# Markov generators as non-hermitian supersymmetric quantum Hamiltonians: spectral properties via bi-orthogonal basis and Singular Value Decompositions 

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#### Abstract

Continuity equations associated to continuous-time Markov processes can be considered as Euclidean Schrödinger equations, where the non-hermitian quantum Hamiltonian $\mathbf{H}=\operatorname{div} \mathbf{J}$ is naturally factorized into the product of the divergence operator div and the current operator J. For non-equilibrium Markov jump processes in a space of $N$ configurations with $M$ links between them and $C=M-(N-1) \geq 1$ independent cycles, this factorization of the $N \times N$ Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ involves the incidence matrix $\mathbf{I}$ and the current matrix $\mathbf{J}$ that are both of size $M \times N$, so that the supersymmetric partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ governing the dynamics of the currents living on the $M$ links is a priori of size $M \times M$. To better understand the relations between the spectral decompositions of these two Hamiltonians $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ and $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ with respect to their bi-orthogonal basis of right and left eigenvectors that characterize the relaxation dynamics towards the steady state and the steady currents, it is useful to analyze the properties of the Singular Value Decompositions of the two rectangular matrices $\mathbf{I}$ and $\mathbf{J}$ of size $M \times N$ and the interpretations in terms of discrete Helmholtz decompositions. This general framework concerning Markov jump processes can be adapted to non-equilibrium diffusion processes governed by Fokker-Planck equations in dimension $d$, where the number $N$ of configurations, the number $M$ of links and the number $C=M-(N-1)$ of independent cycles become infinite, while the two matrices $\mathbf{I}$ and $\mathbf{J}$ of size $M \times N$ become first-order differential operators acting on scalar functions to produce vector fields.


## I. INTRODUCTION

While Markov processes satisfying detailed-balance are well understood, in particular via the similarity transformation of the opposite Markov generator into a supersymmetric quantum Hamiltonian of the form $H=Q^{\dagger} Q$ with a real energy spectrum (see the textbooks $[1-3]$ and the applications to various models [4-16]), non-equilibrium Markov processes involving steady currents remain challenging and require the introduction of many new ideas (see the reviews with different scopes [17-25], the PhD Theses [26-31], the Habilitation Thesis [32], and the recent lecture notes [33]). Among the various perspectives that have emerged to analyze non-equilibrium Markov jump processes, the appropriate graph theory based on the $N$ configurations, on the $M$ links existing between them, and on the number $C=M-(N-1)$ of independent cycles has proven to be very useful (see [33-38] and references therein) since the pioneering work of Schnakenberg [39]. In the present paper, we will keep the idea that the incidence matrix $\mathbf{I}$ of size $M \times N$ between links and sites plays an essential role, but we will consider on the same footing the current matrix $\mathbf{J}$ of size $M \times N$, that can be considered as a deformation of the incidence matrix $\mathbf{I}$ via the transition rates. We will focus on the natural rewriting of the opposite Markov generator as the non-hermitian supersymmetric Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ in order to analyze the consequences for its spectral properties, in particular in relation with the Singular Value Decompositions of the two rectangular matrices $\mathbf{I}$ and $\mathbf{J}$ and the interpretations in terms of discrete Helmholtz decompositions. Note that independently of the present motivations coming from Markov processes, the field of non-hermitian physics has become relevant in many areas (see the review [40] and references therein), while the interest into hermitian supersymmetric quantum hamiltonians of the form $H=Q^{\dagger} Q$ (see the review [41] and references therein) has expanded towards various non-hermitian cases [42-45].

The paper is organized as follows. The main text is devoted to non-equilibrium Markov jump processes in a space of $N$ configurations. In section II, we recall that the Markov generator can be considered as the opposite of a nonhermitian quantum $N \times N$ Hamiltonian $\mathbf{H}$, and that the spectral decomposition in the bi-orthogonal basis of its right and left eigenvectors is useful to analyze the convergence of configurations probabilities and configurations observables towards their steady values. In section III, we describe the properties of the currents that are defined on the $M$ links existing between the $N$ configurations and we analyze the consequences of the supersymmetric factorization of the $N \times N$ non-hermitian Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ in terms of the incidence matrix $\mathbf{I}$ and the current matrix $\mathbf{J}$ that are both of size $M \times N$, so that the supersymmetric partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ governing the dynamics of the currents is a priori of size $M \times M$. The relations between the spectral properties of the two Hamiltonians $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ and $\hat{\mathbf{H}}=\mathbf{J} \mathbf{I}^{\dagger}$ are analyzed via the Singular Value Decompositions of the two rectangular matrices $\mathbf{I}$ and $\mathbf{J}$ that are discussed in sections IV and V respectively. Our conclusions are summarized in section VI. In Appendix A, we describe how the general framework described in the main text concerning Markov jump processes can be adapted to non-equilibrium FokkerPlanck generators in dimension $d=3$. The main technical differences are that the number $N$ of configurations, the number $M$ of links and the number $C=M-(N-1)$ of independent cycles become infinite, while the two matrices

I and $\mathbf{J}$ of size $M \times N$ become first-order differential operators acting on scalar functions to produce vector fields.

## II. NON-EQUILIBRIUM MARKOV JUMP PROCESSES IN A SPACE OF $N$ CONFIGURATIONS

In this section, we introduce the general notations for Markov jump processes in a space of $N$ configurations. We recall that the Markov generator can be considered as the opposite of a non-hermitian quantum Hamiltonian $\mathbf{H}$ of size $N \times N$, and that its spectral decomposition in the bi-orthogonal basis of right and left eigenvectors is useful to analyze the convergence of configurations probabilities and configurations observables towards their steady values.

## A. Markov generator $w(.,$.$) of size N \times N$ in the space of the $N$ configurations

For a Markov jump process over $N$ configurations $x$, the master equation for the probability to be at configuration $x$ at time $t$

$$
\begin{equation*}
\partial_{t} p_{t}(x)=\sum_{x^{\prime}} w\left(x, x^{\prime}\right) p_{t}\left(x^{\prime}\right)=w(x, x) p_{t}\left(x^{\prime}\right)+\sum_{x^{\prime} \neq x} w\left(x, x^{\prime}\right) p_{t}\left(x^{\prime}\right) \tag{1}
\end{equation*}
$$

involves the $N \times N$ Markov matrix $w(.,$.$) , where the off-diagonal element w\left(x, x^{\prime}\right) \geq 0$ represents the transition rate from $x^{\prime}$ towards $x$, while the diagonal element is negative and determined by the off-diagonal elements

$$
\begin{equation*}
w(x, x)=-\sum_{x^{\prime} \neq x} w\left(x^{\prime}, x\right)<0 \tag{2}
\end{equation*}
$$

The steady state $p_{*}(x)$ satisfies

$$
\begin{equation*}
0=\partial_{t} p_{*}(x)=\sum_{x^{\prime}} w\left(x, x^{\prime}\right) p_{*}\left(x^{\prime}\right) \tag{3}
\end{equation*}
$$

## B. Parametrization of the transition rates $w\left(x, x^{\prime}\right)$

For each pair of configurations $\left(x \neq x^{\prime}\right)$ related by strictly positive transition rates $w\left(x^{\prime}, x\right) w\left(x, x^{\prime}\right)>0$, it is thus useful to introduce the parametrization

$$
\begin{array}{r}
w\left(x^{\prime}, x\right)=D\left(x^{\prime}, x\right) e^{A\left(x^{\prime}, x\right)} \\
w\left(x, x^{\prime}\right)=D\left(x^{\prime}, x\right) e^{-A\left(x^{\prime}, x\right)} \tag{4}
\end{array}
$$

in terms of the positive symmetric function

$$
\begin{equation*}
D\left(x^{\prime}, x\right)=\sqrt{w\left(x, x^{\prime}\right) w\left(x^{\prime}, x\right)}=D\left(x, x^{\prime}\right)>0 \tag{5}
\end{equation*}
$$

and of the antisymmetric function

$$
\begin{equation*}
A\left(x^{\prime}, x\right)=\frac{1}{2} \ln \left(\frac{w\left(x^{\prime}, x\right)}{w\left(x, x^{\prime}\right)}\right)=-A\left(x, x^{\prime}\right) \tag{6}
\end{equation*}
$$

## C. Opposite of the Markov generator $(-\mathbf{w})=\mathbf{H}$ as a non-hermitian quantum Hamiltonian $\mathbf{H} \neq \mathbf{H}^{\dagger}$

It is useful to consider that the master Eq. 1 is an Euclidean Schrödinger equation

$$
\begin{equation*}
-\partial_{t}\left|p_{t}\right\rangle=\mathbf{H}\left|p_{t}\right\rangle \tag{7}
\end{equation*}
$$

where the quantum Hamiltonian $\mathbf{H}$ is simply the opposite of the Markov matrix $\mathbf{H}=-\mathbf{w}$. With the parametrization of Eq. 4 for the transition rates, the off-diagonal elements read

$$
\begin{align*}
& \mathbf{H}\left(x^{\prime}, x\right)=-w\left(x^{\prime}, x\right)=-D\left(x^{\prime}, x\right) e^{A\left(x^{\prime}, x\right)} \\
& \mathbf{H}\left(x, x^{\prime}\right)=-w\left(x, x^{\prime}\right)=-D\left(x^{\prime}, x\right) e^{-A\left(x^{\prime}, x\right)} \tag{8}
\end{align*}
$$

The symmetric function $D\left(x^{\prime}, x\right)=D\left(x, x^{\prime}\right)$ of Eq. 5 represents the symmetric part of the hopping amplitude between the two sites $x$ and $x^{\prime}$, while the antisymmetric function $A\left(x, x^{\prime}\right)=-A\left(x^{\prime}, x\right)$ of Eq. 6 plays the role of an imaginary vector potential that is responsible for the non-hermitian character $\mathbf{H} \neq \mathbf{H}^{\dagger}$. This correspondence between continuous-time Markov generators and non-hermitian quantum Hamiltonians involving imaginary vector potentials has been already emphasized for diffusion processes [46] and for Markov jump processes on hypercubic lattices [47]. Note that non-hermitian quantum Hamiltonians with imaginary vector potentials have been much studied in various contexts since the pioneering works of Hatano and Nelson [48-50]. However the specificity of Markov models is that the on-site potential $\mathbf{H}(x, x)$ is not chosen independently of the off-diagonal matrix elements of Eq. 8 but is determined in terms of the off-diagonal matrix elements by Eq. 2

$$
\begin{equation*}
\mathbf{H}(x, x)=-w(x, x)=-\sum_{x^{\prime} \neq x} \mathbf{H}\left(x^{\prime}, x\right)=\sum_{x^{\prime} \neq x} D\left(x^{\prime}, x\right) e^{A\left(x^{\prime}, x\right)} \tag{9}
\end{equation*}
$$

## D. Spectral decomposition of the Hamiltonian H in the bi-orthogonal basis of right and left eigenvectors

The spectral decomposition of the non-hermitian Hamiltonian $\mathbf{H} \neq \mathbf{H}^{\dagger}$

$$
\begin{equation*}
\mathbf{H}=\sum_{n=0}^{N-1} E_{n}\left|r_{n}\right\rangle\left\langle l_{n}\right| \tag{10}
\end{equation*}
$$

involves its $N$ eigenvalues $E_{n}$ with the corresponding right and left eigenvectors

$$
\begin{align*}
E_{n}\left|r_{n}\right\rangle & =\mathbf{H}\left|r_{n}\right\rangle \\
E_{n}\left\langle l_{n}\right| & =\left\langle l_{n}\right| \mathbf{H} \tag{11}
\end{align*}
$$

that form a bi-orthogonal basis with the orthonormalization and closure relations

$$
\begin{align*}
\delta_{n, n^{\prime}} & =\left\langle l_{n} \mid r_{n^{\prime}}\right\rangle=\sum_{x}\left\langle l_{n} \mid x\right\rangle\left\langle x \mid r_{n^{\prime}}\right\rangle \\
\mathbf{1}_{N} & =\sum_{n=0}^{N-1}\left|r_{n}\right\rangle\left\langle l_{n}\right| \tag{12}
\end{align*}
$$

The vanishing eigenvalue $E_{0}=0$ is associated to the left eigenvector unity $l_{0}(x)=1$ as a consequence of Eq. 2, while the right eigenvector corresponds to the steady state $r_{0}(x)=p_{*}(x)$ of Eq. 3

$$
\begin{align*}
E_{0} & =0 \\
l_{0}(x) & =1 \\
r_{0}(x) & =p_{*}(x) \tag{13}
\end{align*}
$$

The other $(N-1)$ eigenvalues $E_{n=1, . ., N-1}$ with strictly positive real parts

$$
\begin{equation*}
\operatorname{Re}\left(E_{n}\right)>0 \quad \text { for } n=1,2, . ., N-1 \tag{14}
\end{equation*}
$$

govern the relaxation towards the steady state $p_{*}(x)$ of the propagator $p_{t}\left(x \mid x_{0}\right)$ when one starts at position $x_{0}$ at time $t=0$

$$
\begin{align*}
p_{t}\left(x \mid x_{0}\right)=\langle x| e^{-t \mathbf{H}}\left|x_{0}\right\rangle & =\sum_{n=0}^{N-1} e^{-t E_{n}}\left\langle x \mid r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \\
& =p_{*}(x)+\sum_{n=1}^{N-1} e^{-t E_{n}}\left\langle x \mid r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \tag{15}
\end{align*}
$$

So the $(N-1)$ right eigenvectors $r_{n}(x)=\left\langle x \mid r_{n}\right\rangle$ represent the relaxation modes associated to the excited eigenvalues $E_{n=1, . ., N}$.

The spectral decomposition of the propagator of Eq. 15 is also useful to obtain the relaxation of the average $\overline{O(x(t))}$ of any observable $O(x)$ of the configuration $x$ towards its steady state value $O_{*}=\sum_{x} O(x) p_{*}(x)$.

$$
\begin{align*}
\overline{O(x(t))} \equiv \sum_{x} O(x) p_{t}\left(x \mid x_{0}\right) & =\sum_{n=0}^{N-1} e^{-t E_{n}}\left(\sum_{x} O(x)\left\langle x \mid r_{n}\right\rangle\right)\left\langle l_{n} \mid x_{0}\right\rangle \\
& =\left(\sum_{x} O(x) p_{*}(x)\right)+\sum_{n=1}^{N-1} e^{-t E_{n}}\left(\sum_{x} O(x)\left\langle x \mid r_{n}\right\rangle\right)\left\langle l_{n} \mid x_{0}\right\rangle \tag{16}
\end{align*}
$$

When the observable $O(x)$ coincides with an excited left eigenvector $\left\langle l_{m} \mid x\right\rangle=l_{m}^{*}(x)$, the dynamics of Eq. 16 reduces to a single term

$$
\begin{align*}
\overline{l_{m}^{*}(x(t))} \equiv \sum_{x}\left\langle l_{m} \mid x\right\rangle p_{t}\left(x \mid x_{0}\right) & =\sum_{n=0}^{N-1} e^{-t E_{n}}\left\langle l_{m} \mid r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \\
& =\sum_{n=0}^{N-1} e^{-t E_{n}} \delta_{m, n}\left\langle l_{n} \mid x_{0}\right\rangle=e^{-t E_{m}} l_{m}^{*}\left(x_{0}\right) \tag{17}
\end{align*}
$$

So the excited left eigenvector $\left\langle l_{m} \mid x\right\rangle=l_{m}^{*}(x)$ for $m \in\{1,2, . ., N-1\}$ represents a very simple observable whose dynamics reduces to the relaxation towards zero with the single exponential governed by the excited eigenvalue $E_{m}$.

## III. DYNAMICS OF THE CURRENTS $j_{t}\left(x^{\prime}, x\right)$ DEFINED ON THE $M$ ORIENTED LINKS

In this section, we describe the dynamical properties of the currents that are defined on the $M$ links existing between the $N$ configurations.

## A. Rewriting the master equation as a continuity equation involving the currents defined on the $M$ links

On each link between two configurations $\left(x, x^{\prime}\right)$ that are related by the strictly positive transitions rates $w\left(x, x^{\prime}\right) w\left(x, x^{\prime}\right)>$ 0 with the parametrization of Eq. 4, it is useful to introduce the antisymmetric current

$$
\begin{align*}
j_{t}\left(x^{\prime}, x\right)=-j_{t}\left(x,^{\prime} x\right) & \equiv w\left(x^{\prime}, x\right) p_{t}(x)-w\left(x, x^{\prime}\right) p_{t}\left(x^{\prime}\right) \\
& =D\left(x^{\prime}, x\right)\left[e^{A\left(x^{\prime}, x\right)} p_{t}(x)-e^{-A\left(x^{\prime}, x\right)} p_{t}\left(x^{\prime}\right)\right] \tag{18}
\end{align*}
$$

Then the master Eq. 1 can be rewritten as the discrete continuity equation

$$
\begin{equation*}
-\partial_{t} p_{t}(x)=\sum_{x^{\prime} \neq x} j_{t}\left(x^{\prime}, x\right) \tag{19}
\end{equation*}
$$

where the right handside corresponds to the sum over $x^{\prime}$ of all the currents $j_{t}\left(x^{\prime}, x\right)$ flowing out of the configuration $x$, i.e. to the discrete divergence at position $x$ of the current.

