

# THE QUASI-ISOMETRY INVARIANCE OF THE COSET INTERSECTION COMPLEX

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**ABSTRACT.** For a pair  $(G, \mathcal{P})$  consisting of a group and finite collection of subgroups, we introduce a simplicial  $G$ -complex  $\mathcal{K}(G, \mathcal{P})$  called the coset intersection complex. We prove that the quasi-isometry type and the homotopy type of  $\mathcal{K}(G, \mathcal{P})$  are quasi-isometric invariants of the group pair  $(G, \mathcal{P})$ . Classical properties of  $\mathcal{P}$  in  $G$  correspond to topological or geometric properties of  $\mathcal{K}(G, \mathcal{P})$ , such as having finite height, having finite width, being almost malnormal, admitting a malnormal core, or having thickness of order one. As applications, we obtain that a number of algebraic properties of  $\mathcal{P}$  in  $G$  are quasi-isometry invariants of the pair  $(G, \mathcal{P})$ . For a certain class of right-angled Artin groups and their maximal parabolic subgroups, we show that the complex  $\mathcal{K}(G, \mathcal{P})$  is quasi-isometric to the Deligne complex; in particular, it is hyperbolic.

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## 1. INTRODUCTION

A *group pair*  $(G, \mathcal{P})$  is a pair consisting of a finitely generated group  $G$  and a finite collection  $\mathcal{P}$  of infinite subgroups. The collection  $\mathcal{P}$  is called the *peripheral structure* of the pair. A map  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a quasi-isometry of group pairs if  $q: G \rightarrow H$  is a quasi-isometry such that for each  $P \in \mathcal{P}$ , the image  $q(P)$  is uniform finite Hausdorff distance from a left coset of some  $Q \in \mathcal{Q}$  and the quasi-inverse of  $q$  satisfies the analogous property; see Definition 2.4.

Quasi-isometries of group pairs have been used implicitly in many works [DS05, BDM09, MSW11, FLS15, KL97], typically to prove strong rigidity results about a group. For example, Kapovich and Leeb used similar ideas to prove the quasi-isometry invariance of the geometric decomposition of Haken manifolds [KL97, §5.1], as did Drutu and Sapir and Behrstock, Drutu, and Mosher to prove quasi-isometric rigidity results for relatively hyperbolic groups [DS05, BDM09]. More recently, quasi-isometries of group pairs have been used explicitly by a number of researchers in group theory; see, e.g., [HH20, HH23, GT21, MPSSn22, HMP23, Gen21], and [HMPS21] for a recent survey.

This article introduces the *coset intersection complex* of a group pair  $(G, \mathcal{P})$ , which we denote  $\mathcal{K}(G, \mathcal{P})$ . It is a simplicial complex whose quasi-isometry type and homotopy type are quasi-isometry invariants of the group pair (Theorem 1.4). We reformulate several natural algebraic properties of group pairs into geometric and combinatorial properties of  $\mathcal{K}(G, \mathcal{P})$  (Theorem 1.6). This reformulation allows us to prove that certain natural finiteness properties of group pairs are quasi-isometry invariants (Theorem 1.5). In this way, the coset intersection complex reinforces the central paradigm of geometric group theory: it is fruitful to study algebraic properties from geometric and topological perspectives.

**Definition 1.1** (Coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ ). Let  $(G, \mathcal{P})$  be a group pair, and let  $G/\mathcal{P}$  be the set of all cosets  $gP$  with  $g \in G$  and  $P \in \mathcal{P}$ . The *coset intersection complex* is the simplicial  $G$ -complex with vertex set  $G/\mathcal{P}$  and such that a set  $\{g_1P_1, \dots, g_kP_k\} \subset G/\mathcal{P}$  defines a simplex if  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is an infinite subgroup of  $G$ . Observe that the  $G$ -action on  $G/\mathcal{P}$  by left multiplication extends to a cellular  $G$ -action on  $\mathcal{K}(G, \mathcal{P})$ .

For example, if  $G$  is hyperbolic relative to a collection of subgroups  $\mathcal{P}$ , then  $\mathcal{K}(G, \mathcal{P})$  is a discrete graph. At the opposite extreme, if  $G$  is abelian and  $\mathcal{P} = \{P\}$  is a single infinite subgroup, then  $\mathcal{K}(G, \mathcal{P})$  is a complete graph. We give several examples of coset intersection complexes for particular group pairs in Example 4.2, but we mention the more involved example of right-angled Artin groups here. The (modified) Deligne complex was defined by Charney and Davis in [CD95], building on work of Deligne [Del72].

**Theorem 1.2.** *Suppose  $G$  is a right-angled Artin group whose defining graph  $\Gamma$  is connected, triangle-free, and has no vertices of valence  $< 2$ . Let  $\mathcal{P}$  be the collection of maximal standard flats in  $G$ . Then  $\mathcal{K}(G, \mathcal{P})$  is quasi-isometric to the (modified) Deligne complex.*

Charney and Crisp showed that, under certain assumptions on  $\Gamma$ , the Deligne complex is hyperbolic [CC07], and so we obtain the following corollary.

**Corollary 1.3.** *Under the assumptions of Theorem 1.2 and the additional assumption that  $\Gamma$  has no squares, then  $\mathcal{K}(G, \mathcal{P})$  is hyperbolic.*

Given a quasi-isometry of group pairs  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ , there is an induced map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$ . Roughly, the image of a vertex  $gP$  in  $\mathcal{K}(G, \mathcal{P})$  is a vertex  $hQ$  in  $\mathcal{K}(H, \mathcal{Q})$  such that  $hQ$  and  $q(gP)$  are at finite Hausdorff distance. The map  $\dot{q}$  is not unique: there may be many cosets  $hQ$  at finite Hausdorff distance from  $q(gP)$ . Any choice of such  $hQ$  will result in an induced map; by an abuse of notation, we denote any such induced map by  $\dot{q}$ . The following theorem shows that the induced maps  $\dot{q}$  have particularly nice properties.

**Theorem 1.4.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of group pairs and  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  an induced map.*

- (1) *The induced map  $\dot{q}$  is a simplicial map and a homotopy equivalence.*
- (2) *All choices of  $\dot{q}$  are homotopic as continuous maps  $\mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$ .*
- (3) *The induced map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is a quasi-isometry of simplicial complexes.*

By a *simplicial map*  $f: X \rightarrow Y$  we mean a function between the sets of vertices of the simplicial complexes  $X$  and  $Y$  with the property that the images of the vertices of a simplex always span a simplex. The quasi-isometry in (3) is with respect to a natural length metric on the complexes viewed as piecewise Euclidean complexes; see the discussion before Corollary 4.8 for details. We also give conditions on the group pairs  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  to ensure that  $\dot{q}$  is a simplicial isomorphism; see Proposition 4.9.

We now turn our attention to properties of group pairs satisfying strong rigidity properties in relation with the coset intersection complex. A property of group pairs is called *geometric* if every group pair quasi-isometric to a group pair satisfying the property is virtually isomorphic to a group pair satisfying the property. Here, a *virtual isomorphism* is a generalization of commensurability to group pairs; see Definition 2.8. Let us note that if a property of group pairs is preserved by quasi-isometries then it is geometric, but the converse does not hold, as some of the properties studied in this article illustrate.

There are a number of geometric properties of group pairs  $(G, \mathcal{P})$  that have been studied in the literature, in some cases implicitly. Examples include having a well-defined relative Dehn function [HMPSSn23, Osi06]; having an almost malnormal peripheral structure [HMPSSn23]; being relatively hyperbolic with respect to the peripheral structure [DS05, BDM09]; and having a finite number of filtered ends (Bowditch's coends) with respect to the peripheral structure [MPSSn22]. The coset intersection complex encodes several algebraic finiteness properties of the group which turn out to be geometric.

**Theorem 1.5.** *The following define geometric properties of group pairs  $(G, \mathcal{P})$ :*

- (1)  *$\mathcal{P}$  has finite height in  $G$ .*
- (2)  *$\mathcal{P}$  has finite width in  $G$ .*
- (3)  *$\mathcal{P}$  has fenced infinite intersections in  $G$ .*
- (4)  *$\mathcal{P}$  is an almost malnormal collection in  $G$ .*
- (5)  *$G$  is a network with respect to  $\mathcal{P}$ .*
- (6)  *$G$  is thick of order 1.*
- (7) *Every  $P \in \mathcal{P}$  satisfies  $P = \text{Comm}_G(P)$ .*
- (8) *Every  $P \in \mathcal{P}$  has finite index in  $\text{Comm}_G(P)$ .*
- (9)  *$\mathcal{P}$  has bounded packing in  $G$ .*
- (10)  *$\mathcal{P}$  has finite packing in  $G$ .*

We discuss each of these algebraic properties in more depth in the main body of the article, where we also recall their definitions. Here, we give a brief overview of where they appear in the literature. The study of *height* and *width* of subgroups can be traced back to the work of Gitik, MJ, Rips and Sageev [GMR98] in the late nineties, where they proved that quasi-convex subgroups of hyperbolic groups have finite width and, in particular, finite height. It is a well-known open question of Swarup whether, in a hyperbolic group, a subgroup with finite height and finite index in its commensurator is quasiconvex [Bes04, Question 1.8.]. Parts (1) and (9) of Theorem 1.5 could be regarded as evidence of a positive answer to Swarup's question in the sense that the geometry of such subgroup in the ambient group is preserved by quasi-isometry. The literature on the study of height and width of subgroups also includes the work of Hruska and Wise [HW09] on finite width of strongly quasi-convex subgroups of relatively hyperbolic groups, and the work of Antolin, MJ, Sisto and Taylor on finite width of stable subgroups [AST19]. *Fenced infinite intersections* for group pairs involves uniform bounds on the distance between elements of  $G/\mathcal{P}$  when the intersection of the corresponding conjugates is infinite; see Definition 5.24. This notion has appeared implicitly in the study of non-positively curved groups in relation to height and width, for example, [GMR98, Lemma 1.2] and [HW09, Lemma 8.4]. The *commensurated core* of a group pair is a generalization of the notion of a malnormal core. The latter was introduced by Agol, Groves and Manning in the context of group pairs  $(G, P)$  where  $G$  is a hyperbolic group and  $P$  is a quasi-convex subgroup [AGM09]. *Almost malnormality* has been studied in many contexts, including relative Dehn functions of group pairs, peripheral subgroups in relatively hyperbolic groups and, more generally, hyperbolically embedded subgroups in acylindrically hyperbolic groups [Osi06, DGO17]. That almost malnormality is a geometric property of group pairs is a result of Hughes, Sanchez-Saldana and the second author [HMPSSn23, Theorem D]; in this article we provide an alternative argument. *Networks* were introduced by Behrstock, Druţu, and Mosher in connection with the notion of *thickness* of a group, which is an obstruction to relative hyperbolicity [BDM09]. *Bounded packing* was introduced by Hruska and Wise in relation to the study of width and height of subgroups and actions on CAT(0) cubical complexes [HW09]. The notion of *finite packing* is introduced in this article; see Definition 5.8. It relates to height and bounded packing, and is equivalent to cocompactness of the coset intersection complex.

Each part of Theorem 1.5 is proven by first using Theorem 1.6, below, to translate the algebraic property into a geometric property of an associated (extended) metric space, and then using the geometry of that space to prove that the property is geometric. For most of the properties, the space is the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ . For parts (4), (8), and (9), though, we use a different space, which we call the *coset space*. This is the extended metric space  $(G/\mathcal{P}, \text{hdist}_G)$ , where  $\text{hdist}_G$  is the Hausdorff distance on subsets of  $G$  with respect to a word metric.

**Theorem 1.6.** *Let  $(G, \mathcal{P})$  be a group pair and  $\mathcal{K}(G, \mathcal{P})$  the coset intersection complex.*

- (1)  *$\mathcal{K}(G, \mathcal{P})$  is finite dimensional if and only if  $\mathcal{P}$  has finite height in  $G$ .*
- (2) *The cardinality of cliques in  $\mathcal{K}(G, \mathcal{P})$  is bounded if and only if  $\mathcal{P}$  has finite width in  $G$ .*
- (3)  *$\mathcal{K}(G, \mathcal{P})$  has  $G$ -cocompact 1-skeleton if and only if  $\mathcal{P}$  has fenced infinite intersections in  $G$ .*

- (4)  $\mathcal{K}(G, \mathcal{P})$  is  $G$ -cocompact if and only if  $(G, \mathcal{P})$  has finite height and a finite commensurated core.
- (5)  $\mathcal{K}(G, \mathcal{P})$  is zero dimensional if and only if  $\mathcal{P}$  is almost malnormal in  $G$ .
- (6)  $\mathcal{P}$  is non-empty and  $\mathcal{K}(G, \mathcal{P})$  has a connected and  $G$ -cocompact subgraph with vertex set  $G/\mathcal{P}$  if and only if  $(G, \text{dist}_G)$  is a network with respect to the collection  $G/\mathcal{P}$ .

From our perspective, there are two natural classes of groups/group pairs that merit deeper study. First, one could consider group pairs for which the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  is a quasi-isometry invariant of the *group*, rather than of the group pair. This will be the case, for example, when the collection of subgroups  $\mathcal{P}$  is *qi-characteristic* in  $G$ . A collection of infinite subgroups  $\mathcal{P}$  of a finitely generated group  $G$  is *qi-characteristic* if the collection of subspaces  $G/\mathcal{P}$  is preserved up to uniform finite Hausdorff distance by every quasi-isometry of  $G$ . Containing a finite collection of *qi-characteristic* subgroups is a quasi-isometry invariant of finitely generated groups [MPSSn22, Theorem 1.1]. Specifically, if  $(G, \mathcal{P})$  is a group pair and  $H$  is a finitely generated group quasi-isometric to  $G$ , then there is a *qi-characteristic* collection of subgroups  $\mathcal{Q}$  of  $H$  such that the group pairs  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are quasi-isometric pairs. The class of finitely generated groups that contain a finite collection of *qi-characteristic* subgroups is large and includes hyperbolic groups relative to non-relatively hyperbolic groups [BDM09], atomic right-angled Artin groups [BKS08], lamplighter-like groups [GT21, GT24], and mapping class groups [BKMM12]. Theorem 1.4 implies that the (quasi-isometry class of the) coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  of a group pair  $(G, \mathcal{P})$  with  $\mathcal{P}$  a *qi-characteristic* collection is a quasi-isometry invariant of the group  $G$ .

Second, since actions on hyperbolic spaces have proven to be a powerful tool in geometric group theory, one could consider group pairs  $(G, \mathcal{P})$  such that  $\mathcal{K}(G, \mathcal{P})$  is hyperbolic. Theorem 1.2 provides one example of such a group pair, but there are certainly others. In fact, the maximal flats  $\mathcal{P}$  in atomic right-angled Artin groups (which satisfy the conditions of Theorem 1.2) are *qi-characteristic* [BKS08, Theorems 1.1 & 1.6], and so these groups fall into the intersection of the two classes. In this case, the Gromov boundary of  $\mathcal{K}(G, \mathcal{P})$  is a quasi-isometry invariant of the group  $G$ .

**Question 1.7.** What are examples of finitely generated groups (not commensurable to right-angled Artin groups) that have a collection of *qi-characteristic* subgroups with respect to which the coset intersection complex is hyperbolic?

**Organization.** Section 2 provides background on and basic properties of quasi-isometries of group pairs. In Section 3, we introduce the coset space associated to a group pair and use it to prove Theorem 1.5 (4), (7), (8), and (9).

In Section 4, we introduce the coset intersection complex. We give some basic examples and properties of the complex, proving Theorem 1.6 (1), (2), and (5). We then show that a quasi-isometry of pairs induces a simplicial map of the associated coset intersection complexes and prove Theorem 1.4, which describes several properties of this induced map.

In Section 5, we translate several algebraic finiteness conditions into geometric properties of the coset intersection complex and prove these properties are geometric. This proves the remaining parts of Theorems 1.5 and 1.6.

Finally, in Section 6, we give a detailed example of the coset intersection complex for certain right-angled Artin groups and discuss the relationship between this complex and other complexes associated to the group, proving Theorem 1.2.

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## 2. QUASI-ISOMETRIES OF PAIRS

We begin by setting up some language and notation for the rest of the article. A *pseudo-metric* is a metric such that the distance between two distinct points can be zero. An *extended (pseudo-)metric* on a set  $X$  is a function  $\text{dist}: X \times X \rightarrow [0, \infty]$  that satisfies the usual properties of a (pseudo-)metric with the triangle inequality interpreted in the natural way. In particular, we consider  $[0, \infty]$  with the standard order where  $\infty$  is the maximum value. A set together with an extended metric is called an *extended metric space*.

Let  $(X, \text{dist}_X)$  be an extended metric space. For a subset  $A \subseteq X$ , its closed  $r$ -neighborhood is denoted as

$$\mathcal{N}_r(A) = \{x \in X \mid \exists a \in A \text{ such that } \text{dist}_X(a, x) \leq r\}.$$

The Hausdorff distance between subsets  $A, B \subseteq X$  is defined as

$$\text{hdist}_X(A, B) = \inf\{r \mid B \subseteq \mathcal{N}_r(A) \text{ and } A \subseteq \mathcal{N}_r(B)\},$$

where, as a convention, the infimum of the empty set is defined as infinity. An important property of the Hausdorff distance is that it satisfies the triangle inequality, in the sense that, if  $A, B, C$  are subsets of  $X$  then

$$\text{hdist}_X(A, C) \leq \text{hdist}_X(A, B) + \text{hdist}_X(B, C).$$

In particular, the power set of  $X$  with the Hausdorff distance,  $(\mathcal{P}(X), \text{hdist}_X)$  is an extended pseudo-metric space. The infimum distance between subsets is

$$\text{dist}_X(A, B) = \inf\{\text{dist}_X(a, b) \mid a \in A \quad b \in B\}.$$

Note that the infimum distance between sets does not satisfy the triangle inequality in general.

The notion of quasi-isometry  $f: X \rightarrow Y$  between metric spaces naturally generalizes to extended pseudo-metric spaces.

**Definition 2.1** (Quasi-isometry). Let  $X$  and  $Y$  be extended pseudo-metric spaces, and let  $L \geq 1$  and  $C \geq 0$  be real numbers. A function  $f: X \rightarrow Y$  is an  $(L, C)$ -quasi-isometry if for any  $a, b \in X$ ,

- (1)  $\text{dist}_X(a, b) = \infty$  if and only if  $\text{dist}_Y(f(a), f(b)) = \infty$ ;
- (2) if  $\text{dist}_X(a, b) < \infty$ , then

$$\frac{1}{L} \text{dist}_X(a, b) - C \leq \text{dist}_Y(f(a), f(b)) \leq L \text{dist}_X(a, b) + C;$$

- (3) for every  $y \in Y$  there is  $x \in X$  such that  $\text{dist}_Y(f(x), y) \leq C < \infty$ .

