A self-improving property of Riesz potentials in BMO

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Abstract

In this paper we prove that for non-negative measurable functions f,

$I_\alpha f \in BMO(\mathbb{R}^n)$ if and only if $I_\alpha f \in BMO^\beta(\mathbb{R}^n)$ for $\beta \in (n-\alpha, n]$.

Here I_α denotes the Riesz potential of order α and BMO^β represents the space of functions of bounded β-dimensional mean oscillation introduced in [10].

1 Introduction

The space of functions of bounded β-dimensional mean oscillation, denoted as $BMO^\beta(\mathbb{R}^n)$, was introduced in [10] recently. This space extends the classical notion of bounded mean oscillation to include the Choquet integral with respect to Hausdorff content. Specifically, for $0 < \beta \leq n \in \mathbb{N}$, a function u is said to be in $BMO^\beta(\mathbb{R}^n)$ if it is locally integrable with respect to the β-dimensional Hausdorff content $\mathcal{H}^\beta_\infty$ (see Section 2 for definition) and satisfies the condition

$$\sup_{Q\subseteq\mathbb{R}^n} \inf_{c\in\mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |u(x)-c| \, d\mathcal{H}^\beta_\infty < \infty.$$

In here, $\ell(Q)$ is the length of the side of the cube and $\mathcal{H}^\beta_\infty$ is the Hausdroff content defined as

$$\mathcal{H}^\beta_\infty(E) := \inf\left\{ \sum_{i=1}^\infty \omega_\beta r_i^\beta : E \subset \bigcup_{i=1}^\infty B(x_i, r_i) \right\},$$

where $\omega_\beta := \pi^{\beta/2}/\Gamma\left(\frac{\beta}{2}+1\right)$ is a normalized constant.

An analogue of the John-Nirenberg inequality for BMO^β is also given in [10], which asserts that the functions of β-dimensional mean oscillation also have exponential integrability with respect to Hausdorff content. That is, if $u \in BMO^\beta(\mathbb{R}^n)$, then there exist constants $c_1 = c_1(\beta)$, $c_2 = c_2(\beta) > 0$ such that for any cube Q and any $\lambda > 0$,

$$\mathcal{H}^\beta_\infty\left(\{x \in Q : |u(x) - c_Q| > \lambda\}\right) \leq c_1 l(Q)^\beta \exp(-c_2\lambda/\|u\|_{BMO^\beta(\mathbb{R}^n)}), \qquad (1.1)$$

where $c_Q \in \mathbb{R}$ is a suitable constant depends on Q. In particular, by integrating respect to λ in (1.1) and utilizing the pointwise estimate $t \leq \exp(t)$ for $0 \leq t$, one obtains the equivalence relation: $u \in BMO^\beta(\mathbb{R}^n)$ if and only if there exists constants $\gamma = \gamma(\beta)$ and $C' = C'(\beta)$ such that

$$\int_Q \exp\left(\frac{\gamma|u - c_Q|}{\|u\|_{BMO^\beta(\mathbb{R}^n)}}\right) d\mathcal{H}^\beta_\infty \leq C'\ell(Q)^\beta \tag{1.2}$$

for every finite subcube $Q \subseteq \mathbb{R}^n$ and suitable constant c_Q. Moreover, an application of inequality (1.1) yields a nesting property for $BMO^\beta(\mathbb{R}^n)$: for $0 < \beta_1 \leq \beta_2 \leq n$, there exists a constant $C = C(\beta_1, \beta_2)$ such that

$$\|u\|_{BMO^{\beta_2}(\mathbb{R}^n)} \leq C\|u\|_{BMO^{\beta_1}(\mathbb{R}^n)} \tag{1.3}$$

(see section 2 for further discussion).

The aim of this paper is to explore the mapping properties of Riesz potentials I_α into $BMO^\beta(\mathbb{R}^n)$ spaces, focusing specifically on the Morrey spaces $\mathcal{M}^\alpha_p(\mathbb{R}^n)$ and weak Lebesgue spaces $L^{n/\alpha,\infty}(\mathbb{R}^n)$. Our motivation is inspired by exponential integrability concerning Radon measures satisfying ball growth conditions. This includes several inequalities analogous to (1.1) have been established (see, for example, [1, Theorem 3], [4, p. 210] , [18, Corollary 1.4.1 and Corollary 8.6.2], [12, Theorem 2.2 and Theorem 2.5], and especially the results of Adams [2], Adams and Xiao [7] (see also [6]), and Spector [17, Corollary 1.6]. To state their result, we recall that for $0 < \alpha < n$ and $1 \leq p \leq \frac{n}{\alpha}$, the Morrey space $\mathcal{M}^\alpha_p(\mathbb{R}^n)$ denotes the collection of all the measurable functions f in $\mathbb{R}^n$ satisfying

$$\|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)} := \sup_{Q\subseteq\mathbb{R}^n} \ell(Q)^\alpha \left(\frac{1}{\ell(Q)^n}\int_Q |f|^p\, dy\right)^{\frac{1}{p}} < \infty. \tag{1.4}$$

Moreover, a locally finite Radon measures μ is said to be belong to the Morrey space $\mathcal{M}^\beta(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{|\mu(B(x,r))|}{r^\beta} < \infty,$$

where $|\mu|$ denotes the total variation of μ. As the topological dual of $L^1(\mathcal{H}^\beta_\infty;\mathbb{R}^n)$ can be identified with the Morrey space $\mathcal{M}^\beta(\mathbb{R}^n)$(see (2.6)), the result of Adams and Xiao [7, Theorem 0.1 (iii)] can be stated as

Theorem 1.1 (Adams and Xiao [7])**.** *Let $0 < \alpha < n$, $1 \leq p \leq \frac{n}{\alpha}$ and f be a non-negative function with support in a bounded domain $\Omega \subseteq \mathbb{R}^n$. If $f \in \mathcal{M}^\alpha_p(\mathbb{R}^n)$, then there exist constants γ, $C'' > 0$ depend on α, β, n, p and Ω such that*

$$\int_\Omega \exp\left(\frac{\gamma|I_\alpha f|}{\|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)}}\right) d\mathcal{H}^\beta_\infty < C'', \tag{1.5}$$

holds for $\beta \in (n - \alpha p, n]$.

Here $I_\alpha f$ denotes the Riesz potential of order α of the function f, which is defined as

$$I_\alpha f(x) := \frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

where $\gamma(\alpha)$ is a normalization constant (see [24, p. 117]). An interesting connection to earlier findings by Adams [2, Corollary (i)] emerges when $p = 1$ in Theorem 1.1. It gives a necessary and sufficient condition for the Riesz potential $I_\alpha f$ of a non-negative function f lies in the $BMO(\mathbb{R}^n)$ space:

Theorem 1.2 (Adams [2])**.** *Let f be a non-negative measurable function. Then*

$$I_\alpha f \in BMO(\mathbb{R}^n) \text{ if and only if } f \in \mathcal{M}_1^\alpha(\mathbb{R}^n) \text{ and } I_\alpha f \in L^1_{loc}(\mathbb{R}^n), \tag{1.6}$$

and in this case we have

$$\|I_\alpha f\|_{BMO(\mathbb{R}^n)} \cong \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)}.$$

Remark 1.3. In here, we note that the assumption $I_\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n)$ on the left-hand side of (1.6) is omitted in the statement of [2, Corollary (i)]. Yet there exists a function $f \in L^{n/\alpha}(\mathbb{R}^n) \subseteq \mathcal{M}_1^\alpha(\mathbb{R}^n)$ for which $I_\alpha f = \infty$ everywhere in $\mathbb{R}^n$(see [25, p. 432] for example), one sees that such an assumption is needed. In fact, the proof of [2, Proposition 3.4] regarding sufficiency is unaffected, and the assumption $I_\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is introduced in [2, Proposition 3.2] to establish the necessary condition for (1.6).

