Two-state transfer: a generalization of pair and plus state transfer

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Abstract

In the study of quantum state transfer, one is interested in being able to transmit a quantum state with high fidelity within a quantum spin network. In most of the literature, the state of interest is taken to be associated with a standard basis vector; however, more general states have recently been considered. Here, we consider a general linear combination of two vertex states, which encompasses the definitions of pair states and plus states in connected weighted graphs. A *two-state* in a graph X is a quantum state of the form $\mathbf{e}_u + s\mathbf{e}_v$, where u and v are two vertices in X and s is a non-zero real number. If $s = -1$ or $s = 1$, then such a state is called a *pair state* or a *plus state*, respectively.

In this paper, we investigate quantum state transfer between two-states, where the Hamiltonian is taken to be the adjacency, Laplacian or signless Laplacian matrix of the graph. By analyzing the spectral properties of the Hamiltonian, we characterize strongly cospectral two-states built from strongly cospectral vertices. This allows us to characterize perfect state transfer (PST) between two-states in complete graphs, cycles and hypercubes. We also produce infinite families of graphs that admit strong cospectrality and PST between two-states that are neither pair nor plus states. Using singular values and singular vectors, we show that vertex PST in the line graph of X implies PST between the plus states formed by corresponding edges in X. Furthermore, we provide conditions such that the converse of the previous statement holds. As an application, we characterize strong cospectrality and PST between vertices in line graphs of trees, unicyclic graphs and Cartesian products.

Keywords: strong cospectrality, perfect state transfer, two-states, pair states, plus states, line graph

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1 Introduction

The first implementation of a continuous-time quantum walk through a quantum computer was considered almost three decades ago [FG98]. This was done by evolving a quantum state through a decision tree. Later work [Bos03] used spin chains to transmit a quantum state within a quantum computer. Decision trees, chains, and more general quantum spin networks can be modelled by graphs whose vertices and edges represent qubits and their interactions in the network, respectively [God12].

To any vertex u of a graph, we can associate a vector $\mathbf{e}_u$, called a *vertex state*, with all entries equal to zero except the u^{th} entry being equal to one. Typically one is interested in quantifying the accuracy of quantum state transfer between vertices u and v at time t; this is measured by the squared modulus

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of the $(u, v)^{\text{th}}$ entry of the time-dependent transition matrix $U(t) = e^{-itM}$ of the graph, where M is the Hamiltonian representing the physical dynamics of the system, often taken to be the adjacency matrix or the Laplacian matrix, and to a lesser extent the signless Laplacian or normalized Laplacian. There is *perfect state transfer* (PST) between vertices u and v at time t if there exists a unit complex number γ such that $U(t)\mathbf{e}_v = \gamma\mathbf{e}_u$. In other words, if $|\mathbf{e}_u^T U(t)\mathbf{e}_v|^2 = 1$. A relaxation of perfect state transfer is *pretty good state transfer*: for any $\varepsilon > 0$ there exists a corresponding time t_ε such that $|\mathbf{e}_u^T U(t_\varepsilon)\mathbf{e}_v|^2 = 1 - \varepsilon$.

Note that $\mathbf{e}_u$ is a pure quantum state (a normalized vector in $\mathbb{C}^n$), which is often identified with the outer product $\mathbf{e}_u\mathbf{e}_u^T$, called a *density matrix*. However, there are many other possible pure states: one may consider pure states associated to vertices u and v of the form $\mathbf{s} = \alpha\mathbf{e}_u + \beta\mathbf{e}_v$, for some complex coefficients satisfying $|\alpha|^2 + |\beta|^2 = 1$ and then consider the corresponding density matrix $\mathbf{s}\mathbf{s}^T$. More generally, the density matrix associated with a mixed quantum state cannot be described by a single vector: it can only be written as a convex combination of density matrices of pure states.

Most work on quantum state transfer focused on vertex states. Recently, quantum state transfer between pair states (quantum states of the form $\mathbf{e}_u - \mathbf{e}_v$) relative to the Laplacian matrix was explored in [CG20]. In the same paper, quantum state transfer between plus states (quantum states of the form $\mathbf{e}_u + \mathbf{e}_v$) relative to the signless Laplacian matrix was also investigated. Apart from vertex, pair and plus states, there are many density matrices arising from pure states that can be taken to be the initial state of the system in the continuous quantum walk on the graph. For instance, for the path P_5 on five vertices with middle vertex u and end vertices v and w, PST occurs between $\mathbf{e}_u - 2\mathbf{e}_v$ and $\mathbf{e}_u - 2\mathbf{e}_w$ (Theorem 42). This particular example motivates us to extend the analysis of quantum state transfer between pair or plus states to the more general concept of two-states.

In this paper, we define two-states—states of the form $\mathbf{e}_u + s\mathbf{e}_v$ for some $0 \neq s \in \mathbb{R}$. We develop tools for studying problems arising from quantum state transfer between two-states, with particular emphasis on strong cospectrality between two-states, PST between two-states, and the relation between perfect plus state transfer of edges in a graph relative to the signless Laplacian and PST between corresponding vertices in the line graph relative to the adjacency matrix. In Section 2, we analyze strong cospectrality between two-states. In particular, we explore the basic properties of eigenvalue supports of two-states and give a lower bound on the size of the eigenvalue support of an arbitrary two-state (Proposition 6). Moreover, we provide necessary and sufficient conditions for a transitivity phenomenon to occur between strongly cospectral pairs of two-states (Theorems 7 and 8). We also investigate the role of automorphisms in strong cospectrality, and prove a property of strongly cospectral two-states analogous to that of strongly cospectral vertices, which is that any automorphism that fixes one must also fix the other (Lemma 9). Strong cospectrality is a prerequisite for high probability state transfer, and so we anticipate that our results on strongly cospectral two-states will underpin future work on topics such as perfect and pretty good state transfer, besides being of independent interest. In Section 3, we characterize strongly cospectral two-states built from strongly cospectral vertices, and provide infinite families of graphs that admit or do not admit strong cospectrality between specific pairs of two-states. By adapting known results about periodicity and PST between vertex states, we obtain a characterization of periodicity and PST between two-states in Section 4. We also present basic constructions of two-states that admit PST (Propositions 36, 37 and 39). In Section 5, we characterize PST between two-states in complete graphs, cycles and hypercubes. For cycles, only those of order four and eight admit PST between two-states. In particular, we show that for a cycle order four, there are infinitely many pairs of two-states, that are neither pair nor plus states, that admit PST (Theorem 50(ii)). Meanwhile for a cycle of order eight, only plus states admit PST (Theorem 50(iv)). This complements the result of Chen and Godsil about pair state transfer in cycles [CG20, Theorem 7.3]. We also show that pairs of two-states exhibiting PST are only obtained from pairs of antipodal vertices in a hypercube (Theorem 56). Section 6 is devoted to exploring the relationship between the existence of PST between plus states formed by edges in a graph relative to the signless Laplacian matrix, and the existence of PST between the corresponding vertices in the line graph relative to the adjacency matrix. We utilize the singular values and singular vectors of the incidence matrix of a graph to characterize strong cospectrality (Proposition 60) and PST (Theorem 67) in the line graph. As an application, we completely characterize strong cospectrality and PST between vertices in line graphs of trees and unicyclic graphs. Finally, we characterize strong cospectrality (Theorem 82) and PST (Theorem 84) between vertices in the line graphs of Cartesian products in Section 7. Taken together, our results broaden the literature on pair and plus states, establishing new instances of PST between two-states, while developing techniques that will facilitate future research on this topic.

2 Strong cospectrality

Throughout, we assume that X is a connected undirected weighted graph with positive real edge weights, possible loops, vertex set $V(X)$ and edge set $E(X)$. We denote an edge in X incident to u and v by $\{u, v\}$. If X has no loops, then we say that X is *simple*, and if all edge weights are equal to one, then we say that X is *unweighted*. We denote the adjacency, Laplacian, and signless Laplacian matrices of X by $A(X)$, $L(X)$, and $Q(X)$, respectively. If X is clear from the context, we will simply write these matrices as A, L and Q. We also let M be either A, L or Q. Denote the set of distinct eigenvalues of M by $\sigma(M)$. Since M is a real symmetric matrix, we can write M in its spectral decomposition as

$$M = \sum_{\lambda \in \sigma(M)} \lambda E_\lambda, \tag{1}$$

where E_λ is the orthogonal projection matrix onto the eigenspace associated with λ.

Let u and v be two vertices in X. The *eigenvalue support* of u, denoted $\sigma_u(M)$, is the set

$$\sigma_u(M) = \{\lambda \in \sigma(M) : E_\lambda \mathbf{e}_u \neq \mathbf{0}\},$$

where $\mathbf{0}$ is the all-zeros vector. Vertices u and v are said to be *cospectral* if $(E_\lambda)_{u,u} = (E_\lambda)_{v,v}$ for each $\lambda \in \sigma_u(M)$, *parallel* if $E_\lambda \mathbf{e}_u$ and $E_\lambda \mathbf{e}_w$ are parallel vectors for each $\lambda \in \sigma_u(M)$, and *strongly cospectral* if $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ for each $\lambda \in \sigma_u(M)$. In the case of strong cospectrality, one can define the sets

$$\sigma^+_{u,v}(M) = \{\lambda \in \sigma(M) : E_\lambda \mathbf{e}_u = E_\lambda \mathbf{e}_v \neq \mathbf{0}\} \quad \text{and} \quad \sigma^-_{u,v}(M) = \{\lambda \in \sigma(M) : E_\lambda \mathbf{e}_u = -E_\lambda \mathbf{e}_v \neq \mathbf{0}\}.$$

If u and v are cospectral, then they have the same eigenvalue support and $U(t)_{u,u} = U(t)_{v,v}$ for all t. Two vertices are strongly cospectral if and only if they are cospectral and parallel [GS17, Lemma 4.1]. Thus, if u and v are strongly cospectral, then we have $\sigma_u(M) = \sigma_v(M) = \sigma^+_{u,v}(M) \cup \sigma^-_{u,v}(M)$. Strong cospectrality is a necessary condition for PST [Cou14] and for pretty good state transfer [God12], and has therefore become an important property of interest.

A *two-state* is a state of the form $\mathbf{e}_u + s\mathbf{e}_w$, where $u \neq w$ and $s \in \mathbb{R}\backslash\{0\}$. Throughout this paper, we implicitly assume that $s \neq 0$ for any vector of form $\mathbf{e}_u + s\mathbf{e}_w$. In particular, a two-state is a *plus state* if $s = 1$ and a *pair state* if $s = -1$. Note that if $q, r \in \mathbb{R}\backslash\{0\}$, then we may write the state $q\mathbf{e}_u + r\mathbf{e}_w = q(\mathbf{e}_u + s\mathbf{e}_w)$, where $s = \frac{r}{q}$. Hence, to analyze states of the form $q\mathbf{e}_u + r\mathbf{e}_w$, it suffices to consider the two-states of the form $\mathbf{e}_u + s\mathbf{e}_w$. Similar to the vertex case, the *eigenvalue support* of the two-state $\mathbf{e}_u + s\mathbf{e}_w$, denoted $\sigma_{uw}(M)$, is the set

$$\sigma_{uw}(M) = \{\lambda \in \sigma(M) : E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) \neq \mathbf{0}\}.$$

Definition 1. *Let $s \in \mathbb{R}\backslash\{0\}$ and $u, v, w, x \in V(X)$ be such that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are linearly independent. The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral in X if for each $\lambda \in \sigma_{uw}(M)$,*

$$E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = \pm E_\lambda(\mathbf{e}_v + s\mathbf{e}_x).$$

Note that if $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral, then they have the same eigenvalue support, which can be partitioned into the following sets

$$\sigma^+_{uw,vx}(M) = \{\lambda \in \sigma(M) : E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = E_\lambda(\mathbf{e}_v + s\mathbf{e}_x) \neq \mathbf{0}\}$$

and

$$\sigma^-_{uw,vx}(M) = \{\lambda \in \sigma(M) : E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = -E_\lambda(\mathbf{e}_v + s\mathbf{e}_x) \neq \mathbf{0}\}.$$

The linear independence of $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ ensures that $\sigma^+_{uw,vx}(M)$ and $\sigma^-_{uw,vx}(M)$ are non-empty.

For vertex states, $|\sigma_u(M)| \geqslant 2$ for any $u \in V(X)$. If we add that u is involved in strong cospectrality with another vertex, then $|\sigma_u(M)| \geqslant 3$ whenever $|V(X)| \geqslant 3$. For two-states, $|\sigma_{uv}(M)| \geqslant 1$, and $|\sigma_{uv}(M)| = 1$ if and only if $\mathbf{e}_u + s\mathbf{e}_w$ is an eigenvector for M. If we add that $\mathbf{e}_u + s\mathbf{e}_w$ is involved in strong cospectrality with another two-state, then Definition 1 guarantees that $|\sigma_{uv}(M)| \geqslant 2$.

The union of eigenvalue supports of all vertex states in a graph equals $\sigma(M)$. However, this is not true for two-states. For instance, if X is simple, then 0 is an eigenvalue of L with associated orthogonal projection matrix $E_0 = \frac{1}{|V(X)|}J_{|V(X)|}$ where J_n denotes the $n \times n$ all-ones matrix, and so $E_0(\mathbf{e}_u - \mathbf{e}_v) = \mathbf{0}$ for any $u, v \in V(X)$. Hence, the union of the eigenvalue supports of all pair states in a graph relative to L is contained in $\sigma(L)\backslash\{0\}$. The same applies to M whenever the all-ones vector $\mathbf{1}$ is an eigenvector of M associated with a simple eigenvalue.

From the two observations above, the size and union of eigenvalue supports are two properties that highlight some fundamental differences between vertex states and two-states.

2.1 Eigenvalue supports

We denote the automorphism group of a graph X by $\mathrm{Aut}(X)$. The next proposition displays properties of eigenvalue supports when an automorphism is applied on the vertices of the graph.

Proposition 2. *Let $\psi \in \mathrm{Aut}(X)$. The following hold.*

(i) The eigenvalue support of $\mathbf{e}_u + s\mathbf{e}_w$ is given by

$$\sigma_{uw}(M) = \{\lambda \in \sigma_u(M) \cup \sigma_w(M) : E_\lambda \mathbf{e}_u \neq -sE_\lambda \mathbf{e}_w\}. \tag{2}$$

(ii) If $\psi(u, w) = (v, x)$, then the eigenvalue supports of $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are equal.

(iii) If u and w are cospectral, then the eigenvalue support of $\mathbf{e}_u + s\mathbf{e}_w$ is a subset of $\sigma_w(M)$. In particular, if $\psi(u) = w$ and $s \notin \{\pm 1\}$, then $\sigma_{uw}(M) = \sigma_w(M)$.

Proof. Since (i) and (ii) are immediate, we only prove (iii). Cospectrality between u and w implies that $\sigma_u(M) = \sigma_w(M)$. Thus, (i) yields $\sigma_{uw}(M) \subseteq \sigma_w(M)$. Now, let P be the permutation matrix that represents ψ. Since $E_\lambda = P^T E_\lambda P$ and $P\mathbf{e}_u = \mathbf{e}_w$, we obtain $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = \mathbf{0}$ if and only if

$$P^T E_\lambda P \mathbf{e}_u + sE_\lambda \mathbf{e}_w = P^T E_\lambda \mathbf{e}_w + sE_\lambda \mathbf{e}_w = (P^T + sI)E_\lambda \mathbf{e}_w \overset{(*)}{=} \mathbf{0}.$$

Since P^T is a permutation matrix, all eigenvalues of P^T are unit complex numbers. Thus, if $s \notin \{\pm 1\}$, then $P^T + sI$ is a full rank matrix, so $(*)$ holds if and only if $E_\lambda \mathbf{e}_w = \mathbf{0}$. Equivalently, $\sigma_{uw}(M) = \sigma_w(M)$. $\square$

By Proposition 2(iii), if X is vertex-transitive, then the eigenvalue support of $\mathbf{e}_u + s\mathbf{e}_w$ is equal to $\sigma(M)$ for any two of vertices u and w in X whenever $s \notin \{\pm 1\}$.

For a distance-transitive graph with pairs of vertices $\{u, w\}$ and $\{v, x\}$ having the same distances, the eigenvalue supports of $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are equal by Proposition 2(ii). In particular, for an edge-transitive graph, the eigenvalue supports of $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are equal for edges $\{u, w\}$ and $\{v, x\}$.

We now prove the following result that generalizes [CG20, Theorem 8.3].

Proposition 3. *Let X be a bipartite graph with bipartition B_1 and B_2 with $u, v \in B_1$ and $w, x \in B_2$. Then $\mathbf{e}_u - s\mathbf{e}_w$ and $\mathbf{e}_v - s\mathbf{e}_x$ are Laplacian strongly cospectral if and only if $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are signless Laplacian strongly cospectral. In this case, $\sigma^+_{uw,vx}(L) = \sigma^+_{uw,vx}(Q)$ and $\sigma^-_{uw,vx}(L) = \sigma^-_{uw,vx}(Q)$.*

Proof. If S be the diagonal matrix with $(S)_{a,a} = 1$ if $a \in B_1$ and $(S)_{a,a} = -1$ otherwise, then $Q = S^{-1}LS$. Thus, E_λ is an orthogonal projection matrix for the eigenvalue λ of L if and only if $S^{-1}E_\lambda S$ is an orthogonal projection matrix for the eigenvalue λ of Q. Since $S(\mathbf{e}_u - s\mathbf{e}_w) = \mathbf{e}_u + s\mathbf{e}_w$ and $S(\mathbf{e}_v - s\mathbf{e}_x) = \mathbf{e}_v + s\mathbf{e}_x$, we have $E_\lambda(\mathbf{e}_u - s\mathbf{e}_w) = \pm E_\lambda(\mathbf{e}_v - s\mathbf{e}_x)$ if and only if $E_\lambda S(\mathbf{e}_u + s\mathbf{e}_w) = \pm E_\lambda S(\mathbf{e}_v + s\mathbf{e}_x)$. □

Remark 4. *From Proposition 3, it follows that in a bipartite regular graph with bipartition B_1 and B_2, strong cospectrality (resp., PST) between $\mathbf{e}_u - s\mathbf{e}_w$ and $\mathbf{e}_v - s\mathbf{e}_x$ is equivalent to strong cospectrality (resp., PST) between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ whenever $u, v \in B_1$ and $w, x \in B_2$.*

Proposition 3 does not hold when $u, w \in B_1$. For the following graph, the plus states $(\mathbf{e}_u + \mathbf{e}_w)$ and $(\mathbf{e}_v + \mathbf{e}_x)$ are signless Laplacian strongly cospectral where the pair states $(\mathbf{e}_u - \mathbf{e}_w)$ and $(\mathbf{e}_v - \mathbf{e}_x)$ are not Laplacian strongly cospectral.