Two extended pseudo-metric spaces are *quasi-isometric* if there is a quasi-isometry between them. As in the case of metric spaces, it is straight forward to verify that if  $f: X \rightarrow Y$  is quasi-isometry of extended pseudo-metric spaces, then there is a quasi-isometry  $g: Y \rightarrow X$  such that  $g \circ f$  and  $f \circ g$  are uniformly close to the identity functions on  $X$  and  $Y$  respectively; such function  $g$  is called a *quasi-inverse* of  $f$ .

**Definition 2.2** (Uniformly finite-to-one function). A function is *uniformly finite-to-one* if there is a uniform finite bound on the cardinality of the pre-image of every element of the codomain.

**Definition 2.3** (Uniformly locally finite metric space). A metric space is *uniformly locally finite* if for every  $r$  there is a uniform bound of the cardinality of balls of radius  $r$ .

The notion of a quasi-isometry of pairs was introduced in [MPSSn22].

**Definition 2.4** (Quasi-isometry of pairs). Let  $X$  and  $Y$  be metric spaces, and let  $\mathcal{A}$  and  $\mathcal{B}$  be collections of subspaces of  $X$  and  $Y$  respectively. A quasi-isometry  $q: X \rightarrow Y$  is a *quasi-isometry of pairs*  $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  if there is  $M > 0$  such that:

- (1) for any  $A \in \mathcal{A}$ , the set  $\{B \in \mathcal{B}: \text{hdist}_Y(q(A), B) < M\}$  is non-empty; and
- (2) for any  $B \in \mathcal{B}$ , the set  $\{A \in \mathcal{A}: \text{hdist}_Y(q(A), B) < M\}$  is non-empty.

In this case, if  $q: X \rightarrow Y$  is a  $(L, C)$ -quasi-isometry, then  $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is called a  $(L, C, M)$ -*quasi-isometry of pairs*. If there is a quasi-isometry of pairs  $(X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  we say that  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  are *quasi-isometric pairs*. One can verify that if  $f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is quasi-isometry of pairs and  $g$  is a quasi-inverse of  $f: X \rightarrow Y$  then  $g: (Y, \mathcal{B}) \rightarrow (X, \mathcal{A})$  is a quasi-isometry of pairs.

Given a quasi-isometry of pairs, there is an induced quasi-isometry on the corresponding sets of subspaces equipped with the Hausdorff distance.

**Proposition 2.5** (Induced quasi-isometry  $\dot{q}$ ). *If  $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is a  $(L, C, M)$ -quasi-isometry of pairs, then there is a quasi-isometry of extended metric spaces*

$$\dot{q}: (\mathcal{A}, \text{hdist}_X) \rightarrow (\mathcal{B}, \text{hdist}_Y)$$

*such that*

- (1)  $\text{hdist}_Y(\dot{q}(A), q(A)) \leq M$  for every  $A \in \mathcal{A}$ .
- (2) If  $(\mathcal{A}, \text{hdist}_X)$  is (uniformly) locally finite, then  $\dot{q}$  is (uniformly) finite-to-one.
- (3) If  $p: (Y, \mathcal{B}) \rightarrow (X, \mathcal{A})$  is a quasi-inverse of the quasi-isometry of pairs  $q$ , then  $\dot{p}$  is a quasi-inverse of  $\dot{q}$ .
- (4) For every  $A_1$  and  $A_2$  in  $\mathcal{A}$ ,

$$\text{dist}_Y(\dot{q}(A_1), \dot{q}(A_2)) \leq L \text{dist}_X(A_1, A_2) + C + 2M$$

*and*

$$\text{dist}_X(A_1, A_2) \leq L \text{dist}_Y(\dot{q}(A_1), \dot{q}(A_2)) + (C + 2M)L.$$

*Proof.* The definition of quasi-isometry of pairs implies that there is a function  $\dot{q}: \mathcal{A} \rightarrow \mathcal{B}$  such that  $\text{hdist}_Y(\dot{q}(A), q(A)) \leq M$  for every  $A \in \mathcal{A}$ . It follows that

$$\frac{1}{L} \text{hdist}_X(A_1, A_2) - C - 2M \leq \text{hdist}_Y(\dot{q}(A_1), \dot{q}(A_2)) \leq L \text{hdist}_X(A_1, A_2) + C + 2M.$$

On the other hand, for every  $B \in \mathcal{B}$  there is  $A \in \mathcal{A}$  such that  $\text{hdist}_Y(\dot{q}(A), B) \leq 2M$ . Hence  $\dot{q}: (\mathcal{A}, \text{hdist}_X) \rightarrow (\mathcal{B}, \text{hdist}_Y)$  is an  $(L, C+2M)$ -quasi-isometry. For the second statement, note that if  $\dot{q}(A_1) = \dot{q}(A_2)$  then  $\text{hdist}_X(A_1, A_2) \leq L(2M + C)$ ; hence, if  $(\mathcal{A}, \text{hdist}_X)$  is (uniformly) locally finite, then  $\dot{q}$  is (uniformly) finite to one. For the third statement, by increasing  $M$ , suppose that

$$\text{hdist}_X(\dot{p}(B), p(B)) \leq M, \quad \text{hdist}_X(p \circ q(x), x) \leq M, \quad \text{and} \quad \text{hdist}_Y(q \circ p(y), y) \leq M$$

for all  $B \in \mathcal{B}$ ,  $x \in X$  and  $y \in Y$ . Hence if  $A \in \mathcal{A}$  then

$$\begin{aligned} \text{hdist}_X(\dot{p} \circ \dot{q}(A), A) &\leq \text{hdist}_X(\dot{p} \circ \dot{q}(A), p \circ \dot{q}(A)) + \text{hdist}_X(p \circ \dot{q}(A), A) \\ &\leq M + \text{hdist}_X(p \circ \dot{q}(A), A) \\ &\leq M + \text{hdist}_X(p \circ \dot{q}(A), p \circ q(A)) + \text{hdist}_X(p \circ q(A), A) \\ &\leq M + L \text{hdist}_Y(\dot{q}(A), q(A)) + C + M \\ &\leq 2M + LM + C =: M'. \end{aligned}$$

Analogously,  $\text{hdist}_Y(\dot{q} \circ \dot{p}(B), B) \leq M'$  for every  $B \in \mathcal{B}$ . Hence  $\dot{p}$  and  $\dot{q}$  are quasi-inverses. The last statement is also a direct verification and is left to the reader.  $\square$

**Remark 2.6.** The quasi-isometry  $\dot{q}: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  induced by a quasi-isometry of pairs  $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is not unique. However any two such maps are at uniform  $L_\infty$ -distance by the first item of the previous proposition. We will refer to any of those  $\dot{q}$  as an *induced quasi-isometry*  $\dot{q}$ .

For a group  $G$  and a collection  $\mathcal{P}$  of subgroups of  $G$ , let  $G/\mathcal{P}$  denote the set of all cosets  $gP$  with  $g \in G$  and  $P \in \mathcal{P}$ . A *group pair*  $(G, \mathcal{P})$  is a pair consisting of a finitely generated group  $G$  and a finite collection of **infinite** subgroups  $\mathcal{P}$ . The collection  $\mathcal{P}$  is called the *peripheral structure* of the pair  $(G, \mathcal{P})$ . When considering a group pair  $(G, \mathcal{P})$  we assume that  $G$  is endowed with a chosen word metric  $\text{dist}_G$ , while the Hausdorff distance between subsets of  $G$  is denoted as  $\text{hdist}_G$ .

**Definition 2.7** (Quasi-isometry of group pairs). Consider two group pairs  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$ . An  $(L, C, M)$ -quasi-isometry of group pairs  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a  $(L, C, M)$ -quasi-isometry of pairs  $q: (G, G/\mathcal{P}) \rightarrow (H, H/\mathcal{Q})$  in the sense of Definition 2.4.

Two group pairs  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are called *quasi-isometric* if there is a quasi-isometry of pairs between them. This is an equivalence relation on the class of group pairs.

Our convention is that any finitely generated group  $G$  is identified with the group pair  $(G, \emptyset)$ . Under this convention, the notion of quasi-isometric group pairs extends the notion of quasi-isometric finitely generated groups.

We also consider the stronger notion of virtually isomorphic pairs.

**Definition 2.8** (Virtually isomorphic pairs). Two group pairs are  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are *virtually isomorphic* if one can transform  $(G, \mathcal{P})$  to  $(H, \mathcal{Q})$  by a finite sequence of the following three operations and their inverses:

- (1) (Replace subgroups with commensurable subgroups) Replace  $\mathcal{P}$  with a finite collection  $\mathcal{Q}$  such that each  $Q \in \mathcal{Q}$  is commensurable to a conjugate of some  $P \in \mathcal{P}$  and viceversa.
- (2) (Take a quotient by a finite normal subgroup) Replace the pair  $(G, \mathcal{P})$  with  $(G/N, \{PN/N \mid P \in \mathcal{P}\})$ , where  $N$  is a finite normal subgroup of  $G$ .

- (3) (Pass to a finite index subgroup) Replace  $(G, \mathcal{P})$  with  $(H, \mathcal{Q})$ , where  $H$  is a finite index subgroup of  $G$  and  $\mathcal{Q}$  is defined as follows. If  $\{g_i P_i : i \in I\}$  is a collection of representatives of the orbits of the  $H$ -action on  $G/\mathcal{P}$  by left multiplication, then  $\mathcal{Q} = \{g_i P_i g_i^{-1} \cap H \mid i \in I\}$ .

By definition, virtual isomorphism of pairs is an equivalence relation in the class of group pairs. In the first item of the definition, the condition on  $\mathcal{Q}$  is equivalent to the requirement that the identity map on  $G$  is a quasi-isometry of pairs  $(G, \mathcal{P}) \rightarrow (G, \mathcal{Q})$ ; see Lemma 3.7. In the second item, it is straightforward to verify that the quotient map  $G \rightarrow G/N$  is a quasi-isometry of pairs  $(G, \mathcal{P}) \rightarrow (G/N, \{PN/N \mid P \in \mathcal{P}\})$ . In the third item, the inclusion  $H \hookrightarrow G$  is a quasi-isometry of pairs  $(H, \mathcal{Q}) \rightarrow (G, \mathcal{P})$ ; a verification of this fact can be found in [MPSSn22, Proposition 2.13]. Therefore, we have:

**Proposition 2.9.** *Virtually isomorphic pairs are quasi-isometric pairs.*

The notions of quasi-isometries and virtual isomorphisms of group pairs leads to the notion of a geometric property of a group pair.

**Definition 2.10** (Geometric properties). A property  $\mathbb{P}$  of a group pair is *geometric* if for every group pair  $(G, \mathcal{P})$  which is quasi-isometric to some pair  $(H, \mathcal{Q})$  satisfying  $\mathbb{P}$ , there exists a pair  $(K, \mathcal{L})$  satisfying  $\mathbb{P}$  that is virtually isomorphic to  $(G, \mathcal{P})$ .

Most (but not all) of the properties we consider in this paper are actually quasi-isometry invariants. Note that if a property is a quasi-isometry invariant of group pairs, then it is geometric, but the converse does not necessarily hold; see, for example, Theorem 3.21.

### 3. THE COSET SPACE

In this section, we introduce the *coset space* and use it to give some first examples of geometric properties of group pairs.

**Definition 3.1** (The coset space). Let  $(G, \mathcal{P})$  be a group pair. The *coset space* is the extended metric space  $(G/\mathcal{P}, \text{hdist}_G)$ . The action of  $G$  on  $(G/\mathcal{P}, \text{hdist}_G)$  by left multiplication is by isometries, and since  $\mathcal{P}$  is finite, there are only finitely many distinct  $G$ -orbits.

**Example 3.2.** Let  $A$  and  $B$  be infinite finitely generated groups.

- (1) In the coset space of  $(A * B, A)$ , any two distinct points are at infinite distance.
- (2) The coset space of  $(A \times B, A)$  is quasi-isometric to  $B$ .
- (3) The coset space of  $(\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2, P)$ , where  $P$  is the edge group of the splitting, is quasi-isometric to the Cayley graph of the free group of rank two.

**Lemma 3.3** (The coset space is uniformly locally finite). *For every  $r > 0$  there is a uniform bound on the cardinality of balls of radius  $r$  in  $(G/\mathcal{P}, \text{hdist}_G)$ .*

*Proof.* Since  $G$  is finitely generated,  $(G/\mathcal{P}, \text{hdist}_G)$  is locally finite. Since  $\mathcal{P}$  is finite, there are only finitely many  $G$ -orbits of points on  $G/\mathcal{P}$  and the statement follows.  $\square$

Proposition 2.5 and Lemma 3.3 immediately imply that the coset space is a quasi-isometry invariant of group pairs:

**Proposition 3.4.** *If  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a quasi-isometry of group pairs, then any induced  $\dot{q}: (G/\mathcal{P}, \text{hdist}_G) \rightarrow (H/\mathcal{Q}, \text{hdist}_H)$  is a uniformly finite-to-one quasi-isometry of extended metric spaces.*

**3.1. Bounded packing.** Our first application of the coset space is that bounded packing, originally defined in [HW09], is a geometric property of group pairs, proving Theorem 1.5(9).

**Definition 3.5** (Bounded packing). A group pair  $(G, \mathcal{P})$  has *bounded packing* if, for each constant  $D$ , there is a number  $N$  such that for any collection of  $N$  distinct cosets  $g_1P_1, \dots, g_NP_N$  in  $G/\mathcal{P}$ , at least two are separated by a distance of at least  $D$  in  $(G, \text{dist}_G)$ .

**Theorem 3.6.** *Bounded packing is a quasi-isometry invariant of group pairs.*

*Proof.* Suppose that  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a  $(L, C, M)$ -quasi-isometry of pairs and  $(H, \mathcal{Q})$  has bounded packing. Consider an induced function  $\dot{q}: G/\mathcal{P} \rightarrow H/\mathcal{Q}$ . By Proposition 2.5, there is  $K > 0$  such that  $\dot{q}^{-1}(hQ)$  has cardinality at most  $K$  for any  $hQ \in H/\mathcal{Q}$ .

Let  $D > 0$ . Since  $(H, \mathcal{Q})$  has bounded packing, there exists  $N' > 0$  such that if  $S' \subset H/\mathcal{Q}$  has the property that any two left cosets in  $S'$  are at infimum distance at most  $D' = LD + C + 2M$  in  $(H, \text{dist}_H)$  then  $S'$  has cardinality less than  $N'$ . Let  $N = N' \cdot K$ .

Suppose that  $\mathcal{S} \subset G/\mathcal{P}$  has the property that any two left cosets in  $\mathcal{S}$  are at distance at most  $D$ . Let  $\mathcal{S}' = \{\dot{q}(gP) \mid gP \in \mathcal{S}\}$ . Then any two left cosets of  $\mathcal{S}'$  are at distance at most  $D'$  and hence  $\mathcal{S}'$  has cardinality at most  $N'$ . Since  $\dot{q}$  is a  $K$ -to-one function, it follows that  $\mathcal{S} \subset \dot{q}^{-1}(\mathcal{S}')$  has at most  $N$  elements. Therefore  $(G, \mathcal{P})$  has bounded packing.  $\square$

**3.2. Reducible pairs.** Our second application of the coset space is a geometric proof that being *reducible* is invariant under quasi-isometry of group pairs. A different argument proving this result is implicit in [HMPSSn23]; see Proposition 6.3 and Lemmas 6.9 and 6.11 in that paper.

We begin with a discussion of the commensurator of a subgroup. The *commensurator*  $\text{Comm}_G(P)$  of a subgroup  $P$  of a group  $G$  is defined as

$$\text{Comm}_G(P) = \{g \in G \mid P \cap gPg^{-1} \text{ has finite index in } P \text{ and in } gPg^{-1}\}.$$

Observe that  $\text{Comm}_G(P)$  is a subgroup of  $G$ . Commensurability of subgroups has the following well-known geometric interpretation; for a proof, see, for example, [HMPSSn23, Lemma 5.14].

**Lemma 3.7.** *Let  $G$  be a finitely generated group with a word metric  $\text{dist}_G$ , let  $P$  and  $Q$  be subgroups, and let  $g \in G$ . Then  $P$  and  $gQg^{-1}$  are commensurable subgroups if and only if  $\text{hdist}_G(P, gQ) < \infty$ . In particular,*

$$\text{Comm}_G(P) = \{g \in G \mid \text{hdist}_G(P, gP) < \infty\}.$$

**Definition 3.8** (Reducible pair). A group pair  $(G, \mathcal{P})$  is *reducible* if every  $P \in \mathcal{P}$  has finite index in its commensurator  $\text{Comm}_G(P)$ .

The following theorem proves Theorem 1.5(8).

**Theorem 3.9.** *Being reducible is quasi-isometry invariant of group pairs.*

Our argument proceeds by showing that being reducible can be characterized geometrically in terms of the coset space. We use the notion of bounded galaxies.

**Definition 3.10** (Bounded galaxies). An extended metric space  $X$  is said to have *bounded galaxies* if for every  $x \in X$  the subspace  $B(x, \infty) = \{y \in X \mid \text{dist}_X(x, y) < \infty\}$  is bounded. We call  $B(x, \infty)$  the *galaxy containing  $x$* . Note that if  $B(x) \cap B(y)$  is non-empty then  $B(x) = B(y)$ .

**Lemma 3.11.** *Having bounded galaxies is a quasi-isometry invariant of extended metric spaces.*

*Proof.* Let  $f: X \rightarrow Y$  be a quasi-isometry of extended metric spaces and suppose that  $Y$  has bounded galaxies. Let  $x \in X$  and observe that  $f(B(x, \infty)) \subset B(f(x), \infty)$ . Since  $Y$  has bounded galaxies,  $B(f(x), \infty)$  is a bounded subset of  $Y$ . Since  $f$  is a quasi-isometry, it follows that  $B(x)$  is bounded.  $\square$

Using Lemma 3.7, we now give a geometric characterization of reducibility.

**Lemma 3.12.** *Let  $(G, \mathcal{P})$  be a group pair. The following statements are equivalent.*

- (1)  *$(G, \mathcal{P})$  is reducible.*
- (2) *For every  $Q \in G/\mathcal{P}$ , the set  $\{R \in G/\mathcal{P} \mid \text{hdist}_G(R, Q) < \infty\}$  is bounded.*
- (3) *There is constant  $N > 0$  such that for every  $Q \in G/\mathcal{P}$ , the set  $\{R \in G/\mathcal{P} \mid \text{hdist}_G(R, Q) < \infty\}$  has cardinality at most  $N$ .*

*Proof.* Since  $(G/\mathcal{P}, \text{hdist}_G)$  is locally finite, a subset of  $G/\mathcal{P}$  is bounded if and only if it is finite. Hence all the galaxies of  $(G/\mathcal{P}, \text{hdist}_G)$  are bounded if and only if all galaxies are finite. Since the  $G$ -action on  $(G/\mathcal{P}, \text{hdist}_G)$  has only finitely distinct  $G$ -orbits of points, there are finitely orbits of galaxies. Therefore all galaxies of  $(G/\mathcal{P}, \text{hdist}_G)$  are bounded if and only if there is  $N > 0$  such that every galaxy has at most  $N$  points. This proves the equivalence of (2) and (3).