The first result in this paper is a futher improvement of exponetial integrability in Theorem 1.1, which shows the estimate (1.5) can actually be derived from John-Nirenberg inequality (1.2) for bounded β-dimensional mean oscillation functions.

Theorem 1.4. *Let $0 < \alpha < n$ and $1 \le p < \frac{n}{\alpha}$. If $f \in \mathcal{M}_p^\alpha(\mathbb{R}^n)$ and $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$, then for $\beta \in (n - \alpha p,\ n]$, there exists a constant C such that*

$$\|I_\alpha f\|_{BMO^\beta(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_p^\alpha(\mathbb{R}^n)}.$$

In particular, there exist constants γ and C_0 such that

$$\int_Q \exp\left(\frac{\gamma\,|I_\alpha f - c_Q|}{\|f\|_{\mathcal{M}_p^\alpha(\mathbb{R}^n)}}\right)\, d\mathcal{H}^\beta_\infty \le C_0 \ell(Q)^\beta$$

holds for every cube $Q \subseteq \mathbb{R}^n$.

As $f \in \mathcal{M}_p^\alpha(\mathbb{R}^n)$ having compact support implies $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ (see [25, p. 436] for example), one easily sees that Theorem 1.4 refines Theorem 1.1.

Motivated by Theorem 1.4, one may inquire into the specific function space conditions under which a function f belongs to, leading to $I_\alpha f \in BMO^\beta(\mathbb{R}^n)$ for any $\beta \in (0, n]$. Inspired by [17, Corollary 1.6], which asserts that for every open bounded set $\Omega \subseteq \mathbb{R}^n$ and $\beta \in (0, n]$, there exist constants $\gamma,\ C > 0$ that depend on n, β, α and Ω such that

$$\int_\Omega \exp\left(\frac{\gamma|I_\alpha f|}{\|f\|_{L^{n/\alpha,\infty}(\Omega)}}\right) d\mathcal{H}^\beta_\infty \le C, \tag{1.7}$$

holds for $f \in L^{n/\alpha,\infty}(\Omega)$ with $\operatorname{supp} f \subseteq \Omega$, we introduce our second result. This result is a theorem analogous to Theorem 1.4 but concerns the scale of $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$. The proof can be closely modeled after that of Theorem 1.4.

Theorem 1.5. *Suppose that $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$ and $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. Then there exists a constant C such that*

$$\|I_\alpha f\|_{BMO^\beta(\mathbb{R}^n)} \leq C\|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)}$$

In particular, there exist constants γ and C_0 such that

$$\int_Q \exp\left(\frac{\gamma\,|I_\alpha f - c_Q|}{\|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)}}\right) d\mathcal{H}^\beta_\infty \leq C_0 \ell(Q)^\beta$$

holds for every cube $Q \subseteq \mathbb{R}^n$.

We now introduce the main result of this paper, which establishes a new necessary and sufficient condition for non-negative functions f such that $I_\alpha f$ is in $BMO(\mathbb{R}^n)$.

Theorem 1.6. *Let f be a non-negative measurable function. Then*

$$I_\alpha f \in BMO(\mathbb{R}^n) \text{ if and only if } I_\alpha f \in BMO^\beta(\mathbb{R}^n) \text{ for } \beta \in (n-\alpha, n].$$

In this case, we have

$$\|I_\alpha f\|_{BMO(\mathbb{R}^n)} \cong \|I_\alpha f\|_{BMO^\beta(\mathbb{R}^n)} \text{ for } \beta \in (n-\alpha, n].$$

This result builds on three key pieces: Adams' necessary and sufficient condition for $I_\alpha f \in BMO(\mathbb{R}^n)$ in Theorem 1.2, the inclusion property (1.3) from [10], and the embedding given in Theorem 1.4 when $p = 1$. The connections between these elements are illustrated in the diagram below:

$$\begin{tikzcd}
& \boxed{I_\alpha f \in BMO^\beta(\mathbb{R}^n) \text{ for } \beta \in (n-\alpha, n]} \arrow[dd, Rightarrow, "(1.3)"'] \\
\boxed{f \in M^\alpha_1(\mathbb{R}^n) \text{ and } I_\alpha f \in L^1_{loc}(\mathbb{R}^n)} \arrow[ur, Rightarrow, "Theorem\ 1.4"'] \arrow[dr, Leftrightarrow, "Theorem\ 1.2"] & \\
& \boxed{I_\alpha f \in BMO(\mathbb{R}^n)}
\end{tikzcd}$$

Theorem 1.6, together with Theorem 1.2 and the characterization for $BMO^\beta(\mathbb{R}^n)$ discovered in [10], readily yields the following theorem:

Theorem 1.7. *Let f be a non-negative measurable function. Then the following statements are equivalent.*

(i) $I_\alpha f \in BMO(\mathbb{R}^n)$*;*

(ii) $I_\alpha f \in BMO^\beta(\mathbb{R}^n)$ *for any* $\beta \in (n-\alpha, n]$*;*

(iii) For any $\beta \in (n-\alpha, n]$, inequality (1.1) holds with $u = I_\alpha f$;

(iv) For any $\beta \in (n-\alpha, n]$, inequality (1.2) holds with $u = I_\alpha f$;

(v) For any $\beta \in (n-\alpha, n]$, inequality

$$\|I_\alpha f\|_{BMO^{\beta,p}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \inf_{c\in\mathbb{R}} \left(\frac{1}{l(Q)^\beta}\int_Q |I_\alpha f - c|^p d\mathcal{H}^\beta_\infty\right)^{1/p} < \infty$$

holds for $p \in [1, \infty)$;

(vi) $f \in \mathcal{M}^\alpha_1(\mathbb{R}^n)$ and $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$.

This paper is organized as follows: Section 2 reviews the necessary preliminaries for defining the spaces $BMO^\beta(\mathbb{R}^n)$ and discusses their basic properties. This section also covers the mapping properties of the fractional maximal operator $\mathcal{M}_\eta$, which are important in the sequel. In Section 3, we establish a variant of technical results concerning the estimates of Riesz potentials, to be utilized in the subsequent sections. Finally, Section 4 presents the proofs of our main results.

2 Preliminaries

This section introduces the foundational concepts necessary for our discussion, focusing on the Choquet integral respect to the Hausdorff content, the β-dimensional BMO space, and the fractional maximal function. Additionally, we outline their basic properties, which are crucial for the proof of our results.

