An extension of Gauss congruences for Apéry numbers

Ji-Cai Liu

Department of Mathematics, Wenzhou University, Wenzhou 325035, PR China

jcliu2016@gmail.com

Abstract. Osburn, Sahu and Straub introduced the numbers:

$$A_n^{(r,s,t)}=\sum_{k=0}^{n}\binom{n}{k}^r\binom{n+k}{k}^s\binom{2k}{n}^t,$$

for non-negative integers n,r,s,t with $r\ge 2$, which includes two kinds of Apéry numbers and four kinds of Apéry-like numbers as special cases, and showed that the numbers $\{A_n^{(r,s,t)}\}_{n\ge 0}$ satisfy the Gauss congruences of order 3. We establish an extension of Osburn–Sahu–Straub congruence through Bernoulli numbers, which is one step deep congruence of the Gauss congruence for $A_n^{(r,s,t)}$.

Keywords: Gauss congruences; Apéry numbers; Bernoulli numbers

MR Subject Classifications: 11B50, 11B65, 11B68

1 Introduction

In 1979, Apéry [2] introduced the two sequences $\{a_n\}_{n\ge 0}$ and $\{b_n\}_{n\ge 0}$ through 3-term recurrences in the proof of the irrationality of $\zeta(3)$ and $\zeta(2)$:

$$(n+1)^3a_{n+1}-(2n+1)(17n^2+17n+5)a_n+n^3a_{n-1}=0,\quad (a_0=1,a_1=5),\tag{1.1}$$

$$(n+1)^2b_{n+1}-(11n^2+11n+3)b_n-n^2b_{n-1}=0,\quad (b_0=1,b_1=3).\tag{1.2}$$

The two sequences $\{a_n\}_{n\ge 0}$ and $\{b_n\}_{n\ge 0}$ are known as the famous Apéry numbers, which possess the binomial sum formulae:

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2,$$

$$b_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

Zagier [11] considered the following recurrence related to (1.2):

$$(n+1)^2u_{n+1}-(An^2+An+\lambda)u_n+Bn^2u_{n-1}=0,\quad (u_{-1}=0,u_0=1),\tag{1.3}$$

and searched for triples $(A,B,\lambda)\in\mathbb{Z}^3$ such that the solution of the recurrence (1.3) is an integer sequence $\{u_n\}_{n\geq 0}$. Six sporadic sequences are found in Zagier's search, which include the desired solution $\{b_n\}_{n\geq 0}$. The six sporadic sequences are listed in the following table.

(A,B,λ)	Name	Other names	Formula
$(7,-8,2)$	**A**	Franel numbers	$u_n=\sum_{k=0}^n\binom{n}{k}^3$
$(9,27,3)$	**B**		$u_n=\sum_{k=0}^n(-1)^k3^{n-3k}\binom{n}{3k}\binom{3k}{2k}\binom{2k}{k}$
$(10,9,3)$	**C**		$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}$
$(11,-1,3)$	**D**	Apéry numbers	$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}$
$(12,32,4)$	**E**		$u_n=\sum_{k=0}^n4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2$
$(17,72,6)$	**F**		$u_n=\sum_{k=0}^n(-1)^k8^{n-k}\binom{n}{k}\sum_{j=0}^k\binom{k}{j}^3$

Almkvist and Zudilin [1] studied the other recurrence related to (1.1):

$$(n+1)^3u_{n+1}-(2n+1)(an^2+an+b)u_n+cn^3u_{n-1}=0,\quad (u_{-1}=0,u_0=1),\tag{1.4}$$

and searched for triples $(a,b,c)\in\mathbb{Z}^3$ such that the solution of the recurrence (1.4) is an integer sequence $\{u_n\}_{n\geq 0}$. They also found six sporadic sequences, which include the desired solution $\{a_n\}_{n\geq 0}$.

Cooper [3] considered a more general recurrence:

$$(n+1)^3u_{n+1}-(2n+1)(an^2+an+b)u_n+n(cn^2+d)u_{n-1}=0,\tag{1.5}$$

with $u_{-1}=0$ and $u_0=1$. Note that the case $d=0$ of (1.5) reduces to (1.4). He found three additional sporadic sequences s_7,s_{10} and s_{18}. The nine sporadic sequences are listed in the following table.

(a,b,c,d)	Name	Other names	Formula
$(7,3,81,0)$	(δ)	Almkvist-Zudilin numbers	$u_n=\sum_{k=0}^n(-1)^k3^{n-3k}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k}$
$(11,5,125,0)$	(η)		$u_n=\sum_{k=0}^n(-1)^k\binom{n}{k}^3(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n})$
$(10,4,64,0)$	(α)	Domb numbers	$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$
$(12,4,16,0)$	(ϵ)		$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27,0)$	(ζ)		$u_n=\sum_{k=0}^n\sum_{l=0}^n\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1,0)$	(γ)	Apéry numbers	$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2$
$(13,4,-27,3)$	s_7		$u_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$
$(6,2,-64,4)$	s_{10}	Yang-Zudilin numbers	$u_n=\sum_{k=0}^n\binom{n}{k}^4$
$(14,6,192,-12)$	s_{18}		$u_n=\sum_{k=0}^n(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}(\binom{2n-3k-1}{n}+\binom{2n-3k}{n})$

An integer sequence $\{u_n\}_{n\geq 1}$ is said to satisfy the Gauss congruences of order r if $u_{np^k}\equiv u_{np^{k-1}}\pmod{p^{rk}}$ for all positive integers n,k and all primes $p\geq r+1$. Let $\mathbb{N}$

denote the set of non-negative integers, $\mathbb{Z}^+$ denote the set of positive integers, and $\mathbb{Q}$ denote the set of rational numbers.

As early as 1982, Gessel [5] proved that for all primes $p \geq 5$ and $n \in \mathbb{Z}^+$,

$$a_{np} \equiv a_n \pmod{p^3}.$$

Coster [4] further showed that the Apéry numbers $\{a_n\}_{n\geq 0}$ satisfy the Gauss congruences of order 3, namely,

$$a_{np^m} \equiv a_{np^{m-1}} \pmod{p^{3m}},$$

for primes $p \geq 5$ and $n, m \in \mathbb{Z}^+$.

In 2016, Osburn, Sahu and Straub [7] defined

$$A_n^{(r,s,t)} = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s \binom{2k}{n}^t,$$

for $n, r, s, t \in \mathbb{N}$ with $r \geq 2$. Note that $A_n^{(r,s,t)}$ includes the sequences $\mathbf{A}, \mathbf{D}, (\epsilon), (\gamma), s_7, s_{10}$ as special cases.

Special cases of $A_n^{(r,s,t)}$

(r,s,t)	Name	Other names	Formula
$(3,0,0)$	$\mathbf{A}$	Franel numbers	$u_n = \sum_{k=0}^n \binom{n}{k}^3$
$(2,1,0)$	$\mathbf{D}$	Apéry numbers	$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}$
$(2,0,2)$	(ϵ)		$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n}^2$
$(2,2,0)$	(γ)	Apéry numbers	$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$
$(2,1,1)$	s_7		$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}$
$(4,0,0)$	s_{10}	Yang-Zudilin numbers	$u_n = \sum_{k=0}^n \binom{n}{k}^4$

Osburn, Sahu and Straub [7] also showed that the numbers $\{A_n^{(r,s,t)}\}_{n\geq 0}$ also satisfy the Gauss congruences of order 3, namely,

$$A_{np^m}^{(r,s,t)} \equiv A_{np^{m-1}}^{(r,s,t)} \pmod{p^{3m}}, \tag{1.6}$$

for all primes $p \geq 5$, $n, m \in \mathbb{Z}^+$ and $r, s, t \in \mathbb{N}$ with $r \geq 2$.

The Bernoulli numbers $\{B_n\}_{n\geq 0}$ are defined by the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

In 2020, Sun [10, Conjectures 5.1 and 5.3] conjectured a series of supercongruences concerning u_p, u_{2p} and u_{3p} modulo p^4, where $\{u_n\}_{n\geq 0}$ is one of the sequences $\mathbf{A}, \mathbf{D}, (\alpha), (\gamma), (\epsilon)$.

