

Generalized cyclic symmetric decompositions for the matrix multiplication tensor

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Abstract

In this paper we propose a new generalized cyclic symmetric structure in the factor matrices of polyadic decompositions of matrix multiplication tensors for non-square matrix multiplication to reduce the number of variables in the optimization problem and in this way improve the convergence.

Keywords: matrix multiplication, tensors, optimization, canonical polyadic decomposition

1. Introduction

In this paper we consider the *fast matrix multiplication (FMM)* problem which rewrites matrix multiplication, which is a set of bilinear equations, i.e.,

$$C(i, j) = \sum_{k=1}^p A(i, k)B(k, n), \quad (1)$$

where $C := AB$, as a tensor equation:

$$\begin{aligned} C &= \sum_{i,j,k=1}^{m,p,n} \langle E_{ik}^{m \times p}, A \rangle_{\mathbb{F}} \langle E_{kj}^{p \times n}, B \rangle_{\mathbb{F}} E_{ij}^{m \times n} \\ &= \left(\sum_{i,j,k=1}^{m,p,n} E_{ik}^{m \times p} \otimes E_{kj}^{p \times n} \otimes E_{ij}^{m \times n} \right) \cdot_{\mathbb{F}} (A, B), \end{aligned} \quad (2)$$

where $E_{ij}^{m \times p}$, for $i = 1, \dots, m$ and $j = 1, \dots, p$, is a basis of matrices in $\mathbb{R}^{m \times p}$ such that $E_{ij}^{m \times p}(i_1, i_2) = 1$ if $(i_1, i_2) = (i, j)$ and zero else, ' $\otimes$ ' denotes the tensor product and $\langle \cdot \rangle_{\text{F}}$ and ' $\cdot_{\text{F}}$ ' denote the Frobenius inner product. We add a transpose to (2) such that the tensor that is defined has additional interesting properties, such as *cyclic symmetry (CS)*, which will be discussed in more detail in Section 2.2.2:

$$\begin{aligned} C^{\top} &= \left(\sum_{i,j,k=1}^{m,p,n} E_{ik}^{m \times p} \otimes E_{kj}^{p \times n} \otimes (E_{ij}^{m \times n})^{\top} \right) \cdot_{\text{F}} (A, B) \\ &= \left(\sum_{i,j,k=1}^{m,p,n} E_{ik}^{m \times p} \otimes E_{kj}^{p \times n} \otimes E_{ji}^{n \times m} \right) \cdot_{\text{F}} (A, B). \end{aligned} \quad (3)$$

Remark that the Frobenius inner product is a linear function in the elements of a matrix. The rank of the bilinear equation is the minimal r such that the equation can be written as

$$C^{\top} = \sum_{i=1}^r \langle U_i, A \rangle_{\text{F}} \langle V_i, B \rangle_{\text{F}} W_i, \quad (4)$$

for some matrices $U_i \in \mathbb{R}^{m \times p}$, $V_i \in \mathbb{R}^{p \times n}$, and $W_i \in \mathbb{R}^{n \times m}$. It can be shown that minimizing r minimizes the computational complexity of matrix multiplication [1, Proposition 15.1], [2]. More specifically, the number of arithmetic operations needed to compute two matrices in $\mathbb{R}^{n \times n}$ is $\mathcal{O}(n^{\omega})$, where $\omega := \log_n r$. Since $r < n^3$, it holds that $\omega < 3$. Additionally, it is well known that the exponent is bounded from below by two, since you need at least n^2 operations to compute all n^2 elements of the matrix multiplication.

Minimizing the rank of the bilinear equation corresponds to finding a (*canonical*) *polyadic decomposition ((C)PD)* of the so called *matrix multiplication tensor (MMT)* [3] defined implicitly in (3):

$$T_{mpn} := \sum_{i,j,k=1}^{m,p,n} e_{ik}^{mp} \otimes e_{kj}^{pn} \otimes e_{ji}^{mn}, \quad (5)$$

where $e_{ik}^{mp} := \text{vec}(E_{ik}^{m \times p})$ and T_{mpn} is the MMT of dimension $mp \times pn \times mn$.

Consequently, (3) can also be formulated as

$$\text{vec}(C^\top) = T_{mpn} \cdot_1 \text{vec}(A) \cdot_2 \text{vec}(B), \quad (6)$$

where $\text{vec}(\cdot)$ denotes the column-wise vectorization operator, and $\cdot_1$ and $\cdot_2$ denote the multiplication along the first and second dimension of T_{mpn} respectively.

Because of this definition, T_{mpn} is a sparse tensor consisting of mpn ones:

$$\begin{aligned} & T_{mpn}(i_1 + (i_2 - 1)m, j_1 + (j_2 - 1)p, k_1 + (k_2 - 1)n) \\ &= \begin{cases} 1 & \text{if } i_1 = k_2, i_2 = j_1, j_2 = k_1, \\ 0 & \text{else} \end{cases}, \end{aligned} \quad (7)$$

for all $i_1, k_2 = 1, \dots, m$, $i_2, j_1 = 1, \dots, p$, and $j_2, k_1 = 1, \dots, n$. Remark that a MMT only depends of the size of the matrices that are multiplied and not on the values.

A PD of T_{mpn} of rank r is a decomposition of T_{mpn} into r rank-1 tensors:

$$T_{mpn} = \sum_{i=1}^r u_i \otimes v_i \otimes w_i, \quad (8)$$

where ' $\otimes$ ' denotes the tensor product and u_i, v_i , and w_i are vectors in $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^{n_3}$ respectively. These vectors are collected in three *factor matrices* U , V , and W : $U := [u_1, \dots, u_r]$, $V := [v_1, \dots, v_r]$, and $W := [w_1, \dots, w_r]$. A PD of rank r is written in short as PD_r and a PD of rank r of a tensor T as $\text{PD}_r(T)$. The minimal r for which a PD of a certain tensor exists is called the rank r^* of this tensor or $r^*(T)$ and a PD of this rank a CPD or $\text{PD}_{r^*}(T)$.

With a PD_r (8) of T_{mpn} , we obtain the following *base algorithm* to multiply any two matrices $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$:

$$\text{vec}(C^\top) = \sum_{i=1}^r \langle u_i, \text{vec}(A) \rangle \langle v_i, \text{vec}(B) \rangle w_i, \quad (9)$$

which is the same as (4) when $U_i = \text{reshape}(u_i, [m, p])$, $V_i = \text{reshape}(v_i, [p, n])$, and $W_i = \text{reshape}(w_i, [n, m])$, for all $i = 1, \dots, r$. The number of multiplications between (linear combinations of) elements of A and B in (9) is reduced to $r < mpn$, where the upper bound holds for standard matrix multiplica-

tion. Such multiplications are called ‘active’ because, when these algorithms are applied recursively, they determine the asymptotic complexity of matrix multiplication [4].