In the following context, the notation Ω always denotes either the entire Euclidean space $\mathbb{R}^n$ or an open cube Q within $\mathbb{R}^n$. For a non-negative function f, the Choquet integral of f over Ω with respect to the Hausdorff content, is defined by:

$$\int_\Omega f\, d\mathcal{H}^\beta_\infty := \int_0^\infty \mathcal{H}^\beta_\infty\left(\{x\in\Omega : f(x) > t\}\right)\, dt. \tag{2.1}$$

A function $f : \Omega \to \mathbb{R}$ is said to be $\mathcal{H}^\beta_\infty$-quasicontinuous if, for any $\epsilon > 0$, there exists an open set $O \subseteq \Omega$ with $\mathcal{H}^\beta_\infty(O) < \epsilon$ where f restricted to the complement of O is continuous. This framework allows for the establishment of the L^p norm for any signed $\mathcal{H}^\beta_\infty$-quasicontinuous function f, given $p \geq 1$, as:

$$\|f\|_{L^p(\Omega;\mathcal{H}^\beta_\infty)} := \left(\int_\Omega |f|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}. \tag{2.2}$$

With these preparations, We introduce the vector space:

$$L^p(\Omega;\mathcal{H}^\beta_\infty) := \left\{f\ \mathcal{H}^\beta_\infty\text{-quasicontinuous } : \|f\|_{L^p(\Omega;\mathcal{H}^\beta_\infty)} < +\infty\right\}.$$

The norm defined in (2.2) behaves as a quasi-norm, and there exists an equivalent norm defined via Choquet integral respect to dyadic Hausdorff content such that $L^1(\Omega;\mathcal{H}^\beta_\infty)$ forms a Banach space. To be more precise, let $\mathcal{D}(Q)$

denote the set of all dyadic cube generalize by the cube Q and we adopt the notation $\mathcal{D}(\mathbb{R}^n) = \mathcal{D}(\prod_{i=1}^n [0,1))$, the dyadic Hausdorff content $\mathcal{H}_\infty^{\beta,\Omega}$ subordinate to Ω of $E \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{H}_\infty^{\beta,\Omega}(E) := \inf\left\{ \sum_{i=1}^\infty \ell(Q_i)^\beta : E \subset \bigcup_{i=1}^\infty Q_i \text{ and } Q_i \in \mathcal{D}(\Omega) \right\}. \tag{2.3}$$

It has been shown that for non-negative functions f, there exists a constant C_β such that

$$\frac{1}{C_\beta} \int_\Omega f \, d\mathcal{H}_\infty^{\beta,\Omega} \leq \int_\Omega f \, d\mathcal{H}_\infty^\beta \leq C_\beta \int_\Omega f \, d\mathcal{H}_\infty^{\beta,\Omega}. \tag{2.4}$$

In here, $\int_\Omega f \, d\mathcal{H}_\infty^{\beta,\Omega}$ denotes the Choquet integral of f over Ω with respect to dyadic Hausdorff content $\mathcal{H}_\infty^{\beta,\Omega}$, defined as

$$\int_\Omega f \, d\mathcal{H}_\infty^{\beta,\Omega} := \int_0^\infty \mathcal{H}_\infty^{\beta,\Omega}\left(\{x \in \Omega : f(x) > t\}\right) \, dt.$$

Moreover, the vector space $L^1(\Omega; \mathcal{H}_\infty^\beta)$ equipped with norm

$$\|f\|_{L^1(\Omega, \mathcal{H}_\infty^{\beta,\Omega})} := \int_\Omega |f| \, d\mathcal{H}_\infty^{\beta,\Omega} \tag{2.5}$$

forms a Banach space (see [26, Proposition 2.3], [22, Proposition 3.2 and 3.5] and the survey presented in [10, Section 2]).

The class of continuous and bounded functions in Ω, denoted by $C_b(\Omega)$, for which the norm in equation (2.2) remains finite, is proven to be dense in the vector space $L^1(\Omega; \mathcal{H}_\infty^\beta)$ with quasi-norm $\|\cdot\|_{L^1(\Omega;\mathcal{H}_\infty^\beta)}$ and thus also dense in the vector space $L^1(\Omega; \mathcal{H}_\infty^\beta)$ with norm $\|\cdot\|_{L^1(\Omega;\mathcal{H}_\infty^{\beta,\Omega})}$ (see [3] and [20]). In addition, a function $f : \Omega \to \mathbb{R}^n$ is said to be locally integrable with respect to the Hausdorff content $\mathcal{H}_\infty^\beta$ in Ω, denoted as $f \in L^1_{\text{loc}}(\Omega; \mathcal{H}_\infty^\beta)$, if f multiplied by the characteristic function of any cube $Q \subseteq \Omega$ belongs to $L^1(Q; \mathcal{H}_\infty^\beta)$.

Next, we outline the basic properties of the Choquet integral with respect to the Hausdorff content. The following lemma can be found in [15, p. 5], and its proof is referred to [5] and [8, Chapter 4].

Lemma 2.1. *(i) For $0 \leq a$ and non-negative functions f, we have*

$$\int_\Omega a \, f(x) \, d\mathcal{H}_\infty^\beta = a \int_\Omega f(x) \, d\mathcal{H}_\infty^\beta;$$

(ii) For non-negative functions f_1 and f_2, we have

$$\int_\Omega f_1(x) + f_2(x) \, d\mathcal{H}_\infty^\beta \leq 2 \left(\int_\Omega f_1(x) \, d\mathcal{H}_\infty^\beta + \int_\Omega f_2(x) \, d\mathcal{H}_\infty^\beta \right);$$

(iii) Let $1 \leq p,\ p'$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then for non-negative functions f_1 and f_2, we have

$$\int_\Omega f_1(x) f_2(x) \, d\mathcal{H}_\infty^\beta \leq 2 \left(\int_\Omega f_1(x)^p d\mathcal{H}_\infty^\beta \right)^{\frac{1}{p}} \left(\int_\Omega f_2(x)^{p'} d\mathcal{H}_\infty^\beta \right)^{\frac{1}{p'}}.$$

It has been shown in [3] that the dual space of $L^1(\mathbb{R}^n;\mathcal{H}^\beta_\infty)$ is the Morrey space $\mathcal{M}^\beta(\mathbb{R}^n)$ and thus

$$\int_\Omega f\, d\mathcal{H}^\beta_\infty \asymp \sup_{\|\mu\|_{\mathcal{M}^\beta}\le 1} \int_\Omega f\, d\mu \tag{2.6}$$

holds for every non-negative function $f \in L^1(\Omega;\mathcal{H}^\beta_\infty)$. In particular, (2.6) also holds if f is a non-negative lower semi-continuous function (see Adams [3, p. 118]).

We now introduce the definition of the space of functions with bounded β-dimensional mean oscillation, denoted as $BMO^\beta(\Omega)$. The definition for the case where Ω is a cube has been provided in [10], and the definition for $\Omega = \mathbb{R}^n$ can be readily modified with those in [10]. For a function $u \in L^1_{loc}(\Omega;\mathcal{H}^\beta_\infty)$, u is said to be has bounded β-dimensional mean oscillation in Ω provided

$$\sup_{Q\subseteq\Omega} \inf_{c\in\mathbb{R}} \frac{1}{l(Q)^\beta} \int_Q |u-c|\, d\mathcal{H}^\beta_\infty < +\infty, \tag{2.7}$$

where the supremum is taken over all finite subcubes $Q \subseteq \Omega$. We then define the space of functions of bounded β-dimensional mean oscillation in Ω:

$$BMO^\beta(\Omega) := \left\{u \in L^1(\Omega;\mathcal{H}^\beta_\infty) : \|u\|_{BMO^\beta(\Omega)} < +\infty\right\},$$

where

$$\|u\|_{BMO^\beta(\Omega)} := \sup_{Q\subseteq\Omega} \inf_{c\in\mathbb{R}} \frac{1}{l(Q)^\beta} \int_\Omega |u-c|\, d\mathcal{H}^\beta_\infty.$$

The following two lemmas presented below, and their associated proofs for the case when Ω is a cube in $\mathbb{R}^n$, are given in [10, Corollary 1.5 and Corollary 1.6] and are derived from the extended John-Nirenberg inequality (1.1). The proofs for the lemmas when $\Omega = \mathbb{R}^n$ are trivially adapted to the proofs of case when Ω is a cube.