## B. Reminder on the space of the $M$ links

The number $M$ of links that should connect the $N$ configurations has for maximal value $M_{\max }=\frac{N(N-1)}{2}$ in fullyconnected models when any configuration is connected to the $(N-1)$ other configurations, and for minimal value $M_{\min }=N-1$, when the graph is a tree-like structures without any loop. In the general case, the difference between the number $M$ of links and the minimal value $M_{\min }=(N-1)$ needed to connect the $N$ configurations via a spanning tree (see [33-39] and references therein)

$$
\begin{equation*}
C \equiv M-(N-1) \tag{20}
\end{equation*}
$$

represents the number of independent cycles $\gamma=1,2, . ., C$ with the following notations :
a cycle $\gamma$ is a directed self-avoiding closed path of configurations $x^{[\gamma]}\left(1 \leq l \leq l^{[\gamma]}\right)$
involving at least $l^{[\gamma]} \geq 3$ distinct configurations
and the same number $l^{[\gamma]}$ of oriented links $\left[x^{[\gamma]}(l) \rightarrow x^{[\gamma]}(l+1)\right]$ with $x^{[\gamma]}\left(l^{[\gamma]}+1\right) \equiv x^{[\gamma]}(1)$.

These cycles are essential to characterize the equilibrium or non-equilibrium nature of the steady state as we now recall.

## C. Reminder on the properties of the steady currents $j_{*}(.,$.

The steady current $j_{*}\left(x^{\prime}, x\right)$ associated to the steady state $p_{*}($.$) of Eq. 19$

$$
\begin{align*}
j_{*}\left(x^{\prime}, x\right)=-j_{*}\left(x,^{\prime} x\right) & \equiv w\left(x^{\prime}, x\right) p_{*}(x)-w\left(x, x^{\prime}\right) p_{*}\left(x^{\prime}\right) \\
& =D\left(x, x^{\prime}\right)\left[e^{A\left(x, x^{\prime}\right)} p_{*}(x)-e^{-A\left(x, x^{\prime}\right)} p_{*}\left(x^{\prime}\right)\right] \tag{22}
\end{align*}
$$

should be divergenceless, i.e. for any configuration x , the sum over $x^{\prime}$ of the steady currents $j_{*}\left(x^{\prime}, x\right)$ out of $x$ should vanish

$$
\begin{equation*}
0=\sum_{x^{\prime} \neq x} j_{*}\left(x^{\prime}, x\right) \tag{23}
\end{equation*}
$$

The vanishing or non-vanishing of the steady currents $j_{*}\left(x^{\prime}, x\right)$ define the equilibrium or non-equilibrium character of the dynamics as we now recall.

## 1. Equilibrium steady state $p_{*}^{e q}($.$) with vanishing steady currents j_{*}^{e q}(.,)=$.

At equilibrium, the steady current $j_{*}(.,$.$) of Eq. 22$ vanishes on any link

$$
\begin{equation*}
0=j_{*}^{e q}\left(x_{2}, x_{1}\right)=w\left(x_{2}, x_{1}\right) p_{*}^{e q}\left(x_{1}\right)-w\left(x_{1}, x_{2}\right) p_{*}^{e q}\left(x_{2}\right)=D\left(x_{2}, x_{1}\right)\left[e^{A\left(x_{2}, x_{1}\right)} p_{*}^{e q}\left(x_{1}\right)-e^{-A\left(x_{1}, x_{2}\right)} p_{*}^{e q}\left(x_{2}\right)\right] \tag{24}
\end{equation*}
$$

This vanishing is possible only if the transition rates $w(.,$.$) satisfy$

$$
\begin{equation*}
1=\frac{w\left(x_{2}, x_{1}\right) p_{*}^{e q}\left(x_{1}\right)}{w\left(x_{1}, x_{2}\right) p_{*}^{e q}\left(x_{2}\right)}=e^{2 A\left(x_{2}, x_{1}\right)} \frac{p_{*}^{e q}\left(x_{1}\right)}{p_{*}^{e q}\left(x_{2}\right)} \tag{25}
\end{equation*}
$$

i.e. only if the antisymmetric function $A\left(x_{2}, x_{1}\right)$ of Eq. 6 corresponds to the discrete gradient

$$
\begin{equation*}
A\left(x_{2}, x_{1}\right)=\frac{1}{2}\left[\ln \left(p_{*}^{e q}\left(x_{2}\right)\right)-\ln \left(p_{*}^{e q}\left(x_{1}\right)\right)\right] \tag{26}
\end{equation*}
$$

On a tree-like structure without any cycle $C=0$, the requirement of vanishing divergence of Eq. 23 for the steady currents can be taken into account iteratively starting from the leaves to obtain that the steady current should vanish on every link

$$
\begin{equation*}
C=0: j_{*}\left(x^{\prime}, x\right)=0 \tag{27}
\end{equation*}
$$

On a graph with $C \geq 1$ independent cycles $\gamma=1, . ., C$ introduced around Eq. 21, one needs to check whether the gradient form of Eq. 25 is compatible along each cycle $\gamma$ : these compatibility conditions can be written either in terms of the transition rates $w(.,$.$) in order to recover the famous Kolmogorov criterion for reversibility$

$$
\begin{equation*}
\prod_{l=1}^{l[\gamma]} \frac{w\left(x^{[\gamma]}(l+1), x^{[\gamma]}(l)\right)}{w\left(x^{[\gamma]}(l), x^{[\gamma]}(l+1)\right)}=1 \tag{28}
\end{equation*}
$$

or in terms of the antisymmetric function $A(.,$.$) whose total circulation \Gamma^{[\gamma]}[A(.,)$.$] around each cycle should vanish$

$$
\begin{equation*}
\Gamma^{[\gamma]}[A(., .)] \equiv \sum_{l=1}^{l[\gamma]} A\left(x^{[\gamma]}(l+1), x^{[\gamma]}(l)\right)=0 \tag{29}
\end{equation*}
$$

2. Non-equilibrium steady state $p_{*}($.$) with nonvanishing steady currents j_{*}(.,) \neq$.

In all the other cases where the Kolmogorov criterion of Eq. 28 is not satisfied on the $C$ independent cycles $\gamma=1, . ., C$, i.e. equivalently when the antisymmetric function $A(.,$.$) displays some non-vanishing circulations around$ cycles that prevent its rewriting as a discrete gradient, then the steady state $p_{*}($.$) will be out-of-equilibrium with$ non-vanishing steady currents

$$
\begin{equation*}
j_{*}(., .) \neq 0 \tag{30}
\end{equation*}
$$

The requirement that the discrete divergence should vanish (Eq. 23) yields that the steady current $j_{*}$ can be written as a superposition of the $C$ cycle-currents $j_{*}^{C y c l e[\gamma]}$ that flow around the $C$ cycles $\gamma=1,2, . ., C$. These cycle-currents $j_{*}^{\text {Cycle }[\gamma]}$ can be obtained from the circulation $\Gamma^{[\gamma]}\left[j_{*}(.,).\right]$ of the steady current $j_{*}(.,$.$) around each cycle \gamma$ of Eq. 21

$$
\begin{equation*}
\Gamma^{[\gamma]}\left[j_{*}(., .)\right] \equiv \sum_{l=1}^{l[\gamma]} j_{*}\left(x^{[\gamma]}(l+1), x^{[\gamma]}(l)\right)=l^{[\gamma]} j_{*}^{C y c l e[\gamma]} \tag{31}
\end{equation*}
$$

Then the steady current $j_{*}\left(x_{2}, x_{1}\right)$ on each oriented link can be rewritten as a linear combination of these cycle-currents $j_{*}^{C y c l e[\gamma]}$

$$
\begin{equation*}
j_{*}\left(x_{2}, x_{1}\right)=\sum_{\gamma=1}^{C} j_{*}^{C y c l e[\gamma]} \epsilon^{[\gamma]}\left(x_{2}, x_{1}\right) \tag{32}
\end{equation*}
$$

where the coefficient $\epsilon^{[\gamma]}\left(x_{2}, x_{1}\right)$ takes into account whether the oriented cycle $\gamma$ contains this oriented link $\left(x_{1}, x_{2}\right)$ with the same orientation or with the opposite orientation

$$
\begin{aligned}
\epsilon^{[\gamma]}\left(x_{2}, x_{1}\right) & \equiv \sum_{l=1}^{l^{[\gamma]}}\left(\delta_{x_{2}, x^{[\gamma]}(l+1)} \delta_{x_{1}, x^{[\gamma]}(l)}-\delta_{x_{2}, x^{[\gamma]}(l)} \delta_{x_{1}, x^{[\gamma]}(l+1)}\right) \\
& =\left\{\begin{array}{l}
+1 \text { if the oriented link }\left(x_{1} \rightarrow x_{2}\right) \text { appears in the oriented cycle } \gamma \text { with the same orientation } \\
-1 \text { if the oriented link }\left(x_{1} \rightarrow x_{2}\right) \text { appears in the oriented cycle } \gamma \text { with the opposite orientation(33) } \\
0 \text { otherwise }
\end{array}\right.
\end{aligned}
$$

Once the steady current $j_{*}(.,$.$) has been parametrized by the C$ coefficients of Eq. 32, the steady state $p_{*}($.$) should$ satisfy on each of the $M$ links

$$
\begin{equation*}
w\left(x_{2}, x_{1}\right) p_{*}\left(x_{1}\right)-w\left(x_{2}, x_{1}\right) p_{*}\left(x_{1}\right)=j_{*}\left(x_{2}, x_{1}\right)=\sum_{\gamma=1}^{C} j_{*}^{C y c l e[\gamma]} \epsilon^{[\gamma]}\left(x_{2}, x_{1}\right) \tag{34}
\end{equation*}
$$

This gives a system of $M$ equations for the $M=C+(N-1)$ variables that are the $C$ coefficients $j_{*}^{C y c l e}[\gamma]$ of the steady current of Eq. 32 and the $(N-1)$ independent coefficients of the normalized steady state $p_{*}($.$) .$

In the present paper, we will always assume that the dynamics is out-of-equilibrium with non-vanishing steady current $j_{*}(.,) \neq$.0 with $C=M-(N-1) \geq 1$ cycles.

## D. Factorization of the non-hermitian Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ in terms of two $M \times N$ matrices $\mathbf{I}$ and $\mathbf{J}$

In order to analyze the properties of functions $v\left(x^{\prime}, x\right)$ that are antisymmetric when the two configurations $x$ and $x^{\prime}$ at the ends of a link are exchanged

$$
\begin{equation*}
v\left(x^{\prime}, x\right)=-v\left(x, x^{\prime}\right) \tag{35}
\end{equation*}
$$

it will be useful to introduce the double-ket notation $\left|\begin{array}{l}x_{2} \\ x_{1}\end{array}\right\rangle$ for the space of the $M$ oriented links between two configurations $x_{1}<x_{2}$ and to write

$$
\begin{equation*}
\left\langle\left\langle x_{x_{1}}^{x_{2}} \mid v\right\rangle\right\rangle \equiv v\left(x_{2}, x_{1}\right) \tag{36}
\end{equation*}
$$

## 1. Current matrix $\mathbf{J}$ of size $M \times N$

The current matrix $\mathbf{J}$ of size $M \times N$ with the matrix elements

$$
\begin{equation*}
\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{J} \mid x\right\rangle=w\left(x_{2}, x_{1}\right) \delta_{x, x_{1}}-w\left(x_{1}, x_{2}\right) \delta_{x, x_{2}} \tag{37}
\end{equation*}
$$

can be applied to the probability ket $\left|p_{t}\right\rangle$

$$
\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{J} \mid p_{t}\right\rangle=\sum_{x}\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{J} \mid x\right\rangle\left\langle x \mid p_{t}\right\rangle=w\left(x_{2}, x_{1}\right) p_{t}\left(x_{1}\right)-w\left(x_{1}, x_{2}\right) p_{t}\left(x_{2}\right)=j_{t}\left(x_{2}, x_{1}\right) \equiv\left\langle\left\langle\left.\begin{array}{l}
x_{2}  \tag{38}\\
x_{1}
\end{array} \right\rvert\, j_{t}\right\rangle\right\rangle
$$

to reproduce the current $\left.j_{t}\left(x_{2}, x_{1}\right) \equiv\left\langle\left\langle x_{1}\right| x_{2} \mid j_{t}\right\rangle\right\rangle$ of Eq. 18 flowing from $x_{1}$ towards $x_{2}$, so that one can write at the matrix level

$$
\begin{equation*}
\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle \tag{39}
\end{equation*}
$$

The adjoint matrix $\mathbf{J}^{\dagger}$ of size $N \times M$ acts on $\left.|v\rangle\right\rangle$ to produce at configuration $x$

$$
\begin{align*}
\left.\langle x| \mathbf{J}^{\dagger}|v\rangle\right\rangle & \left.\left.=\left.\sum_{\binom{x_{2}}{x_{1}}}\langle x| \mathbf{J}^{\dagger}\right|_{x_{1}} ^{x_{2}}\right\rangle\right\rangle\left\langle\left\langle\left.\begin{array}{l}
x_{2} \\
x_{1}
\end{array} \right\rvert\, v\right\rangle\right\rangle=\sum_{\binom{x_{2}}{x_{1}}}\left(w\left(x_{2}, x_{1}\right) \delta_{x, x_{1}}-w\left(x_{1}, x_{2}\right) \delta_{x, x_{2}}\right) v\left(x_{2}, x_{1}\right) \\
& =\sum_{x_{2}>x} w\left(x_{2}, x\right) v\left(x_{2}, x\right)-\sum_{x_{1}<x} w\left(x_{1}, x\right) v\left(x, x_{1}\right)=\sum_{x_{2}>x} w\left(x_{2}, x\right) v\left(x_{2}, x\right)+\sum_{x_{1}<x} w\left(x_{1}, x\right) v\left(x_{1}, x\right) \\
& =\sum_{x^{\prime} \neq x} w\left(x^{\prime}, x\right) v\left(x^{\prime}, x\right) \tag{40}
\end{align*}
$$

## 2. Incidence matrix $\mathbf{I}$ of size $M \times N$

When the transition rates are replaced by unity

$$
\begin{equation*}
w\left(x^{\prime}, x\right) \rightarrow 1 \tag{41}
\end{equation*}
$$

the current matrix $\mathbf{J}$ of Eq. 37 reduces to the incidence matrix $\mathbf{I}$ that keeps only the information on the geometry of the existing links between configurations

$$
\begin{equation*}
\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{I} \mid x\right\rangle=\delta_{x, x_{1}}-\delta_{x, x_{2}} \tag{42}
\end{equation*}
$$

The analog of Eq. 38

$$
\begin{equation*}
\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{I} \mid f\right\rangle=f\left(x_{1}\right)-f\left(x_{2}\right) \equiv-\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{g r a d} \mid f\right\rangle \tag{43}
\end{equation*}
$$

represents the difference of the function $f$ between the two ends of the oriented link, so the incidence matrix $\mathbf{I}$ represents the opposite of the discrete gradient

$$
\begin{equation*}
\mathbf{I}=-\operatorname{grad} \tag{44}
\end{equation*}
$$

The analog of Eq. 40

$$
\begin{equation*}
\left.\left.\langle x| \mathbf{I}^{\dagger}|v\rangle\right\rangle=\sum_{x^{\prime} \neq x} v\left(x^{\prime}, x\right) \equiv\langle x| \operatorname{div}|v\rangle\right\rangle \tag{45}
\end{equation*}
$$

represents the discrete divergence div of the vector $|v\rangle\rangle$ at configuration $x$

$$
\begin{equation*}
\mathbf{I}^{\dagger}=\operatorname{div} \tag{46}
\end{equation*}
$$

3. Supersymmetric factorization of the non-hermitian Hamiltonian into $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$

The two matrices $\mathbf{J}$ and $\mathbf{I}$ of size $M \times N$ introduced above are useful to factorize the $N \times N$ non-hermitian Hamiltonian $\mathbf{H}$ into

$$
\begin{equation*}
\mathbf{H}=\operatorname{div} \mathbf{J}=\mathbf{I}^{\dagger} \mathbf{J} \tag{47}
\end{equation*}
$$

This factorization corresponds to the natural splitting of the Euclidean Schödiner Eq. 7 into the pair of matrix equations

$$
\begin{align*}
-\partial_{t}\left|p_{t}\right\rangle & \left.\left.=\mathbf{I}^{\dagger}\left|j_{t}\right\rangle\right\rangle=\operatorname{div}\left|j_{t}\right\rangle\right\rangle \\
\left.\left|j_{t}\right\rangle\right\rangle & =\mathbf{J}\left|p_{t}\right\rangle \tag{48}
\end{align*}
$$