Now we argue that (1) and (2) are equivalent. The coset space  $(G/\mathcal{P}, \text{hdist}_G)$  has a natural partition into a finite number of subspaces, namely,  $G/\mathcal{P} = \bigcup_{P \in \mathcal{P}} G/P$ . The intersection of a galaxy of  $G/\mathcal{P}$  with a subspace  $G/P$  is either a galaxy of  $G/P$  or is empty. Hence  $(G/\mathcal{P}, \text{hdist}_G)$  has bounded galaxies if and only if  $(G/P, \text{hdist}_G)$  has bounded galaxies for every  $P \in \mathcal{P}$ . To conclude that (1) and (2) are equivalent, observe that Lemma 3.7 implies that  $P$  has finite index in  $\text{Comm}_G(P)$  if and only if  $(G/P, \text{hdist}_G)$  has finite (and hence bounded) galaxies.  $\square$

We are now ready to prove Theorem 3.9.

*Proof of Theorem 3.9.* Since quasi-isometric group pairs have quasi-isometric coset spaces by Proposition 3.4, it suffices to prove that a group pair is reducible if and only if its coset space has bounded galaxies. This follows immediately from Lemma 3.12.  $\square$

**3.3. Reduced pairs.** In this section, we show that being a reduced pair is a geometric property of group pairs; see Theorem 3.19.

**Definition 3.13** (Reduced pair). A group pair  $(G, \mathcal{P})$  is *reduced* if  $P = \text{Comm}_G(P)$  for each  $P \in \mathcal{P}$  and no pair of subgroups in  $\mathcal{P}$  are conjugate.

Notice that a group pair  $(G, \mathcal{P})$  is reduced if for any  $P, Q \in \mathcal{P}$  and  $g \in G$ , if  $P$  and  $gQg^{-1}$  are commensurable subgroups then  $P = Q$  and  $g \in P$ .

**Example 3.14.** Let  $A$  and  $B$  be infinite cyclic groups, and let  $2A$  be the index two subgroup of  $A$ . Then the pairs  $(A * B, A)$  and  $(A * B, 2A)$  are quasi-isometric,  $(A * B, A)$  is a reduced pair, and  $(A * B, 2A)$  is reducible but is not reduced.

**Lemma 3.15.** *Let  $P$  and  $Q$  be subgroups of a finitely generated group  $G$ . If  $\text{hdist}_G(P, Q) < \infty$  then  $\text{Comm}_G(P) = \text{Comm}_G(Q)$ .*

*Proof.* Suppose  $\text{hdist}_G(P, Q) < \infty$ . Since  $\text{hdist}_G$  is invariant under multiplication on the left, the triangle inequality in  $(G/\mathcal{P}, \text{hdist}_G)$  implies that  $\text{hdist}_G(P, gP) < \infty$  if and only if  $\text{hdist}_G(Q, gQ) < \infty$  for every  $g \in G$ . The conclusion then follows directly from Lemma 3.7.  $\square$

Reduced pairs can be characterized geometrically as follows. We denote by  $A^g$  the conjugate  $gAg^{-1}$ .

**Proposition 3.16** (Reduced pair characterization). *A group pair is reduced if and only if the distance in the coset space only take the values zero and infinity.*

*Proof.* Suppose  $(G, \mathcal{P})$  is reduced. Let  $g_1P_1, g_2P_2 \in G/\mathcal{P}$  satisfy  $\text{hdist}_G(g_1P_1, g_2P_2)$  is finite. Lemma 3.15 implies that  $\text{Comm}_G(P_1^{g_1}) = \text{Comm}_G(P_2^{g_2})$ . Since  $(G, \mathcal{P})$  is reduced, we have that  $P_1^{g_1} = \text{Comm}_G(P_1^{g_1})$  and  $\text{Comm}_G(P_2^{g_2}) = P_2^{g_2}$ , and so  $g_1P_1g_1^{-1} = g_2P_2g_2^{-1}$ . Since  $(G, \mathcal{P})$  is reduced, no pair of subgroups in  $\mathcal{P}$  are conjugate, and hence  $P_1 = P_2$  and  $g_2^{-1}g_1 \in \text{Comm}_G(P_1) = P_1$ . Therefore  $g_1P_1 = g_2P_2$  and  $\text{hdist}_G(g_1P_1, g_2P_2) = 0$ .

Conversely, suppose that  $\text{hdist}_G$  only takes the values zero and infinity on  $G/\mathcal{P}$ . By Lemma 3.7, it follows that  $P = \text{Comm}_G(P)$  for every  $P \in \mathcal{P}$ . If  $P, Q \in \mathcal{P}$  and  $P = gQg^{-1}$  then  $\text{hdist}_G(P, gQ) < \infty$  and hence  $P = gQ$ . Hence  $P = Q$  and  $g \in P$ , and so  $(G, \mathcal{P})$  is reduced.  $\square$

To prove that being a reduced pair is a geometric property, we need the notion of a refinement.

**Definition 3.17** (Refinement  $(G, \mathcal{P}^*)$ ). A *refinement*  $\mathcal{P}^*$  of a collection of subgroups  $\mathcal{P}$  of  $G$  is a set of representatives of conjugacy classes of the collection of subgroups  $\{\text{Comm}_G(gPg^{-1}) : P \in \mathcal{P} \text{ and } g \in G\}$ .

It is straightforward from the definition of virtually isomorphic group pairs (see Definition 2.8) to see that if  $(G, \mathcal{P})$  is reducible, then it is virtually isomorphic to its refinement  $(G, \mathcal{P}^*)$ ; see [HMPS21, Proposition 6.3].

**Remark 3.18** (Refinements are reduced). Since  $\text{Comm}_G(\text{Comm}_G(P)) = \text{Comm}_G(P)$  and  $\text{Comm}_G(gPg^{-1}) = g\text{Comm}_G(P)g^{-1}$ , it follows that any refinement  $(G, \mathcal{P}^*)$  is reduced.

We now give an alternative proof of Theorem 1.5(7), originally proven in [HMPSSn23].

**Theorem 3.19.** *Being reduced is a geometric property of group pairs.*

*Proof.* First, observe that a reduced pair is reducible. Since being reducible is geometric by Theorem 3.9, a group pair quasi-isometric to a reduced pair is reducible. Finally, every reducible pair is virtually isomorphic to its refinement.  $\square$

Note that the virtual isomorphism in the proof is necessary. In particular, the property of being reduced is *not* a quasi-isometry invariant of group pairs; see Example 3.14.

**3.4. Malnormality.** We now show that almost malnormality is a geometric property of group pairs, proving Theorem 1.5(4).

**Definition 3.20** (Almost malnormal). A collection of subgroups  $\mathcal{Q}$  of a group  $G$  is *almost malnormal* if for any  $Q, Q' \in \mathcal{Q}$  and  $g \in G$ , either  $gQg^{-1} \cap Q'$  is finite or  $Q = Q'$  and  $g \in Q$ . A group pair  $(G, \mathcal{P})$  is called *almost malnormal* if  $\mathcal{P}$  is an almost malnormal collection in  $G$ .

**Theorem 3.21.** *Being almost malnormal is a geometric property of group pairs.*

*Proof.* Suppose that  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are quasi-isometric group pairs and  $(H, \mathcal{Q})$  is almost malnormal. Let  $(G, \mathcal{P}^*)$  be a refinement of  $(G, \mathcal{P})$ . By [HMPSSn23, Theorem D], we have that  $(G, \mathcal{P}^*)$  is almost malnormal. It remains to show that  $(G, \mathcal{P})$  is virtually isomorphic to  $(G, \mathcal{P}^*)$ . For this, note that  $(H, \mathcal{Q})$  is reducible, as it is almost malnormal. Thus Theorem 3.9 implies that  $(G, \mathcal{P})$  is reducible, as well, and so  $(G, \mathcal{P})$  is virtually isomorphic to  $(G, \mathcal{P}^*)$ .  $\square$

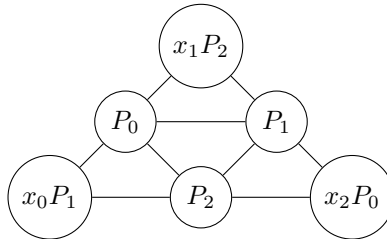
#### 4. THE COSET INTERSECTION COMPLEX

In this section, we introduce our primary object of study, the *coset intersection complex*  $\mathcal{K}(G, \mathcal{P})$  associated to a group pair  $(G, \mathcal{P})$ . In the first two subsections, we study geometric properties and the homotopy type of  $\mathcal{K}(G, \mathcal{P})$ , respectively. In the third subsection, we study the behavior of  $\mathcal{K}(G, \mathcal{P})$  under quasi-isometries of pairs.

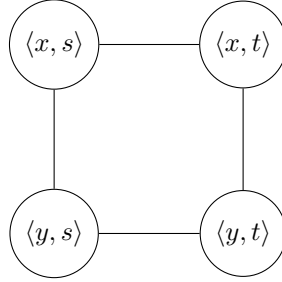
**Definition 4.1** (Coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ ). Let  $(G, \mathcal{P})$  be a group pair. The *coset intersection complex* is the simplicial complex with vertex set  $G/\mathcal{P}$  and such that a set  $\{g_1P_1, \dots, g_kP_k\} \subset G/\mathcal{P}$  defines a simplex if  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is an infinite subgroup of  $G$ .

Observe that the  $G$ -action on  $G/\mathcal{P}$  extends to a cellular  $G$ -action on  $\mathcal{K}(G, \mathcal{P})$  with finitely many  $G$ -orbits of 0-cells; however the action is not necessarily cocompact.

**Example 4.2.** (1) The group pair  $(G, \mathcal{P})$  is almost malnormal if and only if the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  is zero dimensional. This is Theorem 1.6(5).  
 (2) The coset intersection complex  $\mathcal{K}(\mathbb{Z}^2, \mathbb{Z})$  is the flag complex with 1-skeleton the complete graph with vertex set  $\mathbb{Z}$ . In particular,  $\mathcal{K}(\mathbb{Z}^2, \mathbb{Z})$  is an infinite-dimensional simplex, and the  $\mathbb{Z}^2$ -action on its 1-skeleton is not cocompact.  
 (3) Consider the free group  $G$  with free generating set  $\{x_0, x_1, x_2\}$  and the collection of subgroups  $\mathcal{P}$  consisting of  $P_i = \langle x_i, x_{(i+1) \bmod 3} \rangle$  for  $i = 0, 1, 2$ . In this case,  $\mathcal{K}(G, \mathcal{P})$  is cocompact connected  $G$ -graph. The figure below illustrates a part of this graph:



- (4) If  $Q$  is a quasi-convex subgroup of a hyperbolic group  $G$ , then  $\mathcal{K}(G, Q)$  is a  $G$ -cocompact simplicial complex. In particular  $\mathcal{K}(G, Q)$  is finite dimensional. This follows from an argument of [GMRS98]; see Example 5.11 for details.
- (5) If  $G = BS(1, k) = \langle a, t \mid tat^{-1} = a^k \rangle$  is a solvable Baumslag-Solitar group, then  $\mathcal{K}(G, \langle t \rangle)$  is an infinite edgeless graph.
- (6) Consider the group  $G = F_2 \times \mathbb{Z} = \langle x, y, t \mid [x, t], [y, t] \rangle$  and the collection of subgroups  $\mathcal{P} = \{\langle x, t \rangle, \langle y, t \rangle\}$ . Since any conjugate of a subgroup in  $\mathcal{P}$  contains the subgroup  $\langle t \rangle$ , the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  is the flag complex with 1-skeleton the complete graph with vertex set  $G/\mathcal{P}$ . In particular,  $\mathcal{K}(G, \mathcal{P})$  is an infinite-dimensional simplex.
- (7) The coset intersection complex  $\mathcal{K}$  of a right-angled Artin group  $A(\Gamma)$  with respect to standard flats is infinite dimensional if the defining graph  $\Gamma$  contains a 2-path as the previous example illustrates. However, in general,  $\mathcal{K}$  is not an infinite dimensional simplex. Consider  $F_2 \times F_2 = \langle x, y, s, t \mid [x, s], [x, t], [y, s], [y, t] \rangle$  and the collection of subgroups  $\mathcal{P} = \{\langle x, s \rangle, \langle x, t \rangle, \langle y, s \rangle, \langle y, t \rangle\}$ . Then  $\mathcal{K}(G, \mathcal{P})$  contains the 4-cycle in the figure



and the diagonals are not part of the complex. We expect that if  $\Gamma$  is not simply-connected, then neither is  $\mathcal{K}$ .

- (8) For any atomic right-angled Artin group, its coset intersection complex with respect to maximal standard flats is connected and quasi-isometric to the modified Deligne complex of Charney and Davis [CD95] and to the flat space of Bestvina, Kleiner and Sageev [BKS08]. See Section 6 for details.

**4.1. Simplicial maps between coset intersection complexes.** In this section, we illustrate the geometric nature of the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ . We continue to use the notation of Proposition 2.5, where it was shown that a quasi-isometry  $q$  of pairs induces a quasi-isometry  $\hat{q}$  of coset spaces. Recall that a *simplicial map*  $f: X \rightarrow Y$  is a function between the sets of vertices of the simplicial complexes  $X$  and  $Y$  with the property that the images of the vertices of a simplex always span a simplex.

**Proposition 4.3.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs. Then any induced quasi-isometry  $\hat{q}: (G/\mathcal{P}, \text{hdist}_G) \rightarrow (H/\mathcal{Q}, \text{hdist}_H)$  is simplicial map  $\hat{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$ . Moreover,  $\hat{q}: \mathcal{K}(G, \mathcal{P})^{(1)} \rightarrow \mathcal{K}(H, \mathcal{Q})^{(1)}$  is also a quasi-isometry.*

The proof of the proposition relies on some well-known facts summarized in the lemmas below. The first lemma has appeared in a number of articles, see, for example, [MSW11, Lemma 2.2], [MP09, Lemma 4.7], [Hru10, Proposition 9.41]

and [HW09, Corollary 4.6]. For a subset  $A \subset G$ , let  $\mathcal{N}_r(A)$  denote the closed  $r$ -neighborhood of  $A$  in  $(G, \text{dist}_G)$ .

**Lemma 4.4** (Fundamental lemma). *Let  $G$  be a finitely generated group with word metric  $\text{dist}_G$ , and let  $g_1P_1, \dots, g_kP_k$  be arbitrary left cosets of subgroups of  $G$ . For any  $r > 0$ , there exists  $R > 0$  such that*

$$\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i) \subseteq \mathcal{N}_R\left(\bigcap_{i=1}^k g_iP_i g_i^{-1}\right).$$

The fundamental lemma provides the following geometric interpretation of simplices of the coset intersection complex.

**Lemma 4.5** ( $\mathcal{K}(G, \mathcal{P})$  is geometric). *For a group pair  $(G, \mathcal{P})$  and any finite subset  $\{g_1P_1, g_2P_2, \dots, g_kP_k\}$  of  $G/\mathcal{P}$ , the following statements are equivalent.*

- (1) *The group  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is infinite.*
- (2) *The set  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  is unbounded in  $(G, \text{dist}_G)$  for some  $r > 0$ .*

*Proof.* Observe that the subgroup  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is a subset of  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  for  $r > \max_i \{\text{dist}_G(e, g_i)\}$ . Since  $(G, \text{dist}_G)$  is a locally finite metric space, if  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is infinite then is unbounded, and hence  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  is unbounded as well.

Conversely, suppose  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  is unbounded. By Lemma 4.4, there exists  $R > 0$  such that  $\mathcal{N}_R\left(\bigcap_{i=1}^k g_iP_i g_i^{-1}\right)$  is unbounded. Since  $(G, \text{dist}_G)$  is locally finite, it follows that  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is an infinite subgroup.  $\square$

The next lemma relates intersections of conjugates of a subgroup and the corresponding intersections in the image under an induced quasi-isometry  $\dot{q}$ .

**Lemma 4.6.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs, and let  $\dot{q}$  be an induced quasi-isometry of coset spaces. Let  $g_iP_i \in G/\mathcal{P}$  and  $h_iQ_i = \dot{q}(g_iP_i)$  for  $i = 1, \dots, k$ . Then  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is infinite if and only if  $\bigcap_{i=1}^k h_iQ_i h_i^{-1}$  is infinite.*

*Proof.* Since  $q: (G, \text{dist}_G) \rightarrow (H, \text{dist}_H)$  is a quasi-isometry, the set  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  is unbounded for some  $r$  if and only if  $\bigcap_{i=1}^k \mathcal{N}_{r'}(q(g_iP_i))$  is unbounded for some  $r'$ . Since  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a quasi-isometry of pairs, the Hausdorff distance  $\text{hdist}_H(h_iQ_i, q(g_iP_i))$  is uniformly bounded by Proposition 2.5. Hence  $\bigcap_{i=1}^k \mathcal{N}_r(g_iP_i)$  is unbounded for some  $r$  if and only if  $\bigcap_{i=1}^k \mathcal{N}_{r'}(h_iQ_i)$  is unbounded for some  $r'$ . Therefore, it follows from Lemma 4.5 that  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is infinite if and only if  $\bigcap_{i=1}^k h_iQ_i h_i^{-1}$  is infinite.  $\square$

The next lemma shows that when the induced map  $\dot{q}$  is a simplicial map, it restricts to a quasi-isometry on the 1-skeleton.

**Lemma 4.7.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs with quasi-inverse  $p: (H, \mathcal{Q}) \rightarrow (G, \mathcal{P})$ . If  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  and  $\dot{p}: \mathcal{K}(H, \mathcal{Q}) \rightarrow \mathcal{K}(G, \mathcal{P})$  are simplicial maps, then they restrict to quasi-isometries of the 1-skeletons of the complexes.*

*Proof.* Since  $\dot{q}$  and  $\dot{p}$  are simplicial maps, they are also Lipschitz maps with respect to the combinatorial metric on 1-skeleta. Let  $r = p \circ q$ . We will show that the simplicial map  $\dot{r}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P})$  is at bounded distance from the identity map  $\text{id}$  on  $\mathcal{K}(G, \mathcal{P})$  and hence, by symmetry,  $\dot{p}$  and  $\dot{q}$  are quasi-isometries.

