CONFORMAL PARA QUATERNIONIC CONTACT CURVATURE AND THE LOCAL FLATNESS THEOREM

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Abstract. A tensor invariant is defined on a para quaternionic contact manifold in terms of the curvature and torsion of the canonical para quaternionic connection involving derivatives up to third order of the contact form. This tensor, called para quaternionic contact conformal curvature, is similar to the Weyl conformal curvature in Riemannian geometry, the Chern-Moser tensor in CR geometry, the para contact curvature in para CR geometry and to the quaternionic contact conformal curvature in quaternionic contact geometry.

It is shown that a para quaternionic contact manifold is locally para quaternionic contact conformal to the standard flat para quaternionic contact structure on the para quaternionic Heisenberg group, or equivalently, to the standard para 3-Sasakian structure on the para quaternionic pseudo-sphere iff the para quaternionic contact conformal curvature vanishes.

Contents

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1. Introduction

We develop a tensor invariant in the sub-Riemannian geometry of para contact structures on a $4n+3$-dimensional differentiable manifold related to the algebra of para quaternions, known also as split quaternions [6], quaternions of the second kind [15], and complex product structures [2]. The para quaternionic contact structures, introduced in [5], turns out to be a generalization of the para 3-Sasakian geometry developed in [1, 6].

Para quaternionic contact geometry is a topic with some analogies with the quaternionic contact geometry introduced by O.Biquard [3] and its developments in connection with finding the extremals and the best constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group and related quaternionic contact Yamabe problem [8, 9, 11, 10, 13], but also with differences mainly because the para quaternionic contact structure lead to sub-hyperbolic PDE instead of sub-elliptic PDE in the quaternionic contact case.

A para quaternionic contact (pqc) manifold $(M,[g],\mathbb{PQ})$ is a $4n+3$-dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form $\eta=(\eta_1,\eta_2,\eta_3)$ with values in $\mathbb{R}^3$. In addition, H has a conformal $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$ structure, i.e. it is equiped with a conformal class of neutral metrics $[g]$ of signature $(2n,2n)$ and a rank-three bundle $\mathbb{PQ}$ consisting of (1,1)-tensors on H, locally generated by two almost para complex structures I_1,I_2 and an almost complex structure I_3 on H, satisfying the identities of the imaginary unit para quaternions, $I_1^2=I_2^2=id_H, \quad I_3^2=-id_H, \quad I_1I_2=-I_2I_1=I_3$, such that $-\epsilon_i 2g(I_iX,Y) \;=\; d\eta_i(X,Y), \quad \epsilon_1=\epsilon_2=-\epsilon_3=1, \quad g\in[g]$.

The 1-form η is determined up to a conformal factor. Hence, H becomes equipped with a conformal class [g] of neutral Riemannian metrics of signature (2n,2n). Transformations preserving a given pqc structure η, i.e. $\bar\eta=\mu\Phi\eta$ for a non-vanishing smooth function μ and a $SO(1,2)$ valued smooth matrix Φ, are called *para quaternionic contact conformal (pqc conformal) transformations*. To every metric in the fixed conformal class one can associate a linear connection ∇ preserving the pqc structure, introduced in [5], the (canonical) pqc connection.

A basic example is provided by any para 3-Sasakian manifold, which can be defined as a $(4n+3)$-dimensional pseudo Riemannian manifold, whose Riemannian cone is a hypersymplectic manifold.

The para quaternionic Heisenberg group pQH with its "standard" left-invariant pqc structure is the unique (up to a $SO(1,2)$-action) example of a pqc structure with flat canonical connection [5]. As a manifold $pQH \;=pH^n\times \operatorname{Im} pH$, while the group multiplication is given by

$$(q',\omega')\;=\;(q_o,\omega_o)\circ(q,\omega)\;=\;(q_o\;+\;q,\omega\;+\;\omega_o\;+\;2\ \mathrm{Im}\ q_o\,\bar q),$$

where $q,\ q_o\in pH^n$ and $\omega,\omega_o\in\operatorname{Im} pH$. The standard flat para quaternionic contact structure is defined by the left-invariant para quaternionic contact form

$$\tilde\Theta\;=\;(\tilde\Theta_1,\ \tilde\Theta_2,\ \tilde\Theta_3)\;=\;\frac12\,(d\omega\;-\;q\cdot d\bar q\;+\;dq\;\cdot\bar q),$$

where . denotes the para quaternion multiplication.

The aim of this paper is to find a tensor invariant on the tangent bundle, characterizing locally the pqc structures, which are para quaternionic contact conformally equivalent to the flat pqc structure on the para quaternionic Heisenberg group pQH . With this goal in mind, we describe a curvature-type tensor W^{pqc} defined in terms of the curvature and torsion of the canonical connection by (4.20), involving derivatives up to second order of the horizontal metric, whose form is similar to the Weyl conformal curvature in Riemannian geometry (see e.g. [7]), to the Chern-Moser invariant in CR geometry [4], see also [16], the para contact conformal curvature dveloped in [14] and to the quaternionic contact conformal curvature described by Ivanov-Vassilev in [12], see also [13]. We call W^{pqc} the *para quaternionic contact conformal curvature*, or *pqc conformal curvature*. The main purpose of this article is to prove the following two facts.

Theorem 1.1. *The pqc conformal curvature W^{pqc} is invariant under pqc conformal transformations.*

Theorem 1.2. *A pqc structure on a (4n+3)-dimensional smooth pqc manifold is locally pqc conformal to the standard flat pqc structure on the paraquaternionic Heisenberg group pQH if and only if the pqc conformal curvature vanishes, $W^{pqc} = 0$.*

We define a local map between the para quaternionic Heisenberg group and the para 3-Sasakian pseudo-sphere (called the para quaternionic Cayley transform) and show in Proposition 3.1 below that the para quaternionic Cayley transform establishes a conformal para quaternionic contact automorphism between the standard para 3-Sasaki structure on the para quaternionic pseudo-sphere pS^{4n+3} and the standard flat pqc structure on pQH. As a consequence of Theorem 1.2 and Proposition 3.1, we obtain

Corollary 1.3. *A pQC manifold is locally para quaternionic contact conformal to the para quaternionic pseudosphere pS^{4n+3} if and only if the pqc conformal curvature vanishes, $W^{pqc} = 0$.*

Our investigations follow the classical approach used by H.Weyl, see e.g. [7] and are close to [14] and [12] while [4], and [16] follow the Cartan method of equivalence.

Remark 1.4. *Following the work of Cartan and Tanaka, a pqc structure can be considered as an example of what has become known as a parabolic geometry. The para quaternionic Heisenberg group, as well as the para quaternionic pseudo-sphere, provide the flat models of such a geometry due to the para quaternionic Cayley transform. It is well known that the curvature of the corresponding regular Cartan connection is the obstruction for the local flatness. However, the Cartan curvature is not a tensor field on the tangent bundle and it is highly nontrivial to extract a tensor field involving the lowest order derivatives of the structure, which implies the vanishing of the obstruction. Theorem 1.2 suggests that a necessary and sufficient condition for the vanishing of the Cartan curvature of a pqc structure is the vanishing of the pqc conformal curvature tensor, $W^{pqc} = 0$.*

Organization of the paper. The paper relies heavily on the canonical connection introduced in [5] and the properties of its torsion and curvature described in [5]. In order to make the present paper self-contained, in Section 2 we give a review of the notion of a para quaternionic contact structure and collect formulas and results from [5] that will be used in the subsequent sections.

Convention 1.5. *We use the following conventions:*

a) *We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$;*
b) *$\{e_1, \dots, e_n, I_1e_1, \dots, I_1e_n, Ie_2, \dots, I_2e_n, I_3e_1, \dots, I_3e_n\}$ denotes an adapted orthonormal basis of the horizontal space $\mathbb{H}$.;*
c) *The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used,*

$$P(e_b, e_b) = \sum_{b=1}^{4n} g(e_b, e_b)R(e_b, e_b) = \sum_{b=1}^{n} \Big[P(e_b, e_b) - P(I_1e_b, I_1e_b) - P(I_2e_b, I_2e_b) + P(I_3e_b, I_3e_b)\Big];$$

d) *The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$. In particular, any equation involving i, j, k holds for any such permutation;*
e) *s and t will be any numbers from the set $\{1, 2, 3\}$, $\quad s, t \in \{1, 2, 3\}$.*

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2. Paraquaternionic contact manifolds and the canonical connection

In this section we will briefly review the basic notions of paraquaternionic contact geometry and recall some results from [5].

The algebra pQ of para quaternions (sometimes called split quaternions [6]) is a four-dimensional real vector space with basis $1, r_1, r_2, r_3$, satisfying $r_1^2 = r_2^2 = 1, \quad r_3^2 = -1, \quad r_1r_2 = -r_2r_1 = r_3$.

This carries a natural indefinite inner product given by $< p, q >= Re(\bar{p}q)$, where $p = t+r_3x+r_1y+r_2z$ has $\bar{p} = t - r_3x - r_1y - r_2z$. We have $||p||^2 = t^2 + x^2 - y^2 - z^2$, so a metric of signature (2,2). This norm is multiplicative, $||pq||^2 = ||p||^2||q||^2$, but the presence of elements of length zero means that pQ contains zero divisors.

A para quaternionic contact (pqc) manifold (M, g, pQ) is a $4n + 3$ dimensional manifold M with a codimension three distribution H equipped with a metric g of neutral signature (2n,2n) and a $Sp(n, \mathbb{R})Sp(1, \mathbb{R})$ structure, i.e., we have

i) a rank-three bundle $\mathbb{PQ}$, consisting of (1,1)-tensors on H, locally generated by two almost para complex structures I_1, I_2 and an almost complex structure I_3 on H, satisfying the identities of the imaginary unit para quaternions,

$$I_s^2 = \epsilon_s, \quad I_iI_j = -I_jI_i = -\epsilon_kI_k, \quad where \quad \epsilon_1 = \epsilon_2 = -\epsilon_3 = 1, \tag{2.1}$$

which are para quaternionic compatible with the neutral metric g on H,

$$g(I_s., I_s.) = -\epsilon_s g(., .). \tag{2.2}$$

ii) H is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3, H = \cap_{s=1}^3 Ker\, \eta_s$ and the following compatibility condition holds

$$-2\epsilon_s g(I_sX, Y) \;=\; d\eta_s(X, Y), \quad X, Y \in H. \tag{2.3}$$

A pqc manifold $(M, \bar{g}, \mathbb{PQ})$ is called para quaternionic contact conformal (pqc conformal) to $(M, g, \mathbb{Q})$ if $\bar{g} = \nu g$, for a nowhere vanishing smooth function ν. In that case, if $\bar{\eta}$ is a corresponding associated one-form with para quaternionic structure $\bar{I}_s$, we have $\bar{\eta} \;=\; \nu\, \Psi\, \eta$ for some $\Psi \in SO(1, 2)$. In particular, starting with a pqc manifold (M, η) and defining $\bar{\eta} \;=\; \nu\, \eta$ we obtain a pqc manifold $(M, \bar{\eta})$, pqc conformal to the original one.

On a paraquaternionic contact manifold there exists a canonical connection defined in [5].

Theorem 2.1. [5] *Let (M, g, pQ) be a para quaternionic contact manifold of dimension $4n + 3 > 7$ with a fixed neutral metric g on H. Then there exists a unique supplementary subspace V to H in TM, locally generated by a vector fields $\{\xi_1, \xi_2, \xi_3\}$, satisfying the conditions*

$$\begin{aligned} \eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \\ (\xi_j \lrcorner d\eta_i)_{|H} = \epsilon_k(\xi_i \lrcorner d\eta_j)_{|H}, \end{aligned} \tag{2.4}$$

and a unique connection ∇ with torsion T on M^{4n+3}, such that:

i) ∇ preserves the splitting $H \oplus V$ and the $Sp(n, \mathbb{R})Sp(1, \mathbb{R})$-structure on H, $\nabla g = 0, \nabla pQ \subset pQ$;
ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;
iii) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R}))^{\perp} \subset gl(4n)$;
iv) the connection on V is induced by the natural identification φ of V with the subspace $sp(1, \mathbb{R})$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.

We shall call the above connection *the canonical pqc connection.* The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields.

The fundamental 2-forms ω_s of the pqc structure pQ are defined by

$$-2\epsilon_s\omega_{s|H} \;=\; d\eta_{s|H}, \qquad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

The second condition ii) yields that the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = -2\sum_{s=1}^{3}\epsilon_s\omega_s(X,Y)\xi_s = -2\omega_1(X,Y)\xi_1 - 2\omega_2(X,Y)\xi_2 + 2\omega_3(X,Y)\xi_3. \tag{2.5}$$

The fourth condition iv) implies

$$\nabla I_i = -\alpha_j\otimes I_k + \epsilon_k\alpha_k\otimes I_j, \quad \nabla\xi_i = -\alpha_j\otimes\xi_k + \epsilon_k\alpha_k\otimes\xi_j, \tag{2.6}$$

where the $sp(1,\mathbb{R})$-connection 1-forms α_s are determined by the pqc structure [5, Theorem 3.8].

If the dimension of M is seven, the conditions (2.4) do not always hold. It is shown in [5] that if we additionally assume the existence of Reeb vector fields as in (2.4), then Theorem 2.1 holds. Henceforth, by a pqc structure in dimension 7 we shall mean a pqc structure satisfying (2.4).

Notice that equations (2.4) are invariant under the natural $SO(1,2)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1,\xi_2,\xi_3\} = V\perp H$ and $g(\xi_s,\xi_k) = -\epsilon_s\delta_{sk}$. The extended metric does not depend on the action of $SO(1,2)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the canonical pqc connection preserves the extended metric on TM, $\nabla g = 0$.

2.1. **The torsion endomorphism.** The properties of the canonical pqc connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi,.) : H\to H, \quad \xi\in V$. Recall that any endomorphism Ψ of H can be decomposed with respect to the para quaternionic structure (pQ,g) uniquely into $Sp(n,\mathbb{R})$-invariant parts as follows $\Psi = \Psi^{+++}+\Psi^{+--}+\Psi^{-+-}+\Psi^{--+}$,where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc.

The two $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--}+\Psi^{-+-}+\Psi^{--+}. \tag{2.7}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger \;=\; I_1\otimes I_1 \;+\; I_2\otimes I_2 \;+\; I_3\otimes I_3, \tag{2.8}$$

corresponding to the eigenvalues 3 and -1, respectively. If $n=1$ then the space of symmetric endomorphisms commuting with all $I_i, i=1,2,3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = \frac{Tr(\Psi)}{4}Id_{|H}$.

Decomposing the torsion endomorphism $T_\xi\in(sp(n)+sp(1))^\perp$ into symmetric part T^{sym}_ξ and anti-symmetric part T^a_ξ, we define the tensors $\tau(X,Y)$ and $\mu(X,Y)$ on H by

$$\begin{aligned}\tau(X,Y) = -\epsilon_iT^{sym}(\xi_i,I_iX,Y) - \epsilon_jT^{sym}(\xi_j,I_jX,Y) - \epsilon_kT^{sym}(\xi_k,I_kX,Y);\\ \mu(X,Y) = \epsilon_sT^a(\xi_s,I_sX,Y).\end{aligned} \tag{2.9}$$

The tensors τ and μ do not depend on the particular choice of the Reeb vector fields and are invariant under the natural action of $SO(1,2)$.

We summarize the description of the torsion from [5] in the following Proposition.

Proposition 2.2. [5] *The tensor τ on H is symmetric, trace-free, belongs to the [-1]- component and determines the symmetric part of the torsion endomorphism, i.e. it satisfies the relations*

$$\tau(X,Y) = \tau(Y,X), \quad \tau(e_a,e_a) = \tau(I_se_a,e_a) = 0; \tag{2.10}$$

$$\tau(X,Y) - \tau(I_1,X,I_1,Y) - \tau(I_2X,I_2Y) + \tau(I_3X,I_3Y) = 0; \tag{2.11}$$

$$T^{sym}(\xi_s,X,Y) = -\frac{1}{4}\Big[\tau(I_sX,Y)+\tau(X,I_sY)\Big]. \tag{2.12}$$

The tensor μ is symmetric trace-free, has the properties

$$\mu(I_sX, I_sY) = -\epsilon_s \mu(X, Y), \tag{2.13}$$

belongs to the [3]- component and determines the skew-symmetric part of the torsion endomorphism

$$T^a(\xi_s, X, Y) = \mu(I_sX, Y). \tag{2.14}$$

If the dimension is seven then $\mu = 0$*.*

2.2. **The curvature and the Ricci type tensors.** Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇ and denote the curvature tensor of type (0,4) by the same letter. The first Bianchi identity for the canonical pqc connection reads

$$\sum_{(A,B,C)} \Big\{ R(A, B, C, D) \Big\} = \sum_{(A,B,C)} \Big\{ (\nabla_A T)(B, C, D) + T(T(A, B), C, D) \Big\} = b(A, B, C, D), \tag{2.15}$$

where $\sum_{(A,B,C)}$ denotes the cyclic sum and $A, B, C, D \in \Gamma(TM)$.