The current $\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle$ follows the dynamics

$$
\begin{equation*}
\left.\left.\left.-\partial_{t}\left|j_{t}\right\rangle\right\rangle \quad=\mathbf{J}\left(-\partial_{t}\left|p_{t}\right\rangle\right)=\mathbf{J I}^{\dagger}\left|j_{t}\right\rangle\right\rangle \equiv \hat{\mathbf{H}}\left|j_{t}\right\rangle\right\rangle \tag{49}
\end{equation*}
$$

governed by supersymmetric partner

$$
\begin{equation*}
\hat{\mathbf{H}} \equiv \mathbf{J I}^{\dagger} \text { of dimension } M \times M \tag{50}
\end{equation*}
$$

of the Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ of dimension $N \times N$.
However, the currents $\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle$ do not live in the full space of dimension $M$ but in a smaller subspace since they can be computed from the ket $\left|p_{t}\right\rangle$ of dimension $N$ given by the spectral decomposition of the propagator of Eq. 15

$$
\begin{equation*}
\left|p_{t}\right\rangle=\sum_{n=0}^{N-1} e^{-t E_{n}}\left|r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle=\left|p_{*}\right\rangle+\sum_{n=1}^{N-1} e^{-t E_{n}}\left|r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \tag{51}
\end{equation*}
$$

So the relaxation of the current

$$
\begin{align*}
\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle & =\mathbf{J}\left|p_{*}\right\rangle+\sum_{n=1}^{N-1} e^{-t E_{n}} \mathbf{J}\left|r_{n}\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \\
& \left.\left.\equiv\left|j_{*}\right\rangle\right\rangle+\sum_{n=1}^{N-1} e^{-t E_{n}}\left|j_{n}\right\rangle\right\rangle\left\langle l_{n} \mid x_{0}\right\rangle \tag{52}
\end{align*}
$$

towards the steady current $\left.\left|j_{*}\right\rangle\right\rangle$ associated to the steady state $\left|p_{*}\right\rangle$

$$
\begin{equation*}
\left.\left|j_{*}\right\rangle\right\rangle \equiv \mathbf{J}\left|p_{*}\right\rangle \tag{53}
\end{equation*}
$$

involves the same $(N-1)$ excited eigenvalues $E_{n=1, . ., N-1}$ as the Hamiltonian $\mathbf{H}$. The corresponding relaxation modes $\left.\left|j_{n}\right\rangle\right\rangle$ are the currents associated to the excited right eigenvectors $r_{n}$

$$
\begin{equation*}
\left.\left|j_{n}\right\rangle\right\rangle \equiv \mathbf{J}\left|r_{n}\right\rangle \tag{54}
\end{equation*}
$$

while the eigenvalue Eq. 11 for the right eigenvectors $\left|r_{n}\right\rangle$ gives the counterpart of Eq. 54

$$
\begin{equation*}
\left.E_{n}\left|r_{n}\right\rangle=\mathbf{I}^{\dagger} \mathbf{J}\left|r_{n}\right\rangle=\mathbf{I}^{\dagger}\left|j_{n}\right\rangle\right\rangle \tag{55}
\end{equation*}
$$

Eqs 54 and 55 yield that $\left.\left|j_{n}\right\rangle\right\rangle$ is a right eigenvector of the partner $\hat{\mathbf{H}} \equiv \mathbf{J I}^{\dagger}$ of Eq. 50 associated to the eigenvalue $E_{n}$

$$
\begin{equation*}
\left.\left.\left.E_{n}\left|j_{n}\right\rangle\right\rangle=\mathbf{J I}^{\dagger}\left|j_{n}\right\rangle\right\rangle=\hat{\mathbf{H}}\left|j_{n}\right\rangle\right\rangle \tag{56}
\end{equation*}
$$

including the case $n=0$ since the non-vanishing steady current $\left.\left.\left|j_{n=0}\right\rangle\right\rangle=\left|j_{*}\right\rangle\right\rangle \neq 0$ is annihilated by $\mathbf{I}^{\dagger}=\operatorname{div}$

$$
\begin{equation*}
\left.\left.\mathbf{I}^{\dagger}\left|j_{*}\right\rangle\right\rangle=0=\hat{\mathbf{H}}\left|j_{*}\right\rangle\right\rangle \tag{57}
\end{equation*}
$$

Similarly, the eigenvalue Eq. 11 for the excited left eigenvector $\left\langle l_{n}\right|$ can be split into the two matrix equations

$$
\begin{align*}
E_{n}\left\langle\left\langle i_{n}\right|\right. & =\left\langle l_{n}\right| \mathbf{I}^{\dagger} \\
\left\langle l_{n}\right| & =\left\langle\left\langle i_{n}\right| \mathbf{J}\right. \tag{58}
\end{align*}
$$

involving the bra $\left\langle\left\langle i_{n}\right|\right.$ which is a left eigenvector of the partner $\hat{\mathbf{H}} \equiv \mathbf{J I}^{\dagger}$ of Eq. 50 associated to the eigenvalue $E_{n}$

$$
\begin{equation*}
E_{n}\left\langle\left\langle i_{n}\right|=\left\langle\left\langle i_{n}\right| \mathbf{J I}^{\dagger}=\left\langle\left\langle i_{n}\right| \hat{\mathbf{H}}\right.\right.\right. \tag{59}
\end{equation*}
$$

For $n=0$ where the left eigenvector is unity $\left\langle l_{(n=0)} \mid x\right\rangle=1$, the bra $\left\langle\left\langle i_{(n=0)}\right|\right.$ satisfies

$$
\begin{align*}
\left\langle l_{0}\right| & =\left\langle\left\langle i_{0}\right| \mathbf{J} \neq 0\right. \\
0=\left\langle l_{0}\right| \mathbf{I}^{\dagger} & =\left\langle\left\langle i_{0}\right| \mathbf{J}^{\dagger}=\left\langle\left\langle i_{0}\right| \hat{\mathbf{H}}\right.\right. \tag{60}
\end{align*}
$$

The scalar products between the bra $\left\langle\left\langle i_{n}\right|\right.$ and the kets $\left.\left.\mid j_{n^{\prime}}\right\rangle\right\rangle$ satisfy the orthonormalisation inherited from the orthonormalization of Eq. 12 concerning the left eigenvectors $l_{n}$ and the right eigenvectors $r_{n^{\prime}}$

$$
\begin{equation*}
\left\langle\left\langle i_{n} \mid j_{n^{\prime}}\right\rangle\right\rangle=\left\langle\left\langle i_{n}\right| \mathbf{J} \mid r_{n^{\prime}}\right\rangle=\left\langle l_{n} \mid r_{n^{\prime}}\right\rangle=\delta_{n^{\prime}, n} \tag{61}
\end{equation*}
$$

Note that for the $(N-1)$ excited eigenvalues $E_{n=1, . ., N-1} \neq 0$, the relations involving the incidence matrix $\mathbf{I}=-\mathbf{g r a d}$ of Eq. 44 and its adjoint $\mathbf{I}^{\dagger}=\mathbf{d i v}$ of Eq. 46 mean that the right eigenvector $r_{n}($.$) can be rewritten in terms of the$ discrete divergence of the $\left.\left|j_{n}\right\rangle\right\rangle$

$$
\begin{equation*}
\left.r_{n}(x)=\left\langle x \mid r_{n}\right\rangle=\frac{1}{E_{n}}\langle x| \mathbf{I}^{\dagger}\left|j_{n}\right\rangle\right\rangle=\frac{\left.\langle x| \operatorname{div}\left|j_{n}\right\rangle\right\rangle}{E_{n}} \tag{62}
\end{equation*}
$$

in constrast to the steady state $\left|r_{n=0}\right\rangle=\left|p_{*}\right\rangle$ associated to the divergenceless steady current $\left.\left.\left|j_{n=0}\right\rangle\right\rangle=\left|j_{*}\right\rangle\right\rangle$

$$
\begin{equation*}
\left.\operatorname{div}\left|j_{*}\right\rangle\right\rangle=0 \tag{63}
\end{equation*}
$$

while the $\left.\left|i_{n}\right\rangle\right\rangle$ for $n=1, . ., N-1$ can be rewritten as the discrete gradient of the left eigenvector $l_{n}($.

$$
\begin{equation*}
\left\langle\left\langle{ }_{x_{1}}^{x_{2}} \mid i_{n}\right\rangle\right\rangle=\frac{1}{E_{n}^{*}}\left\langle\left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{I} \mid l_{n}\right\rangle=-\frac{1}{E_{n}^{*}}\left\langle\left\langle{ }_{\left\langle x_{1}\right.}^{x_{2}}\right| \operatorname{grad} \mid l_{n}\right\rangle=\frac{l_{n}\left(x_{1}\right)-l_{n}\left(x_{2}\right)}{E_{n}^{*}} \tag{64}
\end{equation*}
$$

in contrast to $\left.\left|i_{n=0}\right\rangle\right\rangle$ associated to the unity left eigenvector $l_{0}(x)=1$.
Since the initial current $\left.\left|j_{t=0}\right\rangle\right\rangle$ associated to the initial condition $\left|x_{0}\right\rangle$ is

$$
\begin{equation*}
\left.\left|j_{t=0}\right\rangle\right\rangle=\mathbf{J}\left|x_{0}\right\rangle \tag{65}
\end{equation*}
$$

one can rewrite the scalar products $\left\langle l_{n} \mid x_{0}\right\rangle$ appearing in Eq. 52 as

$$
\begin{equation*}
\left\langle l_{n} \mid x_{0}\right\rangle=\left\langle\left\langle i_{n}\right| \mathbf{J} \mid x_{0}\right\rangle=\left\langle\left\langle i_{n} \mid j_{t=0}\right\rangle\right\rangle \tag{66}
\end{equation*}
$$

including the case $n=0$ with

$$
\begin{equation*}
1=l_{0}\left(x_{0}\right)=\left\langle l_{0} \mid x_{0}\right\rangle=\left\langle\left\langle i_{0}\right| \mathbf{J} \mid x_{0}\right\rangle=\left\langle\left\langle i_{0} \mid j_{t=0}\right\rangle\right\rangle \tag{67}
\end{equation*}
$$

in order to rewrite the spectral decomposition of Eq. 52

$$
\begin{equation*}
\left.\left.\left.\left.\left|j_{t}\right\rangle\right\rangle=\left|j_{*}\right\rangle\right\rangle+\sum_{n=1}^{N-1} e^{-t E_{n}}\left|j_{n}\right\rangle\right\rangle\left\langle\left\langle i_{n} \mid j_{t=0}\right\rangle\right\rangle=\left(\sum_{n=0}^{N-1} e^{-t E_{n}}\left|j_{n}\right\rangle\right\rangle\left\langle\left\langle i_{n}\right|\right)\left|j_{t=0}\right\rangle\right\rangle \tag{68}
\end{equation*}
$$

In conclusion, the current $\left.\left|j_{t}\right\rangle\right\rangle$ lives in the subspace of dimension $N$ spanned by the bi-orthogonal basis of the bras $\left\langle\left\langle i_{n=0, . ., N-1}\right|\right.$ and the kets $\left.\left.\mid j_{n=0,1, . ., N-1}\right\rangle\right\rangle$ that are left and right eigenvectors of the partner $\hat{\mathbf{H}}$ of Eq. 50 associated to the eigenvalues $E_{n=0, ., N-1}$. However since the partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ of Eq. 50 is a matrix of size $M \times M$ bigger than $N \times N$, it is useful in the two following sections to write the Singular Values Decompositions of the two rectangular matrices $\mathbf{I}$ and $\mathbf{J}$ of size $M \times N$ and to analyze the consequences for the spectral properties of the two Hamiltonians $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ and $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$.

## IV. SINGULAR VALUE DECOMPOSITION OF THE INCIDENCE MATRIX I OF SIZE $M \times N$

This section is devoted to the properties of the Singular Value Decomposition of the Incidence matrix I of Eq. 42 of size $M \times N$ with $M \geq N$.
A. SVD of the Incidence matrix I involving $(N-1)$ positive singular values $I_{\alpha=1, \ldots, N-1}>0$ and $I_{(\alpha=0)}=0$

The Singular Value Decomposition of the Incidence matrix $\mathbf{I}$ of size $M \times N$ with $M \geq N$ involves ( $N-1$ ) strictly positive singular values $I_{\alpha}>0$ with $\alpha=1, . ., N-1$

$$
\begin{align*}
\mathbf{I} & \left.=\sum_{\alpha=1}^{N-1} I_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle I_{\alpha}^{L}\right| \\
\mathbf{I}^{\dagger} & =\sum_{\alpha=1}^{N-1} I_{\alpha}\left|I_{\alpha}^{L}\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{69}
\end{align*}
$$

while the vanishing singular value

$$
\begin{equation*}
I_{(\alpha=0)}=0 \tag{70}
\end{equation*}
$$

is associated to the uniform normalized eigenvector

$$
\begin{equation*}
\left\langle x \mid I_{(\alpha=0)}^{L}\right\rangle=\frac{1}{\sqrt{N}} \tag{71}
\end{equation*}
$$

that is annihilated by the matrix I that represents the opposite of the discrete gradient (Eq. 43)

$$
\begin{equation*}
\mathbf{I}\left|I_{(\alpha=0)}^{L}\right\rangle=-\operatorname{grad} \mid I_{(\alpha=0)}^{L}=0 \tag{72}
\end{equation*}
$$

The $N$ left singular kets $\left|I_{\alpha=0,1 . ., N-1}^{L}\right\rangle$ form an orthonormal basis of the space of the $N$ configurations

$$
\begin{align*}
\delta_{\alpha, \alpha^{\prime}} & =\left\langle I_{\alpha}^{L} \mid I_{\alpha^{\prime}}^{L}\right\rangle=\sum_{x}\left\langle I_{\alpha}^{L} \mid x\right\rangle\left\langle x \mid I_{\alpha^{\prime}}^{L}\right\rangle \\
\mathbf{1}_{N} & =\sum_{\alpha=0}^{N-1}\left|I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L}\right|=\sum_{\alpha=1}^{N-1}\left|I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L}\right|+\left|I_{(\alpha=0)}^{L}\right\rangle\left\langle I_{(\alpha=0)}^{L}\right| \tag{73}
\end{align*}
$$

while the $(N-1)$ right singular kets $\left.\left|I_{\alpha=1, \ldots, N-1}^{R}\right\rangle\right\rangle$ that are associated to the $(N-1)$ strictly positive singular values $I_{\alpha=1, . ., N-1}>0$ in Eq. 69 should be supplemented by $M-(N-1)=C$ other kets $\left|I_{\alpha=0,-1,-2 . .,-(C-1)}^{R}\right\rangle$ in order to obtain an orthonormal basis of the space of the $M$ links

$$
\begin{align*}
\delta_{\alpha, \alpha^{\prime}} & \left.=\left\langle\left\langle I_{\alpha}^{R} \mid I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle=\sum_{\binom{x_{2}}{x_{1}}}\left\langle\left\langle I_{\alpha}^{R} \left\lvert\, \begin{array}{l}
x_{2} \\
x_{1}
\end{array}\right.\right\rangle\right\rangle\left\langle\left.\begin{array}{l}
x_{2} \\
x_{1}
\end{array} \right\rvert\, I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle \\
\mathbf{1}_{M} & \left.\left.\left.=\sum_{\alpha=-(C-1)}^{N-1}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle I_{\alpha}^{R}\right|=\sum_{\alpha=1}^{N-1}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|+\sum_{\alpha=-(C-1)}^{0} \mid I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{74}
\end{align*}
$$

## B. Relation with the spectral decomposition of the discrete Laplacian in the space of the $N$ configurations

Since the incidence matrix $\mathbf{I}$ represents the opposite of the discrete gradient (Eq. 44) and since the adjoint $\mathbf{I}^{\dagger}$ represents the discrete divergence (Eq. 46), the supersymmetric matrix $\mathbf{I}^{\dagger} \mathbf{I}$ of size $N \times N$ corresponds to the opposite of the discrete Laplacian $\boldsymbol{\Delta}$ in the space of the $N$ configurations

$$
\begin{equation*}
\mathbf{I}^{\dagger} \mathbf{I}=\operatorname{div}(-\operatorname{grad})=-\boldsymbol{\Delta} \tag{75}
\end{equation*}
$$

as can be checked via the evaluation of the matrix elements using Eq. 42

$$
\begin{align*}
\langle x| \mathbf{I}^{\dagger} \mathbf{I}\left|x^{\prime}\right\rangle & \left.=\sum_{\binom{x_{2}}{x_{1}}}\left\langle x^{\prime}\right| \mathbf{I} \right\rvert\, \begin{array}{c}
\left.x_{x_{1}}\right\rangle \\
\\
\end{array} \\
& \left.\left.=\sum_{\substack{x_{2} \\
x_{1}}}\left(\delta_{x, x_{1}} \delta_{x^{\prime}, x_{1}}+\delta_{x, x_{2}} \delta_{x_{1}, x_{2}}^{x_{2}}\right\rangle\right\rangle\right\rangle=\sum_{x, x_{2}}\left(\delta_{x, x_{1}}-\delta_{x, x_{2}}\right)\left(\delta_{x^{\prime}, x_{1}}-\delta_{x^{\prime}, x_{2}}\right) \\
& =\left\{\begin{array}{l}
z(x) \text { if } x=x^{\prime} \\
-1 \text { if } x \text { and } x^{\prime} \\
0 \text { otherwise }
\end{array} \text { are the two ends of a link } \quad \equiv-\langle x| \boldsymbol{\Delta}\left|x^{\prime}\right\rangle\right. \tag{76}
\end{align*}
$$

where $z(x)$ is the number of links connected to the configuration $x$.
On the other hand, the evaluation of the supersymmetric matrix $\mathbf{I}^{\dagger} \mathbf{I}$ via the Singular Value Decompositions of Eq. 69 for the incidence matrix $\mathbf{I}$ and its adjoint $\mathbf{I}^{\dagger}$

$$
\begin{equation*}
-\boldsymbol{\Delta}=\mathbf{I}^{\dagger} \mathbf{I}=\left(\sum_{\alpha=1}^{N-1} I_{\alpha}\left|I_{\alpha}^{L}\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right)\left(\sum_{\alpha^{\prime}=1}^{N-1} I_{\alpha^{\prime}}\left|I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle\left\langle I_{\alpha^{\prime}}^{L}\right|\right)=\sum_{\alpha=1}^{N-1} I_{\alpha}^{2}\left|I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L}\right| \tag{77}
\end{equation*}
$$

gives its spectral decomposition in terms of the $(N-1)$ strictly positive eigenvalues $I_{\alpha=1, \ldots, N-1}^{2}>0$ with their associated orthonormalized eigenvectors $\left|I_{\alpha=1, . ., N-1}^{L}\right\rangle$, while the vanishing eigenvalue $I_{(\alpha=0)}^{2}=0$ of Eq. 70 is associated to the uniform normalized eigenvector $\left|I_{(\alpha=0)}^{L}\right\rangle$ of Eq. 71.