One of the difficulties is that the rank of a CPD of T_{mpn} , denoted by $r^*(T_{mpn})$, is not known for most combinations of (m, p, n) . Counterexamples are T_{222} , for which the set of PDs is completely understood and r^* is known to be seven [5], T_{223} , for which the rank is known to be 11 [6], and T_{224} , for which the rank is known to be 14 [7]. We call $\text{PD}_7(T_{222})$ originally discovered by Strassen $\text{PD}_{\text{Strassen}}$ [3]. For other combinations of m , p , and n , lower bounds on the rank exist but none of these lower bounds are attained in practice. For example, the lower bound on $r^*(T_{333})$ is proven to be 19 [8], and the upper bound is 23 [9, 10, 11, 12]. We call the lowest rank for which an exact PD of T_{mpn} is known in the literature $\tilde{r}(T_{mpn})$. For an overview of $\tilde{r}(T_{mpn})$ for various m , p , and n , we refer to [11, Table 1] and [13, Fig. 1]. Remark that the rank of T_{mpn} is equal to the rank of T_{pnm} or any other permutation of m , p , and n . This is true because we can obtain a PD_r of T_{pnm} and T_{nmp} by cyclically permuting the factors in the PD:

$$T_{pnm} = \sum_{i=1}^r v_i \otimes w_i \otimes u_i, \quad T_{nmp} = \sum_{i=1}^r w_i \otimes u_i \otimes v_i,$$

and additionally, it can be shown that $T_{mnp} = \sum_{i=1}^r \text{vec}(W_i^\top) \otimes \text{vec}(V_i^\top) \otimes \text{vec}(U_i^\top)$, and similarly for T_{npm} and T_{pmn} .

To find a PD_r of T_{mpn} , we use the following *nonlinear least squares (NLS)* cost function, which is also frequently used in the literature for the FMM problem [11, 14, 10, 15]:

$$\min_x \underbrace{\frac{1}{2} \|F(x) - \text{vec}(T_{mpn})\|^2}_{=:f(x)}, \quad (10)$$

where $\|\cdot\|$ denotes the ℓ_2 -norm, and $F(x)$ is a vector function defined as

$$F(x) := \text{vec} \left(\sum_{r=1}^r u_r \otimes v_r \otimes w_r \right), \quad x := \begin{bmatrix} \text{vec}(U) \\ \text{vec}(V) \\ \text{vec}(W) \end{bmatrix}. \quad (11)$$

Remark that $f(x^*) = 0$, if and only if x^* is a PD_r of T_{mpn} . However, most

standard optimization algorithms fail to converge to a global minimum of (10). One of the reasons is that PDs of MMT have additional continuous invariances compared to generic PDs, which we will review in Section 2.2.1 [16].

In practice, the minimal length or rank r^* is approximated experimentally by the lowest length $\tilde{r}$ for which a solution x^* with $f(x^*) = 0$ can be obtained using numerical optimization. However, no globally convergent algorithm for (10) is known to us, and thus we can never be sure that no solution with a lower rank exists.

Since the total number of elements of the factor matrices grows rapidly with the size of the matrices, solving (10) is only feasible for relatively small values of m , p , and n , e.g., $m, p, n \leq 5$. In this paper we give a new structure that can be enforced in PDs of MMTs to reduce the number of variables in the optimization problem and hereby improve the convergence.

To solve (10), in the literature the alternating least squares (ALS) method is frequently used [11, 17, 15, 10] but the convergence is usually slow and a lot of starting points are needed to obtain a solution in a reasonable amount of time. In [14] and [11], a constrained optimization problem and corresponding method are proposed to improve the convergence. More specifically, a Levenberg-Marquardt (LM) method with Lagrange parameters and a quadratic penalty (QP) term in combination with the ALS method are used respectively. However, both methods were still not able to find PDs of rank 49 of MMTs for 4×4 matrix multiplication whereas these PDs are known to exist. That is why we proposed in [18] an augmented Lagrangian (AL) method and a new constrained optimization problem to find PDs of MMTs:

$$\min_x f(x), \quad \text{s.t.} \quad h(x) = 0, \quad l \leq x \leq u. \quad (12)$$

Different equality constraints $h(x)$ were proposed. The AL method can be considered as a combination of the LM and QP method since the AL objective function is:

$$\min_{x, y_1, y_2, z} \underbrace{f(x) + \langle y_1, h(x) \rangle + \langle y_2, x - z \rangle + \frac{\beta}{2} (\|h(x)\|^2 + \|x - z\|^2)}_{=:\mathcal{L}_A(x, y_1, y_2, z; \beta)},$$

subject to (s.t.) $l \leq z \leq u$, where y_1 and y_2 are vectors of Lagrange multipliers, β is the regularization parameter, and z is a vector of slack variables

that enables us to rewrite the inequality constraint as an equality constraint. Because $f(x)$ is an NLS function, the AL objective can also be rewritten as an NLS function. The variables in x and the auxiliary parameters are updated successively according to [19, Section 17.4]. We use the LM method for the minimization to x . The pseudo-code of the algorithm is shown in [18, Algorithm 4.2] and the MATLAB implementation is publicly available¹. The advantage compared to the QP method is that the constraints are satisfied more accurately during optimization. Using this method, we were able to obtain new PD₄₉ (T_{444})s and also different new CS PDs. We will use this method in the numerical experiments in Section 4.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries concerning the FMM problem. In Section 3, the new generalized CS structure is proposed, and in Section 4, numerical experiments are given to demonstrate the use and advantages of including this structure in the optimization problem (10).

2. Preliminaries

In this section, we first discuss what we mean with practical PDs and FMM algorithms. Afterwards, we give some well known properties of MMTs.

2.1. Practical algorithms

As discussed in the introduction, a $\text{PD}_r(T_{mpn})$ can be applied recursively to, e.g., multiply matrices of size $m^k \times p^k$ and $p^k \times n^k$. Such a recursive algorithm requires $\mathcal{O}(cr^k)$ operations, where the constant c depends on the number of scalar multiplications and nonzeros in the factor matrices. When k is sufficiently large and c sufficiently small, fewer operations are required compared to standard matrix multiplication. To decrease the constant c , we search for sparse PDs, with elements in a discrete set, e.g., $\{-1, 0, 1\}$. This set can be extended with powers of 2 since this is not a costly operation when implemented in hardware. Such PDs are called *practical PDs* as they result in practical FMM algorithms.

Consequently, if we obtain a numerical PD of T_{mpn} using numerical optimization, it still has to be transformed into a practical one. The *invariance transformations* or *inv-transformations* discovered in [16], which are further

¹https://github.com/CharlotteVermeylen/ALM_FMM_stability

discussed in Section 2.2.1, can be used to obtain such PDs [14]. However, not all PDs of T_{mpn} can be transformed into a practical PD in this way [20]. In [20], this process is called *discretization*.

A disadvantage of (10) is that the solutions have floating point elements and thus extra steps have to be taken or constraints have to be added to the optimization problem to obtain practical PDs. That is why we proposed a new constrained problem formulation in [18] with equality constraint:

$$h_{\text{discr}}(x) := x \cdot (x - 1) \cdot (x + 1), \quad (13)$$

where ‘ $\cdot$ ’ denotes element-wise multiplication.

2.2. Properties of the matrix multiplication tensor

In this section, first the invariance transformations that are present for all PDs of T_{mpn} are discussed. Additionally, a short overview of what is known in the literature about the use of these invariance transformations to obtain practical PDs is given. Afterwards, cyclic symmetry and recursive PDs are discussed in more detail.