The curvature of a metric connection is skew-symmetric with respect to the last two arguments, $R(A, B, C, D) = -R(A, B, D, C)$. It follows directly from first Bianchi identity (2.15) that

$$\begin{aligned} 2Zam(A, B, C, D) &= 2R(A, B, C, D) - 2R(C, D, A, B) \\ &= b(A, B, C, D) + b(B, C, D, A) - b(A, C, D, B) - b(A, B, D, C). \end{aligned} \tag{2.16}$$

The horizontal Ricci tensor and the scalar curvature $Scal$ of the canonical pqc connection, called *pqc Ricci tensor* and *pqc scalar curvature*, respectively, are defined by

$$Ric(X, Y) = R(e_a, X, Y, e_a), Scal = Ric(e_a, e_a).$$

The curvature of the canonical pqc connection admits several traces, defined in [5] by

$$4n\rho_s(A, B) = R(A, B, e_a, I_se_a),\ 4n\varrho_s(A, B) = R(e_a, I_se_a, A, B),\ 4n\zeta_s(A, B) = R(e_a, A, B, I_se_a).$$

The curvature operator $R(B, C)$ preserves the pqc structure on M since the connection ∇ preserves it. In particular, $R(B, C)$ preserves the splitting $H \oplus V$ and the paraquaternionic structure on H, so $R(B, C) \in sp(n, \mathbb{R}) \oplus sp(1, \mathbb{R})$ on H. We use the following result established in [5].

Lemma 2.3. [5] *On a pqc manifold the next identities hold*

$$\epsilon_i R(A, B, I_iX, I_iY) + R(A, B, X, Y) = -2\epsilon_j \rho_j(A, B)\omega_j(X, Y) - 2\epsilon_k \rho_k(A, B)\omega_k(X, Y); \tag{2.17}$$

$$R(A, B)\xi_i = -2\epsilon_i \rho_k(A, B)\xi_j + 2\epsilon_i \rho_j(A, B)\xi_k, \qquad \rho_i = \frac{1}{2}\Big[\epsilon_k d\alpha_i - \epsilon_j \alpha_j \wedge \alpha_k\Big]; \tag{2.18}$$

It is shown in [5] that all horizontal Ricci type contractions of the curvature of the canonical pqc connection can be expressed in terms of the torsion endomorhism. We utilize the following from [5].

Theorem 2.4. [5] *On a* $(4n + 3)$*-dimensional pqc manifold the horizontal Ricci tensors Ric and* $\zeta_s(X, I_sY)$ *are symmetric, the horizontal Ricci tensors* $\rho_s(X, I_sY), \varrho_s(X, I_sY)$ *are symmetric (1,1) tensors with respect to* I_s *and the next formulas hold*

$$Ric(X,Y) = \frac{Scal}{4n} g(X,Y) + (2n+2)\tau(X,Y) + (4n+10)\mu(X,Y); \tag{2.19}$$

$$\rho_s(X, I_sY) = \epsilon_s \frac{Scal}{8n(n+2)} g(X,Y) + \frac{1}{2}\Big[\epsilon_s \tau(X,Y) - \tau(I_sX, I_sY)\Big] + 2\epsilon_s \mu(X,Y); \tag{2.20}$$

$$\varrho_s(X, I_sY) = \epsilon_s \frac{Scal}{8n(n+2)} g(X,Y) + \frac{n+2}{2n}\Big[\epsilon_s \tau(X,Y) - \tau(I_sX, I_sY)\Big]; \tag{2.21}$$

$$\epsilon_s \zeta_s(X, I_sY) \;=\; -\frac{Scal}{16n(n+2)} g(X,Y) - \frac{2n+1}{4n}\tau(X,Y) + \epsilon_s \frac{1}{4n}\tau(I_sX, I_sY) - \frac{2n+1}{2n}\mu(X,Y); \tag{2.22}$$

$$T(\xi_i, \xi_j) = \epsilon_k \frac{Scal}{8n(n+2)} \xi_k - [\xi_i, \xi_j]_H; \tag{2.23}$$

$$T(\xi_i, \xi_j, I_kX) = \rho_k(I_jX, \xi_i) = -\rho_k(I_iX, \xi_j) = \omega_k([\xi_i, \xi_j], X); \tag{2.24}$$

$$-\epsilon_i \rho_i(\xi_i, \xi_j) - \epsilon_k \rho_k(\xi_k, \xi_j) = \frac{1}{16n(n+2)} \xi_j(Scal). \tag{2.25}$$

For $n = 1$ the above formulas hold with $\mu = 0$.

Clearly, the condition $Ric = \frac{Scal}{4n} g$ is equivalent to $\tau = \mu = 0$. It was shown in [5] that the torsion endomorphism of a pqc manifold vanishes exactly when it is locally para 3-Sasakian provided the pqc scalar curvature is nonvanishing and the dimension is bigger than seven.

It is established in [5] that the whole curvature is determined from the horizontal curvature. We will use the next result proved in [5].

Theorem 2.5. [5] *On a pqc manifold the curvature of the canonical connection satisfies the equalities:*

$$\begin{aligned} R(\xi_i, X, Y, Z) = &-(\nabla_X \mu)(I_iY, Z) - \frac{1}{4}\Big[(\nabla_Y \tau)(I_iZ, X) + (\nabla_Y \tau)(Z, I_iX)\Big] \\ &+ \frac{1}{4}\Big[(\nabla_Z \tau)(I_iY, X) + (\nabla_Z \tau)(Y, I_iX)\Big] + \omega_j(X,Y)\rho_k(I_iZ, \xi_i) - \omega_k(X,Y)\rho_j(I_iZ, \xi_i) \\ &- \omega_j(X,Z)\rho_k(I_iY, \xi_i) + \omega_k(X,Z)\rho_j(I_iY, \xi_i) - \omega_j(Y,Z)\rho_k(I_iX, \xi_i) + \omega_k(Y,Z)\rho_j(I_iX, \xi_i). \end{aligned} \tag{2.26}$$

$$\begin{aligned} R(\xi_i, \xi_j, X, Y) = &(\nabla_{\xi_i} \mu)(I_jX, Y) - (\nabla_{\xi_j} \mu)(I_iX, Y) + \epsilon_j (\nabla_X \rho_k)(I_iY, \xi_i) \\ &- \frac{1}{4}\Big[(\nabla_{\xi_i} \tau)(I_jX, Y) + (\nabla_{\xi_i} \tau)(X, I_jY)\Big] + \frac{1}{4}\Big[(\nabla_{\xi_j} \tau)(I_iX, Y) + (\nabla_{\xi_j} \tau)(X, I_iY)\Big] \\ &+ \epsilon_k \frac{Scal}{8n(n+2)} T(\xi_k, X, Y) - T(\xi_j, X, e_a) T(\xi_i, e_a, Y) + T(\xi_j, e_a, Y) T(\xi_i, X, e_a), \end{aligned} \tag{2.27}$$

where the Ricci 2-forms are given by

$$\begin{aligned} 3(2n+1)\rho_i(\xi_i, X) = &-\epsilon_i \frac{1}{4}(\nabla_{e_a} \tau)(e_a, X) - \frac{3}{4}(\nabla_{e_a} \tau)(I_ie_a, I_iX) + \epsilon_i (\nabla_{e_a} \mu)(X, e_a) \\ &-\epsilon_i \frac{2n+1}{16n(n+2)} X(Scal); \end{aligned} \tag{2.28}$$

$$\begin{aligned} 3(2n+1)\rho_i(I_kX, \xi_j) = -3(2n+1)\rho_i(I_jX, \xi_k) = &-\frac{(2n+1)(2n-1)}{16n(n+2)} X(Scal) \\ &+2(n+1)(\nabla_{e_a} \mu)(X, e_a) + \frac{4n+1}{4}(\nabla_{e_a} \tau)(e_a, X) - \epsilon_i \frac{3}{4}(\nabla_{e_a} \tau)(I_ie_a, I_iX); \end{aligned} \tag{2.29}$$

$$(n-1)(\nabla_{e_a} \tau)(e_a, X) + 2(n+2)(\nabla_{e_a} \mu)(e_a, X) - \frac{(n-1)(2n+1)}{8n(n+2)} d(Scal)(X) = 0. \tag{2.30}$$

As a consequence of Theorem 2.5 one gets the next result originally proved in [5].

Proposition 2.6. [5] *A pqc manifold is locally isomorphic to the paraquaternionic Heisenberg group exactly when the curvature of the canonical connection restricted to H vanishes, $R_{|_H} = 0$.*

3. Para quaternionic Heisenberg group and the para quaternionic Cayley transform

Since our goal is to classify para quaternionic contact manifolds locally conformal to the para quaternionic Heisenberg group, we recall its definition from [5], define the para quaternionic Cayley transform and show that it is a local para quaternionic contact authomrphism between the para 3-Sasakian structure on the pseudo-sphere and the para quaternionic Heisenberg group.

3.1. **The para quaternionic Heisenberg group.** As a manifold the para quaternionic Heisenberg group of topological dimension $4n+3$ is $G(pH) = pH^n \times Im(pH)$ with the group law given by

$$(q', \omega') = (q_o, \omega_o)_o(q, \omega) = (q_o + q, \omega_o + \omega + 2Im(q_o\bar{q})),$$

where $q, q_o \in pH^n$ and $\omega, \omega_o \in Im(pH)$.

On $G(pH)$ we define the para quaternionic contact form in para quaternionic variables as follows

$$\tilde{\Theta} = (\tilde{\Theta}_3, \tilde{\Theta}_1\tilde{\Theta}_2) = \frac{1}{2}(d\omega - q d\bar{q} + dq\bar{q}).$$

In real coordinates we get

$$\begin{aligned} \tilde{\Theta}_3 &= \frac{1}{2}dx - x^a dt^a + t^a dx^a - z^a dy^a + y^a dz^a; \\ \tilde{\Theta}_1 &= \frac{1}{2}dy - y^a dt^a - z^a dx^a + t^a dy^a + x^a dz^a; \\ \tilde{\Theta}_2 &= \frac{1}{2}dz - z^a dt^a + y^a dx^a - x^a dy^a + t^a dz^a. \end{aligned} \tag{3.1}$$

The structure equations of $G(pH)$ are

$$\begin{aligned} d\tilde{\Theta}_3 &= 2(dt^a \wedge dx^a + dy^a \wedge dz^a); \\ d\tilde{\Theta}_1 &= 2(dt^a \wedge dy^a + dx^a \wedge dz^a); \\ d\tilde{\Theta}_2 &= 2(dt^a \wedge dz^a - dx^a \wedge dy^a). \end{aligned}$$

The left-invariant horizontal vector fields $T_a, X_a = J_3T_a, Y_a = -J_1T_a, Z_a = -J_2T_a$ are given by

$$\begin{aligned} T_a &= \partial/\partial t_a + 2x^a\partial/\partial x + 2y^a\partial/\partial y + 2z^a\partial/\partial z; \qquad & X_a &= \partial/\partial x_a + 2t^a\partial/\partial x - 2z^a\partial/\partial y + 2y^a\partial/\partial z; \\ Y_a &= \partial/\partial y_a + 2z^a\partial/\partial x - 2t^a\partial/\partial y - 2x^a\partial/\partial z; \qquad & Z_a &= \partial/\partial z_a - 2y^a\partial/\partial x + 2x^a\partial/\partial y - 2t^a\partial/\partial z. \end{aligned}$$

The horizontal metric of signature (2n,2n) is defined by

$$g(T_a, T_a) = g(X_a, X_a) = -g(Y_a, Y_a) = -g(Z_a, Z_a) = 1.$$

The central (left-invariant vertical) Reeb vector fields are $\xi_3 = 2\partial/\partial x, \quad \xi_1 = 2\partial/\partial y, \quad \xi_2 = 2\partial/\partial z$ and a straightforward calculation shows the following commutation relations

$$[J_iT_a, T_a] = -2\epsilon_i\xi_i, \qquad [J_iT_a, J_jT_a] = 2\epsilon_k\xi_k.$$

It is easy to verify that the left-invariant flat connection on $G(pH)$ coincides with the canonical pqc connection of the pqc manifold $(G(pH), \tilde{\Theta})$.

3.2. **An embeding of the paraquaternionic Heisenberg group $G(pH)$.** Consider the hypersurface

$$\Sigma \subset pH^n \times pH : \Sigma = (q', p') \in pH \times pH : Re(p') = -|q'|^2.$$

Clearly, Σ is the 0-level set of $\rho = |q'|^2 + t$ and

$$d\rho = q'd\bar{q'} + dq'\bar{q'} + dt = 2(t^a dt^a + x^a dx^a - y^a dy^a - z^a dz^a) + dt. \tag{3.2}$$

The standard para quaternionic structure J_3, J_1, J_2 on $\mathbb{R}^{4n+4}$, induced by the multiplication on the right by the para quaternions $r_3, r_1, r_2 \in pH^{n+1}$, is

$$\begin{aligned} J_3 dt^a = dx^a, \quad J_1 dt^a = dy^a, \quad J_2 dt^a = dz^a, \\ J_3 dy^a = -dz^a, \quad J_1 dx^a = dz^a, \quad J_2 dx^a = -dy^a, \\ J_3 dt = dx, \quad J_1 dt = dy, \quad J_2 dt = dz. \end{aligned} \tag{3.3}$$

Combining (3.2) with (3.3) and comparing with (3.1) we get

$$\begin{aligned} J_3 d\rho &= 2(t^a dx^a - x^a dt^a + y^a dz^a - z^a dy^a) + dx = 2\tilde{\Theta}_3; \\ J_1 d\rho &= 2(t^a dy^a + x^a dz^a - y^a dt^a - z^a dx^a) + dy = 2\tilde{\Theta}_1; \\ J_2 d\rho &= 2(t^a dz^a - x^a dy^a + y^a dx^a - z^a dt^a) + dz = 2\tilde{\Theta}_2. \end{aligned}$$

We identify $G(pH)$ with Σ by $(q', \omega') \to (q', p' = -|q'|^2 + \omega')$. Since $dp' = -q'd\bar{q'} - dq'\bar{q'} + d\omega'$, we write

$$\tilde{\Theta} = \frac{1}{2}(d\omega - q'd\bar{q'} + dq'\bar{q'}) = \frac{1}{2}dp' + dq'\bar{q'}.$$

Taking into account that $\tilde{\Theta}$ is pure imaginary we can write the last equation in the form

$$\tilde{\Theta} = \frac{1}{4}(dp' - d\bar{p'}) + \frac{1}{2}(dq'\bar{q'} - q'd\bar{q'}). \tag{3.4}$$

3.3. **The para 3-Sasakian pseudo sphere and the para quaternionic Cayley transform.** The second explicit example is the pqc-structure on the para 3-Sasakian pseudo-sphere. The para 3-Sasakian structure on the pseudo-sphere (hyperboloid) $pS^{4n+3} = \{|q|^2 + |p|^2 = 1\} \subset pH^n \times pH$ is inherited from the standard flat hypersymplectic structure on $\mathbb{R}^{4n+4} = pH^n \times pH$. In para quaternionic variables, the pqc 1-form on the pseudo sphere $pS^{4n+3} = \{|q|^2 + |p|^2 = 1\} \subset pH^n \times pH$ is defined as follows

$$\tilde{\eta} = dq.\bar{q} + dp.\bar{p} - q.d\bar{q} - p.d\bar{p}. \tag{3.5}$$

We consider the map from the pseudo-sphere pS^{4n+3} minus the points Σ_0,

$$\Sigma_0 = (q, p) \in pS^{4n+3} : |p-1|^2 = (t-1)^2 + x^2 - y^2 - z^2 = 0$$

to the paraquaternionic Heisenberg group $G(pH) \cong \Sigma$, defined by

$$\mathbb{C} : \Big(pS^{4n+3} - \Sigma_0\Big) \to \Sigma, \quad (q', p') = \mathbb{C}\Big((q, p)\Big), \quad q' = (p-1)^{-1}q, \quad p' = (p-1)^{-1}(p+1)$$

since $Re(p') = Re\Big(\frac{(\bar{p}-1)(p+1)}{|p-1|^2}\Big) = \frac{|p|^2-1}{|p-1|^2} = -\frac{|q|^2}{|p-1|^2} = -|q'|^2$.

