_i\left(\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta\right)\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right) + \left(\partial_k\eta\partial_\ell\eta\partial_{k\ell}\eta\right)\partial_i\left(\nabla_*\eta\cdot\nabla_*\partial_j\eta\right)}{\left(1+\left|\nabla_*\eta\right|^2\right)^{5/2}}\mathcal{N} + \frac{2\partial_{ikj}\eta\partial_\ell\eta\partial_{k\ell}\eta}{\left(1+\left|\nabla_*\eta\right|^2\right)^{3/2}}\mathcal{N}.
\end{aligned}
$$

We can now obtain the following result.

Proposition 4.4.

$$
\left\|F^1\right\|_0 + \left\|F^2\right\|_0 + \left\|F^3\right\|_{-1/2} + \left\|F^4\right\|_{1/2} \lesssim \left(\mathcal{E}^{1/2} + \mathcal{E}\right)\mathcal{D}^{1/2}.
$$

Proof. To prove the estimates of forcing terms, it suffices to consider the case of twice spatial derivatives, since the once differentiable case is similar yet much easier. It is easy to see that

$$\begin{aligned}\left\|F^1\right\|_{H^0(\Omega)}^2 \lesssim & \int_\Omega \left|\nabla^2\partial_t\bar\eta\right|^2|\nabla u|^2+|\nabla\partial_t\bar\eta|^2\left|\nabla^2\bar\eta\right|^2|\nabla u|^2+|\nabla\partial_t\bar\eta|^2\left|\nabla^2 u\right|^2+|\partial_t\bar\eta|^2\left|\nabla^3\bar\eta\right|^2|\nabla u|^2\\ &+\int_\Omega |\partial_t\bar\eta|^2\left|\nabla^2\bar\eta\right|^2\left|\nabla^2 u\right|^2+\left|\nabla^2\bar\eta\right|^2|\nabla u|^4+|u|^2\left|\nabla^3\bar\eta\right|^2|\nabla u|^2+|u|^2\left|\nabla^2\bar\eta\right|^2\left|\nabla^2 u\right|^2\\ &+\int_\Omega |\nabla u|^2\left|\nabla^2 u\right|^2+\left|\nabla^3\bar\eta\right|^2|\nabla p|^2+\left|\nabla^2\bar\eta\right|^2\left|\nabla^2 p\right|^2+\left|\nabla^3\bar\eta\right|^2\left(\left|\nabla^2\bar\eta\right|^2|\nabla u|^2+\left|\nabla^2 u\right|^2\right)\\ &+\int_\Omega \left|\nabla^2\bar\eta\right|^4\left|\nabla^2 u\right|^2+\left|\nabla^2\bar\eta\right|^2\left|\nabla^2 u\right|^2+\left|\nabla^4\bar\eta\right|^2|\nabla u|^2.\end{aligned}$$

Then by the Hölder inequality, the Sobolev inequality and usual trace theory, we have

$$\begin{aligned}\left\|F^1\right\|_{H^0}^2 \lesssim & \left(\left\|\nabla^2\partial_t\bar\eta\right\|_{L^2}^2+\|\nabla\partial_t\bar\eta\|_{L^4}^2\left\|\nabla^2\bar\eta\right\|_{L^4}^2\right)\|\nabla u\|_{L^\infty}^2+\|\nabla\partial_t\bar\eta\|_{L^4}^2\left\|\nabla^2 u\right\|_{L^4}^2\\ &+\|\partial_t\bar\eta\|_{L^\infty}^2\left\|\nabla^3\bar\eta\right\|_{L^2}^2\|\nabla u\|_{L^\infty}^2+\|\partial_t\bar\eta\|_{L^6}^2\left\|\nabla^2\bar\eta\right\|_{L^6}^2\|\nabla u\|_{L^6}^2+\left\|\nabla^4\bar\eta\right\|_{L^2}^2\|\nabla u\|_{L^\infty}^2\\ &+\left\|\nabla^2\bar\eta\right\|_{L^4}^2\|\nabla u\|_{L^4}^2\|\nabla u\|_{L^\infty}^2+\|u\|_{L^\infty}^2\left\|\nabla^3\bar\eta\right\|_{L^2}^2\|\nabla u\|_{L^\infty}^2+\|u\|_{L^\infty}^2\left\|\nabla^2\bar\eta\right\|_{L^\infty}^2\left\|\nabla^2 u\right\|_{L^2}^2\\ \lesssim & \left(\mathcal{E}+\mathcal{E}^2\right)\mathcal{D}.\end{aligned}$$

Similarly, for F^2, we have

$$\begin{aligned}\left\|F^2\right\|_0^2 &\lesssim \int_\Omega \left|\nabla_*^2\nabla\bar\eta\right|^2|\nabla u|^2+|\nabla_*\nabla\bar\eta|^2|\nabla_*\nabla u|^2\\ &\lesssim \left\|\left|\nabla_*^2\nabla\bar\eta\right|^2\right\|_{L^3}^2\|\nabla u\|_{L^6}^2+\left\|\left|\nabla_*\nabla\bar\eta\right|^2\right\|_{L^\infty}^2\|\nabla\nabla_* u\|_{L^2}^2\lesssim\|\eta\|_3^2\|u\|_2^2+\|\eta\|_{7/2}^2\|u\|_2\lesssim\mathcal{E}\mathcal{D}.\end{aligned}$$