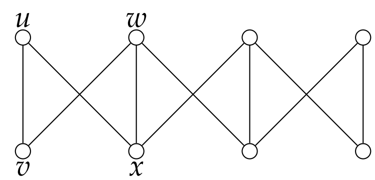

Figure 1: A counter-example of Proposition 3 when $u, w \in B_1$

Proposition 5. *If $\mathbf{e}_u + s\mathbf{e}_w$ is an eigenvector for M, then $\mathbf{e}_u + s\mathbf{e}_w$ cannot exhibit strong cospectrality with another two-state in X.*

Proof. If $\mathbf{e}_u + s\mathbf{e}_w$ is a θ-eigenvector for M, then $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = 0$ for each $\lambda \in \sigma_{uv}(M)\backslash\{\theta\}$. Thus, $|\sigma_{uv}(M)| = 1$, and so $\mathbf{e}_u + s\mathbf{e}_w$ cannot be strongly cospectral with another two-state by Definition 1. □

We also include a lower bound on the size of the eigenvalue support of $\mathbf{e}_u + s\mathbf{e}_v$.

Proposition 6. *If $\mathbf{e}_u + s\mathbf{e}_v$ is not an eigenvector for M and $d(u, v)$ is the distance between u and v in X, then*

$$|\sigma_{uv}(M)| \geqslant \left\lceil \frac{d(u, v)}{2} \right\rceil.$$

Proof. Let M be $I + A$, Q or $rI - L$ for some r greater than the maximum degree in X. Then M is a non-negative matrix with positive diagonal entries. Since $\mathbf{e}_u + s\mathbf{e}_v$ is not an eigenvector for M, the supports of the vectors $M^j(\mathbf{e}_u + s\mathbf{e}_v)$ are strictly increasing for $j \in \{0, \ldots, \lceil \frac{d(u,v)}{2} \rceil - 1\}$, and are linearly independent. Finally, since $M^j(\mathbf{e}_u + s\mathbf{e}_v) = \sum_j \lambda^j E_j(\mathbf{e}_u + s\mathbf{e}_v)$, it follows that the span of $\{E_\lambda(\mathbf{e}_u + s\mathbf{e}_v) : \lambda \in \sigma(M)\}$ has dimension at least $\lceil \frac{d(u,v)}{2} \rceil$. □

2.2 Transitivity

In [CG20, Theorem 6.1], it was shown that if $\mathbf{e}_a - \mathbf{e}_u$ and $\mathbf{e}_b - \mathbf{e}_v$ admit Laplacian PST and $\mathbf{e}_u - \mathbf{e}_w$ and $\mathbf{e}_v - \mathbf{e}_x$ admit Laplacian PST, then $\{\mathbf{e}_a - \mathbf{e}_w, \mathbf{e}_b - \mathbf{e}_x\}$ also admit Laplacian PST. Therefore, if $\mathbf{e}_a - \mathbf{e}_u$ and $\mathbf{e}_b - \mathbf{e}_v$ are strongly cospectral, and $\mathbf{e}_u - \mathbf{e}_w$ and $\mathbf{e}_v - \mathbf{e}_x$ are strongly cospectral, then it may happen that $\mathbf{e}_a - \mathbf{e}_w$ and $\mathbf{e}_b - \mathbf{e}_x$ are also strongly cospectral. We call this *transitivity* of strongly cospectral pair states. The following result characterizes this phenomenon.

Theorem 7. *Let $\{\mathbf{e}_a - \mathbf{e}_u, \mathbf{e}_b - \mathbf{e}_v\}$ and $\{\mathbf{e}_u - \mathbf{e}_w, \mathbf{e}_v - \mathbf{e}_x\}$ be strongly cospectral pairs. Then $\mathbf{e}_a - \mathbf{e}_w$ and $\mathbf{e}_b - \mathbf{e}_x$ are strongly cospectral if and only if*

(i) $\sigma^+_{au,bv}(M) = \sigma^+_{uw,vx}(M)$ *and* $\sigma^-_{au,bv}(M) = \sigma^-_{uw,vx}(M)$*, and*

(ii) $E_\lambda \mathbf{e}_a \neq E_\lambda \mathbf{e}_w$ *for at least one* λ *in* $\sigma^+_{au,bv}(M)$ *and at least one* $\lambda \in \sigma^-_{au,bv}(M)$.

In this case,

$$\sigma^+_{aw,bx}(M) = \{\lambda \in \sigma^+_{au,bv}(M) : E_\lambda \mathbf{e}_a \neq E_\lambda \mathbf{e}_w\} \quad \text{and} \quad \sigma^-_{aw,bx}(M) = \{\lambda \in \sigma^-_{au,bv}(M) : E_\lambda \mathbf{e}_a \neq E_\lambda \mathbf{e}_w\}.$$

Proof. Suppose $\mathbf{e}_a - \mathbf{e}_w$ and $\mathbf{e}_b - \mathbf{e}_x$ are strongly cospectral, so that $\sigma_{aw}(M) = \sigma_{bx}(M)$. We first show that Condition (i) is necessary. By way of contradiction, suppose $\lambda \in \sigma^+_{au,bv}(M) \cap \sigma^-_{uw,vx}(M)$. Then $E_\lambda(\mathbf{e}_a - \mathbf{e}_u) = E_\lambda(\mathbf{e}_b - \mathbf{e}_v) \neq \mathbf{0}$ and $E_\lambda(\mathbf{e}_u - \mathbf{e}_w) = -E_\lambda(\mathbf{e}_v - \mathbf{e}_x) \neq \mathbf{0}$, and so adding these two equations yields

$$E_\lambda(\mathbf{e}_a - \mathbf{e}_w) = E_\lambda(\mathbf{e}_b - \mathbf{e}_x) - 2E_\lambda(\mathbf{e}_v - \mathbf{e}_x) = -E_\lambda(\mathbf{e}_b - \mathbf{e}_x) + 2E_\lambda(\mathbf{e}_b - \mathbf{e}_v).$$

As $E_\lambda(\mathbf{e}_v - \mathbf{e}_x) \neq \mathbf{0}$, we get $E_\lambda(\mathbf{e}_a - \mathbf{e}_w) \neq E_\lambda(\mathbf{e}_b - \mathbf{e}_x)$, and since $E_\lambda(\mathbf{e}_b - \mathbf{e}_v) \neq \mathbf{0}$, it follows that $E_\lambda(\mathbf{e}_a - \mathbf{e}_w) \neq -E_\lambda(\mathbf{e}_b - \mathbf{e}_x)$. Thus, $\mathbf{e}_a - \mathbf{e}_w$ and $\mathbf{e}_b - \mathbf{e}_x$ are not strongly cospectral, a contradiction. Thus, (i) holds, and (ii) is immediate from the fact that $\sigma^+_{aw,bx}(M)$ and $\sigma^-_{aw,bx}(M)$ are non-empty sets.

Conversely, suppose (i) and (ii) hold. By our assumption in (i), $\sigma_{au}(M) = \sigma_{bv}(M) = \sigma_{uw}(M) = \sigma_{vx}(M)$. Hence, if $\lambda \notin \sigma_{au}(M)$, then $E_\lambda(\mathbf{e}_a - \mathbf{e}_u) = E_\lambda(\mathbf{e}_b - \mathbf{e}_v) = E_\lambda(\mathbf{e}_u - \mathbf{e}_w) = E_\lambda(\mathbf{e}_v - \mathbf{e}_x) = \mathbf{0}$, which in turn implies that $E_\lambda(\mathbf{e}_a - \mathbf{e}_w) = E_\lambda(\mathbf{e}_b - \mathbf{e}_x) = \mathbf{0}$. Thus, $\sigma_{aw}(M) \subseteq \sigma_{au}(M)$. Now, note that if $\sigma^+_{au,bv}(M) = \sigma^+_{uw,vx}(M)$, then $E_\lambda(\mathbf{e}_a - \mathbf{e}_w) = E_\lambda(\mathbf{e}_b - \mathbf{e}_x) \neq \mathbf{0}$ provided $E_\lambda \mathbf{e}_a \neq E_\lambda \mathbf{e}_w$. Similarly, if $\sigma^-_{au,bv}(M) = \sigma^-_{uw,vx}(M)$, then $E_\lambda(\mathbf{e}_a - \mathbf{e}_w) = -E_\lambda(\mathbf{e}_b - \mathbf{e}_x) \neq \mathbf{0}$ provided $E_\lambda \mathbf{e}_a \neq E_\lambda \mathbf{e}_w$. This shows that $\mathbf{e}_a - \mathbf{e}_w$ and $\mathbf{e}_b - \mathbf{e}_x$ are strongly cospectral. □

For two-states other than pair states, we present a sufficient condition for transitivity.

Theorem 8. *Let $\{\mathbf{e}_a + s\mathbf{e}_u, \mathbf{e}_b + s\mathbf{e}_v\}$ and $\{\mathbf{e}_u + s\mathbf{e}_w, \mathbf{e}_v + s\mathbf{e}_x\}$ be strongly cospectral pairs of two-states with $s \neq -1$. If all of the following conditions hold, then $\mathbf{e}_a + s\mathbf{e}_w$ and $\mathbf{e}_b + s\mathbf{e}_x$ are strongly cospectral.*

(i) $\sigma^+_{au,bv}(M) = \sigma^+_{uw,vx}(M)$ *and* $\sigma^-_{au,bv}(M) = \sigma^-_{uw,vx}(M)$.

(ii) $E_\lambda \mathbf{e}_u = E_\lambda \mathbf{e}_v$ *for all* $\lambda \in \sigma^+_{au,bv}(M)$*, and* $E_\lambda \mathbf{e}_u = -E_\lambda \mathbf{e}_v$ *for all* $\lambda \in \sigma^-_{au,bv}(M)$.

(iii) $E_\lambda \mathbf{e}_a \neq -sE_\lambda \mathbf{e}_w$ *for at least one* λ *in* $\sigma^+_{au,bv}(M)$ *and at least one* $\lambda \in \sigma^-_{au,bv}(M)$.

In this case,

$$\sigma^+_{aw,bx}(M) = \{\lambda \in \sigma^+_{au,bv}(M) : E_\lambda \mathbf{e}_a \neq -sE_\lambda \mathbf{e}_w\} \quad \text{and} \quad \sigma^-_{aw,bx}(M) = \{\lambda \in \sigma^-_{au,bv}(M) : E_\lambda \mathbf{e}_a \neq -sE_\lambda \mathbf{e}_w\}.$$

Proof. Conditions (i)-(iii) and the fact that $E_\lambda(\mathbf{e}_a + s\mathbf{e}_w) = E_\lambda(\mathbf{e}_a + s\mathbf{e}_u) + E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) - (s+1)E_\lambda \mathbf{e}_u$ yield the desired result. □

2.3 Automorphisms

The following result is an analogue of [GS17, Corollary 6.4].

Lemma 9. *Suppose $\psi \in \mathrm{Aut}(X)$ and $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral. If $s \neq 1$ and ψ fixes u and w, then it also fixes v and x. If $s = 1$ and ψ fixes the set $\{u, w\}$, then it fixes the set $\{v, x\}$.*

Proof. By assumption, $E_j(\mathbf{e}_u + s\mathbf{e}_w) = \pm E_j(\mathbf{e}_v + s\mathbf{e}_x)$. Let P be a permutation matrix that represents $\psi \in \mathrm{Aut}(X)$ so that $P(\mathbf{e}_u + s\mathbf{e}_w) = \mathbf{e}_u + s\mathbf{e}_w$. Since $P^T E_j P = E_j$ for each j, we have

$$E_j(\mathbf{e}_v + s\mathbf{e}_x) = \pm E_j(\mathbf{e}_u + s\mathbf{e}_w) = \pm P^T E_j P(\mathbf{e}_u + s\mathbf{e}_w) = \pm P^T E_j(\mathbf{e}_u + s\mathbf{e}_w) = P^T E_j(\mathbf{e}_v + s\mathbf{e}_x).$$

Since the E_j's sum to identity, the above equation yields $P(\mathbf{e}_v + s\mathbf{e}_x) = \mathbf{e}_v + s\mathbf{e}_x$. □

Lemma 9 tells that for a pair of strongly cospectral two-states, any automorphism that fixes one two-state must fix the other one too.

Next, we show that graph automorphisms can help build pairs of two-states that exhibit transitivity using pairs of strongly cospectral vertices. The following result follows from the fact that $P^T E_j P = E_j$ for each j whenever P represents an automorphism ψ of a graph.

Proposition 10. *Let $\psi \in \mathrm{Aut}(X)$ such that $\psi(a,b) = (u,v)$. If a and b are strongly cospectral vertices in X, then $\{\mathbf{e}_a + s\mathbf{e}_u, \mathbf{e}_b + s\mathbf{e}_v\}$ are strongly cospectral pairs with $\sigma^+_{au,bv}(M) = \sigma^+_{a,b}(M)$ and $\sigma^-_{au,bv}(M) = \sigma^-_{a,b}(M)$.*

Example 11. *Let X be an even cycle or a hypercube. Then two vertices are strongly cospectral if and only if they are antipodal in X. As X is distance transitive, for any pairs $\{a,b\}$ and $\{u,v\}$ of strongly cospectral vertices, the two-states $\mathbf{e}_a + s\mathbf{e}_u$ and $\mathbf{e}_b + s\mathbf{e}_v$ are strongly cospectral.*

3 Strongly cospectral two-states from strongly cospectral vertices

In this section, we characterize strong cospectrality between two-states of the form $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$, where either $\{u,w\}$, $\{u,v\}$, $\{u,x\}$ or $\{w,x\}$ are strongly cospectral pairs of vertices.

3.1 The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$

We begin by determining strong cospectrality between two-states that are linear combinations of strongly cospectral vertices.

Theorem 12. *Let $\{u,v\}$ and $\{w,x\}$ be disjoint pairs of strongly cospectral vertices. Then $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral if and only if for each $\lambda \in \sigma_u(M)$, there is some $\delta \in \{\pm 1\}$ such that*

$$\begin{cases} E_\lambda \mathbf{e}_u = \delta E_\lambda \mathbf{e}_w, & \text{if } \lambda \in \sigma^+_{u,v}(M) \cap \sigma^+_{w,x}(M) \text{ and } s \neq -1, \text{ or } \lambda \in \sigma^-_{u,v}(M) \cap \sigma^-_{w,x}(M) \text{ and } s \neq 1 \\ E_\lambda \mathbf{e}_u = \frac{\delta(1+s)}{1-s} E_\lambda \mathbf{e}_w, & \text{if } \lambda \in \sigma^-_{u,v}(M) \cap \sigma^+_{w,x}(M) \text{ and } s \notin \{\pm 1\} \\ E_\lambda \mathbf{e}_u = \frac{\delta(1-s)}{1+s} E_\lambda \mathbf{e}_w, & \text{if } \lambda \in \sigma^+_{u,v}(M) \cap \sigma^-_{w,x}(M) \text{ and } s \notin \{\pm 1\}. \end{cases} \tag{3}$$

Moreover, if $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral, then $\sigma_u(M) = \sigma_w(M)$ and $\sigma^+_{uv,wx}(M)$ (resp., $\sigma^-_{uv,wx}(M)$) is the set of all λ's in (3) that yield $\delta = 1$ (resp., $\delta = -1$). The following also hold.

(i) Suppose $s \in \{\pm 1\}$ and $\sigma^s_{uv}(M)$ denote $\sigma^+_{u,v}(M)$ if $s = 1$ and $\sigma^-_{u,v}(M)$ if $s = -1$. Then the two-states $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral if and only if $\sigma^s_{uv}(M) = \sigma^s_{wx}(M)$ and for some non-empty proper subset $\mathcal{S}$ of $\sigma^s_{uv}(M)$,

$$E_\lambda \mathbf{e}_u = E_\lambda \mathbf{e}_w \quad \text{for all } \lambda \in \mathcal{S} \quad \text{and} \quad E_\lambda \mathbf{e}_u = -E_\lambda \mathbf{e}_w \quad \text{for all } \lambda \in \sigma^s_{uv}(M) \backslash \mathcal{S}. \tag{4}$$

In this case, $\sigma^+_{uv,wx}(M) = \mathcal{S}$ and $\sigma^-_{uv,wx}(M) = \sigma^s_{uv}(M) \backslash \mathcal{S}$.

(ii) Suppose $s \notin \{\pm 1\}$ and $\sigma^+_{u,v}(M) = \sigma^+_{w,x}(M)$. Then the two-states $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral if and only if u, v, w and x are pairwise strongly cospectral vertices. In this case, $\sigma^+_{uv,wx}(M) = \sigma^+_{uw}(M)$ and $\sigma^-_{uv,wx}(M) = \sigma^-_{uw}(M)$.

(iii) Suppose $s \notin \{\pm 1\}$ and $\sigma^+_{u,v}(M) \neq \sigma^+_{w,x}(M)$. If the two-states $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral, then u and w are not strongly cospectral.

Proof. Since u and v are strongly cospectral, we get

$$E_\lambda(\mathbf{e}_u + s\mathbf{e}_v) = \begin{cases} (1+s)E_\lambda \mathbf{e}_u, & \text{if } \lambda \in \sigma^+_{u,v}(M) \\ (1-s)E_\lambda \mathbf{e}_u, & \text{if } \lambda \in \sigma^-_{u,v}(M). \end{cases} \tag{5}$$

Thus, $E_\lambda(\mathbf{e}_u + s\mathbf{e}_v) = \delta E_\lambda(\mathbf{e}_w + s\mathbf{e}_x)$ if and only if (3) holds. The second statement is also immediate from (3). To prove (i), it suffices to show the case $s = -1$, and the case $s = 1$ follows similarly. If $s = -1$, then the eigenvalue support of $\mathbf{e}_u - \mathbf{e}_v$ is $\sigma^-_{u,v}(M)$. For $\mathbf{e}_u - \mathbf{e}_v$ and $\mathbf{e}_w - \mathbf{e}_x$ to be strongly cospectral, their eigenvalue supports must coincide, so that $\sigma^-_{u,v}(M) = \sigma^-_{w,x}(M)$, and the existence of the set $\mathcal{S}$ in (4) is necessary to guarantee that both $\sigma^+_{uv,wx}(M)$ and $\sigma^-_{uv,wx}(M)$ are non-empty. The converse is straightforward. Next, the assumption in (ii) implies that $\sigma^-_{u,v}(M) \cap \sigma^+_{w,x}(M) = \sigma^+_{u,v}(M) \cap \sigma^-_{w,x}(M) = \varnothing$, and so (3) implies that $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral if and only if $E_\lambda \mathbf{e}_u = \delta E_\lambda \mathbf{e}_v$ for all $\lambda \in \sigma_u(M)$. This yields (ii). Finally, since $s \neq 0$, we get $\frac{1+s}{1-s} \neq \pm 1$, and so (iii) holds. □

Remark 13. *For disjoint pairs of strongly cospectral vertices $\{u, v\}$ and $\{w, x\}$, Theorem 12 requires that u and w are parallel for $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ to be strongly cospectral.*

We utilize Theorem 12 to rule out strong cospectrality of two-states in distance-regular graphs.