The motivation of the paper is to establish an extension of the Gauss congruence (1.6), which involves the Bernoulli numbers. The main result is stated as follows.

Theorem 1.1 *Let $p \geq 5$ be a prime, $n, m \in \mathbb{Z}^+$, and $r, s, t \in \mathbb{N}$ with $r \geq 2$. Then*

$$A_{np^m}^{(r,s,t)} \equiv A_{np^{m-1}}^{(r,s,t)} + p^{3m} B_{p-3} \mathcal{A}_n^{(r,s,t)} \pmod{p^{3m+1}},$$

where $\mathcal{A}_n^{(r,s,t)}$, independent of m and p, are given by

$$\begin{aligned}\mathcal{A}_n^{(2,s,t)} &= -\frac{1}{3}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t nk(sn+sk+2n-2k+4tk-2tn) \\ &\quad + \frac{1}{6}\sum_{k=0}^{n-1} \binom{n}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t (n-k)^2 (9nt-3ns+24k-18n+14) \\ &\quad + \frac{1}{6}\sum_{k=0}^{n-1} \binom{n}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t (n-k)^2 (9ns+15nt-24k+6n-10),\end{aligned}$$

$$\begin{aligned}\mathcal{A}_n^{(3,s,t)} &= -\frac{1}{3}\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}^s \binom{2k}{n}^t nk(sn+sk+3n-3k+4tk-2tn) \\ &\quad + \frac{1}{4}\sum_{k=0}^{n-1} (n-k)^3 \binom{n}{k}^3 \binom{n+k}{k}^s \left(\binom{2k}{n}^t + \binom{2k+1}{n}^t\right),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{A}_n^{(r,s,t)} \\ &= -\frac{1}{3}\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s \binom{2k}{n}^t nk(sn+sk+rn-rk+4tk-2tn) \quad \textit{for } r \geq 4.\end{aligned}$$

The rest of the paper is organized as follows. We establish some preliminary results in the next section. The proof of Theorem 1.1 is presented in Section 3, which is divided into two parts.

2 Preliminaries

Throughout the paper, let $\sum'$ denote the sum over indices not divisible by p, $\lfloor x \rfloor$ denote the integral part of real x, and $\{k/p^m\}$ denote the remainder of k divided by p^m, namely,

$$\{k/p^m\} = k - p^m \lfloor k/p^m \rfloor.$$

Lemma 2.1 *(See [6].) Let $p \geq 5$ be a prime, and $n, k \in \mathbb{Z}^+$ with $n \geq k$. Then*

$$\binom{np}{kp} \Big/ \binom{n}{k} \equiv 1 - \frac{nk(n-k)}{3} p^3 B_{p-3} \pmod{p^{ord_p(nk(n-k))+4}}, \tag{2.1}$$

where $ord_p(x)$ denotes the exponent of p in $x \in \mathbb{Q}$.

Lemma 2.2 *Let $p \geq 5$ be a prime, $a, b, c, r, s, t \in \mathbb{N}$ and $n, m \in \mathbb{Z}^+$. Then*

$$\begin{aligned}
&\binom{np^m-1}{a}^r \binom{np^m+b}{b}^s \binom{c}{np^m}^t \\
&\equiv (-1)^{r(a+\lfloor a/p \rfloor)} \binom{np^{m-1}-1}{\lfloor a/p \rfloor}^r \binom{np^{m-1}+\lfloor b/p \rfloor}{\lfloor b/p \rfloor}^s \binom{\lfloor c/p \rfloor}{np^{m-1}}^t \\
&\times \left(1 - rnp^m \sum_{j=1}^{a}{}' \frac{1}{j} + snp^m \sum_{j=1}^{b}{}' \frac{1}{j} + tnp^m \sum_{j=1}^{c}{}' \frac{1}{j}\right) \pmod{p^{m+1}}. \qquad (2.2)
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
\binom{np^m-1}{a} &= \prod_{j=1}^{a} \frac{np^m-j}{j} \\
&= \prod_{\substack{j=1 \\ p\nmid j}}^{a} \frac{np^m-j}{j} \prod_{i=1}^{\lfloor a/p \rfloor} \frac{np^{m-1}-i}{i} \\
&= \binom{np^{m-1}-1}{\lfloor a/p \rfloor} \prod_{\substack{j=1 \\ p\nmid j}}^{a} \frac{np^m-j}{j} \\
&\equiv (-1)^{a+\lfloor a/p \rfloor} \binom{np^{m-1}-1}{\lfloor a/p \rfloor} \left(1 - np^m \sum_{j=1}^{a}{}' \frac{1}{j}\right) \pmod{p^{m+1}}. \qquad (2.3)
\end{aligned}$$

By (2.3), we have

$$\begin{aligned}
\binom{np^m+b}{b} &= (-1)^b \binom{-np^m-1}{b} \\
&\equiv \binom{np^{m-1}+\lfloor b/p \rfloor}{\lfloor b/p \rfloor} \left(1 + np^m \sum_{j=1}^{b}{}' \frac{1}{j}\right) \pmod{p^{m+1}}, \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
\binom{c}{np^m} &= \binom{np^m + c - np^m}{c - np^m} \\
&\equiv \binom{np^{m-1} + \lfloor c/p \rfloor - np^{m-1}}{\lfloor c/p \rfloor - np^{m-1}} \left(1 + np^m \sum_{j=1}^{c-np^m}{}' \frac{1}{j}\right) \pmod{p^{m+1}}. \qquad (2.5)
\end{aligned}$$

Since $\sum_{j=1}^{p-1} \frac{1}{j^d} \equiv 0 \pmod{p}$ for $d \in \mathbb{Z}^+$, we have

$$\sum_{j=1}^{a}{}' \frac{1}{j^d} \equiv \sum_{j=1}^{b}{}' \frac{1}{j^d} \pmod{p}, \quad \text{for } d \in \mathbb{Z}^+ \text{ and } a \equiv b \pmod{p}. \qquad (2.6)$$

Then we rewrite (2.5) as

$$\binom{c}{np^m} \equiv \binom{\lfloor c/p \rfloor}{np^{m-1}} \left(1 + np^m \sum_{j=1}^{c}{}' \frac{1}{j}\right) \pmod{p^{m+1}}. \tag{2.7}$$

Combining (2.3), (2.4) and (2.7), we arrive at the desired result (2.2). □

Lemma 2.3 *Let $p \geq 5$ be a prime, $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. Then*

$$\sum_{\substack{\lfloor k/p^m \rfloor = n \\ \{k/p^m\} < p^m/2}}{}' \frac{1}{k^2} \equiv \frac{12n+7}{3} p^m B_{p-3} \pmod{p^{m+1}}, \tag{2.8}$$

$$\sum_{\substack{\lfloor k/p^m \rfloor = n \\ \{k/p^m\} > p^m/2}}{}' \frac{1}{k^2} \equiv -\frac{12n+5}{3} p^m B_{p-3} \pmod{p^{m+1}}. \tag{2.9}$$

Proof. By [8, Theorem 1.2], we have

$$\sum_{k=1}^{p^m-1}{}' \frac{1}{k^2} \equiv \frac{2}{3} p^m B_{p-3} \pmod{p^{m+1}}. \tag{2.10}$$

Since

$$\frac{1}{(p^m-k)^2} \equiv \frac{1}{k^2} + \frac{2}{k^3} p^m \pmod{p^{m+1}},$$

we have

$$\begin{aligned} \sum_{k=1}^{p^m-1}{}' \frac{1}{k^2} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^2} + \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{(p^m-k)^2} \\ &\equiv 2 \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^2} + 2p^m \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^3} \pmod{p^{m+1}}. \end{aligned} \tag{2.11}$$

By (2.6) and the fact $(p^m-1)/2 \equiv (p-1)/2 \pmod{p}$, we have

$$\begin{aligned} \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^3} &\equiv \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \\ &\equiv -2B_{p-3} \pmod{p}, \end{aligned} \tag{2.12}$$

where we have used the result [9, Corollary 5.2] in the last step.