We now turn to the estimate of term F^3. We first focus on the terms of η with derivatives of 4-th order. For $v\in H^1(\Omega)$ with trace $v|_\Sigma\in H^{1/2}$, we will use the dual form and product of estimates in Sobolev spaces, and see

$$\begin{aligned}&\left|\int_\Sigma\left(\frac{\partial_k\eta\partial_\ell\eta\partial_{ijk\ell}\eta}{(1+|\nabla_*\eta|^2)^{3/2}}+\frac{|\nabla_*\eta|^2\Delta_*\partial_{ij}\eta}{\sqrt{1+|\nabla_*\eta|^2}(1+\sqrt{1+|\nabla_*\eta|^2})}\right)\mathcal{N}\cdot v\right|\\ &\lesssim\left|\int_\Sigma\left(\partial_k\eta\partial_\ell\eta\partial_{ijk\ell}\eta+|\nabla_*\eta|^2\Delta_*\partial_{ij}\eta\right)\mathcal{N}\cdot v\right|\lesssim\left\|\nabla_*^4\eta\right\|_{-1/2,\Sigma}\|\nabla_*\eta\nabla_*\eta\,(v\cdot\mathcal{N})\|_{1/2,\Sigma}\lesssim\|\eta\|_{7/2}\|\eta\|_3^2\|v\|_{1/2,\Sigma}.\end{aligned}$$

The other terms in F^3 can be bounded by

$$\begin{aligned}&\int_\Sigma|\nabla\nabla_*u|^2|\nabla_*\nabla\bar\eta|^2+|\nabla u|^2\left|\nabla_*^2\nabla\bar\eta\right|^2+\left(|\nabla_*p|^2+|\nabla_*\nabla\bar\eta|^2|\nabla u|^2+|\nabla\nabla_*u|^2\right)\left|\nabla_*^2\eta\right|^2\\ &+\int_\Sigma\left(|p|^2+|\nabla u|^2\right)\left|\nabla_*^3\eta\right|^2+\int_\Sigma|\nabla_*\eta|^2\left|\nabla_*^2\eta\right|^2+\left(|\eta|^2+|\nabla_*\eta|^4+\left|\nabla_*^2\eta\right|^2\right)\left|\nabla_*^3\eta\right|^2\\ &+\int_\Sigma\left(|\nabla_*\eta|^2+|\nabla_*\eta|^6\right)\left|\nabla_*^2\eta\right|^2\left|\nabla_*^3\eta\right|^2+|\nabla_*\eta|^4\left|\nabla_*^3\eta\right|^4+\left(1+|\nabla_*\eta|^2+|\nabla_*\eta|^6\right)\left|\nabla_*^2\eta\right|^6\\ &+\int_\Sigma\left|\nabla_*^2\eta\right|^4\left|\nabla_*^3\eta\right|^2+\left(|\nabla_*\eta|^8+|\nabla_*\eta|^4\right)\left|\nabla_*^2\eta\right|^6+|\nabla_*\eta|^4\left|\nabla_*^4\eta\right|^2\left|\nabla_*^3\eta\right|^2.\end{aligned}$$

It turns out that the above terms can be bounded further by $(\mathcal{E}+\mathcal{E}^2)\mathcal{D}$.

Finally, we apply similar arguments as in the estimate of F^3 to estimate F^4,

$$\begin{aligned}\left\|F^4\right\|_{1/2}^2 &\lesssim\left\|\nabla u_*\cdot\nabla_*^2\eta\right\|_{1/2,\Sigma}^2+\left\|u_*\cdot\nabla_*^3\eta\right\|_{1/2,\Sigma}^2\\ &\lesssim\|\nabla u\|_{H^{3/2}(\Sigma)}^2\|\eta\|_3^2+\|u\|_{L^\infty(\Sigma)}^2\|\eta\|_{7/2}^2+\|u\|_{W^{1/2,\infty}(\Sigma)}\left\|\nabla_*^3\eta\right\|_{L^2}\\ &\lesssim\|u\|_3^2\|\eta\|_3^2+\|u\|_2^2\|\eta\|_{7/2}^2\lesssim\mathcal{E}\mathcal{D}.\end{aligned}$$

□

4.2. **Elliptic estimates.** In order to close the a priori estimates, we need to employ (3.17) and (3.18). Since

$$G^1 = -\partial_t u + \partial_t \bar{\eta} K W \partial_3 u - u \cdot \nabla_{\mathcal{A}} u + \nabla_{I-\mathcal{A}} p - \operatorname{div}_{\mathcal{A}} \mathbb{D}_{I-\mathcal{A}} u - \operatorname{div}_{I-\mathcal{A}} \mathbb{D} u,$$
$$G^2 = \operatorname{div}_{I-\mathcal{A}} u, \text{ and } G^3 = -\mathbb{D}_{I-\mathcal{A}} u \mathcal{N} \cdot \mathcal{T} - \mathbb{D} u (e_3 - \mathcal{N}) \cdot \mathcal{T} - \mathbb{D} u e_3 \cdot (\tau - \mathcal{T}),$$

we have the following estimate in order to use the elliptic theory in Theorem 3.3.