Lemma 2.2. *Let $0 < \beta_1 \le \beta_2 \le n$ and suppose $u \in BMO^{\beta_1}(\Omega)$. Then $u \in BMO^{\beta_2}(\Omega)$ and*

$$\|u\|_{BMO^{\beta_2}(\Omega)} \le C\|u\|_{BMO^{\beta_1}(\Omega)}$$

for some constant $C = C(\alpha,\beta) > 0$ independent of u.

Lemma 2.3. *Let $\beta \in (0,n]$. There exists a constant $C = C(\beta) > 0$ such that*

$$\frac{1}{C}\|u\|_{BMO^\beta(\Omega)} \le \|u\|_{BMO^{\beta,p}(\Omega)} \le Cp\|u\|_{BMO^\beta(\Omega)}$$

for all $u \in BMO^\beta(\Omega)$.

We now introduce the definition of the fractional maximal operator. Let $0 < \eta < n$, where $n \in \mathbb{N}$. The fractional maximal operator, associated with a locally integrable function f of order η, is defined as

$$\mathcal{M}_\eta f(x) := \sup_{r>0} \frac{1}{r^{n-\eta}} \int_{B(x,r)} |f(y)|\, dy.$$

In particular, a locally integrable function f belongs to the Morrey space $\mathcal{M}_1^\alpha(\mathbb{R}^n)$ if and only if

$$\sup_{x\in\mathbb{R}^n} \mathcal{M}_\alpha f(x) < \infty.$$

Furthermore, the following estimates of weak and strong types hold:

Lemma 2.4. *Let $0 < \eta < n$.*

(i) If $1 < p < \frac{n}{\eta}$ and $\beta = n - \eta p$, then

$$\|\mathcal{M}_\eta f\|_{L^p(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} \leq A_1 \|f\|_{L^p(\mathbb{R}^n)};$$

(ii) If $p = \frac{\beta}{n-\eta}$ with $n - \eta \leq \beta \leq n$, then

$$\|\mathcal{M}_\eta f\|_{L^{p,\infty}(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} \leq A_2 \|f\|_{L^1(\mathbb{R}^n)}.$$

In here, $L^{p,\infty}(\mathbb{R}^n;\mathcal{H}^\beta_\infty)$ denotes the vector spaces of all the functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\|u\|_{L^{p,\infty}(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} := \sup_{\lambda>0} \lambda^{\frac{1}{p}} \mathcal{H}^\beta_\infty \left(\{x \in \mathbb{R}^n : |u(x)| > \lambda\}\right) < \infty.$$

The proof of (i) follows from the one weight inequality in [21, Theorem B]. The proof of (ii), when $\beta = n-\eta$, can be found in [9, Lemma 3.2]. Furthermore, both (i) and (ii) are given in [5, Theorem 7], where they are presented in terms of the Choquet integral concerning Hausdorff content, as in Lemma 2.4 (see also [14] for the results concerning the weak Choquet spaces).

3 Auxiliary Results

The fist lemma in this section is an improvement of the result in [25, p. 439] as the Morrey space $\mathcal{M}_1^\alpha(\mathbb{R}^n)$ is a suitable subspace of $L^{n/\alpha}(\mathbb{R}^n)$.

Lemma 3.1. *Let $0 < N$, $0 < \alpha < n \in \mathbb{N}$, $f \in \mathcal{M}_1^\alpha(\mathbb{R}^n)$ and $\tilde{I_\alpha^N} f$ be defined as*

$$\tilde{I_\alpha^N} f(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{B_N}(y)}{|y|^{n-\alpha}} \right) f(y)\, dy,$$

where B_N denotes the complement of closed unit ball, that is, $B_N := \{x \in \mathbb{R}^n : |x| > N\}$. Then $\tilde{I_\alpha^N} f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. It is sufficient to show that

$$\int_{|x|<M} \tilde{I_\alpha^N} f(x)\, dx < \infty \text{ for every } N < M < \infty. \tag{3.1}$$

To this end, we split the integral into two pieces:

$$\begin{aligned}
\int_{|x|<M} \tilde{I_\alpha^N} f(x)\, dx &\leq \frac{1}{\gamma(\alpha)} \int_{|x|<M} \int_{|y|<2M} \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{B_N}(y)}{|y|^{n-\alpha}} \right| |f(y)|\, dy\, dx \\
&\leq \frac{1}{\gamma(\alpha)} \int_{|x|<M} \int_{|y|\geq 2M} \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{B_N}(y)}{|y|^{n-\alpha}} \right| |f(y)|\, dy\, dx \\
&:= I + II.
\end{aligned}$$

For I, we have $\frac{\chi_{B_N}(y)}{|y|^{n-\alpha}} \leq N^{\alpha-n}$ for every $y \in \mathbb{R}^n$ and thus Tonelli's theorem gives

$$\begin{aligned}
I &\leq \frac{1}{\gamma(\alpha)} \int_{|x|<M} \int_{|y|<2M} \left| \frac{1}{|x-y|^{n-\alpha}} + N^{\alpha-n} \right| |f(y)| \, dy \, dx \\
&= \frac{1}{\gamma(\alpha)} \int_{|y|<2M} \int_{|x|<M} \left| \frac{1}{|x-y|^{n-\alpha}} + N^{\alpha-n} \right| \, dx \, |f(y)| \, dy.
\end{aligned}$$

Since f is locally finite and

$$\int_{|x|<M} \left| \frac{1}{|x-y|^{n-\alpha}} + N^{\alpha-n} \right| \, dx \leq C \left(M^\alpha + N^{\alpha-n} M^n \right) < \infty,$$

we obtain $I < \infty$.

For II, using mean value theorem, we deduce that for $|x| < M$ and $|y| > 2M$, there exists a constant C depending on n and α such that

$$\begin{aligned}
\left| \frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{B_N}(y)}{|y|^{n-\alpha}} \right| &= \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right| \\
&\leq C \frac{|x|}{|y|^{n-\alpha+1}}
\end{aligned}$$

and again by Tonelli's theorem one deduces that

$$\begin{aligned}
II &\leq C \int_{|x|<M} |x| \, dx \, \int_{|y|\geq 2M} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\
&\leq C M^{n+1} \int_{|y|\geq 2M} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy. \qquad (3.2)
\end{aligned}$$

Moreover, a dyadic splitting yields that

$$\begin{aligned}
\int_{|y|\geq 2M} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy &= \sum_{k=0}^{\infty} \int_{2^k M \leq |y| < 2^{k+1} M} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\
&\leq \sum_{k=0}^{\infty} \left(2^k M \right)^{-(n-\alpha+1)} \int_{|y|<2^{k+1}M} |f(y)| \, dy \\
&\leq \sum_{k=0}^{\infty} \left(2^k M \right)^{-(n-\alpha+1)} \left(2^{k+1} M \right)^{n-\alpha} \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \\
&\leq \frac{C}{M} \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)}. \qquad (3.3)
\end{aligned}$$

Combining (3.2) and (3.3), we obtain $II < \infty$ and therefore the proof is complete. ☐

We next use an argument based on [25, p. 443] to give the following result.