The inverse map $(q, p) = \mathbb{C}^{-1}\Big((q', p')\Big)$ is given by

$$q = 2(p'-1)^{-1}q', \quad p = (p'-1)^{-1}(p'+1).$$

We call this map *para quaternionic Cayley transform.*

An easy calculation gives

$$dp' = -2(p-1)^{-1}.dp.(p-1)^{-1}; \quad dq' = (p-1)^{-1}.\Big[dq - dp.(p-1)^{-1}.q\Big]. \tag{3.6}$$

Using (3.4) together with (3.6), we calculate

$$
\begin{aligned}
2\mathbb{C}^*\tilde{\Theta} &= -(p-1)^{-1}.dp.(p-1)^{-1} + (\bar{p}-1)^{-1}.d\bar{p}.(\bar{p}-1)^{-1} \\
+(p-1)^{-1}.\Big[dq - dp.(p-1)^{-1}.q\Big].\bar{q}.(\bar{p}-1)^{-1} &\quad - (p-1)^{-1}.q.\Big[d\bar{q} - \bar{q}.(\bar{p}-1)^{-1}.d\bar{p}\Big].(\bar{p}-1)^{-1} \\
&= (p-1)^{-1}.\Big[dq.\bar{q} - q.d\bar{q}\Big].(\bar{p}-1)^{-1} \\
&-(p-1)^{-1}.\Big[dp.(p-1)^{-1}(\bar{p}-1) + |q|^2 dp.(p-1)^{-1}\Big].(\bar{p}-1)^{-1} \\
&+(p-1)^{-1}.\Big[(p-1)(\bar{p}-1)^{-1}.d\bar{p} + |q|^2.(\bar{p}-1)^{-1}.d\bar{p}\Big].(\bar{p}-1)^{-1} \\
&= (p-1)^{-1}.\Big[dq.\bar{q} - q.d\bar{q}\Big].(\bar{p}-1)^{-1} - (p-1)^{-1}.\Big[dp.(p-1)^{-1}(\bar{p}-\bar{p}.p)\Big].(\bar{p}-1)^{-1} \\
&+(p-1)^{-1}.\Big[(p-p\bar{p})(\bar{p}-1)^{-1}.d\bar{p}\Big].(\bar{p}-1)^{-1} \\
&= (p-1)^{-1}.\Big[dq.\bar{q} - q.d\bar{q} + dp.\bar{p} - p.d\bar{p}\Big].(\bar{p}-1)^{-1} = \frac{1}{|p-1|^2}\lambda.\tilde{\eta}.\bar{\lambda},
\end{aligned}
\tag{3.7}
$$

where $\lambda = \frac{|p-1|}{p-1}$ is a unit paraquaternion and $\tilde{\eta}$ is the standard paraquaternionic contact form on the pseudo-sphere pS^{4n+3}, given by (3.5).

Since $p - 1 = 2(p' - 1)^{-1}$ we have $\lambda = \frac{1}{|p'-1|}(p' - 1)$ and we can put (3.7) into the form

$$
\lambda.(\mathbb{C}^*)^{-1}.\bar{\lambda} = \frac{8}{|p'-1|^2}\tilde{\Theta}. \tag{3.8}
$$

Thus, we prove the next

Proposition 3.1. *The para quaternionic Heisenberg group $G(pH)$ and the para 3-Sasakian pseudo-sphere pS^{4n+3} are locally pqc conformally equivalent via the para quaternionic Cayley transform*

4. Para quaternionic contact conformal curvature. Proof of Theorem 1.1

In this section we define the para quaternionic contact conformal curvature and prove Theorem 1.1.

4.1. **Conformal transformations.** A conformal para quaternionic contact transformation between two para quaternionic contact manifold is a diffeomorphism Φ, which satisfies $\Phi^*\eta = \mu\ \Psi\cdot\eta$ for some positive smooth function μ and some matrix $\Psi \in SO(1,2)$ with smooth functions as entries, where $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$. The canonical pqc connection introduced in [5] does not change under the action of $SO(1,2)$, i.e., the canonical connection of $\Psi\cdot\eta$ and η coincides. Hence, as we study pqc conformal transformations we may consider only transformations $\Phi^*\eta = \mu\,\eta$.

4.2. **Para quaternionic conformal transformations.** Let h be a positive smooth function on a pqc manifold (M, η). Let $\bar{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of the pqc structure η. We will denote the objects related to $\bar{\eta}$ by over-lining the same object corresponding to η. Thus,

$$
d\bar{\eta} = -(2h^2)^{-1}dh\wedge\eta + (2h)^{-1}d\eta, \qquad \bar{g} = (2h)^{-1}g.
$$

The new triple $\{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\}$, determined by the conditions (2.4) defining the Reeb vector fields, is

$$
\bar{\xi}_s = 2h\,\xi_s + I_s\nabla h,
$$

where ∇h is the horizontal gradient defined by $\nabla h = dh(e_a)e_a, \quad g(\nabla h, X) = dh(X)$.

The horizontal hyperbolic sub-Laplacian and the norm of the horizontal gradient are defined by $\triangle_h h = tr^g_H(\nabla^2 h) = g(e_a,e_a)\ \nabla^2 h(e_a,e_a)$, $|\nabla h|^2 = g(e_a,e_a)dh(e_a)\,dh(e_a)$, respectively. The canonical pqc connections ∇ and $\bar{\nabla}$ are connected by a (1,2)-tensor S,

$$
\bar{\nabla}_A B = \nabla_A B + S_A B, \qquad A, B \in \Gamma(TM). \tag{4.1}
$$

Condition (2.5) yields $g(S_XY,Z)-g(S_YX,Z)=h^{-1}\sum_{s=1}^3\epsilon_s\omega_s(X,Y)dh(I_sZ)$, while $\bar\nabla\bar g=0$ implies $g(S_XY,Z)+g(S_XZ,Y)=-h^{-1}dh(X)g(Y,Z)$. The last two equations determine $g(S_XY,Z)$,

$$\begin{aligned}g(S_XY,Z)=-(2h)^{-1}\{&dh(X)g(Y,Z)+\sum_{s=1}^3\epsilon_s dh(I_sX)\omega_s(Y,Z)\\&+dh(Y)g(Z,X)-\sum_{s=1}^3\epsilon_s dh(I_sY)\omega_s(Z,X)-dh(Z)g(X,Y)-\sum_{s=1}^3\epsilon_s dh(I_sZ)\omega_s(X,Y)\}.\end{aligned}\tag{4.2}$$

Using Theorem 2.1 and after some calculations, we obtain

$$\begin{aligned}&g(\bar T_{\bar\xi_i}X,Y)-2hg(T_{\xi_i}X,Y)-g(S_{\bar\xi_i}X,Y)\\&\qquad=-\nabla dh(X,I_iY)-\epsilon_ih^{-1}[dh(I_kX)dh(I_jY)-dh(I_jX)dh(I_kY)].\end{aligned}\tag{4.3}$$

The identity $d^2=0$ yields $\nabla^2h(X,Y)-\nabla^2h(Y,X)=-dh(T(X,Y))$. Applying (2.5), we have

$$\nabla^2h(X,Y)=[\nabla^2h]_{[sym]}(X,Y)+\sum_{s=1}^3\epsilon_s dh(\xi_s)\omega_s(X,Y),\tag{4.4}$$

where $[.]_{[sym]}$ denotes the symmetric part of the corresponding (0,2)-tensor.

Decompose (4.3) into [3] and [-1] parts according to (2.7), use the properties of the torsion tensor T_{ξ_i} and (2.9) to come to the next transformation formulas

Proposition 4.1. *Let $\bar\eta=\frac{1}{2h}\eta$ be a pqc conformal transformation of a given pqc structure η. Then the two parts of the torsion endomorphism transforms as follows*

$$\begin{aligned}\bar\tau&=\tau+h^{-1}[\nabla^2h]_{[sym][-1]},\\\bar\mu&=\mu+(2h)^{-1}[\nabla^2h-2h^{-1}dh\otimes dh]_{[3][0]},\end{aligned}\tag{4.5}$$

where $[\nabla^2h]_{[sym][-1]}$ denotes the (-1)-component of the symmetric part of the horizontal Hessian and $[\nabla^2h-2h^{-1}dh\otimes dh]_{[3][0]}$ is the trace-free part of the (3)-component. Explicitly,

$$\begin{aligned}&[\nabla^2h]_{[sym][-1]}(X,Y)=\frac14\Big[3\nabla^2h(X,Y)+\sum_{s=1}^3\epsilon_s\nabla^2h(I_sX,I_sY)-4\sum_{s=1}^3\epsilon_s dh(\xi_s)\omega_s(X,Y)\Big];\\&[\nabla^2h-2h^{-1}dh\otimes dh]_{[3][0]}(X,Y)=\frac14\Big[\nabla^2h(X,Y)-\sum_{s=1}^3\epsilon_s\nabla^2h(I_sX,I_sY)\\&\qquad-\frac2h\Big(dh(X)dh(Y)-\sum_{s=1}^3\epsilon_s dh(I_sX)dh(I_sY)\Big)\Big]-\frac1{4n}\Big(\triangle_hh-\frac2h|\nabla h|^2\Big)g(X,Y).\end{aligned}\tag{4.6}$$

The tensor $g(S_{\bar\xi_i}X,Y)$ is given by

$$\begin{aligned}g(S_{\bar\xi_i}X,Y)\ =\ &\frac14\Big[\nabla^2h(X,I_iY)-\nabla^2h(I_iX,Y)-\epsilon_i\nabla^2h(I_jX,I_kY)+\epsilon_i\nabla^2h(I_kX,I_jY)\Big]\\&+(2h)^{-1}\Big[\epsilon_idh(I_kX)dh(I_jY)-\epsilon_idh(I_jX)dh(I_kY)-dh(I_iX)dh(Y)+dh(X)dh(I_iY)\Big]\\&+\frac1{4n}\left(-\triangle h+2h^{-1}|\nabla h|^2\right)\omega_i(X,Y)+\epsilon_idh(\xi_k)\omega_j(X,Y)-\epsilon_idh(\xi_j)\omega_k(X,Y).\end{aligned}\tag{4.7}$$

4.3. **Para quaternionic contact conformal curvature.** Let $(M,g,\mathbb{Q})$ be a (4n+3)-dimensional pqc manifold. We consider the symmetric (0,2) tensor L defined on H by the equality

$$L\ =\ \Big(\frac1{4(n+1)}Ric_{[-1]}+\frac1{2(2n+5)}Ric_{[3][0]}+\frac1{32n(n+2)}Scal\,g\Big)\ =\ \frac12\tau+\mu+\frac{Scal}{32n(n+2)}\,g,\tag{4.8}$$

where $Ric_{[-1]}$ is the [-1]-part of the horizontal Ricci tensor, $Ric_{[3][0]}$ is the trace-free [3]-part of Ric and we use the identities in Theorem 2.4 to obtain the second equality.

Let us denote the trace-free part of L with L_0, hence,

$$L_0 = \frac{1}{4(n+1)}Ric_{[-1]} + \frac{1}{2(2n+5)}Ric_{[3][0]} = \frac{1}{2}\tau + \mu. \tag{4.9}$$

The Kulkarni-Nomizu product of two (not necessarily symmetric) tensors is defined by

$$(A \owedge B)(X,Y,Z,V) := A(X,Z)B(Y,V) + A(Y,V)B(X,Z) - A(Y,Z)B(X,V) - A(X,V)B(Y,Z).$$

We also note explicitly that following usual conventions $I_sL\,(X,Y) = g(I_sLX,Y) = -L(X,I_sY)$.

Now, define the (0,4) tensor PWR on H as follows

$$\begin{aligned} PWR(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L)(X,Y,Z,V) - \sum_{s=1}^{3}\epsilon_s(\omega_s \owedge I_sL)(X,Y,Z,V) \\ &+ \frac{1}{2}\sum_{(i,j,k)}\epsilon_i\omega_i(X,Y)\Big[L(Z,I_iV) - L(I_iZ,V) - \epsilon_iL(I_jZ,I_kV) + \epsilon_iL(I_kZ,I_jV)\Big] \\ &+ \sum_{s=1}^{3}\epsilon_s\omega_s(Z,V)\Big[L(X,I_sY) - L(I_sX,Y)\Big] - \frac{1}{2n}(trL)\sum_{s=1}^{3}\epsilon_s\omega_s(X,Y)\omega_s(Z,V), \end{aligned} \tag{4.10}$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum.

A substitution of (4.8) and (4.9) in (4.10), invoking Proposition 2.2 gives

$$\begin{aligned} PWR(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L_0)(X,Y,Z,V) - \sum_{s=1}^{3}\epsilon_s(\omega_s \owedge I_sL_0)(X,Y,Z,V) \\ &+ \frac{1}{2}\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(X,Y)\Big\{\tau(Z,I_sV) - \tau(I_sZ,V)\Big\} + \omega_s(Z,V)\Big\{\tau(X,I_sY) - \tau(I_sX,Y) + 4\mu(X,I_sY)\Big\}\Big] \\ &+ \frac{Scal}{32n(n+2)}\Big[(g \owedge g)(X,Y,Z,V) - \sum_{s=1}^{3}\epsilon_s\Big((\omega_s \owedge \omega_s)(X,Y,Z,V) + 4\omega_s(X,Y)\omega_s(Z,V)\Big)\Big]. \end{aligned} \tag{4.11}$$

Proposition 4.2. *The tensor PWR is completely trace-free, i.e.*

$$Ric(PWR) = \rho_s(PWR) = \varrho_s(PWR) = \zeta_s(PWR) = 0.$$

Proof. Proposition 2.2 and (4.8) imply the following identities

$$\tau(X,Y) = \frac{1}{2}\Big[3L(X,Y) + L(I_1X,I_1Y) + L(I_2X,I_2Y) - L(I_3X,I_3Y)\Big]; \tag{4.12}$$

$$\mu(X,Y) = \frac{1}{4}\Big[L(X,Y) - L(I_1X,I_1Y) - L(I_2X,I_2Y) + L(I_3X,I_3Y) - \frac{1}{n}tr\,L\,g(X,Y)\Big]; \tag{4.13}$$

$$\begin{aligned} T(\xi_i,X,Y) &= -\frac{1}{2}\Big[L(I_iX,Y) + L(X,I_iY)\Big] + \mu(I_iX,Y) \\ &= -\frac{1}{4}L(I_iX,Y) - \frac{3}{4}L(X,I_iY) + \frac{1}{4}\epsilon_iL(I_kX,I_jY) - \frac{1}{4}\epsilon_iL(I_jX,I_kY) - \frac{1}{4n}(tr\,L)\,\omega_i(X,Y). \end{aligned} \tag{4.14}$$

After substituting (4.12) and (4.13) into the first four equations of Theorem 2.4, we derive

$$\begin{aligned}
Ric(X,Y) &= \frac{2n+3}{2n} tr\, L\, g(X,Y) \\
&\quad + \frac{8n+11}{2} L(X,Y) - \frac{3}{2}\Big[\epsilon_i L(I_iX,I_iY) + \epsilon_j L(I_jX,I_jY) + \epsilon_k L(I_kX,I_kY)\Big]; \\
\rho_i(X,Y) &= L(X,I_iY) - L(I_iX,Y) - \frac{1}{2n} tr L\, \omega_i(X,Y); \\
\varrho_i(X,Y) &= -\frac{1}{n} tr\, L\, \omega_i(X,Y) \\
&\quad - \frac{n+2}{2n}\Big[L(I_iX,Y) - L(X,I_iY) - \epsilon_i L(I_kX,I_jY) + \epsilon_i L(I_jX,I_kY)\Big]; \\
\zeta_i(X,Y) &= \frac{2n-1}{8n^2} tr\, L\, \omega_i(X,Y) \\
&\quad + \frac{3}{8n} L(I_iX,Y) - \frac{8n+3}{8n} L(X,I_iY) - \frac{1}{8n}\epsilon_i\Big[L(I_kX,I_jY) - L(I_jX,I_kY)\Big].
\end{aligned} \tag{4.15}$$

Take the corresponding traces in (4.10) and use (4.15) to verify the claim. □

We outline the following

Theorem 4.3. *On a pQC manifold the curvature of the canonical connection satisfies the equalities:*

$$\begin{aligned}
R(X,Y,Z,V) - R(Z,V,X,Y) &= -2\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(X,Y)\mu(I_sZ,V) - \omega_s(Z,V)\mu(I_sX,Y)\Big] \\
&+ \frac{1}{2}\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(Y,Z)\Big(\tau(I_sX,V)+\tau(X,I_sV)\Big) + \omega_s(X,V)\Big(\tau(I_sZ,Y)+\tau(Z,I_sY)\Big)\Big] \\
&- \frac{1}{2}\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(X,Z)\Big(\tau(I_sY,V)+\tau(Y,I_sV)\Big) + \omega_s(Y,V)\Big(\tau(I_sZ,X)+\tau(Z,I_sX)\Big)\Big].
\end{aligned} \tag{4.16}$$

The [3]-componenet of the horizontal curvature with respect to the first two arguments is given by

$$\begin{aligned}
&3R(X,Y,Z,V) + \sum_{s=1}^{3}\epsilon_s R(I_sX,I_sY,Z,V) \\
&= 2\Big[g(Y,Z)\tau(X,V) + g(X,V)\tau(Z,Y)\Big] - 2\Big[g(Z,X)\tau(Y,V) + g(V,Y)\tau(Z,X)\Big] \\
&+ 2\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(Y,Z)\tau(X,I_sV) + \omega_s(X,V)\tau(Y,I_sZ)\Big] - 2\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(X,Z)\tau(Y,I_sV) + \omega_s(Y,V)\tau(X,I_sZ)\Big] \\
&-2\sum_{s=1}^{3}\epsilon_s\Big[\omega_s(X,Y)\Big(\tau(Z,I_sV)-\tau(I_sZ,V)\Big) - 4\omega_s(Z,V)\mu(I_sX,Y)\Big] + \frac{Scal}{2n(n+2)}\sum_{s=1}^{3}\epsilon_s\omega_s(X,Y)\omega_s(Z,V).
\end{aligned} \tag{4.17}$$

Proof. The first Bianchi identity (2.15), its consequence (2.16) together with the help of Proposition 2.2 and (2.5) imply the identity (4.16) in a straightforward way.