Proposition 4.5.

$$\left\|G^1\right\|_0^2 + \left\|G^2\right\|_1^2 + \left\|G^3\right\|_{1/2,\Sigma}^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2,$$

and

$$\left\|G^1\right\|_1^2 + \left\|G^2\right\|_2^2 + \left\|G^3\right\|_{3/2,\Sigma}^2 \lesssim \mathcal{D}_{\shortparallel} + \left(\mathcal{E} + \mathcal{E}^2\right) \mathcal{D}.$$

Proof. From the expression of G^1, we use the product estimates in Sobolev spaces to deduce

$$\left\|G^1\right\|_0^2 \lesssim \|\partial_t u\|_0^2 + \|\partial_t \eta\|_{3/2}^2 \|u\|_1^2 + \|u\|_1^2 \|u\|_2^2 + \|\eta\|_3^2 \|p\|_1^2 + \|\eta\|_3^2 \|u\|_2^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2.$$

Following the same argument, we also have

$$\begin{aligned}
\left\|G^1\right\|_1^2 \lesssim{}& \|\partial_t u\|_1^2 + \|\partial_t \eta\|_{3/2}^2 \|\eta\|_3^2 \|u\|_3^2 + \|u\|_2^2 \|u\|_3^2 + \|\nabla \bar{\eta}\|_{L^\infty}^2 \|p\|_2^2 \\
&+ \|\nabla \bar{\eta}\|_{L^4(\Omega)}^2 \|\nabla p\|_{L^4(\Omega)}^2 + \|\nabla \bar{\eta}\|_{L^\infty}^2 \|u\|_3^2 + \left\|\nabla^2 \bar{\eta}\right\|_{L^4(\Omega)}^2 \left\|\nabla^2 u\right\|_{L^4(\Omega)}^2 + \left\|\nabla^3 \bar{\eta}\right\|_{L^4(\Omega)}^2 \|\nabla u\|_{L^4(\Omega)}^2 \\
\lesssim{}& \|\partial_t u\|_1^2 + \left(1 + \|\partial_t \eta\|_{3/2}^2\right) \|\eta\|_3^2 \|u\|_3^2 + \|u\|_2^2 \|u\|_3^2 + \|\eta\|_3^2 \|p\|_2^2 \lesssim \mathcal{D}_{\shortparallel} + \left(\mathcal{E} + \mathcal{E}^2\right) \mathcal{D}.
\end{aligned}$$

Since $G^2 = \operatorname{div}_{I-\mathcal{A}} u$, we directly have

$$\left\|G^2\right\|_1^2 \lesssim \|\nabla \bar{\eta}\|_{L^\infty} \|u\|_2^2 + \left\|\nabla^2 \bar{\eta}\right\|_{L^4(\Omega)}^2 \|\nabla u\|_{L^4(\Omega)}^2 \lesssim \|\eta\|_3^2 \|u\|_2^2 + \|\eta\|_{5/2}^2 \|u\|_2^2 \lesssim \mathcal{E}^2,$$

and

$$\left\|G^2\right\|_2^2 \lesssim \|\nabla \bar{\eta}\|_{L^\infty} \|u\|_3^2 + \left\|\nabla^3 \bar{\eta}\right\|_2^2 \|\nabla u\|_{L^\infty}^2 \lesssim \|\eta\|_3^2 \|u\|_3^2 \lesssim \mathcal{E}\mathcal{D}.$$

From the explicit form of G^3, we apply the Hölder inequality and Sobolev inequality and get

$$\begin{aligned}
\left\|G^3\right\|_{1/2,\Sigma}^2 \lesssim{}& \left(\|\nabla \eta\|_{L^\infty(\Sigma)}^2 + \|\nabla \bar{\eta}\|_{L^\infty(\Sigma)}^2\right) \|\nabla u\|_{1/2,\Sigma}^2 + \left(\|\nabla \eta\|_{W^{1/2,4}(\Sigma)}^2 + \|\nabla \bar{\eta}\|_{W^{1/2,4}(\Sigma)}^2\right) \|\nabla u\|_{L^4(\Sigma)}^2 \\
\lesssim{}& \|\eta\|_3^2 \|u\|_2^2 \lesssim \mathcal{E}^2.
\end{aligned}$$

The same argument enables us to have

$$\left\|G^3\right\|_{3/2,\Sigma}^2 \lesssim \left(\|\nabla \eta\|_{3/2}^2 + \|\nabla \bar{\eta}\|_{3/2,\Sigma}^2\right) \|\nabla u\|_{3/2,\Sigma}^2 \lesssim \|\eta\|_3^2 \|u\|_3^2 \lesssim \mathcal{E}\mathcal{D}.$$

Combining the above estimates, we get the results. □

Proposition 4.5 allows us to control the remainder terms in energy $\mathcal{E}$ and dissipation $\mathcal{D}$.