Example 14. *Let X be an antipodal distance-regular graph such that for each $u \in V(X)$, there is a unique $v \in V(X)$ at maximum distance from u. Now, let $\{u, v\}$ and $\{w, x\}$ be pairs of vertices in X that are at maximum distance, where $u \notin \{w, x\}$. From* [CG16, Lemma 3.2.2], *$\{u, v\}$ and $\{w, x\}$ are strongly cospectral pairs of vertices. Since u and w are not antipodal vertices, they are not strongly cospectral, and because they are cospectral, it follows that they are not parallel. By Remark 13, we conclude that the two-states $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are not strong cospectral .*

The next example provides an infinite families of graphs containing disjoint pairs of strongly cospectral vertices $\{u, v\}$ and $\{w, x\}$ such that $\mathbf{e}_u + s\mathbf{e}_v$ and $\mathbf{e}_w + s\mathbf{e}_x$ are strongly cospectral.

Example 15. *Consider P_n with end vertices a and b such that $\lambda - \theta \neq 2$ for any two eigenvalues λ and θ of A. Let c and d be the vertices of P_2 and consider $X = P_n \square P_2$. Since all eigenvalues of P_n are simple, all eigenvalues of X are also simple. Consider the corner vertices $u = (a, c)$, $v = (a, d)$, $w = (b, c)$, and $x = (b, d)$ of X. As the automorphisms of X act transitively on its corners, it follows that these vertices are cospectral, and hence pairwise strongly cospectral. Since there exist $\psi_1, \psi_2, \psi_3 \in \mathrm{Aut}(X)$ such that $\psi_1(u, v) = (w, x)$, $\psi_2(u, w) = (v, x)$ and $\psi_3(u, x) = (w, v)$, we get $\sigma^+_{u,v}(A) = \sigma^+_{w,x}(A)$, $\sigma^+_{u,w}(A) = \sigma^+_{v,x}(A)$ and $\sigma^+_{u,x}(A) = \sigma^+_{w,v}(A)$. Applying (i) and (ii) of Theorem 12, we conclude that $\{\mathbf{e}_u + s\mathbf{e}_v, \mathbf{e}_w + s\mathbf{e}_x\}$, $\{\mathbf{e}_u + s\mathbf{e}_w, \mathbf{e}_v + s\mathbf{e}_x\}$ and $\{\mathbf{e}_u + s\mathbf{e}_x, \mathbf{e}_w + s\mathbf{e}_v\}$ are strongly cospectral pairs of two-states. This fact also holds for L, and because X is bipartite, it also holds for Q.*

Next, we examine strong cospectrality between the two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ where $\{u, v\}$ and $\{w, x\}$ are strongly cospectral pairs of vertices.

Theorem 16. *Let $\{u,v\}$ and $\{w,x\}$ be disjoint pairs of strongly cospectral vertices. The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral if and only if the sets $\sigma^+_{u,v}(M) \cap \sigma^-_{w,x}(M)$ and $\sigma^-_{u,v}(M) \cap \sigma^+_{w,x}(M)$ are both empty. Moreover, if $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral, then*

$$\sigma^+_{uw,vx}(M) = \sigma^+_{u,v}(M) \cup \sigma^+_{w,x}(M) \quad \textit{and} \quad \sigma^-_{uw,vx}(M) = \sigma^-_{u,v}(M) \cup \sigma^-_{w,x}(M).$$

Proof. By assumption, $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ for $\lambda \in \sigma_u(M)$, and $sE_\lambda \mathbf{e}_w = \pm sE_\lambda \mathbf{e}_x$ for $\lambda \in \sigma_w(M)$. If $\lambda \in \sigma^+_{u,v}(M) \cup \sigma^+_{w,x}(M)$, then $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = E_\lambda(\mathbf{e}_v + s\mathbf{e}_x) \neq \mathbf{0}$. If $\lambda \in \sigma^-_{u,v}(M) \cup \sigma^-_{w,x}(M)$, then $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = -E_\lambda(\mathbf{e}_v + s\mathbf{e}_x) \neq \mathbf{0}$. If $\lambda \in \sigma^+_{u,v}(M) \cap \sigma^-_{w,x}(M)$, then $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) \neq \pm E_\lambda(\mathbf{e}_v + s\mathbf{e}_x)$; otherwise, we would have $E_\lambda \mathbf{e}_v = \mathbf{0}$ or $E_\lambda \mathbf{e}_x = \mathbf{0}$. From these observations, the result is immediate. □

Corollary 17. *Let $\{u,v\}$ and $\{w,x\}$ be disjoint pairs of strongly cospectral vertices with $\sigma_u(M) = \sigma_w(M)$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral if and only if $\sigma^+_{u,v}(M) = \sigma^+_{w,x}(M)$.*

We end this subsection with infinite families of graphs containing disjoint pairs of strongly cospectral vertices $\{u,v\}$ and $\{w,x\}$ such that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are also strongly cospectral.

Example 18. *Let $\{u,v\}$ and $\{w,x\}$ be disjoint pairs of strongly cospectral vertices in Q_n. As Q_n is distance-transitive, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral by Proposition 10.*

Example 19. *Consider $X = P_n \square P_2$ in Example 15 with corners $u = (a,c)$, $v = (a,d)$, $w = (b,c)$, and $x = (b,d)$ that are pairwise strongly cospectral with $\sigma^+_{u,v}(M) = \sigma^+_{w,x}(M)$, where a,b and c,d are end vertices of P_n and P_2, respectively. As $\sigma_u(M) = \sigma_w(M)$, Corollary 17 yields strong cospectrality between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$.*

3.2 The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$

We now characterize strong cospectrality between two-states whose corresponding tails are the same vertex and one of the two-states is a linear combination of strongly cospectral vertices.

Theorem 20. *Let u and w be strongly cospectral vertices and suppose $v \in V(X)\backslash\{u,w\}$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are strongly cospectral if and only if the following two conditions hold.*

(i) Either $E_\lambda \mathbf{e}_v = E_\lambda \mathbf{e}_w$ or $E_\lambda \mathbf{e}_v = (-2s-1)E_\lambda \mathbf{e}_w$, whenever $\lambda \in \sigma^+_{u,w}(M)$.

(ii) Either $E_\lambda \mathbf{e}_v = -E_\lambda \mathbf{e}_w$ or $E_\lambda \mathbf{e}_v = (-2s+1)E_\lambda \mathbf{e}_w$, whenever $\lambda \in \sigma^-_{u,w}(M)$.

Moreover, if $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are strongly cospectral, then

$$\sigma^+_{uw,vw}(M) = \{\lambda \in \sigma_w(M) : E_\lambda \mathbf{e}_v = \pm E_\lambda \mathbf{e}_w\} \quad \textit{and} \quad \sigma^-_{uw,vw}(M) = \{\lambda \in \sigma_w(M) : E_\lambda \mathbf{e}_v = (-2s \pm 1)E_\lambda \mathbf{e}_w\}.$$

Proof. Since u and w are strongly cospectral, (5) yields $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = \begin{cases} (1+s)E_\lambda \mathbf{e}_w, & \text{if } \lambda \in \sigma^+_{u,w}(M) \\ (s-1)E_\lambda \mathbf{e}_w, & \text{if } \lambda \in \sigma^-_{u,w}(M) \end{cases}$. Combining this with the fact that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are strongly cospectral if and only if $E_\lambda(\mathbf{e}_v + s\mathbf{e}_w) = \pm E_\lambda(\mathbf{e}_u + s\mathbf{e}_w)$ yields the desired result. □

Now, if we add that v and w are strongly cospectral in Theorem 20, then $\sigma^-_{uw,vw}(M) = \varnothing$ whenever $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are strongly cospectral. Since this cannot happen, we get the following result.

Corollary 21. *If u, v and w are three distinct pairwise strongly cospectral vertices, then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are not strongly cospectral.*

Next, we investigate strong cospectrality between two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ with u strongly cospectral to v.

Corollary 22. *Let u and v be strongly cospectral vertices and suppose $w \in V(X)\backslash\{u, v\}$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are strongly cospectral if and only if $\sigma^-_{u,v}(M) \cap \sigma_w(M) = \varnothing$. In this case,*

$$\sigma^+_{uw,vw}(M) = \{\lambda \in \sigma^+_{u,v}(M) \cup \sigma_w(M) : E_\lambda \mathbf{e}_u \neq -sE_\lambda \mathbf{e}_w\} \quad \textit{and} \quad \sigma^-_{uw,vw}(M) = \sigma^-_{u,v}(M).$$

Proof. Applying Theorem 16 with $w = x$ and noting that $\sigma^+_{w,x}(M) = \sigma_w(M)$ and $\sigma^-_{w,x}(M) = \varnothing$ yields the desired result. □

The following result is a consequence of Corollary 22.

Corollary 23. *Let u and v be strongly cospectral vertices. If $w \in V(X)\backslash\{u, v\}$ such that $\sigma_u(M) = \sigma_w(M)$, then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are not strongly cospectral.*

Proof. The condition that $\sigma_u(M) = \sigma_w(M)$ violates Corollary 22, and so the result is immediate. □

To illustrate Corollary 23, recall that a *weighted walk-regular graph* X is a simple weighted-regular graph such that $(A^k)_{u,u} = (A^k)_{v,v}$ for any positive integer k and for any two vertices u and v in X. Thus, the vertices in a weighted walk-regular graph are pairwise cospectral. Vertex-transitive and distance-regular graphs are well-known examples of unweighted walk-regular graphs.

Corollary 24. *Let X be a weighted walk-regular graph so that $\sigma_w(M) = \sigma(M)$ for each $w \in V(X)$. If u and v are strongly cospectral vertices in X and $w \in V(X)\backslash\{u, v\}$, then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ are not strongly cospectral by Corollary 23. This applies to $M = A, L, Q$ because X is regular.*

3.3 The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_u$

The following result is straightforward.

Proposition 25. *Let $u, v, w \in V(X)$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_u$ are strongly cospectral if and only if for each $\lambda \in \sigma_{uw}(M)$, either (i) $(1 - s)E_\lambda \mathbf{e}_u = E_\lambda(\mathbf{e}_v - s\mathbf{e}_w)$ or (ii) $(1 + s)E_\lambda \mathbf{e}_u = -E_\lambda(\mathbf{e}_v + s\mathbf{e}_w)$, and the equations in (i) and (ii) hold for at least one $\lambda \in \sigma_{uw}(M)$. Moreover, $\sigma^+_{uw,vu}(M)$ and $\sigma^-_{uw,vu}(M)$ are the subsets of $\sigma_{uw}(M)$ such that (i) and (ii) hold, respectively.*

If $v = w$ in Proposition 25, then we get the following result.

Proposition 26. *Let $s \notin \{\pm 1\}$. Then the two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ are strongly cospectral if and only if u and w are strongly cospectral. In this case, $\sigma^+_{uw,wu}(M) = \sigma^+_{u,w}(M)$ and $\sigma^-_{uw,wu}(M) = \sigma^-_{u,w}(M)$.*

If we add that u and w are strongly cospectral in Proposition 25, then obtain the following.

Corollary 27. *Let u and w be strongly cospectral vertices. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_u$ are strongly cospectral if and only if the following two conditions hold.*

(i) Either (a) $E_\lambda \mathbf{e}_w = E_\lambda \mathbf{e}_v$ or (b) $(1 + 2s)E_\lambda \mathbf{e}_w = -E_\lambda \mathbf{e}_v$, whenever $\lambda \in \sigma^+_{u,w}(M)$.

(ii) Either (a) $(-1 + 2s)E_\lambda \mathbf{e}_w = -E_\lambda \mathbf{e}_v$ or (b) $E_\lambda \mathbf{e}_w = E_\lambda \mathbf{e}_v$, whenever $\lambda \in \sigma^-_{u,w}(M)$.

Moreover, if $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_u$ are strongly cospectral, then $\sigma^+_{uw,vu}(M)$ (resp., $\sigma^-_{uw,vu}(M)$) is the set of all λ's satisfying (i)(a) and (ii)(b) (resp., (i)(b) and (ii)(b)).

The following remark is similar in spirit to Remark 13.

Remark 28. *If u and w are strongly cospectral vertices, then Theorem 20 and Corollary 27 require that u and v are parallel for the pairs $\{\mathbf{e}_u + s\mathbf{e}_w, \mathbf{e}_v + s\mathbf{e}_w\}$ and $\{\mathbf{e}_u + s\mathbf{e}_w, \mathbf{e}_v + s\mathbf{e}_u\}$ to be strong cospectral.*

Example 29. *Let X be an antipodal distance-regular graph such that for each $u \in V(X)$, there is a unique $v \in V(X)$ at maximum distance from u. Now, let $\{u, v\}$ and $\{w, x\}$ be pairs of vertices in X that are at maximum distance, where $u \notin \{w, x\}$. By Example 14, u and w are not parallel. Applying Remark 28, we get that $\{\mathbf{e}_u + s\mathbf{e}_v, \mathbf{e}_w + s\mathbf{e}_v\}$ and $\{\mathbf{e}_u + s\mathbf{e}_v, \mathbf{e}_w + s\mathbf{e}_u\}$ are not strongly cospectral pairs .*

4 State transfer between two-states

A *continuous-time quantum walk* with respect to M is determined by the unitary matrix

$$U_M(t) = e^{-itM},$$

called the *transition matrix* relative to M. Using (1), we may write the above equation as

$$U_M(t) = \sum_{\lambda \in \sigma(M)} e^{-it\lambda} E_\lambda.$$

Let $u, v, w, x \in V(X)$ such that $u \neq w$, $v \neq x$ and $(u, w) \neq (v, x)$. We say that *perfect state transfer* (PST) occurs between the two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ if

$$U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_v + s\mathbf{e}_x) \tag{6}$$

for some time $\tau > 0$ and unit $\gamma \in \mathbb{C}$. If $u = v$ and $w = x$ in (6), then we say that the two-state $\mathbf{e}_v + s\mathbf{e}_x$ is *periodic*. The minimum positive time such that PST occurs between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ is called the *minimum PST time*, and the minimum positive time such that $\mathbf{e}_u + s\mathbf{e}_w$ is periodic is called the *minimum period*. Note that PST between two-states at time τ implies that both are periodic at time 2τ. Moreover, the minimum PST time between a pair of two-states is half of their minimum period.

We say that $\mathbf{e}_u + s\mathbf{e}_w$ is a *fixed two-state* if $\mathbf{e}_u + s\mathbf{e}_w$ is an eigenvector for M. Note that $\mathbf{e}_u + s\mathbf{e}_w$ is fixed if and only if $\sigma_{uw}(M)$ has size one. Moreover, if $\mathbf{e}_u + s\mathbf{e}_w$ is fixed, then

$$U(t)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_u + s\mathbf{e}_w) \quad \text{for all } t,$$

where $\gamma = e^{-it\lambda}$ and λ is the eigenvalue of M with associated eigenvector $\mathbf{e}_u + s\mathbf{e}_w$. Consequently, a fixed two-state cannot be involved in periodicity and PST. Thus, we limit our discussion of periodicity and PST to two-states that are not eigenvectors associated with M. Since we deal with connected graphs, a standard basis vector cannot be an eigenvector M, and so $\mathbf{e}_u$ for any $u \in V(X)$ cannot be a fixed state. This fact highlights the difference between two-state transfer and vertex state transfer.

4.1 Periodicity

We say that a set $\mathcal{S} \subset \mathbb{R}$ with at least two elements satisfies the *ratio condition* if

$$\frac{\lambda_j - \lambda_k}{\lambda_\ell - \lambda_m} \in \mathbb{Q}$$

for any $\lambda_j, \lambda_k, \lambda_\ell, \lambda_m \in \mathcal{S}$ with $\lambda_\ell \neq \lambda_m$. If $|\mathcal{S}| = 2$, then $\mathcal{S}$ automatically satisfies the ratio condition. In what follows, we denote the characteristic polynomial of M in the variable x by $\phi(M, x)$. The following theorem follows directly from Theorems 7.6.1, 9.1.1 and 9.5.1 in [CG21].

Theorem 30. *The two-state $\mathbf{e}_u + s\mathbf{e}_w$ is periodic in X if and only if $\sigma_{uw}(M)$ satisfies the ratio condition. If we add that $\sigma_{uw}(M)$ is closed under taking algebraic conjugates, then $\mathbf{e}_u + s\mathbf{e}_w$ is periodic in X if and only if either*

(i) all eigenvalues in $\sigma_{uw}(M)$ are integers, or

(ii) there is a square-free integer $\Delta > 1$ such that all eigenvalues in $\sigma_{uw}(M)$ are quadratic integers in $\mathbb{Q}(\sqrt{\Delta})$, and the difference of any two eigenvalues in $\sigma_{uw}(M)$ is an integer multiple of $\sqrt{\Delta}$.

In particular, if s is rational and $\phi(M, x)$ has integer coefficients, then $\mathbf{e}_u + s\mathbf{e}_w$ is periodic in X if and only if conditions (i) or (ii) hold.

Corollary 31. *If $\sigma_u(M) \cup \sigma_w(M)$ satisfies the ratio condition, then $\mathbf{e}_u + s\mathbf{e}_w$ is periodic in X for all s.*

Proof. The corollary follows immediately from Proposition 2(i) and Theorem 30. □

In particular, if both u and w are periodic at time τ then so is $\mathbf{e}_u + s\mathbf{e}_w$ for all s. We now give an infinite family of graphs that have periodic pair states but no periodic vertices. Let X be a conference graph on n vertices, where $\sqrt{n} \notin \mathbb{Z}$. Now X is regular with valency $k = \frac{n-1}{2}$ and the (adjacency) eigenvalue support of each vertex is

$$\left\{k, \frac{-1+\sqrt{n}}{2}, \frac{-1-\sqrt{n}}{2}\right\}.$$

Hence X has no periodic vertices. Since $E_k = \frac{1}{n}J$, we have $E_k(\mathbf{e}_u - \mathbf{e}_w) = \mathbf{0}$ and σ_{uw} satisfies the ratio condition. Therefore the pair state $\mathbf{e}_u - \mathbf{e}_w$ is periodic. This also applies to L and Q because X is regular.

From Theorem 30, the following is immediate.