Set $A = Z, B = V$ in (2.17) and take a cyclic sum to get

$$\begin{aligned}
\sum_{s=1}^{3}\epsilon_s R(Z,V,I_sX,I_sY) + 3R(Z,V,X,Y) &= -4\sum_{s=1}^{3}\epsilon_s\rho_j(Z,V)\omega_s(X,Y) \\
&= \sum_{s=1}^{3}\epsilon_s\Big[\frac{Scal}{2n(n+2)}\omega_s(Z,V) + 2\tau(I_sZ,V) - 2\tau(Z,I_sV) - 8\mu(Z,I_sV)\Big]\omega_s(X,Y).
\end{aligned} \tag{4.18}$$

where we used (2.20) to obtain the last equality. Furthermore, we have applying (4.18)

$$
\begin{aligned}
(4.19)\quad & 3R(X,Y,Z,V)+\sum_{s=1}^{3}\epsilon_s R(I_sX,I_sY,Z,V)=3R(Z,V,X,Y)+3Zam(X,Y,Z,V)\\
&+\sum_{s=1}^{3}\epsilon_s\Big[R(Z,V,I_sX,I_sY)+Zam(I_sX,I_sY,Z,V)\Big]=3R(Z,V,X,Y)+3Zam(X,Y,Z,V)-3R(Z,V,X,Y)\\
&+\sum_{s=1}^{3}\epsilon_s\Big[\frac{Scal}{2n(n+2)}\omega_s(Z,V)+2\tau(I_sZ,V)-2\tau(Z,I_sV)-8\mu(Z,I_sV)\Big]\omega_s(X,Y)+\sum_{s=1}^{3}\epsilon_s Zam(I_sX,I_sY,Z,V)\\
&=3Zam(X,Y,Z,V)+\sum_{s=1}^{3}\epsilon_s Zam(I_sX,I_sY,Z,V)\\
&+\sum_{s=1}^{3}\epsilon_s\Big[\frac{Scal}{2n(n+2)}\omega_s(Z,V)+2\tau(I_sZ,V)-2\tau(Z,I_sV)-8\mu(Z,I_sV)\Big]\omega_s(X,Y).
\end{aligned}
$$

Next, after substituting (4.16) into (4.19) we obtain (4.17) by series of straightforward calculations. □

Comparing (4.11) with (4.17) we obtain the next

Proposition 4.4. *On a pqc manifold the [-1]-part with respect to the first two arguments of the tensor PWR vanishes identically,*

$$PWR_{[-1]}(X,Y,Z,V)=\frac{1}{4}\Big[3PWR(X,Y,Z,V)+\sum_{s=1}^{3}\epsilon_s PWR(I_sX,I_sY,Z,V)\Big]=0.$$

The [3]-component with respect to the first two arguments of the tensor PWR is determined completely by the torsion and the pqc scalar curvature as follows

$$
\begin{aligned}
(4.20)\quad & PWR_{[3]}(X,Y,Z,V)=\frac{1}{4}\Big[PWR(X,Y,Z,V)-\sum_{s=1}^{3}\epsilon_s PWR(I_sX,I_sY,Z,V)\Big]\\
&=\frac{1}{4}\Big[R(X,Y,Z,V)-\sum_{s=1}^{3}\epsilon_s R(I_sX,I_sY,Z,V)\Big]+\frac{1}{2}\sum_{s=1}^{3}\epsilon_s\omega_s(Z,V)\Big[\tau(X,I_sY)-\tau(I_sX,Y)\Big]\\
&+\frac{Scal}{32n(n+2)}\Big[(g\owedge g)(X,Y,Z,V)-\sum_{s=1}^{3}\epsilon_s(\omega_s\owedge\omega_s)(X,Y,Z,V)\Big]\\
&+(g\owedge\mu)(X,Y,Z,V)-\sum_{s=1}^{3}\epsilon_s(\omega_s\owedge I_s\mu)(X,Y,Z,V).
\end{aligned}
$$

Definition 4.5. *We denote the [3]-part of the tensor PWR described in (4.20) by W^{pqc}, $W^{pqc}:=PWR_{[3]}$ and call it the para quaternionic contact conformal curvature.*

4.4. **Proof of Theorem 1.1.** The significance of the tensor PWR is partially justified by the following

Theorem 4.6. *The tensor PWR is invariant under pqc conformal transformations, i.e. if*

$$\bar\eta=(2h)^{-1}\Psi\eta \quad \textit{then} \qquad 2hPWR_{\bar\eta}=PWR_{\eta},$$

for any smooth positive function h and any $SO(1,2)$-matrix Ψ.

Proof. After a series of standard computation based on (4.1), (4.2), (4.7) and a careful study of the structure of the obtained equation we put it in the following form

$$
\begin{aligned}
(4.21)\quad 2hg(\bar R(X,Y)Z,V) - g(R(X,Y)Z,V) &= -g \owedge M(X,Y,Z,V) + \sum_{s=1}^{3}\epsilon_s\omega_s \owedge (I_sM)(X,Y,Z,V)\\
&- \frac{1}{2}\sum_{(i,j,k)}\epsilon_i\omega_i(X,Y)\Big[M(Z,I_iV) - M(I_iZ,V) - \epsilon_i M(I_jZ,I_kV) + \epsilon_i M(I_kZ,I_jV)\Big]\\
&- g(Z,V)\Big[M(X,Y) - M(Y,X)\Big] - \sum_{s=1}^{3}\epsilon_s\omega_s(Z,V)\Big[M(X,I_sY) - M(Y,I_sX)\Big]\\
&+ \frac{1}{2n}(trM)\sum_{s=1}^{3}\epsilon_s\omega_s(X,Y)\omega_s(Z,V) + \frac{1}{2n}\sum_{(i,j,k)}M_i\Big[\omega_j(X,Y)\omega_k(Z,V) - \omega_k(X,Y)\omega_j(Z,V)\Big],
\end{aligned}
$$

where the (0,2) tensor M is given by

$$
(4.22)\quad M(X,Y) = \frac{1}{2h}\Big(\nabla^2 h(X,Y) - \frac{1}{2h}\Big[dh(X)dh(Y) - \sum_{s=1}^{3}\epsilon_s dh(I_sX)dh(I_sY) + \frac{1}{2}g(X,Y)|\nabla h|^2\Big]\Big),
$$

and $trM = M(e_a,e_a), M_s = M(e_a,I_se_a)$ are its traces. Using (4.22) and (4.4), we obtain

$$
(4.23)\qquad trM = (2h)^{-1}\Big(\triangle_h h - (n+2)h^{-1}|\nabla h|^2\Big), \qquad M_s = -2n\, h^{-1}dh(\xi_s).
$$

After taking the traces in (4.21), using (4.22) and the fact that the [3]-component $(\nabla^2 h)_{[3]}$ of the horisontal hessian $\nabla^2 h$ is symmetric, we obtain

$$
(4.24)\qquad \overline{Ric} - Ric = 4(n+1)M_{[sym]} + 6M_{[3]} + \frac{2n+3}{2n}trM\, g, \qquad \frac{\overline{Scal}}{2h} - Scal = 8(n+2)trM.
$$

The $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-invariant, [-1] and [3], parts of (4.24) are

$$
(4.25)\qquad (\overline{Ric} - Ric)_{[-1]} = 4(n+1)M_{[sym][-1]}, \quad (\overline{Ric} - Ric)_{[3]} = 2(2n+5)M_{[3]} + \frac{2n+3}{2n}(tr\, M)\, g.
$$

The identities in Theorem 2.4 together with equations (4.24), (4.25) and (4.8) yield

$$
\begin{aligned}
(4.26)\quad M_{[sym]} &= \Big(\frac{1}{4(n+1)}\overline{Ric}_{[-1]} + \frac{1}{2(2n+5)}\overline{Ric}_{[3]} - \frac{2n+3}{32n(n+2)(2n+5)}\overline{Scal}\,\overline{g}\Big)\\
&\quad - \Big(\frac{1}{4(n+1)}Ric_{[-1]} + \frac{1}{2(2n+5)}Ric_{[3]} - \frac{2n+3}{32n(n+2)(2n+5)}Scal\, g\Big)\\
&= \Big[\frac{1}{2}\overline{\tau} + \overline{\mu} + \frac{\overline{Scal}}{32n(n+2)}\,\overline{g}\Big] - \Big[\frac{1}{2}\tau + \mu + \frac{Scal}{32n(n+2)}\, g\Big] = \bar L - L.
\end{aligned}
$$

Next, from (4.22) and (4.4) we obtain

$$
(4.27)\qquad M(X,Y) = M_{[sym]}(X,Y) + \frac{1}{2h}\sum_{s=1}^{3}\epsilon_s dh(\xi_s)\omega_s(X,Y).
$$

Substitute (4.26) into (4.27), insert the obtained equality into (4.21) and use (4.23) to complete the proof of Theorem 4.6. □

At this point, a combination of Theorem 4.6 and Proposition 4.4 ends the proof of Theorem 1.1.

The second equality in (4.24) combined with the first equation in (4.23) yield

Corollary 4.7. *On a pqc manifold the pqc scalar curvature transforms under the pqc conformal transformation $\bar\eta = (2h)^{-1}\Psi\eta$ according to the equation*

$$
(4.28)\qquad \overline{Scal} = 2hScal - 8(n+2)^2h^{-1}|\nabla h|^2 + 8(n+2)\triangle_h h.
$$

The equation (4.28) constitutes the sub-hyperbolic Yamabe equation.

5. Converse problem. Proof of Theorem 1.2

Suppose $W^{pqc} = 0$, hence $PWR = 0$ by Proposition 4.4. In order to prove Theorem 1.2 we search for a conformal factor, such that after a pqc conformal transformation using this factor the new pqc structure has flat pqc canonical connection, when restricted to the horizontal space H. After we achieve this task we can invoke Proposition 2.6 and conclude that the given structure is locally pqc conformal to the flat pqc structure on the para quaternionic Heisenberg group $G(pH)$.

With this considerations in mind, it is then sufficient to find (locally) a solution h of equation (4.27) with $M_{[sym]} = -L$. In fact, a substitution of (4.27) in (4.21) and an application of the condition $W^{pqc} = 0 = PWR$ allows us to see that the pqc structure $\bar\eta = \frac{1}{2h}\eta$ has flat pqc canonical connection.

Let us consider the following overdetermined system of partial differential equations with respect to an unknown function u

$$\nabla^2 u(X,Y) = -du(X)du(Y) - \sum_{s=1}^{3}\epsilon_s\Big[du(I_sX)du(I_sY) - du(\xi_s)\omega_s(X,Y)\Big] + \frac{1}{2}g(X,Y)|\nabla u|^2 - L(X,Y), \tag{5.1}$$

$$\nabla^2 u(X,\xi_i) = \mathbb{B}(X,\xi_i) - L(X,I_idu) + \frac{1}{2}du(I_iX)|\nabla u|^2 - du(X)du(\xi_i) + \epsilon_i du(I_jX)du(\xi_k) - \epsilon_i du(I_kX)du(\xi_j), \tag{5.2}$$

$$\nabla^2 u(\xi_i,\xi_i) = -\mathbb{B}(\xi_i,\xi_i) + \mathbb{B}(I_idu,\xi_i) - \epsilon_i\frac{1}{4}|\nabla u|^4 - (du(\xi_i))^2 - \epsilon_k(du(\xi_j))^2 - \epsilon_j(du(\xi_k))^2, \tag{5.3}$$

$$\nabla^2 u(\xi_j,\xi_i) = -\mathbb{B}(\xi_j,\xi_i) + \mathbb{B}(I_idu,\xi_j) - 2du(\xi_i)du(\xi_j) + \epsilon_k\frac{Scal}{16n(n+2)}du(\xi_k), \tag{5.4}$$

$$\nabla^2 u(\xi_k,\xi_i) = -\mathbb{B}(\xi_k,\xi_i) + \mathbb{B}(I_idu,\xi_k) - 2du(\xi_i)du(\xi_k) - \epsilon_j\frac{Scal}{16n(n+2)}du(\xi_j). \tag{5.5}$$

Here the tensor L is given by (4.8), while the tensors $\mathbb{B}(X,\xi_i)$ and $\mathbb{B}(\xi_i,\xi_j)$ do not depend on the unknown function u and will be determined later in (5.11) and (5.28), respectively. If we make the substitution

$$2u = \ln h, \qquad 2hdu = dh, \qquad \nabla^2 h = 2h\nabla^2 u + 4hdu\otimes du,$$

in (4.22) we recognize that (4.27) transforms into (5.1). Therefore, it is sufficient to show that the system (5.1)-(5.5) admits (locally) a smooth solution.

The integrability condition of the over-determined system (5.1)-(5.5) is the Ricci identity

$$\nabla^3 u(A,B,C) - \nabla^3 u(B,A,C) = -R(A,B,C,du) - \nabla^2 u((T(A,B)),C), \quad A,B,C\in\Gamma(TM). \tag{5.6}$$

The proof of Theorem 1.2 will be achieved by considering all possible cases of (5.6). It will be presented as a sequel of subsections, which occupy the rest of this section. The goal is to show that the vanishing of the pqc conformal tensor W^{pqc} implies the validity of (5.6), which guaranties the existence of a local smooth solution to the system (5.1)-(5.5).

We start with the next Lemma, which is an application of a standard result in differential geometry.

Lemma 5.1. *In a neighborhood of any point $p\in M^{4n+3}$ and a $p\mathbb{Q}$-orthonormal basis*

$$\{X_1(p), X_2(p) = I_1X_1(p)\dots, X_{4n}(p) = I_3X_{4n-3}(p), \xi_1(p), \xi_2(p), \xi_3(p)\}$$

of the tangential space at p, there exists a $p\mathbb{Q}$ - orthonormal frame field

$$\{X_1, X_2 = I_1X_1, \dots, X_{4n} = I_3X_{4n-3}, \xi_1,\xi_2,\xi_3\},\ X_{a_{|p}} = X_a(p), \xi_{s_{|p}} = \xi_s(p),$$

such that the connection 1-forms of the canonical connection are all zero at the point p, i.e., we have

$$(\nabla_{X_a}X_b)_{|}p = (\nabla_{\xi_i}X_b)_{|}p = (\nabla_{X_a}\xi_t)_{|}p = (\nabla_{\xi_t}\xi_s)_{|}p = 0, \tag{5.7}$$

for $a, b = 1, \dots, 4n$, $s, t, r = 1, 2, 3$. *In particular,*

$$((\nabla_{X_a}I_s)X_b)_{|}p = ((\nabla_{X_a}I_s)\xi_t)_{|}p = ((\nabla_{\xi_t}I_s)X_b)_{|}p = ((\nabla_{\xi_t}I_s)\xi_r)_{|}p = 0.$$

Proof. Since ∇ preserves the splitting $H \oplus V$ we can apply the standard arguments for the existence of a normal frame with respect to a metric connection (see e.g. [17]). We sketch the proof for completeness.

Let $\{\tilde X_1, \dots, \tilde X_{4n}, \tilde\xi_1, \tilde\xi_2, \tilde\xi_3\}$ be a $p\mathbb{Q}$-orthonormal basis around p such that $\tilde X_{a|p} = X_a(p)$, $\quad \tilde\xi_{i|p} = \xi_i(p)$. We want to find a modified frame $X_a = o_a^b \tilde X_b$, $\quad \xi_i = o_i^j \tilde\xi_j$, which satisfies the normality conditions of the lemma.