Since  $p$  is a quasi-inverse of  $q$ , there is a constant  $D$  such that  $r = p \circ q$  is at distance at most  $D$  from the identity map on  $G$ , and hence

$$\text{hdist}_G(\dot{r}(gP), \dot{\mathbb{I}}(gP)) = \text{hdist}_G(\dot{r}(gP), gP) \leq D < \infty.$$

In particular,  $\dot{r}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P})$  satisfies that for every vertex  $gP$  of  $\mathcal{K}(G, \mathcal{P})$ , either  $gP$  and  $\dot{r}(gP)$  are the same vertex, or  $\{gP, \dot{r}(gP)\}$  is a 1-cell of  $\mathcal{K}(G, \mathcal{P})$ . This implies that the maps  $\dot{r}$  and  $\dot{\mathbb{I}}$  are at distance at most one.  $\square$

We are now ready to prove Proposition 4.3.

*Proof of Proposition 4.3.* Let  $g_i P_i \in G/\mathcal{P}$  and  $h_i Q_i = \dot{q}(g_i P_i)$  for  $i = 1, \dots, k$ . Suppose that  $\{g_1 P_1, \dots, g_k P_k\}$  is a simplex of  $\mathcal{K}(G, \mathcal{P})$ . By definition, the intersection  $\bigcap_{i=1}^k g_i P_i g_i^{-1}$  is infinite. Lemma 4.6 implies that  $\bigcap_{i=1}^k h_i Q_i h_i^{-1}$  is infinite, and hence  $\{h_1 Q_1, \dots, h_k Q_k\}$  is a simplex of  $\mathcal{K}(H, \mathcal{Q})$ . Therefore  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is a simplicial map, and, by Lemma 4.7, it restricts to a quasi-isometry  $\dot{q}: \mathcal{K}(G, \mathcal{P})^{(1)} \rightarrow \mathcal{K}(H, \mathcal{Q})^{(1)}$ .  $\square$

Considering  $\mathcal{K}(G, \mathcal{P})$  as a regular piecewise Euclidean complex, that is, one in which each simplex is Euclidean and has unit edge length, it admits a natural length metric; see [BH99, Definition I.7.4 & Corollary I.7.10].

**Corollary 4.8.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs. Any induced map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is a quasi-isometry of regular piecewise Euclidean complexes.*

*Proof.* Each simplex  $\Delta$  has uniformly bounded diameter, independent of the dimension of  $\Delta$ , and so each  $\Delta$  is uniformly quasi-isometric to its 1-skeleton. It is straightforward to verify that these quasi-isometries extend to a quasi-isometry  $\mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P})^{(1)}$  using the length metric above. Combining this with Proposition 4.3 completes the proof.  $\square$

**4.2. Isomorphisms between coset intersection complexes.** Recall that a pair  $(G, \mathcal{P})$  is reduced if  $P = \text{Comm}_G(P)$  for each  $P \in \mathcal{P}$  and no pair of subgroups in  $\mathcal{P}$  are conjugate; see Definition 3.13. In our context, the relevance of reduced group pairs is that quasi-isometries between reduced group pairs induce isomorphisms of the corresponding coset intersection complexes as the following proposition shows.

**Proposition 4.9.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs.*

- (1) *If  $(H, \mathcal{Q})$  is reduced, then the induced simplicial map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is surjective, and its restriction to the zero-skeleton is uniformly finite-to-one.*
- (2) *If  $(G, \mathcal{P})$  is reduced, then  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is an embedding of simplicial complexes.*
- (3) *If  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are both reduced, then  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is an isomorphism of simplicial complexes.*

*Proof.* Let  $p: (H, \mathcal{Q}) \rightarrow (G, \mathcal{P})$  be a quasi-inverse of  $q$ . By Proposition 2.5, the induced maps  $\dot{q}: (G/\mathcal{P}, \text{hdist}_G) \rightarrow (H/\mathcal{Q}, \text{hdist}_H)$  and  $\dot{p}: (H/\mathcal{Q}, \text{hdist}_H) \rightarrow (G/\mathcal{P}, \text{hdist}_G)$  are quasi-isometries and quasi-inverses. Moreover,  $(G/\mathcal{P}, \text{hdist}_G)$  and  $(H/\mathcal{Q}, \text{hdist}_H)$  are uniformly locally finite by Lemma 3.3, and so  $\dot{p}$  and  $\dot{q}$  are uniformly finite-to-one functions.

If  $(H, \mathcal{Q})$  is reduced, then any two distinct points of  $(H/\mathcal{Q}, \text{hdist}_H)$  are at infinite distance. Since  $\dot{q} \circ \dot{p}: (H/\mathcal{Q}, \text{hdist}_H) \rightarrow (H/\mathcal{Q}, \text{hdist}_H)$  is a quasi-isometry, it follows that  $\dot{q} \circ \dot{p}: (H/\mathcal{Q}, \text{hdist}_H) \rightarrow (H/\mathcal{Q}, \text{hdist}_H)$  has to be a bijection. In fact,  $\dot{q} \circ \dot{p}$  must be the identity, since  $\text{hdist}_G(\dot{q} \circ \dot{p}(hQ), hQ) < \infty$  implies  $\dot{q} \circ \dot{p}(hQ) = hQ$ . On the other hand, by Proposition 4.3, we have that  $\dot{q}$  and  $\dot{p}$  are simplicial maps  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  and  $\dot{p}: \mathcal{K}(H, \mathcal{Q}) \rightarrow \mathcal{K}(G, \mathcal{P})$ . Hence  $\dot{q} \circ \dot{p}: \mathcal{K}(H, \mathcal{Q}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is a simplicial map which is the identity at the level of the zero skeleton, and therefore  $\dot{q} \circ \dot{p}$  is the identity map on  $\mathcal{K}(H, \mathcal{Q})$ . This implies that  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is surjective and  $\dot{p}: \mathcal{K}(H, \mathcal{Q}) \rightarrow \mathcal{K}(G, \mathcal{P})$  is an embedding. This completes the proof of the first two statements; the third statement is immediate.  $\square$

**4.3. Homotopy equivalences between coset intersection complexes.** In this section, we prove that a quasi-isometry of pairs induces a homotopy equivalence of coset intersection complexes. This, along with Corollary 4.8, proves Theorem 1.4.

**Theorem 4.10.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs. An induced simplicial map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  is a homotopy equivalence. Moreover, any two such induced simplicial maps are homotopic.*

We begin with a few general lemmas about simplicial complexes. For a finite set  $X$ , let  $\Delta(X)$  denote the simplex with vertex set  $X$ . Abusing notation, we will not distinguish the abstract simplex  $\Delta(X)$  and its topological realization. In particular, every point of  $\Delta(X)$  is a formal sum  $\sum_{x \in X} \alpha_x x$  where  $\alpha_x \in [0, 1]$  and  $\sum_x \alpha_x = 1$ .

**Lemma 4.11.** *Let  $\mathcal{L}$  be a subcomplex of a simplicial complex  $\mathcal{K}$ . Let  $r: \mathcal{K}^{(0)} \rightarrow \mathcal{L}^{(0)}$  be a function such that*

- (1)  $r(x) = x$  for all  $x \in \mathcal{L}^{(0)}$ , and
- (2) *for every simplex  $\Delta(X)$  of  $\mathcal{K}$ ,  $\Delta(X \cup r(X))$  is a simplex of  $\mathcal{K}$ .*

*Then  $r$  extends to a deformation retraction  $r: \mathcal{K} \times I \rightarrow \mathcal{L}$  such that its restriction to any simplex  $\Delta(X)$  of  $\mathcal{K}$  is given by*

$$(1) \quad r\left(\sum_{x \in X} \alpha_x x, t\right) = \sum_{x \in X \cap \mathcal{L}} \alpha_x x + \sum_{x \in X - \mathcal{L}} \alpha_x (tr(x) + (1-t)x)$$

*Sketch of the proof.* Every simplex  $\Delta(X)$  of  $\mathcal{K}$  is a face of the simplex  $\Delta(X \cup r(X))$  of  $\mathcal{K}$  by the second assumption. Hence the right hand side of (1) defines a point of  $\mathcal{K}$ . Then, using the first assumption, it is an observation that (1) defines a deformation retraction of  $\mathcal{K}$  onto  $\mathcal{L}$ .  $\square$

The next lemma gives a sufficient condition for two simplicial maps to be homotopic.

**Lemma 4.12.** *Let  $\mathcal{K}$  and  $\mathcal{K}'$  be simplicial complexes, and let  $r: \mathcal{K} \rightarrow \mathcal{K}'$  and  $s: \mathcal{K} \rightarrow \mathcal{K}'$  be simplicial maps. Suppose that for any finite-dimensional simplex  $\Delta(X)$  of  $\mathcal{K}$ ,  $\Delta(r(X) \cup s(X))$  is a simplex of  $\mathcal{K}'$ . Then  $r$  and  $s$  are homotopic.*

*Proof.* Define a homotopy  $F: \mathcal{K} \times I \rightarrow \mathcal{K}'$  such that its restriction to any finite-dimensional simplex  $\Delta(X)$  of  $\mathcal{K}$  is given by

$$F\left(\sum_{x \in X} \alpha_x x, t\right) = \sum_{x \in X} \alpha_x (ts(x) + (1-t)r(x)).$$

The right-hand side is well-defined by the assumption that  $\Delta(r(X) \cup s(X))$  is a simplex of  $\mathcal{K}'$ . It is clearly continuous on every finite subcomplex, and so by the weak topology it is continuous. Hence  $F$  is a homotopy from  $r$  to  $s$ .  $\square$

We now apply Lemma 4.12 to two induced maps of coset intersection complexes.

**Lemma 4.13.** *Let  $p$  and  $q$  be quasi-isometries of pairs  $(G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ , and let  $\dot{p}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  and  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  be the corresponding induced simplicial maps. If  $\text{hdist}_H(\dot{p}(gP), \dot{q}(gP)) < \infty$  for every  $gP \in G/\mathcal{P}$ , then  $\dot{p}$  and  $\dot{q}$  are homotopic.*

*Proof.* It is enough to verify the hypothesis of Lemma 4.12 for  $\dot{p}$  and  $\dot{q}$ . Consider a finite-dimensional simplex  $\{g_1P_1, \dots, g_kP_k\}$  of  $\mathcal{K}(G, \mathcal{P})$ . We will show that  $\{\dot{p}(g_iP_i) \mid 1 \leq i \leq k\} \cup \{\dot{q}(g_iP_i) \mid 1 \leq i \leq k\}$  is a simplex of  $\mathcal{K}(H, \mathcal{Q})$ . By assumption  $\text{hdist}_H(\dot{p}(g_iP_i), \dot{q}(g_iP_i))$  is finite and so there exists  $r > 0$  such that  $\dot{p}(g_iP_i) \subset \mathcal{N}_r(\dot{q}(g_iP_i))$  for  $1 \leq i \leq k$ . It follows that

$$\bigcap_{i=1}^k \mathcal{N}_r(\dot{p}(g_iP_i)) \subset \bigcap_{i=1}^k \mathcal{N}_{2r}(\dot{q}(g_iP_i)),$$

and hence

$$\bigcap_{i=1}^k \mathcal{N}_r(\dot{p}(g_iP_i)) \subset \bigcap_{i=1}^k \mathcal{N}_{2r}(\dot{q}(g_iP_i)) \cap \mathcal{N}_{2r}(\dot{p}(g_iP_i)).$$

Since  $\{g_1P_1, \dots, g_kP_k\}$  is a simplex of  $\mathcal{K}(G, \mathcal{P})$  and  $\dot{p}$  is simplicial, we have that  $\{\dot{p}(g_1P_1), \dots, \dot{p}(g_kP_k)\}$  is a simplex of  $\mathcal{K}(H, \mathcal{Q})$ . By increasing  $r$  if necessary, Lemma 4.5 implies that  $\bigcap_{i=1}^k \mathcal{N}_r(\dot{p}(g_iP_i))$  is unbounded in  $(H, \text{dist}_H)$ , and therefore  $\bigcap_{i=1}^k \mathcal{N}_{2r}(\dot{q}(g_iP_i)) \cap \mathcal{N}_{2r}(\dot{p}(g_iP_i))$  is unbounded in  $(G, \text{dist})$ . By Lemma 4.5,  $\{\dot{p}(g_iP_i) \mid 1 \leq i \leq k\} \cup \{\dot{q}(g_iP_i) \mid 1 \leq i \leq k\}$  is therefore a simplex of  $\mathcal{K}(H, \mathcal{Q})$ , which completes the proof.  $\square$

**Example 4.14.** Let  $q: (G, \mathcal{P}) \rightarrow (G, \mathcal{P})$  be a self-quasi-isometry of pairs. Clearly, the identity map  $(G, \mathcal{P}) \rightarrow (G, \mathcal{P})$  is also a self-quasi-isometry of pairs. However, the induced simplicial map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P})$  is not necessarily homotopic to the identity map on  $\mathcal{K}(G, \mathcal{P})$ . Consider the case of a free group of rank two  $G = \langle x, y \rangle$ , where  $\mathcal{P} = \{\langle x \rangle\}$  and  $q: G \rightarrow G$  is given by  $q(g) = yg$ . In this case  $q$  is a self-quasi-isometry of pairs  $q: (G, \mathcal{P}) \rightarrow (G, \mathcal{P})$ ; in fact, it is an isometry of  $G$ . The complex  $\mathcal{K}(G, \mathcal{P})$  is an infinite discrete set, and so a map homotopic to the identity must be the identity. The induced  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P})$ , however, is given by  $\dot{q}(g\langle x \rangle) = yg\langle x \rangle$ , which is not the identity.

**Proposition 4.15.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  and  $p: (H, \mathcal{Q}) \rightarrow (K, \mathcal{R})$  be quasi-isometries of pairs and let  $r = p \circ q$ . Let  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$ ,  $\dot{p}: \mathcal{K}(H, \mathcal{Q}) \rightarrow \mathcal{K}(K, \mathcal{R})$  and  $\dot{r}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(K, \mathcal{R})$  be corresponding induced simplicial maps. Then  $\dot{r}$  and  $\dot{p} \circ \dot{q}$  are homotopic.*

*Proof.* It is enough to verify the hypothesis of Lemma 4.13 for  $\dot{p} \circ \dot{q}$  and  $\dot{r}$ . Let  $A \in G/\mathcal{P}$ . We show below that  $\text{hdist}_K(\dot{r}(A), \dot{p} \circ \dot{q}(A)) < \infty$ .

Suppose that  $p$  is an  $(L, C, M)$ -quasi-isometry of pairs. The definition of  $\dot{p}$  and  $\dot{q}$  implies

$$\begin{aligned}
 \text{hdist}_K(\dot{p} \circ \dot{q}(A), p \circ q(A)) &\leq \\
 (2) \quad &\leq \text{hdist}_K(\dot{p} \circ \dot{q}(A), p(\dot{q}(A))) + \text{hdist}_K(p(\dot{q}(A)), p \circ q(A)) \\
 &\leq \text{hdist}_K(\dot{p} \circ \dot{q}(A), p(\dot{q}(A))) + L \text{hdist}_H(\dot{q}(A), q(A)) + C \\
 &< \infty
 \end{aligned}$$

Analogously,

$$(3) \quad \text{hdist}_K(\dot{r}(A), r(A)) < \infty.$$

Combining (2), (3), and the fact that  $r = p \circ q$  yields

$$\text{hdist}_K(\dot{r}(A), \dot{p} \circ \dot{q}(A)) \leq \text{hdist}_K(\dot{r}(A), r(A)) + \text{hdist}_K(p \circ q(A), \dot{p} \circ \dot{q}(A)) < \infty,$$

which concludes the proof.  $\square$

We now turn to proof of the main result of this section, that the induced simplicial map is a homotopy equivalence.

*Proof of Theorem 4.10.* Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs. A direct consequence of Lemma 4.13 is that all choices for the induced simplicial map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  are homotopic.

Let  $p: (H, \mathcal{Q}) \rightarrow (G, \mathcal{P})$  be a quasi-inverse of  $q$ . We need to prove that  $\dot{p} \circ \dot{q}$  and  $\dot{q} \circ \dot{p}$  are homotopic to the identity maps on  $\mathcal{K}(G, \mathcal{P})$  and  $\mathcal{K}(H, \mathcal{Q})$  respectively. The arguments are analogous, so we only prove the statement for  $\dot{p} \circ \dot{q}$ .

Let  $\mathbb{I}$  denote the identity map on  $G$ . Note that we can assume that  $\mathbb{I}$  is the identity map on  $\mathcal{K}(G, \mathcal{P})$ . Let  $r = p \circ q$ , and suppose that it is an  $(L, C, M)$ -quasi-isometry of pairs. By Proposition 4.15, the induced maps  $\dot{r}$  and  $\dot{p} \circ \dot{q}$  are homotopic. Hence it is enough to show that  $\dot{r}$  is homotopic to the identity map  $\mathbb{I}$  on  $\mathcal{K}(G, \mathcal{P})$ .

Since  $p$  is a quasi-inverse of  $q$ , there is a constant  $D$  such that  $r = p \circ q$  is at distance at most  $D$  from the identity map  $\mathbb{I}$  in the  $L^\infty$ -metric; that is,

$$(4) \quad \text{dist}_G(r(x), \mathbb{I}_G(x)) = \text{dist}_G(r(x), x) \leq D$$

for all  $x \in G$ . Then we have that

$$\begin{aligned}
 \text{hdist}_G(\dot{r}(gP), \dot{\mathbb{I}}(gP)) &= \text{hdist}_G(\dot{r}(gP), gP) \\
 (5) \quad &\leq \text{hdist}_G(\dot{r}(gP), r(gP)) + \text{hdist}_G(r(gP), gP) \\
 &\leq M + D < \infty.
 \end{aligned}$$

This verifies the hypothesis of Lemma 4.13 for  $r$  and  $\mathbb{I}$ , and hence  $\dot{r}$  is homotopic to the identity map on  $\mathcal{K}(G, \mathcal{P})$ .  $\square$

## 5. GEOMETRIC PROPERTIES OF GROUP PAIRS

In this section, we use the coset intersection complex to prove that several algebraic properties are quasi-isometry invariants of group pairs. We discuss finite height and width in Section 5.1. We next consider two conditions which we call *finite packing* and having a *commensurated core* in Sections 5.2 and 5.3, respectively. Finally, we discuss thickness in Section 5.5.

**5.1. Height and Width.** We first show that finite height and finite width, properties defined by Gitik, Mj, Rips, and Sageev in [GMR98], are quasi-isometry invariants of group pairs.

**Definition 5.1** (Height and width of subgroups). Let  $\mathcal{P}$  be a collection of subgroups of a group  $G$ .

- (1) The *height* of  $\mathcal{P}$  in  $G$  is the maximal  $m \in \mathbb{N}$  such that there exists a collection of  $m$  distinct cosets  $\{g_1P_1, \dots, g_mP_m\} \subset G/\mathcal{P}$  whose intersection  $\bigcap_{i=1}^m g_iP_i g_i^{-1}$  is infinite.
- (2) The *width* of  $\mathcal{P}$  in  $G$  is the maximal  $m \in \mathbb{N}$  such that there exists a collection of  $m$  distinct cosets  $\{g_1P_1, \dots, g_mP_m\} \subset G/\mathcal{P}$  whose pairwise intersections  $g_iP_i g_i^{-1} \cap g_jP_j g_j^{-1}$  are infinite for all  $1 \leq i < j \leq m$ .