Corollary 32. *Let $\mathbf{e}_u + s\mathbf{e}_w$ be periodic in X. If $\sigma_{uw}(M)$ is closed under taking algebraic conjugates, then $|\lambda - \mu| \geqslant 1$, for $\lambda, \mu \in \sigma_{uw}(M)$.*

Corollary 7.7.1 of [CG21] states that, when $M = A$, graphs with periodic vertices are rare. We are ready to show a similar statement about periodic two-states.

Corollary 33. *For $M = A$ and $s > 0$, there are only finitely many connected integer-weighted graphs with maximum valency at most k that contain a periodic two-state $\mathbf{e}_u + s\mathbf{e}_w$ which is not an eigenvector of A.*

Proof. Let r be the covering radius of $\{u, w\}$ in X. Then $\{(I + A)^j(\mathbf{e}_u + s\mathbf{e}_w) : j = 0, \dots, r\}$ is a linearly independent set in the span of $\{E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) : \lambda \in \sigma_{uw}\}$. Hence $r < |\sigma_{uw}|$. The spectral radius of A is at most k and it follows from Corollary 32 that $|\sigma_{uw}| \leqslant 2k + 1$. There are only finitely many graphs with maximum degree k and the covering radius of $\{u, w\}$ bounded above by $2k$. □

Corollary 33 does not apply when $s < 0$. The following infinite family of trees $\{T_n : n \geqslant 0\}$ is a special case of Pal's construction [Pal24]. Each tree in this family has maximum degree 3 with PST from $\mathbf{e}_u - \mathbf{e}_w$ to $\mathbf{e}_v - \mathbf{e}_x$, hence $\mathbf{e}_u - \mathbf{e}_w$ and $\mathbf{e}_v - \mathbf{e}_x$ are periodic two-states.

We close this subsection with the following result, which will be used in Section 5.

Proposition 34. *Let u and w be cospectral vertices in X. If $\mathbf{e}_u + s\mathbf{e}_w$ is periodic at time τ, then either $s \in \{\pm 1\}$ or both u and w are periodic in X at time τ.*

Proof. By assumption, $U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_u + s\mathbf{e}_w)$ for some unit $\gamma \in \mathbb{C}$. Comparing the u^{th} and w^{th} entries of this equation gives us $U(\tau)_{u,u} + sU(\tau)_{u,w} = \gamma$ and $sU(\tau)_{w,w} + U(\tau)_{u,w} = s\gamma$, respectively. Therefore, $U(\tau)_{u,u} = U(\tau)_{w,w}$ implies $U(\tau)_{u,w}(s^2 - 1) = 0$, from which the conclusion is immediate. □

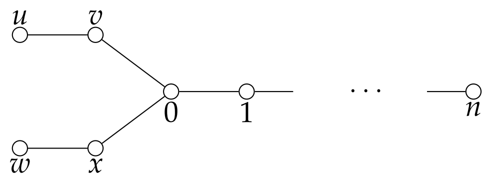

Figure 2: T_n

4.2 Perfect state transfer

Adapting [Cou14, Theorem 2.4.4] yields a characterization of PST between two-states.

Theorem 35. *Let X be a weighted graph with possible loops and suppose $\sigma_{uw}(M)$ is closed under taking algebraic conjugates. The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ exhibit perfect state transfer in X if and only if all of the following conditions hold.*

(i) The two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral. Let $\lambda_1 \in \sigma^+_{uw,vx}(M)$.

(ii) The elements in $\sigma_{uw}(M)$ are either all integers or all quadratic integers. Moreover, there is a square-free integer $\Delta \geqslant 1$ such that all eigenvalues in $\sigma_{uw}(M)$ are quadratic integers in $\mathbb{Q}(\sqrt{\Delta})$, and the difference of any two eigenvalues in $\sigma_{uw}(M)$ is an integer multiple of $\sqrt{\Delta}$.

(iii) Let $g = \gcd\left(\left\{\frac{\lambda_1-\lambda_j}{\sqrt{\Delta}} : \lambda_j \in \sigma_{uw}(M)\right\}\right)$. Then $\lambda_j \in \sigma^+_{uw,vx}(M)$ if and only if $\frac{\lambda_1-\lambda_j}{g\sqrt{\Delta}}$ is even.

Moreover, the minimum PST time between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ is $\tau_0 = \frac{\pi}{g\sqrt{\Delta}}$.

The following result can be viewed as a strengthening of Proposition 10.

Proposition 36. *Suppose perfect state transfer occurs between u and v. If $\psi \in \operatorname{Aut}(X)$ such that $\psi(u,v) = (w,x)$, then perfect state transfer occurs between vertices w and x, and between the pairs of two-states $\{\mathbf{e}_u + s\mathbf{e}_w, \mathbf{e}_v + s\mathbf{e}_x\}$ and $\{\mathbf{e}_u + s\mathbf{e}_x, \mathbf{e}_v + s\mathbf{e}_w\}$, at the same time.*

Proof. By assumption, $U(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$ for some unit $\gamma \in \mathbb{C}$. If P represents ψ, then $U(t) = P^T U(t) P$, $P\mathbf{e}_u = \mathbf{e}_w$ and $P\mathbf{e}_v = \mathbf{e}_x$, which then gives us $U(\tau)\mathbf{e}_w = \gamma\mathbf{e}_x$ so that $U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_v + s\mathbf{e}_x)$. Since $U(\tau)\mathbf{e}_w = \gamma\mathbf{e}_x$ is equivalent to $U(\tau)\mathbf{e}_x = \gamma\mathbf{e}_w$, we also have $U(\tau)(\mathbf{e}_u + s\mathbf{e}_x) = \gamma(\mathbf{e}_v + s\mathbf{e}_w)$. □

The following is an analogue of Corollary 22 for PST.

Proposition 37. *Suppose perfect state transfer occurs between u and v at time τ and w is periodic at τ. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ admit perfect state transfer at time τ if and only if $\sigma^+_{u,v}(M) \cap \sigma_w(M) \neq \varnothing$.*

Proof. By assumption, $U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = e^{-it\lambda}\mathbf{e}_v + se^{-it\mu}\mathbf{e}_w$ for some $\lambda \in \sigma^+_{u,v}(M)$ and $\mu \in \sigma_w(M)$. Now, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ admit perfect state transfer at time τ if and only if $U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_v + s\mathbf{e}_w)$ for some unit $\gamma \in \mathbb{C}$. Equivalently, $\gamma(\mathbf{e}_v + s\mathbf{e}_w) = e^{-i\tau\lambda}(\mathbf{e}_v + se^{i\tau(\lambda-\mu)}\mathbf{e}_w)$. Since $\mathbf{e}_v + s\mathbf{e}_w$ is real, the preceding equation holds if and only if $\gamma = e^{-i\tau\lambda}$ and $\lambda = \mu$; that is, $\sigma^+_{u,v}(M) \cap \sigma_w(M) \neq \varnothing$. □

Example 38. *Consider P_3 with vertices u, v and w, where w is the degree-two vertex. With respect to A, w is periodic at $\tau = \frac{\pi}{\sqrt{2}}$ and PST occurs between u and v at time τ. Since $\sigma^+_{u,v}(A) = \sigma_w(A) = \{\pm\sqrt{2}\}$, we see from Proposition 37 that PST occurs between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_w$ at time τ. As a generalisation, we also consider $G = P_3^{\square k}$, the k^{th} Cartesian power of P_3, for any $k \geqslant 1$. According to the fact that* [CDD+05]

$U_G(t) = \bigotimes_{i=1}^{k} U_{P_3}(t)$, for any time t, we deduce that the vertices $(u,\ldots,u)$ and $(v,\ldots,v)$ exhibit PST and $(w,\ldots,w)$ is periodic at time τ. Furthermore, since $k\sqrt{2}$ is the largest simple eigenvalue of G, it follows that

$$k\sqrt{2} \in \sigma^{+}_{(u,\ldots,u),(v,\ldots,v)}(A) \cap \sigma_{(w,\ldots,w)}(A).$$

Therefore, according to Proposition 37, the two-states $\mathbf{e}_{(u,\ldots,u)} + s\mathbf{e}_{(w,\ldots,w)}$ and $\mathbf{e}_{(v,\ldots,v)} + s\mathbf{e}_{(w,\ldots,w)}$ admit PST at τ. This provides an infinite family of graphs admitting PST between two-states.

Next, we have the following result which can be viewed as a stronger version of Proposition 26.

Proposition 39. *Suppose u and w are two vertices in X. The following hold.*

(i) Let $s \in \{\pm 1\}$. Perfect state transfer occurs between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ if and only if they are periodic at the same time.

(ii) Let $s \notin \{\pm 1\}$. Perfect state transfer occurs between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ if and only if it occurs between u and w at the same time.

Proof. Note that PST between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ is equivalent to $U(\tau)(\mathbf{e}_u + s\mathbf{e}_w) = \gamma(\mathbf{e}_w + s\mathbf{e}_u)$ for some unit $\gamma \in \mathbb{C}$. Since $s \in \{\pm 1\}$, the statement (i) is straightforward. Since the converse of (ii) is straightforward, it suffices to prove the forward direction. Suppose PST occurs between $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$. Then Theorem 35(i)-(iii) hold. In particular, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ are strongly cospectral, and so Proposition 26 implies that u and w are strongly cospectral with $\sigma^{+}_{uw,wu}(M) = \sigma^{+}_{u,w}(M)$ and $\sigma^{-}_{uw,wu}(M) = \sigma^{-}_{u,w}(M)$. Since Theorem 35(ii)-(iii) applies to $\sigma^{+}_{uw,wu}(M)$ and $\sigma^{-}_{uw,wu}(M)$, they also apply to $\sigma^{+}_{u,w}(M)$ and $\sigma^{-}_{u,w}(M)$. Applying [Cou14, Theorem 2.4.4] yields PST between u and w at the same time. This proves (ii). $\square$

Note that in Proposition 39(i), the states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ are either equal or opposite in sign whenever $s \in \{\pm 1\}$. Hence, to avoid conflating PST and periodicity, we will not consider PST between the two-states $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ whenever $s \in \{\pm 1\}$.

5 Special classes

We now investigate PST between two-states in complete graphs, cycles and hypercubes.

5.1 Complete graphs

Recall that for a complete graph K_n, we have $\sigma(L) = \{0, n\}$, where 0 is simple with eigenvector $\mathbf{1}$ and n has multiplicity $n-1$ with eigenvectors $\mathbf{e}_1 - \mathbf{e}_j$ for all $j \in \{1,\ldots,n\}$. Consequently, $\sigma_{uw}(L) = \{n\}$ whenever $s = -1$, or $n = 2$ and $s = 1$, and $\sigma_{uw}(L) = \{0, n\}$ otherwise. Thus, the following is straightforward.

Lemma 40. *Let $n \geqslant 2$. The two-state $\mathbf{e}_u + s\mathbf{e}_w$ is periodic in K_n if and only if either (i) $n = 2$ and $s \neq \pm 1$ or (ii) $n \geqslant 3$ and $s \neq -1$. In both cases, $\rho = \frac{2\pi}{n}$.*

Since $E_n = I - \frac{1}{n}J_n$ in K_n, we get $E_n(\mathbf{e}_u + s\mathbf{e}_w) = \mathbf{e}_u + s\mathbf{e}_w - \frac{1+s}{n}\mathbf{1}$. Thus, if $\{u, w\} \neq \{v, x\}$, then $E_n(\mathbf{e}_u + s\mathbf{e}_w) \neq \pm E_n(\mathbf{e}_v + s\mathbf{e}_x)$. Invoking Theorem 35 yields the following result.

Theorem 41. *No pair of two-states in K_n of the form $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ exhibit perfect state transfer.*

5.2 Weighted P_5

Let $\omega > 0$ and consider the weighted path $P_5(\omega)$ with vertex set $\{1,2,3,4,5\}$ and edges $\{j, j+1\}$ for $j \in \{1,2,3,4\}$ (see Figure 3). The eigenvalues of $A(P_5(\omega))$ are

$$\lambda_1 = 0, \;\; \lambda_2 = \omega, \;\; \lambda_3 = -\omega, \;\;\; \lambda_4 = \sqrt{\omega(\omega+2)} \;\text{ and }\; \lambda_5 = -\sqrt{\omega(\omega+2)}. \tag{7}$$

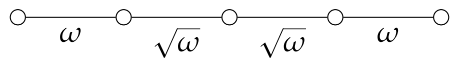

Figure 3: The weighted path $P_5(\omega)$

Theorem 42. *In $P_5(\omega)$, perfect state transfer occurs between $\mathbf{e}_3 - \frac{2}{\sqrt{\omega}}\mathbf{e}_1$ and $\mathbf{e}_3 - \frac{2}{\sqrt{\omega}}\mathbf{e}_5$ at time $\frac{\pi}{\omega}$.*

Proof. The associated eigenvectors of the eigenvalues in (7) are $\mathbf{v}_1 = [1, 0, -\sqrt{\omega}, 0, 1]^T$, $\mathbf{v}_2 = [-1, -1, 0, 1, 1]^T$, $\mathbf{v}_3 = [-1, 1, 0, -1, 1]^T$, $\mathbf{v}_4 = [\sqrt{\omega}, \sqrt{\omega+2}, 2, \sqrt{\omega+2}, \sqrt{\omega}]^T$ and $\mathbf{v}_5 = [\sqrt{\omega}, -\sqrt{\omega+2}, 2, -\sqrt{\omega+2}, \sqrt{\omega}]^T$. Now, let $s = -\frac{2}{\sqrt{\omega}}$. From these eigenvectors, one can check that the two-states $\mathbf{e}_3 + s\mathbf{e}_1$ and $\mathbf{e}_3 + s\mathbf{e}_5$ are strongly cospectral with $\sigma^+_{31,35}(A) = \{0\}$ and $\sigma^-_{31,35}(A) = \{\pm\omega\}$. Consequently,

$$\begin{aligned} U(\pi/\omega)(\mathbf{e}_3 + s\mathbf{e}_1) &= \left(E_1 + e^{i(\frac{\pi}{\omega})\omega}E_2 + e^{i(\frac{\pi}{\omega})(-\omega)}E_3\right)(\mathbf{e}_3 + s\mathbf{e}_1) \\ &= (E_1 - E_2 - E_3)(\mathbf{e}_3 + s\mathbf{e}_1) = (E_1 + E_2 + E_3)(\mathbf{e}_3 + s\mathbf{e}_5) = \mathbf{e}_3 + s\mathbf{e}_5, \end{aligned}$$

which yields the desired result. □

5.3 Cycles

In this subsection, we completely charaterize PST between two-states in C_n for $n \geqslant 4$. Without loss of generality, we deal with the Laplacian case. The eigenvalues of $L(C_n)$ are

$$\lambda_j = 2 - 2\cos(2j\pi/n)$$

for $j \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ with associated eigenvectors $\mathbf{v}_j$ and $\mathbf{v}_{n-j}$, where

$$\mathbf{v}_k = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & \left(e^{2\pi i/n}\right)^k & \left(e^{2\pi i/n}\right)^{2k} & \cdots & \left(e^{2\pi i/n}\right)^{(n-1)k} \end{bmatrix}^T, \quad \text{for } k = 0, \ldots, n-1.$$

Here we recall a useful result from [CDE+21] that if u and v are strongly cospectral in C_n, then n is even, $\sigma^+_{u,v}(L) = \{\lambda_k \in \sigma(L) \mid k \text{ even}\}$ and $\sigma^-_{u,v}(L) = \{\lambda_k \in \sigma(L) \mid k \text{ odd}\}$. Furthermore, we also remark the following fact that is used throughout this subsection.

Remark 43. *Let $4 \leqslant n \leqslant 17$. We see from* [WZ93] *that the degree of the minimal polynomial of* $\cos(2\pi/n)$ *is one only for $n = 4, 6$, and is two only for $n = 5, 8, 10, 12$.*

Since a two-state involved in PST is necessarily periodic, we narrow our focus on periodic two-states. Suppose that $\mathbf{e}_u + s\mathbf{e}_w$ is periodic. Since any pair of two vertices is a cospectral pair, we see from Proposition 34 that (i) $s = \pm 1$ or (ii) $s \neq \pm 1$ implies that two vertices are periodic.

Let us first consider the latter case.

Proposition 44. *Let $s \neq \pm 1$. A two-state $\mathbf{e}_u + s\mathbf{e}_w$ is periodic C_n if and only if $n = 4, 6$. In particular, $\mathbf{e}_u + s\mathbf{e}_w$ is periodic with $\rho = \pi$ whenever $n = 4$, and $\rho = 2\pi$ whenever $n = 6$.*

Proof. Since C_n is vertex-transitive, $\sigma_u(L) = \sigma(L) = \lfloor \frac{n}{2} \rfloor + 1$. We find from Corollary 32 that $\lfloor \frac{n}{2} \rfloor + 1 \leqslant 5$ implies $n \leqslant 9$. Examining $\sigma_u(L)$ satisfying Theorem 30, the conclusion follows. $\square$

The characterization of PST between pair states or plus states is given in [CG20] for the case that at least one of the states corresponds to an edge. In this case, C_4 is the only one allowing PST between pair states and plus states. We shall complete the characterization. We first consider periodicity of pair states and plus states.

Lemma 45. *Let $\mathbf{e}_1 + s\mathbf{e}_w$ be a two-state in C_n for $2 \leqslant w \leqslant \frac{n}{2} + 1$. Let $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$. If $s = -1$ and $(w-1)j/n \notin \mathbb{Z}$, then $\lambda_j \in \sigma_{1w}(M)$. Furthermore, if $s = 1$ and $(w-1)j/n - \frac{1}{2} \notin \mathbb{Z}$, then $\lambda_j \in \sigma_{1w}(M)$.*

Proof. We will prove it by contrapositive. Suppose that $E_{\lambda_j}(\mathbf{e}_1 - \mathbf{e}_w) = \mathbf{0}$, which is equivalent to $(\mathbf{v}_j\mathbf{v}_j^T + \mathbf{v}_{n-j}\mathbf{v}_{n-j}^T)(\mathbf{e}_1 - \mathbf{e}_w) = \mathbf{0}$. Then $\mathbf{e}_1^T(\mathbf{v}_j\mathbf{v}_j^T + \mathbf{v}_{n-j}\mathbf{v}_{n-j}^T)(\mathbf{e}_1 - \mathbf{e}_w) = 0$, which is equivalent to $\cos(2\pi(w-1)j/n) = 1$. Hence, $(w-1)j/n \in \mathbb{Z}$. Similarly, if $E_{\lambda_j}(\mathbf{e}_1 + \mathbf{e}_w) = \mathbf{0}$, then $(w-1)j/n - \frac{1}{2} \in \mathbb{Z}$. $\square$

Lemma 46. *Let $\mathbf{e}_u + s\mathbf{e}_w$ be a two-state in C_n. Suppose that u and w are not strongly cospectral. Then $\lambda_0, \lambda_1 \in \sigma_{uw}(L)$ for $s = 1$, and $\lambda_1, \lambda_2 \in \sigma_{uw}(L)$ for $s = -1$.*

Proof. Clearly, $\lambda_0 \in \sigma_{uw}(L)$ for $s = 1$, and $\lambda_0 \notin \sigma_{uw}(L)$ for $s = -1$. We may assume $u = 1$ and $2 \leqslant w < \frac{n}{2} + 1$. Since $0 < \frac{w-1}{n} < \frac{1}{2}$ and $-\frac{1}{2} < \frac{w-1}{n} - \frac{1}{2} < 0$, the conclusion follows from Lemma 45. $\square$

Proposition 47. *A two-state $\mathbf{e}_u + \mathbf{e}_w$ is periodic at time ρ in C_n if and only if one of the following hold:*

(i) If u and w are not antipodal, then $(n, \rho) = (4, \pi)$ or $(n, \rho) = (6, 2\pi)$.