Theorem 4.6. *Under the same assumption in Theorem 4.1, we have*

$$\|u\|_2^2 + \|p\|_1^2 + \|\partial_t \eta\|_{3/2}^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2,$$

and

$$\|u\|_3^2 + \|p\|_2^2 + \|\eta\|_{7/2} + \|\partial_t \eta\|_{5/2}^2 + \left\|\partial_t^2 \eta\right\|_{1/2}^2 \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}.$$

Proof. By $G^4 = u \cdot \mathcal{N}$, the Sobolev inequality and Poincaré inequality, we get

$$\begin{aligned}
\|u \cdot \mathcal{N}\|_{3/2,\Sigma}^2 \lesssim{}& \|u_3\|_{3/2,\Sigma}^2 + \|u_* \cdot \nabla_* \eta\|_{3/2}^2 \lesssim \|u_3\|_{0,\Sigma}^2 + \|\nabla_* u_3\|_{1/2,\Sigma}^2 + \|u\|_2^2 \|\eta\|_{5/2}^2 \\
\lesssim{}& \|\partial_3 u_3\|_0^2 + \|\nabla_* u\|_1^2 + \|u\|_2^2 \|\eta\|_{5/2}^2.
\end{aligned}$$

Then we use $\operatorname{div}_{\mathcal{A}} u = 0$, i.e.,

$$K \partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2 + A K \partial_3 u_1 + B K \partial_3 u_2.$$

By the smallness of η, we can obtain that there exists a constant $\delta_1 > 0$ such that $1 \gtrsim \|J\|_{W^{1,\infty}} \geq \delta_1$, so that we have

$$\|K\partial_3 u_3\|_0^2 \gtrsim \|\partial_3 u_3\|_0^2 .$$

By the Hölder inequality and Sobolev inequality, we have

$$\|AK\partial_3 u_1 + BK\partial_3 u_2\|_0^2 \lesssim \|u\|_2^2 \|\eta\|_{5/2}^2.$$

Thus

$$\|\partial_3 u_3\|_0^2 \lesssim \|\nabla_* u_*\|_0^2 + \|u\|_2^2 \|\eta\|_{5/2}^2.$$

Therefore, we have

$$\|u \cdot \mathcal{N}\|_{3/2,\Sigma}^2 \lesssim \|\nabla_* u\|_1^2 + + \|u\|_2^2 \|\eta\|_{5/2}^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2.$$

Similarly, we have

$$\begin{aligned}
\|u \cdot \mathcal{N}\|_{5/2,\Sigma}^2 &\lesssim \|u_3\|_{5/2,\Sigma}^2 + \|u_*\|_{5/2,\Sigma}^2 \|\eta\|_{5/2}^2 + \|u\|_{3/2,\Sigma}^2 \|\eta\|_{7/2}^2 \\
&\lesssim \|u_3\|_{1,\Sigma}^2 + \left\|\nabla_*^2 u_3\right\|_{1/2,\Sigma}^2 + \|u\|_3^2 \|\eta\|_{5/2}^2 + \|u\|_2^2 \|\eta\|_{7/2}^2 \\
&\lesssim \|\partial_3 u_3\|_1^2 + \left\|\nabla_*^2 u\right\|_1^2 + \|u\|_3^2 \|\eta\|_{5/2}^2 + \|u\|_2^2 \|\eta\|_{7/2}^2.
\end{aligned}$$

We still use the divergence condition $\operatorname{div}_{\mathcal{A}} u = 0$ to get

$$\|\partial_3 u_3\|_1^2 \lesssim \|K\partial_3 u_3\|_1^2 \lesssim \|\nabla_* u_*\|_1^2 + \|u\|_3^2 \|\eta\|_{5/2}^2 + \|u\|_2^2 \|\eta\|_{7/2}^2,$$

which implies

$$\|u \cdot \mathcal{N}\|_{5/2,\Sigma}^2 \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}.$$

Hence, applying the elliptic theory in Theorem 3.3 and Proposition 4.5, we have

$$\|u\|_2^2 + \|p\|_1^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2, \text{ and } \|u\|_3^2 + \|p\|_2^2 + \|\eta\|_{7/2}^2 \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}.$$

Since $\partial_t \eta = u \cdot \mathcal{N}$, by the estimates of $u \cdot \mathcal{N}$, we get

$$\|\partial_t \eta\|_{3/2}^2 \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2, \text{ and } \|\partial_t \eta\|_{5/2}^2 \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}.$$

Finally, we take ∂_t on both sides of $\partial_t \eta = u \cdot \mathcal{N}$ and obtain

$$\partial_t^2 \eta = \partial_t u \cdot \mathcal{N} + u \cdot \partial_t \mathcal{N}.$$

Therefore, we use the horizontal dissipation to get

$$\left\|\partial_t^2 \eta\right\|_{1/2}^2 \lesssim \|\partial_t u\|_{1/2,\Sigma}^2 \left(1 + \|\eta\|_3^2\right) + \|\partial_t \nabla_* \eta\|_{1/2}^2 \|u\|_{5/2,\Sigma}^2 \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}.$$

□

5. Global existence and decay

In this section, we establish the *a priori estimates* and obtain the well-posedness. Our strategy to prove the global well-posedness is the usual continuity argument. We will first need to prove the uniform boundedness for any finite time interval $[0, T]$, which is concluded in our a priori estimates.