Furthermore, combining (2.10), (2.11) and (2.12) gives

$$\sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^2} \equiv \frac{7}{3} p^m B_{p-3} \pmod{p^{m+1}}. \tag{2.13}$$

Thus, by (2.12) and (2.13) we have

$$\begin{aligned}
\sum_{\substack{\lfloor k/p^m\rfloor=n\\ \{k/p^m\}<p^m/2}}{}' \frac{1}{k^2} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{(np^m+k)^2}\\
&\equiv \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^2} - 2np^m \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^3}\\
&\equiv \frac{12n+7}{3}p^m B_{p-3} \pmod{p^{m+1}},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{\lfloor k/p^m\rfloor=n\\ \{k/p^m\}>p^m/2}}{}' \frac{1}{k^2} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{((n+1)p^m-k)^2}\\
&\equiv \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^2} + 2(n+1)p^m \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^3}\\
&\equiv -\frac{12n+5}{3}p^m B_{p-3} \pmod{p^{m+1}},
\end{aligned}$$

as desired. □

Lemma 2.4 *Let $p \ge 5$ be a prime and $n, l \in \mathbb{N}$. Then*

$$\sum_{\substack{\lfloor k/p^{l+1}\rfloor=n\\ \{k/p^{l+1}\}<p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^l\rfloor}{}' \frac{1}{j} \equiv \frac{1}{3}p^l B_{p-3} \pmod{p^{l+1}}, \tag{2.14}$$

$$\sum_{\substack{\lfloor k/p^{l+1}\rfloor=n\\ \{k/p^{l+1}\}>p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^l\rfloor}{}' \frac{1}{j} \equiv \frac{1}{3}p^l B_{p-3} \pmod{p^{l+1}}, \tag{2.15}$$

$$\sum_{\substack{\lfloor k/p^{l+1}\rfloor=n\\ \{k/p^{l+1}\}<p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor 2k/p^l\rfloor}{}' \frac{1}{j} \equiv \frac{4}{3}p^l B_{p-3} \pmod{p^{l+1}}, \tag{2.16}$$

$$\sum_{\substack{\lfloor k/p^{l+1}\rfloor=n\\ \{k/p^{l+1}\}>p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor 2k/p^l\rfloor}{}' \frac{1}{j} \equiv \frac{4}{3}p^l B_{p-3} \pmod{p^{l+1}}. \tag{2.17}$$

Proof. Note that

$$\sum_{\substack{\lfloor k/p^{l+1}\rfloor=n\\ \{k/p^{l+1}\}<p^{l+1}/2}}{}' \frac{1}{k^2}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j} = \sum_{k=np^{l+1}+1}^{np^{l+1}+(p^{l+1}-1)/2}{}'\frac{1}{k^2}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j}$$

$$= \sum_{m=np}^{np+\frac{p-3}{2}}\sum_{\lfloor k/p^l\rfloor=m}{}'\frac{1}{k^2}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j} + \sum_{\substack{\lfloor k/p^l\rfloor=np+(p-1)/2\\ \{k/p^l\}<p^l/2}}{}'\frac{1}{k^2}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j}. \tag{2.18}$$

By [8, Theorem 1.2], we have

$$\sum_{\lfloor k/p^l\rfloor=m}{}'\frac{1}{k^2} \equiv \frac{2}{3}p^l B_{p-3} \pmod{p^{l+1}}.$$

It follows that

$$\begin{aligned}
&\sum_{m=np}^{np+\frac{p-3}{2}}\sum_{\lfloor k/p^l\rfloor=m}{}'\frac{1}{k^2}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j}\\
&\equiv \frac{2}{3}p^l B_{p-3}\sum_{m=np}^{np+\frac{p-3}{2}}\sum_{j=1}^{m}{}'\frac{1}{j}\\
&\equiv \frac{2}{3}p^l B_{p-3}\sum_{k=1}^{(p-3)/2}\sum_{j=1}^{k}\frac{1}{j} \pmod{p^{l+1}},
\end{aligned} \tag{2.19}$$

where we have used (2.6) in the last step.

On the other hand, by (2.6) and (2.8) we have

$$
\begin{aligned}
&\sideset{}{'}\sum_{\substack{\lfloor k/p^l\rfloor = np+(p-1)/2\\ \{k/p^l\}<p^l/2}} \frac{1}{k^2} \sideset{}{'}\sum_{j=1}^{\lfloor k/p^l\rfloor} \frac{1}{j}\\
&= \sideset{}{'}\sum_{j=1}^{np+(p-1)/2} \frac{1}{j} \sideset{}{'}\sum_{\substack{\lfloor k/p^l\rfloor = np+(p-1)/2\\ \{k/p^l\}<p^l/2}} \frac{1}{k^2}\\
&\equiv \frac{1}{3}p^l B_{p-3} \sideset{}{'}\sum_{j=1}^{np+(p-1)/2} \frac{1}{j}\\
&\equiv \frac{1}{3}p^l B_{p-3} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^{l+1}}.
\end{aligned}
\tag{2.20}
$$

It follows from (2.18), (2.19) and (2.20) that

$$
\begin{aligned}
&\sideset{}{'}\sum_{\substack{\lfloor k/p^{l+1}\rfloor = n\\ \{k/p^{l+1}\}<p^{l+1}/2}} \frac{1}{k^2} \sideset{}{'}\sum_{j=1}^{\lfloor k/p^l\rfloor} \frac{1}{j}\\
&\equiv \frac{2}{3}p^l B_{p-3} \sum_{k=1}^{(p-3)/2} \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{3}p^l B_{p-3} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^{l+1}}.
\end{aligned}
\tag{2.21}
$$

Note that

$$
\begin{aligned}
&2\sum_{k=1}^{(p-3)/2} \sum_{j=1}^{k} \frac{1}{j} + \sum_{k=1}^{(p-1)/2} \frac{1}{k}\\
&= 2\sum_{k=1}^{(p-3)/2} \frac{1}{k}\left(\frac{p-1}{2} - k\right) + \sum_{k=1}^{(p-1)/2} \frac{1}{k}\\
&\equiv -\sum_{k=1}^{(p-3)/2} \frac{1}{k} + \sum_{k=1}^{(p-1)/2} \frac{1}{k} + 3\\
&= \frac{2}{p-1} + 3\\
&\equiv 1 \pmod{p}.
\end{aligned}
\tag{2.22}
$$

Then the proof of (2.14) follows from (2.21) and (2.22). The proof of (2.15)–(2.17) runs analogously, and we omit the details. □

Lemma 2.5 *Let $p \geq 5$ be a prime and $m \in \mathbb{Z}^+$. Then*

$$\sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^k}{k^3} \equiv -\frac{1}{4} B_{p-3} \pmod p. \tag{2.23}$$

Proof. By (2.12), we have

$$\begin{aligned}\sum_{k=1}^{p^m-1}{}' \frac{(-1)^k}{k^3} &= \frac{1}{4} \sum_{k=1}^{(p^m-1)/2}{}' \frac{1}{k^3} - \sum_{k=1}^{p^m-1}{}' \frac{1}{k^3} \\ &\equiv -\frac{1}{2} B_{p-3} \pmod p. \end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned}\sum_{k=1}^{p^m-1}{}' \frac{(-1)^k}{k^3} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^k}{k^3} + \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^{p^m-k}}{(p^m-k)^3} \\ &\equiv 2 \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^k}{k^3} \pmod p. \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25), we complete the proof of (2.23). □

3 Proof of Theorem 1.1

Let

$$S_n^{(r,s,t)}(k) = \binom{n}{k}^r \binom{n+k}{k}^s \binom{2k}{n}^t.$$

Note that

$$\begin{aligned} A_{np^m}^{(r,s,t)} &= \sum_{k=0}^{np^m} S_{np^m}^{(r,s,t)}(k) \\ &= \sum_{l\geq 1} \sum_k{}' S_{np^m}^{(r,s,t)}(kp^l) + \sum_{k=0}^{np^m}{}' S_{np^m}^{(r,s,t)}(k). \end{aligned} \tag{3.1}$$