Let ϖ be the $sp(n,\mathbb{R}) \oplus sp(1,\mathbb{R})$-valued connection 1-forms with respect to the frame $\{\tilde X_1, \dots, \tilde X_{4n}, \tilde\xi_1, \tilde\xi_2, \tilde\xi_3\}$,

$$\nabla \tilde X_b = \varpi_b^c \tilde X_c, \quad \nabla \tilde\xi_s = \varpi_s^t \tilde\xi_t, \quad B \in \{\tilde X_1, \dots, \tilde X_{4n}, \tilde\xi_1, \tilde\xi_2, \tilde\xi_3\}.$$

Let $\{x^1, \dots, x^{4n+3}\}$ be a coordinate system around p, such that

$$\frac{\partial}{\partial x^a}(p) = X_a(p), \quad \frac{\partial}{\partial x^{4n+t}}(p) = \xi_t(p), \quad a = 1, \dots, 4n, \quad t = 1, 2, 3.$$

One can easily check that the matrices with entries

$$o_a^b = exp\left(- \sum_{c=1}^{4n+3} \varpi_a^b (\frac{\partial}{\partial x^c})_{|p} x^c \right) \in Sp(n,\mathbb{R}), \; o_t^s = exp\left(- \sum_{c=1}^{4n+3} \varpi_t^s (\frac{\partial}{\partial x^c})_{|p} x^c \right) \in Sp(1,\mathbb{R})$$

are the desired matrices making the identities (5.7) true.

Next, the last identity in the lemma is a consequence of the fact that the choice of the orthonormal basis of V does not depend on the action of $SO(1,2)$ on V combined with (2.6). □

Definition 5.2. *We refer to the orthonormal frame constructed in Lemma 5.1 as a* pqc-normal frame.

Since (5.6) is $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-invariant it is sufficient to check it in a pqc-normal frame.

From now on the frame $\{e_1, e_2 = I_1 e_1, e_3 = I_2 e_1, e_4 = I_3 e_1 \dots, e_{4n} = I_3 e_{4n-3}, \xi_1, \xi_2, \xi_3\}$ is a fixed pqc-normal frame at a fixed point $p \in M$.

5.1. Case 1, $X, Y, Z \in H$. Integrability condition (5.10).

The equation (5.6) on H takes the form

$$\begin{aligned} \nabla^3 u(Z,X,Y) - \nabla^3 u(X,Z,Y) &= -R(Z,X,Y,du) \\ &+ 2\omega_1(Z,X)\nabla^2 u(\xi_1,Y) + 2\omega_2(Z,X)\nabla^2 u(\xi_2,Y) - 2\omega_3(Z,X)\nabla^2 u(\xi_3,Y), \end{aligned} \tag{5.8}$$

where we have used (2.5). The identity $d^2u = 0$ gives

$$\nabla^2 u(X,\xi_s) - \nabla^2 u(\xi_s,X) = du(T(\xi_s,X)) = T(\xi_s,X,du). \tag{5.9}$$

After we take a covariant derivative of (5.1) along $Z \in H$, substitute the derivatives from (5.1) and (5.2), then anti-commute the covariant derivatives, substitute the result in (5.8), use (4.10) with $PWR = 0$, (5.9), (4.8) and the properties of the torsion described in Proposition 2.2, we obtain by series of standard calculations, that the integrability condition in this case is

$$\begin{aligned} &(\nabla_Z L)(X,Y) - (\nabla_X L)(Z,Y) \\ &= -\sum_{s=1}^{3} \epsilon_s \Big[\omega_s(Z,Y)\mathbb{B}(X,\xi_s) - \omega_s(X,Y)\mathbb{B}(Z,\xi_s) + 2\omega_s(Z,X)\mathbb{B}(Y,\xi_s) \Big]. \end{aligned} \tag{5.10}$$

Now, we determine the tensors $\mathbb{B}(X,\xi_s)$. The traces in (5.10) give the next sequence of equalities

$$(5.11)\qquad \begin{aligned} &(\nabla_{e_a}L)(I_ie_a,I_iX)=(4n+1)\mathbb{B}(I_iX,\xi_i)+\epsilon_k\mathbb{B}(I_jX,\xi_j)+\epsilon_j\mathbb{B}(I_kX,\xi_k);\\ &\sum_{s=1}^3\epsilon_s\mathbb{B}(I_sX,\xi_s)=\frac13\Big[(\nabla_{e_a}L)(e_a,X)-\nabla_X tr\,L\Big]=\frac{1}{4n-1}\sum_{s=1}^3\epsilon_s(\nabla_{e_a}L)(I_se_a,I_sX);\\ &\mathbb{B}(X,\xi_i)=\frac{1}{2(2n+1)}\Big[(\nabla_{e_a}L)(I_ie_a,X)+\frac13\Big((\nabla_{e_a}L)(e_a,I_iX)-\nabla_{I_iX}tr\,L\Big)\Big],\end{aligned}$$

where the second equality in (5.11) is precisely equivalent to (2.30).

Lemma 5.3. *The condition* (5.10) *is equivalent to*

$$(5.12)\qquad (\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)=0\qquad mod\quad g,\omega_1,\omega_2,\omega_3.$$

Proof. Observe the cyclic sum $\sum_{(Z,X,Y)}[(\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)]=0$. The condition (5.12) implies

$$(5.13)\qquad \begin{aligned}(\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)&=g(Z,Y)C(X)-g(X,Y)C(Z)\\ &-\sum_{s=1}^3\epsilon_s\Big[\omega_s(Z,Y)\mathbb{B}(X,\xi_s)-\omega_s(X,Y)\mathbb{B}(Z,\xi_s)+2\omega_s(Z,X)B(Y,\xi_s)\Big],\end{aligned}$$

for some tensors $C(X),B(X,\xi_s)$. The traces in (5.13) yield

$$(\nabla_{e_a}L)(I_ie_a,I_iX)=(4n+1)\mathbb{B}(I_iX,\xi_i)+\epsilon_k\mathbb{B}(I_jX,\xi_j)+\epsilon_j\mathbb{B}(I_kX,\xi_k)+\epsilon_iC(X)\quad \text{implying}$$

$$\sum_{s=1}^3\epsilon_s(\nabla_{e_a}L)(I_se_a,I_sX)=\sum_{s=1}^3\epsilon_s(4n-1)\mathbb{B}(I_sX,\xi_s)+3C(X).$$

The latter together with the second equality in (5.11) shows $C(X)=0$. ∎

Proposition 5.4. *If $W^{pqc}=0$ then the condition* (5.10) *holds.*

Proof. The second Bianchi identity

$$(5.14)\qquad \sum_{(A,B,C)}\Big\{(\nabla_A R)(B,C,D,E)+R(T(A,B),C,D,E)\Big\}=0,$$

combined with (2.5) yields

$$(5.15)\qquad \sum_{(X,Y,Z)}\Big[(\nabla_X R)(Y,Z,V,W)-2\sum_{s=1}^3\epsilon_s\omega_s(X,Y)R(\xi_s,Z,V,W)\Big]=0.$$

The trace in (5.15) leads to

$$(5.16)\qquad \begin{aligned}(\nabla_{e_a}R)(X,Y,Z,e_a)&=(\nabla_X Ric)(Y,Z)-(\nabla_Y Ric)(X,Z)\\ &+2\sum_{s=1}^3\epsilon_s\Big[R(\xi_s,Y,Z,I_sX)-R)\xi_s,X,Z,I_sY)+\omega_s(X,Y)Ric(\xi_s,Z)\Big].\end{aligned}$$

We use $W^{pqc}=0$, (5.16) and apply (4.10) to calculate

$$(5.17)\qquad (\nabla_{e_a}R)(X,Y,Z,e_a)=-(\nabla_Y L)(X,Z)+(\nabla_X L)(Y,Z)$$
$$+\sum_{s=1}^3\epsilon_s\Big[(\nabla_{I_sX}L)(Y,I_sZ)-(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_{I_sZ}L)(X,I_sY)+(\nabla_{I_sZ}L)(I_sX,Y)\Big]\quad mod\quad g,\omega_s.$$

By substituting (4.12), (4.13) into (2.26) we get

$$
\begin{aligned}
(5.18)\quad & 2\sum_{s=1}^{3}\epsilon_s\Big[R(\xi_s,Y,Z,I_sX)-R(\xi_s,X,Z,I_sY)\Big]\\
&=-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)+(\nabla_{I_sY}L)(I_sX,Z)-(\nabla_{I_sX}L)(I_sY,Z)\Big]\\
&+\frac{3}{2}\sum_{s=1}^{3}\Big[(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)-\epsilon_s(\nabla_YL)(I_sX,I_sZ)+\epsilon_s(\nabla_XL)(I_sY,I_sZ)\Big]\quad mod\quad g,\omega_s.
\end{aligned}
$$

The second Bianchi identity gives $\sum_{(X,Y,Z)}(\nabla_X\rho_i)(Y,Z)=0 \quad mod \quad g,\omega_s$. Use (4.15) to see

$$
\begin{aligned}
(5.19)\quad & 3\Big((\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)\Big)-\sum_{s=1}^{3}\epsilon_s\Big((\nabla_YL)(I_sX,I_sZ)-(\nabla_XL)(I_sY,I_sZ)\Big)\\
&-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sZ}L)(X,I_sY)-(\nabla_{I_sZ}L)(I_sX,Y)\Big]=0\quad mod\quad g,\omega_s.
\end{aligned}
$$

A substitution of (5.17), (5.18), (5.19) and (4.15) in (5.16) shows, after standard calculations, that

$$
\begin{aligned}
(5.20)\quad & (4n+3)\Big[(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)\Big]-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sY}L)(I_sX,Z)-(\nabla_{I_sX}L)(I_sY,Z)\Big]\\
&-2\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_YL)(I_sX,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)+(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_XL)(I_sY,I_sZ)\Big]=0\quad mod\quad g,\omega_s.
\end{aligned}
$$

Taking the [3]-component with respect to X,Y in (5.20), we obtain

$$
(5.21)\quad (\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sY}L)(I_sX,Z)-(\nabla_{I_sX}L)(I_sY,Z)\Big]=0\quad mod\quad g,\omega_s.
$$

A substitution of (5.21) in (5.20) gives

$$
\begin{aligned}
(5.22)\quad & 2n\Big[(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)\Big]-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_XL)(I_sY,I_sZ)\Big]\\
&+(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_YL)(I_sX,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)\Big]=0\quad mod\quad g,\omega_s.
\end{aligned}
$$

Taking the [-1]-component with respect to X,Z of (5.22), calculated with the help of (5.21), yields

$$
\begin{aligned}
(5.23)\quad & (6n-1)\Big[(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)\Big]-4\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_{I_sY}L)(X,I_sZ)-(\nabla_XL)(I_sY,I_sZ)\Big]\\
&+(2n+1)\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_YL)(I_sX,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)\Big]=0\quad mod\quad g,\omega_s.
\end{aligned}
$$

The equations (5.22) and (5.23) lead to

$$
(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)-\sum_{s=1}^{3}\epsilon_s\Big[(\nabla_YL)(I_sX,I_sZ)-(\nabla_{I_sX}L)(Y,I_sZ)\Big]=0\quad mod\quad g,\omega_s.
$$

The latter and (5.22) imply

$$
(5.24)\qquad (2n-1)\Big[(\nabla_YL)(X,Z)-(\nabla_XL)(Y,Z)\Big]=0\quad mod\quad g,\omega_s
$$

and Lemma 5.3 completes the proof of (5.10). □

5.2. **Case 2,** $Z, X \in H, \quad \xi_i \in V$**. Integrability condition (5.27).**
In this case (5.6) turns into

$$\begin{aligned}(5.25)\quad \nabla^3 u(Z,X,\xi_i) - \nabla^3 u(X,Z,\xi_i) &= -R(Z,X,\xi_i,du) - \nabla^2 u(T(Z,X),\xi_i) = \\ &- 2\epsilon_i du(\xi_j)\rho_k(Z,X) + 2\epsilon_i du(\xi_k)\rho_j(Z,X) \\ &+ 2\epsilon_i\omega_i(Z,X)\nabla^2 u(\xi_i,\xi_i) + 2\epsilon_j\omega_j(Z,X)\nabla^2 u(\xi_j,\xi_i) + 2\epsilon_k\omega_k(Z,X)\nabla^2 u(\xi_k,\xi_i),\end{aligned}$$

after using (2.5) and (2.18). Take a covariant derivative of (5.2) along $Z \in H$, substitute into the obtained equality (5.1) and (5.2) and anti-commute the covariant derivatives to get

$$\begin{aligned}(5.26)\quad \nabla^3 u(Z,X,\xi_i) - \nabla^3 u(X,Z,\xi_i) &= (\nabla_Z\mathbb{B})(X,\xi_i) - (\nabla_X\mathbb{B})(Z,\xi_i) \\ &- (\nabla_Z L)(X,I_i du) + (\nabla_X L)(Z,I_i du) - L(X,\nabla_Z I_i du) + L(Z,\nabla_X I_i du) \\ &+ other \quad terms \quad comming \quad from \quad the \quad use \quad of \quad (5.1) \quad and \quad (5.2).\end{aligned}$$

Next, we substitute (5.26) into (5.25), use (5.10), already proved in Proposition 5.4, together with (5.3), (5.4), (5.5) and the second equation in (4.15), perform some basic calculations, we arrive at

$$(5.27)\qquad (\nabla_Z\mathbb{B})(X,\xi_t) - (\nabla_X\mathbb{B})(Z,\xi_t) - L(Z,I_tL(X)) + L(X,I_tL(Z)) = -2\sum_{s=1}^{3}\epsilon_s\mathbb{B}(\xi_s,\xi_t)\omega_s(Z,X),$$

which is the integrability condition in this case. The functions $\mathbb{B}(\xi_s,\xi_t)$ are uniquely determined by

$$(5.28)\qquad \mathbb{B}(\xi_s,\xi_t) = \frac{1}{4n}\Big[(\nabla_{e_a}\mathbb{B})(I_s e_a,\xi_t) + L(e_a,e_b)L(I_t e_a,I_s e_b)\Big].$$

Proposition 5.5. *If $W^{qc} = 0$ then the condition (5.27) holds.*

Proof. To prove the assertion it is sufficient to show that the left hand side of (5.27) vanishes $mod \quad \omega_s$. Differentiating (5.10) and taking the corresponding traces yields

$$\begin{aligned}(5.29)\quad (\nabla^2_{e_a,I_ie_a}L)(X,Y) - (\nabla^2_{e_a,X}L)(I_ie_a,Y) &= -(\nabla_Y\mathbb{B})(X,\xi_i) - 2(\nabla_X\mathbb{B})(Y,\xi_i) \\ &- \epsilon_i\Big[(\nabla_{I_kY}\mathbb{B})(X,\xi_j) + 2(\nabla_{I_kX}\mathbb{B})(Y,\xi_j) - (\nabla_{I_jY}\mathbb{B})(X,\xi_k) - 2(\nabla_{I_jX}\mathbb{B})(Y,\xi_k)\Big];\end{aligned}$$

$$\begin{aligned}(5.30)\quad (\nabla^2_{e_a,X}L)(I_ie_a,Y) - (\nabla^2_{e_a,Y}L)(I_ie_a,X) &= (\nabla_X\mathbb{B})(Y,\xi_i) - (\nabla_Y\mathbb{B})(X,\xi_i) \\ &- \epsilon_i\Big[(\nabla_{I_kY}\mathbb{B})(X,\xi_j) - (\nabla_{I_kX}\mathbb{B})(Y,\xi_j) - (\nabla_{I_jY}\mathbb{B})(X,\xi_k) + (\nabla_{I_jX}\mathbb{B})(Y,\xi_k)\Big] \quad mod \quad \omega_s;\end{aligned}$$

$$(5.31)\quad (\nabla^2_{X,e_a}L)(I_ie_a,Y) = (4n+1)(\nabla_X\mathbb{B})(Y,\xi_i) + \epsilon_i(\nabla_X\mathbb{B})(I_kY,\xi_j) - \epsilon_i(\nabla_X\mathbb{B})(I_jY,\xi_k) \quad mod \quad \omega_s;$$

$$(5.32)\quad -\nabla^2_{X,I_iY}tr\,L + (\nabla^2_{X,e_a}L)(e_a,I_iY) = 3(\nabla_X\mathbb{B})(Y,\xi_i) - 3\epsilon_i(\nabla_X\mathbb{B})(I_kY,\xi_j) + 3\epsilon_i(\nabla_X\mathbb{B})(I_jY,\xi_k).$$

We obtain from equalities (5.30) and (5.31) that

$$\begin{aligned}(5.33)\quad \Big[\nabla^2_{X,e_a} - \nabla^2_{e_a,X}\Big]L(I_ie_a,Y) + \Big[\nabla^2_{e_a,Y} - \nabla^2_{Y,e_a}\Big]L(I_ie_a,X) \\ = 4n\Big[(\nabla_X\mathbb{B})(Y,\xi_i) - (\nabla_Y\mathbb{B})(X,\xi_i)\Big] \\ + \epsilon_i\Big[(\nabla_X\mathbb{B})(I_kY,\xi_j) + (\nabla_{I_kY}\mathbb{B})(X,\xi_j) - (\nabla_Y\mathbb{B})(I_kX,\xi_j) - (\nabla_{I_kX}\mathbb{B})(Y,\xi_j)\Big] \\ - \epsilon_i\Big[(\nabla_X\mathbb{B})(I_jY,\xi_k) + (\nabla_{I_jY}\mathbb{B})(X,\xi_k) - (\nabla_Y\mathbb{B})(I_jX,\xi_k) - (\nabla_{I_jX}\mathbb{B})(Y,\xi_k)\Big] \quad mod \quad \omega_s.\end{aligned}$$