5.1. **A priori estimates.**

Our a priori estimates is one ingredient required to develop the global well-posedness. It contains the decay of energy and the finite integral of dissipation.

Theorem 5.1. *Suppose (u, p, η) satisfies (2.1). Then for any finite interval $[0, T]$, there exists a universal constant $\delta \in (0, 1)$ such that, if*

$$\sup_{0 \leq t \leq T} \mathcal{E}(t) \leq \delta,$$

there exists a universal constant $\sigma > 0$ so that

$$\sup_{0 \leq t \leq T} e^{\sigma t} \mathcal{E}(t) + \int_0^T \mathcal{D}(t) \, \mathrm{d}t \lesssim \mathcal{E}(0).$$

Here, δ and σ are independent of T.

Proof. From the definition of horizontal energy $\mathcal{E}_{\shortparallel}$ and dissipation $\mathcal{D}_{\shortparallel}$, and Theorem 4.6, we see $\mathcal{E} \lesssim \mathcal{E}_{\shortparallel} + \mathcal{E}^2$ and $\mathcal{D} \lesssim \mathcal{D}_{\shortparallel} + \mathcal{E}\mathcal{D}$. Consequently, after taking δ sufficiently small, the terms $\mathcal{E}^2$ and $\mathcal{E}\mathcal{D}$ can be absorb into $\mathcal{E}$ and $\mathcal{D}$, respectively. Consequently, we have $\mathcal{E} \lesssim \mathcal{E}_{\shortparallel}$ and $\mathcal{D} \lesssim \mathcal{D}_{\shortparallel}$. We plunge this structure into the horizontal energy-dissipation inequality in Theorem 4.1 and obtain

$$\frac{d}{dt}\left(\mathcal{E}_{\shortparallel} - \mathcal{P}\right) + \mathcal{D}_{\shortparallel} \lesssim \mathcal{E}^{1/2}\mathcal{D}_{\shortparallel}.$$

After restricting δ small enough, we get

$$\frac{d}{dt}\left(\mathcal{E}_{\shortparallel} - \mathcal{P}\right) + \frac{1}{2}\mathcal{D}_{\shortparallel} \leq 0. \tag{5.1}$$

So we need to ensure that the terms differentiated is positive. By the Hölder inequality, Sobolev inequalities and usual trace theory, we have

$$\mathcal{P} = \int_\Omega p \operatorname{div}_{\partial_t \mathcal{A}} uJ \lesssim \|p\|_1 \|u\|_2 \, \|\partial_t \eta\|_{3/2} \lesssim \mathcal{E}^{1/2}\mathcal{E}_{\shortparallel}.$$

As a result, $\mathcal{P} \lesssim \delta^{1/2}\mathcal{E}_{\shortparallel}$, and hence

$$\mathcal{E}_{\shortparallel} - \mathcal{P} \gtrsim \left(1 - \delta^{1/2}\right)\mathcal{E}_{\shortparallel},$$

provided that we might restrict δ necessarily again. Also, it is trivial to see that $\mathcal{E}_{\shortparallel} - \mathcal{P} \lesssim \mathcal{E}_{\shortparallel}$. Therefore, $\mathcal{E}_{\shortparallel} - \mathcal{P}$ is equivalent to $\mathcal{E}_{\shortparallel}$.

From the definition of $\mathcal{E}$ and $\mathcal{D}$ in (1.9) and (1.10), it is obvious that

$$\mathcal{D} \gtrsim \mathcal{E}_{\shortparallel} \gtrsim \mathcal{E}_{\shortparallel} - \mathcal{P} > 0.$$

Consequently, we can choose a universal constant $\sigma > 0$ such that (5.1) becomes

$$\frac{d}{dt}\left(\mathcal{E}_{\shortparallel} - \mathcal{P}\right) + \sigma\left(\mathcal{E}_{\shortparallel} - \mathcal{P}\right) \leq 0.$$

Then Gronwall's inequality implies

$$\mathcal{E}_{\shortparallel}(t) \lesssim \mathcal{E}_{\shortparallel}(t) - \mathcal{P}(t) \lesssim e^{-\sigma t}\left(\mathcal{E}_{\shortparallel}(0) - \mathcal{P}(0)\right) \lesssim e^{-\sigma t}\mathcal{E}(0).$$

Therefore, we obtain a uniform bound

$$\sup_{0 \leq t \leq T} e^{\sigma t}\mathcal{E}(t) \lesssim \mathcal{E}(0).$$

(5.1) also implies

$$\frac{d}{dt}\left(\mathcal{E}_{\shortparallel} - \mathcal{P}\right) + C\mathcal{D} \leq 0,$$

for some constant $C > 0$. Then integrating over $[0, T]$ implies

$$\int_0^T \mathcal{D} \lesssim \mathcal{E}_{\shortparallel}(0) \lesssim \mathcal{E}(0).$$