2.2.1. Invariance transformations

The representation of a PD via factor matrices is not unique. For example when permuting the rank-1 tensors, and consequently the columns of the factor matrices according to

$$(U, V, W) \rightarrow (U', V', W'),$$

where $u'_i = u_{\sigma(i)}$, for all $i = 1, \dots, r$, and $\sigma = (\sigma(1) \ \sigma(2) \ \sigma(3) \ \dots \ \sigma(r))$ is an element of the permutation group of r elements, then the same tensor is obtained:

$$\sum_{i=1}^r u_i \otimes v_i \otimes w_i = \sum_{i=1}^r u'_i \otimes v'_i \otimes w'_i.$$

Additionally, because of the multi-linearity of the tensor product, the columns of each rank-1 tensor can be scaled as: $\alpha_i u_i, \beta_i v_i, \frac{1}{\alpha_i \beta_i} w_i$, for all $i = 1, \dots, r$, and α_i, β_i in $\mathbb{R}_0$, without changing the rank-1 tensors because

$$\alpha_i u_i \otimes \beta_i v_i \otimes \frac{1}{\alpha_i \beta_i} w_i = u_i \otimes v_i \otimes w_i.$$

These two invariances hold for all PDs. Remark that only the scaling invariance is a continuous invariance transformation (of dimension $2r$).

For PDs of T_{mpn} , additional invariances hold [16]. More specifically, it is well known that these PDs are invariant under the following *PQR-transformation*:

$$u'_i \leftarrow \text{vec}(PU_iQ^{-1}), \quad v'_i \leftarrow \text{vec}(QV_iR^{-1}), \quad w'_i \leftarrow \text{vec}(RW_iP^{-1}), \quad (14)$$

for $i = 1, \dots, r$, and where $P \in GL(m)$, $Q \in GL(p)$, and $R \in GL(n)$, where $GL(i)$ represents the *general linear group* of invertible matrices in $\mathbb{R}^{i \times i}$. This invariance can easily be proven using the fact that $C := AB = (P^\top)^{-1} (P^\top A (Q^\top)^{-1}) (Q^\top B (R^\top)^{-1}) R^\top$. When we substitute this in (9), we obtain:

$$\begin{aligned} C &= \sum_{i=1}^r \text{trace} \left(U_i \left(P^\top A (Q^\top)^{-1} \right)^\top \right) \\ &\quad \text{trace} \left(V_i \left(Q^\top B (R^\top)^{-1} \right)^\top \right) (P^\top)^{-1} W_i^\top R^\top \\ &= \sum_{i=1}^r \text{trace} (PU_iQ^{-1}A^\top) \text{trace} (QV_iR^{-1}B^\top) (RW_iP^{-1})^\top, \end{aligned}$$

where we made use of the fact that the trace is invariant under cyclic permutation of the factors. And thus indeed another base algorithm or PD_r of T_{mpn} can be obtained using (14). Additionally, when $m = n = p$, PDs of T_{mmm} are invariant under the following *transpose-transformation*:

$$u'_i \leftarrow \text{vec}(V_i^\top), \quad v'_i \leftarrow \text{vec}(U_i^\top), \quad w'_i \leftarrow \text{vec}(W_i^\top),$$

for $i = 1, \dots, r$, which can be proven using the fact that $C = (B^\top A^\top)^\top$. Lastly, still for $m = n = p$, the PDs are invariant under cyclic permutation of the factor matrices:

$$\sum_{i=1}^r u_i \otimes v_i \otimes w_i = \sum_{i=1}^r v_i \otimes w_i \otimes u_i = \sum_{i=1}^r w_i \otimes u_i \otimes v_i,$$

which holds because T_{mmm} is a CS tensor.

Remark that concerning the specific invariances of PDs of T_{mpn} , only the

PQR-invariance is a continuous invariance and it overlaps with the scaling invariance when $P = aI$, $Q = bI$, and $R = cI$, where I is the identity matrix of the appropriate size and a, b , and c are scaling factors in $\mathbb{R}_0$, because in this case the PQR-transformation scales the columns of the factor matrix U with $\frac{a}{b}$, all columns of V with $\frac{b}{c}$, and all columns of W with $\frac{c}{a}$, and indeed the product of these factors equals one as for the scaling invariance. Thus, the combination of the scaling and PQR-transformation has at most dimension $2r + m^2 + p^2 + n^2 - 3$ for all PDs of T_{mpn} .

We call the combination of the invariance transformations described above *inv-transformations*. Two PDs of T_{mpn} that can be obtained using inv-transformations are called *inv-equivalent*. Only for T_{222} , it is known that all PDs are inv-equivalent [5].

2.2.2. Cyclic symmetry

Because of its definition, T_{mpn} is a structured and more specifically cyclic symmetric (CS) tensor when $m = p = n$. This means that $T_{mmm}(i, j, k) = T_{mmm}(j, k, i) = T_{mmm}(k, i, j)$, for all $i, j, k = 1, \dots, m^2$.

We can make use of this property for constructing a PD by requiring that the rank-1 tensors are symmetric or occur in CS pairs [21]:

$$T_{mmm} = \sum_{i=1}^s a_i \otimes a_i \otimes a_i + \sum_{j=1}^t (b_j \otimes d_j \otimes c_j + c_j \otimes b_j \otimes d_j + d_j \otimes c_j \otimes b_j),$$

where a_i, b_j, c_j , and d_j are vectors in $\mathbb{R}^{m^2}$, for all $i = 1, \dots, s$ and $j = 1, \dots, t$, where s and t are parameters denoting respectively the number of symmetric and CS pairs of asymmetric rank-1 tensors. The length then equals $r = s + 3t$. If we define the matrices $A := [a_1 \cdots a_s]$, $B := [b_1 \cdots b_t]$, $C := [c_1 \cdots c_t]$, and $D := [d_1 \cdots d_t]$, the factor matrices can be written in function of these matrices:

$$U = [A \ B \ C \ D], \quad V = [A \ D \ B \ C], \quad W = [A \ C \ D \ B].$$

Consequently, the number of unknowns is reduced by a factor 3, which makes this CS structure very useful for larger problem parameters, e.g., $m, p, n \geq 4$.

Furthermore, the cost function can then be changed to

$$\begin{aligned}
\min_{A,B,C,D} \frac{1}{2} & \left(\sum_{i=1}^{m^2} \sum_{j=i}^{m^2} \sum_{k=i+1}^{m^2} \left(T_{mmm}(i, j, k) - \sum_{i'=1}^s a_{ii'} a_{ji'} a_{ki'} \right. \right. \\
& \left. \left. - \sum_{j'=1}^t (b_{ij'} d_{jj'} c_{kj'} + c_{ij'} b_{jj'} d_{kj'} + d_{ij'} c_{jj'} b_{kj'}) \right) \right)^2 \\
& + \sum_{i=1}^{m^2} \left(T_{mmm}(i, i, i) - \sum_{i'=1}^s a_{ii'}^3 - 3 \sum_{j'=1}^t (b_{ij'} d_{ij'} c_{ij'}) \right)^2,
\end{aligned} \tag{15}$$

where $a_{ii'} := A(i, i')$, $b_{ii'} := B(i, i')$ and similarly for $c_{ii'}$ and $d_{ii'}$, which reduces the number of rows in the Jacobian matrix used in NLS optimization methods to solve (15) from m^6 to $\frac{1}{3}(m^6 - m^2) + m^2$. Remark that this cost function only includes one element for each CS pair of points because the CS structure already ensures that these elements are equal. A disadvantage of including this structure may be that by restricting the search space we might not be able to find the most sparse or stable PDs of T_{mmm} . Furthermore, it is not known if the rank of a PD of T_{mmm} with this structure equals the canonical rank.