On the other hand, the Ricci identities

$$\Big[\nabla^2_{X,e_a}-\nabla^2_{e_a,X}\Big]L(I_ie_a,Y)=-R(X,e_a,Y,e_b)L(e_b,I_ie_a)-4n\zeta_i(X,e_a)L(Y,e_a)\\ +2(\nabla_{\xi_i}L)(X,Y)+2\epsilon_i(\nabla_{\xi_j}L)(I_kX,Y)-2\epsilon_i(\nabla_{\xi_k}L)(I_jX,Y), \tag{5.34}$$

the first Bianchi identity (2.15) and Proposition 2.2 imply

$$\begin{aligned}&\Big[\nabla^2_{X,e_a}-\nabla^2_{e_a,X}\Big]L(I_ie_a,Y)+\Big[\nabla^2_{e_a,Y}-\nabla^2_{Y,e_a}\Big]L(I_ie_a,X)=\\ &2\epsilon_i\Big[(\nabla_{\xi_j}L)(I_kX,Y)-(\nabla_{\xi_j}L)(X,I_kY)\Big]-2\epsilon_i\Big[(\nabla_{\xi_k}L)(I_jX,Y)-(\nabla_{\xi_k}L)(X,I_jY)\Big]\\ &+2T(\xi_i,Y,e_a)L(X,e_a)+2\epsilon_iT(\xi_j,Y,e_a)L(I_kX,e_a)-2\epsilon_iT(\xi_k,Y,e_a)L(I_jX,e_a)\\ &-2T(\xi_i,X,e_a)L(Y,e_a)-2\epsilon_iT(\xi_j,X,e_a)L(I_kY,e_a)+2\epsilon_iT(\xi_k,X,e_a)L(I_jY,e_a)\\ &-R(X,Y,e_a,e_b)L(e_b,I_ie_a)-4n[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)]\qquad mod\quad \omega_s.\end{aligned}\tag{5.35}$$

The second equality in (4.15) and a suitable contraction in the second Bianchi identity give the next two equations valid mod ω_s

$$\begin{aligned}(\nabla_{\xi_j}L)(X,I_kY)-(\nabla_{\xi_j}L)(I_kX,Y)&=(\nabla_{\xi_j}\rho_k)(X,Y)\\ &=(\nabla_X\rho_k)(\xi_j,Y)-(\nabla_Y\rho_k)(\xi_j,X)-\rho_k(T(\xi_j,X),Y)+\rho_k(T(\xi_j,Y),X);\\ (\nabla_{\xi_k}L)(X,I_jY)-(\nabla_{\xi_k}L)(I_jX,Y)&=(\nabla_{\xi_k}\rho_j)(X,Y)\\ &=(\nabla_X\rho_j)(\xi_k,Y)-(\nabla_Y\rho_j)(\xi_k,X)-\rho_j(T(\xi_k,X),Y)+\rho_j(T(\xi_k,Y),X).\end{aligned}\tag{5.36}$$

A substitution of (4.12), (4.13) into (2.28) together with (5.10) and an application of (5.11) give

Lemma 5.6. *We have the following formulas for the Ricci 2-forms*

$$\begin{aligned}&\rho_k(\xi_i,X)=-\epsilon_j\mathbb{B}(X,\xi_j)-\mathbb{B}(I_kX,\xi_i), &&\rho_i(\xi_k,X)=\epsilon_j\mathbb{B}(X,\xi_j)-\mathbb{B}(I_iX,\xi_k),\\ &\rho_i(X,\xi_i)=\epsilon_i\frac{1}{4n}d(trL)(X)+\mathbb{B}(I_iX,\xi_i).\end{aligned}\tag{5.37}$$

The identity $\nabla^2u(\xi_i,\xi_j)-\nabla^2u)\xi_j,\xi_i)=-T(\xi_i,\xi_j,du)$, (5.4), (4.8), (2.23), (2.24) and (5.37) imply

Lemma 5.7. *The tensors $B(\xi_i,\xi_j)$ are symmetric, $B(\xi_i,\xi_j)=B(\xi_j,\xi_i)$.*

When we take the covariant derivative of (5.37), substitute the obtained equalities together with (5.35), (5.36) in (5.33), we derive the formula

$$(4n+2)\Big[(\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)\Big]-\epsilon_j\Big[(\nabla_{I_jX}\mathbb{B})(I_jY,\xi_i)-(\nabla_{I_jY}\mathbb{B})(I_jX,\xi_i)\Big]\\ -\epsilon_k\Big[(\nabla_{I_kX}\mathbb{B})(I_kY,\xi_i)-(\nabla_{I_kY}\mathbb{B})(I_kX,\xi_i)\Big]=F(X,Y)\quad mod\quad \omega_s, \tag{5.38}$$

where the (0,2)-tensor F is defined by

$$\begin{aligned}F(X,Y)=&-R(X,Y,e_a,e_b)L(e_b,I_ie_a)-4n\Big[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)\Big]\\ &+2T(\xi_i,Y,e_a)L(X,e_a)+2\epsilon_i\Big[T(\xi_j,Y,e_a)L(I_kX,e_a)-T(\xi_k,Y,e_a)L(I_jX,e_a)\Big]\\ &-2T(\xi_i,X,e_a)L(Y,e_a)-2\epsilon_i\Big[T(\xi_j,X,e_a)L(I_kY,e_a)-T(\xi_k,X,e_a)L(I_jY,e_a)\Big]\\ &-\epsilon_i\Big[\rho_j(T(\xi_k,X),Y)-\rho_j(T(\xi_k,Y),X)\Big]-\epsilon_k\Big[\rho_j(T(\xi_k,I_jX),I_jY)-\rho_j(T(\xi_k,I_jY),I_jX)\Big]\\ &+\epsilon_i\Big[\rho_k(T(\xi_j,X),Y)-\rho_k(T(\xi_j,Y),X)\Big]+\epsilon_j\Big[\rho_k(T(\xi_j,I_kX),I_kY)-\rho_k(T(\xi_j,I_kY),I_kX)\Big].\end{aligned}\tag{5.39}$$

Solving for $(\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)$ from (5.38) we obtain

$$\begin{aligned}(5.40)\quad & 16n(n+1)(2n+1)\Big[(\nabla_X\mathbb{B})(Y,\xi_i)-(\nabla_Y\mathbb{B})(X,\xi_i)\Big]\\ &=(8n^2+8n+1)F(X,Y)-\epsilon_iF(I_iX,I_iY)+(2n+1)\Big[\epsilon_jF(I_jX,I_jY)+\epsilon_kF(I_kX,I_kY)\Big]\quad mod\quad \omega_s.\end{aligned}$$

The condition $W^{pqc}=0$ and (4.10) give

$$\begin{aligned}(5.41)\quad & -R(X,Y,e_a,e_b)L(I_ie_a,e_b)=4L(X,e_a)L(Y,I_ie_a)-2L(X,e_a)L(I_iY,e_a)\\ &+2L(I_iX,e_a)L(Y,e_a)-2\epsilon_i\Big[L(X,e_a)L(I_jY,I_ke_a)-L(I_kX,e_a)L(Y,I_je_a)\\ &-L(X,e_a)L(I_kY,I_je_a)+L(I_jX,e_a)L(Y,I_ke_a)\Big]-tr\,L\Big[L(X,I_iY)-L(I_iX,Y)\Big].\end{aligned}$$

Using (4.15), we obtain from (5.41)

$$\begin{aligned}(5.42)\quad & -R(X,Y,e_a,e_b)L(I_ie_a,e_b)-4n\Big[\zeta_i(X,e_a)L(Y,e_a)-\zeta_i(Y,e_a)L(X,e_a)\Big]\\ &=-(8n-1)L(X,e_a)L(Y,I_ie_a)-\frac{1}{2n}(trL)\Big[L(X,I_iY)-L(I_iX,Y)\Big]\\ &+\frac12\Big[L(Y,e_a)L(I_iX,e_a)-L(X,e_a)L(I_iY,e_a)\Big]+\frac32\epsilon_i\Big[L(Y,e_a)L(I_jX,I_ke_a)-L(X,e_a)L(I_jY,I_ke_a)\Big]\\ &-\frac32\epsilon_i\Big[L(Y,e_a)L(I_kX,I_je_a)-L(X,e_a)L(I_kY,I_je_a)\Big].\end{aligned}$$

Since ρ_s is a (1,1)-form with respect to I_s (see Proposition 2.4), we have

$$\rho_j(T(\xi_k,I_jX),I_jY)=\rho_j(e_a,I_jY)T(\xi_k,I_jX,e_a)=\rho_j(e_a,Y)T(\xi_k,I_jX,I_je_a).$$

Using (4.15) we obtain

$$\begin{aligned}(5.43)\quad & -\epsilon_i\rho_j(T(\xi_k,X),Y)-\epsilon_k\rho_j(T(\xi_k,I_jX),I_jY)+\epsilon_i\rho_k(T(\xi_j,X),Y)+\epsilon_j\rho_k(T(\xi_j,I_kX),I_kY)\\ &-2T(\xi_i,X,e_a)L(Y,e_a)-2\epsilon_i\Big[T(\xi_j,X,e_a)L(I_kY,e_a)-2T(\xi_k,X,e_a)L(I_jY,e_a)\Big]\\ &=L(e_a,Y)\Big[-\epsilon_iT(\xi_k,X,I_je_a)+\epsilon_iT(\xi_k,I_jX,e_a)+\epsilon_iT(\xi_j,X,I_ke_a)-\epsilon_iT(\xi_j,I_kX,e_a)-2T(\xi_i,X,e_a)\Big]\\ &+L(e_a,I_jY)\Big[-\epsilon_kT(\xi_k,I_jX,I_je_a)+\epsilon_iT(\xi_k,X,e_a)\Big]-L(e_a,I_kY)\Big[-\epsilon_jT(\xi_j,I_kX,I_ke_a)+\epsilon_iT(\xi_j,X,e_a)\Big]\\ &+\frac{\epsilon_i}{2n}tr\,L\Big[-T(\xi_k,X,I_jY)+T(\xi_k,I_jX,Y)-T(\xi_j,I_kX,Y)+T(\xi_j,X,I_kY)\Big]\\ &=\frac{1}{2n}tr\,L.\,L(X,I_iY)+\frac12L(Y,e_a)\Big[5L(X,I_ie_a)-L(I_iX,e_a)-\epsilon_iL(I_jX,I_ke_a)+\epsilon_iL(I_kX,I_je_a)\Big]\\ &-\epsilon_iL(X,e_a)\Big[L(I_kY,I_je_a)-L(I_jY,I_ke_a)\Big]+\epsilon_jL(I_jX,e_a)L(I_jY,I_ie_a)+\epsilon_kL(I_kX,e_a)L(I_kY,I_ie_a).\end{aligned}$$

The last four lines in (5.39) equal the skew symmetric sum of (5.43), which is given by

$$\begin{aligned}(5.44)\quad & -5L(X,e_a)L(Y,I_ie_a)-\frac12\Big[L(Y,e_a)L(I_iX,e_a)-L(X,e_a)L(I_iY,e_a)\Big]\\ &-\frac32\epsilon_i\Big[L(Y,e_a)L(I_jX,I_ke_a)-L(X,e_a)L(I_jY,I_ke_a)\Big]+\frac32\epsilon_i\Big[L(Y,e_a)L(I_kX,I_je_a)-L(X,e_a)L(I_kY,I_je_a)\Big]\\ &+\frac{1}{2n}tr\,L\Big[L(X,I_iY)-L(I_iX,Y)\Big]+2\epsilon_jL(I_jX,e_a)L(I_jY,I_ie_a)+2\epsilon_kL(I_kX,e_a)L(I_kY,I_ie_a).\end{aligned}$$

A substitution of (5.42) and (5.44) in (5.39) yields

$$(5.45)\quad F(X,Y)=2\epsilon_jL(I_jX,e_a)L(I_jY,I_ie_a)+2\epsilon_kL(I_kX,e_a)L(I_kY,I_ie_a)-4(2n+1)L(X,e_a)L(Y,I_ie_a).$$

Inserting (5.45) into (5.40) completes the proof of (5.27). □

5.3. **Case 3, $\xi \in V,\ X, Y \in H$. Integrability condition(5.49).**

In this case (5.6) reads

$$\nabla^3 u(\xi_i, X, Y) - \nabla^3 u(X, \xi_i, Y) = -R(\xi_i, X, Y, du) - \nabla^2 u(T(\xi_i, X), Y). \tag{5.46}$$

When we take a covariant derivative along a Reeb vector field of (5.1) and a covariant derivative along a horizontal direction of (5.2), use (5.2), (5.1), (5.3), (5.4), (5.5), (5.9), we obtain

$$\begin{aligned}
&\nabla^3 u(\xi_i, X, Y) - \nabla^3 u(X, \xi_i, Y) + \nabla^2 u(T(\xi_i, X), Y) \\
&= -du(I_iY)\left[\epsilon_i \mathbb{B}(I_iX, \xi_i) + \frac{1}{4n} d(tr\, L)(X)\right] - du(I_jY)\left[\epsilon_j \mathbb{B}(I_jX, \xi_i) + \epsilon_i \mathbb{B}(X, \xi_k)\right] \\
&\quad - du(I_kY)\left[\epsilon_k \mathbb{B}(I_kX, \xi_i) - \epsilon_i \mathbb{B}(X, \xi_j)\right] + g(X, Y)\mathbb{B}(du, \xi_i) \\
&\quad + \epsilon_i \omega_i(X, Y)\mathbb{B}(I_i du, \xi_i) + \epsilon_j \omega_j(X, Y)\mathbb{B}(I_j du, \xi_i) + \epsilon_k \omega_k(X, Y)\mathbb{B}(I_k du, \xi_i) \\
&\quad - du(X)\mathbb{B}(Y, \xi_i) - \epsilon_i du(I_iX)\mathbb{B}(I_iY, \xi_i) - \epsilon_j du(I_jX)\mathbb{B}(I_jY, \xi_i) - \epsilon_k du(I_kX)\mathbb{B}(I_kY, \xi_i) \\
&\quad + \frac{1}{4}\Big[(\nabla_X L)(Y, I_i du) - (\nabla_X L)(I_iY, du) + \epsilon_i (\nabla_X L)(I_kY, I_j du) - \epsilon_i (\nabla_X L)(I_jY, I_k du)\Big] \\
&\quad - (\nabla_{\xi_i} L)(X, Y) \quad - (\nabla_X \mathbb{B})(Y, \xi_i) + L(X, I_i LY) - T(\xi_i, X, LY) - T(\xi_i, Y, LX) \\
&\quad - \epsilon_i \omega_i(X, Y)\mathbb{B}(\xi_i, \xi_i) - \epsilon_j \omega_j(X, Y)\mathbb{B}(\xi_i, \xi_j) - \epsilon_k \omega_k(X, Y)\mathbb{B}(\xi_i, \xi_k).
\end{aligned} \tag{5.47}$$

On the other hand, a substitution of (4.12) and (4.13) in (2.26), an application of (5.37) together with the already proven (5.10) and (5.27) shows after a series of standard calculations that

$$\begin{aligned}
R(\xi_i, X, Y, Z) &= -\epsilon_j \mathbb{B}(I_jZ, \xi_i)\omega_j(X, Y) - \epsilon_k \mathbb{B}(I_kZ, \xi_i)\omega_k(X, Y) \\
&\quad + \omega_i(Y, Z)\left[\epsilon_i \mathbb{B}(I_iX, \xi_i) + \frac{1}{4n} d(trL)(X)\right] + \omega_j(Y, Z)\Big[\epsilon_i \mathbb{B}(X, \xi_k) + \epsilon_j \mathbb{B}(I_jX, \xi_i)\Big] \\
&\quad - \omega_k(Y, Z)\Big[\epsilon_i \mathbb{B}(X, \xi_j) - \epsilon_k \mathbb{B}(I_kX, \xi_i)\Big] + g(X, Z)\mathbb{B}(Y, \xi_i) + \epsilon_i \omega_i(X, Z)\mathbb{B}(I_iY, \xi_i) \\
&\quad + \epsilon_j \omega_j(X, Z)\mathbb{B}(I_jY, \xi_i) + \epsilon_k \omega_k(X, Z)\mathbb{B}(I_kY, \xi_i) - g(X, Y)\mathbb{B}(Z, \xi_i) - \epsilon_i \mathbb{B}(I_iZ, \xi_i)\omega_i(X, Y) \\
&\quad + \frac{1}{4}\Big[(\nabla_X L)(I_iY, Z) - (\nabla_X L)(Y, I_iZ) - \epsilon_i (\nabla_X L)(I_kY, I_jZ) + \epsilon_i (\nabla_X L)(I_jY, I_kZ)\Big].
\end{aligned} \tag{5.48}$$

In the derivation of the above equation we used (5.11) and Lemma 5.7.