In particular, whenever an orthonormalized basis of eigenvectors $\left|I_{\alpha}^{L}\right\rangle$ of the opposite Laplacian ( $-\boldsymbol{\Delta}$ ) of Eq. 77 is known, this is the orthonormalized basis of left singular vectors $\left|I_{\alpha}^{L}\right\rangle$ that appear in the SVD of the incidence matrix I of Eq. 69, while the corresponding singular values $I_{\alpha}$ of the incidence matrix $\mathbf{I}$ are given by the square-roots of the eigenvalues $I_{\alpha}^{2}$. The corresponding $(N-1)$ right singular vectors $\left|I_{\alpha=1, \ldots, N-1}^{R}\right\rangle$ of Eq. 69 can be then obtained via the application of the incidence matrix $\mathbf{I}$ on the left singular vectors $\left|I_{\alpha}^{L}\right\rangle$

$$
\begin{equation*}
\left.\left.\mathbf{I}\left|I_{\alpha}^{L}\right\rangle \quad=\left(\sum_{\alpha^{\prime}=1}^{N-1} I_{\alpha^{\prime}}\left|I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle\left\langle I_{\alpha^{\prime}}^{L}\right|\right)\left|I_{\alpha}^{L}\right\rangle=I_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \tag{78}
\end{equation*}
$$

The further application of the adjoint matrix $\mathbf{I}^{\dagger}$ to these right singular vectors $\left.\left|I_{\alpha=1, \ldots, N-1}^{R}\right\rangle\right\rangle$

$$
\begin{equation*}
\left.\mathbf{I}^{\dagger}\left|I_{\alpha}^{R}\right\rangle\right\rangle=\left(\sum_{\alpha^{\prime}=1}^{N-1} I_{\alpha^{\prime}}\left|I_{\alpha^{\prime}}^{L}\right\rangle\left\langle\left\langle I_{\alpha^{\prime}}^{R}\right|\right)\left|I_{\alpha}^{R}\right\rangle\right\rangle=I_{\alpha}\left|I_{\alpha}^{L}\right\rangle \tag{79}
\end{equation*}
$$

then reproduce the left singular vectors $\left|I_{\alpha}^{L}\right\rangle$.

## C. Relation with the spectral decomposition of the supersymmetric partner $\mathbf{I I}^{\dagger}$ of size $M \times M$

The matrix elements of the supersymmetric partner $\mathbf{I I}^{\dagger}$ of size $M \times M$ read using Eq. 42

$$
\begin{aligned}
& \left\langle{ }_{x_{1}}^{x_{2}}\right| \mathbf{I I}^{\dagger}\left|\begin{array}{c}
x_{x_{1}^{\prime}}^{\prime}
\end{array}\right\rangle=\sum_{x}\left\langle\begin{array}{l}
x_{1}^{x_{2}} \\
x_{1}
\end{array}\right| \mathbf{I}|x\rangle\left\langle\left.\begin{array}{l}
x_{x_{1}^{\prime}}^{\prime} \\
x_{2}^{\prime}
\end{array} \mathbf{I} \right\rvert\, x\right\rangle=\sum_{x}\left(\delta_{x, x_{1}}-\delta_{x, x_{2}}\right)\left(\delta_{x, x_{1}^{\prime}}-\delta_{x, x_{2}^{\prime}}\right) \\
& =\delta_{x_{1}, x_{1}^{\prime}}+\delta_{x_{2}, x_{2}^{\prime}}-\delta_{x_{1}, x_{2}^{\prime}}-\delta_{x_{1}^{\prime}, x_{2}}
\end{aligned}
$$

$=\left\{\begin{array}{l}2 \text { if the two oriented links coincide, i.e. } x_{1}=x_{1}^{\prime} \text { and } x_{2}=x_{2}^{\prime} \\ 1 \text { if the two oriented links are different but share the same starting-point } x_{1}=x_{1}^{\prime} \text { or the same end-point } x_{2}=x_{2}^{\prime} \\ -1 \text { if the starting-point of one oriented link coincide with the end-point of the other link, i.e. } x_{2}=x_{1}^{\prime} \text { or } x_{2}^{\prime}=x_{1}\end{array}\right.$
On the other hand, the evaluation of the supersymmetric matrix $\mathbf{I I}^{\dagger}$ of size $M \times M$ via the Singular Value Decompositions of Eq. 69 for the incidence matrix $\mathbf{I}$ and its adjoint $\mathbf{I}^{\dagger}$

$$
\begin{equation*}
\left.\mathbf{I I}^{\dagger}=\sum_{\alpha=1}^{N-1} I_{\alpha}^{2}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{81}
\end{equation*}
$$

gives its spectral decomposition that involves the same $(N-1)$ strictly positive eigenvalues $I_{\alpha=1, . ., N-1}^{2}>0$ as the opposite-Laplacian of Eq. 77, while the corresponding eigenvectors $\left.\left|I_{\alpha=1, ., N-1}^{R}\right\rangle\right\rangle$ are related to the eigenvectors $\left|I_{\alpha=1, . ., N-1}^{L}\right\rangle$ of the opposite-Laplacian via Eq. 78.

So here the vanishing eigenvalue $I_{0}^{2}=0$ is degenerate and associated to the subspace of dimension $M-(N-1)=C$ with the orthonormalized basis $\left|I_{\alpha=0,-1, \ldots,-(C-1)}^{R}\right\rangle$, as discussed in more details in the next subsection.

## D. Relation with the discrete Helmholtz decomposition for an arbitrary vector $|v\rangle\rangle$ of dimension $M$

An arbitrary vector $|v\rangle\rangle$ in the space of dimension $M$ can be decomposed with respect to the orthonormalized basis $\left.\left|I_{\alpha}^{R}\right\rangle\right\rangle$ of Eq. 74 in terms of its $M$ coefficients

$$
\begin{equation*}
v_{\alpha} \equiv\left\langle\left\langle I_{\alpha}^{R} \mid v\right\rangle\right\rangle \text { for } \alpha=-(C-1), \ldots,-1,0,+1, . ., N-1 \tag{82}
\end{equation*}
$$

It is useful to separate these $M=(N-1)+C$ terms into two orthogonal contributions of dimensions $(N-1)$ and $C$ respectively

$$
\begin{align*}
|v\rangle\rangle & \left.\left.\left.=\sum_{\alpha=-(C-1)}^{N-1} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \equiv\left|v^{[I .>0]}\right\rangle\right\rangle+\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle \\
\left.\left|v^{[I .>0]}\right\rangle\right\rangle & \left.\equiv \sum_{\alpha=1}^{N-1} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \\
\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle & \left.\equiv \sum_{\alpha=-(C-1)}^{0} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \tag{83}
\end{align*}
$$

with the following properties.

1. The component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$ of dimension $C$ associated to the vanishing singular value $I_{0}=0$

The vanishing singular value $I_{0}=0$ is degenerate and associated to the subspace of dimension $C=M-(N-1)$ that is annihilated by the adjoint operator $\mathbf{I}^{\dagger}=\operatorname{div}$ that represents the discrete divergence (see Eq. 46). As a consequence, the component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$ of Eq. 83 can be characterized by its vanishing divergence

$$
\begin{equation*}
\left.\left.0=\mathbf{I}^{\dagger}\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle=\operatorname{div}\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle \tag{84}
\end{equation*}
$$

As already discussed around Eq. 31 on the special case of the steady current $\left.\left|j_{*}\right\rangle\right\rangle$ whose divergence vanishes $\left.\mathbf{d i v}\left|j_{*}\right\rangle\right\rangle=$ 0 , the divergenceless component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$ can be similarly parametrized by its $C$ circulations around the $C$ independent cycles $\gamma=1,2, . ., C$.

$$
\begin{equation*}
\Gamma^{[\gamma]}\left[v^{\left[I_{0}=0\right]}\right] \equiv \sum_{l=1}^{l[\gamma]}\left\langle\left\langle x_{x}^{[\gamma](l)}(l+1) \mid v^{\left[I_{0}=0\right]}\right\rangle\right\rangle \tag{85}
\end{equation*}
$$

2. The component $\left.\left|v^{[I .>0]}\right\rangle\right\rangle$ of dimension $(N-1)$ associated to the strictly positive singular values $I_{\alpha=1,2, . ., N-1}>0$

The component $\left.\left|v^{[I .>0]}\right\rangle\right\rangle$ of Eq. 83 associated to the $(N-1)$ strictly positive singular values $I_{\alpha=1,2, . ., N-1}>0$ can be rewritten using Eq. 78

$$
\begin{equation*}
\left.\left.\left|v^{[I .>0]}\right\rangle\right\rangle=\sum_{\alpha=1}^{N-1} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle=\sum_{\alpha=1}^{N-1} \frac{v_{\alpha}}{I_{\alpha}} \mathbf{I}\left|I_{\alpha}^{L}\right\rangle=\mathbf{I}\left(\sum_{\alpha=1}^{N-1} \frac{v_{\alpha}}{I_{\alpha}}\left|I_{\alpha}^{L}\right\rangle\right) \equiv-\operatorname{grad}\left|g^{[I .>0]}\right\rangle \tag{86}
\end{equation*}
$$

as the opposite discrete gradient $\mathbf{I} \equiv-\mathbf{g r a d}$ of Eq. 44 applied to the ket $\left|g^{[I .>0]}\right\rangle$ that lives in the space of the $N$ configurations

$$
\begin{equation*}
\left|g^{[I .>0]}\right\rangle \equiv \sum_{\alpha=1}^{N-1} \frac{v_{\alpha}}{I_{\alpha}}\left|I_{\alpha}^{L}\right\rangle \tag{87}
\end{equation*}
$$

and that belongs to the subspace spanned by the $(N-1)$ left singular vectors $\left|I_{\alpha=1,2, . ., N-1}^{L}\right\rangle$ orthogonal to the constant ket $\left|I_{(\alpha=0)}\right\rangle$ of Eq. 71. As a consequence, the circulation of $\left.\left|v^{[I .>0]}\right\rangle\right\rangle=-\operatorname{grad}\left|g^{[I .>0]}\right\rangle$ of Eq. 86 around any of the $C$ independent cycle $\gamma=1,2, . ., C$ vanishes

$$
\begin{equation*}
\left.\Gamma^{[\gamma]}\left[v^{[I .>0]}\right] \equiv \sum_{l=1}^{l[\gamma]}\left\langle_{x}^{x^{[\gamma]}(l+1)}(l) \mid v^{[I .>0]}\right\rangle\right\rangle=0 \tag{88}
\end{equation*}
$$

The application of $\mathbf{I}^{\dagger}=\boldsymbol{d i v}$ that represents the discrete divergence (see Eq. 46) to the component $\left.\left|v^{[I .>0]}\right\rangle\right\rangle=$ $-\operatorname{grad}\left|g^{[I .>0]}\right\rangle$ of Eq. 86 corresponds to the application of the opposite Laplacian of Eq. 77 to the ket $\left|g^{[I .>0]}\right\rangle$ of Eq. 87

$$
\begin{equation*}
\left.\mathbf{I}^{\dagger}\left|v^{[I .>0]}\right\rangle\right\rangle \quad=\mathbf{I}^{\dagger} \mathbf{I}\left|g^{[I .>0]}\right\rangle=-\boldsymbol{\Delta}\left|g^{[I .>0]}\right\rangle \tag{89}
\end{equation*}
$$

with the following decomposition in the orthonormalized basis $\left|I_{\alpha}^{L}\right\rangle$ of the Laplacian associated to the eigenvalues $I_{\alpha}^{2}$

$$
\begin{equation*}
\left.\mathbf{I}^{\dagger}|v\rangle\right\rangle=-\boldsymbol{\Delta}\left(\sum_{\alpha=1}^{N-1} \frac{v_{\alpha}}{I_{\alpha}}\left|I_{\alpha}^{L}\right\rangle\right)=\sum_{\alpha=1}^{N-1} v_{\alpha} I_{\alpha}\left|I_{\alpha}^{L}\right\rangle \tag{90}
\end{equation*}
$$

## 3. Conclusion on the discrete Helmholtz decomposition associated to the SVD of the incidence matrix $\mathbf{I}$

In conclusion, the decomposition of Eq. 83 for arbitrary vector $|v\rangle\rangle$ of the space of $M$ links into the gradient component $\left.\left|v^{[I .>0]}\right\rangle\right\rangle=-\operatorname{grad}\left|g^{[I .>0]}\right\rangle$ of Eq. 86 of dimension $(N-1)$ and into the divergenceless component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$ of Eq. 84 of dimension $C$ corresponds to the discrete Helmholtz decomposition, with its two important properties:
(1) The application of the discrete divergence $\mathbf{I}^{\dagger}=\operatorname{div}$ on $\left.|v\rangle\right\rangle$ only involves the divergence of the gradient component $\left.\left|v^{[I .>0]}\right\rangle\right\rangle=-\operatorname{grad}\left|g^{[I .>0]}\right\rangle$

$$
\begin{equation*}
\left.\left.\operatorname{div}|v\rangle\rangle=\mathbf{I}^{\dagger}|v\rangle\right\rangle \quad=\mathbf{I}^{\dagger}\left|v^{[I .>0]}\right\rangle\right\rangle+0=-\boldsymbol{\Delta}\left|g^{[I .>0]}\right\rangle \tag{91}
\end{equation*}
$$

and thus only the opposite Laplacian of the ket $\left|g^{[I .>0]}\right\rangle$.
(2)The circulation of $|v\rangle\rangle$ along the any of the $C$ independent closed cycles $\gamma=1, . ., C$ only involves the circulation of the component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$

$$
\begin{equation*}
\left.\Gamma^{[\gamma]}[v] \equiv \sum_{l=1}^{l[\gamma]}\left\langle x_{x^{[\gamma]}(l)}^{[\gamma]}(l+1) \mid v\right\rangle\right\rangle=0+\Gamma^{[\gamma]}\left[v^{\left[I_{0}=0\right]}\right] \tag{92}
\end{equation*}
$$

## 4. Application to the definition of the pseudo-inverse $\mathbf{I}^{\text {pseudo }[-1]}$ of the incidence matrix $\mathbf{I}$

The linear system

$$
\begin{equation*}
\mathbf{I}|g\rangle=|v\rangle\rangle \tag{93}
\end{equation*}
$$

for the unknown ket $|g\rangle$ of dimension $N$ when the ket $|v\rangle\rangle$ of dimension $M$ is given can be analyzed via the SVD of Eq. 69 for the incidence matrix

$$
\begin{equation*}
\left.\left.\sum_{\alpha=1}^{N-1} I_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle I_{\alpha}^{L} \mid g\right\rangle=|v\rangle\right\rangle \tag{94}
\end{equation*}
$$

and via the Helmholtz decomposition of Eq. 83 for the arbitrary vector $|v\rangle\rangle$ in the space of dimension $M$ with the following discussion :
(i) The component $\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle$ spanned by the kets $\left.\left|I_{\alpha=0,-1, \ldots,-(C-1)}^{R}\right\rangle\right\rangle$ that do not appear on the left hand side of Eq. 94 should vanish

$$
\begin{equation*}
\left.\left.\left|v^{\left[I_{0}=0\right]}\right\rangle\right\rangle \equiv \sum_{\alpha=-(C-1)}^{0} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle=0 \tag{95}
\end{equation*}
$$

i.e. the $C$ coefficients $v_{\alpha=0,-1, . .,-(C-1)}$ should vanish

$$
\begin{equation*}
v_{\alpha} \equiv\left\langle\left\langle I_{\alpha}^{R} \mid v\right\rangle\right\rangle=0 \quad \text { for } \alpha=0,-1, . .,-(C-1) \tag{96}
\end{equation*}
$$

(ii) The identification of the coefficients of the $(N-1)$ eigenvectors $\left.\left|I_{\alpha=1,2, \ldots, N-1}^{R}\right\rangle\right\rangle$ yields the coefficients of $|g\rangle$ in the basis of the left singular vectors $\left|I_{\alpha}^{L}\right\rangle$ are given by

$$
\begin{equation*}
\left\langle I_{\alpha}^{L} \mid g\right\rangle=\frac{v_{\alpha}}{I_{\alpha}}=\frac{v_{\alpha}}{I_{\alpha}} \quad \text { for } \alpha=1,2, . ., N-1 \tag{97}
\end{equation*}
$$

so that the solution for the ket $|g\rangle$

$$
\begin{equation*}
\left.\left.|g\rangle=\sum_{\alpha=1}^{N-1}\left|I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L} \mid g\right\rangle=\sum_{\alpha=1}^{N-1}\left|I_{\alpha}^{L}\right\rangle \frac{\left\langle\left\langle I_{\alpha}^{R} \mid v\right\rangle\right\rangle}{I_{\alpha}}=\left(\sum_{\alpha=1}^{N-1} \frac{\left|I_{\alpha}^{L}\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right.}{I_{\alpha}}\right)|v\rangle\right\rangle \equiv \mathbf{I}^{p \text { seudo }[-1]}|v\rangle\right\rangle \tag{98}
\end{equation*}
$$

corresponds to the application to the ket $|v\rangle\rangle$ of the pseudo-inverse $\mathbf{I}^{\text {pseudo }}{ }^{\text {[1] }}$ of the incidence matrix $\mathbf{I}$ with the SVD of Eq. 69

$$
\begin{equation*}
\mathbf{I}^{\text {pseudo }[-1]}=\sum_{\alpha=1}^{N-1} \frac{\left|I_{\alpha}^{L}\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right.}{I_{\alpha}} \tag{99}
\end{equation*}
$$

## V. SINGULAR VALUE DECOMPOSITION OF THE CURRENT MATRIX J OF SIZE $M \times N$

This section is devoted to the properties of the SVD of the current matrix $\mathbf{J}$ of Eq. 37 of size $M \times N$ with $M \geq N$. Since the current matrix J of Eq. 37 can be considered as a deformation by the transition rates $w(.,$.$) of the incidence$ matrix I of Eq. 42, whose SVD was discussed in detail in the previous section V, many properties are very similar, so we will mainly emphasize the important differences.