(ii) If u and w are antipodal, then $(n, \rho) = (4, \pi/2)$, $(n, \rho) = (6, 2\pi/3)$, $(n, \rho) = (8, \pi)$ or $(n, \rho) = (12, 2\pi)$.

Proof. Suppose that $\mathbf{e}_1 + \mathbf{e}_w$ is periodic in C_n at time ρ for some $2 \leqslant w \leqslant \frac{n}{2} + 1$. Assume that 1 and w are not strongly cospectral. From Lemma 46, $\lambda_0, \lambda_1 \in \sigma_{uw}(L)$. Then $\lambda_1 = 2\cos\left(\frac{2\pi}{n}\right)$ must be an integer. So, $n = 4$ or $n = 6$. Now assume that u and w are strongly cospectral. Then $\lambda_2 \in \sigma_{uw}(L) = \sigma_{u,w}^+(L)$. Since λ_2 is an integer, we have $n \in \{4, 6, 8, 12\}$.

Let us examine times for periodicity for $n = 6$. If $w = 2$ then $\sigma_{uw}(L) = \{0, 1, 3\}$ and $\rho = 2\pi$. If $w = 3$ then $\sigma_{uw}(L) = \{0, 1, 3, 4\}$ which yields $\rho = 2\pi$. If $w = 4$ then $\sigma_{uw}(L) = \{0, 3\}$ which yields $\rho = 2\pi/3$. One can verify the remaining cases. $\square$

Proposition 48. *A two-state $\mathbf{e}_u - \mathbf{e}_w$ is periodic in C_n if and only if one of the following conditions hold.*

n	ρ	*distance between u and w*
4	π	2
5	$2\pi/\sqrt{5}$	1 *or* 2
6	2π	1
6	π	2
6	$2\pi/3$	3
8	$\pi/\sqrt{2}$	4

Proof. Suppose that u and w are not strongly cospectral. From Lemma 46, $\lambda_1, \lambda_2 \in \sigma_{uw}(L)$. From Corollary 32, $1 \leqslant \lambda_1 - \lambda_2 = 2\left(\cos\left(2\pi/n\right) - \cos\left(4\pi/n\right)\right)$. This yields $n < 12$. From Theorem 30, λ_1 and λ_2 both are either integers or quadratic integers. It follows from Remark 43 that $n \in \{4, 5, 6\}$. (For $n = 8$, $\lambda_1 = \sqrt{2}$ and $\lambda_2 = 0$; for $n = 10$, $\lambda_1 - \lambda_2 = 1$.)

Assume that u and w are strongly cospectral. Then n is even and $\sigma_{uw}(L) = \sigma^-_{u,w}(L)$. Since $\sigma_{uw}(L) \geqslant 2$, we have $n \geqslant 6$. Then $\lambda_1, \lambda_3 \in \sigma_{uw}(L)$. We see that $1 \leqslant \lambda_1 - \lambda_3 = 2\left(\cos\left(2\pi/n\right) - \cos\left(6\pi/n\right)\right)$. This yields $n < 18$. As done above, it can be seen that $n \in \{6, 8\}$.

One can find the times for periodicity. □

We introduce some result for strongly cospectral states in C_n.

Lemma 49. *Let u and w be antipodal in C_n for n even and $n \geqslant 4$. There is no two-state of form $\mathbf{e}_v - \mathbf{e}_x$ that is strongly cospectral with $\mathbf{e}_u - \mathbf{e}_w$. Moreover, $\mathbf{e}_u + \mathbf{e}_w$ and $\mathbf{e}_v + \mathbf{e}_x$ are strongly cospectral if and only if v and x are antipodal and the distance between u and v is $n/4$; in particular, $\sigma^+_{uw,vx}(L) = \{\lambda_k \in \sigma_{uw}(L) \mid k \equiv 0 \pmod 4\}$ and $\sigma^-_{uw,vx}(L) = \{\lambda_k \in \sigma_{uw}(L) \mid k \equiv 2 \pmod 4\}$.*

Proof. We may assume that $u = 1$ and $w = \frac{n}{2} + 1$. Let $s = -1$. Note that $\lambda_1 \in \sigma_{1w}(L)$. Suppose $E_{\lambda_1}(\mathbf{e}_1 - s\mathbf{e}_w) = \pm E_{\lambda_1}(\mathbf{e}_v - s\mathbf{e}_x)$. Taking $\mathbf{e}_1^T$ on both sides, $2 = \pm\left(\cos\left(2\pi(v-1)/n\right) - \cos\left(2\pi(x-1)/n\right)\right)$. It follows that $\{v, x\} = \{u, w\}$.

Let $s = 1$. If $\mathbf{e}_u + \mathbf{e}_w$ and $\mathbf{e}_v + \mathbf{e}_x$ are strongly cospectral, then it follows from Lemma 9 that without loss of generality, $v = \frac{n}{4} + 1$ and $x = \frac{3n}{4} + 1$. Note that $\sigma_{uw}(L) = \sigma^+_{u,w}(L) = \{\lambda_0, \lambda_2, \ldots, \lambda_{2\lfloor \frac{n}{4} \rfloor}\}$. For the converse, one can verify that given $\lambda_j \in \sigma_{uw}(L)$, $\mathbf{e}_k^T E_{\lambda_j} \mathbf{e}_1 = \pm \mathbf{e}_k^T E_{\lambda_j} \mathbf{e}_{\frac{n}{4}+1}$ for $1 \leqslant k \leqslant n$, and establish the remaining result. □

We now characterize two-states in C_n that exhibit PST. We recall that regarding PST between pair or plus states, it is enough to consider non-edges.

Theorem 50. *Let $u \in V(C_n)$. Perfect state transfer occurs between two-states in C_n if and only if*

(i) $n = 4$, *between* $\mathbf{e}_u + s\mathbf{e}_{u+1}$ *and* $\mathbf{e}_{u+2} + s\mathbf{e}_{u+3}$ *with minimum PST time* $\tau_0 = \frac{\pi}{2}$,

(ii) $n = 4$, *between* $\mathbf{e}_u + s\mathbf{e}_{u+2}$ *and* $\mathbf{e}_{u+2} + s\mathbf{e}_u$ *for all* $s \notin \{\pm 1\}$ *with minimum PST time* $\tau_0 = \frac{\pi}{2}$,

(iii) $n = 4$, *between* $\mathbf{e}_u + \mathbf{e}_{u+2}$ *and* $\mathbf{e}_{u+1} + \mathbf{e}_{u+3}$ *with minimum PST time* $\tau_0 = \frac{\pi}{4}$, *or*

(iv) $n = 8$, *between* $\mathbf{e}_u + \mathbf{e}_{u+4}$ *and* $\mathbf{e}_{u+2} + \mathbf{e}_{u+6}$ *with minimum PST time* $\tau_0 = \frac{\pi}{2}$.

Proof. Since periodicity is a requirement for PST, it suffices to check which cases in Propositions 44, 47 and 48 yield PST. We may assume that $u = 1$ and $2 \leqslant w \leqslant \lfloor \frac{n}{2} \rfloor + 1$.

Case 1: $n = 4$. By examining the orthogonal projection matrices for C_4, one can verify that the folloiwng is the only strongly cospectral two-states: (a) $\mathbf{e}_1 + s\mathbf{e}_2$ and $\mathbf{e}_3 + s\mathbf{e}_4$; (b) $\mathbf{e}_1 + s\mathbf{e}_3$ and $s\mathbf{e}_1 + \mathbf{e}_3$ for $s \notin \{\pm 1\}$; or (c) $\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{e}_2 + \mathbf{e}_4$. For the case (a), we have the following subcases: $\sigma^-_{12,34}(L) = \{2\}$; $\sigma^+_{12,34}(L) = \{4\}$ if $s = -1$; $\sigma^+_{12,34}(L) = \{0\}$ if $s = 1$; and $\sigma^+_{12,34}(L) = \{0, 4\}$ otherwise. For the case (b), $\sigma^+_{13,31}(L) = \{0, 4\}$ and $\sigma^-_{13,31}(L) = \{2\}$. For the last case (c), $\sigma^+_{13,24}(L) = \{0\}$ and $\sigma^-_{13,24}(L) = \{4\}$. Applying Theorem 35 yields the results (i)–(iii).

Case 2: $n = 5$. Examining the orthogonal projection matrices, it can be seen that pair states corresponding to non-edges are not strongly cospectral.

Case 3: $n = 8$ and $n = 12$. Consider C_8. From Lemma 49, it suffices to consider pair of plus states $\mathbf{e}_1 + \mathbf{e}_5$ and $\mathbf{e}_3 + \mathbf{e}_7$. Moreover, $\sigma^+_{15,37}(L) = \{0, 4\}$ and $\sigma^+_{15,37}(L) = \{2\}$. Invoking Theorem 35 yields PST at $\tau_0 = \frac{\pi}{2}$. In this way, it can be seen that there is no PST between two-states in C_{12}.

Case 4: $n = 6$. As done in Case 3, we can find that there is no PST between pair states and between plus states. It suffices to show that $\mathbf{e}_1 + s\mathbf{e}_2$, $\mathbf{e}_1 + s\mathbf{e}_3$, and $\mathbf{e}_1 + s\mathbf{e}_4$ are not involved in PST.

- By Proposition 10, $\mathbf{e}_1 + s\mathbf{e}_2$ and $\mathbf{e}_4 + s\mathbf{e}_5$ are strongly cospectral . In this case, $\sigma^+_{12,45}(L) = \{0, 3\}$ and $1 \in \sigma^-_{12,45}(L)$. Applying Theorem 35(iii), there is no PST between $\mathbf{e}_1 + s\mathbf{e}_2$ and $\mathbf{e}_4 + s\mathbf{e}_5$. Examining the orthogonal projection matrices associated with $\lambda \in \{3, 4\}$ yields no other two-state is strongly cospectral with $\mathbf{e}_1 + s\mathbf{e}_2$.

- The same argument above yields $\mathbf{e}_1 + s\mathbf{e}_3$ as the only two-state that is strongly cospectral with $\mathbf{e}_4 + s\mathbf{e}_6$. Since $3 \in \sigma^+_{13,46}(L)$ and $1 \in \sigma^-_{12,45}(L)$, there is no PST between $\mathbf{e}_1 + s\mathbf{e}_3$ and $\mathbf{e}_4 + s\mathbf{e}_6$.

- Since vertices 1 and 4 are antipodal, the only two-state strongly cospectral with $\mathbf{e}_1 + s\mathbf{e}_4$ is $\mathbf{e}_4 + s\mathbf{e}_1$. But as C_6 has no PST, Proposition 39(ii) implies that PST does not occur between these two-states.

Combining all subcases above yields no PST between two-states in C_6.

Combining the above cases completes the proof. □

5.4 Hypercubes

In this subsection, we completely characterize strongly cospectrality and PST between two-states.

The hypercube Q_n of order n is a Cayley graph for $\mathbb{Z}_2^n$ with connection set C that consists of standard basis vectors. Note the vertex set is $\mathbb{Z}_2^n$. For each $a \in \mathbb{Z}_2^n$, the corresponding eigenvector $\mathbf{x}_a$ is given by

$$(\mathbf{x}_a)_v = (-1)^{a^T v}, \quad \text{for } v \in \mathbb{Z}_2^n. \tag{8}$$

with eigenvalue $n - 2\mathrm{wt}(a)$ where $\mathrm{wt}(a)$ is the number of ones in a. These eigenvectors are orthogonal.

Proposition 51. *Let X be a graph, and let E_λ be the orthogonal projection matrix associated to eigenvalue λ. Suppose that $\mathbf{v}_1, \dots \mathbf{v}_k$ are linearly independent orthogonal eigenvectors where k is the multiplicity of λ. Then $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = E_\lambda(\mathbf{e}_v + s\mathbf{e}_x)$ if and only if $\mathbf{v}_j^T(\mathbf{e}_u + s\mathbf{e}_w) = \mathbf{v}_j^T(\mathbf{e}_v + s\mathbf{e}_x)$ for $1 \leqslant j \leqslant k$. Furthermore, $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = -E_\lambda(\mathbf{e}_v + s\mathbf{e}_x)$ if and only if $\mathbf{v}_j^T(\mathbf{e}_u + s\mathbf{e}_w) = \mathbf{v}_j^T(\mathbf{e}_v + s\mathbf{e}_x)$ for $1 \leqslant j \leqslant k$.*

Proof. Let $\mathbf{u}_j$ be the normalized vector of $\mathbf{v}_j$ for $1 \leqslant j \leqslant k$. Then $E_\lambda = \sum_{j=1}^k \mathbf{u}_j \mathbf{u}_j^T$. Since $\mathbf{u}_r^T \mathbf{u}_s = 0$ for $r \neq s$, the conclusion follows. □

Lemma 52. *Suppose that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral in Q_n. Then $u + w + v + x = 0$.*

Proof. Suppose that $E_\lambda(\mathbf{e}_u + s\mathbf{e}_w) = \pm E_\lambda(\mathbf{e}_v + s\mathbf{e}_x)$ for each eigenvalue λ of A. Then

$$\mathbf{x}_a^T(\mathbf{e}_u + s\mathbf{e}_w) = \pm \mathbf{x}_a^T(\mathbf{e}_v + s\mathbf{e}_x), \quad \text{for } a \in \mathbb{Z}_2^n. \tag{9}$$

The equation must be of the form $(1 + s) = \pm(1 + s)$ or $(1 - s) = \pm(1 - s)$. It follows that $a^T u + a^T w$ and $a^T v + a^T x$ have the same parity for all $a \in \mathbb{Z}_2^n$, that is, $a^T(u + w + v + x) = 0$. This completes the proof. □

Note that if two vertices are strongly cospectral in Q_n, then they are antipodal.

Corollary 53. *Suppose that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral in Q_n. If two distinct vertices among u, w, v, x are not antipodal, then u, w, v, x are distinct and they are not pairwise antipodal. If two vertices are antipodal, then the other vertices are identical or antipodal.*

Lemma 54. *Let u, w, v, x be distinct vertices in Q_n for $n \geqslant 3$. Suppose that no two vertices among them are strongly cospectral. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral if and only if $s = 1$, $n = 3$ and without loss of generality, $u = 000$, $w = 110$, $v = 100$ and $x = 010$.*

Proof. Suppose that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral. By Lemma 52, we may assume that by abuse of notation, $u = 0 \cdots 0$ and $\{w, v, x\} = \{c_1, c_2, c_3\}$ where

$$c_1 = 100\ \ 1\cdots1\ \ 0\cdots0\ \ 0\cdots0, \qquad c_2 = 010\ \ 0\cdots0\ \ 1\cdots1\ \ 0\cdots0, \qquad c_3 = 110\ \ 1\cdots1\ \ 1\cdots1\ \ 0\cdots0$$

and $c_1 + c_2 + c_3 = 0$. For $k = 0, \ldots, n$, it follows from Proposition 51 with (8) and (9) that we have

$$1 + s(-1)^{a^T w} = (-1)^{a^T v} + s(-1)^{a^T x} \text{ for all } a \in \mathbb{Z}_2^n \text{ with } \mathrm{wt}(a) = k, \text{ or} \tag{10}$$

$$1 + s(-1)^{a^T w} = -((-1)^{a^T v} + s(-1)^{a^T x}) \text{ for all } a \in \mathbb{Z}_2^n \text{ with } \mathrm{wt}(a) = k. \tag{11}$$

Let $s \neq \pm 1$. Then we necessarily have either $1 = (-1)^{a^T v}$ for $a \in \mathbb{Z}_2^n$ with $\mathrm{wt}(a) = 1$, or $1 = -(-1)^{a^T v}$ for $a \in \mathbb{Z}_2^n$ with $\mathrm{wt}(a) = 1$. Since v contains 0 and 1 both, there exist x and y with weight 1 such that $(-1)^{x^T v} = 1$ and $(-1)^{y^T v} = -1$, which is a contradiction.

Let $s = 1$ and $n \geqslant 4$. Choose $a_1 = 1000\cdots0$ and $a_2 = 00010\cdots0$. If $w = c_2$, then $1 + (-1)^{a_1^T w} = -((-1)^{a_1^T v} + (-1)^{a_1^T x}) = 2$, but $1 + (-1)^{a_1^T w} = (-1)^{a_1^T v} + (-1)^{a_1^T x} = 2$. Pick $b_1 = 11100\cdots0$, $b_2 = 11010\cdots0$ and $b_3 = 10110\cdots0$. Similarly, one can find contradictions for $w = c_1$ through b_2 and b_3 and for $w = c_3$ through b_1 and b_3.

Let $s = -1$. Choose $x_1 = 1100\cdots0$, $x_2 = 1010\cdots0$ and $x_3 = 0110\cdots0$. One can verify that for any choices of w, v, x from c_1, c_2, c_3, there exist two elements in $\{a_1, a_2, a_3\}$ such that one satisfies (10) and the other does (11). This is a contradiction.

Finally, let $s = 1$ and $n = 3$. One can check that $w = 110$ is the only case satisfying (10) and (11). $\square$

Proposition 55. *Let u, w, v, x be vertices in Q_n for $n \geqslant 3$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral if and only if one of the following holds:*

(i) *$n = 3$ and without loss of generality, $u = 000$, $w = 110$, $v = 100$ and $x = 010$.*

(ii) *$s \neq \pm 1$, u and w are antipodal, $v = w$ and $x = u$.*

(iii) *u and v are antipodal, and w and x are antipodal.*

Proof. From Corollary 53, we only need to consider two cases: u, w, v, x are not pairwise antipodal, and there are two antipodal vertices. The former case is complete in Lemma 54. We shall complete the latter.