Firstly, we establish a preliminary result on $\sum_{l\geq 1}\sum_k{}' S_{np^m}^{(r,s,t)}(kp^l)$ modulo p^{3m+1}. Letting $n \to np^{m-1}$ and $k \to kp^{l-1}$ in (2.1), we obtain

$$\begin{aligned} &\binom{np^m}{kp^l} \Big/ \binom{np^{m-1}}{kp^{l-1}} \\ &\equiv 1 - \frac{nk(np^{m-1}-kp^{l-1})}{3} p^{m+l+1} B_{p-3} \pmod{p^{m+l+\min(m,l)+1}}. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\binom{2kp^l}{np^m}\Big/\binom{2kp^{l-1}}{np^{m-1}}$$
$$\equiv 1-\frac{2nk(2kp^{l-1}-np^{m-1})}{3}p^{m+l+1}B_{p-3} \pmod{p^{m+l+\min(m,l)+1}}, \tag{3.3}$$

and

$$\binom{np^m+kp^l}{kp^l}\Big/\binom{np^{m-1}+kp^{l-1}}{kp^{l-1}}$$
$$\equiv 1-\frac{nk(np^{m-1}+kp^{l-1})}{3}p^{m+l+1}B_{p-3} \pmod{p^{m+l+\min(m,l)+1}}. \tag{3.4}$$

It follows from (3.2)–(3.4) that

$$\frac{S^{(r,s,t)}_{np^m}(kp^l)}{S^{(r,s,t)}_{np^{m-1}}(kp^{l-1})}$$
$$\equiv 1-\frac{1}{3}B_{p-3}\left(kn^2p^{2m+l}(r+s-2t)+nk^2p^{m+2l}(s-r+4t)\right) \pmod{p^{m+l+\min(m,l)+1}}. \tag{3.5}$$

Next, we shall prove that for $p\nmid k$,

$$S^{(r,s,t)}_{np^m}(kp^l)\equiv S^{(r,s,t)}_{np^{m-1}}(kp^{l-1})$$
$$-\frac{1}{3}B_{p-3}\left(kn^2p^{2m+l}(r+s-2t)+nk^2p^{m+2l}(s-r+4t)\right)S^{(r,s,t)}_{np^{m-1}}(kp^{l-1}) \pmod{p^{3m+1}}. \tag{3.6}$$

For $l\geq m$, we have $m+l+\min(m,l)+1\geq 3m+1$, and so (3.6) follows from (3.5) directly. For $l<m$, we have

$$m+l+\min(m,l)+1=m+2l+1. \tag{3.7}$$

Noting that for $p\nmid k$ and $r\geq 2$,

$$\binom{np^{m-1}}{kp^{l-1}}^r=\left(\frac{n}{k}\right)^r\binom{np^{m-1}-1}{kp^{l-1}-1}^r p^{r(m-l)}\equiv 0 \pmod{p^{2m-2l}},$$

we have

$$S^{(r,s,t)}_{np^{m-1}}(kp^{l-1})\equiv 0 \pmod{p^{2m-2l}}. \tag{3.8}$$

Then (3.6) follows from (3.5), (3.7) and (3.8).

By (3.6), we have

$$\begin{aligned}
&\sum_{l\geq 1}\sum_k{}' S_{np^m}^{(r,s,t)}(kp^l)\\
&\equiv \sum_{l\geq 1}\sum_k{}' S_{np^{m-1}}^{(r,s,t)}(kp^{l-1})\\
&-\frac{1}{3}n^2(r+s-2t)B_{p-3}\sum_{l\geq 1}\sum_k{}' p^{2m+l}kS_{np^{m-1}}^{(r,s,t)}(kp^{l-1})\\
&-\frac{1}{3}n(s-r+4t)B_{p-3}\sum_{l\geq 1}\sum_k{}' p^{m+2l}k^2S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) \pmod{p^{3m+1}}.
\end{aligned}$$

Noting that

$$\sum_{l\geq 1}\sum_k{}' S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) = A_{np^{m-1}}^{(r,s,t)},$$

and $2m+l, m+2l \geq 3m+1$ for $l > m$, we obtain

$$\begin{aligned}
&\sum_{l\geq 1}\sum_k{}' S_{np^m}^{(r,s,t)}(kp^l)\\
&\equiv A_{np^{m-1}}^{(r,s,t)}\\
&-\frac{1}{3}n^2(r+s-2t)B_{p-3}\sum_{l=1}^{m}\sum_k{}' p^{2m+l}kS_{np^{m-1}}^{(r,s,t)}(kp^{l-1})\\
&-\frac{1}{3}n(s-r+4t)B_{p-3}\sum_{l=1}^{m}\sum_k{}' p^{m+2l}k^2S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) \pmod{p^{3m+1}}.
\end{aligned} \tag{3.9}$$

For $p \nmid k$, $r \geq 2$ and $l < m$, we have

$$p^{2m+l}\binom{np^{m-1}}{kp^{l-1}}^r = p^{(r+2)m+(1-r)l}\left(\frac{n}{k}\right)^r\binom{np^{m-1}-1}{kp^{l-1}-1}^r \equiv 0 \pmod{p^{3m+1}},$$

and so

$$\sum_k{}' p^{2m+l}kS_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) \equiv 0 \pmod{p^{3m+1}}.$$

It follows that

$$\begin{aligned}
&\sum_{l=1}^{m}\sum_k{}' p^{2m+l}kS_{np^{m-1}}^{(r,s,t)}(kp^{l-1})\\
&\equiv p^{3m}\sum_k{}' kS_{np^{m-1}}^{(r,s,t)}(kp^{m-1})\\
&\equiv p^{3m}\sum_{k=0}^{n}{}' kS_{n}^{(r,s,t)}(k)\\
&\equiv p^{3m}\sum_{k=0}^{n} kS_{n}^{(r,s,t)}(k) \pmod{p^{3m+1}},
\end{aligned} \tag{3.10}$$

where we have used the modulo p version of (2.1).

Combining (3.9) and (3.10) gives

$$\begin{aligned}
&\sum_{l\geq 1}\sum_{k}{}' S_{np^m}^{(r,s,t)}(kp^l)\\
&\equiv A_{np^{m-1}}^{(r,s,t)}\\
&-\frac{1}{3}p^{3m}n^2(r+s-2t)B_{p-3}\sum_{k=0}^{n} kS_n^{(r,s,t)}(k)\\
&-\frac{1}{3}n(s-r+4t)B_{p-3}\sum_{l=1}^{m}\sum_{k}{}' p^{m+2l}k^2S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) \pmod{p^{3m+1}}.
\end{aligned} \tag{3.11}$$

Secondly, we rewrite $\sum_k{}' S_{np^m}^{(r,s,t)}(k)$ as

$$\sum_{k=0}^{np^m}{}' S_{np^m}^{(r,s,t)}(k) = n^r p^{rm}\sum_{k}{}'\frac{1}{k^r}\binom{np^m-1}{k-1}^r\binom{np^m+k}{k}^s\binom{2k}{np^m}^t. \tag{3.12}$$

Let $Y_{m,n,p}^{(r,s,t)}$ denote the last double sum on the right-hand side of (3.11) and $Z_{m,n,p}^{(r,s,t)}$ denote the right-hand side of (3.12), namely,

$$\begin{aligned}
Y_{m,n,p}^{(r,s,t)} &= \sum_{l=1}^{m}\sum_{k}{}' p^{m+2l}k^2S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}),\\
Z_{m,n,p}^{(r,s,t)} &= n^r p^{rm}\sum_{k}{}'\frac{1}{k^r}\binom{np^m-1}{k-1}^r\binom{np^m+k}{k}^s\binom{2k}{np^m}^t.
\end{aligned}$$

It follows from (3.1), (3.11) and (3.12) that

$$\begin{aligned}
A_{np^m}^{(r,s,t)} &\equiv A_{np^{m-1}}^{(r,s,t)}\\
&\quad-\frac{1}{3}p^{3m}n^2(r+s-2t)B_{p-3}\sum_{k=0}^{n} kS_n^{(r,s,t)}(k)\\
&\quad-\frac{1}{3}n(s-r+4t)B_{p-3}Y_{m,n,p}^{(r,s,t)}+Z_{m,n,p}^{(r,s,t)} \pmod{p^{3m+1}}.
\end{aligned} \tag{3.13}$$

In order to establish the Theorem 1.1, it suffices to determine $Y_{m,n,p}^{(r,s,t)}$ and $Z_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1}. Substitution of the computed results $Y_{m,n,p}^{(r,s,t)}$ and $Z_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1} into (3.13) sets, after simplifications, up the Theorem 1.1. The rest of the proof is divided into two parts.