Lemma 3.2. *Let $0 < \alpha < n$ and $f \in \mathcal{M}_1^\alpha(\mathbb{R}^n)$. Then $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$ if and only if*

$$\int_{|y|>N} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy < \infty \text{ for any } N > 0. \qquad (3.4)$$

Proof. Suppose that $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. It follows from Lemma 3.1 that $\tilde{I_\alpha^N} f \in L^1_{loc}(\mathbb{R}^n)$ for any $N > 0$ and thus for every $N > 0$ there exists $x_0 \in \mathbb{R}^n$ such that both $I_\alpha f(x_0)$ and $\tilde{I_\alpha^N} f(x_0)$ are finite. In particular, this implies that

$$\frac{1}{\gamma(\alpha)} \int_{|y|>N} \frac{f(y)}{|y|^{n-\alpha}} \, d(y) = I_\alpha f(x_0) - \tilde{I_\alpha^N} f(x_0) < \infty,$$

and therefore

$$\int_{|y|>N} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy < \infty \text{ for any } N > 0.$$

Now suppose that (3.4) holds. Since $\tilde{I_\alpha^N} f \in L^1_{loc}(\mathbb{R}^n)$ by Lemma 3.1 and the identity

$$\begin{aligned} I_\alpha f(x) &= \tilde{I_\alpha^N} f(x) + \frac{1}{\gamma(\alpha)} \int_{|y|>N} \frac{f(y)}{|y|^{n-\alpha}} \, dy \\ &\leq \tilde{I_\alpha^N} f(x) + \frac{1}{\gamma(\alpha)} \int_{|y|>N} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy \end{aligned}$$

holds if $\tilde{I_\alpha^N} f(x)$ is well-defined and finite (and thus holds for Lebesgue almost every x), we obtain $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. ◻

Lemma 3.3. *Let $0 < \beta_1 \leq \beta_2 \leq n$ and $Q \subseteq \mathbb{R}^n$ be a cube. If $f : Q \to \mathbb{R}$ is $\mathcal{H}^{\beta_1}_\infty$-quasicontinuous in Q, then f is $\mathcal{H}^{\beta_2}_\infty$-quasicontinuous in Q.*

Proof. Without loss of generality, we may assume $\mathcal{H}^{\beta_2}_\infty(Q) > 0$. Suppose that f is $\mathcal{H}^{\beta_1}_\infty$-quasicontinuous in Q. Then there exists a sequence of functions $\{\phi_j\}_j^\infty \subseteq C_b(Q)$ such that

$$\lim_{j\to\infty} \int_Q |f - \phi_j| \, d\mathcal{H}^{\beta_1}_\infty = 0.$$

Yet an application of Lemma 2.2 in [11] and Hölder inequality in Lemma 2.1 yields the estimate

$$\begin{aligned} \int_Q |f - \phi_j| \, d\mathcal{H}^{\beta_2}_\infty &\leq \omega_{\beta_1} \left(\frac{1}{\omega_{\beta_2}}\right)^{\frac{\beta_2}{\beta_1}} \left(\int_Q |f - \phi_j|^{\frac{\beta_1}{\beta_2}} \, d\mathcal{H}^{\beta_1}_\infty\right)^{\frac{\beta_2}{\beta_1}} \\ &\leq \omega_{\beta_1} \left(\frac{1}{\omega_{\beta_2}}\right)^{\frac{\beta_2}{\beta_1}} \left(2 \left(\int_Q |f - \phi_j| \, \mathcal{H}^{\beta_1}_\infty\right)^{\frac{\beta_2}{\beta_1}} \mathcal{H}^{\beta_1}_\infty(Q)^{1-\frac{\beta_1}{\beta_2}}\right)^{\frac{\beta_2}{\beta_1}} \\ &\leq 2^{\frac{\beta_2}{\beta_1}} \omega_{\beta_1} \left(\frac{1}{\omega_{\beta_2}}\right)^{\frac{\beta_2}{\beta_1}} \mathcal{H}^{\beta_1}_\infty(Q)^{\frac{\beta_2}{\beta_1}-1} \int_Q |f - \phi_j| \, d\mathcal{H}^{\beta_1}_\infty \end{aligned}$$

which converges to 0 as j goes to ∞. This completes the proof. ◻

We next present a lemma concerning the Riesz potential of a function $\phi \in \mathcal{M}^\alpha_1(\mathbb{R}^n)$ with compact support. This lemma will help us verify the quasicontinuity of $I_\alpha f$ with respect to the Hausdorff content in Lemma 3.5.

Lemma 3.4. *Let $0 < \alpha < n$, $0 < \varepsilon \le \alpha$, Q be a cube in $\mathbb{R}^n$ and ϕ be a real-valued function in $\mathbb{R}^n$. If $\phi \in \mathcal{M}_1^\alpha(\mathbb{R}^n)$ and* $\operatorname{supp}\phi \subseteq 2Q$*, then*

$$\int_Q |I_\alpha \phi| \, d\mu \le C' \ell(Q)^\varepsilon \|\phi\|_{L^1(\mathbb{R}^n)} \le C'' \ell(Q)^{n-\alpha+\varepsilon} \|\phi\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \tag{3.5}$$

holds for every $\mu \in \mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)$ with $\|\mu\|_{\mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)} \le 1$.

Proof. Let $\mu \in \mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)$ satisfying $\|\mu\|_{\mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)} \le 1$. We first note that an usual dyadic splitting gives

$$\begin{aligned} |I_\alpha \phi(x)| &\le \frac{1}{\gamma(\alpha)} \int_{2Q} \frac{|\phi(y)|}{|x-y|^{n-\alpha}} \, dy \\ &\le \frac{1}{\gamma(\alpha)} \sum_{k=1}^{\infty} \frac{1}{(2^{-(k+1)} l(Q))^{n-\alpha}} \int_{2^{-(k+1)}Q \setminus 2^{k+1}Q} |\phi(y)| \, dy \\ &\le \frac{C_\eta}{\gamma(\alpha)} \ell(Q)^{\alpha-\eta} \mathcal{M}_\eta \phi(x) \end{aligned} \tag{3.6}$$

holds for every $\eta \in [0, \alpha)$. Moreover, Lemma 2.4 (ii) yields that for any $0 \le \varepsilon < \alpha$,

$$\|\mathcal{M}_\eta \phi\|_{L^{q,\infty}(\mathbb{R}^n; \mathcal{H}_\infty^{n-\alpha+\varepsilon})} \le C \|\phi\|_{L^1(\mathbb{R}^n)},$$

where $q = \frac{n-\alpha+\varepsilon}{n-\eta}$. Now choosing $\eta = \alpha - \frac{\varepsilon}{2}$, we have $q > 1$ and therefore by Hölder inequality in Lorentz scale (see Theorem 1.4.16 in [13] for example) we have the estimate

$$\begin{aligned} \int_Q \mathcal{M}_\eta \phi(x) \, d\mu(x) &\le C \|\chi_{2Q}\|_{L^{q',1}(\mu)} \|\mathcal{M}_\eta \phi\|_{L^{q,\infty}(\mu)} \\ &\le C \ell(Q)^{\varepsilon+\eta-\alpha} \|\mathcal{M}_\eta \phi\|_{L^{q,\infty}(\mathbb{R}^n; \mathcal{H}_\infty^{n-\alpha+\varepsilon})} \\ &\le C \ell(Q)^{\varepsilon+\eta-\alpha} \|\phi\|_{L^1(\mathbb{R}^n)}, \end{aligned} \tag{3.7}$$

where the second inequality follows by

$$\|\chi_{2Q}\|_{L^{q',1}(\mu)} \le C \ell(Q)^{\frac{n-\alpha+\varepsilon}{q'}} = C \ell(Q)^{\varepsilon+\eta-\alpha}.$$