We now suppose that u and w are strongly cospectral. From Corollary 53, v and x are strongly spectral. If one of v and x is identical with one of u and w, then three vertices are pairwise antipodal, which does not occur in Q_n. Hence, we only need to consider two cases: (a) $v, x \notin \{u, w\}$ and (b) $v = w$ and $x = u$ with $s \neq \pm 1$. For (a), by Example 14, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are not strongly cospectral. For (b), by Proposition 26, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_w + s\mathbf{e}_u$ are strongly cospectral.

Assume that u and v are strongly cospectral. Then w and x are strongly cospectral. By Example 18, $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ are strongly cospectral. $\square$

Theorem 56. *Let u, w, v, x be vertices in Q_n for $n \geqslant 3$. Then $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ exhibit PST at time τ if and only if one of the following hold:*

(i) *$\tau = \frac{\pi}{2}$, $s \neq \pm 1$, u and w are antipodal, $v = w$ and $x = u$.*

(ii) *$\tau = \frac{\pi}{2}$, u and v are antipodal, and w and x are antipodal.*

Proof. Suppose that $\mathbf{e}_u + s\mathbf{e}_w$ and $\mathbf{e}_v + s\mathbf{e}_x$ exhibit PST. Then they are necessarily strongly cospectral. Consider the first case in Proposition 55. From $\sigma^+_{uw,vx}(A) = \{3, 1\}$ and $\sigma^-_{uw,vx}(A) = \{-1, -3\}$, we find that there is no PST between them. Since there is vertex PST between antipodal vertices in Q_n at time $\frac{\pi}{2}$, the conclusion follows. □

6 Line graphs

In this section, we assume that X is a simple connected unweighted graph. The *line graph* $\ell(X)$ of X is the graph whose vertex set consists of edges of X, and two vertices are adjacent if and only if corresponding edges in X are incident. When X exhibits perfect plus state transfer between $\mathbf{e}_a + \mathbf{e}_b$ and $\mathbf{e}_c + \mathbf{e}_d$ for some $\{a, b\}, \{c, d\} \in E(X)$, a natural question arises: does the line graph $\ell(X)$ exhibit PST between the corresponding vertices, and vice versa? Our goal is to provide an answer to this question.

Suppose X has n vertices and m edges. The incidence matrix of X is an $n \times m$ matrix $R = [R_{i,j}]$, where $R_{i,j}$ is 1 if vertex v_i is incident to edge e_j, and 0 otherwise. Then the signless Laplacian of X is $Q = A + D = RR^T$, where D is the degree matrix of X, and the adjacency matrix of $\ell(X)$ is $A_\ell = R^T R - 2I$. We can see that spectral properties such as singular values and singular vectors of Q and A_ℓ are completely determined by the matrix R. Moreover, each column of R can be written as a plus state form. Thus, it is natural to study the relation between perfect plus state transfer of edges in X in terms of Q and perfect state transfer of vertices in $\ell(X)$ in terms of A_ℓ.

Consider

$$U_{\ell(X)}(t) = \exp(-itA_\ell) = \exp(-it(R^T R - 2I)) = \exp(2it) \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} (R^T R)^k.$$

It follows that

$$RU_{\ell(X)}(t) = \alpha(t) U_X(t) R, \tag{12}$$
$$U_{\ell(X)}(t) R^T = \alpha(t) R^T U_X(t), \tag{13}$$

where $\alpha(t) = \exp(2it)$. Note $|\alpha(t)| = 1$.

We say that two edges $\{a, b\}$ and $\{c, d\}$ in X are *strongly cospectral* if the plus states $\mathbf{e}_a + \mathbf{e}_b$ and $\mathbf{e}_c + \mathbf{e}_d$ are strongly cospectral in X.

To avoid confusion, for an edge $\{a, b\}$ of X, we use $\mathbf{f}_{ab}$ to denote the vector of order m whose entry at the vertex of $\ell(X)$ corresponding to $\{a, b\}$ is 1, and is zero elsewhere. Note that $R\mathbf{f}_{ab} = \mathbf{e}_a + \mathbf{e}_b$.

Theorem 57. *Let $\{a, b\}, \{c, d\} \in E(X)$. If $\ell(X)$ exhibits perfect state transfer between two vertices $\{a, b\}$ and $\{c, d\}$, then X exhibits perfect plus state transfer between $\mathbf{e}_a + \mathbf{e}_b$ and $\mathbf{e}_c + \mathbf{e}_d$.*

Proof. Since $\ell(X)$ admits perfect state transfer between $\{a, b\}$ and $\{c, d\}$, we have

$$U_{\ell(X)}(t)\mathbf{f}_{ab} = \gamma \mathbf{f}_{cd} \tag{14}$$

for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. Let R be the incidence matrix of X, and $\alpha(t) = \exp(2it)$. Note $R\mathbf{f}_{ab} = \mathbf{e}_a + \mathbf{e}_b$ and $R\mathbf{f}_{cd} = \mathbf{e}_c + \mathbf{e}_d$. From (14), we have $RU_{\ell(X)}(t)\mathbf{f}_{ab} = \gamma R\mathbf{f}_{cd}$. Using (12), we find that

$$U_X(t)(\mathbf{e}_a + \mathbf{e}_b) = \alpha(t)^{-1} \gamma (\mathbf{e}_c + \mathbf{e}_d).$$

Therefore, our desired result is obtained. □

Remark 58. *The converse of Theorem 57 does not hold in general. It was found in* [Cao21] *that the* 3*-cube, which is bipartite, exhibits perfect plus state transfer between two distinct edges at time* $\frac{\pi}{2}$*. However, its line graph does not exhibit strong cospectrality and therefore there is no time for which PST occurs (see Example 81).*

We can, however, characterize when the converse of Theorem 57 holds. To do this, we begin by investigating strong cospectrality. Let E_λ be the orthogonal projection matrix onto the eigenspace corresponding to eigenvalue λ of Q. Since $\lambda - 2$ is an eigenvalue of the adjacency matrix A_ℓ, we define $F_{\lambda-2}$ to be the orthogonal projection matrix onto the corresponding eigenspace. Suppose that $\lambda > 0$. A left-singular vector $\mathbf{u}$ of R for $\sqrt{\lambda}$, which is an eigenvector of Q associated to λ, can be expressed as $R\mathbf{v} = \sqrt{\lambda}\mathbf{u}$ for some right singular vector $\mathbf{v}$, which is an eigenvector of A_ℓ associated to $\lambda - 2$. Moreover, $R^T\mathbf{u} = \sqrt{\lambda}\mathbf{v}$. This implies that

$$\lambda F_{\lambda-2} = R^T E_\lambda R, \quad \lambda E_\lambda = R F_{\lambda-2} R^T.$$

Remark 59. *To avoid confusion, we use a slightly different notation for eigenvalue supports of vertices in line graphs. Consider an edge* $\{a,b\}$ *of a graph* X*. We use* $\sigma_{\{a,b\}}(A_\ell)$ *to denote the eigenvalue support of the vertex* $\{a,b\}$ *in* $\ell(X)$*. We shall also use* $\sigma_{\mathbf{f}_{ab}}(A_\ell)$ *to denote the same eigenvalue support.*

Proposition 60. *Let* $\{a,b\},\{c,d\} \in E(X)$*. For a non-zero eigenvalue* λ *of* Q*, we have* $E_\lambda(\mathbf{e}_a + \mathbf{e}_b) = \pm E_\lambda(\mathbf{e}_c + \mathbf{e}_d)$ *if and only if* $F_{\lambda-2}\mathbf{f}_{ab} = \pm F_{\lambda-2}\mathbf{f}_{cd}$*. This implies that if* $v = \{a,b\}$ *and* $w = \{c,d\}$ *are strongly cospectral, then*

$$\sigma^+_{ab,cd}(Q) - \{0\} = \sigma^+_{v,w}(A_\ell + 2I) - \{0\}, \quad \sigma^-_{ab,cd}(Q) - \{0\} = \sigma^-_{v,w}(A_\ell + 2I) - \{0\}.$$

Proof. Let λ be a non-zero eigenvalue of Q. Suppose that $E_\lambda(\mathbf{e}_a + \mathbf{e}_b) = \pm E_\lambda(\mathbf{e}_c + \mathbf{e}_d)$. We can see that

$$\lambda F_{\lambda-2}\mathbf{f}_{ab} = R^T E_\lambda R \mathbf{f}_{ab} = R^T E_\lambda(\mathbf{e}_a + \mathbf{e}_b) = \pm R^T E_\lambda(\mathbf{e}_c + \mathbf{e}_d) = \pm R^T E_\lambda R \mathbf{f}_{cd} = \pm \lambda F_{\lambda-2}\mathbf{f}_{cd}.$$

Conversely, suppose that $F_{\lambda-2}\mathbf{f}_{ab} = \pm F_{\lambda-2}\mathbf{f}_{cd}$. Then,

$$R^T E_\lambda(\mathbf{e}_a + \mathbf{e}_b) = R^T E_\lambda R\mathbf{f}_{ab} = F_{\lambda-2}\mathbf{f}_{ab} = \pm F_{\lambda-2}\mathbf{f}_{cd} = \pm R^T E_\lambda R \mathbf{f}_{cd} = \pm R^T E_\lambda(\mathbf{e}_c + \mathbf{e}_d).$$

Pre-multiplying both sides by R, we have $QE_\lambda(\mathbf{e}_a + \mathbf{e}_b) = \pm QE_\lambda(\mathbf{e}_c + \mathbf{e}_d)$. Considering the spectral decomposition of Q, we obtain $E_\lambda(\mathbf{e}_a + \mathbf{e}_b) = \pm E_\lambda(\mathbf{e}_c + \mathbf{e}_d)$. Therefore, the conclusion follows. $\square$

From Proposition 60, we need only check if $E_0(\mathbf{e}_a + \mathbf{e}_b) = \pm E_0(\mathbf{e}_c + \mathbf{e}_d)$ and $F_{-2}\mathbf{e}_{\{a,b\}} = \pm F_{-2}\mathbf{e}_{\{c,d\}}$, in order to see whether strong cospectrality between edges in X is determined by strong cospectrality between the corresponding vertices in $\ell(X)$, and vice versa. The next remark will help with this task.

Remark 61. *The matrix* F_{-2} *is the orthogonal projection matrix onto the null space of* R*, which is the same as the column space of* NN^T*, where* N *is a matrix whose columns are linearly independent right null vectors of* R*. It follows that for two vertices* x *and* y *of* $\ell(X)$*,* $NN^T(\mathbf{e}_x \pm \mathbf{e}_y) = \mathbf{0}$ *if and only if* $F_{-2}(\mathbf{e}_x \pm \mathbf{e}_y) = \mathbf{0}$*. Therefore, we do not need to consider orthogonality of null vectors of* R*. A similar argument can be applied to* E_0*.*

If a graph is not a tree, then the nullity of R is at least $m - n$, making it difficult to analyze F_{-2} when m becomes large relative to n. Hence, it is natural to first consider trees or unicyclic graphs as they have fewer edges than other connected graphs. We require the following basic facts.

Lemma 62. [Bap10, Lemma 2.17] *Let* X *be a connected bipartite graph on* n *vertices, and let* $\{a,b\}$ *and* $\{c,d\}$ *be edges of* X*. Then,* $\operatorname{rank}(R) = n - 1$ *and* $E_0(\mathbf{e}_a + \mathbf{e}_b) = \pm E_0(\mathbf{e}_c + \mathbf{e}_d) = \mathbf{0}$*.*

Lemma 63. *Let X be a unicyclic graph on n vertices. Then, either X has an odd cycle and* $\operatorname{rank}(R) = n$*, or X has an even cycle and* $\operatorname{rank}(R) = n - 1$.

Proof. The matrix R is a square. Hence, from [Bap10, Lemma 2.18], R is singular if and only if the length of the cycle is even, that is, X is bipartite. Hence, if R is singular, then R must be of rank $n - 1$. □

The following result is well-known.

Proposition 64. *We have* $\operatorname{nullity}(R) = 0$ *if and only if X is either a tree or unicyclic with an odd cycle.*

Now, for the case of trees and unicyclic graphs, we present relationships between strong cospectrality of edges of X and the corresponding vertices in $\ell(X)$ in the following two theorems.

Theorem 65. *Let X be a tree, and let $\{a, b\}$ and $\{c, d\}$ be edges of X. Then, $\{a, b\}$ and $\{c, d\}$ are strongly cospectral edges in X if and only if $\{a, b\}$ and $\{c, d\}$ are strongly cospectral vertices in $\ell(X)$.*

Proof. As trees are bipartite, we observe $E_0(\mathbf{e}_a + \mathbf{e}_b) = \pm E_0(\mathbf{e}_c + \mathbf{e}_d)$ from Lemma 62. Since $\operatorname{nullity}(R) = 0$, F_{-2} does not exist. Now the conclusion follows from Proposition 60. □

Theorem 66. *Let X be a unicyclic graph on n vertices, and let $\{a, b\}$ and $\{c, d\}$ be edges of X.*

(i) Suppose that the cycle of X is of odd length. Then, $\{a, b\}$ and $\{c, d\}$ are strongly cospectral edges in X if and only if $\{a, b\}$ and $\{c, d\}$ are strongly cospectral vertices in $\ell(X)$.

(ii) Suppose that the cycle of X is of even length. Then, $\{a, b\}$ and $\{c, d\}$ are strongly cospectral edges in X and both are either on the cycle or outside of the cycle if and only if $\{a, b\}$ and $\{c, d\}$ are strongly cospectral vertices in $\ell(X)$.

Proof. If the cycle is of odd length, then R is of full rank. Thus from Proposition 60, the first statement is true. Suppose that the cycle is of even length. Then $\operatorname{rank}(R) = n - 1$. Since X is bipartite, $E_0(\mathbf{e}_a + \mathbf{e}_b) = \pm E_0(\mathbf{e}_c + \mathbf{e}_d)$. Let $(v_1, \ldots, v_\ell, v_1)$ be the cycle in X for some $\ell \geqslant 4$. Let $\mathbf{x} = (x_k)_{1 \leqslant k \leqslant n}$ be given by

$$x_k = \begin{cases} (-1)^k \mathbf{f}_{v_k\, v_{k+1}} & \text{if } 1 \leqslant k \leqslant \ell - 1, \\ \mathbf{f}_{v_\ell\, v_1} & \text{if } k = \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Then, $\mathbf{x}$ is a right-null vector of R. It follows that $F_{-2}\mathbf{f}_{ab} = \pm F_{-2}\mathbf{f}_{cd}$ if and only if the corresponding edges in X are either on the cycle or outside of the cycle. Therefore, the second statement follows. □

We are now ready to present the main result of this section.

Theorem 67. *Let $\{a, b\}, \{c, d\} \in E(X)$. Then $\ell(X)$ exhibits perfect state transfer between two vertices $\{a, b\}$ and $\{c, d\}$ if and only if all of the following conditions hold:*

(i) X exhibits perfect plus state transfer between edges $\{a, b\}$ and $\{c, d\}$.

(ii) $F_{-2}\mathbf{f}_{ab} = \pm F_{-2}\mathbf{f}_{cd}$ if F_{-2} exists.

(iii) Let $v = \{a, b\}$, $w = \{c, d\}$ and $\sigma_{ab}(Q) = \{\lambda_1, \ldots, \lambda_k\}$. Suppose that λ_1 is the largest eigenvalue. One of the following holds:

(iii-1) $0 \notin \sigma_{ab}(Q)$ and $-2 \notin \sigma_v(A_\ell)$.

(iii-2) $0 \in \sigma_{ab}(Q)$, $-2 \notin \sigma_v(A_\ell)$ *and* $g \equiv h \pmod 2$, *where* $g = \gcd\left(\{\lambda_1 - \lambda_r\}_{r=1}^{k}\right)$ *and* $h = \gcd(g, \lambda_1)$. *Moreover, either* $\frac{\lambda_1}{h}$ *is even and* $0 \in \sigma^{+}_{ab,cd}(Q)$, *or* $\frac{\lambda_1}{h}$ *is odd and* $0 \in \sigma^{-}_{ab,cd}(Q)$.

(iii-3) $0 \notin \sigma_{ab}(Q)$, $-2 \in \sigma_v(A_\ell)$ *and* $g \equiv h \pmod 2$, *where* $g = \gcd\left(\{\lambda_1 - \lambda_r\}_{r=1}^{k}\right)$ *and* $h = \gcd(g, \lambda_1)$. *Moreover, either* $\frac{\lambda_1}{h}$ *is even and* $-2 \in \sigma^{+}_{v,w}(A_\ell)$, *or* $\frac{\lambda_1}{h}$ *is odd and* $-2 \in \sigma^{-}_{v,w}(A_\ell)$.

(iii-4) $0 \in \sigma^{+}_{ab,cd}(Q)$, $-2 \in \sigma^{+}_{v,w}(A_\ell)$.

(iii-5) $0 \in \sigma^{-}_{ab,cd}(Q)$, $-2 \in \sigma^{-}_{v,w}(A_\ell)$.

Proof. Suppose that $\ell(X)$ exhibits perfect state transfer between two vertices $\{a,b\}$ and $\{c,d\}$. Obviously, (ii) follows. From Theorem 57, we obtain (i) and hence, Theorem 35(iii), the so-called *parity condition*, holds in terms of Q and A_ℓ. We claim that those two parity conditions being true implies (iii). From Proposition 60, we may only consider the case that 0 belongs to only one of $\sigma_{ab}(Q)$ and $\sigma_v(A_\ell + 2I)$, which is for (iii-2) and (iii-3). Clearly, $\lambda_1, \ldots, \lambda_k$ are integers so g and h are well-defined. There must be $1 \leqslant r_1 \leqslant k$ such that $(\lambda_1 - \lambda_{r_1})/g$ is odd; otherwise, g would not be the greatest common divisor. If $g = hc$ for some even $c \geqslant 2$, then $(\lambda_1 - \lambda_{r_1})/h$ is even and thus one of the parity conditions for Q and A_ℓ fails to hold. Hence, $g = hc$ for some odd $c \geqslant 1$. That is, g and h have the same parity. Therefore, our desired claim is established.