3.1 $Y_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1}

We shall distinguish three cases to determine $Y_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1}.

Case 1 $m = 1$**.**

It is easy to see that

$$\begin{aligned} Y_{1,n,p}^{(r,s,t)} &= p^3 \sum_{k=0}^{n}{}' k^2 S_n^{(r,s,t)}(k) \\ &\equiv p^3 \sum_{k=0}^{n} k^2 S_n^{(r,s,t)}(k) \pmod{p^4}. \end{aligned}$$

Case 2 $m \geq 2$ **and** $r = 2$**.**

Note that

$$\begin{aligned} Y_{m,n,p}^{(2,s,t)} &= p^{3m} n^2 \sum_{l=1}^{m} \sum_{k}{}' \binom{np^{m-1}-1}{kp^{l-1}-1}^2 \binom{np^{m-1}+kp^{l-1}}{kp^{l-1}}^s \binom{2kp^{l-1}}{np^{m-1}}^t \\ &= p^{3m} n^2 \sum_{k}{}' \binom{np^{m-1}-1}{kp^{m-1}-1}^2 \binom{np^{m-1}+kp^{m-1}}{kp^{m-1}}^s \binom{2kp^{m-1}}{np^{m-1}}^t \\ &+ p^{3m} n^2 \sum_{l=1}^{m-1} \sum_{k}{}' \binom{np^{m-1}-1}{kp^{l-1}-1}^2 \binom{np^{m-1}+kp^{l-1}}{kp^{l-1}}^s \binom{2kp^{l-1}}{np^{m-1}}^t. \end{aligned} \tag{3.14}$$

By a repeated use of the modulo p version of (2.2), we obtain

$$\begin{aligned} & p^{3m} n^2 \sum_{k}{}' \binom{np^{m-1}-1}{kp^{m-1}-1}^2 \binom{np^{m-1}+kp^{m-1}}{kp^{m-1}}^s \binom{2kp^{m-1}}{np^{m-1}}^t \\ &\equiv p^{3m} n^2 \sum_{k}{}' \binom{n-1}{k-1}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \\ &= p^{3m} \sum_{k}{}' k^2 S_n^{(2,s,t)}(k) \\ &\equiv p^{3m} \sum_{k=0}^{n} k^2 S_n^{(2,s,t)}(k) \pmod{p^{3m+1}}. \end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} & p^{3m} n^2 \sum_{l=1}^{m-1} \sum_{k}{}' \binom{np^{m-1}-1}{kp^{l-1}-1}^2 \binom{np^{m-1}+kp^{l-1}}{kp^{l-1}}^s \binom{2kp^{l-1}}{np^{m-1}}^t \\ &= p^{3m} n^2 \sum_{p^{m-1} \nmid k} \binom{np^{m-1}-1}{k-1}^2 \binom{np^{m-1}+k}{k}^s \binom{2k}{np^{m-1}}^t \\ &\equiv p^{3m} n^2 \sum_{p^{m-1} \nmid k} \binom{n-1}{\lfloor k/p^{m-1} \rfloor}^2 \binom{n+\lfloor k/p^{m-1} \rfloor}{\lfloor k/p^{m-1} \rfloor}^s \binom{\lfloor 2k/p^{m-1} \rfloor}{n}^t \pmod{p^{3m+1}}, \end{aligned} \tag{3.16}$$

where we have used the modulo p version of (2.2). Note that for $m \in \mathbb{Z}^+$,

$$\lfloor 2k/p^m \rfloor = \begin{cases} 2\lfloor k/p^m \rfloor & \text{if } \{k/p^m\} < p^m/2, \\ 2\lfloor k/p^m \rfloor + 1 & \text{if } \{k/p^m\} > p^m/2. \end{cases} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\begin{aligned} & p^{3m} n^2 \sum_{l=1}^{m-1} \sum_k{}' \binom{np^{m-1}-1}{kp^{l-1}-1}^2 \binom{np^{m-1}+kp^{l-1}}{kp^{l-1}}^s \binom{2kp^{l-1}}{np^{m-1}}^t \\ & \equiv p^{3m} n^2 \sum_N \sum_{\substack{p^{m-1} \nmid k \\ \lfloor k/p^{m-1} \rfloor = N \\ \{k/p^{m-1}\} < p^{m-1}/2}} \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N}{n}^t \\ & + p^{3m} n^2 \sum_N \sum_{\substack{p^{m-1} \nmid k \\ \lfloor k/p^{m-1} \rfloor = N \\ \{k/p^{m-1}\} > p^{m-1}/2}} \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N+1}{n}^t \\ & \equiv -\frac{1}{2} p^{3m} n^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \\ & - \frac{1}{2} p^{3m} n^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t \pmod{p^{3m+1}}, \end{aligned} \tag{3.18}$$

where we have used the following fact:

$$\sum_{\substack{p^{m-1} \nmid k \\ \lfloor k/p^{m-1} \rfloor = N \\ \{k/p^{m-1}\} < p^{m-1}/2}} 1 = \sum_{\substack{p^{m-1} \nmid k \\ \lfloor k/p^{m-1} \rfloor = N \\ \{k/p^{m-1}\} > p^{m-1}/2}} 1 = \frac{p^{m-1}-1}{2} \equiv -\frac{1}{2} \pmod{p}.$$

It follows from (3.14), (3.15) and (3.18) that

$$\begin{aligned} Y_{m,n,p}^{(2,s,t)} & \equiv p^{3m} \sum_{k=0}^{n} k^2 S_n^{(2,s,t)}(k) \\ & - \frac{1}{2} p^{3m} n^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \\ & - \frac{1}{2} p^{3m} n^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t \pmod{p^{3m+1}}. \end{aligned}$$

Case 3 $m \geq 2$ **and** $r \geq 3$**.**

For $p \nmid k$ and $l < m$, we have

$$p^{m+2l} k^2 \binom{np^{m-1}}{kp^{l-1}}^r = p^{(r+1)m+(2-r)l} k^2 \left(\frac{n}{k}\right)^r \binom{np^{m-1}-1}{kp^{l-1}-1}^r \equiv 0 \pmod{p^{3m+1}},$$

and so

$$p^{m+2l} k^2 S_{np^{m-1}}^{(r,s,t)}(kp^{l-1}) \equiv 0 \pmod{p^{3m+1}}.$$

It follows that

$$\begin{aligned} Y_{m,n,p}^{(r,s,t)} &\equiv p^{3m} \sum_k{}' k^2 S_{np^{m-1}}^{(r,s,t)}(kp^{m-1}) \\ &\equiv p^{3m} \sum_{k=0}^{n}{}' k^2 S_n^{(r,s,t)}(k) \\ &\equiv p^{3m} \sum_{k=0}^{n} k^2 S_n^{(r,s,t)}(k) \pmod{p^{3m+1}}, \end{aligned}$$

where we have used the modulo p version of (2.1).

3.2 $Z_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1}

We shall distinguish four cases to determine $Z_{m,n,p}^{(r,s,t)}$ modulo p^{3m+1}.