Note that finite width implies finite height. We say that the pair  $(G, \mathcal{P})$  has finite height (resp. width) if  $\mathcal{P}$  has finite height (resp. width) in  $G$ .

Recall that a collection distinct cosets  $\{g_1P_1, \dots, g_mP_m\} \subset G/\mathcal{P}$  spans a simplex in  $\mathcal{K}(G, \mathcal{P})$  if and only if  $\bigcap_{i=1}^m g_iP_i g_i^{-1}$  is infinite. We immediately obtain the following lemma, which is Theorem 1.6(1) and (2).

**Lemma 5.2.** *Let  $(G, \mathcal{P})$  be a group pair.*

- (1)  $\mathcal{P}$  has height  $m \in \mathbb{Z}_+$  if and only if  $\mathcal{K}(G, \mathcal{P})$  has dimension  $m$ .
- (2)  $\mathcal{P}$  has width  $m \in \mathbb{Z}_+$  if and only if the largest clique in the 1-skeleton of  $\mathcal{K}(G, \mathcal{P})$  has  $m$  vertices.

The main goal of this section is to prove Theorem 1.5(1) and (2):

**Theorem 5.3.** *Having finite width (resp. finite height) is a quasi-isometry invariant of group pairs.*

The proof of the theorem relies on a sequence of lemmas.

**Lemma 5.4.** *A group pair with finite height is reducible.*

*Proof.* If  $(G, \mathcal{P})$  has finite height, then  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional. If  $P$  has infinite index in  $\text{Comm}_G(P)$ , then  $\mathcal{K}(G, \mathcal{P})$  has simplices of arbitrarily large dimension. Hence every  $P \in \mathcal{P}$  has finite index in its commensurator, and so  $(G, \mathcal{P})$  is reducible.  $\square$

**Lemma 5.5.** *Suppose  $(G, \mathcal{P})$  is reducible and let  $(G, \mathcal{P}^*)$  be a refinement. Then  $(G, \mathcal{P})$  has finite height (resp. finite width) if and only if  $(G, \mathcal{P}^*)$  has finite height (resp. finite width).*

*Proof.* Since  $(G, \mathcal{P})$  is reducible, [HMP21, Proposition 6.3] implies that the identity map on  $G$  is a quasi-isometry of pairs  $(G, \mathcal{P}) \rightarrow (G, \mathcal{P}^*)$ . Since  $(G, \mathcal{P}^*)$  is reduced, Proposition 4.9 implies that there is a simplicial embedding  $\hat{p}: \mathcal{K}(G, \mathcal{P}^*) \rightarrow \mathcal{K}(G, \mathcal{P})$ , and a surjective simplicial map  $\hat{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(G, \mathcal{P}^*)$  that is uniformly finite-to-one at the level of zero skeletons. Suppose that  $\hat{q}^{-1}(gP)$  has cardinality at most  $n$  for every  $gP \in G/\mathcal{P}$ .

Using the embedding  $\hat{p}$ , we observe that the maximal number of vertices of a clique in  $\mathcal{K}(G, \mathcal{P})$  is an upper bound on for the maximal number of vertices of a clique in  $\mathcal{K}(G, \mathcal{P}^*)$ . Hence if  $(G, \mathcal{P})$  has finite width then  $(G, \mathcal{P}^*)$  has finite width. The argument for finite height is analogous, replacing “number of vertices in a clique” with “dimension of a simplex.”

For the other implication, observe that  $\dot{q}$  maps cliques to cliques. Suppose that  $d$  is an upper bound for the maximal number of vertices of a clique in  $\mathcal{K}(G, \mathcal{P}^*)$ . Note that if a clique of  $\mathcal{K}(G, \mathcal{P})$  has  $m$  vertices, then its image in  $\mathcal{K}(G, \mathcal{P}^*)$  is a clique with at least  $m/n$  vertices. Since  $m/n \leq d$ , it follows that  $m \leq d \cdot n$ . Hence if  $(G, \mathcal{P}^*)$  has finite width, then  $(G, \mathcal{P})$  has finite width. The argument for height is again completely analogous.  $\square$

**Lemma 5.6.** *Let  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  be quasi-isometric pairs. Suppose that  $(G, \mathcal{P})$  is reducible. Then  $(G, \mathcal{P})$  has finite height (resp. finite width) if and only if  $(H, \mathcal{Q})$  has finite height (resp. finite width).*

*Proof.* By Theorem 3.19, being reducible is a quasi-isometric invariant, and hence  $(H, \mathcal{Q})$  is reducible. Let  $(G, \mathcal{P}^*)$  and  $(H, \mathcal{Q}^*)$  be refinements of  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  respectively. Then [HMPS21, Proposition 6.3] implies that the pairs  $(G, \mathcal{P})$ ,  $(H, \mathcal{Q})$ ,  $(G, \mathcal{P}^*)$  and  $(H, \mathcal{Q}^*)$  are all quasi-isometric pairs. By Remark 3.18,  $(G, \mathcal{P}^*)$  and  $(H, \mathcal{Q}^*)$  are quasi-isometric reduced pairs. Then Proposition 4.9 implies that the coset intersection complexes  $\mathcal{K}(G, \mathcal{P}^*)$  and  $\mathcal{K}(H, \mathcal{Q}^*)$  are isomorphic. Therefore  $(G, \mathcal{P}^*)$  has finite height (resp. width) if and only if  $(H, \mathcal{Q}^*)$  has finite height (resp. width). Then the conclusion follows from Lemma 5.5.  $\square$

We are now ready to prove Theorem 5.3.

*Proof of Theorem 5.3.* Suppose that  $(G, \mathcal{P})$  and  $(H, \mathcal{Q})$  are a quasi-isometric pairs and that  $(G, \mathcal{P})$  has finite height or finite width. Since finite width implies finite height, in either case,  $(G, \mathcal{P})$  has finite height. Lemma 5.4 then implies that  $(G, \mathcal{P})$  is reducible, and the conclusion of the theorem follows directly from Lemma 5.6.  $\square$

Huang and Wise introduced the notion of *finite stature* for a group pair  $(G, \mathcal{P})$ , a property closely related to finite height [HW19]. To encode stature using the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ , one needs to record information not just about the simplices, as was the case for height, but also about the action of elements of  $\mathcal{P}$  on  $\mathcal{K}(G, \mathcal{P})$ . Additional assumptions would be needed to apply the methods of this paper to show that finite stature is geometric.

**Question 5.7.** Is having finite stature a geometric property of group pairs?

**5.2. Finite packing (cocompactness).** In this subsection, we introduce the notion of finite packing.

**Definition 5.8** (Finite packing). A collection  $\mathcal{P}$  of subgroups of a group  $G$  has *finite packing* if there is a finite subset of left cosets  $D$  in  $G/\mathcal{P}$  such that for any finite subset  $g_1 P_1, \dots, g_d P_d$  of  $G/\mathcal{P}$ , if  $\bigcap_{i=1}^d g_i P_i g_i^{-1}$  is infinite, then there is  $g \in G$  such that  $g g_i P_i \in D$  for each  $i \in \{1, \dots, m\}$ .

**Remark 5.9** (Cocompactness). A group pair  $(G, \mathcal{P})$  has finite packing if and only if the  $G$ -action on the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  has finitely many  $G$ -orbits of simplices, or equivalently, the  $G$ -action on  $\mathcal{K}(G, \mathcal{P})$  is cocompact.

Note that  $(G, \mathcal{P})$  has finite packing if and only if  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional and has finitely many  $G$ -orbits of  $d$ -simplices for every  $d$ . In particular, finite packing implies finite height.

**Proposition 5.10.** *If  $(G, \mathcal{P})$  is a group pair, the following statements are equivalent.*

- (1)  $(G, \mathcal{P})$  has finite packing.
- (2) There is  $r > 0$  such that for any simplex  $\{g_1 P_1, \dots, g_k P_k\}$  of  $\mathcal{K}(G, \mathcal{P})$ , the intersection  $\bigcap_{i=1}^k \mathcal{N}_r(g_i P_i)$  is non-empty.

*Proof.* Suppose (2) holds. Let  $\mathcal{S}$  be an arbitrary simplex of  $\mathcal{K}(G, \mathcal{P})$ , and let  $\mathcal{S}' = \{g_1 P_1, \dots, g_k P_k\}$  be a finite subset of  $\mathcal{S}$ . Let  $g \in \bigcap_{i=1}^k \mathcal{N}_r(g_i P_i)$ . Then there are elements  $\{h_i \mid i = 1, \dots, k\}$  of  $G$  such that  $gh_i P_i = g_i P_i$  and  $\text{dist}_G(1, h_i) \leq r$ , so that  $g^{-1}\mathcal{S}' = \{h_i P_i \mid i = 1, \dots, k\}$ . Since  $\text{dist}_G$  is locally finite (as  $G$  is finitely generated), then there is a uniform bound  $M$  on the number of distinct possibilities for  $h_i$ . As  $\mathcal{P}$  is finite,  $g^{-1}\mathcal{S}'$  contains at most  $M \cdot |\mathcal{P}|$  distinct elements. In particular,  $\mathcal{S}$  is a finite simplex, and there are finitely many possibilities for  $g^{-1}\mathcal{S}$ . Thus  $(G, \mathcal{P})$  has finite packing.

Conversely, suppose that (1) holds. Then  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional. For each simplex  $\mathcal{S} = \{g_1 P_1, \dots, g_k P_k\}$ , let

$$\mu(\mathcal{S}) = \inf \left\{ m \mid \bigcap_{i=1}^k \mathcal{N}_m(g_i P_i) \neq \emptyset \right\}.$$

Observe that  $\mu(\mathcal{S})$  is well-defined since  $\bigcap_{i=1}^k g_i P_i g_i^{-1} \subseteq \bigcap_{i=1}^k \mathcal{N}_m(g_i P_i)$  for any  $m > \max_{i=1, \dots, k} \{\text{dist}_G(1, g_i)\}$ , and hence  $\mu(\mathcal{S})$  is an infimum of non-empty set. On the other hand,  $\mu$  is  $G$ -invariant, that is,  $\mu(g\mathcal{S}) = \mu(\mathcal{S})$ . Since there are finitely many  $G$ -orbits of simplices,

$$r = \sup \{\mu(\mathcal{S}) \mid \mathcal{S} \text{ is a simplex of } \mathcal{K}(G, \mathcal{P})\} < \infty$$

satisfies the required property.  $\square$

**Example 5.11** (Quasiconvex subgroups of hyperbolic groups have finite packing). If  $G$  is a hyperbolic group and  $\mathcal{P}$  is a finite collection of infinite quasi-convex subgroups, then  $(G, \mathcal{P})$  has finite packing. The argument by Gitik, MJ, Rips and Sageev in [GMR98] proving that  $(G, \mathcal{P})$  has finite width actually shows that Proposition 5.10 (2) holds. Let us briefly recall the argument. Assume that the Cayley graph  $\Gamma$  of  $G$  has  $\delta$ -slim triangles and that all subgroups in  $\mathcal{P}$  are  $\sigma$ -quasi-convex. Suppose  $\{g_1 P_1, \dots, g_k P_k\}$  is a simplex  $\mathcal{K}(G, \mathcal{P})$ . Then there is  $g \in \bigcap_{i=1}^k g_i P_i g_i^{-1}$  such that  $\text{dist}_G(e, g) > 3 \max_i \{\text{dist}_G(e, g_i)\} + 3\delta$ . Let  $\gamma = [e, g]$  be a geodesic in  $\Gamma$  from  $e$  to  $g$ , and let  $x$  be the middle vertex of  $\gamma$ . Consider the geodesic rectangle in  $\Gamma$  with geodesic sides  $[e, g_i]$ ,  $[g_i, gg_i^{-1}]$ ,  $[gg_i^{-1}, g]$  and  $\gamma$ . Since rectangles are  $2\delta$ -slim, and  $\gamma$  is long enough, the vertex  $x$  is at distance  $2\delta$  from a point in  $[g_i, gg_i^{-1}]$ . This geodesic has endpoints in the  $\sigma$ -quasiconvex left coset  $g_i P_i$ , and hence  $x$  is at distance  $r = 2\delta + \sigma$  from a vertex in  $g_i P_i$ . It follows that  $x \in \bigcap_{i=1}^k \mathcal{N}_r(g_i P_i)$ . By Proposition 5.10,  $(G, \mathcal{P})$  has finite packing.

**Proposition 5.12.** *Having finite packing is a quasi-isometry invariant of group pairs.*

*Proof.* Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be an  $(L, C, M)$ -quasi-isometry of group pairs and suppose that  $(H, \mathcal{Q})$  has finite packing. We use the characterization of finite packing provided by Proposition 5.10 to prove that  $(G, \mathcal{P})$  has finite packing.

Let  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  be an induced simplicial map; see Proposition 4.3. Recall that  $\text{hdist}_G(\dot{q}(gP), q(gP)) \leq M$  for all  $gP \in G/\mathcal{P}$ . Since  $(H, \mathcal{Q})$  has finite packing, the complex  $\mathcal{K}(H, \mathcal{Q})$  is finite dimensional and there is  $r > 0$  such that if  $\{h_1 Q_1, \dots, h_k Q_k\}$  is a simplex of  $\mathcal{K}(H, \mathcal{Q})$  then  $\bigcap_{i=1}^k \mathcal{N}_r(h_i Q_i)$  is non-empty. Let

$r' = L(r + M + 2C)$ , and let  $\{g_1P_1, \dots, g_kP_k\}$  be an arbitrary simplex of  $\mathcal{K}(G, \mathcal{P})$ . Let  $h_iQ_i = q(g_iP_i)$ , and let  $h \in \bigcap_{i=1}^k \mathcal{N}_r(h_iQ_i)$ . Then there exists  $g \in G$  such that  $\text{dist}_H(q(g), h) \leq C$ . Since  $\text{dist}_H(h, h_iQ_i) \leq r$ , it follows that

$$\begin{aligned} \text{dist}_G(g, g_iP_i) &\leq L \text{dist}_H(q(g), q(g_iP_i)) + LC \\ &\leq L \text{dist}_H(h, q(g_iP_i)) + 2LC \\ &\leq L(\text{dist}_H(h, h_iQ_i) + \text{dist}_H(h_iQ_i, q(g_iP_i))) + 2LC \\ &\leq Lr + LM + 2LC = r'. \end{aligned}$$

Thus  $\bigcap_{i=1}^k \mathcal{N}_{r'}(g_iP_i) \neq \emptyset$ , and so the pair  $(G, \mathcal{P})$  has finite packing by Proposition 5.10.  $\square$

**Question 5.13.** Let  $(G, \mathcal{P})$  be a group pair. Suppose that  $\mathcal{K}(G, \mathcal{P})$  has finitely many  $G$ -orbits of  $d$ -simplices for all  $d$ . Is  $\mathcal{K}(G, \mathcal{P})$   $G$ -cocompact?

**5.3. Commensurated cores.** In this section, we introduce the notion of a commensurated core of a group pair. After giving geometric and algebraic definitions of a commensurated core in Definition 5.14 and Proposition 5.16, respectively, we turn to the main result of the section, Proposition 5.21, which uses commensurated cores and height to describe exactly when the action of  $G$  on  $\mathcal{K}(G, \mathcal{P})$  is cocompact.

**Definition 5.14** (Commensurated core). Let  $(G, \mathcal{P})$  be a group pair. A *commensurated core*  $\mathcal{Q}$  of  $(G, \mathcal{P})$  is a collection of subgroups representing the conjugacy classes of setwise  $G$ -stabilizers of maximal simplices in the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$ .

Observe that a maximal simplex of  $\mathcal{K}(G, \mathcal{P})$  is a maximal subset  $\mathcal{S}$  of  $G/\mathcal{P}$  such that for any finite subset  $\{g_1P_1, \dots, g_kP_k\} \subset \mathcal{S}$  the subgroup  $\bigcap_{i=1}^k g_iP_i g_i^{-1}$  is infinite. Zorn's lemma guarantees that every simplex of  $\mathcal{K}(G, \mathcal{P})$  is a face of a maximal simplex, but this maximal simplex might be infinite dimensional. Moreover, the collection of maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  is naturally a  $G$ -set, and it turns out that the isotropies of this  $G$ -set are commensurated subgroups of  $G$  as we explain below.

Given a collection  $\mathcal{S} \subset G/\mathcal{P}$ , let

$$(6) \quad K(\mathcal{S}) = \bigcap \{gPg^{-1} \mid gP \in \mathcal{S}\}.$$

If  $\mathcal{S}$  is a simplex of  $\mathcal{K}(G, \mathcal{P})$ , then  $K(\mathcal{S})$  is the pointwise stabilizer of the simplex  $\mathcal{S}$ . If  $K(\mathcal{S})$  is an infinite subgroup of  $G$ , then  $\mathcal{S}$  is a simplex of  $\mathcal{K}(G, \mathcal{P})$ . However, the converse does not always hold. If  $\mathcal{S}$  is a finite-dimensional simplex, then  $K(\mathcal{S})$  is always infinite. On the other hand, if  $\mathcal{S}$  is an infinite-dimensional simplex, then  $K(\mathcal{S})$  may be a finite group; see Example 5.15.

We are interested in group pairs  $(G, \mathcal{P})$  with the property that pointwise stabilizers of maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  are infinite subgroups, that is,  $K(\mathcal{S})$  is infinite for all maximal simplices  $\mathcal{S}$ . An algebraic characterization of this property is as follows: for any collection  $\mathcal{S} \subset G/\mathcal{P}$  such that  $K(\mathcal{S}')$  is infinite for any finite  $\mathcal{S}' \subseteq \mathcal{S}$ , we have  $K(\mathcal{S})$  is infinite. If this property holds for a group pair  $(G, \mathcal{P})$ , we say the pair satisfies the *infinite intersection property*.

**Example 5.15.** Consider the Baumslag-Solitar group  $B(1, k) = \langle a, t \mid tat^{-1} = a^k \rangle$ . The group pair  $(BS(1, k), \langle a \rangle)$  does not satisfy the infinite intersection property. For

instance,  $\mathcal{S} = \{t^n \langle a \rangle \mid n \in \mathbb{Z}\}$  is an infinite dimensional simplex of  $\mathcal{K}(BS(1, k), \langle a \rangle)$  with trivial pointwise stabilizer.