## A. SVD of the current matrix $\mathbf{J}$ involving $N$ strictly positive singular values $\lambda_{\beta=1, \ldots, N}>0$

For a non-equilibrium steady state with non-vanishing steady current (Eq. 30), the Singular Value Decomposition of the current matrix $\mathbf{J}$ of size $M \times N$ involves $N$ strictly positive singular values $\lambda_{\beta=1, \ldots, N}>0$

$$
\begin{align*}
\mathbf{J} & \left.=\sum_{\beta=1}^{N} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L}\right| \\
\mathbf{J}^{\dagger} & =\sum_{\beta=1}^{N} \lambda_{\beta}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{100}
\end{align*}
$$

where the $N$ right singular vectors $\left.\left|\lambda_{\beta=1, . ., N}^{R}\right\rangle\right\rangle$ should be supplemented by $M-N=(C-1)$ other kets $\left.\left|\lambda_{\beta=0,-1 .,-(C-2)}^{R}\right\rangle\right\rangle$ in order to obtain an orthonormal basis of the space of the $M$ links

$$
\begin{align*}
\delta_{\beta, \beta^{\prime}} & =\left\langle\left\langle\lambda_{\beta}^{R} \mid \lambda_{\beta^{\prime}}^{R}\right\rangle\right\rangle=\sum_{\binom{x_{2} 2}{x_{1}}}\left\langle\left\langle\left.\lambda_{\beta}^{R}\right|_{x_{1}} ^{x_{1}}\right\rangle\right\rangle\left\langle\left\langle\left\langle_{x_{1}}^{x_{2}} \mid \lambda_{\beta^{\prime}}^{R}\right\rangle\right\rangle\right. \\
\mathbf{1}_{M} & \left.\left.\left.=\sum_{\beta=-(C-2)}^{N}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|=\sum_{\beta=1}^{N} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|+\sum_{\beta=-(C-2)}^{0} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{101}
\end{align*}
$$

while the $N$ left singular vectors $\left|\lambda_{\beta=1,2 \ldots, N}^{L}\right\rangle$ form an orthonormal basis of the space of the $N$ configurations

$$
\begin{align*}
\delta_{\beta, \beta^{\prime}} & =\left\langle\lambda_{\beta}^{L} \mid \lambda_{\beta^{\prime}}^{L}\right\rangle=\sum_{x}\left\langle\lambda_{\beta}^{L} \mid x\right\rangle\left\langle x \mid \lambda_{\beta^{\prime}}^{L}\right\rangle \\
\mathbf{1}_{N} & =\sum_{\beta=1}^{N}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\lambda_{\beta}^{L}\right| \tag{102}
\end{align*}
$$

that can be found from the diagonalization of the supersymmetric matrix of size $N \times N$

$$
\begin{equation*}
\mathbf{J}^{\dagger} \mathbf{J}=\sum_{\beta=1}^{N} \lambda_{\beta}^{2}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\lambda_{\beta}^{L}\right| \tag{103}
\end{equation*}
$$

whose matrix elements read

$$
\begin{align*}
& \left.\left.\left.\left.\langle x| \mathbf{J}^{\dagger} \mathbf{J}\left|x^{\prime}\right\rangle=\left.\sum_{\binom{x_{2}}{x_{1}}}\langle x| \mathbf{J}\right|_{x_{1}} ^{x_{2}}\right\rangle\right\rangle\left.\left\langle x^{\prime}\right| \mathbf{J}\right|_{x_{1}} ^{x_{2}}\right\rangle\right\rangle=\sum_{\binom{x_{2}}{x_{1}}}\left(w\left(x_{2}, x_{1}\right) \delta_{x, x_{1}}-w\left(x_{1}, x_{2}\right) \delta_{x, x_{2}}\right)\left(w\left(x_{2}, x_{1}\right) \delta_{x^{\prime}, x_{1}}-w\left(x_{1}, x_{2}\right) \delta_{x^{\prime}, x_{2}}\right) \\
& =\sum_{\binom{x_{2}}{x_{1}}}\left(w^{2}\left(x_{2}, x_{1}\right) \delta_{x, x_{1}} \delta_{x^{\prime}, x_{1}}+w^{2}\left(x_{1}, x_{2}\right) \delta_{x, x_{2}} \delta_{x^{\prime}, x_{2}}-w\left(x_{2}, x_{1}\right) w\left(x_{1}, x_{2}\right) \delta_{x, x_{2}} \delta_{x^{\prime}, x_{1}}-w\left(x_{2}, x_{1}\right) w\left(x_{1}, x_{2}\right) \delta_{x, x_{1}} \delta_{x^{\prime}, x_{2}}\right) \\
& = \begin{cases}\sum_{x^{\prime \prime} \neq x} w^{2}\left(x^{\prime \prime}, x\right) & \text { if } x=x^{\prime} \\
-w\left(x, x^{\prime}\right) w\left(x^{\prime}, x\right)=-D^{2}\left(x, x^{\prime}\right) & \text { if } x \text { and } x^{\prime} \text { are the two ends of a link } \\
0 \text { otherwise }\end{cases} \tag{104}
\end{align*}
$$

If the spectral decomposition of Eq. 103 is known for the this matrix $\mathbf{J}^{\dagger} \mathbf{J}$, then the $N$ singular values $\lambda_{\beta=1, \ldots, N}>0$ and the $N$ left singular vectors $\left|\lambda_{\beta=1, . ., N}^{L}\right\rangle$ of the matrix $\mathbf{J}$ are known, while the corresponding $N$ right singular vectors $\left.\left|\lambda_{\beta=1, . ., N}^{R}\right\rangle\right\rangle$ can be obtained via the application of the matrix $\mathbf{J}$

$$
\begin{equation*}
\left.\mathbf{J}\left|\lambda_{\beta}^{L}\right\rangle \quad=\lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \tag{105}
\end{equation*}
$$

## B. Analog of the Helmholtz decomposition when the incidence matrix $I$ is replaced by the current matrix $J$

As discussed in detail in subsection IV D, the discrete Helmholtz decomposition is directly related to the SVD of the incidence matrix $\mathbf{I}$. In the present subsection, it is thus useful to describe the analog of the discrete Helmholtz decomposition when the incidence matrix $\mathbf{I}$ is replaced by the current matrix $\mathbf{J}$ using its SVD decomposition of Eq. 100 as follows.

An arbitrary vector $|u\rangle\rangle$ in the space of dimension $M$ can be decomposed with respect to the orthonormalized basis $\left.\left|\lambda_{\beta}^{R}\right\rangle\right\rangle$ of Eq. 101 in terms of its $M$ coefficients

$$
\begin{equation*}
u_{\beta} \equiv\left\langle\left\langle\lambda_{\beta}^{R} \mid u\right\rangle\right\rangle \text { for } \beta=-(C-2), \ldots,-1,0,+1, . ., N \tag{106}
\end{equation*}
$$

Let us now analyze the separation of these $M$ terms into two orthogonal contributions of dimensions $N$ and ( $C-1$ ) respectively

$$
\begin{align*}
|u\rangle\rangle & \left.\left.\left.=\sum_{\beta=-(C-2)}^{N} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \equiv\left|u^{[\lambda .>0]}\right\rangle\right\rangle+\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle \\
\left.\left|u^{[\lambda .>0]}\right\rangle\right\rangle & \left.\equiv \sum_{\beta=1}^{N} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \\
\left.\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle & \left.\equiv \sum_{\beta=-(C-2)}^{0} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \tag{107}
\end{align*}
$$

$$
\text { 1. The component } \left.\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle \text { of dimension }(C-1) \text { annihilated by } \mathbf{J}^{\dagger}
$$

The component $\left.\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle$ of Eq. 107 is annihilated by the adjoint $\mathbf{J}^{\dagger}$ with the SVD of Eq. 100

$$
\begin{equation*}
\left.0=\mathbf{J}^{\dagger}\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle \tag{108}
\end{equation*}
$$

2. The component $\left.\left|u^{[\lambda .>0]}\right\rangle\right\rangle$ of dimension $N$ associated to the strictly positive singular values $\lambda_{\beta}^{R}>0$

The component $\left.\left|u^{[\lambda>0]}\right\rangle\right\rangle$ of Eq. 107 associated to the $N$ strictly positive singular values $\lambda_{\beta}^{R}>0$ can be rewritten using Eq. 105

$$
\begin{equation*}
\left.\left.\left|u^{[\lambda .>0]}\right\rangle\right\rangle \equiv \sum_{\beta=1}^{N} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle=\sum_{\beta=1}^{N} \frac{u_{\beta}}{\lambda_{\beta}} \mathbf{J}\left|\lambda_{\beta}^{L}\right\rangle=\mathbf{J}\left(\sum_{\beta=1}^{N} \frac{u_{\beta}}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\right) \equiv \mathbf{J}\left|k^{[\lambda .>0]}\right\rangle \tag{109}
\end{equation*}
$$

as the application of the current matrix $\mathbf{J}$ of Eq. 44 to the ket $\left|k^{[\lambda .>0]}\right\rangle$ that lives in the space of the $N$ configurations

$$
\begin{equation*}
\left|k^{[\lambda .>0]}\right\rangle \equiv \sum_{\beta=1}^{N} \frac{u_{\beta}}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle \tag{110}
\end{equation*}
$$

The application of $\mathbf{J}^{\dagger}$ on $\left.|u\rangle\right\rangle$ of Eq. 107 only involves the application on the component $\left.\left|u^{[\lambda .>0]}\right\rangle\right\rangle=\mathbf{J}\left|k^{[\lambda .>0]}\right\rangle$ of Eq. 109, and thus the application of the matrix $\mathbf{J}^{\dagger} \mathbf{J}$ with the spectral decomposition of Eq. 103 on the ket $\left|k^{[\lambda .>0]}\right\rangle$

$$
\begin{equation*}
\left.\left.\mathbf{J}^{\dagger}|u\rangle\right\rangle=\mathbf{J}^{\dagger}\left|u^{[\lambda .>0]}\right\rangle\right\rangle+0=\mathbf{J}^{\dagger} \mathbf{J}\left|k^{[\lambda .>0]}\right\rangle=\sum_{\beta=1}^{N} u_{\beta} \lambda_{\beta}\left|\lambda_{\beta}^{L}\right\rangle \tag{111}
\end{equation*}
$$

## 3. Conclusion on the subspace of dimension $N$ for the physical currents $\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle$

The above discussion shows that the component $\left.\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle$ of dimension $(C-1)$ corresponds to the unphysical subspace orthogonal to the physical space for the currents $\left.\left|j_{t}\right\rangle\right\rangle=\mathbf{J}\left|p_{t}\right\rangle$ of Eq. 39 that are obtained from the application of the current matrix $\mathbf{J}$ to a ket $\left|p_{t}\right\rangle$ of the configuration space. This explains why the spectral decomposition of Eq. 68 only involves the bi-orthogonal basis of the $\left\langle\left\langle i_{n}\right|\right.$ and the $\left.\left.\mid j_{n}\right\rangle\right\rangle$ for $n=0,1, \ldots,(N-1)$, even if the partner $\hat{\mathbf{H}}$ that governs the dynamics of the current is a matrix of size $M \times M$.

In conclusion, the projector $\mathcal{P}^{\text {PhysicalSpaceCurrents }}$ onto the subspace of dimension $N$ for the physical currents can be written either with the bi-orthogonal basis of the $\left\langle\left\langle i_{n}\right|\right.$ and the $\left.\left.\mid j_{n}\right\rangle\right\rangle$ for $n=0,1, \ldots,(N-1)$ or with the orthonormalized basis $\left.\left|\lambda_{\beta=1, \ldots, N}^{R}\right\rangle\right\rangle$ of right singular vectors of the current matrix $\mathbf{J}$ associated to the subspace associated to the strictly positive singular values $\lambda_{\beta=1, \ldots, N}^{R}>0$

$$
\begin{equation*}
\left.\left.\mathcal{P}^{\text {PhysicalSpaceCurrents }}=\sum_{n=0}^{N-1}\left|j_{n}\right\rangle\right\rangle\left\langle\left\langle i_{n}\right|=\sum_{\beta=1}^{N} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{112}
\end{equation*}
$$

## 4. Application to the definition of the pseudo-inverse $\mathbf{J}^{\text {pseudo[-1] }}$ of the current matrix $\mathbf{J}$

The linear system

$$
\begin{equation*}
\left.|u\rangle\rangle=\mathbf{J}|k\rangle \equiv \sum_{\beta=1}^{N} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L} \mid k\right\rangle \tag{113}
\end{equation*}
$$

for the unknown ket $|k\rangle$ of dimension $N$ when the ket $|u\rangle\rangle$ of dimension $M$ is given, can be analyzed via the following adaptation of the discussion of subsection IV D 4 :
(i) The $(C-1)$ coefficients $u_{\beta=0,-1, . .,-(C-2)}$ should vanish

$$
\begin{equation*}
u_{\beta} \equiv\left\langle\left\langle\lambda_{\beta}^{R} \mid u\right\rangle\right\rangle=0 \quad \text { for } \beta=0,-1, . .,-(C-2) \tag{114}
\end{equation*}
$$

(ii) The solution for the ket $|k\rangle$

$$
\begin{equation*}
\left.|k\rangle=\mathbf{J}^{p \text { seudo }[-1]}|u\rangle\right\rangle \tag{115}
\end{equation*}
$$

corresponds to the application to the ket $|u\rangle\rangle$ of the pseudo-inverse $\mathbf{J}^{\text {pseudo }[-1]}$ of the current matrix $\mathbf{J}$ with the SVD of Eq. 100

$$
\begin{equation*}
\mathbf{J}^{\text {pseudo }[-1]}=\sum_{\beta=1}^{N} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{116}
\end{equation*}
$$

## C. Solving the linear system $\left.\left|j_{*}\right\rangle\right\rangle=\mathbf{J}\left|p_{*}\right\rangle$ to obtain the steady state $p_{*}$ and the current $j_{*}$

Let us consider the application of the previous subsection to the linear system of Eq. 34

$$
\begin{equation*}
\left.\left.\left|j_{*}\right\rangle\right\rangle=\mathbf{J}\left|p_{*}\right\rangle \equiv \sum_{\beta=1}^{N} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L} \mid p_{*}\right\rangle \tag{117}
\end{equation*}
$$

for the unknown steady state $\left|p_{*}\right\rangle$ in the space of the $N$ configurations, once the steady current $\left.\left|j_{*}\right\rangle\right\rangle$ has been parametrized by its $C$ coefficients in Eq. 32 :
(i) Eq. 114 gives $(C-1)$ equations for the $C$ coefficients parametrizing the steady current $\left.\left|j_{*}\right\rangle\right\rangle$

$$
\begin{equation*}
\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{*}\right\rangle\right\rangle=0 \quad \text { for } \beta=0,-1, . .,-(C-2) \tag{118}
\end{equation*}
$$

(ii) Eq. 115 gives the steady state of dimension $N$

$$
\begin{equation*}
\left.\left|p_{*}\right\rangle=\mathbf{J}^{p \text { seudo }[-1]}\left|j_{*}\right\rangle\right\rangle=\sum_{\beta=1}^{N} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{*}\right\rangle\right\rangle \tag{119}
\end{equation*}
$$

in terms of the projections of the steady current $\left.\left|j_{*}\right\rangle\right\rangle$ on the $N$ singular eigenvectors $\left\langle\left\langle\lambda_{\beta=1, ., N}^{R}\right|\right.$, Then the normalization of the steady state $p_{*}($.$) of Eq. 119$ determines the last remaining unknown coefficient for the steady current.