Case 1 $r \geq 4$**.**

In this case, we have $p^{rm} \equiv 0 \pmod{p^{3m+1}}$, and so

$$Z_{m,n,p}^{(r,s,t)} \equiv 0 \pmod{p^{3m+1}}.$$

Case 2 $r = 3$**.**

By a repeated use of the modulo p version of (2.2), we obtain

$$\begin{aligned} &Z_{m,n,p}^{(3,s,t)} \\ &\equiv -n^3 p^{3m} \sum_k{}' \frac{(-1)^{k+\lfloor k/p^m \rfloor}}{k^3} \binom{n-1}{\lfloor k/p^m \rfloor}^3 \binom{n+\lfloor k/p^m \rfloor}{\lfloor k/p^m \rfloor}^s \binom{\lfloor 2k/p^m \rfloor}{n}^t \\ &= -n^3 p^{3m} \sum_N \binom{n-1}{N}^3 \binom{n+N}{N}^s \binom{2N}{n}^t \sum_{\substack{\lfloor k/p^m \rfloor = N \\ \{k/p^m\} < p^m/2}}{}' \frac{(-1)^{N+k}}{k^3} \\ &\quad - n^3 p^{3m} \sum_N \binom{n-1}{N}^3 \binom{n+N}{N}^s \binom{2N+1}{n}^t \sum_{\substack{\lfloor k/p^m \rfloor = N \\ \{k/p^m\} > p^m/2}}{}' \frac{(-1)^{N+k}}{k^3} \pmod{p^{3m+1}}, \end{aligned} \tag{3.19}$$

where we have used the fact that $\lfloor (k-1)/p^m \rfloor = \lfloor k/p^m \rfloor$ for $p \nmid k$.

By (2.23), we have

$$\begin{aligned}
\sum_{\substack{\lfloor k/p^m \rfloor = N \\ \{k/p^m\} < p^m/2}}{}' \frac{(-1)^{N+k}}{k^3} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^{N+Np^m+k}}{(Np^m+k)^3} \\
&\equiv \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^k}{k^3} \\
&\equiv -\frac{1}{4} B_{p-3} \pmod{p}, \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{\lfloor k/p^m \rfloor = N \\ \{k/p^m\} > p^m/2}}{}' \frac{(-1)^{N+k}}{k^3} &= \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^{N+(N+1)p^m-k}}{((N+1)p^m-k)^3} \\
&\equiv \sum_{k=1}^{(p^m-1)/2}{}' \frac{(-1)^k}{k^3} \\
&\equiv -\frac{1}{4} B_{p-3} \pmod{p}. \qquad (3.21)
\end{aligned}$$

It follows from (3.19)–(3.21) that

$$\begin{aligned}
&Z_{m,n,p}^{(3,s,t)} \\
&\equiv \frac{1}{4} p^{3m} B_{p-3} \sum_k \binom{n-1}{k}^3 \binom{n+k}{k}^s \binom{2k}{n}^t n^3 \\
&+ \frac{1}{4} p^{3m} B_{p-3} \sum_k \binom{n-1}{k}^3 \binom{n+k}{k}^s \binom{2k+1}{n}^t n^3 \pmod{p^{3m+1}}.
\end{aligned}$$

Case 3 $r=2$ and $m=1$.

By (2.2), we have

$$\begin{aligned}
Z_{1,n,p}^{(2,s,t)} &= n^2 p^2 \sum_k{}' \frac{1}{k^2} \binom{np-1}{k-1}^2 \binom{np+k}{k}^s \binom{2k}{n}^t \\
&\equiv n^2 p^2 \sum_k{}' \frac{1}{k^2} \binom{n-1}{\lfloor k/p \rfloor}^2 \binom{n+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{n}^t \\
&\times \left(1 - 2np \sum_{j=1}^{k-1}{}' \frac{1}{j} + snp \sum_{j=1}^{k}{}' \frac{1}{j} + tnp \sum_{j=1}^{2k}{}' \frac{1}{j}\right) \pmod{p^4}, \qquad (3.22)
\end{aligned}$$

where we have used the fact that $\lfloor (k-1)/p \rfloor = \lfloor k/p \rfloor$ for $p \nmid k$.

By (2.8), (2.9) and (3.17), we have

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2}\binom{n-1}{\lfloor k/p \rfloor}^2 \binom{n+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{n}^t \\
&= \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N}{n}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} < p/2}}{}' \frac{1}{k^2} \\
&+ \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N+1}{n}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} > p/2}}{}' \frac{1}{k^2} \\
&\equiv \frac{1}{3} p B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t (12k+7) \\
&- \frac{1}{3} p B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t (12k+5) \pmod{p^2}.
\end{aligned} \tag{3.23}$$

By (3.17), we have

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2}\binom{n-1}{\lfloor k/p \rfloor}^2 \binom{n+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{n}^t \left(-2 \sum_{j=1}^{k-1}{}' \frac{1}{j} + s \sum_{j=1}^{k}{}' \frac{1}{j} + t \sum_{j=1}^{2k}{}' \frac{1}{j} \right) \\
&= \sum_k{}' \frac{1}{k^2}\binom{n-1}{\lfloor k/p \rfloor}^2 \binom{n+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{n}^t \left(\frac{2}{k} + (s-2) \sum_{j=1}^{k}{}' \frac{1}{j} + t \sum_{j=1}^{2k}{}' \frac{1}{j} \right) \\
&= \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N}{n}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} < p/2}}{}' \frac{1}{k^2} \left(\frac{2}{k} + (s-2) \sum_{j=1}^{k}{}' \frac{1}{j} + t \sum_{j=1}^{2k}{}' \frac{1}{j} \right) \\
&+ \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N+1}{n}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} > p/2}}{}' \frac{1}{k^2} \left(\frac{2}{k} + (s-2) \sum_{j=1}^{k}{}' \frac{1}{j} + t \sum_{j=1}^{2k}{}' \frac{1}{j} \right).
\end{aligned} \tag{3.24}$$

By the result [9, Corollary 5.2] and the fact $\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv 0 \pmod p$, we have

$$\sum_{\substack{\lfloor k/p\rfloor=N\\ \{k/p\}<p/2}}{}' \frac{1}{k^3}\equiv -2B_{p-3}\pmod p, \tag{3.25}$$

$$\sum_{\substack{\lfloor k/p\rfloor=N\\ \{k/p\}>p/2}}{}' \frac{1}{k^3}\equiv 2B_{p-3}\pmod p. \tag{3.26}$$

It follows from (2.14)–(2.17) and (3.24)–(3.26) that

$$\begin{aligned}&\sum_k{}' \frac{1}{k^2}\binom{n-1}{\lfloor k/p\rfloor}^2\binom{n+\lfloor k/p\rfloor}{\lfloor k/p\rfloor}^s\binom{\lfloor 2k/p\rfloor}{n}^t\left(-2\sum_{j=1}^{k-1}{}'\frac{1}{j}+s\sum_{j=1}^{k}{}'\frac{1}{j}+t\sum_{j=1}^{2k}{}'\frac{1}{j}\right)\\ &\equiv \frac{1}{3}B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t(4t+s-14)\\ &+\frac{1}{3}B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t(4t+s+10)\pmod p. \end{aligned}\tag{3.27}$$

Finally, combining (3.22), (3.23) and (3.27) gives

$$\begin{aligned}Z_{1,n,p}^{(2,s,t)}\equiv&\frac{1}{3}p^3B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t n^2(12k+7)\\ &-\frac{1}{3}p^3B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t n^2(12k+5)\\ &+\frac{1}{3}p^3B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t n^3(4t+s-14)\\ &+\frac{1}{3}p^3B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t n^3(4t+s+10)\pmod{p^4}.\end{aligned}$$

Case 4 $r=2$ **and** $m\geq 2$**.**

By (2.2), we have

$$\begin{aligned}Z_{m,n,p}^{(2,s,t)}\equiv n^2p^{2m}&\sum_k{}'\frac{1}{k^2}\binom{np^{m-1}-1}{\lfloor k/p\rfloor}^2\binom{np^{m-1}+\lfloor k/p\rfloor}{\lfloor k/p\rfloor}^s\binom{\lfloor 2k/p\rfloor}{np^{m-1}}^t\\ &\times\left(1-2np^m\sum_{j=1}^{k-1}{}'\frac{1}{j}+snp^m\sum_{j=1}^{k}{}'\frac{1}{j}+tnp^m\sum_{j=1}^{2k}{}'\frac{1}{j}\right)\pmod{p^{3m+1}},\end{aligned}\tag{3.28}$$

where we have used the fact that $\lfloor (k-1)/p \rfloor = \lfloor k/p \rfloor$ for $p \nmid k$.