In Lemma 5.19 below, we point out that group pairs with finite height satisfy the infinite intersection property. On the other hand, there are interesting group pairs with infinite height that also satisfy the infinite intersection property, for example pairs consisting of a right-angled Artin group together with maximal uniform flats, as described in Section 6.

**Proposition 5.16** (Commensurated core, algebraic characterization). *Let  $(G, \mathcal{P})$  be a group pair that satisfies the infinite intersection property. Then a collection  $\mathcal{Q}$  of subgroups of  $G$  is a commensurated core if and only if the following three conditions hold.*

- (1) *If  $\mathcal{S} \subset G/\mathcal{P}$  is maximal with respect to the property that  $K(\mathcal{S})$  is an infinite subgroup, then  $\text{Comm}_G(K(\mathcal{S}))$  is conjugate in  $G$  to some  $Q \in \mathcal{Q}$ .*
- (2) *If  $Q \in \mathcal{Q}$ , then there is a collection  $\mathcal{S} \subset G/\mathcal{P}$  that is maximal with respect to the property that  $K(\mathcal{S})$  is infinite and satisfies  $Q = \text{Comm}_G(K(\mathcal{S}))$ .*
- (3) *No two distinct subgroups in  $\mathcal{Q}$  are conjugate in  $G$ .*

Before the proof of the proposition, we need a lemma.

**Lemma 5.17.** *Let  $(G, \mathcal{P})$  be a group pair, and let  $\mathcal{S} \subset G/\mathcal{P}$  a maximal simplex of  $\mathcal{K}(G, \mathcal{P})$  such that  $K(\mathcal{S})$  is an infinite subgroup. Then  $\text{Comm}_G(K(\mathcal{S}))$  is the setwise  $G$ -stabilizer of the simplex  $\mathcal{S}$ .*

*Proof.* Suppose  $g \in G$  fixes the simplex  $\mathcal{S} = \{g_i P_i \mid i \in I\}$  setwise. Then the simplex  $g\mathcal{S} = \{gg_i P_i \mid g_i P_i \in \mathcal{S}\}$  equals the simplex  $\mathcal{S}$ , and so  $K(\mathcal{S}) = K(g\mathcal{S}) = gK(\mathcal{S})g^{-1}$ . Hence  $g$  is in the normalizer of  $K(\mathcal{S})$  and, in particular, in the commensurator  $\text{Comm}_G(K(\mathcal{S}))$ .

Conversely, suppose that  $g \in \text{Comm}_G(K(\mathcal{S}))$ . Then the subgroup  $K(g\mathcal{S})$  is commensurable to  $K(\mathcal{S})$ , and hence  $K(\mathcal{S}) \cap K(g\mathcal{S})$  is infinite. It follows that  $\mathcal{S} \cup g\mathcal{S}$  is a simplex of  $\mathcal{K}(G, \mathcal{P})$ . Since  $\mathcal{S}$  is a maximal simplex, we have that  $g\mathcal{S}$  is a maximal simplex as well. By maximality,  $g\mathcal{S} = \mathcal{S}$  and so  $g$  stabilizes  $\mathcal{S}$  setwise.  $\square$

*Proof of Proposition 5.16.* Since  $(G, \mathcal{P})$  satisfies the infinite intersection property, a subset  $\mathcal{S} \subset G/\mathcal{P}$  is a maximal simplex if and only if  $\mathcal{S}$  is maximal with respect to the property that  $K(\mathcal{S})$  is infinite. Thus when  $\mathcal{S}$  is a maximal simplex, Lemma 5.17 implies that  $\text{Comm}_G(K(\mathcal{S}))$  is the setwise stabilizer of  $\mathcal{S}$ .

Using these facts, we see that Property (1) holds if and only if  $\mathcal{Q}$  contains representatives of all conjugacy classes of setwise stabilizers of maximal simplices of  $\mathcal{K}(G, \mathcal{P})$ . Similarly, Property (2) holds if and only if every subgroup in  $\mathcal{Q}$  is the setwise stabilizer of a maximal simplex of  $\mathcal{K}(G, \mathcal{P})$ . Therefore,  $\mathcal{Q}$  satisfies (1), (2), and (3) if and only if it is a commensurated core.  $\square$

It was pointed out by Agol, Groves and Manning that for pairs  $(G, P)$  where  $G$  is a hyperbolic group and  $P$  is a quasi-convex subgroup, the commensurated core is an almost malnormal collection [AGM09]. The following proposition generalizes their result.

**Proposition 5.18.** *A commensurated core of a group pair with finite height is an almost malnormal collection.*

The proof of this proposition relies on the following lemma which, in particular, shows that if a pair  $(G, \mathcal{P})$  has finite height, then pointwise stabilizers of maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  are infinite. Therefore the commensurated core of pairs with finite height can be characterized by Proposition 5.16.

**Lemma 5.19.** *Suppose  $(G, \mathcal{P})$  has finite height, and let  $\mathcal{S} \subset G/\mathcal{P}$  be a maximal simplex of  $\mathcal{K}(G, \mathcal{P})$ . Then*

- (1)  $K(\mathcal{S})$  is infinite and is the pointwise  $G$ -stabilizer of the simplex  $\mathcal{S}$ .
- (2)  $\mathcal{S}$  is a maximal subset of  $G/\mathcal{P}$  with the property that  $K(\mathcal{S})$  is infinite.
- (3)  $K(\mathcal{S})$  is finite index in  $\text{Comm}_G(K(\mathcal{S}))$ .

*Proof.* Since  $(G, \mathcal{P})$  has finite height, the simplex  $\mathcal{S}$  is a finite subset of  $G/\mathcal{P}$  and, in particular,  $K(\mathcal{S})$  is an infinite subgroup. The first statement is then immediate, while the second statement re-states that  $\mathcal{S}$  is a maximal simplex. Since  $\mathcal{S}$  is finite subset of  $G/\mathcal{P}$ , its pointwise stabilizer  $K(\mathcal{S})$  is a finite index subgroup of its setwise stabilizer which is  $\text{Comm}_G(K(\mathcal{S}))$  by Lemma 5.17.  $\square$

*Proof of Proposition 5.18.* Let  $Q, Q' \in \mathcal{Q}$  and  $g \in G$ . There are maximal simplices  $\mathcal{S}$  and  $\mathcal{S}'$  of  $\mathcal{K}(G, \mathcal{P})$  with setwise stabilizers  $gQg^{-1}$  and  $Q'$  respectively. Suppose  $gQg^{-1} \cap Q' = \text{Comm}_G(K(\mathcal{S})) \cap \text{Comm}_G(K(\mathcal{S}'))$  is an infinite subgroup. By the third item of Lemma 5.19, we have that  $K(\mathcal{S}) \cap K(\mathcal{S}') = K(\mathcal{S} \cup \mathcal{S}')$  is infinite. By maximality, it follows that  $\mathcal{S} = \mathcal{S}'$  and hence  $gQg^{-1} = Q'$ . It follows that  $Q = Q'$  by definition of  $\mathcal{Q}$ , and so  $g \in \text{Comm}_G(Q) = Q$ .  $\square$

**Example 5.20** (Commensurated cores of quasi-convex subgroups in hyperbolic groups are almost malnormal). Let  $G$  be a hyperbolic group and  $\mathcal{P}$  a collection of infinite quasiconvex subgroups. Then  $(G, \mathcal{P})$  has finite height; see Example 5.11. Therefore, by Proposition 5.18, if  $\mathcal{Q}$  is a commensurated core of  $(G, \mathcal{P})$ , then  $\mathcal{Q}$  is an almost malnormal collection of subgroups of  $G$ , recovering a result from Agol, Groves and Manning [AGM09]. In particular  $\mathcal{K}(G, \mathcal{Q})$  is a 0-dimensional simplicial  $G$ -complex. In this case, the commensurated core coincides with the notion of the malnormal core from [AGM09].

The following proposition exhibits a relationship between the new properties that we have introduced: finite packing and a finite commensurated core. It also proves Theorem 1.6(4), as finite packing is equivalent to cocompactness of the coset intersection complex; see Remark 5.9.

**Proposition 5.21.** *A group pair  $(G, \mathcal{P})$  has finite packing if and only if  $(G, \mathcal{P})$  has finite height and a finite commensurated core.*

We begin with a lemma.

**Lemma 5.22.** *If  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional, then the number of  $G$ -orbits of maximal simplices is finite if and only if the number of conjugacy classes of  $G$ -setwise stabilizers of maximal simplices is finite.*

*Proof.* The only if part of the lemma is trivial. For the if part, first recall that  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional if and only if  $(G, \mathcal{P})$  has finite height. By Lemma 5.19, if  $\Delta$  and  $\Delta'$  are maximal simplices with the same setwise stabilizer, then their pointwise stabilizers are commensurable and then maximality implies that  $\Delta = \Delta'$ . Hence if two maximal simplices have conjugate setwise  $G$ -stabilizers, then they are in the same  $G$ -orbit, and the conclusion follows.  $\square$

*Proof of Proposition 5.21.* The pair  $(G, \mathcal{P})$  having finite height is equivalent to  $\mathcal{K}(G, \mathcal{P})$  being finite dimensional. Hence it is enough to prove that if  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional, then  $\mathcal{K}(G, \mathcal{P})$  is  $G$ -cocompact if and only if there are finitely many conjugacy classes of setwise stabilizers of maximal simplices. Since a finite dimensional simplicial  $G$ -complex is cocompact if and only if there are finitely many  $G$ -orbits of maximal simplices, the statement of the proposition follows from Lemma 5.22.  $\square$

If  $\mathcal{K}(G, \mathcal{P})$  is infinite-dimensional, then the  $G$ -action is not cocompact. However, the converse is unknown.

**Question 5.23.** Suppose  $\mathcal{K}(G, \mathcal{P})$  is finite dimensional. Is the  $G$ -action on  $\mathcal{K}(G, \mathcal{P})$  cocompact? Equivalently, is there an example of a group pair  $(G, \mathcal{P})$  that has finite height but does not have a finite commensurated core?

#### 5.4. Fenced infinite intersections.

**Definition 5.24.** Let  $\tau \geq 0$ . A group pair  $(G, \mathcal{P})$  has  $\tau$ -fenced infinite intersections if for any  $g_1 P_1, g_2 P_2 \in G/\mathcal{P}$  such that  $g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1}$  is an infinite subgroup,  $\text{dist}_G(g_1 P_1, g_2 P_2) \leq \tau$ . If  $(G, \mathcal{P})$  has  $\tau$ -fenced infinite intersections for some  $\tau$ , we say that  $(G, \mathcal{P})$  has *fenced infinite intersections*.

**Lemma 5.25.** Let  $(G, \mathcal{P})$  be a group pair. For any  $\tau \geq 0$ ,

$$E_\tau(G, \mathcal{P}) = \{ \{g_1 P_1, g_2 P_2\} \mid g_1 P_1, g_2 P_2 \in G/\mathcal{P} \text{ and } 0 < \text{dist}_G(g_1 P_1, g_2 P_2) \leq \tau \}$$

is a  $G$ -set with  $G$ -action given by  $g \cdot \{g_1 P_1, g_2 P_2\} = \{gg_1 P_1, gg_2 P_2\}$ . Moreover,  $E_\tau(G, \mathcal{P})$  has only finitely many distinct  $G$ -orbits.

*Proof.* That  $E_\tau(G, \mathcal{P})$  is a  $G$ -set follows from the observation that  $\text{dist}_G(g_1 P_1, g_2 P_2) = \text{dist}_G(gg_1 P_1, gg_2 P_2)$  for any  $\{g_1 P_1, g_2 P_2\} \in E_\tau(G, \mathcal{P})$  and any  $g \in G$ . On the other hand, any  $G$ -orbit has a representative of the form  $\{P_1, g_2 P_2\}$  such that  $\text{dist}_G(P_1, g_2 P_2) = \text{dist}_G(e, g_2)$ . Since  $(G, \text{dist}_G)$  is locally finite and  $\mathcal{P}$  is finite, there are finitely many choices for triples  $(P_1, g_2, P_2) \in \mathcal{P} \times G \times \mathcal{P}$  such that  $\text{dist}_G(P_1, g_2 P_2) = \text{dist}_G(e, g_2) \leq \tau$ . Therefore  $E_\tau(G, \mathcal{P})$  has only finitely many distinct  $G$ -orbits.  $\square$

The following proposition proves Theorem 1.6(3).

**Proposition 5.26** (Geometric characterization of fenced infinite intersections). *A group pair  $(G, \mathcal{P})$  has fenced infinite intersections if and only if the  $G$ -action on the 1-skeleton of  $\mathcal{K}(G, \mathcal{P})$  is cocompact.*

*Proof.* By definition  $\{g_1 P_1, g_2 P_2\}$  is a 1-cell of  $\mathcal{K}(G, \mathcal{P})$  if and only if the intersection  $g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1}$  is infinite. If  $(G, \mathcal{P})$  has  $\tau$ -fenced intersections, then the  $G$ -set of 1-cells of  $\mathcal{K}(G, \mathcal{P})$  is a  $G$ -subset of the  $G$ -set  $E_\tau(G, \mathcal{P})$  which has only finitely many distinct  $G$ -orbits by Lemma 5.25. Since  $\mathcal{P}$  is finite,  $\mathcal{K}(G, \mathcal{P})$  also has only finitely many distinct  $G$ -orbits of 0-cells, and so its 1-skeleton is  $G$ -cocompact. Conversely, if there are finitely many  $G$ -orbits of 1-cells, then we let  $\sigma$  be the maximum of  $\text{dist}_G(g_1 P_1, g_2 P_2)$  over all possible 1-cells  $\{g_1 P_1, g_2 P_2\}$  and observe that  $(G, \mathcal{P})$  has  $\sigma$ -fenced infinite intersections.  $\square$

The following theorem proves Theorem 1.5(3).

**Theorem 5.27.** *Having fenced infinite intersections is a quasi-isometry invariant of group pairs.*

*Proof.* Suppose that  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a  $(L, C, M)$ -quasi-isometry of pairs and  $(H, \mathcal{Q})$  has  $\sigma$ -fenced infinite intersections. Consider an induced function  $\dot{q}: G/\mathcal{P} \rightarrow H/\mathcal{Q}$  given by Proposition 2.5.

Let  $g_1 P_1, g_2 P_2 \in G/\mathcal{P}$  be such that  $g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1}$  is an infinite subgroup, and let  $h_i Q_i = \dot{q}(g_i P_i)$  for  $i = 1, 2$ . By Lemma 4.6,  $h_1 Q_1 h_1^{-1} \cap h_2 Q_2 h_2^{-1}$  is infinite. Since  $(H, \mathcal{Q})$  has  $\sigma$ -fenced infinite intersections, we have that  $\text{dist}_H(h_1 Q_1, h_2 Q_2) \leq \sigma$ . By Proposition 2.5, we have that  $\text{dist}_G(g_1 P_1, g_2 P_2) \leq L\sigma + (C + 2M)L$ . Hence  $(G, \mathcal{P})$  has  $(L\sigma + (C + 2M)L)$ -fenced infinite intersections.  $\square$

### 5.5. Networks and thickness.

**Definition 5.28** ( $\tau$ -Network). Let  $X$  be a metric space and  $\mathcal{L}$  a collection of subsets of  $X$ . Given  $\tau \geq 0$ , the space  $X$  is a  $\tau$ -network with respect to  $\mathcal{L}$  if:

- (1)  $X = \bigcup_{L \in \mathcal{L}} \mathcal{N}_\tau(L)$ , and
- (2) For any  $L, L' \in \mathcal{L}$  there exists a sequence  $L_1 = L, L_2, \dots, L_n = L'$  of elements of  $\mathcal{L}$  such that  $\mathcal{N}_\tau(L_i) \cap \mathcal{N}_\tau(L_{i+1})$  is an unbounded subspace for all  $i$ .

**Definition 5.29.** A group pair  $(G, \mathcal{P})$  is a  $\tau$ -network if  $\mathcal{P}$  is non-empty and  $(G, \text{dist}_G)$  is a  $\tau$ -network with respect to  $G/\mathcal{P}$ .

**Definition 5.30** (The  $\tau$ -coset intersection complex  $\mathcal{K}_\tau(G, \mathcal{P})$ ). Let  $(G, \mathcal{P})$  be a group pair and let  $\tau \geq 0$ . The  $\tau$ -coset intersection complex  $\mathcal{K}_\tau(G, \mathcal{P})$  is the simplicial  $G$ -complex with vertex set  $G/\mathcal{P}$  and such that a finite subset  $\{g_1 P_1, \dots, g_k P_k\}$  of  $G/\mathcal{P}$  defines a simplex if and only if  $\bigcap_{i=1}^k \mathcal{N}_\tau(g_i P_i)$  is unbounded in  $(G, \text{dist}_G)$ .

The  $\tau$ -coset intersection complexes define a filtration indexed by  $\mathbb{R}_+$  of the coset intersection complex. The next lemma shows that each term in this filtration is a  $G$ -complex with cocompact 1-skeleton.

**Lemma 5.31.** Let  $(G, \mathcal{P})$  be a group pair and  $\tau \geq 0$ .

- (1) The complex  $\mathcal{K}_\tau(G, \mathcal{P})$  is a  $G$ -subcomplex of  $\mathcal{K}(G, \mathcal{P})$ .
- (2) The 1-skeleton of  $\mathcal{K}_\tau(G, \mathcal{P})$  is  $G$ -cocompact.

*Proof.* That  $\mathcal{K}_\tau(G, \mathcal{P})$  is a  $G$ -subcomplex of  $\mathcal{K}(G, \mathcal{P})$  is a consequence of the metric  $\text{dist}_G$  being  $G$ -equivariant and Lemma 4.5, which characterizes simplices of the coset intersection complex  $\mathcal{K}(G, \mathcal{P})$  geometrically. For the second statement, observe that the  $G$ -set of edges of  $\mathcal{K}_\tau(G, \mathcal{P})$  is a  $G$ -subset of the  $G$ -set  $E_\tau(G, \mathcal{P})$  of Lemma 5.25, which has only finitely many distinct  $G$ -orbits. Hence  $\mathcal{K}_\tau(G, \mathcal{P})$  has only finitely many  $G$ -orbits of edges and, since  $\mathcal{P}$  is finite, only finitely many  $G$ -orbits of vertices. Therefore the 1-skeleton of  $\mathcal{K}_\tau(G, \mathcal{P})$  is  $G$ -cocompact.  $\square$

**Proposition 5.32.** Let  $(G, \mathcal{P})$  be a group pair with  $\mathcal{P}$  a non-empty collection. Then  $(G, \mathcal{P})$  is a  $\tau$ -network if and only if  $\mathcal{K}_\tau(G, \mathcal{P})$  is connected.