For the proof of the converse, (i) and (ii) imply the strong cospectrality between two vertices $\{a,b\}$ and $\{c,d\}$ in $\ell(X)$; and the parity condition for X together with (iii) implies that for $\ell(X)$. □

Remark 68. *If conditions (ii) and (iii) hold, then $\ell(X)$ exhibits perfect state transfer between two vertices $\{a,b\}$ and $\{c,d\}$ if and only if X exhibits perfect plus state transfer between edges $\{a,b\}$ and $\{c,d\}$.*

Corollary 69. *Let X be a tree or a unicyclic graph with odd cycle and let $\{a,b\}$ and $\{c,d\}$ be edges of X. Then, $\ell(X)$ exhibits perfect state transfer between two vertices $\{a,b\}$ and $\{c,d\}$ if and only if X exhibits perfect plus state transfer between edges $\{a,b\}$ and $\{c,d\}$.*

Proof. Clearly, F_{-2} does not exist. Since $E_0(\mathbf{e}_a + \mathbf{e}_b) = E_0(\mathbf{e}_c + \mathbf{e}_d) = \mathbf{0}$, the eigenvalue 0 is not in the eigenvalue support of $\mathbf{e}_a + \mathbf{e}_b$. Thus the result follows. □

Corollary 70. *Let X be a unicyclic graph with even cycle and let $\{a,b\}$ and $\{c,d\}$ be edges of X. Then, $\ell(X)$ exhibits perfect state transfer between two vertices $\{a,b\}$ and $\{c,d\}$ if and only if one of the following hold:*

(i) *X exhibits perfect plus state transfer between edges $\{a,b\}$ and $\{c,d\}$ both outside of the cycle.*

(ii) *X exhibits perfect plus state transfer between adjacent edges $\{a,b\}$ and $\{c,d\}$ both on the cycle, and $\frac{\lambda_1}{h}$ is odd, where λ_1 and h are defined as in (c-3) of Theorem 67.*

(iii) *X exhibits perfect plus state transfer between non-adjacent edges $\{a,b\}$ and $\{c,d\}$ both on the cycle, and $\frac{\lambda_1}{h}$ is even, where λ_1 and h are defined as in (c-3) of Theorem 67.*

Proof. Since $E_0(\mathbf{e}_a + \mathbf{e}_b) = E_0(\mathbf{e}_c + \mathbf{e}_d) = \mathbf{0}$, the eigenvalue 0 is not in the eigenvalue support of $\mathbf{e}_a + \mathbf{e}_b$. It follows from (15) that if two edges are outside of the cycle, then $F_{-2}\mathbf{f}_{ab} = F_{-2}\mathbf{f}_{cd} = \mathbf{0}$; and if two edges on the cycle are adjacent, then $F_{-2}\mathbf{f}_{ab} = -F_{-2}\mathbf{f}_{cd}$; and if two edges on the cycle are not adjacent, then $F_{-2}\mathbf{f}_{ab} = F_{-2}\mathbf{f}_{cd}$. This completes the proof. □

The complete bipartite graph $K_{2,4n}$ with $n \geqslant 1$ exhibits perfect plus state transfer between two edges sharing one of two vertices with degree $4n$ in common [Che19]. We will show that $\ell(K_{2,4n})$ exhibits PST between corresponding vertices by using Theorem 67.

Corollary 71. *Let $n \geqslant 1$, and let u and v be the vertices of degree $4n$ in $K_{2,4n}$. For any $w \in V(K_{2,4n})$ with $w \neq u, v$, $\ell(K_{2,4n})$ admits perfect state transfer between vertices $\{u, w\}$ and $\{v, w\}$.*

Proof. Since $K_{2,4n}$ is bipartite, we have $0 \notin \sigma_{uw}(Q)$ by Lemma 62. From [Che19], $K_{2,4n}$ exhibits perfect plus state transfer between $\mathbf{e}_u + \mathbf{e}_w$ and $\mathbf{e}_v + \mathbf{e}_w$. Now we shall examine the conditions (ii) and (iii) of Theorem 67. Since $\ell(K_{2,4n}) = K_2 \square K_{4n}$, we can find that F_{-2} can be written as

$$F_{-2} = \left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \otimes \left(I - \frac{1}{4n}J\right).$$

It follows that $-2 \in \sigma^-_{x,y}(A_\ell)$ where $x = \{u, w\}$ and $y = \{v, w\}$. Let $g = \gcd(2, 4n)$ and $h = \gcd(2, 4n, 4n+2)$. Then, $g = h = 2$ and $\frac{4n+2}{2} \equiv 1 \pmod 2$. Hence, (ii) and (iii-3) of Theorem 67 holds. Therefore, the conclusion follows. □

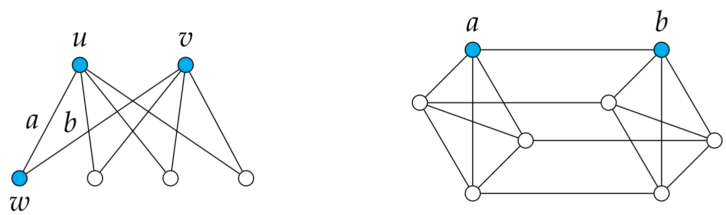

Figure 4: $K_{2,4}$ (left) and its line graph (right)

7 Line graphs of Cartesian products

Similar to the previous section, we assume that X is a simple connected unweighted graph. Our goal in this section is to characterize strong cospectrality and PST in $\ell(X)$ whenever X is a Cartesian product.

Let $X = X_1 \square X_2$ where X_1 and X_2 are connected graphs on n_1 and n_2 vertices with m_1 and m_2 edges, respectively. Let R_1 and R_2 be the incidence matrices of X_1 and X_2, respectively. We use E_λ and $F_{\lambda-2}$ to denote the sums of outer products of left-singular and right-singular vectors for singular value $\sqrt{\lambda}$ of R, respectively. Similarly, we use analogous notation $E^{(i)}_\mu$'s and $F^{(i)}_{\mu-2}$'s to denote the sums of outer products of left-singular and right-singular vectors, respectively, for the singular value $\sqrt{\mu}$ of R_i for $i = 1, 2$. Note that $E^{(i)}_\mu$ (resp. $F^{(i)}_{\mu-2}$) is the orthogonal projection matrix onto the eigenspace of the signless Laplacian matrix Q_i (resp. the adjacency matrix of $\ell(X_i)$) corresponding to eigenvalue μ (resp. $\mu - 2$).

Let N_i be a matrix whose columns are linearly independent right null vectors of R_i for $i = 1, 2$. Recall from Remark 61 that for two vertices x and y of X, $NN^T\mathbf{f}_x = \pm NN^T\mathbf{f}_y$ is equivalent to $F_{-2}\mathbf{f}_x = \pm F_{-2}\mathbf{f}_y$. Hence, we shall use null vectors throughout this section to examine $F_{-2}\mathbf{f}_x = F_{-2}\mathbf{f}_y$.

Proposition 72. *Let $X = X_1 \square X_2$. Then, NN^T can be written as follows:*

$$NN^T = \begin{bmatrix} I_{n_1} \otimes N_2N_2^T + R_1\tilde{I}_{r_1}R_1^T \otimes \tilde{I}_{r_2} & -R_1\tilde{I}_{r_1} \otimes \tilde{I}_{r_2}R_2^T \\ -\tilde{I}_{r_1}R_1^T \otimes R_2\tilde{I}_{r_2} & N_1N_1^T \otimes I_{n_2} + \tilde{I}_{r_1} \otimes R_2\tilde{I}_{r_2}R_2^T \end{bmatrix}, \tag{16}$$

where for $i = 1, 2$, $\tilde{I}_{r_i}$ is an $m_i \times m_i$ diagonal matrix whose diagonal entries consists of $\operatorname{rank}(R_i)$ *ones and $(m_i - \operatorname{rank}(R_i))$ zeros, if $m_i > \operatorname{rank}(R_i)$, so that non-zero columns of $R_i\tilde{I}_{r_i}$ are linearly independent.*

Proof. From [FTL22], R can be written as

$$R = \begin{bmatrix} I_{n_1} \otimes R_2 & R_1 \otimes I_{n_2} \end{bmatrix}. \tag{17}$$

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are null vectors of R_1 and R_2, respectively. Then

$$R \begin{bmatrix} \mathbf{e}_i \otimes \mathbf{x}_2 \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad R \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_1 \otimes \mathbf{e}_j \end{bmatrix} = \mathbf{0}$$

for $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$. Let $\operatorname{rank}(R_1) = r_1$ and $\operatorname{rank}(R_2) = r_2$. Without loss of generality, we assume that the first r_1 columns of R_1 are linearly independent and the first r_2 columns of R_2 are linearly independent. For $1 \leqslant i \leqslant r_1$ and $1 \leqslant j \leqslant r_2$, we have

$$R \begin{bmatrix} -R_1\mathbf{f}_i \otimes \mathbf{e}_j \\ \mathbf{e}_i \otimes R_2\mathbf{f}_j \end{bmatrix} = \mathbf{0}.$$

We claim that $\operatorname{nullity}(R) = s_1n_2 + n_1s_2 + r_1r_2$ where $s_1 = \operatorname{nullity}(R_1)$ and $s_2 = \operatorname{nullity}(R_2)$. From the rank-nullity theorem, we have $m_1 = r_1 + s_1$, $m_2 = r_2 + s_2$. Note that RR^T is the signless Laplacian matrix of X and $RR^T = I_{n_1} \otimes R_2R_2^T + R_1R_1^T \otimes I_{n_2}$. It follows that any eigenvalue of RR^T can be written as $\lambda + \mu$, where λ and μ are eigenvalues of $R_1R_1^T$ and $R_2R_2^T$, respectively. Hence, $\operatorname{rank}(R) = n_1n_2 - a_1a_2$, where $a_1 = \operatorname{nullity}(R_1R_1^T)$ and $a_2 = \operatorname{nullity}(R_2R_2^T)$. Since $n_1 = r_1 + a_1$ and $n_2 = r_2 + a_2$, we have $\operatorname{rank}(R) = n_1r_2 + n_2r_1 - r_1r_2$. Now we can see that

$$\begin{aligned} \operatorname{nullity}(R) =& n_1m_2 + m_1n_2 - \operatorname{rank}(R) \\ =& n_1m_2 + m_1n_2 - n_1r_2 - n_2r_1 + r_1r_2 \\ =& n_1m_2 + m_1n_2 - n_1(m_2 - s_2) - n_2(m_1 - s_1) + r_1r_2 = s_1n_2 + n_1s_2 + r_1r_2. \end{aligned}$$

Therefore, the conclusion follows. □

We will look for an equivalent condition for two vertices x and y of $\ell(X)$ to satisfy $NN^T\mathbf{f}_x = \pm NN^T\mathbf{f}_y$. To this goal, we present several lemmas.

Lemma 73. *Let $n \geqslant 3$. Let X be a connected graph on n vertices, and $\operatorname{rank}(R) = r$. For distinct vertices i and j of X, there exists a matrix $\tilde{R}$ such that it comprises r linearly independent $(0,1)$ column vectors of R and i^{th} and j^{th} rows of $\tilde{R}$ are not identical.*

Proof. Clearly, $r \geqslant 2$. Suppose that i and j are adjacent. Since $n \geqslant 3$, one of i and j has degree more than 1. There exists a 2×2 submatrix of R whose rows are indexed by i and j as one of the following:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Similarly, if i and j are not adjacent, there exists a 2×2 submatrix of R whose rows are indexed by i and j as one of the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, we obtain the result, as desired. □

Lemma 74. *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be $(0,1)$ column vectors of the same size, and let $\mathbf{y}_1$ and $\mathbf{y}_2$ be $(0,1)$ column vectors of the same size. Then, $\mathbf{x}_1 \otimes \mathbf{y}_1 = \mathbf{x}_2 \otimes \mathbf{y}_2$ if and only if $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$.*

Lemma 75. *Suppose X is neither a tree nor a unicyclic graph with odd cycle. Let v be a cut-vertex of X and suppose there is a component C in $X - v$ such that the subgraph X_1 induced in X by $V(C) \cup \{v\}$ is a tree. Then $F_{-2}\mathbf{f}_w = \mathbf{0}$ for any vertex $w \in E(X_1)$.*

Proof. Let X_2 be the subgraph induced by $V(X) - V(C)$, and let R_2 be its incidence matrix. Since X_1 is a tree, the submatrix of R whose rows and columns are indexed by $V(C)$ and $E(X_1)$, respectively, is of full rank. It follows that $\mathrm{nullity}(R_2) = \mathrm{nullity}(R)$. Therefore, F_{-2} can be expressed as a direct sum so that the submatrix of F_{-2} whose rows and columns are indexed by $E(X_1)$ is the zero matrix. $\square$

Let $X = X_1 \square X_2$ and $\{x, y\}$ be an edge of X. Then, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for some $x_1, y_1 \in V(X_1)$ and $x_2, y_2 \in V(X_2)$. Regarding $\{x, y\}$ as a vertex of $\ell(X)$ and labelling the row and column indices of R as in (17), we can see that

$$\mathbf{f}_{xy} = \begin{cases} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{x_1 y_1} \otimes \mathbf{e}_{x_2} \end{bmatrix}, & \text{if } x_1 \sim y_1 \text{ and } x_2 = y_2, \\ \begin{bmatrix} \mathbf{e}_{x_1} \otimes \mathbf{f}_{x_2 y_2} \\ \mathbf{0} \end{bmatrix}, & \text{if } x_1 = y_1 \text{ and } x_2 \sim y_2. \end{cases}$$

Proposition 76. *Let $X = X_1 \square X_2$, and $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be edges of X. Then, $NN^T\mathbf{f}_{v_1w_1} = \pm NN^T\mathbf{f}_{v_2w_2}$ if and only if one of the following holds:*

(i) $X_1 = K_2$, $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{e}_1 \otimes \mathbf{f}_i \\ \mathbf{0} \end{bmatrix}$, $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{e}_2 \otimes \mathbf{f}_i \\ \mathbf{0} \end{bmatrix}$ *for some* $1 \leqslant i \leqslant m_2$, *and* $N_2N_2^T\mathbf{f}_i = \mathbf{0}$.

(ii) $X_2 = K_2$, $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_1 \end{bmatrix}$, $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_2 \end{bmatrix}$ *for some* $1 \leqslant i \leqslant m_1$, *and* $N_1N_1^T\mathbf{f}_i = \mathbf{0}$.

(iii) There exist $1 \leqslant i_1 \leqslant n_1$, $1 \leqslant k_1 \leqslant m_2$, $1 \leqslant i_2 \leqslant m_1$ *and* $1 \leqslant k_2 \leqslant n_2$ *such that* $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{e}_{i_1} \otimes \mathbf{f}_{k_1} \\ \mathbf{0} \end{bmatrix}$, $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} \end{bmatrix}$, $\deg(i_1) = \deg(k_2) = 1$, *and i_2 and k_1 are the edges incident to i_1 and k_2, respectively. Moreover,* $N_1N_1^T\mathbf{f}_{i_2} = \mathbf{0}$ *and* $N_2N_2^T\mathbf{f}_{k_1} = \mathbf{0}$.

Proof. For the proof, we examine the following two cases:

- Without loss of generality, $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{i_1} \otimes \mathbf{e}_{k_1} \end{bmatrix}$ and $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} \end{bmatrix}$ for $1 \leqslant i_1, i_2 \leqslant m_1$ and $1 \leqslant k_1, k_2 \leqslant n_2$ with $i_1 \neq i_2$ or $k_1 \neq k_2$.
- $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{e}_{i_1} \otimes \mathbf{f}_{k_1} \\ \mathbf{0} \end{bmatrix}$ and $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} \end{bmatrix}$ for $1 \leqslant i_1 \leqslant n_1$, $1 \leqslant k_1 \leqslant m_2$, $1 \leqslant i_2 \leqslant m_1$ and $1 \leqslant k_2 \leqslant n_2$.

For the first case, we suppose that $NN^T\mathbf{f}_{v_1w_1} = \pm NN^T\mathbf{f}_{v_2w_2}$. This is equivalent to

$$\begin{bmatrix} -R_1\tilde{I}_{r_1}\mathbf{f}_{i_1} \otimes \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_1} \\ N_1N_1^T\mathbf{f}_{i_1} \otimes \mathbf{e}_{k_1} + \tilde{I}_{r_1}\mathbf{f}_{i_1} \otimes R_2\tilde{I}_{r_2}R_2^T\mathbf{e}_{k_1} \end{bmatrix} = \pm \begin{bmatrix} -R_1\tilde{I}_{r_1}\mathbf{f}_{i_2} \otimes \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2} \\ N_1N_1^T\mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} + \tilde{I}_{r_1}\mathbf{f}_{i_2} \otimes R_2\tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2} \end{bmatrix}.$$

Applying Lemma 74 to the first row blocks on both sides, we see that $R_1\tilde{I}_{r_1}\mathbf{f}_{i_1} = R_1\tilde{I}_{r_1}\mathbf{f}_{i_2}$ and $\tilde{I}_{r_2}R_2^T\mathbf{e}_{k_1} = \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2}$. Since any two columns of R_1 are linearly independent, i_1 must equal i_2; otherwise, we would have $R_1\tilde{I}_{r_1}\mathbf{e}_{i_1} \neq R_1\tilde{I}_{r_1}\mathbf{e}_{i_2}$ by a proper choice of ones in $\tilde{I}_{r_1}$. Then, $k_1 \neq k_2$. If $X_2 \neq K_2$, then from Lemma 73 we would have $\tilde{I}_{r_2}R_2^T\mathbf{e}_{k_1} \neq \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2}$ by an appropriate choice of $\tilde{I}_{r_2}$. Hence, $X_2 = K_2$. Let $N_1N_1^T\mathbf{f}_{i_1} = \mathbf{x}_0$, $\tilde{I}_{r_1}\mathbf{f}_{i_1} = \mathbf{y}_0$, $k_1 = 1$ and $k_2 = 2$. Then, the second row blocks on both sides give

$$\mathbf{x}_0 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{y}_0 \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \pm\left(\mathbf{x}_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mathbf{y}_0 \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

It follows that $N_1N_1^T\mathbf{f}_{i_1} = \mathbf{0}$.