The combination of (3.6) and (3.7) then gives

$$\begin{aligned} \int_Q |I_\alpha \phi(x)| \, d\mu(x) &\le C \ell(Q)^{\alpha-\eta} \int_Q \mathcal{M}_\eta \phi(x) \, d\mu(x) \\ &\le C \ell(Q)^\varepsilon \|\phi\|_{L^1(\mathbb{R}^n)} \\ &\le C \ell(Q)^{n-\alpha+\varepsilon} \|\phi\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)}, \end{aligned} \tag{3.8}$$

which completes the proof. ∎

In the following lemma, we demonstrate that for functions f satisfying the conditions of Lemma 3.4, $I_\alpha f$ must belong to $L^1(Q; \mathcal{H}_\infty^\beta)$. Consequently, an analogue of estimate (3.4), in terms of the Choquet integral with respect to Hausdorff content, is established.

Lemma 3.5. *Let $0 < \alpha < n$, $0 < \varepsilon \leq \alpha$, Q be a cube in $\mathbb{R}^n$ and f be a real-valued function in $\mathbb{R}^n$. If $f \in \mathcal{M}_1^\alpha(\mathbb{R}^n)$ and $\operatorname{supp} f \subseteq 2Q$, then $I_\alpha f \in L^1(Q; \mathcal{H}_\infty^{n-\alpha+\varepsilon})$ and*

$$\int_Q |I_\alpha f(x)| \, d\mathcal{H}_\infty^{n-\alpha+\varepsilon} \leq C\ell(Q)^{n-\alpha+\varepsilon} \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \tag{3.9}$$

for any $0 < \varepsilon \leq \alpha$.

Proof. By Lemma 3.4, we have

$$\int_Q |I_\alpha \phi| \, d\mu \leq C'\ell(Q)^\varepsilon \|\phi\|_{L^1(\mathbb{R}^n)} \leq C''\ell(Q)^{n-\alpha+\varepsilon} \|\phi\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \tag{3.10}$$

holds for every $\phi \in C_b(2Q)$ and $\mu \in \mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)$ with $\|\mu\|_{\mathcal{M}^{n-\alpha+\varepsilon}(\mathbb{R}^n)} \leq 1$. Since $C_b(2Q)$ is dense in $L^1(2Q; \mathbb{R}^n)$ and $f \in L^1(2Q; \mathbb{R}^n)$, there exists a sequence of functions $\{\phi_i\}_{i=1}^\infty \subseteq C_b(2Q)$ such that $\operatorname{supp} \phi_i \subseteq 2Q$ and

$$\int_{2Q} |\phi_i(x) - f(x)| \, dx \leq \frac{1}{4^i} \text{ for each } i \in \mathbb{R}^n.$$

As $|I_\alpha \phi_i - I_\alpha \phi_j|$ is continuous, we may use estimate (3.4) and deduce that for $j \geq i$, there exists a constant $C > 0$ such that

$$\int_Q |I_\alpha \phi_i(x) - I_\alpha \phi_j(x)| \, d\mathcal{H}_\infty^{n-\alpha+\varepsilon}(x) \leq \frac{C\ell(Q)^\varepsilon}{4^i}. \tag{3.11}$$

The combination of (2.4) and (3.11) then yields that there exists a constant $C' > 0$ such that

$$\int_Q |I_\alpha \phi_i(x) - I_\alpha \phi_j(x)| \, d\mathcal{H}_\infty^{n-\alpha+\varepsilon, Q}(x) \leq \frac{C'\ell(Q)^\varepsilon}{4^i}, \tag{3.12}$$

where $\mathcal{H}_\infty^{n-\alpha+\varepsilon,Q}$ denotes the dyadic Hausdorff content given in (2.3). Since the vector space $L^1(Q; \mathcal{H}_\infty^{n-\alpha+\varepsilon})$ equipped with the norm $\|\cdot\|_{L^1(Q;\mathcal{H}_\infty^{n-\alpha+\varepsilon,Q})}$ forms a Banach space, there exists $g \in L^1(Q; \mathcal{H}_\infty^{n-\alpha+\varepsilon})$ such that $I_\alpha \phi_i$ converges to g in $L^1(Q; \mathcal{H}_\infty^{n-\alpha+\varepsilon})$ and

$$\lim_{i\to\infty} I_\alpha \phi_i(x) = g(x) \text{ for } \mathcal{H}_\infty^{n-\alpha+\varepsilon} \text{ almost every } x \in Q. \tag{3.13}$$

We next claim

$$\lim_{i\to\infty} I_\alpha \phi_i(x) = I_\alpha f(x) \text{ for } \mathcal{H}_\infty^{n-\alpha+\varepsilon} \text{ almost every } x \in Q, \tag{3.14}$$

and thus $I_\alpha f \in L^1(Q; \mathcal{H}_\infty^{n-\alpha+\varepsilon})$. To this end, we observe that an argument similar to (3.7) gives that

$$\int_Q \mathcal{M}_\eta(\phi_i - f)(x) \, d\mu(x) \leq C\ell(Q)^{\varepsilon+\eta-\alpha} \|\phi_i - f\|_{L^1(\mathbb{R}^n)}.$$

Moreover, as $\mathcal{M}_\eta(\phi_i - f)$ is a lower semi-continuous function, we obtain by (2.6) that

$$\int_Q \mathcal{M}_\eta(\phi_i - f)(x) \, d\mathcal{H}_\infty^{n-\alpha+\varepsilon}(x) \leq C\ell(Q)^{\varepsilon+\eta-\alpha} \|\phi_i - f\|_{L^1(\mathbb{R}^n)}, \tag{3.15}$$

where $\eta = \alpha - \frac{\varepsilon}{2}$. As Fatou's lemma holds for Choquet integrals with respect to Hausdorff content (see (1.2) in [20] for example), we deduce that

$$
\begin{aligned}
0 &\leq \int_Q \liminf_{i\to 0} \mathcal{M}_\eta(\phi_i - f)(x)\, d\mathcal{H}^{n-\alpha+\varepsilon}_\infty(x) \\
&\leq \liminf_{i\to 0} \int_Q \mathcal{M}_\eta(\phi_i - f)(x)\, d\mathcal{H}^{n-\alpha+\varepsilon}_\infty(x) \\
&\leq \liminf_{i\to 0} C\ell(Q)^{\varepsilon+\eta-\alpha}\|\phi_i - f\|_{L^1(\mathbb{R}^n)} \\
&= 0.
\end{aligned}
$$