After substituting equations (5.48), with $Z = du$, and (5.47) in (5.46), we obtain that the integrability condition (5.46) reduces to

$$\begin{aligned}
&(\nabla_{\xi_i} L)(X, Y) + (\nabla_X \mathbb{B})(Y, \xi_i) + L(Y, I_i L(X)) + L(T(\xi_i, X), Y) + g(T(\xi_i, Y), L(X)) \\
&\qquad = -\sum_{s=1}^{3} \epsilon_s \mathbb{B}(\xi_s, \xi_i)\omega_s(X, Y).
\end{aligned} \tag{5.49}$$

Notice that Case 3 implies Case 2 since (5.27) is the skew-symmetric part of (5.49).

Lemma 5.8. *For the vertical part of the Ricci 2-forms we have the equalities*

$$\begin{aligned}
&\rho_i(\xi_j, \xi_k) = \frac{1}{8n^2}(tr\, L)^2 + \epsilon_j \mathbb{B}(\xi_j, \xi_j) + \epsilon_k \mathbb{B}(\xi_k, \xi_k), \\
&\rho_i(\xi_i, \xi_j) = -\epsilon_i \frac{1}{4n} d(tr\, L)(\xi_j) - \epsilon_k \mathbb{B}(\xi_i, \xi_k), \qquad \rho_i(\xi_i, \xi_k) = -\epsilon_i \frac{1}{4n} d(tr\, L)(\xi_k) + \epsilon_j \mathbb{B}(\xi_i, \xi_j).
\end{aligned} \tag{5.50}$$

Proof. From the formula for the curvature (2.27) and Proposition 2.2 it follows

$$\begin{aligned}
4n\rho_i(\xi_i, \xi_k) &= -\epsilon_k (\nabla_{e_a} \rho_j)(I_je_a, \xi_k) + T(\xi_i, e_a, e_b)T(\xi_k, e_b, I_ie_a) - T(\xi_i, e_b, I_ie_a)T(\xi_k, e_a, e_b); \\
4n\rho_j(\xi_i, \xi_k) &= -(\nabla_{e_a} \rho_j)(I_ie_a, \xi_k) + T(\xi_i, e_a, e_b)T(\xi_k, e_b, I_je_a) - T(\xi_i, e_a, I_je_b)T(\xi_k, e_b, e_a).
\end{aligned}$$

Lemma 5.6 allows us to compute

$$(\nabla e_a \rho_i)(I_k e_a, \xi_j) = \epsilon_k (\nabla_{a_a} \mathbb{B})(I_k e_a, \xi_k) + \epsilon_j (\nabla_{e_a} \mathbb{B})(I_j e_a, \xi_j).$$

After a calculation in which we use the integrability condition (5.27), the preceding paragraphs imply the first equation of (5.50).

For the calculation of $\rho_i(\xi_i, \xi_k)$ we use again (5.27) to obtain

$$(\nabla_{e_a} \mathbb{B})(I_i e_a, \xi_k) = -L(I_i e_a, I_k e_b) L(e_a, e_b) + 4n \mathbb{B}(\xi_i, \xi_k).$$

Letting $A = \xi_i, B = X, C = Y, D = e_a, E = I_s e_a$ in the second Bianchi identity (5.14) we get

$$(\nabla_{\xi_i} \rho_s)(X, Y) - (\nabla_X \rho_s)(\xi_i, Y) + (\nabla_Y \rho_s)(\xi_i, X) + \rho_s(T(\xi_i, X), Y) - \rho_s(T(\xi_i, Y), X) - 2 \sum_{t=1}^{3} \epsilon_t \omega_t(X, Y) \rho_s(\xi_t, \xi_i) = 0. \tag{5.51}$$

Setting $s = i, Y = I_i X$ in (5.51), using (4.15), (4.4) with respect to the function $tr\, L$, together with Lemma 5.6 we obtain

$$\epsilon_i \Big[(\nabla_{\xi_i} L)(X, X) + (\nabla_X \mathbb{B})(X, \xi_i) \Big] - \Big[(\nabla_{\xi_i} L)(I_i X, I_i X) + (\nabla_{I_i X} \mathbb{B})(I_i X, \xi_i) \Big] = \rho_i(e_a, X) \Big[T(\xi_i, I_i X, e_a) - T(\xi_i, X, I_i e_a) \Big]. \tag{5.52}$$

We take the trace in (5.52) and use the properties of the torsion listed in Proposition 2.2 to conclude

$$2 \Big[(\nabla_{e_b} \mathbb{B})(e_b, \xi_i) + d(trL)(\xi_i) \Big] = \rho_i(e_a, e_b) \Big[T(\xi_i, I_i e_b, e_a) - T(\xi_i, e_b, I_i e_a) \Big] = 0, \tag{5.53}$$

which implies the formula for $\rho_i(\xi_i, \xi_k)$ after a short computation. Finally, with the help of (2.25) we obtain the formula for $\rho_i(\xi_i, \xi_j)$. □

Proposition 5.9. *If $W^{qc} = 0$ then the condition (5.49) holds.*

Proof. It is sufficient to consider only the symmetric part of (5.49) since its skew-symmetric part is the already established (5.27).

Setting $s = j, Y = I_j X$ in (5.51), using (4.15), Lemma 5.6, Lemma 5.8 and (5.27), we calculate

$$\Big[(\nabla_{\xi_i} L)(X, X) + (\nabla_X \mathbb{B})(X, \xi_i) \Big] - \epsilon_j \Big[(\nabla_{\xi_i} L)(I_j X, I_j X) + (\nabla_{I_j X} \mathbb{B})(I_j X, \xi_i) \Big] = -2\epsilon_i L(X, I_k e_a) L(I_j X, e_a) - \epsilon_j \rho_j(e_a, X) \Big[T(\xi_i, X, I_j e_a) - T(\xi_i, I_j X, e_a) \Big]. \tag{5.54}$$

Similarly, we take $s = k, Y = I_k X$ in (5.51), use (4.15), Lemma 5.6, Lemma 5.8 and (5.27) to obtain

$$\Big[(\nabla_{\xi_i} L)(X, X) + (\nabla_X \mathbb{B})(X, \xi_i) \Big] - \epsilon_k \Big[(\nabla_{\xi_i} L)(I_k X, I_k X) + (\nabla_{I_k X} \mathbb{B})(I_k X, \xi_i) \Big] = -2\epsilon_i L(I_k X, I_j e_a) L(X, e_a) - \epsilon_k \rho_k(e_a, X) \Big[T(\xi_i, X, I_k e_a) - T(\xi_i, I_k X, e_a) \Big].$$

We replace X with $I_i X$, subtract the obtained equality from (5.54) and add the result to (5.52) to get

$$2 \Big[(\nabla_{\xi_i} L)(X, X) + (\nabla_X \mathbb{B})(X, \xi_i) \Big] = -2\epsilon_i L(X, I_k e_a) L(I_j X, e_a) + 2\epsilon_j L(I_j X, I_j e_a) L(I_i X, e_a) + \epsilon_i \rho_i(e_a, X) \Big[T(\xi_i, I_i X, e_a) - T(\xi_i, X, I_i e_a) \Big] - \epsilon_j \rho_j(e_a, X) \Big[T(\xi_i, X, I_j e_a) - T(\xi_i, I_j X, e_a) \Big] + \rho_k(e_a, I_i X) \Big[T(\xi_i, I_j X, e_a) + \epsilon_j T(\xi_i, I_i X, I_k e_a) \Big]. \tag{5.55}$$

Next, using (4.14) and the second equality in (4.15) applied to (5.55) together with some standard calculations concludes the proof of (5.49). □

5.4. **Cases 4 and 5, $\xi_i, \xi_j \in V, X \in H$. Integrability conditions (5.61), (5.59) and (5.56).**

Case 4, $\xi_i, \xi_j \in V, \quad X \in H$. In this case (5.6) reads

$$\nabla^3 u(\xi_i, \xi_j, X) - \nabla^3 u(\xi_j, \xi_i, X) = -R(\xi_i, \xi_j, X, du) - \nabla^2 u(T(\xi_i, \xi_j), X). \tag{5.56}$$

Working as in the previous case, using (5.3),(5.4), (5.5), substituting (4.12), (4.13) (4.15) into (2.27), applying the already proven (5.10), (5.27), (5.49) and performing a series of standard calculations we conclude that (5.56) is equivalent to

$$\begin{aligned}
(\nabla_{\xi_i}\mathbb{B})(X,\xi_j) - (\nabla_{\xi_j}\mathbb{B})(X,\xi_i) &= L(X, I_j e_a)\mathbb{B}(e_a,\xi_i) - L(X, I_i e_a)\mathbb{B}(e_a,\xi_j)\\
&+ \epsilon_j L(e_a, X)\rho_k(I_i e_a, \xi_i) - T(\xi_i, X, e_a)\mathbb{B}(e_a,\xi_j) + T(\xi_j, X, e_a)\mathbb{B}(e_a,\xi_i) - \epsilon_k \frac{trL}{n}\mathbb{B}(X,\xi_k)\\
&= \Big[2L(X, I_j e_a) + T(\xi_j, X, e_a)\Big]\mathbb{B}(e_a,\xi_i) - \Big[2L(X, I_i e_a) + T(\xi_i, X, e_a)\Big]\mathbb{B}(e_a,\xi_j) - \epsilon_k \frac{trL}{n}\mathbb{B}(X,\xi_k).
\end{aligned} \tag{5.57}$$

where we used Lemma 5.6 to derive the second equality.

Case 5_a**,** $X \in H, \quad \xi_i, \xi_j \in V$. In this case (5.6) becomes

$$\begin{aligned}
\nabla^3 u(X, \xi_i, \xi_j) - \nabla^3 u(\xi_i, X, \xi_j) &= -R(X, \xi_i, \xi_j, du) + \nabla^2 u(T(\xi_i, X), \xi_j) =\\
&- 2\epsilon_j du(\xi_i)\rho_k(X,\xi_i) + 2\epsilon_j du(\xi_k)\rho_i(X,\xi_i) + T(\xi_i, X, e_a)\nabla^2 u(e_a, \xi_j).
\end{aligned} \tag{5.58}$$

With a similar calculations as in the previous cases, we see that (5.58) is equivalent to

$$\begin{aligned}
&(\nabla_{\xi_i}\mathbb{B})(X,\xi_j) + (\nabla_X\mathbb{B})(\xi_i,\xi_j)\\
&\qquad - 2L(X, I_j e_a)\mathbb{B}(e_a,\xi_i) + T(\xi_i, X, e_a)\mathbb{B}(e_a,\xi_j) + \epsilon_k \frac{trL}{2n}\mathbb{B}(X,\xi_k) = 0.
\end{aligned} \tag{5.59}$$

Case 5_b**,** $X \in H, \quad \xi_j, \xi_j \in V$. In this case (5.6) reads

$$\begin{aligned}
\nabla^3 u(X, \xi_j, \xi_j) - \nabla^3 u(\xi_j, X, \xi_j) &= -R(X, \xi_j, \xi_j, du) + \nabla^2 u(T(\xi_j, X), \xi_j) =\\
&- 2\epsilon_j du(\xi_i)\rho_k(X,\xi_j) + 2\epsilon_j du(\xi_k)\rho_i(X,\xi_j) + T(\xi_j, X, e_a)\nabla^2 u(e_a, \xi_j).
\end{aligned} \tag{5.60}$$

and (5.60) is equivalent to

$$(\nabla_{\xi_j}\mathbb{B})(X,\xi_j) + (\nabla_X B)(\xi_j,\xi_j) - 2\mathbb{B}(e_a,\xi_j)L(X, I_j e_a) + T(\xi_j, X, e_a)\mathbb{B}(e_a,\xi_j) = 0. \tag{5.61}$$

Proposition 5.10. *If $W^{qc} = 0$ then the conditions (5.61), (5.59) and (5.56) hold.*

Proof. Differentiating the already proven (5.27) and taking the corresponding traces we get

$$(\nabla^2_{X,e_a}\mathbb{B})(I_i e_a, \xi_t) + 2(\nabla_X L)(e_a, e_b)L(I_i e_a, I_t e_b) = 4n(\nabla_X\mathbb{B})(\xi_i,\xi_t); \tag{5.62}$$

$$\begin{aligned}
&(\nabla^2_{e_a,X}\mathbb{B})(I_i e_a, \xi_t) - (\nabla^2_{e_a,I_i e_a}\mathbb{B})(X,\xi_t) - 2(\nabla_{e_b} L)(X, I_t e_a)L(I_i e_b, e_a)\\
&-2(\nabla_{e_b} L)(I_i e_b, e_a)L(X, I_t e_a) = 2(\nabla_X\mathbb{B})(\xi_i,\xi_t) + 2\epsilon_i(\nabla_{I_k X}\mathbb{B})(\xi_j,\xi_t) - 2\epsilon_i(\nabla_{I_j X}\mathbb{B})(\xi_k,\xi_t).
\end{aligned} \tag{5.63}$$

Subtracting (5.63) from (5.62), we obtain

$$\begin{aligned}
&\Big[\nabla^2_{X,e_a} - \nabla^2_{e_a,X}\Big]\mathbb{B}(I_i e_a, \xi_t) + (\nabla^2_{e_a,I_i e_a}\mathbb{B})(X,\xi_t) + 2(\nabla_{e_b} L)(I_i e_b, e_a)L(X, I_t e_a)\\
&\qquad + 2\Big[(\nabla_X L)(e_a, e_b) - (\nabla_{e_b} L)(X, e_a)\Big]L(I_i e_b, I_t e_a)\\
&\qquad\qquad = 2(2n-1)(\nabla_X\mathbb{B})(\xi_i,\xi_t) - 2\epsilon_i(\nabla_{I_k X}\mathbb{B})(\xi_j,\xi_t) + 2\epsilon_i(\nabla_{I_j X}\mathbb{B})(\xi_k,\xi_t).
\end{aligned} \tag{5.64}$$

The Ricci identities and (2.18) yield

$$\begin{aligned}
&\Big[\nabla^2_{X,e_a}-\nabla^2_{e_a,X}\Big]\mathbb{B}(I_ie_a,\xi_i)=2(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+2\epsilon_i(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)-2\epsilon_i(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i) \qquad (5.65)\\
&\qquad -4n\zeta_i(X,e_a)\mathbb{B}(e_a,\xi_i)+2\epsilon_i\rho_k(X,e_a)\mathbb{B}(I_ie_a,\xi_j)-2\epsilon_i\rho_j(X,e_a)\mathbb{B}(I_ie_a,\xi_k).\\
&(\nabla^2_{e_a,I_ie_a}\mathbb{B})(X,\xi_i)=-2n\varrho_i(X,e_a)\mathbb{B}(e_a,\xi_i)-4n(\nabla_{\xi_i}\mathbb{B})(X,\xi_i). \qquad (5.66)
\end{aligned}$$

Next, we apply the already established (5.10) and use the condition $L(e_a,I_se_a)=0$ to get

$$\begin{aligned}
&\Big[(\nabla_XL)(e_a,e_b)-(\nabla_{e_b}L)(X,e_a)\Big]L(I_ie_b,I_ie_a)=-3\mathbb{B}(e_a,\xi_i)L(X,I_ie_a) \qquad (5.67)\\
&\qquad -3\epsilon_i\mathbb{B}(e_a,\xi_j)L(I_kX,I_ie_a)+3\epsilon_i\mathbb{B}(e_a,\xi_k)L(I_jX,I_ie_a).\\
&(\nabla_{e_b}L)(I_ie_b,e_a)L(X,I_ie_a)=(4n+1)\mathbb{B}(e_a,\xi_i)L(X,I_ie_a)+\epsilon_k\mathbb{B}(e_a,\xi_j)L(X,I_je_a) \qquad (5.68)\\
&\qquad +\epsilon_j\mathbb{B}(e_a,\xi_k)L(X,I_ke_a).
\end{aligned}$$