*Proof.* A sequence  $g_1 P_1, \dots, g_n P_n$  in  $G/\mathcal{P}$  is the sequence of vertices of an edge-path in  $\mathcal{K}_\tau(G, \mathcal{P})$  if and only if  $\mathcal{N}_\tau(g_i P_i) \cap \mathcal{N}_\tau(g_{i+1} P_{i+1})$  is an unbounded subspace for  $1 \leq i < n$ . Therefore  $\mathcal{K}_\tau(G, \mathcal{P})$  is connected if and only if for any two  $gP, g'P'$  there is a sequence  $gP = g_1 P_1, g_2 P_2, \dots, g_n P_n = g'P'$  in  $G/\mathcal{P}$  such that  $\mathcal{N}_\tau(g_i P_i) \cap \mathcal{N}_\tau(g_{i+1} P_{i+1})$  is unbounded for  $1 \leq i < n$ .

Since  $\mathcal{P}$  is non-empty, it is always the case that  $G = \bigcup_{L \in G/\mathcal{P}} \mathcal{N}_\tau(L)$ . Therefore  $(G, \text{dist}_G)$  is a  $\tau$ -network with respect to  $G/\mathcal{P}$  if and only if  $\mathcal{K}_\tau(G, \mathcal{P})$  is connected.  $\square$

A consequence of Proposition 5.32 is the following characterization of being a network, which proves Theorem 1.6(6).

**Corollary 5.33.** *Let  $(G, \mathcal{P})$  be a group pair. Then  $(G, \text{dist}_G)$  is a network with respect to the collection  $G/\mathcal{P}$  if and only if  $\mathcal{P}$  is non-empty and  $\mathcal{K}(G, \mathcal{P})$  has a connected and  $G$ -cocompact subgraph  $\Delta$  with vertex set  $G/\mathcal{P}$ .*

*Proof.* If  $(G, \mathcal{P})$  is a  $\tau$ -network, then  $\mathcal{P}$  is non-empty, and so Proposition 5.32 implies that the 1-skeleton of  $\mathcal{K}_\tau(G, \mathcal{P})$  is a connected and cocompact  $G$ -subgraph of  $\mathcal{K}(G, \mathcal{P})$  with vertex set  $G/\mathcal{P}$ .

Conversely, suppose that  $\mathcal{P}$  is non-empty and  $\Delta$  is a connected and  $G$ -cocompact subgraph of  $\mathcal{K}(G, \mathcal{P})$  with vertex set  $G/\mathcal{P}$ . Let  $e_1, \dots, e_k$  be a collection of representatives of  $G$ -orbits of edges of  $\Delta$ . By Lemma 4.5, for each  $e_i = \{g_i P_i, g'_i P'_i\}$  there is  $\tau_i \geq 0$  such that  $\mathcal{N}_{\tau_i}(g_i P_i) \cap \mathcal{N}_{\tau_i}(g'_i P'_i)$  is unbounded in  $(G, \text{dist}_G)$ . Let  $\tau = \max\{\tau_1, \dots, \tau_k\}$ . By  $G$ -equivariance of  $\text{dist}_G$ , it follows that for every edge  $\{gP, g'P'\}$  of  $\Delta$ , the set  $\mathcal{N}_\tau(gP) \cap \mathcal{N}_\tau(g'P')$  is unbounded in  $(G, \text{dist})$ . It follows that  $\Delta$  is a connected subgraph of  $\mathcal{K}_\tau(G, \mathcal{P})$ , and since both graphs have the same vertex set, we have that  $\mathcal{K}_\tau(G, \mathcal{P})$  is connected. Since  $\mathcal{P}$  is non-empty, Proposition 5.32 implies that  $(G, \mathcal{P})$  is a  $\tau$ -network.  $\square$

**Lemma 5.34.** *Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of group pairs. For every  $\sigma \geq 0$  there is  $\tau_0 \geq 0$  such that for every  $\tau \geq \tau_0$  the simplicial map  $\dot{q}: \mathcal{K}(G, \mathcal{P}) \rightarrow \mathcal{K}(H, \mathcal{Q})$  restricts to a simplicial map  $\dot{q}: \mathcal{K}_\sigma(G, \mathcal{P}) \rightarrow \mathcal{K}_\tau(H, \mathcal{Q})$ .*

*Proof.* Let us assume that  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a  $(L, C, M)$ -quasi-isometry of pairs. Let  $\tau_0 = L\sigma + C + M$ . Suppose that  $\{g_1 P_1, \dots, g_k P_k\}$  is simplex of  $\mathcal{K}_\sigma(G, \mathcal{P})$ . It follows that  $\bigcap_{i=1}^k \mathcal{N}_\sigma(g_i P_i)$  is unbounded in  $(G, \text{dist}_G)$ . Hence  $\bigcap_{i=1}^k \mathcal{N}_{L\sigma+C}(q(g_i P_i))$  is unbounded in  $(H, \text{dist}_H)$ . As a consequence,  $\bigcap_{i=1}^k \mathcal{N}_\tau(\dot{q}(g_i P_i))$  is unbounded for all  $\tau \geq \tau_0$ . Therefore  $\{\dot{q}(g_1 P_1), \dots, \dot{q}(g_k P_k)\}$  is a simplex of  $\mathcal{K}_\tau(H, \mathcal{Q})$ .  $\square$

The following theorem proves Theorem 1.5(5).

**Theorem 5.35.** *Being a network is a quasi-isometry invariant of group pairs.*

*Proof.* Suppose that  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a  $(L, C, M)$ -quasi-isometry of pairs and  $(G, \mathcal{P})$  is a  $\sigma$ -network. By Lemma 5.34, there is  $\tau \geq 0$  and an induced simplicial map  $\dot{q}: \mathcal{K}_\sigma(G, \mathcal{P}) \rightarrow \mathcal{K}_\tau(H, \mathcal{Q})$ . By increasing  $\tau$  if necessary, assume that  $\tau \geq M$ .

We claim that  $\mathcal{K}_\tau(H, \mathcal{Q})$  is connected. Let  $hQ, h'Q' \in H/\mathcal{Q}$ . Proposition 2.5 implies that there are  $gP, g'P' \in G/\mathcal{P}$  such that  $\text{hdist}_H(\dot{q}(gP), hQ) \leq M$  and  $\text{hdist}_H(\dot{q}(g'P'), h'Q') \leq M$  and therefore  $\{\dot{q}(gP), hQ\}$  and  $\{\dot{q}(g'P'), h'Q'\}$  are edges of  $\mathcal{K}_\tau(H, \mathcal{Q})$ . Since  $(G, \mathcal{P})$  is a  $\tau$ -network, Proposition 5.32 implies that there is an edge-path in  $\mathcal{K}_\sigma(G, \mathcal{P})$  from  $gP$  to  $g'P'$ . Since  $\dot{q}: \mathcal{K}_\sigma(G, \mathcal{P}) \rightarrow \mathcal{K}_\tau(H, \mathcal{Q})$  is a simplicial map, there is a path from  $\dot{q}(gP)$  to  $\dot{q}(g'P')$ . By considering the edges  $\{\dot{q}(gP), hQ\}$  and  $\{\dot{q}(g'P'), h'Q'\}$  of  $\mathcal{K}_\tau(H, \mathcal{Q})$ , it follows that there is path in  $\mathcal{K}_\tau(H, \mathcal{Q})$  from  $hQ$  to  $h'Q'$ . This shows that  $\mathcal{K}_\tau(H, \mathcal{Q})$  is connected, and Proposition 5.32 then implies that  $(H, \mathcal{Q})$  is a  $\tau$ -network.  $\square$

Networks are essential in the notion of *thickness*, which was introduced by Behrstock, Druţu, and Mosher in [BDM09]. We first define thickness for a metric space, and then, as will be most relevant for this paper, define thickness for a group pair.

**Definition 5.36** ([BDM09, Definition 7.1]). A collection of metric spaces  $\mathcal{Y}$  is *uniformly thick of order 0* if  $\mathcal{Y}$  is uniformly uncontracted, that is, if there is a

constant  $c \geq 0$  such that every point in every space  $Y \in \mathcal{Y}$  is at distance at most  $c$  from a bi-infinite geodesic in  $Y$  and, roughly, every sequence of spaces  $Y_i \in \mathcal{Y}$  has an asymptotic cone without a cut point. See [BDM09, Definition 3.4] for a precise definition.

The space  $X$  is *thick of order at most  $n$*  with respect to a collection of subspaces  $\mathcal{Y}$  if there is a  $\tau \geq 0$  such that  $\mathcal{Y}$  forms a  $\tau$ -network in  $X$  and each  $Y \in \mathcal{Y}$  is thick of order at most  $n - 1$  when endowed with the subspace metric from  $X$ . If  $X$  is thick of order at most  $n$  but not thick of order at most  $n - 1$ , we say  $X$  is *thick of order  $n$* . If the order of thickness is not important, we simply say  $X$  is *thick*.

Finally, we say a group pair  $(G, \mathcal{P})$  is *thick of order  $n$*  if  $(G, d_G)$  is thick of order  $n$  with respect to the collection of subspaces  $G/\mathcal{P}$ .

This definition of thickness is also called *metric thickness* in [BDM09]. Our definition of thickness for a group pair  $(G, \mathcal{P})$  is slightly stronger than simply saying that  $G$  is (metrically) thick of order  $n$ , because we require the subspaces to be (cosets of) subgroups. Behrstock, Druţu, and Mosher also define the notion of *algebraic thickness* for a group, which requires that the subspaces in the network be subgroups that satisfy certain additional algebraic conditions. These additional conditions are not necessarily preserved by quasi-isometries of pairs, so we do not consider algebraic thickness in this paper.

The following corollary of Theorem 5.35 proves Theorem 1.5(6).

**Corollary 5.37.** *Being thick of order one is a quasi-isometry invariant of group pairs.*

*Proof.* Let  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  be a quasi-isometry of pairs. Suppose that  $(G, \mathcal{P})$  is thick of order one. By the definition of thickness,  $G/\mathcal{P}$  is uniformly thick of order 0, and thus uniformly unconstricted. Since each element of  $H/\mathcal{Q}$  is uniformly quasi-isometric to an element of  $G/\mathcal{P}$ , it follows that  $H/\mathcal{Q}$  is also uniformly thick of order 0; see also [BDM09, Remark 3.5]. By Theorem 5.35,  $(H, \mathcal{Q})$  is  $\tau$ -network for some  $\tau \geq 0$ . Therefore,  $(H, \mathcal{Q})$  is thick of order at most one. As  $H$  is quasi-isometric to  $G$ , and  $G$  is not thick of order 0 by assumption,  $H$  is thick of order one.  $\square$

A collection of subgroups  $\mathcal{P}$  are qi characteristic if the collection  $G/\mathcal{P}$  is preserved up to uniform finite Hausdorff distance by every quasi-isometry of  $G$ , see [MPSSn22] for a precise definition.

**Question 5.38.** Let  $G$  be a finitely generated group which is thick of order  $n$ . Is there a finite qi-characteristic collection of subgroups which respect to which  $G$  is thick of order  $n$ ?

A positive answer to this question is first step towards using induction to prove that the class of group pairs that are thick of order  $n$  is geometric. The following example gives a blueprint for how such an argument might go.

**Example 5.39.** Suppose  $(G, \mathcal{P})$  is thick of order 2, that  $G$  and each  $P \in \mathcal{P}$  is finitely generated, and that Question 5.38 has a positive answer. If  $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$  is a quasi-isometry of pairs, then  $(H, \mathcal{Q})$  is thick of order 2.

To prove this, note that by Theorem 5.35, the collection  $H/\mathcal{Q}$  forms a network in  $H$ , and thus it suffices to show that each  $Q \in \mathcal{Q}$  is thick of order 1. Let  $Q \in \mathcal{Q}$ . Then  $Q$  is at finite Hausdorff distance from  $q(gP)$  for some  $gP \in \mathcal{P}$ . By [MPSSn22, Proposition 2.16] there is a quasi-isometry  $q': P \rightarrow Q$ . By assumption,

$P$  is a finitely generated group that is thick of order one. A positive answer to Question 5.38 thus gives a finite qi-characteristic collection of subgroups  $K_1, \dots, K_m \leq P$  so that  $(P, \{K_i\})$  is thick of order one. Since this collection is qi-characteristic, there is a collection  $\{J_i\}$  of subgroups of  $Q$  such that  $q' : (P, \{K_i\}) \rightarrow (Q, \{J_i\})$  is a quasi-isometry of pairs. It follows from Corollary 5.37 that  $(Q, \{J_i\})$  is thick of order 1, which completes the argument.

## 6. TWO DIMENSIONAL RIGHT-ANGLED ARTIN GROUPS

Let  $G$  be a right-angled Artin group whose defining graph  $\Gamma$  has no vertices of valence  $< 2$  and no triangles. Abusing notation, we identify the set of vertices of  $\Gamma$  with the standard generating set of  $G$ . For a standard generator  $a$ , let

$$\text{Link}(a) = \{b \mid b \text{ is a standard generator, } a \neq b, \text{ and } a \text{ and } b \text{ commute}\}$$

and

$$\text{Star}(a) = \text{Link}(a) \cup \{a\}.$$

A subgroup of the form  $\langle \text{Star}(a) \rangle$  is called a *standard star subgroup*.

In this section, we will focus on the collection  $\mathcal{P}$  of standard maximal flats, so that the elements in  $\mathcal{P}$  are subgroups generated by maximal cliques in  $\Gamma$ . Our assumption that  $\Gamma$  has no triangles and no vertices of valence  $< 2$  implies that maximal cliques in  $\Gamma$  are edges, and so standard maximal flats are  $\mathbb{Z}^2$ -subgroups of the form

$$\mathcal{P} = \{\langle a, b \rangle \mid a \text{ and } b \text{ are commuting standard generators and } a \neq b\}.$$

**6.1. The Deligne complex.** The (*modified*) *Deligne complex* of Charney and Davis [CD95] for a general Artin group is the geometric realization of the poset of left cosets of finite-type standard parabolic subgroups, that is, those whose corresponding Coxeter group is finite. For right-angled Artin groups, a standard parabolic subgroup is finite type exactly when it is abelian, and so in our context, the Deligne complex is the complex realizing the poset of left cosets of abelian subgroups generated by standard generators.

**Theorem 6.1.** *Suppose  $\Gamma$  is connected, has no vertices of valence  $< 2$ , and no triangles. There is a quasi-isometry between  $\mathcal{K}(G, \mathcal{P})$  and the Deligne complex of  $G$ .*

When  $G$  is an FC-type Artin group whose defining graph has no empty squares, Charney and Crisp prove that the Deligne complex is hyperbolic. FC-type Artin groups are a large class of Artin groups that include RAAGs, and having no empty squares means that every 4-cycle contains at least one of its diagonals. If  $\Gamma$  has no triangles, then it has no empty squares exactly when it has no squares at all. Thus we have the following corollary.

**Corollary 6.2.** *If  $\Gamma$  is a connected graph with no vertices of valence  $< 2$  and no cycles of length  $\leq 4$ , then  $\mathcal{K}(G, \mathcal{P})$  is hyperbolic.*

In the special case that  $G$  is an atomic right-angled Artin group and  $\mathcal{P}$  is the collection of standard flats, the construction of  $\mathcal{K}(G, \mathcal{P})$  is closely related to construction of the *flat space*  $\mathbb{F}(\Gamma)$  defined by Bestvina, Kleiner and Sageev [BKS08]. They note that, in that case,  $\mathbb{F}(\Gamma)$  also coincides with the Deligne complex.

To prove Theorem 6.1, we first compute the commensurated core of  $(G, \mathcal{P})$ .

**Proposition 6.3.** *Suppose  $\Gamma$  has no vertices of valence  $< 2$  and no triangles. The  $G$ -set of maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  is isomorphic to the  $G$ -set of left cosets  $G/\mathcal{Q}$  where*

$$\mathcal{Q} = \{\langle \text{Star}(a) \rangle \mid a \text{ is a standard generator}\}.$$

*In particular,  $\mathcal{Q}$  is the commensurated core of  $(G, \mathcal{P})$ .*

The proposition is an immediate consequence of the following lemma.

**Lemma 6.4.** *Suppose  $\Gamma$  has no vertices of valence  $< 2$  and no triangles. If  $\mathcal{S}$  is a maximal simplex of  $\mathcal{K}(G, \mathcal{P})$ , then there is  $g \in G$  and a standard generator  $a$  such that the following hold.*

- (1)  $\mathcal{S} = \{gw\langle a, c \rangle \mid w \in \langle \text{Star}(a) \rangle \text{ and } c \in \text{Link}(a)\}$ .
- (2) *the pointwise stabilizer of  $\mathcal{S}$  is  $g\langle a \rangle g^{-1}$ .*
- (3) *the setwise stabilizer of  $\mathcal{S}$  is  $\text{Comm}_G(g\langle a \rangle g^{-1})$ , and*

$$\text{Comm}_G(g\langle a \rangle g^{-1}) = g\langle \text{Star}(a) \rangle g^{-1}.$$

- (4)  *$\mathcal{S}$  is infinite dimensional.*

*Proof.* Up to translating  $\mathcal{S}$  by an element  $g$ , assume that  $\langle a, b \rangle \in \mathcal{S}$  where  $a, b$  are commuting standard generators. Observe that  $\mathcal{S}$  has more than one element. Indeed, since  $a$  has valence  $\geq 2$  in  $\Gamma$ , there is  $c \in \text{Link}(a)$  such that  $b \neq c$ . The cosets  $\langle a, b \rangle$  and  $\langle a, c \rangle$  form a 1-dimensional simplex, and hence if  $\mathcal{S}$  has a single element then it is not maximal.

Let  $wP = w\langle x, y \rangle$  be an element of  $\mathcal{S}$  different than  $\langle a, b \rangle$ . The intersection  $\langle a, b \rangle \cap w\langle x, y \rangle w^{-1}$  is an infinite parabolic subgroup, and the only possibilities are  $\langle a \rangle$ ,  $\langle b \rangle$  or  $\langle a, b \rangle$ ; see [DKR07, Corollary 2.5]. Let us argue that  $\langle a, b \rangle \cap w\langle x, y \rangle w^{-1}$  is  $\langle a \rangle$  or  $\langle b \rangle$ . Suppose  $\langle a, b \rangle \cap w\langle x, y \rangle w^{-1} = \langle a, b \rangle$ . Then  $\{x, y\} = \{a, b\}$  by [DKR07, Lemma 2.7], and therefore  $w$  is in the normalizer of  $\langle a, b \rangle$ . In a right-angled Artin group, centralizers of parabolic subgroups coincide with normalizers [God03]. By the Centralizer Theorem [Ser89], the centralizer  $C(u)$  of a standard generator  $u$  is the subgroup generated by  $\text{Star}(u)$ , and therefore the centralizer of  $\langle a, b \rangle$  is  $\langle \text{Star}(a) \rangle \cap \langle \text{Star}(b) \rangle$ . By [DKR07, Lemma 2.7] and since  $\Gamma$  has no triangles, we have  $C(\langle a, b \rangle) = \langle \text{Star}(a) \rangle \cap \langle \text{Star}(b) \rangle = \langle \text{Star}(a) \cap \text{Star}(b) \rangle = \langle a, b \rangle$ . Hence  $w \in \langle a, b \rangle$ , showing that  $\langle a, b \rangle = w\langle x, y \rangle$ , contrary to our assumption.