□

5.2. **Local well-posedness.** Let $u_0 \in H^2(\Omega)$, $\eta_0 \in H^3(\Sigma)$ and also satisfy the compatibility conditions, we need to construct $\partial_t u(0)$, $p(0)$ and $\partial_t \eta(0)$ appearing in $\mathcal{E}(0)$. We set $\partial_t \eta(0) = u_0 \cdot \mathcal{N}_0$. From the Sobolev embedding theory, we see that $\partial_t \eta(0) \in H^{1/2}(\Sigma)$. Then we set $p(0) \in H^1(\Omega)$ is the weak solution of

$$\begin{cases} -\operatorname{div}_{\mathcal{A}_0}\left(\nabla_{\mathcal{A}_0} p(0) - \partial_t \bar{\eta}(0) K(0) W \partial_3 u_0\right) = \operatorname{div}_{\mathcal{A}_0}\left(R(0) u_0\right) \in H^0(\Omega), \\ p(0) = \left[\left(\eta_0 - \mathcal{H}(0)\right)\mathcal{N}_0 + \mathbb{D}_{\mathcal{A}_0} u_0 \mathcal{N}_0\right] \cdot \dfrac{\mathcal{N}_0}{|\mathcal{N}_0|} \in H^{1/2}(\Sigma), \\ \left(\nabla_{\mathcal{A}_0} p(0) - \partial_t \bar{\eta}(0) K(0) W \partial_3 u_0\right) \cdot \nu = \Delta_{\mathcal{A}_0} u_0 \cdot \nu \in H^{1/2}(\Sigma_b), \end{cases} \tag{5.2}$$

where $R = \partial_t M M^{-1}$ with the matrix $M = J\nabla\Phi$ that guarantees that $\partial_t u - Ru$ is $\operatorname{div}_{\mathcal{A}}$ free. We also set $\partial_t u(0) = \Delta_{\mathcal{A}_0} u_0 - \nabla_{\mathcal{A}_0} p(0) + \partial_t \bar{\eta}(0) K(0) W \partial_3 u_0 \in H^0(\Omega)$. The elliptic equations (5.2) are well solved by the theory of Poisson equations.

Then the local well-posedness is stated as follows.

Theorem 5.2. *Assume that $\eta_0+1>\delta_0>0$ for some $\delta_0>0$. Suppose (u_0,η_0) satisfies $\|u_0\|^2_{H^2(\Omega)}+\|\eta_0\|^2_{H^3(\Sigma)}<\varepsilon$ for some small $\varepsilon>0$ and the compatibility condition (1.11). Then there exists a positive constant $\bar{T}<1$ such that for $0<T<\bar{T}$, the system (1.8) admits a unique strong solution (u,p,η) achieving the initial data and*

$$\sup_{0\le t\le T}\mathcal{E}(t)+\int_0^T\mathcal{D}(t)\,\mathrm{d}t+\left\|\partial_t^2 u\right\|^2_{(\mathcal{X}_T)^*}\le\varepsilon, \tag{5.3}$$

where the space is defined to be

$$\mathcal{X}_T=L^2([0,T];\mathcal{X}(t)),\ \text{ with } \mathcal{X}(t):=\left\{u\in\mathcal{H}^1(t):\operatorname{div}_{\mathcal{A}(t)}u=0\right\}.$$

Furthermore, the mapping Φ is a C^1 diffeomorphism.

The above theorem can be proved by a standard method based on the idea in [30] coupled with the elliptic theory in Theorem 3.3. So we ignore to iron out the details here and only give a sketch.

Proof. We construct $\left(u^0,\eta^0\right)$ by [13, Theorem A.5] as a start point and then construct the approximate solutions (u^m,p^m,η^m) for $m\ge1$, by the following linear iterated system

$$\begin{cases}\partial_t u^m+\nabla_{\mathcal{A}^{m-1}}p^m-\Delta_{\mathcal{A}^{m-1}}u^m=F^1\left(u^{m-1},\eta^{m-1}\right) & \text{in } \Omega,\\ \operatorname{div}_{\mathcal{A}^{m-1}}u^m=0 & \text{in } \Omega,\\ S_{\mathcal{A}^{m-1}}\left(p^m,u^m\right)\mathcal{N}^{m-1}=\left(\eta^m-\Delta_*\eta^m\right)\mathcal{N}^{m-1}+F^3\left(\eta^{m-1}\right) & \text{on } \Sigma,\\ \partial_t\eta^m=u^m\cdot\mathcal{N}^{m-1} & \text{on } \Sigma,\\ u^m=0 & \text{on } \Sigma_b,\end{cases} \tag{5.4}$$

where

$$F^1\left(u^{m-1},\eta^{m-1}\right)=\partial_t\bar{\eta}^{m-1}WK^{m-1}\partial_3u^{m-1}-u^{m-1}\cdot\nabla_{\mathcal{A}^{m-1}}u^{m-1},$$
$$F^3(\eta^{m-1})=-\left(\nabla_*\cdot\left(\frac{\nabla_*\eta^{m-1}}{\sqrt{1+|\nabla_*\eta^{m-1}|^2}}-\nabla_*\eta^{m-1}\right)\right)\mathcal{N}^{m-1},$$

and $\mathcal{A}^{m-1}$, $\mathcal{N}^{m-1}$, K^{m-1}, $\mathcal{H}^{m-1}$ are determined in terms of η^{m-1}, with the initial data $(u^m(0),\eta^m(0))=(u_0,\eta_0)$. Then we can apply the standard method to prove the limit of (u^m,p^m,η^m) is the solution of (1.8).