Let

$$T_{s,t}(k) = s\sum_{j=1}^{k}{}' \frac{1}{j} + t\sum_{j=1}^{2k}{}' \frac{1}{j} - 2\sum_{j=1}^{k-1}{}' \frac{1}{j}.$$

Firstly, we shall determine the following sum modulo p:

$$\sum_{k}{}' \frac{1}{k^2}\binom{np^{m-1}-1}{\lfloor k/p \rfloor}^2 \binom{np^{m-1}+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{np^{m-1}}^t T_{s,t}(k).$$

Note that

$$\begin{aligned}
&\sum_{k}{}' \frac{1}{k^2}\binom{np^{m-1}-1}{\lfloor k/p \rfloor}^2 \binom{np^{m-1}+\lfloor k/p \rfloor}{\lfloor k/p \rfloor}^s \binom{\lfloor 2k/p \rfloor}{np^{m-1}}^t T_{s,t}(k) \\
&= \sum_{N}\binom{np^{m-1}-1}{N}^2 \binom{np^{m-1}+N}{N}^s \binom{2N}{np^{m-1}}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} < p/2}}{}' \frac{1}{k^2} T_{s,t}(k) \\
&+ \sum_{N}\binom{np^{m-1}-1}{N}^2 \binom{np^{m-1}+N}{N}^s \binom{2N+1}{np^{m-1}}^t \sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} > p/2}}{}' \frac{1}{k^2} T_{s,t}(k). \qquad (3.29)
\end{aligned}$$

By using (2.14)–(2.17), (3.25) and (3.26), we obtain

$$\sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} < p/2}}{}' \frac{1}{k^2} T_{s,t}(k) \equiv \frac{1}{2}(3t-s-6)B_{p-3} \pmod{p}, \qquad (3.30)$$

$$\sum_{\substack{\lfloor k/p \rfloor = N \\ \{k/p\} > p/2}}{}' \frac{1}{k^2} T_{s,t}(k) \equiv \frac{1}{2}(3s+5t+2)B_{p-3} \pmod{p}. \qquad (3.31)$$

By a repeated use of the modulo p version of (2.2), we obtain

$$\begin{aligned}
&\sum_N \binom{np^{m-1}-1}{N}^2 \binom{np^{m-1}+N}{N}^s \binom{2N}{np^{m-1}}^t \\
&\equiv \sum_N \binom{n-1}{\lfloor N/p^{m-1}\rfloor}^2 \binom{n+\lfloor N/p^{m-1}\rfloor}{\lfloor N/p^{m-1}\rfloor}^s \binom{\lfloor 2N/p^{m-1}\rfloor}{n}^t \\
&= \sum_M \sum_{\substack{\lfloor N/p^{m-1}\rfloor = M \\ \{N/p^{m-1}\} < p^{m-1}/2}} \binom{n-1}{M}^2 \binom{n+M}{M}^s \binom{2M}{n}^t \\
&+ \sum_M \sum_{\substack{\lfloor N/p^{m-1}\rfloor = M \\ \{N/p^{m-1}\} > p^{m-1}/2}} \binom{n-1}{M}^2 \binom{n+M}{M}^s \binom{2M+1}{n}^t \\
&\equiv \frac{1}{2}\sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t - \frac{1}{2}\sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t \pmod p,
\end{aligned} \tag{3.32}$$

where we have used the fact that

$$\sum_{\substack{\lfloor N/p^{m-1}\rfloor = M \\ \{N/p^{m-1}\} < p^{m-1}/2}} 1 = \frac{p^{m-1}+1}{2} \equiv \frac{1}{2} \pmod p,$$

$$\sum_{\substack{\lfloor N/p^{m-1}\rfloor = M \\ \{N/p^{m-1}\} > p^{m-1}/2}} 1 = \frac{p^{m-1}-1}{2} \equiv -\frac{1}{2} \pmod p.$$

Similarly, we have

$$\begin{aligned}
&\sum_N \binom{np^{m-1}-1}{N}^2 \binom{np^{m-1}+N}{N}^s \binom{2N+1}{np^{m-1}}^t \\
&\equiv \frac{1}{2}\sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t - \frac{1}{2}\sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \pmod p.
\end{aligned} \tag{3.33}$$

It follows from (3.29)–(3.33) that

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2}\binom{np^{m-1}-1}{\lfloor k/p\rfloor}^2\binom{np^{m-1}+\lfloor k/p\rfloor}{\lfloor k/p\rfloor}^s\binom{\lfloor 2k/p\rfloor}{np^{m-1}}^t T_{s,t}(k)\\
&\equiv \frac{1}{2}B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t(t+2s+4)\\
&-\frac{1}{2}B_{p-3}\sum_k\binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t(t+2s+4) \pmod p. \qquad (3.34)
\end{aligned}$$

For $l<m$, let

$$C^{(2,s,t)}_{n,p,m,l}(k)=\binom{np^{m-l}-1}{\lfloor k/p^l\rfloor}^2\binom{np^{m-l}+\lfloor k/p^l\rfloor}{\lfloor k/p^l\rfloor}^s\binom{\lfloor 2k/p^l\rfloor}{np^{m-l}}^t.$$

Secondly, we shall determine the following sum modulo p^{m+1}:

$$\sum_k{}' \frac{1}{k^2}\binom{np^{m-1}-1}{\lfloor k/p\rfloor}^2\binom{np^{m-1}+\lfloor k/p\rfloor}{\lfloor k/p\rfloor}^s\binom{\lfloor 2k/p\rfloor}{np^{m-1}}^t=\sum_k{}'\frac{1}{k^2}C^{(2,s,t)}_{n,p,m,1}(k).$$

By (2.2), we have

$$\begin{aligned}
&C^{(2,s,t)}_{n,p,m,l}(k)\equiv C^{(2,s,t)}_{n,p,m,l+1}(k)\\
&\times\left(1-2np^{m-l}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j}+snp^{m-l}\sum_{j=1}^{\lfloor k/p^l\rfloor}{}'\frac{1}{j}+tnp^{m-l}\sum_{j=1}^{\lfloor 2k/p^l\rfloor}{}'\frac{1}{j}\right) \pmod{p^{m-l+1}}. \qquad (3.35)
\end{aligned}$$

By (2.8) and (2.9), we have

$$\sum_{\substack{\lfloor k/p^l\rfloor=N\\ \{k/p^l\}<p^l/2}}{}'\frac{1}{k^2}\equiv\sum_{\substack{\lfloor k/p^l\rfloor=N\\ \{k/p^l\}>p^l/2}}{}'\frac{1}{k^2}\equiv 0 \pmod{p^l}. \qquad (3.36)$$

Moreover, we have

$$C^{(2,s,t)}_{n,p,m,l}(k)=\begin{cases}\binom{np^{m-l}-1}{N}^2\binom{np^{m-l}+N}{N}^s\binom{2N}{np^{m-l}}^t & \text{if } \lfloor k/p^l\rfloor=N \text{ and } \{k/p^l\}<p^l/2,\\ \binom{np^{m-l}-1}{N}^2\binom{np^{m-l}+N}{N}^s\binom{2N+1}{np^{m-l}}^t & \text{if } \lfloor k/p^l\rfloor=N \text{ and } \{k/p^l\}>p^l/2.\end{cases} \qquad (3.37)$$