Different practical CS PDs of rank 7 for T_{222} and of rank 23 for T_{333} are known in the literature [15, 10]. These PDs were found using an ALS method and further investigated using algebraic geometry and group theory. Strassen's decomposition satisfies $s = 1$ and $t = 2$. Although all $\text{PD}_7(T_{222})$ s are known to be inv-equivalent, $\text{PD}_{\text{Strassen}}$ can be transformed into a practical PD with CS parameters $(s, t) = (4, 1)$. However, this PD contains more nonzeros than $\text{PD}_{\text{Strassen}}$ and thus is not used in practice.

2.3. Decompositions obtained by recursion

As mentioned in the introduction, an FMM algorithm applies a base algorithm for small m , p , and n , to large matrices, e.g., of size $m^k \times p^k$ and $p^k \times n^k$. In the same way, we can also obtain PDs of rank 49 of T_{444} using a $\text{PD}_7(T_{222})$ one time recursively. The $\text{PD}_{49}(T_{444})$ that is obtained using $\text{PD}_{\text{Strassen}}$ is called $\text{PD}_{\text{Strassen}}^{\text{rec}}$ in the rest of the text. In the following proposition, we give the formulas to obtain the factor matrices of a recursive PD from the factor matrices of the original PD. This result is well known, e.g., from the supplementary material of [13], and in [22], and a proof can be found, e.g., in [23, Section 2.2.3].

Proposition 2.1 (Recursive PD). *If U , V , and W are the factor matrices of a $\text{PD}_r(T_{mpn})$, then the factor matrices U' , V' , and W' , where*

$$u'_{i'} := \text{vec}(U_{i_1} \otimes U_{i_2}), \quad v'_{i'} := \text{vec}(V_{i_1} \otimes V_{i_2}), \quad (16)$$

$$w'_{i'} := \text{vec}(W_{i_1} \otimes W_{i_2}), \quad i' := i_2 + (i_1 - 1)r, \quad (17)$$

for $i_1, i_2 = 1, \dots, r$, are the factor matrices of a $\text{PD}_{r^2}(T_{m^2p^2n^2})$.

The following corollary gives the CS parameters and factor matrices of a recursive PD as a function of the original PD.

Corollary 2.2. *If U , V , and W are CS factor matrices of a $\text{PD}_r(T_{mpn})$ with CS parameters s and t , then the factor matrices U' , V' , and W' of the recursive decomposition $\text{PD}_{r^2}^{\text{rec}}(T_{m^2p^2n^2})$ is also CS with parameters $s' = s^2$ and $t' = t(s + r)$:*

$$U' := [A' \ B' \ C' \ D'], \quad V' := [A' \ D' \ B' \ C'], \quad W' := [A' \ C' \ D' \ B'],$$

where

$$A'_{i'} := A_{i_1} \otimes A_{i_2}, \quad i' := i_2 + (i_1 - 1)s, \quad i_1, i_2 = 1, \dots, s, \quad (18)$$

and

$$B' := [B'_1 \ B'_2], \quad C' := [C'_1 \ C'_2], \quad D' := [D'_1 \ D'_2], \quad (19)$$

where

$$\begin{aligned} B'_{1,j'} &:= A_{i_1} \otimes B_j, & C'_{1,j'} &:= A_{i_1} \otimes C_j, & D'_{1,j'} &:= A_{i_1} \otimes D_j, \\ B'_{2,k'} &:= B_j \otimes U_k, & C'_{2,k'} &:= C_j \otimes W_k, & D'_{2,k'} &:= D_j \otimes V_k, \end{aligned} \quad (20)$$

where $j' := j + (i_1 - 1)t$ and $k' := k + (j - 1)r$, for $j = 1, \dots, t$ and $k = 1, \dots, r$.

Proof. Using Proposition 2.1, we know that $U'_{i'} = U_{i_1} \otimes U_{i_2}$, $V'_{i'} = V_{i_1} \otimes V_{i_2}$, and $W'_{i'} = W_{i_1} \otimes W_{i_2}$, where $i' := i_2 + (i_1 - 1)r$. Thus, the symmetric part of the recursive PD must satisfy: $U_{i_1} = V_{i_1} = W_{i_1}$ and $U_{i_2} = V_{i_2} = W_{i_2}$. Consequently, i_1 and i_2 must be smaller than or equal to s , such that $U_{i_1} = V_{i_1} = W_{i_1} = A_{i_1}$ and $U_{i_2} = V_{i_2} = W_{i_2} = A_{i_2}$. For the other values of i_1 and i_2 , the columns still appear in CS pairs and they can be rearranged as in (20) to satisfy the CS structure. In [23, Appendix A], the different columns

that are obtained by using Proposition 2.1 are written out to see this more clearly. $\square$

Remark that, since $\text{PD}_{\text{Strassen}}$ satisfies $s = 1$ and $t = 2$, the parameters of $\text{PD}_{\text{Strassen}}^{\text{rec}}$ are $s = 1$ and $t = 16$. Another CS $\text{PD}_{49}(T_{444})$ can be obtained with $(s, t) = (16, 11)$ by using a $\text{PD}_7(T_{222})$ with $(s, t) = (4, 1)$ one time recursively.

2.4. Jacobian matrix

In [18], we showed that the Jacobian matrix can be used to investigate the inv-equivalence of PDs of MMTs and to investigate any additional invariances, such as the ones discovered in [18] using parametrizations of decompositions. That is why in Section 4, we will give the size and ranks of the Jacobian matrix at solutions with the structure proposed in the next section.

3. Generalization cyclic symmetry

In this section, we give a generalization of the CS structure discussed in Section 2.2.2, to the case when m , n , and p are not equal. We do this to reduce the number of variables and reduce the search space to hopefully speed up the convergence. Note however that similarly as for the CS structure discussed in Section 2.2.2, it is not known if the rank of PDs with this structure equals the canonical rank. Because we can obtain a PD of T_{pnm} or another permutation of m , p , and n from a PD of T_{mpn} , we assume in this chapter that $m \leq p \leq n$. We know from (7) that:

$$\begin{aligned} & T_{mpn}(i_1 + (i_2 - 1)m, j_1 + (j_2 - 1)p, k_1 + (k_2 - 1)n) \\ &= \sum_{i=1}^r U_i(i_1, i_2) V_i(j_1, j_2) W_i(k_1, k_2) = \begin{cases} 1 & \text{if } i_1 = k_2, i_2 = j_1, j_2 = k_1 \\ 0 & \text{else} \end{cases}, \end{aligned}$$

for all $i_1, k_2 = 1, \dots, m$, $i_2, j_1 = 1, \dots, p$, and $j_2, k_1 = 1, \dots, n$.