Hence the solvable of (5.4) is reduced to solving the linear problem

$$\begin{cases}\partial_t v+\nabla_{\mathcal{A}}q-\Delta_{\mathcal{A}}v=F^1 & \text{in } \Omega,\\ \operatorname{div}_{\mathcal{A}}v=0 & \text{in } \Omega,\\ S_{\mathcal{A}}(q,v)\mathcal{N}=\left(\xi-\Delta_*\xi\right)\mathcal{N}+F^3 & \text{on } \Sigma,\\ \partial_t\xi=v\cdot\mathcal{N} & \text{on } \Sigma,\\ v=0 & \text{on } \Sigma_b,\end{cases} \tag{5.5}$$

where $\mathcal{A}$, $\mathcal{N}$ and J determined in terms of η are given and F^i, $i=1,3,5$ are given functions. Then as the procedure in [30, Theorem 4.8], we apply the Galerkin method with finite time-dependent basis to construct the sequence of approximate solutions $\{v^n\}\in\mathcal{X}_m$ and $\xi^n=\eta_0+\int_0^t v^n(s)\cdot\mathcal{N}(s)\,\mathrm{d}s$ satisfying

$$(\partial_t v^n,w)_{\mathcal{H}^0}+\frac{1}{2}\left(v^n,w\right)_{\mathcal{H}^1}+(\xi^n,w\cdot\mathcal{N})_{H^1(\Sigma)}=\left(F^1,w\right)_{\mathcal{H}^0}-\left(F^3-\Pi_0\left(F^3(0)+\mathbb{D}_{\mathcal{A}_0}\left(\mathcal{P}_0^m u_0\right)\mathcal{N}_0\right),w\right)_{0,\Sigma} \tag{5.6}$$

with $w\in\mathcal{X}_m$, where $\mathcal{X}_m=\operatorname{span}\left\{w^j\right\}_{j=1}^m$, and $\left\{w^j\right\}_{j=1}^\infty$ is a countable basis of $H^2(\Omega)\cap\mathcal{X}(t)$. $\mathcal{P}_0^m$ is a projection from $H^2(\Omega)$ to $\mathcal{X}_m(0)$. Then (5.6) is reduced to an integral-differential equation

$$\dot{d}^n(t)+A(t)d^n(t)+\int_0^t C(t,s)d^n(s)\,\mathrm{d}s=\mathfrak{F}(t),$$

where $A(t)$ and $C(t,s)$ are $n\times n$ matrix-valued C^1 functions, which can be solved by the theory of integral-differential equations(see for instance [9]).

Then we use the standard argument to bound the v^n and ξ^n in the topology of $\mathcal{E}$ and $\mathcal{D}$, similar to the argument in the proof of [11, Theorem 4.8], to close the proof. □

Remark 5.3. *Although we only need the local theory for small initial data in our global result, we note that the local well-posedness can be established for the large data of (u_0, η_0). The techniques used for large data is just a combination of ε-modification for the mapping Φ, with the modified elliptic estimates in Theorem 3.3 depending on η_0, that is similar as in [28].*

5.3. Global existence and decay.

Proof of Theorem 1.1. The result is a consequence of continuity argument. Let T^* be the supremum of T such that Theorem 5.1 holds. Suppose

$$\sup_{0\le t\le T} e^{\sigma t}\mathcal{E}(t) + \int_0^T \mathcal{D}(t)\,\mathrm{d}t \le 2\varepsilon$$

for ε sufficiently small and any $T < T^*$. Then the a priori estimates in Theorem 5.1 implies

$$\sup_{0\le t\le T} e^{\sigma t}\mathcal{E}(t) + \int_0^T \mathcal{D}(t)\,\mathrm{d}t \le C_1\mathcal{E}(0)$$

for a universal constant C_1. The Local well-posedness of Theorem 5.2 implies $\mathcal{E}(0) \le C_2\left(\|u_0\|_2^2 + \|\eta_0\|_3^2\right)$ for a universal constant C_2. Then we can restrict the initial data (u_0, η_0), such that δ small enough, to ensure that $C_1C_2\left(\|u_0\|_2^2 + \|\eta_0\|_3^2\right) < \varepsilon$, contradicting to the definition of T^*. Hence the continuity argument yields $T^* = \infty$. □

Acknowledgment. Xing Cheng has been supported by the "the Fundamental Research Funds for the Central Universities" (No.B210202147), and Yunrui Zheng is partially supported by NSF of China under Grant 11901350, and The Fundamental Research Funds of Shandong University under Grant 11140079614046.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.*, Comm. Pure Appl. Math. **17** (1964), 35-92.

[2] H. Bae, *Solvability of the free boundary value problem of the Navier-Stokes equations*, Discrete Contin. Dyn. Syst. **29** (2011), no. 3, 769-801.

[3] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math. **34** (1981), no. 3, 359-392.

[4] J. T. Beale, *Large-time regularity of viscous surface waves*, Arch. Rational Mech. Anal. **84** (1983/84), no. 4, 307-352.