## D. Consequences for the excited right eigenstates $\left|r_{n=1, \ldots, N-1}\right\rangle$ and their associated currents $\left.\left|j_{n}\right\rangle\right\rangle=\mathbf{J}\left|r_{n}\right\rangle$

Plugging the SVD of $\mathbf{J}$ of Eq. 100 into Eq. 54

$$
\begin{equation*}
\left.\left.\left|j_{n}\right\rangle\right\rangle=\mathbf{J}\left|r_{n}\right\rangle=\sum_{\beta=1}^{N} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L} \mid r_{n}\right\rangle \tag{120}
\end{equation*}
$$

yields that the excited right eigenstate $\left|r_{n=1, . ., N-1}\right\rangle$ can be also computed from the current $\left.\left|j_{n}\right\rangle\right\rangle$ via the pseudo-inverse as in Eq. 119

$$
\begin{equation*}
\left.\left|r_{n}\right\rangle=\mathbf{J}^{\text {pseudo }[-1]}\left|j_{n}\right\rangle\right\rangle=\sum_{\beta=1}^{N} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{n}\right\rangle\right\rangle \tag{121}
\end{equation*}
$$

E. Consequences for the left eigenstates $\left\langle l_{n=0, . ., N-1}\right|$ and their associated $\left\langle\left\langle i_{n}\right|\right.$ satisfying $\left\langle l_{n}\right|=\left\langle\left\langle i_{n}\right| \mathbf{J}\right.$

Plugging the SVD of $\mathbf{J}$ of Eq. 100 into Eq. 58 yields

$$
\begin{equation*}
\left\langle l_{n}\right|=\left\langle\left\langle i_{n}\right| \mathbf{J}=\sum_{\beta=1}^{N} \lambda_{\beta}\left\langle\left\langle i_{n} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L}\right|\right. \tag{122}
\end{equation*}
$$

Since $\left\langle\left\langle i_{n}\right|\right.$ belongs to the subspace of physical currents of Eq. 112, Eq. 122 can be inversed to compute $\left\langle\left\langle i_{n}\right|\right.$ in terms of $\left\langle l_{n}\right|$ via

$$
\begin{equation*}
\left\langle\left\langle i_{n}\right|=\sum_{\beta=1}^{N}\left\langle\left\langle i_{n}\right| \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|=\sum_{\beta=1}^{N} \frac{\left\langle l_{n} \mid \lambda_{\beta}^{L}\right\rangle}{\lambda_{\beta}}\left\langle\left\langle\lambda_{\beta}^{R}\right|=\left\langle l_{n}\right| \mathbf{J}^{\text {pseudo }[-1]}\right.\right. \tag{123}
\end{equation*}
$$

that involves the pseudo-inverse $\mathbf{J}^{\text {pseudo }[-1]}$.
For $n=0$ with the left eigenvector $l_{0}(x)=1$, Eq. 123 yields that the bra $\left\langle\left\langle i_{0}\right|\right.$ can be evaluated from the pseudoinverse $\mathbf{J}^{\text {pseudo }[-1]}$

$$
\begin{align*}
\left\langle\left\langle i_{0}\right|\right. & =\left\langle l_{0}\right| \mathbf{J}^{\text {pseudo }[-1]}=\sum_{x}\left\langle l_{0} \mid x\right\rangle\langle x| \mathbf{J}^{\text {pseudo }[-1]} \\
& =\sum_{x}\langle x| \mathbf{J}^{\text {pseudo }[-1]}=\sum_{\beta=1}^{N} \frac{\left\langle x \mid \lambda_{\beta}^{L}\right\rangle}{\lambda_{\beta}}\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{124}
\end{align*}
$$

## VI. CONCLUSIONS

In this paper, we have focused on continuity equations associated to continuous-time Markov processes that can be considered as Euclidean Schrödinger equations, where the non-hermitian quantum Hamiltonian $\mathbf{H}=\operatorname{divJ}$ is naturally factorized into the product of the divergence operator div and the current operator $\mathbf{J}$. In the main text devoted to non-equilibrium Markov jump processes in a space of $N$ configurations with $M$ links between them and $C=M-(N-1) \geq 1$ independent cycles, this factorization of the $N \times N$ Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ involves the incidence matrix $\mathbf{I}$ and the current matrix $\mathbf{J}$ that are both of size $M \times N$, so that the supersymmetric partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ governing the dynamics of the currents living on the $M$ links is a priori of size $M \times M$. To better understand the relations between the spectral decompositions of these two Hamiltonians $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ and $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ with respect to their bi-orthogonal basis of right and left eigenvectors that characterize the relaxation dynamics towards the steady state and the steady currents, we have analyzed the properties of the Singular Value Decompositions of the two rectangular matrices $\mathbf{I}$ and $\mathbf{J}$ of size $M \times N$ and the interpretations in terms of discrete Helmholtz decompositions.

As described in the following Appendix, this general framework concerning Markov jump processes can be adapted to non-equilibrium diffusion processes governed by Fokker-Planck equations in dimension $d$, where the number $N$ of configurations, the number $M$ of links and the number $C=M-(N-1)$ of independent cycles become infinite, while the two matrices $\mathbf{I}$ and $\mathbf{J}$ of size $M \times N$ become first-order differential operators acting on scalar functions to produce vector fields.

## Appendix A: Fokker-Planck generators as non-hermitian supersymmetric quantum Hamiltonians

For diffusion processes in dimension $d$, the essential ideas are the same as in the main text concerning Markov jump processes, but there are important technical differences that should be stressed. In particular, the number $N$ of configurations, the number $M$ of links, and the number $C$ of cycles become infinite, while the various Hamiltonians and matrices become differential operators. For concreteness and to simplify the notations, we will focus on the space dimension $d=3$ in order to use the 3D curl operator that is more familiar than its generalization in higher dimensions $d>3$.

## 1. Fokker-Planck equation in terms of probability density $p_{t}(\vec{x})$ and the currents $\vec{j}_{t}(\vec{x})$

The Fokker-Planck equation for the probability density $p_{t}(\vec{x})$ to be around the position $\vec{x}$ at time $t$ can be written as the continuity equation

$$
\begin{equation*}
\partial_{t} p_{t}(\vec{x})=-\operatorname{div} \vec{j}_{t}(\vec{x}) \tag{A1}
\end{equation*}
$$

where the current involves the force $\vec{F}(\vec{x})$ and the diffusion coefficient $D(\vec{x})$

$$
\begin{equation*}
\vec{j}_{t}(\vec{x})=\vec{F}(\vec{x}) p_{t}(x)-D(\vec{x}) \mathbf{\operatorname { g r a d }} p_{t}(\vec{x}) \tag{A2}
\end{equation*}
$$

## 2. Rephrasing with the first-order differential operators $J$ and $I$

The analog of the current matrix $\mathbf{J}$ of size $M \times N$ of Eq. 3739 is the differential operator

$$
\mathbf{J} \equiv \vec{F}(\vec{x})-D(\vec{x}) \mathbf{\text { grad }}=\left(\begin{array}{l}
F_{1}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{1}}  \tag{A3}\\
F_{2}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{2}} \\
F_{3}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{3}}
\end{array}\right)
$$

that acts on the scalar density $p_{t}(x)$ to produce the current $\vec{j}_{t}(\vec{x})$ of Eq. A2 with its $d=3$ components

$$
\begin{equation*}
\vec{j}_{t}(\vec{x})=\mathbf{J} p_{t}(x) \tag{A4}
\end{equation*}
$$

The analog of the incidence matrix I of size $M \times N$ of Eq. 4244 is the opposite of the gradient operator, that can be recovered from the current operator of Eq. A3 for the simplest case where the force vanishes $\vec{F}(\vec{x}) \rightarrow 0$ and where the diffusion coefficient reduces to unity $D(\vec{x}) \rightarrow 1$

$$
\mathbf{I} \equiv-\mathbf{g r a d}=-\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}}  \tag{A5}\\
\frac{\partial}{\partial x_{2}} \\
\frac{\partial}{\partial x_{3}}
\end{array}\right)
$$

The adjoint operator $\mathbf{I}^{\dagger}$ is the divergence operator that acts on three-dimensional vectors to produce a scalar

$$
\begin{equation*}
\mathbf{I}^{\dagger} \equiv \operatorname{div}=\left(\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x_{3}}\right) \tag{A6}
\end{equation*}
$$

which is the analog of the discrete divergence matrix of size $N \times M$ of Eq 4546 .
The Fokker-Planck Eq. A1 can be rewritten as the euclidean Schrödinger equation

$$
\begin{equation*}
-\partial_{t} p_{t}(\vec{x})=\mathbf{H} p_{t}(\vec{x}) \tag{A7}
\end{equation*}
$$

where the second-order differential non-hermitian Hamiltonian $\mathbf{H} \neq \mathbf{H}^{\dagger}$ is factorized into the two first-order differential operators $\mathbf{I}^{\dagger}$ and $\mathbf{J}$

$$
\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}=\left(\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x_{3}}\right)\left(\begin{array}{l}
F_{1}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{1}}  \tag{A8}\\
F_{2}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{2}} \\
F_{3}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{3}}
\end{array}\right)=\sum_{\mu=1}^{3} \frac{\partial}{\partial x_{\mu}}\left(F_{\mu}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{\mu}}\right)
$$

## 3. Electromagnetic quantum interpretation of the non-hermitian Hamiltonian $\mathbf{H}$

The electromagnetic quantum interpretation requires the rewriting of this non-hermitian Hamiltonian $\mathbf{H}$ as

$$
\begin{align*}
\mathbf{H} & =-\sum_{\mu=1}^{3}\left(\frac{\partial}{\partial x_{\mu}}-A_{\mu}(\vec{x})\right) D(\vec{x})\left(\frac{\partial}{\partial x_{\mu}}-A_{\mu}(\vec{x})\right)+V(\vec{x}) \\
& =\sum_{\mu=1}^{3}\left(-i \frac{\partial}{\partial x_{\mu}}+i A_{\mu}(\vec{x})\right) D(\vec{x})\left(-i \frac{\partial}{\partial x_{\mu}}+i A_{\mu}(\vec{x})\right)+V(\vec{x}) \\
& =(-i \vec{\nabla}+i \vec{A}(\vec{x})) D(\vec{x})(-i \vec{\nabla}+i \vec{A}(\vec{x}))+V(\vec{x}) \tag{A9}
\end{align*}
$$

that involves a purely imaginary vector potential $[-i \vec{A}(\vec{x})]$ of real amplitude

$$
\begin{equation*}
\vec{A}(\vec{x}) \equiv \frac{\vec{F}(\vec{x})}{2 D(\vec{x})} \tag{A10}
\end{equation*}
$$

which is the analog of the antisymmetric function $A(.,$.$) of Eq. 6$ that appears in the off-diagonal terms $\mathbf{H}\left(x, x^{\prime}\right)$ of Eq. 8, while the scalar potential

$$
\begin{align*}
V(\vec{x}) & \equiv \sum_{\mu=1}^{3}\left(D(\vec{x}) A_{\mu}^{2}(\vec{x})+\frac{\partial\left[D(\vec{x}) A_{\mu}(\vec{x})\right]}{\partial x_{\mu}}\right) \\
& =\sum_{\mu=1}^{3}\left(\frac{F_{\mu}^{2}(\vec{x})}{4 D(\vec{x})}+\frac{1}{2} \frac{\partial F_{\mu}(\vec{x})}{\partial x_{\mu}}\right) \tag{A11}
\end{align*}
$$

is the analog of the on-site potential $\mathbf{H}(x, x)$ of Eq. 9 .
The magnetic field $\vec{B}(\vec{x})$ associated to the vector potential $\vec{A}(\vec{x})$ of Eq. A10

$$
\vec{B}(\vec{x}) \equiv \operatorname{curl} \vec{A}(\vec{x})=\vec{\nabla} \times \vec{A}(\vec{x})=\left(\begin{array}{c}
\frac{\partial A_{3}}{\partial x_{2}}-\frac{\partial A_{2}}{\partial x_{3}}  \tag{A12}\\
\frac{\partial A_{1}}{\partial x_{3}}-\frac{\partial A_{3}}{\partial x_{1}} \\
\frac{\partial A_{2}}{\partial x_{2}}-\frac{\partial A_{1}}{\partial x_{2}}
\end{array}\right)
$$

is useful to rewrite the circulation of the vector potential $\vec{A}(\vec{x})$ around any closed curve $\gamma$

$$
\begin{equation*}
\Gamma^{[\gamma]}[\vec{A}(.)] \equiv \oint_{\gamma} d \vec{l} \cdot \vec{A}(\vec{x})=\int d^{2} \vec{S} \cdot \vec{B}(\vec{x}) \tag{A13}
\end{equation*}
$$

as the flux of the magnetic field $\vec{B}(\vec{x})$ through the surface enclosed by the closed curve $\gamma$. This magnetic field $\vec{B}(\vec{x})$ determines the equilibrium or non-equilibrium nature of the steady state as we now recall (see [46] for more detailed discussions).
4. Reminder on the properties of the steady current $\vec{j}_{*}($.
a. Equilibrium steady state $p_{*}^{e q}($.$) with vanishing steady currents \vec{j}_{*}^{\text {eq }}(\vec{x})=\overrightarrow{0}$

At equilibrium, the steady current $\vec{j}_{*}^{e q}(\vec{x})$ associated to the steady density $p_{*}^{e q}($.$) vanishes everywhere$

$$
\begin{equation*}
\overrightarrow{0}=\vec{j}_{*}^{e q}(\vec{x})=\vec{F}(\vec{x}) p_{*}^{e q}(x)-D(\vec{x}) \vec{\nabla} p_{*}^{e q}(\vec{x}) \tag{A14}
\end{equation*}
$$

This is possible only if the vector potential $\vec{A} \equiv \frac{\vec{F}(\vec{x})}{2 D(\vec{x})}$ introduced in Eq. A10 can be written as the gradient

$$
\begin{equation*}
\vec{A}(\vec{x})=\frac{1}{2} \vec{\nabla} \ln p_{*}^{e q}(\vec{x}) \tag{A15}
\end{equation*}
$$

This condition is the continuous analog of Eq. 26 and means that the circulation of the vector potential $\vec{A}(\vec{x})$ of Eq. A13 vanishes along any closed curve $\gamma$ as in Eq. 29

$$
\begin{equation*}
\Gamma^{[\gamma]}[\vec{A}(.)] \equiv \oint_{\gamma} d \vec{l} \cdot \vec{A}(\vec{x})=0 \tag{A16}
\end{equation*}
$$

i.e. that the magnetic field $\vec{B}(\vec{x}) \equiv \vec{\nabla} \times \vec{A}(\vec{x})$ introduced in Eq. A12 should vanish everywhere

$$
\begin{equation*}
\vec{B}(\vec{x}) \equiv \vec{\nabla} \times \vec{A}(\vec{x})=\overrightarrow{0} \tag{A17}
\end{equation*}
$$

b. Non-equilibrium steady state $p_{*}($.$) with nonvanishing steady currents \vec{j}_{*}(\vec{x})=\overrightarrow{0}$

When the magnetic field $\vec{B}(\vec{x}) \equiv \vec{\nabla} \times \vec{A}(\vec{x})$ associated to the vector potential $\vec{A}(\vec{x}) \equiv \frac{\vec{F}(\vec{x})}{2 D(\vec{x})}$ of Eq. A10 does not vanish

$$
\begin{equation*}
\vec{B}(\vec{x}) \equiv \vec{\nabla} \times \vec{A}(\vec{x}) \neq \overrightarrow{0} \tag{A18}
\end{equation*}
$$

then the steady current

$$
\begin{equation*}
\vec{j}_{*}(\vec{x}) \equiv \vec{F}(\vec{x}) p_{*}(x)-D(\vec{x}) \vec{\nabla} p_{*}(\vec{x})=2 D(\vec{x}) p_{*}(x)\left[\vec{A}(\vec{x})-\frac{1}{2} \vec{\nabla} \ln p_{*}(\vec{x})\right] \neq \overrightarrow{0} \tag{A19}
\end{equation*}
$$

cannot vanish, but should be divergenceless

$$
\begin{equation*}
\operatorname{div} \vec{j}_{*}(\vec{x})=0 \tag{A20}
\end{equation*}
$$

So the steady current $\vec{j}_{*}(\vec{x})$ can be rewritten as the curl of a divergenceless vector $\vec{\omega}_{*}(\vec{x})$

$$
\begin{align*}
\vec{j}_{*}(\vec{x}) & =\vec{\nabla} \times \vec{\omega}_{*}(\vec{x}) \\
\operatorname{div} \vec{\omega}_{*}(\vec{x}) & \equiv \vec{\nabla} \cdot \vec{\omega}_{*}(\vec{x})=0 \tag{A21}
\end{align*}
$$

which is the analog of the decomposition into cycles-currents of Eq. 32. The curl of the steady current

$$
\begin{equation*}
\vec{\nabla} \times \vec{j}_{*}(\vec{x})=\vec{\nabla} \times\left(\vec{\nabla} \times \vec{\omega}_{*}(\vec{x})\right)=\vec{\nabla}\left(\vec{\nabla} \cdot \vec{\omega}_{*}(\vec{x})\right)-\Delta \vec{\omega}_{*}(\vec{x})=-\Delta \vec{\omega}_{*}(\vec{x}) \tag{A22}
\end{equation*}
$$

corresponds to the opposite Laplacian of the vector $\vec{\omega}_{*}(\vec{x})$.