Consequently, as long as $i_1, i_2, j_1, j_2, k_1, k_2 \leq m$, it holds that

$$\begin{aligned} & T_{mpn}(i_1 + (i_2 - 1)m, j_1 + (j_2 - 1)p, k_1 + (k_2 - 1)n) \\ &= T_{mpn}(k_1 + (k_2 - 1)m, i_1 + (i_2 - 1)p, j_1 + (j_2 - 1)n) \\ &= T_{mpn}(j_1 + (j_2 - 1)m, k_1 + (k_2 - 1)p, k_1 + (k_2 - 1)n). \end{aligned}$$

And thus,

$$\begin{aligned} \sum_{i=1}^r U_i(i_1, i_2) V_i(j_1, j_2) W_i(k_1, k_2) &= \sum_{i=1}^r U_i(k_1, k_2) V_i(i_1, i_2) W_i(j_1, j_2) \\ &= \sum_{i=1}^r U_i(j_1, j_2) V_i(k_1, k_2) W_i(i_1, i_2), \end{aligned}$$

which can be used to include a CS structure in a subpart of the factor matrices. An illustration is shown in Figure 1, where the following submatrices are defined in function of the position in the factor matrices:

$$U_i(i_1, i_2) =: \begin{cases} A_i(i_1, i_2) & \text{if } i_2 \leq m, i \leq s \\ B_{i-s}(i_1, i_2) & \text{if } i_2 \leq m, s < i \leq s+t \\ C_{i-s-t}(i_1, i_2) & \text{if } i_2 \leq m, s+t < i \leq s+2t \\ D_{i-s-2t}(i_1, i_2) & \text{if } i_2 \leq m, s+2t < i \leq r_{\text{CS}} \\ \tilde{U}_i(i_1, i_2-m) & \text{if } i_2 > m, i \leq r_{\text{CS}} \\ \dot{U}_{i-r_{\text{CS}}}(i_1, i_2) & \text{if } r_{\text{CS}} < i \end{cases},$$

where $r_{\text{CS}} := s + 3t$, and for $i = 1, \dots, r$, $i_1 = 1, \dots, m$, and $i_2 = 1, \dots, p$,

$$V_i(j_1, j_2) =: \begin{cases} A_i(j_1, j_2) & \text{if } j_1, j_2 \leq m, i \leq s \\ D_{i-s}(j_1, j_2) & \text{if } j_1, j_2 \leq m, s < i \leq s+t \\ B_{i-s-t}(j_1, j_2) & \text{if } j_1, j_2 \leq m, s+t < i \leq s+2t \\ C_{i-s-2t}(j_1, j_2) & \text{if } j_1, j_2 \leq m, s+2t < i \leq r_{\text{CS}} \\ \hat{V}_i(j_1-m, j_2) & \text{if } j_1 > m, j_2 \leq m, i \leq r_{\text{CS}} \\ \tilde{V}_i(j_1, j_2-m) & \text{if } j_2 > m, i \leq r_{\text{CS}} \\ \dot{V}_{i-r_{\text{CS}}}(j_1, j_2) & \text{if } r_{\text{CS}} < i \end{cases},$$

for $j_1 = 1, \dots, p$, and $j_2 = 1, \dots, n$, and

$$W_i(k_1, k_2) =: \begin{cases} A_i(k_1, k_2) & \text{if } k_1 \leq m, i \leq s \\ B_{i-s}(k_1, k_2) & \text{if } k_1 \leq m, s < i \leq s+t \\ C_{i-s-t}(k_1, k_2) & \text{if } k_1 \leq m, s+t < i \leq s+2t \\ D_{i-s-2t}(k_1, k_2) & \text{if } k_1 \leq m, s+2t < i \leq r_{\text{CS}} \\ \hat{W}_i(k_1 - m, k_2) & \text{if } k_1 > m, i \leq r_{\text{CS}} \\ \dot{W}_{i-r_{\text{CS}}}(k_1, k_2) & \text{if } r_{\text{CS}} < i \end{cases},$$

for $k_1 = 1, \dots, n$, and $k_2 = 1, \dots, m$, where $A_i := \text{reshape}(a_i, m \times m)$, for $i = 1, \dots, s$, $B_j := \text{reshape}(b_j, m \times m)$, for $j = 1, \dots, t$, and similarly for C_j and D_j , and

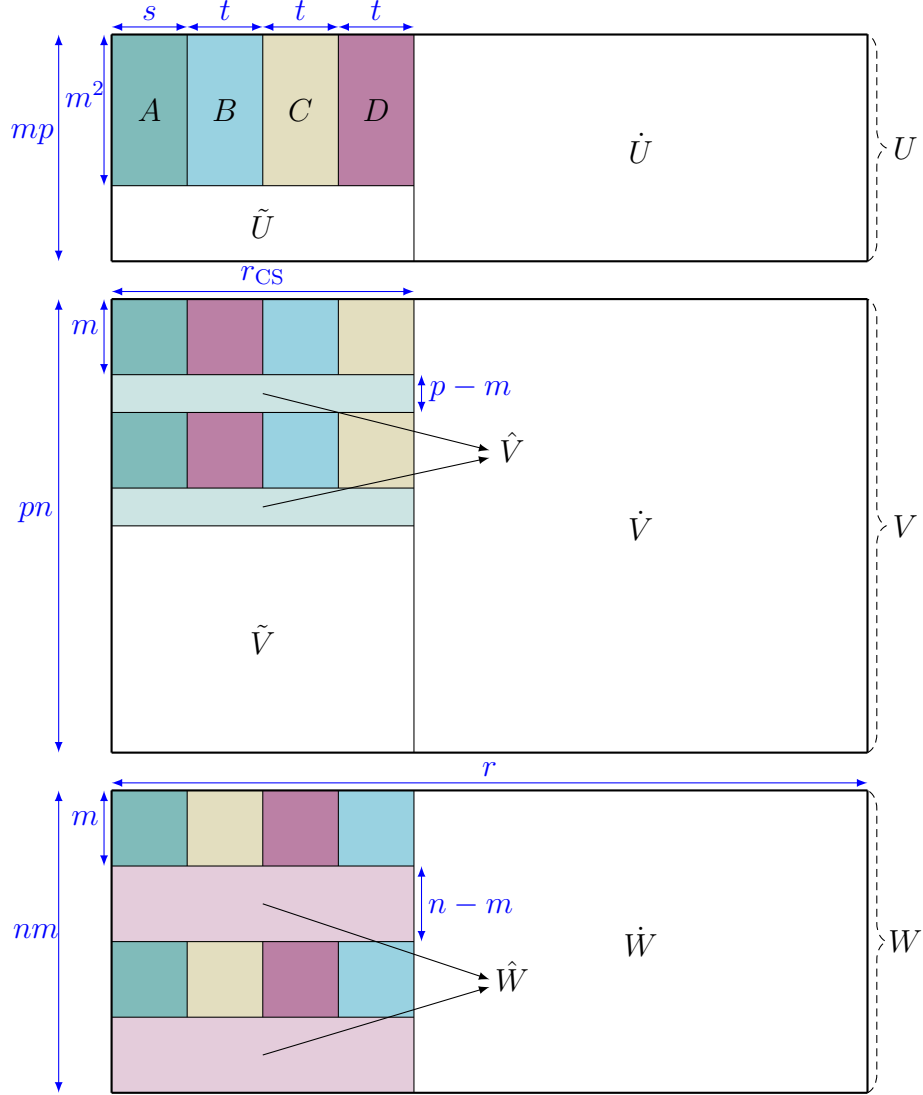
$$\begin{aligned} \tilde{U}_k &:= \text{reshape}(\tilde{U}(:, k), m \times (p-m)), \\ \tilde{V}_k &:= \text{reshape}(\tilde{V}(:, k), p \times (n-m)), \\ \hat{V}_k &:= \text{reshape}(\hat{V}(:, k), (p-m) \times m), \\ \hat{W}_k &:= \text{reshape}(\hat{W}(:, k), (n-m) \times m), \end{aligned}$$