[5] J. T. Beale and T. Nishida, *Large-time behavior of viscous surface waves*, Recent topics in nonlinear PDE, II (Sendai, 1984), 1-14, North-Holland Math. Stud., **128**, Lecture Notes Numer. Appl. Anal., 8, North-Holland, Amsterdam, 1985.

[6] O. Bühler, J. Shatah, S. Walsh, and C. Zeng, *On the wind generation of water waves*, Arch. Ration. Mech. Anal. **222** (2016), no. 2, 827-878.

[7] D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid*, Comm. Pure Appl. Math. **53** (2000), no. 12, 1536-1602.

[8] L. C. Evans, *Partial differential equations. Second edition*, Graduate Studies in Mathematics, **19**. American Mathematical Society, Providence, RI, 2010. xxii+749 pp. ISBN: 978-0-8218-4974-3.

[9] S. I. Grossman and R. K. Miller, *Perturbation theory for Volterra integrodifferential systems*, J. Differential Equations **8** (1970), 457-474.

[10] G. Gui, *Lagrangian approach to global well-posedness of the viscous surface wave equations without surface tension*, Peking Math. J. **4** (2021), no. 1, 1-82.

[11] B. Guo and Y. Zheng, *A steady model on Navier-Stokes equations with a free surface*, Dyn. Partial Differ. Equ. **17** (2020), no. 3, 185-227.

[12] Y. Guo and I. Tice, *Almost exponential decay of periodic viscous surface waves without surface tension*, Arch. Ration. Mech. Anal. **207** (2013), no. 2, 459-531.

[13] Y. Guo and I. Tice, *Local well-posedness of the viscous surface wave problem without surface tension*, Anal. PDE **6** (2013), no. 2, 287-369.
[14] Y. Guo and I. Tice, *Decay of viscous surface waves without surface tension in horizontally infinite domains*, Anal. PDE **6** (2013), no. 6, 1429-1533.
[15] Y. Guo, I. Tice, *Stability of contact lines in fluids: 2D Stokes flow*, Arch. Ration. Mech. Anal. **227** (2018), no. 2, 767-854.
[16] Y. Hataya, *Decaying solution of a Navier-Stokes flow without surface tension*, J. Math. Kyoto Univ. **49** (2009), no. 4, 691-717.
[17] Y. Hataya and S. Kawashima, *Decaying solution of the Navier-Stokes flow of infinite volume without surface tension*, Nonlinear Anal. **71** (2009), no. 12, e2535-e2539.
[18] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow. Second English edition*, revised and enlarged Translated from the Russian by Richard A. Silverman and John Chu Mathematics and its Applications, Vol. 2 Gordon and Breach Science Publishers, New York-London-Paris 1969 xviii+224 pp.
[19] T. Nishida, Y. Teramoto, and H. Yoshihara, *Global in time behavior of viscous surface waves: horizontally periodic motion*, J. Math. Kyoto Univ. **44** (2004), no. 2, 271-323.
[20] X. Ren, Z. Xiang, and Z. Zhang, *Low regularity well-posedness for the viscous surface wave equation*, Sci. China Math. **62** (2019), no. 10, 1887-1924.
[21] J. Shatah and C. Zeng, *A priori estimates for fluid interface problems*, Comm. Pure Appl. Math. **61** (2008), no. 6, 848-876.
[22] J. Shatah and C. Zeng, *Local well-posedness for fluid interface problems*, Arch. Ration. Mech. Anal. **199** (2011), no. 2, 653-705.
[23] D. L. G. Sylvester, *Large time existence of small viscous surface waves without surface tension*, Comm. Partial Differential Equations **15** (1990), no. 6, 823-903.
[24] Z. Tan and Y. Wang, *Zero surface tension limit of viscous surface waves*, Comm. Math. Phys. **328** (2014), no. 2, 733-807.
[25] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001. xiv+408 pp. ISBN: 0-8218-2737-5.
[26] C. Wang, Z. Zhang, W. Zhao, and Y. Zheng, *Local well-posedness and break-down criterion of the incompressible Euler equations with free boundary*, Mem. Amer. Math. Soc. **270** (2021), no. 1318, v+119 pp. ISBN: 978-1-4704-4689-5; 978-1-4704-6524-7.
[27] J. V. Wehausen and E. V. Laitone, *Surface waves*, 1960 Handbuch der Physik, Vol. **9**, Part 3 pp. 446-778 Springer-Verlag, Berlin.
[28] L. Wu, *Well-posedness and decay of the viscous surface wave*, SIAM J. Math. Anal. **46** (2014), no. 3, 2084-2135.
[29] Y. Zheng, *Local well-posedness for the Bénard convection without surface tension*, Commun. Math. Sci. **15** (2017), no. 4, 903-956.
[30] Y. Zheng and I. Tice, *Local well posedness of the near-equilibrium contact line problem in 2-dimensional Stokes flow*, SIAM J. Math. Anal. **49** (2017), no. 2, 899-953.

College of Science, Hohai University, Nanjing 210098, Jiangsu, China.
Email address: `chengx@hhu.edu.cn`

School of Mathematics, Shandong University, Shandong 250100, Jinan, P. R. China
Email address: `yunrui_zheng@sdu.edu.cn`