## 5. Spectral decomposition of the Hamiltonian $H$ in the bi-orthogonal basis of right and left eigenvectors

For concreteness, we will consider that the Fokker-Planck dynamics takes place in a finite domain with reflecting boundary conditions, so that the spectral decomposition of the non-hermitian Hamiltonian $\mathbf{H} \neq \mathbf{H}^{\dagger}$ of Eq. A8 does not involve a continuum but only an infinity of discrete eigenvalues $E_{n}$ (instead the finite number $N$ of Eq. 10 for Markov jump processes considered in the main text)

$$
\begin{equation*}
\mathbf{H}=\sum_{n=0}^{+\infty} E_{n}\left|r_{n}\right\rangle\left\langle l_{n}\right| \tag{A23}
\end{equation*}
$$

The corresponding right eigenvectors $r_{n}(\vec{x})=\left\langle\vec{x} \mid r_{n}\right\rangle$ and left eigenvectors $l_{n}(\vec{x})=\left\langle\vec{x} \mid l_{n}\right\rangle$ of $\mathbf{H}$ (that are equivalently the right eigenvectors of the adjoint differential operator $\mathbf{H}^{\dagger}$ )

$$
\begin{align*}
& E_{n} r_{n}(\vec{x})=\mathbf{H} r_{n}(\vec{x})=\sum_{\mu=1}^{3} \frac{\partial}{\partial x_{\mu}}\left(F_{\mu}(\vec{x}) r_{n}(\vec{x})-D(\vec{x}) \frac{\partial r_{n}(\vec{x})}{\partial x_{\mu}}\right) \\
& E_{n}^{*} l_{n}(\vec{x})=\mathbf{H}^{\dagger} l_{n}(\vec{x})=-\sum_{\mu=1}^{3}\left(F_{\mu}(\vec{x})+\frac{\partial}{\partial x_{\mu}} D(\vec{x})\right) \frac{\partial l_{n}(\vec{x})}{\partial x_{\mu}} \tag{A24}
\end{align*}
$$

form a bi-orthogonal basis where the orthonormalization and closure relations of Eq. 12 become

$$
\begin{align*}
\delta_{n, n^{\prime}} & =\left\langle l_{n} \mid r_{n^{\prime}}\right\rangle=\int d^{d} \vec{x}\left\langle l_{n} \mid \vec{x}\right\rangle\left\langle\vec{x} \mid r_{n^{\prime}}\right\rangle=\int d^{d} \vec{x} l_{n}^{*}(\vec{x}) r_{n}\left(\vec{x}^{\prime}\right) \\
\delta^{(3)}\left(\vec{x}-\vec{x}^{\prime}\right) & =\left\langle\vec{x} \mid \vec{x}^{\prime}\right\rangle=\sum_{n=0}^{+\infty}\left\langle\vec{x} \mid r_{n}\right\rangle\left\langle l_{n} \mid \vec{x}^{\prime}\right\rangle=\sum_{n=0}^{+\infty} r_{n}(\vec{x}) l_{n}^{*}\left(\vec{x}^{\prime}\right) \tag{A25}
\end{align*}
$$

As in Eq. 13, the vanishing eigenvalue $E_{0}=0$ is associated to the positive left eigenvector unity and to the positive right eigenvector given by the steady density $p_{*}$

$$
\begin{align*}
E_{0} & =0 \\
l_{0}(\vec{x}) & =1 \\
r_{0}(\vec{x}) & =p_{*}(\vec{x}) \tag{A26}
\end{align*}
$$

while the other eigenvalues $E_{n=1, . .,+\infty}$ with strictly positive real parts $\operatorname{Re}\left(E_{n}\right)>0$ govern the relaxation towards the steady density $p_{*}(x)$ as in Eqs 15 and 16 .

At the level of the eigenvalues Eqs A24, the factorization of Eq. A8 corresponds for the right eigenvectors $r_{n}(\vec{x})$ to the splitting into a pair of first-order differential equations

$$
\begin{align*}
E_{n} r_{n}(\vec{x}) & =\mathbf{I}^{\dagger} \vec{j}_{n}(\vec{x})=\vec{\nabla} \cdot \vec{j}_{n}(\vec{x}) \\
\vec{j}_{n}(\vec{x}) & =\mathbf{J} r_{n}(\vec{x})=[\vec{F}(\vec{x})-D(\vec{x}) \vec{\nabla}] r_{n}(\vec{x}) \tag{A27}
\end{align*}
$$

that is the analog of Eqs 5455 , while the corresponding splitting for the left eigenvectors $l_{n}(\vec{x})$

$$
\begin{align*}
l_{n}(\vec{x}) & =\mathbf{J}^{\dagger} \vec{i}_{n}(\vec{x})=\vec{F}(\vec{x}) \cdot \vec{i}_{n}(\vec{x})+\vec{\nabla} \cdot\left(D(\vec{x}) \cdot \vec{i}_{n}(\vec{x})\right) \\
E_{n}^{*} \vec{i}_{n}(\vec{x}) & =\mathbf{I} l_{n}(\vec{x})=-\vec{\nabla} l_{n}(\vec{x}) \tag{A28}
\end{align*}
$$

is the analog of Eq 58 .
6. Dynamics of the current $\vec{j}_{t}(\vec{x})$ governed by the supersymmetric partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$

As in Eq. 49, the dynamics of the current $\vec{j}_{t}(\vec{x})$

$$
\begin{equation*}
-\partial_{t} \vec{j}_{t}(\vec{x})=\mathbf{J}\left(-\partial_{t}\left|p_{t}\right\rangle\right)=\mathbf{J I}^{\dagger} \vec{j}_{t}(\vec{x}) \equiv \hat{\mathbf{H}} \vec{j}_{t}(\vec{x}) \tag{A29}
\end{equation*}
$$

involves the supersymmetric partner $\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}$ of the Hamiltonian $\mathbf{H}=\mathbf{I}^{\dagger} \mathbf{J}$ of Eq. A8

$$
\hat{\mathbf{H}}=\mathbf{J I}^{\dagger}=\left(\begin{array}{l}
F_{1}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{1}}  \tag{A30}\\
F_{2}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{2}} \\
F_{3}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{3}}
\end{array}\right)\left(\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x_{3}}\right)=\left[\left(F_{\mu}(\vec{x})-D(\vec{x}) \frac{\partial}{\partial x_{\mu}}\right) \frac{\partial}{\partial x_{\nu}}\right]_{\mu=1,2,3 ; \nu=1,2,3}
$$

that corresponds to a $3 \times 3$ matrix of differential operators acting on the 3 components of the current.
As discussed around Eqs 5268 , the dynamics of the current $\vec{j}_{t}(\vec{x})$ involves the same non-vanishing eigenvalues $E_{n=1, . .,+\infty}$ as the hamiltonian $\mathbf{H}$, where the right and left eigenvectors for $\hat{\mathbf{H}}$ are given by the $\vec{j}_{n}(\vec{x})$ of Eq. A27 and $\vec{i}_{n}(\vec{x})$ of Eq. A28

$$
\begin{align*}
E_{n} \vec{j}_{n}(\vec{x}) & =\mathbf{J I}^{\dagger} \vec{j}_{n}(\vec{x})=\hat{\mathbf{H}} \vec{j}_{n}(\vec{x}) \\
E_{n}^{*} \vec{i}_{n}(\vec{x}) & =\mathbf{I} \mathbf{J}^{\dagger} \vec{i}_{n}(\vec{x})=\hat{\mathbf{H}}^{\dagger} \vec{i}_{n}(\vec{x}) \tag{A31}
\end{align*}
$$

For $n=0$, the steady current $\vec{j}_{n}(\vec{x})=\vec{j}_{*}(\vec{x})$ is annihilated by $\mathbf{I}^{\dagger}=\operatorname{div}$ and thus by $\hat{\mathbf{H}}$

$$
\begin{equation*}
\mathbf{I}^{\dagger} \vec{j}_{*}(\vec{x})=0=\hat{\mathbf{H}} \vec{j}_{*}(\vec{x}) \tag{A32}
\end{equation*}
$$

while $\vec{i}_{0}(\vec{x})$ satisfies Eq. A28 with $l_{0}(\vec{x})=1$ annihilated by $\mathbf{I}=-\operatorname{grad}$

$$
\begin{align*}
1 & =l_{0}(\vec{x})=\mathbf{J}^{\dagger} \vec{i}_{0}(\vec{x})=\vec{F}(\vec{x}) \cdot \vec{i}_{0}(\vec{x})+\vec{\nabla} \cdot\left(D(\vec{x}) \cdot \vec{i}_{0}(\vec{x})\right) \\
0 & =\mathbf{I} \vec{i}_{0}(\vec{x})=\mathbf{I} \mathbf{J}^{\dagger} \vec{i}_{0}(\vec{x})=\hat{\mathbf{H}}^{\dagger} \vec{i}_{0}(\vec{x}) \tag{A33}
\end{align*}
$$

7. Singular Value Decompositions for the differential operator $I=-\overrightarrow{\operatorname{grad}}$ and its adjoint $\mathbf{I}^{\dagger} \equiv \operatorname{div}$

The Singular Value Decompositions for the operator $\mathbf{I}=-\mathbf{g r a d}$ and its adjoint $\mathbf{I}^{\dagger} \equiv \operatorname{div}$ involves an infinite series of strictly positive singular values $I_{\alpha}>0$ (instead of the $(N-1)$ values in Eq. 69 of the main text concerning Markov jump processes)

$$
\begin{align*}
\mathbf{I} & \left.\equiv-\mathbf{g r a d}=\sum_{\alpha=1}^{+\infty} I_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle I_{\alpha}^{L}\right| \\
\mathbf{I}^{\dagger} & \equiv \operatorname{div}=\sum_{\alpha=1}^{+\infty} I_{\alpha}\left|I_{\alpha}^{L}\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{A34}
\end{align*}
$$

As in Eq. 70, the vanishing singular value

$$
\begin{equation*}
I_{(\alpha=0)}=0 \tag{A35}
\end{equation*}
$$

is associated to the uniform normalized eigenvector $\left\langle x \mid I_{(\alpha=0)}^{L}\right\rangle$ over the bounded volume $\vec{x} \in \mathcal{V}$ where the Fokker-Planck dynamics takes place. The operator $\mathbf{I}^{\dagger} \mathbf{I}$ corresponds to the opposite of the Laplacian

$$
\mathbf{I}^{\dagger} \mathbf{I}=\operatorname{div}(-\operatorname{grad})=-\left(\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x_{3}}\right)\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}}  \tag{A36}\\
\frac{\partial}{\partial x_{2}} \\
\frac{\partial}{\partial x_{3}}
\end{array}\right)=-\sum_{\mu=1}^{3} \frac{\partial^{2}}{\partial x_{\mu}^{2}}=-\boldsymbol{\Delta}
$$

while its evaluation from the SVD decompositions of Eq. A34

$$
\begin{equation*}
-\boldsymbol{\Delta}=\mathbf{I}^{\dagger} \mathbf{I}=\sum_{\alpha=1}^{+\infty} I_{\alpha}^{2}\left|I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L}\right| \tag{A37}
\end{equation*}
$$

gives its spectral decomposition in terms of its positive eigenvalues $I_{\alpha}^{2}$ with the corresponding orthonormalized basis of eigenvectors $\left|I_{\alpha}^{L}\right\rangle$. So the left singular kets $\left|I_{\alpha=0,1, . .,+\infty}^{L}\right\rangle$ form an orthonormal basis of the space of scalar functions

$$
\begin{align*}
\delta_{\alpha, \alpha^{\prime}} & =\left\langle I_{\alpha}^{L} \mid I_{\alpha^{\prime}}^{L}\right\rangle=\int_{\mathcal{V}} d^{3} \vec{x}\left\langle I_{\alpha}^{L} \mid \vec{x}\right\rangle\left\langle\vec{x} \mid I_{\alpha^{\prime}}^{L}\right\rangle \\
\delta^{(3)}\left(\vec{x}-\vec{x}^{\prime}\right) & =\left\langle\vec{x} \mid \vec{x}^{\prime}\right\rangle=\sum_{\alpha=0}^{+\infty}\left\langle\vec{x} \mid I_{\alpha}^{L}\right\rangle\left\langle I_{\alpha}^{L} \mid \vec{x}^{\prime}\right\rangle \tag{A38}
\end{align*}
$$

When the basis of eigenvectors $\left|I_{\alpha=0, . .,+\infty}^{L}\right\rangle$ of the opposite Laplacian $[-\boldsymbol{\Delta}]$ of Eq. A37 over the bounded volume $\vec{x} \in \mathcal{V}$ is known, the corresponding right singular vectors $\left|I_{\alpha=1, . .,+\infty}^{R}\right\rangle$ can be obtained via the application of the operator $\mathbf{I}=-\mathbf{g r a d}$

$$
\begin{equation*}
\left.\mathbf{I}\left|I_{\alpha}^{L}\right\rangle \quad=-\operatorname{grad}\left|I_{\alpha}^{L}\right\rangle=I_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \tag{A39}
\end{equation*}
$$

while the application of the adjoint $\mathbf{I}^{\dagger}=\operatorname{div}$ to these right singular vectors $\left|I_{\alpha=1, . .}^{R}\right\rangle$

$$
\begin{equation*}
\left.\mathbf{I}^{\dagger}\left|I_{\alpha}^{R}\right\rangle\right\rangle \quad=\operatorname{div}\left|I_{\alpha=1, . .}^{R}\right\rangle=I_{\alpha}\left|I_{\alpha}^{L}\right\rangle \tag{A40}
\end{equation*}
$$

reproduce the left singular vectors $\left|I_{\alpha}^{L}\right\rangle$.
The supersymmetric partner

$$
\mathbf{I I}^{\dagger}=-\mathbf{g r a d} \operatorname{div}=-\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}}  \tag{A41}\\
\frac{\partial}{\partial x_{2}} \\
\frac{\partial}{\partial x_{3}}
\end{array}\right)\left(\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x_{3}}\right)=-\left[\frac{\partial^{2}}{\partial x_{\mu} \partial x_{\nu}}\right]_{\mu=1,2,3 ; \nu=1,2,3}
$$

corresponds to the opposite of the $3 \times 3$ symmetric matrix of the double derivatives $\frac{\partial^{2}}{\partial x_{\mu} \partial x_{\nu}}$.

The SVD decomposition of Eq. 69 yields that its spectral decomposition

$$
\begin{equation*}
\left.\mathbf{I I}^{\dagger}=\sum_{\alpha=1}^{+\infty} I_{\alpha}^{2}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{A42}
\end{equation*}
$$

involves the same strictly positive eigenvalues $I_{\alpha=1, . .,+\infty}^{2}>0$ as the opposite-Laplacian of Eq. A37, while the corresponding eigenvectors $\left.\left|I_{\alpha=1, \ldots,}^{R}\right\rangle\right\rangle$ are related to the eigenvectors $\left|I_{\alpha=1, . .}^{L}\right\rangle$ of the opposite-Laplacian via Eq. 78. However the eigenvectors $\left.\left|I_{\alpha=1, \ldots,}^{R}\right\rangle\right\rangle$ should be supplemented an infinite number of other kets $\left|I_{\alpha=0,-1,-2 . .,-\infty}^{R}\right\rangle$ (instead of the finite number $C$ of Eq. 74 of the main text concerning Markov jump processes) in order to obtain an orthonormal basis of the space of vector fields

$$
\begin{align*}
\delta_{\alpha, \alpha^{\prime}} & =\left\langle\left\langle I_{\alpha}^{R} \mid I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle \\
\mathbf{1}_{\{\text {VectorFields }\}} & \left.\left.\left.=\sum_{\alpha=-\infty}^{+\infty}\left|I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|=\sum_{\alpha=1}^{+\infty} \mid I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|+\sum_{\alpha=-\infty}^{0} \mid I_{\alpha}^{R}\right\rangle\right\rangle\left\langle\left\langle I_{\alpha}^{R}\right|\right. \tag{A43}
\end{align*}
$$

As discussed in detail in subsection IV D concerning the finite configuration space, the vectors $\left.\left|I_{\alpha^{\prime}}^{R}\right\rangle\right\rangle$ are directly related to the Helmholtz decomposition for an arbitrary three-dimensional vector fields $\vec{v}(\vec{x})$

$$
\begin{align*}
\vec{v}(\vec{x}) & \left.=\sum_{\alpha=-\infty}^{+\infty} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \equiv \vec{v}^{[I .>0]}(\vec{x})+\vec{v}^{\left[I_{0}=0\right]}(\vec{x}) \\
\vec{v}^{[I .>0]}(\vec{x}) & \left.\equiv \sum_{\alpha=1}^{+\infty} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \\
\vec{v}^{\left[I_{0}=0\right]}(\vec{x}) & \left.\equiv \sum_{\alpha=-\infty}^{0} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle \tag{A44}
\end{align*}
$$

into two orthogonal components with the following properties :
(i) the component $\vec{v}^{\left[I_{0}=0\right]}(\vec{x})$ is annihilated by $\mathbf{I}^{\dagger}=\operatorname{div}$

$$
\begin{equation*}
0=\mathbf{I}^{\dagger} \vec{v}^{\left[I_{0}=0\right]}(\vec{x})=\operatorname{div} \vec{v}^{\left[I_{0}=0\right]}(\vec{x}) \tag{A45}
\end{equation*}
$$