for $k = 1, \dots, r_{\text{CS}}$, and

$$\begin{aligned} \dot{U}_{i'} &:= \text{reshape}(\dot{U}(:, i'), m \times p), \\ \dot{V}_{i'} &:= \text{reshape}(\dot{V}(:, i'), p \times n), \\ \dot{W}_{i'} &:= \text{reshape}(\dot{W}(:, i'), n \times m), \end{aligned}$$

for $i' = 1, \dots, (r - r_{\text{CS}})$, and thus $\tilde{U} \in \mathbb{R}^{m(p-m) \times r_{\text{CS}}}$, $\tilde{V} \in \mathbb{R}^{p(n-m) \times r_{\text{CS}}}$, $\hat{V} \in \mathbb{R}^{(p-m)m \times r_{\text{CS}}}$, $\hat{W} \in \mathbb{R}^{(n-m)m \times r_{\text{CS}}}$, $\dot{U} \in \mathbb{R}^{mp \times (r-r_{\text{CS}})}$, $\dot{V} \in \mathbb{R}^{pn \times (r-r_{\text{CS}})}$, and $\dot{W} \in \mathbb{R}^{nm \times (r-r_{\text{CS}})}$. The proportions in Figure 1 are those of T_{234} , $(s, t) := (2, 2)$, and $r := 20$. We show in the experiments that such PDs indeed exist. Remark that, contrary to the case when m , n , and p are equal, the CS rank r_{CS} does not equal the rank r , because this would be too restrictive. Also in the experiments that follow we did not find PDs for which both are equal. This is likely because of the interaction of the CS part with the parts $\tilde{U}$, $\tilde{V}$, $\hat{V}$, and $\hat{W}$. However, including this structure still reduces the number of parameters from $(mp + np + mn)r$ to $(mp + np + mn)r - 2m^2r_{\text{CS}}$.

Figure 1: Illustration of the generalized CS factor matrices of a $\text{PD}_{20}(T_{234})$, with $(s, t) := (2, 2)$.



4. Numerical experiments

In this section we show the results that we obtained with the AL method from [18] to find PDs of MMTs with the generalized CS structure for different combinations of s and t . The AL method takes as input an upper and lower bound on the elements in the factor matrices: $l \leq x \leq u$. To show the

advantage of the new structure, we set $u := -l := 1$ and generate 50 random starting points of size 10^{-2} with the built-in function **randn** of MATLAB:

$$x_0 := 10^{-2} \cdot \mathbf{randn}((mp + np + mn)r - 2m^2r_{CS} \times 1). \quad (21)$$

The number of inner iterations of the LM method was set to 50 and the number of outer iterations to 15. The tolerance on the gradient and on the bound constraint were both set to 10^{-13} .

In Table 1, we show the results that we obtained with the AL method from [18] to find $\text{PD}_{11}(T_{223})$ s for different combinations of s and t . The second to last column (from left to right) indicates the number of exact solutions found for these 50 random starting points. With an exact solution, a solution for which the optimality conditions, i.e., the tolerance on the gradient and constraint, are met, and furthermore for which the cost function is smaller than 10^{-12} . The last column in the table indicates the number of practical solutions that were obtained when starting from the exact numerical PDs and by adding the constraint h_{discr} from (13), scaled with a factor 0.1, to the optimization problem. As can be seen, for many combinations, exact and practical PDs exist. An example practical $\text{PD}_{11}(T_{223})$ with $(s, t) := (2, 2)$ is given by the following equations:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \\ \hat{V} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \tilde{U} = \mathbb{I}, \quad \tilde{V} = \mathbb{I}, \\ \hat{W} &= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \\ \dot{U} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \quad \dot{W} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \dot{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (22)$$

Also the size and minimal, maximal, and number of different ranks of $J_{\text{CS,gen}}$,

i.e., the Jacobian matrix of the new generalized CS structure, are given at the solutions. For a list of all the different ranks, we refer to the detailed results of the experiment, which are publicly available ². Remark that the second dimension of the Jacobian matrix and the ranks decrease when r_{CS} increases.

Table 1: Number of exact (numerical and practical) $\text{PD}_{11}(T_{223})$ s that we obtained for different combinations of s and t using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(x^*))$			$\# x^*$	pract.
				min	max	#		
10	7	1	144×96	82	82	1	10	0
9	6	1	144×104	88	89	2	32	0
8	2	2	144×112	92	97	4	8	1
	5	1		93	98	5	45	0
7	1	2	144×120	85	98	7	21	10
	4	1		87	103	10	46	16
	7	0		101	105	2	15	0
6	0	2	144×128	91	108	14	29	13
	3	1		93	107	13	48	36
	6	0		107	111	5	29	0
5	2	1	144×136	99	117	13	38	25
	5	0		112	117	6	44	0
4	1	1	144×144	105	123	18	43	30
	4	0		107	123	16	48	18
3	0	1	144×152	111	129	16	39	18
	3	0		113	129	15	49	29
2	2	0	144×160	119	133	14	50	21
1	1	0	144×168	123	135	11	49	14
0	0	0	144×176	125	135	9	50	15

As can be seen from Table 1, the highest value for r_{CS} that we were able to obtain is 10 and is obtained for $(s, t) := (7, 1)$. We were not able to transform this PD into a practical one using the constraint h_{discr} .

²https://github.com/CharlotteVermeylen/ALM.FMM_gen_CS

In the last row, as a reference the results for $(s, t) := (0, 0)$, i.e., without the CS structure, are given. Remark that the number of numerical solutions in general decreases when r_{CS} increases but, on the other hand, the number of practical PDs increases for some combinations of s and t .

In Figure 2, the convergence of the cost function is shown for the experiment with $(s, t) := (7, 0)$. As can be seen, 15 out of the 50 starting points converge quickly to a numerically exact PD in less than 100 iterations and the other starting points get stuck in swaps.

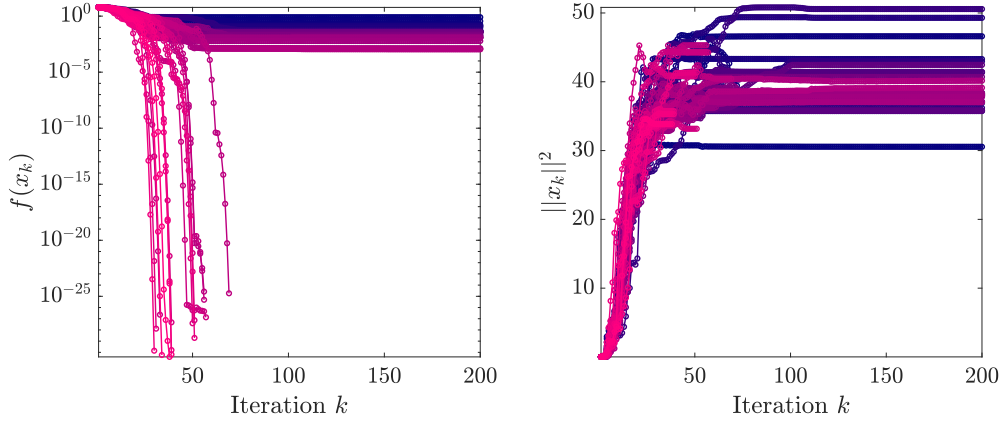


Figure 2: Convergence of the cost function in (10) using the AL method from [18] with $u := -l := 1$ and for 50 random starting points generated with `randn` of size 10^{-2} to find $\text{PD}_{11}(T_{223})$ s, with $(s, t) := (7, 0)$.