As a consequence, we conclude that

$$\liminf_{i\to 0} \mathcal{M}_\eta(\phi_i - f)(x) = 0 \text{ for } \mathcal{H}^{n-\alpha+\varepsilon}_\infty \text{ almost every } x \in Q.$$

Yet a standard dyadic splitting similar to (3.6) yields that

$$|I_\alpha\phi_i(x) - I_\alpha f(x)| \leq C_\eta \ell(Q)^{\frac{\varepsilon}{2}}\mathcal{M}_\eta(\phi_i - f)(x) \text{ for every } x \in Q$$

leading to

$$\liminf_{i\to 0} |I_\alpha\phi_i(x) - I_\alpha f(x)| = 0 \text{ for } \mathcal{H}^{n-\alpha+\varepsilon}_\infty \text{ almost every } x \in Q. \tag{3.16}$$

Combining (3.13) and (3.16), we obtain that

$$\lim_{i\to 0} I_\alpha\phi_i(x) = g(x) = I_\alpha f(x) \text{ for } \mathcal{H}^{n-\alpha+\varepsilon}_\infty \text{ almost every } x \in Q,$$

as claimed.

Finally, since $I_\alpha f = g \in L^1(Q; \mathcal{H}^{n-\alpha+\varepsilon}_\infty)$, the combination of (2.6) and (3.5) yields

$$\int_Q |I_\alpha f|\, d\mathcal{H}^{n-\alpha+\varepsilon}_\infty \leq C\ell(Q)^{n-\alpha+\varepsilon}\|f\|_{\mathcal{M}^\alpha_1(\mathbb{R}^n)}$$

and the proof is complete.

□

The subsequent Lemma improves Lemma 3.5 in the case where $p \geq 1$.

Lemma 3.6. *Let $0 < \alpha < n$, $1 \leq p \leq \frac{n}{\alpha}$, Q be a cube and f be a real-valued function in $\mathbb{R}^n$. If $\operatorname{supp} f \subseteq 2Q$ and $f \in \mathcal{M}^\alpha_p(\mathbb{R}^n)$, then $I_\alpha f \in L^1(Q; \mathcal{H}^\beta_\infty)$ and*

$$\int_Q |I_\alpha f|\, d\mathcal{H}^\beta_\infty \leq C'\ell(Q)^{\beta+\alpha-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)} \leq C''\ell(Q)^\beta\|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)} \tag{3.17}$$

holds for any $\beta \in (n - \alpha p, n]$.

Proof. It is sufficient to prove the case $1 < p \leq \frac{n}{\alpha}$, as Lemma 3.5 yields (3.17) for $p = 1$. To this end, we again use the usual dyadic splitting in (3.6) and obtain that

$$|I_\alpha f(x)| \leq \frac{C_\eta}{\gamma(\alpha)}\ell(Q)^{\alpha-\eta}\mathcal{M}_\eta f(x)$$

holds for any $\eta \in [0, \alpha)$. Moreover, Lemma 2.4 (i) yields that for any $1 < p < \frac{n}{\alpha}$,

$$\|\mathcal{M}_\eta f\|_{L^p(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.18}$$

Applying Hölder's inequality of the Hausdorff content (see Lemma 2.1 (iii)) and estimate (3.18), we obtain

$$\begin{aligned}\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty &\leq C\ell(Q)^{\alpha-\eta} \int_Q \mathcal{M}_\eta f \, d\mathcal{H}^\beta_\infty \\ &\leq C\ell(Q)^{\alpha-\eta} \|\chi_Q\|_{L^{p'}(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} \|\mathcal{M}_\eta f\|_{L^p(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} \\ &\leq C\ell(Q)^{\beta+\alpha-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} \\ &\leq C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)}. \end{aligned} \tag{3.19}$$

Moreover, since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, there exists a sequence of functions $\{\phi_j\}_{j=1}^\infty \subseteq C_c^\infty(\mathbb{R}^n)$ such that for each j,

$$\lim_{j\to\infty} \|f - \phi_j\|_{L^p(\mathbb{R}^n)} = 0.$$

In particular, it follows by (3.19) that

$$\int_Q |I_\alpha f - I_\alpha \phi_j| \, d\mathcal{H}^\beta_\infty \leq C\ell(Q)^{\alpha+\beta-\frac{n}{p}} \|f - \phi_j\|_{L^p(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. Since $I_\alpha \phi_j \in C_b(Q)$ for each $j \in \mathbb{N}$, we have by definition of $\mathcal{H}^\beta_\infty$-quasicontinuity that $I_\alpha f$ is $\mathcal{H}^\beta_\infty$-quasicontinuous and thus $I_\alpha f \in L^1(Q; \mathcal{H}^\beta_\infty)$. □

The following lemma presents an analogue of Lemma 3.4 for when f belongs to the weak Lebesgue space $L^{n/\alpha,\infty}(\mathbb{R}^n)$.

Lemma 3.7. *Let $0 < \alpha < n$, $0 < \beta \leq n$, Q be a cube in $\mathbb{R}^n$ and f be a real-valued function in $\mathbb{R}^n$. If $\operatorname{supp} f \subseteq 2Q$ and $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$, then $I_\alpha f \in L^1(Q; \mathcal{H}^\beta_\infty)$ and for every $1 < p < n/\alpha$,*

$$\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty \leq C'\ell(Q)^{\alpha+\beta-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} \leq C''\ell(Q)^\beta \|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)}. \tag{3.20}$$

Proof. Since $\mathcal{M}^\alpha_p(\mathbb{R}^n) \subseteq L^{n/\alpha,\infty}(\mathbb{R}^n)$ for $1 \leq p \leq \frac{n}{\alpha}$, we have $I_\alpha f \in L^1(Q; \mathcal{H}^\beta_\infty)$ for $0 < \beta \leq n$ by Lemma 3.6.

Choosing $p \in (1, \frac{n}{\alpha})$ so that $n - \alpha p < \beta$, it then follows again by Lemma 3.6 that

$$\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty \leq C'\ell(Q)^{\beta+\alpha-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} \tag{3.21}$$

since $f \in L^{n/\alpha,\infty}(\mathbb{R}^n) \subseteq \mathcal{M}^\alpha_p(\mathbb{R}^n)$. Using Hölder's inequality in Lorentz scale, we get

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)} \|\chi_{2Q}\|_{L^{s,1}(\mathbb{R}^n)}, \tag{3.22}$$

where s satisfying $\frac{1}{p} = \frac{\alpha}{n} + \frac{1}{s}$. Since

$$\|\chi_{2Q}\|_{L^{s,1}(\mathbb{R}^n)} = C\ell(Q)^{\frac{n}{s}}$$

and

$$\|\chi_{2Q}\|_{L^{p'}(\mathbb{R}^n;\mathcal{H}^\beta_\infty)} = C\ell(Q)^{\beta(1-\frac{1}{p})},$$

the combination (3.21) of (3.22) yields

$$\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty \leq C'\ell(Q)^{\alpha+\beta-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)} \leq C''\ell(Q)^\beta\|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)},$$

which completes the proof.

□

We conclude this section with a lemma that provides an auxiliary estimate for the proofs of Theorem 1.4 and Theorem 1.5 and it will be used in conjunction with Lemma 3.7 and Lemma 3.6 in the proofs of Theorem 1.4 and Theorem 1.5, respectively.