We substitute (5.68), (5.67), (5.66), (5.65) into (5.64) and obtain after some calculations that

$$\begin{aligned}
(1-2n)&\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+(\nabla_X\mathbb{B})(\xi_i,\xi_i)\Big]+\epsilon_i\Big[(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)+(\nabla_{I_kX}\mathbb{B})(\xi_j,\xi_i)\Big] \qquad (5.69)\\
&-\epsilon_i\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)+(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_i)\Big]=D_{ijk}(X),
\end{aligned}$$

where $D_{ijk}(X)$ is defined by

$$\begin{aligned}
D_{ijk}(X)=&\Big[2n\zeta_i(X,e_a)+n\varrho_i(X,e_a)-(4n-2)L(X,I_ie_a)\Big]\mathbb{B}(e_a,\xi_i) \qquad (5.70)\\
&+\epsilon_i\Big[\rho_k(X,I_ie_a)+3L(I_kX,I_ie_a)+\epsilon_jL(X,I_je_a)\Big]\mathbb{B}(e_a,\xi_j)\\
&-\epsilon_i\Big[\rho_j(X,I_ie_a)+3L(I_jX,I_ie_a)-\epsilon_kL(X,I_ke_a)\Big]\mathbb{B}(e_a,\xi_k).
\end{aligned}$$

The second Bianchi identity (5.14) taken with respect to $A=\xi_i,B=\xi_j,C=X,D=e_a,E=I_se_a$ and the formulas described in Theorem 2.4 yield

$$\begin{aligned}
&(\nabla_{\xi_i}\rho_s)(\xi_j,X)-(\nabla_{\xi_j}\rho_s)(\xi_i,X)+(\nabla_X\rho_s)(\xi_i,\xi_j) \qquad (5.71)\\
&\quad=\rho_s(T(\xi_i,X),\xi_j)-\rho_s(T(\xi_j,X),\xi_i)-\epsilon_j\rho_s(e_a,X)\rho_k(I_ie_a.\xi_i)-\epsilon_k\frac{tr\,L}{n}\rho_s(\xi_k,X).
\end{aligned}$$

Setting successively $s=1,2,3$ in (5.71), using (5.9) with respect to the function $tr\,L$ and applying Lemma 5.6 and Lemma 5.8, we obtain after some calculations

$$\begin{aligned}
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_iX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_iX,\xi_i)\Big]+\epsilon_k\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_k)+(\nabla_X\mathbb{B})(\xi_i,\xi_k)\Big]=\alpha_{ijk}(X);\\
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_jX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_jX,\xi_i)\Big]+\epsilon_k\Big[(\nabla_{\xi_j}\mathbb{B})(X,\xi_k)+(\nabla_X\mathbb{B})(\xi_j,\xi_k)\Big]=\beta_{ijk}(X); \qquad (5.72)\\
&\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]-\epsilon_i\Big[(\nabla_{\xi_i}\mathbb{B})(X,\xi_i)+(\nabla_X\mathbb{B})(\xi_i,\xi_i)\Big]\\
&\qquad -\epsilon_j\Big[(\nabla_{\xi_j}\mathbb{B})(X,\xi_j)+(\nabla_X\mathbb{B})(\xi_j,\xi_j)\Big]=\gamma_{ijk}(X),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{ijk}(X)&=\rho_i(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_i(e_a,\xi_j)T(\xi_i,X,e_a)+\epsilon_j\rho_i(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad-\epsilon_i\frac{1}{4n}d(tr\,L)(e_a)T(\xi_j,X,e_a)+\epsilon_k\frac{tr\,L}{n}\rho_i(\xi_k,X);\\
\beta_{ijk}(X)&=\rho_j(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_j(e_a,\xi_j)T(\xi_i,X,e_a)+\epsilon_j\rho_j(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad+\epsilon_j\frac{1}{4n}d(tr\,L)(e_a)T(\xi_i,X,e_a)+\epsilon_k\frac{tr\,L}{n}\rho_j(\xi_k,X);\\
\gamma_{ijk}(X)&=\rho_k(e_a,\xi_i)T(\xi_j,X,e_a)-\rho_k(e_a,\xi_j)T(\xi_i,X,e_a)+\epsilon_j\rho_k(e_a,X)\rho_k(I_ie_a,\xi_i)\\
&\qquad+\frac{1}{4n^2}(tr\,L)d(tr\,L)(X)+\epsilon_k\frac{tr\,L}{n}\rho_k(\xi_k,X).
\end{aligned}\tag{5.73}$$

Now, we solve the system consisting of (5.70) and (5.72) with the help of Lemma 5.7. We obtain

$$\begin{aligned}
&2n\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)+(\nabla_{I_jX}\mathbb{B})(\xi_k,\xi_i)\Big]\\
&\quad-\Big[(\nabla_{\xi_k}\mathbb{B})(I_iX,\xi_j)+(\nabla_{I_iX}\mathbb{B})(\xi_k,\xi_j)\Big]=-\epsilon_iD_{ijk}(X)-\epsilon_jD_{jki}(X)+(2n-1)\gamma_{ijk}(X).
\end{aligned}\tag{5.74}$$

The first two equalities in (5.72) together with (5.74) lead to

$$\begin{aligned}
&2(n+1)\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]+\Big[(\nabla_{\xi_k}\mathbb{B})(I_jX,\xi_i)-(\nabla_{\xi_i}\mathbb{B})(I_jX,\xi_k)\Big]+\\
&\qquad\Big[(\nabla_{\xi_j}\mathbb{B})(I_iX,\xi_k)-(\nabla_{\xi_k}\mathbb{B})(I_iX,\xi_j)\Big]=A_{ijk}(X),
\end{aligned}\tag{5.75}$$

where

$$A_{ijk}(X)=-\epsilon_iD_{ijk}(X)-\epsilon_jD_{jki}(X)+(2n-1)\gamma_{ijk}(X)-\epsilon_k\alpha_{ijk}(I_jX)+\epsilon_k\beta_{ijk}(I_iX).\tag{5.76}$$

Consequently, we derive easily that

$$\begin{aligned}
&2(n+2)(2n+1)\Big[(\nabla_{\xi_i}\mathbb{B})(I_kX,\xi_j)-(\nabla_{\xi_j}\mathbb{B})(I_kX,\xi_i)\Big]\\
&\qquad=(2n+3)A_{ijk}(X)-A_{jki}(X)-A_{kij}(X).
\end{aligned}\tag{5.77}$$

The second equality in (4.15) together with (4.14) and Lemma 5.6 applied to (5.73) and (5.70), followed by some standard calculations give expressions of

$$\epsilon_k\beta_{ijk}(I_iX)-\epsilon_k\alpha_{ijk}(I_jX)\quad and\quad -\epsilon_iD_{ijk}(X)-\epsilon_jD_{jki}(X)+(2n-1)\gamma_{ijk}(X)$$

in terms of L and $B(.,\xi)$, which substituted into (5.76) yields the following formula for $A_{ijk}(X)$

$$\begin{aligned}
A_{ijk}(X)&=-\frac{1}{4n}(tr\,L)\Big[(1-2n)\epsilon_i\,\mathbb{B}(I_iX,\xi_i)+(1-2n)\,\epsilon_j\mathbb{B}(I_jX,\xi_j)+(8n+6)\,\epsilon_k\,\mathbb{B}(I_kX,\xi_k)\Big]\\
&-\frac{1}{4}L(X,e_a)\Big[(2n+3)\,\epsilon_iB(I_ie_a,\xi_i)+(2n+3)\,\epsilon_jB(I_je_a,\xi_j)+2\epsilon_kB(I_ke_a,\xi_k)\Big]\\
&+\frac{1}{4}L(I_iX,e_a)\Big[(2n+3)\,\epsilon_iB(e_a,\xi_i)+(2n-3)\,B(I_ke_a,\xi_j)+4B(I_je_a,\xi_k)\Big]\\
&+\frac{1}{4}L(I_jX,e_a)\Big[-(2n-3)\,B(I_ke_a,\xi_i)+(2n+3)\,\epsilon_jB(e_a,\xi_j)-4B(I_ie_a,\xi_k)\Big]\\
&+\frac{1}{4}L(I_kX,e_a)\Big[-(10n+9)\,B(I_je_a,\xi_i)+(10n+9)\,B(I_ie_a,\xi_j)+2\epsilon_kB(e_a,\xi_k)\Big].
\end{aligned}\tag{5.78}$$

Inserting (5.78) into (5.77) and using (4.14) we arrive at the proof of (5.57). We substitute (5.57) into the first equality of (5.72) to obtain (5.59). We insert (5.59) into (5.69) to see that (5.60) holds. $\square$

5.5. **Case 6, $\xi_k,\xi_i,\xi_j\in V$. Integrability conditions (5.80) and (5.82).**

Case 6_a**,** $\xi_k,\xi_i,\xi_j\in V$. In this case the Ricci identity (5.6) becomes

$$\nabla^3u(\xi_k,\xi_i,\xi_j)-\nabla^3u(\xi_i,\xi_k,\xi_j)=-R(\xi_k,\xi_i,\xi_j,du)-\nabla^2u(T(\xi_k,\xi_i),\xi_j)$$
$$=-2\epsilon_j du(\xi_i)\rho_k(\xi_k,\xi_i)+2\epsilon_j du(\xi_k)\rho_i(\xi_k,\xi_i)-\epsilon_i\rho_j(I_ke_a,\xi_k)\nabla du(e_a,\xi_j)-\frac{\epsilon_j}{n}(tr\,L)\,\nabla^2u(\xi_j,\xi_j). \tag{5.79}$$

With the help of Lemma 5.6 and Lemma 5.8 we see after some calculations that the integrability condition (5.79) takes the form

$$(\nabla_{\xi_i}\mathbb{B})(\xi_k,\xi_j)-(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)=-\frac{1}{2n}(tr\,L\,)\left[\epsilon_i\mathbb{B}(\xi_i,\xi_i)-2\epsilon_j\mathbb{B}(\xi_j,\xi_j)+\epsilon_k\mathbb{B}(\xi_k,\xi_k)\right]$$
$$+2\mathbb{B}(e_a,\xi_i)\mathbb{B}(I_je_a,\xi_k)+\mathbb{B}(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_j)+\mathbb{B}(I_ie_a,\xi_k)\mathbb{B}(e_a,\xi_j). \tag{5.80}$$

Case 6_b**,** $\xi_k,\xi_j,\xi_j\in V$. In this case, equation (5.6) reads

$$\nabla^3u(\xi_k,\xi_j,\xi_j)-\nabla^3u(\xi_j,\xi_k,\xi_j)=-R(\xi_k,\xi_j,\xi_j,du)-\nabla^2u(T(\xi_k,\xi_j),\xi_j)$$
$$=-2\epsilon_j du(\xi_i)\rho_k(\xi_k,\xi_j)+2\epsilon_j du(\xi_k)\rho_i(\xi_k,\xi_j)+\epsilon_j\rho_i(I_ke_a,\xi_k)\nabla du(e_a,\xi_j)+\epsilon_i\frac{(tr\,L)}{n}\,\nabla^2u(\xi_i,\xi_j). \tag{5.81}$$

After a series of straightforward calculation similar as above, we show that (5.81) is equivalent to

$$(\nabla_{\xi_j}\mathbb{B})(\xi_k,\xi_j)-(\nabla_{\xi_k}\mathbb{B})(\xi_j,\xi_j)=3\mathbb{B}(I_je_a,\xi_k)\mathbb{B}(e_a,\xi_j)-\frac{3\epsilon_i}{2n}(tr\,L)\,\mathbb{B}(\xi_i,\xi_j). \tag{5.82}$$

Proposition 5.11. *If $W^{qc}=0$ then the conditions 5.80, 5.82 hold.*

Proof. Differentiate (5.59) and take the corresponding trace to get

$$(\nabla^2_{e_a,\xi_i}\mathbb{B})(I_ke_a,\xi_j)+(\nabla^2_{e_a,I_ke_a}\mathbb{B})(\xi_i,\xi_j)=$$
$$2(\nabla_{e_b}L)(I_ke_b,I_je_a)\mathbb{B}(e_a,\xi_i)+2L(I_ke_b,I_je_a)(\nabla_{e_b}\mathbb{B})(e_a,\xi_i)$$
$$-(\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)\mathbb{B}(e_a,\xi_j)-T(\xi_i,I_ke_b,e_a)(\nabla_{e_b}\mathbb{B})(e_a,\xi_j)$$
$$-\frac{\epsilon_k}{2n}d(tr\,L)(e_a)\mathbb{B}(I_ke_a,\xi_k)-\frac{\epsilon_k}{2n}(tr\,L)(\nabla_{e_a}\mathbb{B})(I_ke_a,\xi_k). \tag{5.83}$$

On the other hand, the Ricci identities, (5.28), (2.18) and (4.15) yield

$$(\nabla^2_{e_a,I_ke_a}\mathbb{B})(\xi_i,\xi_j)=-4n(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)-4\epsilon_j(tr\,L)\mathbb{B}(\xi_j,\xi_j)+4\epsilon_i(tr\,L)\mathbb{B}(\xi_i,\xi_i); \tag{5.84}$$

$$(\nabla^2_{e_a,\xi_i}\mathbb{B})(I_ke_a,\xi_j)=4n(\nabla_{\xi_i}\mathbb{B})(\xi_j,\xi_k)-2(\nabla_{\xi_i}L)(e_a,e_b)L(I_je_a,I_ke_b)+4n\zeta_k(\xi_i,e_a)\mathbb{B}(e_a,\xi_j)$$
$$+2\epsilon_j\rho_i(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_k)-2\epsilon_j\rho_k(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_i)+T(\xi_i,e_a,e_b)(\nabla_{e_b}\mathbb{B})(I_ke_a,\xi_j) \tag{5.85}$$

Substituting (5.84) and (5.85) in (5.83) we come to

$$4n\Big[(\nabla_{\xi_i}\mathbb{B})(\xi_j,\xi_k)-(\nabla_{\xi_k}\mathbb{B})(\xi_i,\xi_j)\Big]$$
$$=2(\nabla_{e_b}L)(I_ke_b,I_je_a)\mathbb{B}(e_a,\xi_i)+2\Big[(\nabla_{e_b}\mathbb{B})(e_a,\xi_i)+(\nabla_{\xi_i}L)(e_b,e_a)\Big]L(I_ke_b,I_je_a)$$
$$-\mathbb{B}(e_a,\xi_j)\Big[4n\zeta_k(\xi_i,e_a)+(\nabla_{e_b}T)(\xi_i,I_ke_b,e_a)\Big]-\Big[2\epsilon_j\rho_i(e_a,\xi_i)+\frac{\epsilon_k}{2n}d(tr\,L)(e_a)\Big]\mathbb{B}(I_ke_a,\xi_k)$$
$$+2\epsilon_j\rho_k(e_a,\xi_i)\mathbb{B}(I_ke_a,\xi_i)+T(\xi_i,I_ke_a,e_b)\Big[(\nabla_{e_b}\mathbb{B})(e_a,\xi_j)-(\nabla_{e_a}\mathbb{B})(e_b,\xi_j)\Big]$$
$$-\frac{\epsilon_k}{2n}(tr\,L)(\nabla_{e_a}\mathbb{B})(I_ke_a,\xi_k)+4\epsilon_j(tr\,L)\mathbb{B}(\xi_j,\xi_j)-4\epsilon_i(tr\,L)\mathbb{B}(\xi_i,\xi_i). \tag{5.86}$$

We find with the help of (5.48), the symmetry of L and (5.11) that

$$4n\zeta_k(\xi_i,e_a)=(4n+1)\mathbb{B}(I_ke_a,\xi_i)+\epsilon_j\mathbb{B}(e_a,\xi_j)+\mathbb{B}(I_ie_a,\xi_k)-\frac{\epsilon_j}{4n}d(trL)(I_je_a)$$
$$-\frac{1}{4}\Big[(\nabla_{e_b}L)(I_ie_a,I_ke_b)-\epsilon_j(\nabla_{e_b}L)(e_a,I_je_b)+(\nabla_{e_b}L)(I_ke_a,I_ie_b)-\epsilon_j(\nabla_{e_b}L)I_je_a,e_b)\Big].$$

It follows from (4.14) that

$$(\nabla_{e_b}T)(\xi_i, I_k e_b, e_a) = -\frac{1}{4}\Big[\epsilon_j(\nabla_{e_b}L)(I_j e_b, e_a) + 3(\nabla_{e_b}L)(I_k e_b, I_i e_a) + \epsilon_j(\nabla_{e_b}L)(e_b, I_j e_a) \\ - (\nabla_{e_b}L)(I_i e_b, I_k e_a) - \frac{\epsilon_j}{n} d(trL)(I_j e_a)\Big].$$

The sum of the last two equalities yields

$$4n\zeta_k(\xi_i, e_a) + (\nabla_{e_b}T)(\xi_i, I_k e_b, e_a) = 4nB(I_k e_a, \xi_i) - 4nB(I_i e_a, \xi_k). \tag{5.87}$$

Finally, combining (5.86), (5.87), Lemma 5.6, (5.49), (5.27) with $L(e_b, I_s e_b) = 0$ leads to a series of standart calculations, which at the end imply (5.80).

The other integrability condition in this case, (5.82), can be obtained similarly using (5.61) and the Ricci identities. The proof of Theorem 1.2 is completed. □

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