Since  $\mathcal{S}$  is a simplex, we can assume without loss of generality that  $\langle a, b \rangle \cap wPw^{-1} = \langle a \rangle$  for every  $wP \in \mathcal{S}$  different than  $\langle a, b \rangle$ . Therefore the pointwise stabilizer of  $\mathcal{S}$  satisfies  $K(\mathcal{S}) = \langle a \rangle$  and, by maximality of  $\mathcal{S}$ ,

$$\mathcal{S} = \{wP \mid w \in G, P \in \mathcal{P}, a \in wPw^{-1}\}.$$

By [DKR07, Lemma 2.7], if  $a, x, y$  are standard generators,  $h \in G$ , and  $a \in h\langle x, y \rangle h^{-1}$ , then  $a \in \{x, y\}$  and  $h$  belongs to the product set  $C(a) \cdot \langle x, y \rangle$ . In particular,  $h\langle x, y \rangle h^{-1} = w\langle x, y \rangle w^{-1}$  for some  $w \in C(a) = \langle \text{Star}(a) \rangle$ . Therefore

$$\mathcal{S} = \{w\langle a, c \rangle \mid w \in \langle \text{Star}(a) \rangle \text{ and } c \in \text{Link}(a)\}.$$

The expression above shows that  $\mathcal{S}$  is infinite. More explicitly, since  $\Gamma$  has no vertices of valence  $< 2$ , there are distinct standard generators  $b$  and  $c$  in  $\text{Link}(a)$ . Then, for every  $n \in \mathbb{Z}$ , the coset  $b^n\langle a, c \rangle$  is an element of  $\mathcal{S}$ . Hence  $\mathcal{S}$  is infinite.

It is also a consequence of [DKR07, Lemma 2.7] that the normalizer and commensurator of  $\langle a \rangle$  coincide, and, as stated above, they are also equal the centralizer  $C(a) = \langle \text{Star}(a) \rangle$ . Hence, by Lemma 5.17 and the Centralizer theorem, the setwise stabilizer of  $\mathcal{S}$  is  $\text{Comm}_G(K(\mathcal{S})) = \text{Comm}_G(\langle a \rangle) = \langle \text{Star}(a) \rangle$ .  $\square$

The following lemma shows that removing the  $G$ -orbit of the trivial subgroup from the Deligne complex produces a quasi-isometric complex. The proof uses the assumption that  $\Gamma$  has no triangles, as those are the RAAGS we are considering in this section. However, with slight modifications to the proof, the conclusion holds without this assumption.

**Lemma 6.5.** *Suppose that  $\Gamma$  is connected, has no triangles, and has at least two vertices. There is a quasi-isometry between the Deligne complex of  $G$  and the  $G$ -complex  $\Delta$  realizing the poset of left cosets of non-trivial abelian subgroups generated by standard generators.*

*Proof.* Let  $\mathcal{D}$  be the Deligne complex of the right-angled Artin group with defining graph  $\Gamma$ . We show that the natural embedding  $\Delta \hookrightarrow \mathcal{D}$  is a quasi-isometry.

Let  $CT'$  be the finite graph obtained from coning-off the barycentric subdivision  $\Gamma'$  of  $\Gamma$ . There is an embedding of  $CT'$  into  $\mathcal{D}$  such that the cone vertex is mapped to the vertex of  $\mathcal{D}$  corresponding to the trivial subgroup, a vertex of  $\Gamma$  labelled by a standard generator  $a$  is mapped to the vertex corresponding to the subgroup  $\langle a \rangle$ , and a barycenter of an edge of  $\Gamma$  connecting the vertices labelled by the standard generators  $a$  and  $b$  is mapped to the vertex corresponding to the subgroup  $\langle a, b \rangle$ . We identify  $CT'$  with its image under this embedding. Since  $\Gamma$  has no triangles, each  $G$ -orbit of cells in  $\mathcal{D}$  is represented by a cell of  $CT'$  and no two distinct cells of  $CT'$  are in the same  $G$ -orbit. That is,  $CT'$  is a minimal fundamental domain for the  $G$ -complex  $\mathcal{D}$ .

Enumerate the edges of  $CT'$  that have the cone-point as an endpoint as  $e_1, \dots, e_k$ . Since  $\Gamma$  is not a single vertex, we have that  $k > 1$ . Let  $\mathcal{D}_1$  be the  $G$ -graph obtained by removing removing all (open) edges in the  $G$ -orbit of  $e_i$  for  $1 \leq i < k$ . Let us argue that the embedding  $\mathcal{D}_1 \hookrightarrow \mathcal{D}$  is a quasi-isometry. Since  $\Gamma$  is connected, the endpoints of every edge  $e_i$ , with  $1 \leq i < k$ , are connected by a path in  $CT' - \bigcup_{i=1}^{k-1} e_i \subseteq \mathcal{D}_1$  of length at most  $K = 1 + \text{Diam}(\Gamma')$ , where  $\text{Diam}(\Gamma')$  is the diameter of  $\Gamma'$ . Since all translates of  $\Gamma'$  are contained in  $\mathcal{D}_1$ , it follows that for every pair of vertices  $x, y \in \mathcal{D}_1$  we have that

$$\text{dist}_{\mathcal{D}_1}(x, y) \leq \text{dist}_{\mathcal{D}}(x, y) \leq K \text{dist}_{\mathcal{D}_1}(x, y).$$

Since  $\mathcal{D}$  and  $\mathcal{D}_1$  have the same vertex set, the embedding  $\mathcal{D}_1 \hookrightarrow \mathcal{D}$  is a quasi-isometry.

Now we argue that the embedding  $\Delta \hookrightarrow \mathcal{D}_1$  is a quasi-isometry. Observe that the graph  $CT' - \bigcup_{i=1}^{k-1} e_i$  is a minimal fundamental domain of  $\mathcal{D}_1$  and the cone vertex  $v$  has degree one in  $CT' - \bigcup_{i=1}^{k-1} e_i$ . Since  $v$  has trivial stabilizer in  $\mathcal{D}_1$ , it follows that  $v$  has degree one in  $\mathcal{D}_1$ . On the other hand,  $\Delta$  is obtained from  $\mathcal{D}_1$  by removing the  $G$ -orbit of the cone-vertex  $v$  and the  $G$ -orbit of the open edge  $e_k$ . Since  $v$  is a vertex of degree 1 and an endpoint of  $e_k$  in  $\mathcal{D}_1$ , removing the  $G$ -orbits of  $e_k$  and  $v$  from  $\mathcal{D}_1$  preserves the path-metric on the subgraph  $\Delta$  of  $\mathcal{D}_1$ . Since every vertex of  $\mathcal{D}_1$  is at distance one from a vertex of  $\Delta$ , we conclude that the embedding  $\Delta \hookrightarrow \mathcal{D}_1$  is a quasi-isometry. Hence the composition  $\Delta \hookrightarrow \mathcal{D}_1$  and  $\mathcal{D}_1 \hookrightarrow \mathcal{D}$  is a quasi-isometry.  $\square$

*Proof of Theorem 6.1.* Let  $\mathcal{O}$  be the collection of cyclic groups of the form  $\langle a \rangle$  where  $a$  is a standard generator. Observe that any two vertices  $u\langle a \rangle$  and  $v\langle b \rangle$  of  $\mathcal{K}(G, \mathcal{O})$  are adjacent if and only if  $a = b$  and  $u^{-1}v \in \text{Comm}_G(\langle a \rangle) = \langle \text{Star}(a) \rangle$ . Therefore each connected component of  $\mathcal{K}(G, \mathcal{O})$  is a translate of the complete

graph with vertex set  $\text{Comm}_G(\langle a \rangle) / \langle a \rangle$  where  $a$  is a standard generator. Since  $\text{Comm}_G(\langle a \rangle) = \langle \text{Star}(a) \rangle$ , Proposition 6.3 implies that the  $G$ -set of connected components of  $\mathcal{K}(G, \mathcal{O})$  is isomorphic to the  $G$ -set  $G/\mathcal{Q}$ . Since all connected components of  $\mathcal{K}(G, \mathcal{O})$  have diameter one, this isomorphism implies that the  $G$ -equivariant inclusion  $\mathcal{K}(G, \mathcal{P}) \hookrightarrow \mathcal{K}(G, \mathcal{O} \cup \mathcal{P})$  is a  $G$ -equivariant quasi-isometry. Geometrically, the effect of adding the collection  $\mathcal{O}$  is to add a diameter-one set as the “barycenter” of each maximal simplex in  $\mathcal{K}(G, \mathcal{P})$ .

Consider an edge of  $\mathcal{K}(G, \mathcal{P})$ , say with endpoints  $\langle a, b \rangle$  and  $u\langle b, c \rangle$  where  $u \in \text{Comm}_G(b)$ . In this case there is 3-path of the form  $\langle a, b \rangle, \langle b \rangle, u\langle b \rangle, u\langle b, c \rangle$  in  $\mathcal{K}(G, \mathcal{O} \cup \mathcal{P})$  which contains no edges of  $\mathcal{K}(G, \mathcal{P})$ . Hence the  $G$ -graph  $\Delta$  obtained from  $\mathcal{K}(G, \mathcal{O} \cup \mathcal{P})$  by removing all edges of  $\mathcal{K}(G, \mathcal{P})$  is quasi-isometric to  $\mathcal{K}(G, \mathcal{O} \cup \mathcal{P})$ . Observe that  $\Delta$  is exactly the  $G$ -graph described in Lemma 6.5, and therefore the modified Deligne complex of  $G$  is quasi-isometric to  $\mathcal{K}(G, \mathcal{O} \cup \mathcal{P})$  and hence to  $\mathcal{K}(G, \mathcal{P})$ .  $\square$

So far, we have only considered group pairs consisting of 2-dimensional right-angled Artin groups and their standard maximal flats. By adding additional assumptions on  $\Gamma$ , the arguments of this subsection could be extended to higher-dimensional right-angled Artin groups  $G$  to show that the Deligne complex is quasi-isometric to the coset intersection complex for the group pair  $(G, \mathcal{P})$ , where  $\mathcal{P}$  is the set of maximal standard parabolic subgroups. The idea of the argument would be to start with the Deligne complex  $\mathcal{D}$ , remove the orbits of vertices corresponding to the trivial subgroup, then the cyclic subgroups, then the rank two abelian subgroups, etc., until one obtains the coset intersection complex. Additional assumptions on  $\Gamma$  would be needed to show that this results in quasi-isometric complexes at each stage.

More generally, one could let  $\mathcal{P}$  be the collection of standard parabolic subgroups of rank at least  $k$  for some  $k \geq 1$ . In this case, it is likely that weaker additional assumptions on  $\Gamma$  would yield a quasi-isometry between the Deligne complex and  $\mathcal{K}(G, \mathcal{P})$  by a similar argument.

**6.2. Commensurated cores in right-angled Artin groups.** In general, it not known how replacing a collection  $\mathcal{P}$  of subgroups with its commensurated core will affect the quasi-isometry type of  $\mathcal{K}(G, \mathcal{P})$ . However, for the right-angled Artin groups considered in this section, replacing the collection  $\mathcal{P}$  of maximal standard flats with its commensurated core  $\mathcal{Q}$  actually preserves the quasi-isometry type. It remains to explore this relationship in general.

Recall from Proposition 6.3 that

$$\mathcal{Q} = \{ \langle \text{Star}(a) \rangle \mid a \text{ is a standard generator} \}.$$

**Proposition 6.6.** *Suppose  $\Gamma$  is connected, has no vertices of valence  $< 2$ , and no triangles. There is  $G$ -equivariant quasi-isometry between  $\mathcal{K}(G, \mathcal{P})$  and  $\mathcal{K}(G, \mathcal{Q})$ .*

Before proving the proposition, we gather two lemmas about the structure of  $\mathcal{K}(G, \mathcal{P})$ .

**Lemma 6.7.** *If  $\Gamma$  is connected, then  $\mathcal{K}(G, \mathcal{P})$  is connected.*

*Proof.* Let  $\Delta$  be the graph with vertex set  $\mathcal{P}$  and such that two subgroups are adjacent if they have infinite intersection. Equivalently,  $\Delta$  is the graph with vertex set the set of edges of  $\Gamma$  and such that two edges are adjacent if they have an

endpoint in common. Since  $\Gamma$  is connected,  $\Delta$  is connected. Moreover  $\Delta$  has a natural embedding into  $\mathcal{K}(G, \mathcal{P})$  such that  $\Delta$  contains representatives of all  $G$ -orbits of vertices of  $\mathcal{K}(G, \mathcal{P})$ . Observe that for any standard generator  $a$ , the subgraphs  $\Delta$  and  $a\Delta$  of  $\mathcal{K}(G, \mathcal{P})$  intersect in at least one vertex. It follows that for any  $g \in G$ , there is a path from  $\Delta$  to  $g\Delta$  in  $\mathcal{K}(G, \mathcal{P})$ . Therefore  $\mathcal{K}(G, \mathcal{P})$  is connected.  $\square$

**Lemma 6.8.** *If  $\Gamma$  has no vertices of valence  $< 2$  and no triangles, then any two distinct maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  intersect in at most one vertex.*

*Proof.* Let  $\mathcal{S}$  and  $\mathcal{S}'$  be two distinct maximal simplices. Suppose without loss of generality that  $K(\mathcal{S}) = \langle a \rangle$  and  $K(\mathcal{S}') = w\langle b \rangle w^{-1}$  for some  $w \in G$ . Since  $\mathcal{S} \neq \mathcal{S}'$ , we have that  $\langle a \rangle \cap w\langle b \rangle w^{-1}$  is trivial by Lemma 6.4. If  $\mathcal{S} \cap \mathcal{S}'$  contains two distinct left cosets  $g_1 P_1$  and  $g_2 P_2$ , then  $\langle a \rangle = g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1} = w\langle b \rangle w^{-1}$  which is impossible.  $\square$

We are now ready to prove Proposition 6.6.

*Proof of Proposition 6.6.* We begin by defining two auxiliary quasi-isometric  $G$ -graphs,  $\mathcal{E}$  and its barycentric subdivision  $\mathcal{E}'$ . We will show that  $\mathcal{K}(G, \mathcal{P})$  is quasi-isometric to  $\mathcal{E}'$  and that  $\mathcal{K}(G, \mathcal{Q})$  is quasi-isometric to  $\mathcal{E}$ . All of these quasi-isometries will be  $G$ -equivariant, which will prove the result.

Let  $\mathcal{E}$  be the  $G$ -graph with vertex set in bijection with the set of maximal simplices in  $\mathcal{K}(G, \mathcal{P})$  and an edge between two vertices if the corresponding maximal simplices share a common vertex. Let  $\mathcal{E}'$  be the barycentric subdivision of  $\mathcal{E}$ , so that  $\mathcal{E}$  is  $G$ -equivariantly quasi-isometric to  $\mathcal{E}'$ .

We first show that  $\mathcal{K}(G, \mathcal{P})$  is quasi-isometric to  $\mathcal{E}'$ . Since  $\Gamma$  has no vertices of valence  $< 2$ , the  $G$ -sets  $G/\mathcal{P}$  and  $G/\mathcal{Q}$  are disjoint. By Proposition 6.3, we can equivalently define  $\mathcal{E}'$  to be the bipartite  $G$ -graph with vertex set  $G/\mathcal{P} \sqcup G/\mathcal{Q}$  and edge set given by inclusion of cosets. Using this alternative description of  $\mathcal{E}'$  and the fact that maximal simplices of  $\mathcal{K}(G, \mathcal{P})$  are in bijection with  $G/\mathcal{Q}$  by Proposition 6.3, we see that  $\mathcal{E}'$  is formed from the 1-skeleton of  $\mathcal{K}(G, \mathcal{P})$  by coning off each maximal simplex with a vertex and then removing the original edges of  $\mathcal{K}(G, \mathcal{P})$ . Since maximal simplices have finite diameter, coning off each one does not change the quasi-isometry type. Deleting the edges of  $\mathcal{K}(G, \mathcal{P})$  also does not change the quasi-isometry type, since there is a 2-path in  $\mathcal{E}'$  between the endpoints of each edge removed.

We next show that  $\mathcal{K}(G, \mathcal{Q})$  is quasi-isometric to  $\mathcal{E}$ . By Lemmas 6.4 and 6.8, the graph  $\mathcal{E}$  can be equivalently defined as the graph with vertex set  $G/\mathcal{Q}$  and such that  $g_1 Q_1, g_2 Q_2$  are adjacent if and only if  $Q_1^{g_1} \cap Q_2^{g_2}$  has rank two. To prove  $\mathcal{K}(G, \mathcal{Q})$  and  $\mathcal{E}$  are quasi-isometric, first note that both have the same vertex set and every edge of  $\mathcal{E}$  is an edge of  $\mathcal{K}(G, \mathcal{Q})$ . For distinct vertices  $g_1 Q_1, g_2 Q_2 \in G/\mathcal{Q}$ , the intersection  $Q_1^{g_1} \cap Q_2^{g_2}$  is a free abelian group of rank at most two. If the rank is two, then this intersection is an edge of  $\mathcal{E}$ . If  $Q_1^{g_1} \cap Q_2^{g_2}$  has rank one, then  $g_1 Q_2$  and  $g_2 Q_2$  are vertices at distance two in  $\mathcal{E}$ . Therefore the inclusion  $\mathcal{E} \hookrightarrow \mathcal{K}(G, \mathcal{Q})$  is a  $G$ -equivariant quasi-isometry.  $\square$

We end our discussion of right-angled Artin groups with several general questions about the associated coset intersection complex.

**Question 6.9.** Let  $G$  be a 2-dimensional right-angled Artin group with defining graph  $\Gamma$ , and  $\mathcal{P}$  the collection of standard maximal flats. Suppose the induced flag complex  $\Gamma^\blacktriangle$  is simply-connected. Is  $\mathcal{K}(G, \mathcal{P})$  simply connected?

**Question 6.10.** If  $G$  is a 2-dimensional right-angled Artin group and  $\mathcal{P}$  is the collection of standard maximal flats, is  $\mathcal{K}(G, \mathcal{P})$  contractible?

More generally, one can ask the following.

**Question 6.11.** Let  $G$  be a 2-dimensional RAAG and  $\mathcal{P}$  the collection of standard maximal flats. Let  $\mathcal{F}$  be the family generated by  $\mathcal{P}$  that is closed under conjugation and subgroups. Is  $\mathcal{K}(G, \mathcal{P})$  a classifying space for  $G$  with respect to  $\mathcal{F}$ ?

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