Lemma 3.8. *Let $0 < \alpha < n$ and f be a real-valued function with* $\operatorname{supp} f \subseteq (2Q)^c$. *If $f \in \mathcal{M}^\alpha_1(\mathbb{R}^n)$ and $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$, then $I_\alpha f$ is continuous in Q and there exists $c \in \mathbb{R}$ such that*

$$\int_Q |I_\alpha f - c| \, d\mathcal{H}^\beta_\infty \leq C\ell(Q)^\beta\|f\|_{\mathcal{M}^\alpha_1(\mathbb{R}^n)} \tag{3.23}$$

holds for any $\beta \in (0, n]$.

Proof. Let x_1 and x_2 belong to Q. By Lemma 3.2, we have for every $x \in Q$,

$$|I_\alpha f(x)| \leq \frac{1}{\gamma(\alpha)}\int_{(2Q)^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy \leq \frac{1}{\gamma(\alpha)}\int_{|y|>\frac{\ell(Q)}{2}} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy < \infty.$$

The fact that $y \in (2Q)^c$ yields, by the mean value theorem, the estimate

$$\left|\frac{1}{|x_1-y|^{n-\alpha}} - \frac{1}{|x_2-y|^{n-\alpha}}\right| \leq C\frac{|x_1-x_2|}{|y|^{n-\alpha+1}} \tag{3.24}$$

and thus for every $x_1, x_2 \in Q$, there exists a constant C such that

$$\begin{aligned}
|I_\alpha f(x_1) - I_\alpha f(x_2)| &\leq C|x_1-x_2|\int_{(2Q)^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\
&\leq C|x_1-x_2|\int_{|y|>\ell(Q)} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\
&\leq C\frac{|x_1-x_2|}{\ell(Q)}\|f\|_{\mathcal{M}^\alpha_1(\mathbb{R}^n)} \to 0
\end{aligned}$$

as $|x_1 - x_2| \to 0$. Thus, $I_\alpha f$ is continuous in Q.

Now it remains only to verify (3.23). Without loss of generality we take the center of the cube Q to be the origin. Setting

$$c := I_\alpha f(0) = \frac{1}{\gamma(\alpha)}\int_{(2Q)^c} \frac{1}{|y|^{n-\alpha}} f(y) \, dy,$$

we have $|c| < \infty$ by lemma 3.2. Moreover, inequality (3.24) yields that

$$\begin{aligned}\int_Q |I_\alpha f(x) - c| \, d\mathcal{H}^\beta_\infty(x) &\le \frac{1}{\gamma(\alpha)} \int_Q \int_{(2Q)^c} \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right| |f(y)| \, dy \, d\mathcal{H}^\beta_\infty(x) \\ &\le C \int_Q |x| \, d\mathcal{H}^\beta_\infty(x) \int_{(2Q)^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\ &\le C l(Q)^{\beta+1} \int_{|y| \ge \ell(Q)} \frac{|f(y)|}{|y|^{n-\alpha+1}} \, dy \\ &\le C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_1(\mathbb{R}^n)},\end{aligned}$$

where we use (3.3) in the last inequality. □

4 Proofs of Main Results

We now give the proof of Theorem 1.4.

Proof. Let $Q \subseteq \mathbb{R}^n$ be a cube and

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x),$$

where $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. Then by Lemma 3.5, we have $I_\alpha f_1 \in L^1(Q; \mathcal{H}^\beta_\infty)$ and

$$\int_Q |I_\alpha f_1| \, d\mathcal{H}^\beta_\infty \le C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)}. \tag{4.1}$$

Moreover, Lemma 3.8 yields that $I_\alpha f_2 \in L^1(Q; \mathcal{H}^\beta_\infty)$ and there exists a constant $c \in \mathbb{R}$ such that

$$\begin{aligned}\int_Q |I_\alpha f_2(x) - c| \, d\mathcal{H}^\beta_\infty &\le C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_1(\mathbb{R}^n)} \\ &\le C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)}.\end{aligned} \tag{4.2}$$

The combination of (4.1) and (4.2) then gives

$$\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty \le C\ell(Q)^\beta \|f\|_{\mathcal{M}^\alpha_p(\mathbb{R}^n)},$$

which completes the proof by taking the supremum over all the cubes Q in $\mathbb{R}^n$. □

We now give the proof of Theorem 1.5.

Proof. Let Q be a cube in $\mathbb{R}^n$. Denote $2Q$ be the cube with the same center and twice the side length and set

$$I_\alpha f = I_\alpha f_1 + I_\alpha f_2,$$

where $f_1 := f\chi_{2Q}$ and $f_2 := f - f_1$. Since $\operatorname{supp} f_1 \subseteq 2Q$ and $f_1 \in L^{n/\alpha,\infty}(\mathbb{R}^n)$, we have by Lemma 3.7 that $I_\alpha f \in L^1(Q; \mathcal{H}^\beta_\infty)$ and

$$\int_Q |I_\alpha f_1| \, d\mathcal{H}^\beta_\infty \le C'\ell(Q)^{\alpha+\beta-\frac{n}{p}} \|f_1\|_{L^p(\mathbb{R}^n)} \tag{4.3}$$

for any $1 < p < \frac{n}{\alpha}$.

Moreover, since $f_2 \in L^{n/\alpha,\infty}(\mathbb{R}^n) \subseteq \mathcal{M}_1^\alpha(\mathbb{R}^n)$ and $\operatorname{supp} f_2 \subseteq (2Q)^c$, an application of Lemma 3.8 yields that $I_\alpha f_2 \in L^1(Q;\mathcal{H}^\beta_\infty)$ and there exists a constant $c \in \mathbb{R}$ such that

$$\begin{aligned}\int_Q |I_\alpha f_2 - c| \, d\mathcal{H}^\beta_\infty &\leq C\ell(Q)^\beta \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \\ &\leq C\ell(Q)^\beta \|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)}.\end{aligned}$$

This completes the proof.

$\square$

We now give the proof of Theorem 1.6

Proof. Suppose that $I_\alpha f \in BMO(\mathbb{R}^n)$. An application of Theorem 1.2 and Theorem 1.4 for $p = 1$ yields that for $\beta \in (n-\alpha, n]$,

$$\|I_\alpha f\|_{BMO^\beta(\mathbb{R}^n)} \leq C_1 \|f\|_{\mathcal{M}_1^\alpha(\mathbb{R}^n)} \leq C_2 \|f\|_{BMO(\mathbb{R}^n)}.$$

Now suppose that $I_\alpha f \in BMO^\beta(\mathbb{R}^n)$ for some $\beta \in (n-\alpha, n]$. It follows by Lemma 2.2 that

$$\|I_\alpha f\|_{BMO(\mathbb{R}^n)} \leq C_\beta \|I_\alpha f\|_{BMO^\beta(\mathbb{R}^n)},$$

which completes the proof.

$\square$

Finally, we give the proof of Theorem 1.7.

Proof. The equivalence of (i) and (vi) is due to Theorem 1.1. The equivalence of (i) and (ii) is due to Theorem 1.6. The equivalence of (ii) and (v) is due to Lemma 2.3. Finally, the equivalence of (ii), (iii) and (iv) is due to (1.1) and (1.2). $\square$

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