Note that

$$\sum_k{}'\frac{1}{k^2}C^{(2,s,t)}_{n,p,m,l}(k)=\sum_N\sum_{\substack{\lfloor k/p^l\rfloor=N\\ \{k/p^l\}<p^l/2}}{}'\frac{1}{k^2}C^{(2,s,t)}_{n,p,m,l}(k)+\sum_N\sum_{\substack{\lfloor k/p^l\rfloor=N\\ \{k/p^l\}>p^l/2}}{}'\frac{1}{k^2}C^{(2,s,t)}_{n,p,m,l}(k). \qquad (3.38)$$

Applying (3.35)–(3.37) to the right-hand side of (3.38), we arrive at

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l}(k) \equiv \sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l+1}(k)$$
$$\times \left(1 - 2np^{m-l} \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j} + snp^{m-l} \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j} + tnp^{m-l} \sum_{j=1}^{\lfloor 2k/p^l \rfloor}{}' \frac{1}{j}\right) \pmod{p^{m+1}}. \tag{3.39}$$

Note that

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l+1}(k) \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j}$$
$$= \sum_N \binom{np^{m-l-1}-1}{N}^2 \binom{np^{m-l-1}+N}{N}^s \binom{2N}{np^{m-l-1}}^t \sum_{\substack{\lfloor k/p^{l+1} \rfloor = N \\ \{k/p^{l+1}\} < p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j}$$
$$+ \sum_N \binom{np^{m-l-1}-1}{N}^2 \binom{np^{m-l-1}+N}{N}^s \binom{2N+1}{np^{m-l-1}}^t \sum_{\substack{\lfloor k/p^{l+1} \rfloor = N \\ \{k/p^{l+1}\} > p^{l+1}/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j}. \tag{3.40}$$

By (3.32) and (3.33), for $m > l+1$ we have

$$\sum_N \binom{np^{m-l-1}-1}{N}^2 \binom{np^{m-l-1}+N}{N}^s \binom{2N}{np^{m-l-1}}^t$$
$$+ \sum_N \binom{np^{m-l-1}-1}{N}^2 \binom{np^{m-l-1}+N}{N}^s \binom{2N+1}{np^{m-l-1}}^t \equiv 0 \pmod{p}. \tag{3.41}$$

Applying (2.14), (2.15) and (3.41) to the right-hand side of (3.40), we deduce that for $m > l+1$,

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l+1}(k) \sum_{j=1}^{\lfloor k/p^l \rfloor}{}' \frac{1}{j} \equiv 0 \pmod{p^{l+1}}. \tag{3.42}$$

Similarly, from (2.16), (2.17) and (3.41), we derive that for $m > l+1$,

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l+1}(k) \sum_{j=1}^{\lfloor 2k/p^l \rfloor}{}' \frac{1}{j} \equiv 0 \pmod{p^{l+1}}. \tag{3.43}$$

Combining (3.39), (3.42) and (3.43), we find that for $m > l+1$,

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l}(k) \equiv \sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,l+1}(k) \pmod{p^{m+1}}. \tag{3.44}$$

By a repeated use of (3.44), we have

$$\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,1}(k) \equiv \sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m-1}(k) \pmod{p^{m+1}}. \tag{3.45}$$

Note that

$$\begin{aligned}&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m-1}(k)\\ &= \sum_N \sum_{\substack{\lfloor k/p^{m-1}\rfloor = N\\ \{k/p^{m-1}\} < p^{m-1}/2}}{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m-1}(k) + \sum_N \sum_{\substack{\lfloor k/p^{m-1}\rfloor = N\\ \{k/p^{m-1}\} > p^{m-1}/2}}{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m-1}(k).\end{aligned} \tag{3.46}$$

By (2.8) and (2.9), we have

$$\sum_{\substack{\lfloor k/p^{m-1}\rfloor = N\\ \{k/p^{m-1}\} < p^{m-1}/2}}{}' \frac{1}{k^2} \equiv \sum_{\substack{\lfloor k/p^{m-1}\rfloor = N\\ \{k/p^{m-1}\} > p^{m-1}/2}}{}' \frac{1}{k^2} \equiv 0 \pmod{p^{m-1}}. \tag{3.47}$$

Applying (2.2) and (3.47) to the right-hand side of (3.46), we arrive at

$$\begin{aligned}&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m-1}(k) \equiv \sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m}(k)\\ &\times \left(1 - 2np \sum_{j=1}^{\lfloor k/p^{m-1}\rfloor}{}' \frac{1}{j} + snp \sum_{j=1}^{\lfloor k/p^{m-1}\rfloor}{}' \frac{1}{j} + tnp \sum_{j=1}^{\lfloor 2k/p^{m-1}\rfloor}{}' \frac{1}{j}\right) \pmod{p^{m+1}}.\end{aligned} \tag{3.48}$$

By (2.8) and (2.9), we have

$$\begin{aligned}&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m}(k)\\ &= \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N}{n}^t \sum_{\substack{\lfloor k/p^{m}\rfloor = N\\ \{k/p^{m}\} < p^{m}/2}}{}' \frac{1}{k^2}\\ &+ \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N+1}{n}^t \sum_{\substack{\lfloor k/p^{m}\rfloor = N\\ \{k/p^{m}\} > p^{m}/2}}{}' \frac{1}{k^2}\\ &\equiv \frac{1}{3} p^m B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t (12k+7)\\ &- \frac{1}{3} p^m B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t (12k+5) \pmod{p^{m+1}}.\end{aligned} \tag{3.49}$$

By (2.14) and (2.15), we have

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m}(k) \sum_{j=1}^{\lfloor k/p^{m-1}\rfloor}{}' \frac{1}{j} \\
&= \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N}{n}^t \sum_{\substack{\lfloor k/p^m\rfloor = N\\ \{k/p^m\}<p^m/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^{m-1}\rfloor}{}' \frac{1}{j} \\
&+ \sum_N \binom{n-1}{N}^2 \binom{n+N}{N}^s \binom{2N+1}{n}^t \sum_{\substack{\lfloor k/p^m\rfloor = N\\ \{k/p^m\}>p^m/2}}{}' \frac{1}{k^2} \sum_{j=1}^{\lfloor k/p^{m-1}\rfloor}{}' \frac{1}{j} \\
&\equiv \frac{1}{3} p^{m-1} B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \\
&+ \frac{1}{3} p^{m-1} B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t \pmod{p^m}. \qquad (3.50)
\end{aligned}$$

Similarly, by (2.16) and (2.17) we have

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,m}(k) \sum_{j=1}^{\lfloor 2k/p^{m-1}\rfloor}{}' \frac{1}{j} \\
&\equiv \frac{4}{3} p^{m-1} B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t \\
&+ \frac{4}{3} p^{m-1} B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t \pmod{p^m}. \qquad (3.51)
\end{aligned}$$

Combining (3.45) and (3.48)–(3.51) gives

$$\begin{aligned}
&\sum_k{}' \frac{1}{k^2} C^{(2,s,t)}_{n,p,m,1}(k) \\
&\equiv \frac{1}{3} p^m B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k}{n}^t (12k+7+sn-2n+4tn) \\
&+ \frac{1}{3} p^m B_{p-3} \sum_k \binom{n-1}{k}^2 \binom{n+k}{k}^s \binom{2k+1}{n}^t (-12k-5+sn-2n+4tn) \pmod{p^{m+1}}. \\
&\qquad (3.52)
\end{aligned}$$

It follows from (3.28), (3.34) and (3.52) that

$$\begin{aligned}
&Z^{(2,s,t)}_{m,n,p}\\
&\equiv \frac{1}{3}p^{3m}B_{p-3}\sum_k \binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t n^2(12k+7+sn-2n+4tn)\\
&+\frac{1}{3}p^{3m}B_{p-3}\sum_k \binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t n^2(-12k-5+sn-2n+4tn)\\
&+\frac{1}{2}p^{3m}B_{p-3}\sum_k \binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k+1}{n}^t n^3(t+2s+4)\\
&-\frac{1}{2}p^{3m}B_{p-3}\sum_k \binom{n-1}{k}^2\binom{n+k}{k}^s\binom{2k}{n}^t n^3(t+2s+4) \pmod{p^{3m+1}}.
\end{aligned}$$

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