As already discussed on the special case of the divergencesless steady current around Eq. A20, this divergencesless component $\vec{v}^{\left[I_{0}=0\right]}(\vec{x})$ can be rewritten as the curl of a divergenceless vector $\vec{\omega}(\vec{x})$

$$
\begin{align*}
\vec{v}^{\left[I_{0}=0\right]}(\vec{x}) & =\vec{\nabla} \times \vec{\omega}(\vec{x}) \\
\operatorname{div} \vec{\omega}(\vec{x}) & =0 \tag{A46}
\end{align*}
$$

(ii) the component $\vec{v}^{[I .>0]}(\vec{x})$ can be rewritten using Eq. A39

$$
\begin{equation*}
\left.\vec{v}^{[I .>0]}(\vec{x}) \equiv \sum_{\alpha=1}^{+\infty} v_{\alpha}\left|I_{\alpha}^{R}\right\rangle\right\rangle=-\mathbf{\operatorname { g r a d }}\left(\sum_{\alpha=1}^{+\infty} \frac{v_{\alpha}}{I_{\alpha}}\left|I_{\alpha}^{L}\right\rangle\right) \equiv-\overrightarrow{\operatorname{grad}} g(\vec{x}) \tag{A47}
\end{equation*}
$$

as the opposite gradient of the scalar function

$$
\begin{equation*}
g(\vec{x}) \equiv \sum_{\alpha=1}^{+\infty} \frac{v_{\alpha}}{I_{\alpha}}\left|I_{\alpha}^{L}\right\rangle \tag{A48}
\end{equation*}
$$

(iii) The application of the divergence $\mathbf{I}_{\rightarrow}^{\dagger}=\operatorname{div}$ to the vector field $\vec{v}(\vec{x})$ of Eq. A44 only involves the application to the gradient component $\vec{v}^{[I .>0]}(\vec{x})=-\operatorname{grad} g(\vec{x})$ of Eq. A47

$$
\begin{equation*}
\operatorname{div} \vec{v}(\vec{x}) \quad=\operatorname{div} \vec{v}^{[I .>0]}(\vec{x})+0=-\boldsymbol{\Delta} g(\vec{x}) \tag{A49}
\end{equation*}
$$

and reduces to the opposite Laplacian of the scalar function $g(\vec{x})$. The application of the curl to the vector field $\vec{v}(\vec{x})$ of Eq. A44 only involves the application to the component $\vec{v}^{\left[I_{0}=0\right]}(\vec{x})=\vec{\nabla} \times \vec{\omega}(\vec{x})$ of Eq. A46

$$
\begin{equation*}
\vec{\nabla} \times \vec{v}(\vec{x})=0+\vec{\nabla} \times \vec{v}^{\left[I_{0}=0\right]}(\vec{x}) \vec{\nabla} \times(\vec{\nabla} \times \vec{\omega}(\vec{x}))=\vec{\nabla}(\vec{\nabla} \cdot \vec{\omega}(\vec{x}))-\Delta \vec{\omega}(\vec{x})=-\Delta \vec{\omega}(\vec{x}) \tag{A50}
\end{equation*}
$$

and reduces to the opposite Laplacian of the the field $\vec{\omega}(\vec{x})$.

## 8. Singular Value Decomposition of the current differential operator $\mathbf{J}$ and its adjoint $\mathbf{J}^{\dagger}$

Let us replace the force $\vec{F}(\vec{x})=2 D(\vec{x}) \vec{A}(\vec{x})$ in terms of the vector potential $\vec{A}(\vec{x})$ introduced in Eq. A10 and write the Singular Value Decomposition for the current differential operator $\mathbf{J}$ and its adjoint $\mathbf{J}^{\dagger}$ that involves an infinite series of strictly positive singular values $\lambda_{\beta=1,2, \ldots,+\infty}>0$ (instead of the $N$ values in Eq. 100 of the main text concerning Markov jump processes)

$$
\begin{align*}
\mathbf{J} & \left.=D(\vec{x})(2 \vec{A}(\vec{x})-\vec{\nabla})=\sum_{\beta=1}^{+\infty} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L}\right| \\
\mathbf{J}^{\dagger} & \equiv(2 \vec{A}(\vec{x})+\vec{\nabla}) D(\vec{x}) \cdot=\sum_{\beta=1}^{+\infty} \lambda_{\beta}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{A51}
\end{align*}
$$

For a non-equilibrium steady state with a non-vanishing steady current $\vec{j}_{*}(\vec{x}) \neq 0$, the left singular vectors $\lambda_{\beta=1,2, \ldots,+\infty}^{L}$ form an orthonormal basis of the space of scalar functions

$$
\begin{align*}
\delta_{\beta, \beta^{\prime}} & =\left\langle\lambda_{\beta}^{L} \mid \lambda_{\beta^{\prime}}^{L}\right\rangle=\int_{\mathcal{V}} d^{3} \vec{x}\left\langle\lambda_{\beta}^{L} \mid \vec{x}\right\rangle\left\langle\vec{x} \mid \lambda_{\beta^{\prime}}^{L}\right\rangle \\
\delta^{(3)}\left(\vec{x}-\vec{x}^{\prime}\right) & =\left\langle\vec{x} \mid \vec{x}^{\prime}\right\rangle=\sum_{\beta=1}^{+\infty}\left\langle\vec{x} \mid \lambda_{\beta}^{L}\right\rangle\left\langle\lambda_{\beta}^{L} \mid \vec{x}^{\prime}\right\rangle \tag{A52}
\end{align*}
$$

that appear the spectral decomposition of the supersymmetric operator

$$
\begin{equation*}
\mathbf{J}^{\dagger} \mathbf{J}=(2 \vec{A}(\vec{x})+\vec{\nabla}) D^{2}(\vec{x})(2 \vec{A}(\vec{x})-\vec{\nabla})=\sum_{\beta=1}^{+\infty} \lambda_{\beta}^{2}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\lambda_{\beta}^{L}\right| \tag{A53}
\end{equation*}
$$

The supersymmetric partner

$$
\begin{equation*}
\left.\mathbf{J} \mathbf{J}^{\dagger}=\left[D(\vec{x})\left(2 A_{\mu}(\vec{x})-\frac{\partial}{\partial x_{\mu}}\right)\left(2 A_{\nu}(\vec{x})+\frac{\partial}{\partial x_{\mu}}\right) D(\vec{x})\right]_{\mu=1,2,3 ; \nu=1,2,3}=\sum_{\beta=1}^{+\infty} \lambda_{\beta}^{2}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right.\right. \tag{A54}
\end{equation*}
$$

involves the same strictly positive eigenvalues $\lambda_{\beta=1,2, \ldots,+\infty}^{2}$ as Eq. A53 while the corresponding eigenvectors $\left.\left|\lambda_{\beta}^{R}\right\rangle\right\rangle$ are related to the eigenvectors $\left|\lambda_{\beta}^{L}\right\rangle$ via

$$
\begin{equation*}
\left.\mathbf{J}\left|\lambda_{\beta}^{L}\right\rangle \quad=\lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \tag{A55}
\end{equation*}
$$

These eigenvectors $\left.\left|\lambda_{\beta=1,2, . .,+\infty}^{R}\right\rangle\right\rangle$ should be supplemented an infinite number of other kets $\left.\left|\lambda_{\beta=0,-1, \ldots,-\infty}^{R}\right\rangle\right\rangle$ (instead of the finite number $(C-1)$ of Eq. 101 of the main text concerning Markov jump processes) in order to obtain an orthonormal basis of the space of vector fields

$$
\begin{align*}
\delta_{\beta, \beta^{\prime}} & =\left\langle\left\langle\lambda_{\beta}^{R} \mid \lambda_{\beta^{\prime}}^{R}\right\rangle\right\rangle \\
\mathbf{1}_{\{\text {VectorFields }\}} & \left.\left.\left.=\sum_{\beta=-\infty}^{+\infty}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|=\sum_{\beta=1}^{+\infty} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|+\sum_{\beta=-\infty}^{0} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{A56}
\end{align*}
$$

As discussed in subsection VB of the main text, it is interesting to define the analog of the Helmholtz decomposition when the operator $\mathbf{I}=-\operatorname{grad}$ is replaced by the current matrix $\mathbf{J}$ using its SVD decomposition of Eq. A51 as follows.

An arbitrary three-dimensional vector field $\vec{u}(\vec{x})$ can be decomposed into two orthogonal components

$$
\begin{align*}
\vec{u}(\vec{x}) & =\sum_{\beta=-\infty}^{+\infty} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle \equiv \vec{u}^{[\lambda .>0]}(\vec{x})+\vec{u}^{\left[\lambda_{0}=0\right]}(\vec{x}) \\
\vec{u}^{[\lambda .>0]}(\vec{x}) & \equiv \sum_{\beta=1}^{+\infty} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle \\
\vec{u}^{\left[\lambda_{0}=0\right]}(\vec{x}) & \left.\equiv \sum_{\beta=-\infty}^{0} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle \tag{A57}
\end{align*}
$$

with the following properties :
(i) the component $\vec{u}^{\left[\lambda_{0}=0\right]}(\vec{x})$ of Eq. A57 is annihilated by $\mathbf{J}^{\dagger}$ of Eq. A51

$$
\begin{equation*}
0=\mathbf{J}^{\dagger} \vec{u}^{\left[\lambda_{0}=0\right]}(\vec{x})=(2 \vec{A}(\vec{x})+\vec{\nabla}) \cdot\left(D(\vec{x}) \vec{u}^{\left[\lambda_{0}=0\right]}(\vec{x})\right) \tag{A58}
\end{equation*}
$$

(ii) the component $\vec{u}^{[\lambda .>0]}(\vec{x})$ can be rewritten using Eq. A55

$$
\begin{equation*}
\left.\vec{u}^{[\lambda .>0]}(\vec{x}) \quad=\sum_{\beta=1}^{+\infty} u_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle=\mathbf{J}\left(\sum_{\beta=1}^{+\infty} \frac{u_{\beta}}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\right) \equiv \mathbf{J} k(\vec{x}) \tag{A59}
\end{equation*}
$$

as the application of the current operator $\mathbf{J}$ to the scalar function

$$
\begin{equation*}
k(\vec{x}) \equiv \sum_{\beta=1}^{+\infty} \frac{u_{\beta}}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle \tag{A60}
\end{equation*}
$$

The application of $\mathbf{J}^{\dagger}$ to the vector field $\vec{u}(\vec{x})$ of Eq. A57 only involves the application to the component $\vec{u}^{[\lambda .>0]}(\vec{x})=$ $\mathbf{J} k(\vec{x})$ of Eq. A59 and reduces to the application of the supersymmetric operator $\mathbf{J}^{\dagger} \mathbf{J}$ of Eq. A53 to the scalar function $k(\vec{x})$ of Eq. A60

$$
\begin{equation*}
\mathbf{J}^{\dagger} \vec{u}(\vec{x}) \quad=\mathbf{J}^{\dagger} \vec{u}^{[\lambda .>0]}(\vec{x})+0=\mathbf{J}^{\dagger} \mathbf{J} k(\vec{x})=\sum_{\beta=1}^{+\infty} u_{\beta} \lambda_{\beta}\left|\lambda_{\beta}^{L}\right\rangle \tag{A61}
\end{equation*}
$$

As discussed in subsection V B 3 of the main text, the component $\left.\left|u^{\left[\lambda_{0}=0\right]}\right\rangle\right\rangle$ corresponds to the unphysical subspace unphysical subspace orthogonal to the physical space of the physical currents $\vec{j}_{t}(\vec{x})=\mathbf{J} p_{t}(\vec{x})$ that are obtained from the application of the current matrix $\mathbf{J}$ to a ket $\left|p_{t}\right\rangle$ of the configuration space. The projector $\mathcal{P}^{\text {PhysicalSpaceCurrents }}$ onto the physical subspace of the physical currents can be written either with the bi-orthogonal basis of the $\left\langle\left\langle i_{n}\right|\right.$ and the $\left.\left|j_{n}\right\rangle\right\rangle$ for $n=0,1, \ldots,+\infty$ or with the orthonormalized basis $\left.\left|\lambda_{\beta=1, . .,+\infty}^{R}\right\rangle\right\rangle$ of the right singular vectors of the current matrix $\mathbf{J}$ associated to the strictly positive singular values $\lambda_{\beta=1, . .,+\infty}^{R}>0$

$$
\begin{equation*}
\left.\left.\mathcal{P}^{\text {PhysicalSpaceCurrents }}=\sum_{n=0}^{+\infty}\left|j_{n}\right\rangle\right\rangle\left\langle\left\langle i_{n}\right|=\sum_{\beta=1}^{+\infty} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{A62}
\end{equation*}
$$

with the following consequences :
(a) As in Eqs 117118 119, the decomposition of the steady current on the basis of right singular vectors $\left.\left|\lambda_{\beta}^{R}\right\rangle\right\rangle$

$$
\begin{equation*}
\left.\left.\left|j_{*}\right\rangle\right\rangle=\mathbf{J}\left|p_{*}\right\rangle \equiv \sum_{\beta=1}^{+\infty} \lambda_{\beta}\left|\lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L} \mid p_{*}\right\rangle \tag{A63}
\end{equation*}
$$

yields the vanishing of the components for $\beta=0,-1, . .,-\infty$

$$
\begin{equation*}
\beta=0,-1, . .,-\infty: \quad\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{*}\right\rangle\right\rangle=0 \tag{A64}
\end{equation*}
$$

while the steady state $\left|p_{*}\right\rangle$ can be obtained from the inversion of Eq. A63

$$
\begin{equation*}
\left.\left|p_{*}\right\rangle=\mathbf{J}^{\text {pseudo }[-1]}\left|j_{*}\right\rangle\right\rangle=\sum_{\beta=1}^{N} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{*}\right\rangle\right\rangle \tag{A65}
\end{equation*}
$$

with the pseudo-inverse

$$
\begin{equation*}
\mathbf{J}^{\text {pseudo }[-1]}=\sum_{\beta=1}^{+\infty} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{A66}
\end{equation*}
$$

that can also be used for the excited right eigenvectors $\left|r_{n=1, .,+\infty}\right\rangle$ and their associated currents $\left.\left|j_{n}\right\rangle\right\rangle=\mathbf{J}\left|p_{n}\right\rangle$ as in Eq. 121

$$
\begin{equation*}
\left.\left|r_{n}\right\rangle=\mathbf{J}^{\text {pseudo }[-1]}\left|j_{n}\right\rangle\right\rangle=\sum_{\beta=1}^{+\infty} \frac{1}{\lambda_{\beta}}\left|\lambda_{\beta}^{L}\right\rangle\left\langle\left\langle\lambda_{\beta}^{R} \mid j_{n}\right\rangle\right\rangle \tag{A67}
\end{equation*}
$$

(b) As in Eqs 122 123, the pseudo-inverse $\mathbf{J}^{\text {pseudo }}{ }^{[-1]}$ of Eq. A66 is also useful to invert

$$
\begin{equation*}
\left\langle l_{n}\right|=\left\langle\left\langle i_{n}\right| \mathbf{J}=\sum_{\beta=1}^{+\infty} \lambda_{\beta}\left\langle\left\langle i_{n} \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\lambda_{\beta}^{L}\right|\right. \tag{A68}
\end{equation*}
$$

into

$$
\begin{equation*}
\left\langle\left\langle i_{n}\right|=\sum_{\beta=1}^{+\infty}\left\langle\left\langle i_{n}\right| \mid \lambda_{\beta}^{R}\right\rangle\right\rangle\left\langle\left\langle\lambda_{\beta}^{R}\right|=\sum_{\beta=1}^{+\infty} \frac{\left\langle l_{n} \mid \lambda_{\beta}^{L}\right\rangle}{\lambda_{\beta}}\left\langle\left\langle\lambda_{\beta}^{R}\right|=\left\langle l_{n}\right| \mathbf{J}^{\text {pseudo }[-1]}\right.\right. \tag{A69}
\end{equation*}
$$

For $n=0$ with the left eigenvector $l_{0}(\vec{x})=1$, Eq. A69 yields that the bra $\left\langle\left\langle i_{0}\right|\right.$ is given by

$$
\begin{align*}
\left\langle i_{0}\right| & =\left\langle l_{0}\right| \mathbf{J}^{\text {pseudo }[-1]}=\sum_{\vec{x}}\left\langle l_{0} \mid \vec{x}\right\rangle\langle\vec{x}| \mathbf{J}^{\text {pseudo }[-1]} \\
& =\sum_{\vec{x}}\langle\vec{x}| \mathbf{J}^{\text {pseudo }[-1]}=\sum_{\beta=1}^{+\infty} \frac{\left\langle\vec{x} \mid \lambda_{\beta}^{L}\right\rangle}{\lambda_{\beta}}\left\langle\left\langle\lambda_{\beta}^{R}\right|\right. \tag{A70}
\end{align*}
$$

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