We now consider the second case. The equation $NN^T\mathbf{f}_{v_1w_1} = \pm NN^T\mathbf{f}_{v_2w_2}$ is equivalent to

$$\begin{bmatrix} \mathbf{e}_{i_1} \otimes N_2N_2^T\mathbf{f}_{k_1} + R_1\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} \otimes \tilde{I}_{r_2}\mathbf{f}_{k_1} \\ -\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} \otimes R_2\tilde{I}_{r_2}\mathbf{f}_{k_1} \end{bmatrix} = \pm\begin{bmatrix} -R_1\tilde{I}_{r_1}\mathbf{f}_{i_2} \otimes \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2} \\ N_1N_1^T\mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} + \tilde{I}_{r_1}\mathbf{f}_{i_2} \otimes R_2\tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2} \end{bmatrix}.$$

Consider the first row block. Note that $-\tilde{I}_{r_1}R_1^T\mathbf{f}_{i_2} \otimes R_2\tilde{I}_{r_2}\mathbf{e}_{k_2}$ is a $(0,-1)$ vector. Choosing $\tilde{I}_{r_1}$ and $\tilde{I}_{r_2}$ properly, we have

$$(\mathbf{e}_{i_1} \otimes \mathbf{f}_{k_1})^T(R_1\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} \otimes \tilde{I}_{r_2}\mathbf{f}_{k_1}) \geqslant 1, \tag{18}$$

where the equality holds if and only if i_1 is a vertex of degree 1. If $N_2N_2^T\mathbf{f}_{k_1} \neq \mathbf{0}$ then we would have $(\mathbf{e}_{i_1} \otimes \mathbf{f}_{k_1})^T\left(\mathbf{e}_{i_1} \otimes N_2N_2^T\mathbf{f}_{k_1} + R_1\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} \otimes \tilde{I}_{r_2}\mathbf{f}_{k_1}\right) > 1$. Hence, $N_2N_2^T\mathbf{f}_{k_1} = \mathbf{0}$ and the degree of i_1 is 1. Now we have

$$R_1\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} \otimes \tilde{I}_{r_2}\mathbf{f}_{k_1} = \pm\left(-R_1\tilde{I}_{r_1}\mathbf{f}_{i_2} \otimes \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2}\right).$$

From Lemma 74, $R_1\tilde{I}_{r_1}R_1^T\mathbf{e}_{i_1} = R_1\tilde{I}_{r_1}\mathbf{f}_{i_2}$ and $\tilde{I}_{r_2}\mathbf{f}_{k_1} = \tilde{I}_{r_2}R_2^T\mathbf{e}_{k_2}$. It follows that i_2 is the edge incident to i_1; moreover, k_2 is a vertex of degree 1 and k_1 is the edge incident to k_2. By Lemma 75, $N_1N_1^T\mathbf{f}_{i_2} = \mathbf{0}$. Exhausting all cases, the proof is complete. □

Remark 77. *Examining the proof of Proposition 76, we can see that $NN^T\mathbf{f}_{v_1w_1} = NN^T\mathbf{f}_{v_2w_2}$ if one of (i) and (ii) holds; and $NN^T\mathbf{f}_{v_1w_1} = -NN^T\mathbf{f}_{v_2w_2}$ if (iii) holds.*

Let $X = X_1 \square X_2$. Let $\mathbf{u}$ and $\mathbf{v}$ be left-singular and right-singular vectors of R_1, respectively, corresponding to singular value τ_1. We also let $\mathbf{x}$ and $\mathbf{y}$ be left-singular and right-singular vectors of R_2, respectively, corresponding to singular value τ_2. Then $R_1\mathbf{v} = \tau_1\mathbf{u}$ and $R_2\mathbf{x} = \tau_2\mathbf{y}$. Suppose that $\tau_1 + \tau_2 > 0$. From (17), we have

$$R^T(\mathbf{u} \otimes \mathbf{x}) = \begin{bmatrix} I_{n_1} \otimes R_2^T \\ R_1^T \otimes I_{n_2} \end{bmatrix}(\mathbf{u} \otimes \mathbf{x}) = \sqrt{\tau_1^2 + \tau_2^2}\begin{bmatrix} \frac{\tau_2}{\sqrt{\tau_1^2+\tau_2^2}}(\mathbf{u} \otimes \mathbf{y}) \\ \frac{\tau_1}{\sqrt{\tau_1^2+\tau_2^2}}(\mathbf{v} \otimes \mathbf{x}) \end{bmatrix}.$$

Let $\lambda \in \sigma(Q_1)$ and $\mu \in \sigma(Q_2)$ with $\lambda + \mu > 0$. We use $\Omega^+_{\lambda+\mu}$ to denote the set of pairs (α, β), where $\alpha \in \sigma(Q_1)$ and $\beta \in \sigma(Q_2)$, such that $\alpha, \beta > 0$ and $\alpha + \beta = \lambda + \mu$. It follows that

$$\begin{aligned} F_{\lambda+\mu-2} = & \sum_{(\alpha,\beta)\in\Omega^+_{\lambda+\mu}} \frac{1}{\alpha+\beta}\begin{bmatrix} \frac{\beta}{\alpha}R_1F^{(1)}_{\alpha-2}R_1^T \otimes F^{(2)}_{\beta-2} & R_1F^{(1)}_{\alpha-2} \otimes F^{(2)}_{\beta-2}R_2^T \\ F^{(1)}_{\alpha-2}R_1^T \otimes R_2F^{(2)}_{\beta-2} & \frac{\alpha}{\beta}F^{(1)}_{\alpha-2} \otimes R_2F^{(2)}_{\beta-2}R_2^T \end{bmatrix} \\ & + \mathcal{I}_{\substack{\lambda+\mu\in\sigma(Q_1), \\ 0\in\sigma(Q_2)}}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F^{(1)}_{\lambda+\mu-2} \otimes E^{(2)}_0 \end{bmatrix} + \mathcal{I}_{\substack{0\in\sigma(Q_1), \\ \lambda+\mu\in\sigma(Q_2)}}\begin{bmatrix} E^{(1)}_0 \otimes F^{(2)}_{\lambda+\mu-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{19}$$

where $\mathcal{I}_A$ is the indicator function of an event A.

Proposition 78. *Let $X = X_1 \Box X_2$, and $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be edges of X. If there exist $1 \leqslant i_1 \leqslant n_1$, $1 \leqslant k_1 \leqslant m_2$, $1 \leqslant i_2 \leqslant m_1$ and $1 \leqslant k_2 \leqslant n_2$ such that $\mathbf{f}_{v_1w_1} = \begin{bmatrix} \mathbf{e}_{i_1} \otimes \mathbf{f}_{k_1} \\ \mathbf{0} \end{bmatrix}$, $\mathbf{f}_{v_2w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{i_2} \otimes \mathbf{e}_{k_2} \end{bmatrix}$, then $\mathbf{f}_{v_1w_1}$ and $\mathbf{f}_{v_2w_2}$ are not strongly cospectral.*

Proof. Assume to the contrary that $\mathbf{f}_{v_1w_1}$ and $\mathbf{f}_{v_2w_2}$ are strongly cospectral in $\ell(X)$. It follows from (iii) of Proposition 76 that $R_1^T\mathbf{e}_{i_1} = \mathbf{f}_{i_2}$, $R_2^T\mathbf{e}_{k_2} = \mathbf{f}_{k_1}$. We note that if $F^{(1)}_{\alpha-2}\mathbf{f}_{i_2} \neq \mathbf{0}$ for nonzero $\alpha \in \sigma(Q_1)$, then $R_1F^{(1)}_{\alpha-2}\mathbf{f}_{i_2} \neq \mathbf{0}$; similarly, if $F^{(2)}_{\beta-2}\mathbf{f}_{k_1} \neq \mathbf{0}$ for nonzero $\beta \in \sigma(Q_2)$, then $R_2F^{(2)}_{\beta-2}\mathbf{f}_{k_1} \neq \mathbf{0}$. Note that $R_1^TR_1 - 2I$ and $R_2^TR_2 - 2I$ are the adjacency matrices of $\ell(X_1)$ and $\ell(X_2)$, respectively.

Let $\lambda \in \sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)$ and $\mu \in \sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)$ such that $\lambda + \mu > 0$. Then $F_{\lambda+\mu-2}\mathbf{f}_{v_1w_1} = \pm F_{\lambda+\mu-2}\mathbf{f}_{v_2w_2}$. We can find from (19) that $F_{\lambda+\mu-2}\mathbf{f}_{v_1w_1} = \pm F_{\lambda+\mu-2}\mathbf{f}_{v_2w_2}$ is equivalent to

$$\mathcal{I}_{\substack{0\in\sigma_{\mathbf{e}_{i_1}}(Q_1),\\ \lambda+\mu\in\sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)}} \left(E^{(1)}_0\mathbf{e}_{i_1} \otimes F^{(2)}_{\lambda+\mu-2}\mathbf{f}_{k_1}\right) + \frac{1}{\lambda+\mu} \sum_{\substack{(\alpha,\beta)\in\Omega^+_{\lambda+\mu}\\ \alpha\in\sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)\\ \beta\in\sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)}} \left(\frac{\beta}{\alpha} \mp 1\right) R_1F^{(1)}_{\alpha-2}\mathbf{f}_{i_2} \otimes F^{(2)}_{\beta-2}\mathbf{f}_{k_1} = \mathbf{0},$$

$$-\mathcal{I}_{\substack{\lambda+\mu\in\sigma_{\mathbf{f}_{i_2}}(R_1^TR_1),\\ 0\in\sigma_{\mathbf{e}_{k_2}}(Q_2)}} \left(F^{(1)}_{\lambda+\mu-2}\mathbf{f}_{i_2} \otimes E^{(2)}_0\mathbf{e}_{k_2}\right) + \frac{1}{\lambda+\mu} \sum_{\substack{(\alpha,\beta)\in\Omega^+_{\lambda+\mu}\\ \alpha\in\sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)\\ \beta\in\sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)}} \left(\frac{\beta}{\alpha} \mp 1\right) F^{(1)}_{\alpha-2}\mathbf{f}_{i_2} \otimes R_2F^{(2)}_{\beta-2}\mathbf{f}_{k_1} = \mathbf{0}.$$

It follows from the definition of linearly independent vectors that $0 \notin \sigma_{\mathbf{e}_{i_1}}(Q_1)$ or $\lambda + \mu \notin \sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)$, and $0 \notin \sigma_{\mathbf{e}_{k_2}}(Q_2)$ or $\lambda + \mu \notin \sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)$; and $\lambda = \mu > 0$, and for each $(\alpha, \beta) \in \Omega^+_{\lambda+\mu}$ with $\alpha \neq \beta$, $\alpha \notin \sigma_{\mathbf{e}_{i_1}}(Q_1)$ or $\beta \notin \sigma_{\mathbf{e}_{k_2}}(Q_2)$. (Here we must have $F_{\lambda+\mu-2}\mathbf{f}_{v_1w_1} = F_{\lambda+\mu-2}\mathbf{f}_{v_2w_2}$.)

It follows from (iii) of Proposition 76 that $0 \notin \sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)$. Then $\sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)$ contains at least two elements. Similarly, for Similarly, $\sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)$. Suppose λ_j and μ_j are the smallest elements in $\sigma_{\mathbf{f}_{i_2}}(R_1^TR_1)$ and $\sigma_{\mathbf{f}_{k_1}}(R_2^TR_2)$, respectively, with $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. From the argument above, we must have $\lambda_1 = \mu_1$ and $\lambda_1 = \mu_2$, a contradiction. This proves the proposition. □

From Proposition 78, many line graphs of Cartesian products do not exhibit PST.

Theorem 79. *Let X be a Cartesian product. If X cannot be expressed as a Cartesian product of K_2 and some graph, then $\ell(X)$ has no pairs of strongly cospectral vertices.*

Proof. The conclusion follows from Propositions 76 and 78. □

We now characterize strong cospectrality and PST for the remaining line graphs, that is, line graphs that can be written as $X_1 \Box K_2$ for some graph X_1. We require the following lemma.

Lemma 80. *Let $X = X_1 \Box K_2$ where X_1 is a Cartesian product of non-trivial graphs. Then, $\ell(X)$ has no pairs of strongly cospectral vertices.*

Proof. Since X_1 is a Cartesian product, $N_1N_1^T$ can be written as in (16). Using a similar argument as in (18), one can show that $|\mathbf{f}_i^TN_1N_1^T\mathbf{f}_i| > 0$ for $1 \leqslant i \leqslant m_1$. Therefore, $N_1N_1^T\mathbf{f}_i \neq \mathbf{0}$. From (ii) of Proposition 76, the desired result is established. □

Note that the assumptions in Lemmas 79 and 80 imply that $\ell(X)$ has no PST.

Example 81. *For all $n \geqslant 3$, $\ell(Q_n)$ has no pairs of strongly cospectral vertices. Thus $\ell(Q_n)$ has no PST.*

Theorem 82. *Let $X = X_1 \Box K_2$, and $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be edges of X. Then $\mathbf{f}_{v_1 w_1}$ and $\mathbf{f}_{v_2 w_2}$ are strongly cospectral if and only if $\mathbf{f}_{v_1 w_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_1 \end{bmatrix}$ and $\mathbf{f}_{v_2 w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_2 \end{bmatrix}$ for some $1 \leqslant i \leqslant m_1$ and all of the following conditions hold:*

- *X_1 is not a Cartesian product of graphs.*
- *$F^{(1)}_{-2} \mathbf{f}_i = \mathbf{0}$.*
- *There is no pair of non-zero eigenvalues λ and μ in $\sigma_{ab}(Q_1)$ with $|\lambda - \mu| = 2$, where $\{a, b\}$ is an edge of X_1 corresponding to $\mathbf{f}_i$.*

Furthermore, if $\mathbf{f}_{v_1 w_1}$ and $\mathbf{f}_{v_2 w_2}$ are strongly cospectral, then the following statements hold:

- *$-2 \in \sigma^{+}_{\mathbf{f}_{v_1 w_1}, \mathbf{f}_{v_2 w_2}}(A_\ell)$.*
- *Let $\lambda > 0$. Then $\lambda \in \sigma_{ab}(Q_1)$ if and only if $\lambda - 2 \in \sigma^{-}_{\mathbf{f}_{v_1 w_1} \mathbf{f}_{v_2 w_2}}(A_\ell)$ and $\lambda \in \sigma^{+}_{\mathbf{f}_{v_1 w_1} \mathbf{f}_{v_2 w_2}}(A_\ell)$.*

Proof. From Proposition 76, if $\mathbf{f}_{v_1 w_1}$ and $\mathbf{f}_{v_2 w_2}$ are strongly cospectral, then they necessarily have the particular block structure and $F^{(1)}_{-2} \mathbf{f}_i = \mathbf{0}$. Let S be the set of pairs (α, β) such that $\alpha, \beta \in \sigma(Q_1)$ and $\alpha - \beta = 2$. For $(\lambda, \mu) \in S$, it follows from (19) that

$$F_{\lambda-2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F^{(1)}_{\lambda-2} \otimes \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} + \frac{1}{\mu + 2} \begin{bmatrix} \frac{2}{\mu} R_1 F^{(1)}_{\mu-2} R_1^T \otimes 1 & R_1 F^{(1)}_{\mu-2} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ F^{(1)}_{\mu-2} R_1^T \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T & \frac{\mu}{2} F^{(1)}_{\mu-2} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{bmatrix}. \tag{20}$$

Examining the $(2,2)$ blocks of $F_{\lambda-2}$ together with $\mathbf{f}_i \otimes \mathbf{e}_1$ and $\mathbf{f}_i \otimes \mathbf{e}_2$, we can find that $F_{\lambda-2} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_1 \end{bmatrix} \neq \pm F_{\lambda-2} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_2 \end{bmatrix}$ if and only if $F^{(1)}_{\lambda-2} \mathbf{f}_i \neq \mathbf{0}$ and $F^{(1)}_{\mu-2} \mathbf{f}_i \neq \mathbf{0}$—that is, $\lambda, \mu \in \sigma_{\mathbf{f}_i}(R_1^T R_1)$. For $\gamma \in \sigma(R_1^T R_1)$ with $|\gamma - \alpha| \neq 2$ for all $\alpha \in \sigma(R_1^T R_1)$, it can be easily checked that $F_{\gamma-2} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_1 \end{bmatrix} = \pm F_{\gamma-2} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_2 \end{bmatrix}$.

The remaining conclusion can be established from the examination above, and Remark 77. □

Corollary 83. *Let $X = X_1 \Box K_2$ where X_1 is a tree or a unicyclic graph with odd cycle. Suppose that no pair of non-zero eigenvalues of Q_1 has difference 2. Then, two vertices in $\ell(X)$ are strongly cospectral if and only if they correspond to edges $\{(v_1, w_1), (v_2, w_1)\}$ and $\{(v_1, w_2), (v_2, w_2)\}$ in X, where $\{v_1, v_2\}$ is an edge of X_1 and $V(K_2) = \{w_1, w_2\}$. This implies that $\ell(X)$ has exactly m_1 pairs of strongly cospectral vertices. From Theorem 65, the corresponding edges are strongly cospectral in X.*

We conclude the paper with the following result.

Theorem 84. *Let $X = X_1 \Box K_2$, and $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be edges of X. Then $\ell(X)$ exhibits PST between $\mathbf{f}_{v_1 w_1}$ and $\mathbf{f}_{v_2 w_2}$ if and only if $\mathbf{f}_{v_1 w_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_1 \end{bmatrix}$ and $\mathbf{f}_{v_2 w_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \otimes \mathbf{e}_2 \end{bmatrix}$ for some $1 \leqslant i \leqslant m_1$ and all of the following conditions hold:*

- *X_1 is not a Cartesian product of graphs.*
- *$F^{(1)}_{-2} \mathbf{f}_i = \mathbf{0}$.*

- *There is no pair of non-zero eigenvalues λ and μ in $\sigma_{ab}(Q_1)$ with $|\lambda - \mu| = 2$, where $\{a, b\}$ is an edge of X_1 corresponding to $\mathbf{f}_i$.*
- *$\sigma_{ab}(Q_1)$ consists of integral elements so that for each non-zero $\lambda \in \sigma_{ab}(Q_1)$, $\lambda \equiv 2 \pmod 4$.*

Proof. From Theorem 82, we only need to check the parity condition, recalling the last condition of Theorem 35. Suppose that $\ell(X)$ exhibits PST between $\mathbf{f}_{v_1w_1}$ and $\mathbf{f}_{v_2w_2}$. Let $\sigma_{ab}(Q_1)\backslash\{0\} = \{\lambda_1, \ldots, \lambda_k\}$ for some $k \geqslant 1$. Suppose that λ_1 is the largest eigenvalue of Q_1. From Theorem 82, $-2 \in \sigma^+_{\mathbf{f}_{v_1w_1}\,\mathbf{f}_{v_2w_2}}(A_\ell)$, $\lambda_j - 2 \in \sigma^-_{\mathbf{f}_{v_1w_1}\,\mathbf{f}_{v_2w_2}}(A_\ell)$ and $\lambda_j \in \sigma^+_{\mathbf{f}_{v_1w_1}\,\mathbf{f}_{v_2w_2}}(A_\ell)$ for $1 \leqslant j \leqslant k$. Let $\lambda_0 = -2$ and

$$g = \gcd\left(\{\lambda_1 - \lambda_r\}_{r=0}^{k}, \{\lambda_1 - \lambda_r + 2\}_{r=1}^{k}\right).$$

If $g = 1$, then λ_1 is necessarily odd and so is $(\lambda_1 + 2)/g$, which is a contradiction to $-2 \in \sigma^+_{\mathbf{f}_{v_1w_1}\,\mathbf{f}_{v_2w_2}}(A_\ell)$. Hence, $g = 2$. Since $(\lambda_1 + 2)/g$ is even, $\lambda_1 \equiv 2 \pmod 4$. Moreover, for $2 \leqslant j \leqslant k$, $(\lambda_1 - \lambda_j)/g$ is even and $(\lambda_1 - \lambda_j + 2)/g$ is odd. Therefore, $\lambda_j \equiv 2 \pmod 4$. □

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