On the other hand, the results for $\text{PD}_{14}(T_{224})$ s are shown in Table 2, again for different values of r_{CS} , s and t . The largest CS rank that we were able to obtain is again $r_{\text{CS}} = 10$ but now both PDs with $(s, t) = (4, 2)$ and $(s, t) = (7, 1)$ were found. However, again none of these PDs were discretizable using the constraint h_{discr} .

An example practical $\text{PD}_{14}(T_{224})$ that we obtained for $(s, t) = (3, 1)$ has

as submatrices:

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
\tilde{V} = \hat{W} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{U} = \mathbb{I}, \quad \hat{V} = \mathbb{I}, \\
\dot{U} &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \\
\dot{V} &= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \\
\dot{W} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Again it can be noticed that by including the generalized CS structure, more practical PDs are obtained and furthermore, more different values of the rank of the Jacobian and thus more non-inv-equivalent PDs are obtained as long as the CS rank is not set too high.

Table 2: Number of exact (numerical and practical) $\text{PD}_{14}(T_{224})$ s that we obtained for different combinations of s and t using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the minimal, maximal, and number of different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(\mathbf{x}^*))$			$\# \mathbf{x}^*$	pract.
				min	max	#		
10	4	2	256×200	170	170	1	1	0
	7	1		172	178	2	6	0
9	3	2	256×208	183	183	1	2	0
	6	1		178	185	5	15	0
8	2	2	256×216	187	190	3	6	0
	5	1		185	190	5	38	0
	8	0		184	191	2	3	0
7	1	2	256×224	178	193	6	22	9
	4	1		180	197	10	44	14
	7	0		190	199	5	20	0
6	0	2	256×232	184	198	9	20	9
	3	1		184	201	10	41	21
	6	0		196	205	7	20	0
5	2	1	256×240	188	212	15	36	16
	5	0		205	210	6	36	0
4	1	1	256×248	196	214	13	30	12
	4	0		196	217	15	44	12
3	0	1	256×256	202	221	11	25	4
	3	0		202	221	15	42	11
2	2	0	256×264	214	227	10	49	4
1	1	0	256×272	217	228	9	45	4
0	0	0	256×280	217	228	8	50	5

Thirdly, in Table 3, the results for T_{233} , rank 15 are shown. Now, the maximal value for r_{CS} that we obtained is 9, and one practical PD is obtained

with parameters $(s, t) = (6, 1)$:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
\tilde{U} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\hat{V} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}, \\
\tilde{V} &= \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \\
\hat{W} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \\
\dot{U} &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}, \\
\dot{V} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \dot{W} = \begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix},
\end{aligned}$$

Table 3: Number of exact (numerical and practical) $\text{PD}_{15}(T_{233})$ s that we obtained for different combinations of s and t using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(\mathbf{x}^*))$			$\# \mathbf{x}^*$	pract.
				min	max	#		
9	3	2	324×243	216	216	1	2	0
	6	1		218	218	1	1	1
8	2	2	324×251	223	227	2	2	0
	5	1		225	226	2	4	2
7	1	2	324×259	227	228	2	13	2
	4	1		229	234	5	15	5
6	0	2	324×267	232	234	3	10	1
	3	1		234	239	6	25	10
	6	0		237	238	2	2	1
5	2	1	324×275	240	244	4	12	7
	5	0		244	245	2	2	0
4	1	1	324×283	247	250	3	5	2
	4	0		249	254	5	11	6
3	0	1	324×291	252	256	3	6	2
	3	0		254	259	6	18	6
2	2	0	324×299	254	262	6	20	4
1	1	0	324×307	260	264	5	30	0
0	0	0	324×315	262	264	3	29	0

Fourthly, in Table 4, the results for T_{225} , rank 18, are shown. The maximal CS rank is 10 as in Table 1 and Table 2. However, the highest value of the CS rank for which a practical PD is obtained is 9 and is obtained for $(s, t) = (6, 1)$. The CS submatrices of the PD that we obtained are:

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
\tilde{V} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{U} = \mathbb{I}, \quad \hat{V} = \mathbb{I}, \\
\hat{W} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \\
\dot{U} &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \\
\dot{V} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\dot{W} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Table 4: Number of exact (numerical and practical) $\text{PD}_{18}(T_{225})$ s for different combinations of s and t using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(\mathbf{x}^*))$			$\# \mathbf{x}^*$	pract.
				min	max	#		
10	4	2	400×352	302	315	9	13	0
	7	1		298	310	10	34	0
9	3	2	400×360	309	319	9	17	0
	6	1		288	316	11	44	1
8	2	2	400×368	308	326	11	19	0
	5	1		313	323	9	49	0
	8	0		308	325	12	23	0
7	1	2	400×376	292	331	18	33	4
	4	1		293	327	15	46	3
	7	0		314	332	17	39	0
6	0	2	400×384	297	339	14	23	3
	3	1		299	338	23	46	10
	6	0		319	339	19	41	0
5	2	1	400×392	305	341	19	47	8
	5	0		329	341	13	49	0
4	1	1	400×400	313	352	23	49	4
	4	0		312	351	21	48	5
3	0	1	400×408	327	357	20	46	1
	3	0		322	354	21	48	5
2	2	0	400×416	330	354	17	49	2
1	1	0	400×424	332	356	16	49	2
0	0	0	400×432	336	358	15	50	1

Then in Table 5, the results for T_{234} , rank 20, are shown. The parameters s and t for which solutions are found are similar to the previous experiments but the number of solutions is on average smaller because the problem becomes more difficult as p , n , and the rank r increase. Also, fewer different values of the rank of the Jacobian matrix are found. Remark that the number of different values of the rank in general increases for moderate values of r_{CS} . Furthermore, including the CS structure again helps to find practical solutions compared to the generic case $((s, t) = (0, 0))$ for different combinations of s and t .

Table 5: Number of exact (numerical and practical) $\text{PD}_{20}(T_{234})$ s that we obtained for different combinations of (s, t) using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(\mathbf{x}^*))$			$\# \mathbf{x}^*$	pract.
				min	max	#		
10	7	1	576×440	401	401	1	1	0
9	6	1	576×448	403	408	2	2	0
8	2	2	576×456	409	409	1	2	0
	5	1		408	414	5	4	1
7	1	2	576×464	412	415	3	5	0
	4	1		410	419	6	5	1
6	0	2	576×472	420	420	1	1	0
	3	1		421	424	4	10	0
5	2	1	576×480	425	432	6	7	1
	5	0		424	434	3	4	0
4	1	1	576×488	433	438	3	6	0
	4	0		427	438	3	4	1
3	0	1	576×496	441	443	2	4	0
	3	0		438	447	7	17	0
2	2	0	576×504	441	447	6	15	0
1	1	0	576×512	442	448	7	18	0
0	0	0	576×520	444	448	4	21	0

Lastly in Table 6, the results for T_{245} and rank 33 are shown. Again the problem has become more difficult because p , n , and r have increased. Still, we find more numerical solutions for several CS parameters compared to the generic case $((s, t) = (0, 0))$.

Table 6: Number of exact (numerical and practical) $\text{PD}_{33}(T_{245})$ s that we obtained for different combinations of s and t using the AL method from [18] on 50 random starting points generated as in (21). Also the size and the different ranks of the Jacobian matrix are given.

r_{CS}	s	t	$\text{size}(J_{\text{CS,gen}})$	$\text{rank}(J_{\text{CS,gen}}(\mathbf{x}^*))$			$\# \mathbf{x}^*$	pract.
				min	max	#		
10	4	2	1600×1174	1091	1091	1	1	0
9	3	2	1600×1182	1088	1096	2	2	0
	6	1		1089	1099	3	5	0
	9	0		1096	1096	1	1	0
8	5	1	1600×1190	1096	1105	3	3	0
	8	0		1106	1106	1	1	0
7	1	2	1600×1198	1094	1098	2	2	0
	4	1		1102	1104	2	3	0
	7	0		1105	1105	1	1	0
6	3	1	1600×1206	1102	1116	5	5	0
	6	0		1115	1115	1	2	0
5	2	1	1600×1214	1118	1122	3	3	0
	5	0		1118	1123	4	4	0
4	1	1	1600×1222	1115	1120	4	4	0
	4	0		1118	1121	3	3	0
3	0	1	1600×1230	1123	1127	3	5	0
	3	0		1119	1131	5	7	0
2	2	0	1600×1238	1123	1128	4	7	0
1	1	0	1600×1246	1125	1134	3	6	0
0	0	0	1600×1254	1127	1130	3	4	0

Remark that we can also choose the CS rank smaller than r when $m = p = n$. For example, we are able to enforce 11 symmetric rank-1 tensors in a $\text{PD}_{23}(T_{333})$. The other 12 rank-1 tensors do not have to admit a structure in this case. One of the practical PDs that we found for these parameters has

as symmetric part:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix},$$

and as asymmetric part:

$$\dot{U} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\dot{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\dot{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Conclusion

The proposed generalized cyclic symmetric structure can be useful to include in the optimization problem for the *fast matrix multiplication (FMM)* problem to reduce the number of parameters and to improve the convergence to numerically exact and practical *polyadic decompositions (PDs)* of *matrix multiplication tensors (MMTs)*. Several new *practical* PDs, i.e., sparse PDs with elements that are either one, minus one, or zero, with this new structure are discovered for various problem parameters, i.e., sizes of the MMT.

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7. Competing interests

Declarations of interest: none.

References

- [1] P. Bürgisser, M. Clausen, M. A. Shokrollahi, Algebraic complexity theory, Vol. 315, Springer Science & Business Media, 2013.

- [2] J. M. Landsberg, *Tensors: Geometry and Applications*, Vol. 128, American Mathematical Soc., 2011.
- [3] V. Strassen, Gaussian elimination is not optimal, *Numerische Mathematik* 13 (1969) 354–356. doi:10.1007/BF02165411.
- [4] J. M. Landsberg, *Geometry and Complexity Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2017. doi:10.1017/9781108183192.
- [5] H. F. de Groote, On varieties of optimal algorithms for the computation of bilinear mappings II. Optimal algorithms for 2×2 -matrix multiplication, *Theoretical Computer Science* 7 (1978) 127–148. doi:https://doi.org/10.1016/0304-3975(78)90045-2.
- [6] V. B. Alekseyev, On the complexity of some algorithms of matrix multiplication, *Journal of Algorithms* 6 (1) (1985) 71–85. doi:10.1016/0196-6774(85)90019-7.
- [7] V. B. Alekseev, A. V. Smirnov, On the exact and approximate bilinear complexities of multiplication of 4×2 and 2×2 matrices, Vol. 282, 2013, pp. 123–139.
URL <https://api.semanticscholar.org/CorpusID:120039387>
- [8] M. Bläser, On the complexity of the multiplication of matrices of small formats, *Journal of Complexity* 19 (1) (2003) 43–60. doi:10.1016/S0885-064X(02)00007-9.
- [9] J. D. Laderman, A non commutative algorithm for multiplying 3×3 matrices using 23 multiplications, *Bulletin of the American Mathematical Society* 82 (1) (1976) 126–128. doi:10.1090/S0002-9904-1976-13988-2.
- [10] G. Ballard, C. Ikenmeyer, J. M. Landsberg, N. Ryder, The geometry of rank decompositions of matrix multiplication II: 3×3 matrices, *Journal of Pure and Applied Algebra* 223 (8) (2019) 3205–3224. doi:10.1016/j.jpaa.2018.10.014.
- [11] A. V. Smirnov, The bilinear complexity and practical algorithms for matrix multiplication, *Computational Mathematics and Mathematical Physics* 53 (12) (2013) 1781–1795. doi:10.1134/S0965542513120129.

- [12] M. J. Heule, M. Kauers, M. Seidl, New ways to multiply 3×3 -matrices, *Journal of Symbolic Computation* 104 (2021) 899–916. doi:10.1016/j.jsc.2020.10.003.
- [13] A. Fawzi, M. Balog, A. Huang, T. Hubert, B. Romera-Paredes, M. Barekatin, A. Novikov, F. J. R. Ruiz, J. Schrittwieser, G. Swirszcz, D. Silver, D. Hassabis, P. Kohli, Discovering faster matrix multiplication algorithms with reinforcement learning, *Nature* 610 (7930) (2022) 47–53. doi:10.1038/s41586-022-05172-4.
- [14] P. Tichavský, A. H. Phan, A. Cichocki, Numerical CP decomposition of some difficult tensors, *Journal of Computational and Applied Mathematics* 317 (2017) 362–370. doi:10.1016/j.cam.2016.12.007.
- [15] L. Chiantini, C. Ikenmeyer, J. Landsberg, G. Ottaviani, The geometry of rank decompositions of matrix multiplication I: 2×2 matrices, *Experimental Mathematics* 28 (2019) 322–327. arXiv:1610.08364.
- [16] H. F. de Groote, On varieties of optimal algorithms for the computation of bilinear mappings I. The isotropy group of a bilinear mapping, *Theoretical Computer Science* 7 (1) (1978) 1–24. doi:10.1016/0304-3975(78)90038-5.
- [17] R. W. Johnson, A. M. McLoughlin, Noncommutative bilinear algorithms for 3×3 matrix multiplication, *SIAM Journal on Computing* 15 (2) (1986) 595–603. doi:10.1137/0215043.
- [18] C. Vermeylen, M. V. Barel, Stability improvements for fast matrix multiplication, *Numerical Algorithms* (2024). doi:10.1007/s11075-024-01806-y.
- [19] J. Nocedal, S. J. Wright, *Numerical optimization*, Springer, New York, 1999. doi:10.1007/978-0-387-40065-5.
- [20] G. O. Berger, P.-A. Absil, L. De Lathauwer, R. M. Jungers, M. Van Barel, Equivalent polyadic decompositions of matrix multiplication tensors, *Journal of Computational and Applied Mathematics* 406 (2022) 113941. doi:j.cam.2021.113941.
- [21] G. Ballard, Discovering fast matrix multiplication algorithms via tensor decomposition, in: *SIAM Conference on Computational Science and*

Engineering, Vol. 30, 2017.

URL <https://users.wfu.edu/ballard/pdfs/CSE17.pdf>

- [22] J. Huang, L. Rice, D. A. Matthews, R. A. van de Geijn, Generating families of practical fast matrix multiplication algorithms, in: 2017 IEEE International Parallel and Distributed Processing Symposium (IPDPS), 2017, pp. 656–667. doi:10.1109/IPDPS.2017.56.
- [23] C. Vermeulen, Tensor methods: Fast matrix multiplication and rank adaptation, Ph.D. thesis (2024-03-29).
URL <https://lirias.kuleuven.be/4144594&lang=en>