Bulk flows, general relativity and the fundamental role of the "peculiar" flux

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Abstract

Recent surveys have been reporting bulk peculiar flows considerably faster than expected. Bulk flows are moving matter and matter in motion implies nonzero energy flux. In relativity, as opposed to Newtonian physics, matter fluxes gravitate as well, since they also contribute to the energy-momentum tensor. The gravitational input of the "peculiar" flux survives at the linear level and it can drastically change our understanding of the way bulk flows have evolved in time. By default, the Newtonian analysis of peculiar motions bypasses the relativistic flux-contribution to the gravitational field. The problem is that there are also studies, with an otherwise relativistic profile, which inadvertently do the same. As result, these treatments reduce to Newtonian and they misleadingly reproduce the slow Newtonian growth-rate of linear peculiar velocities. In contrast, by accounting for the flux effects, the proper relativistic analysis arrives at a considerably stronger growth. We show that it is the flux input of the moving matter to gravity that separates the relativistic studies of peculiar motions from the rest and, in so doing, it could provide an answer to the bulk-flow question.

1 Introduction

Surveys have repeatedly confirmed the presence of large-scale peculiar motions, the so-called bulk flows, with sizes of few hundred Mpc and speeds of few hundred km/sec [1]. These are believed to have started as weak peculiar-velocity perturbations around recombination, when structure formation begun in earnest. It is the increasing inhomogeneity of the post-recombination universe that has triggered, sustained and amplified the initial velocity perturbations to the bulk flows observed today. Nevertheless, the picture is incomplete. Several surveys have reported bulk velocities in excess of those predicted by the current cosmological model (e.g. [2]-[7]). Among them is the recent survey of [8], using the *CosmicFlows-4* data to report bulk flows considerably faster than anticipated. Such fast peculiar motions are difficult to explain within the current theoretical framework, without additional free parameters, or without appealing to new physics.

Bulk flows are matter in motion and moving matter implies nonzero flux by default. In general relativity, as opposed to Newtonian physics, moving matter has an extra input to the gravitational field, since its flux also contributes to the energy-momentum tensor (e.g. see [9, 10]). However, most studies of cosmological peculiar velocities (v) are Newtonian. This means that the flux triggered by the peculiar motion of the matter has no gravitational input. As a result, the Newtonian studies have led to a moderate linear growth-rate of $v \propto t^{1/3}$ after recombination (e.g. see [11]-[13]), which seems too slow to reproduce the bulk-velocity values reported in [2]-[8].

In the literature, there are also few quasi-Newtonian treatments that recover the Newtonian growth-rate (e.g. see [14, 15]). Nevertheless, despite their relativistic appearance, these studies reduce to Newtonian in practice. The reasons are multiple and they all stem from the severe mathematical restrictions imposed upon the host spacetime, which inevitably compromise its relativistic nature [10]. Indeed, the quasi-Newtonian approach adopts a reference frame with zero linear shear and vorticity. This switches the gravitational waves off as well. Perturbed relativistic spacetimes without these features are rather unnatural. Moreover, the driving force of the peculiar velocities, namely the 4-acceleration, is given by the gradient of an ad hoc (Newtonian-like) potential, the evolution of which follows from an ansatz. All this may simplify the calculations, but it compromises and blurs the physics. As a result, the relativistic gravitational input of the matter flux is bypassed and the analysis (inadvertently) reduces to Newtonian (see § 3, 4 below).[1] Nevertheless, as will show here, there is no need for any of the aforementioned severe mathematical constraints when studying linear peculiar motions relativistically, which makes the related quasi-Newtonian studies redundant for all practical purposes.

In a proper relativistic analysis of peculiar velocities, there are no restrictions. The linear shear and vorticity are nonzero and the perturbed spacetime has gravitational waves. None of these quantities is involved in the linear calculations, however, which makes them irrelevant. The real difference comes from the 4-acceleration. In a truly relativistic study, the latter is not the gradient of an arbitrary scalar potential, but follows naturally from standard cosmological perturbation theory. Crucially, the relativistic 4-acceleration also carries the gravitational input of the matter flux. The result is a considerably stronger growth-rate of $v \propto t^{4/3}$ for the peculiar-velocity field [16, 17], which seems capable of producing the bulk flows reported in [2]-[8].

The gravitational input of the peculiar flux comes into play from the (fairly well known) fact that, when peculiar motions are present, the cosmic fluid cannot be treated as perfect, even at the linear level. The "imperfection" appears as a nonzero *peculiar flux* due to the moving matter (e.g. see § 2 here and § 5.2.1 in [10]). Less well known is that, when dealing with imperfect media, the 4-acceleration is not zero even in the absence of pressure. In other words, the *peculiar-flux* is always linked to a *peculiar 4-acceleration*. The linear form of the latter follows from the energy and the momentum conservation laws and makes all the difference.

The problematic nature of the highly restrictive quasi-Newtonian setup has been known, though not widely, at least since [10] (see § 6.8.2 there for a discussion and "warning" comments). The extent of the problem was not realised, however, which explains why quasi-Newtonian studies of peculiar-velocity fields (misleadingly termed relativistic) occasionally still appear in the literature. The reason is probably because, so far, there has been no direct comparison between the quasi-Newtonian and the proper relativistic study. It is the aim of this work to make the comparison and in so doing to demonstrate how deeply serious the problem can be.

[1] There more examples of studies involving peculiar motions, which start relativistically and reduce to Newtonian when the flux-input of the moving matter to the gravitational field is bypassed (see § 4 below).

2 Peculiar flux and peculiar 4-acceleration

When studying cosmological peculiar motions, one needs to assume a universal reference system, relative to which it makes sense to define and measure peculiar velocities. Typically, this is the Cosmic Microwave Background (CMB) frame, which is defined as the only coordinate system where the radiation dipole vanishes [10]. This will be our assumption as well, although our analysis still holds if peculiar velocities were to be defined relative to a different reference frame.

Relativistic studies of peculiar motions require "tilted" spacetimes, with two groups of observers in relative motion. Consider a tilted, perturbed Friedmann-Robertson-Walker (FRW) universe with two 4-velocity fields u_a and $\tilde{u}_a$. Identify the former with the reference frame of the universe and the latter with the moving matter. When the peculiar motion is non-relativistic, we have $\tilde{u}_a = u_a + v_a$, where v_a is the relative velocity of the matter. Note that both frames "live" in the perturbed universe and no restrictions are imposed on them [16, 17]. In other words, none of these frames is the quasi-Newtonian and they both have nonzero linear shear and vorticity (recall that the CMB is nearly but not fully isotropic).

Here, we will compare the relativistic studies of [16, 17] to the quasi-Newtonian treatments of [14, 15], because they are all covariant and gauge-invariant. To facilitate the comparison further, we will also adopt the conventions of [14, 15]. We will therefore assume an Einstein-de Sitter background and set the flux and the 4-acceleration to zero in the frame of the matter (i.e. $\tilde{q}_a = 0$ and $\tilde{A}_a = 0$ respectively – tilded variables are measured in the matter frame). Put another way, there is no peculiar flux in the coordinate system of the pressureless matter, which moves along timelike geodesics. Then, to linear order in the reference u_a-frame, we have

$$q_a = \rho v_a \qquad \text{and} \qquad \dot{v}_a + H v_a = -A_a \,, \tag{1}$$

with zero linear pressure in both systems [14, 15]. Note that ρ is the density of the cosmic fluid (with $\tilde{\rho} = \rho$) and $H = \dot{a}/a$ is the Hubble parameter of the Einstein-de Sitter background (with $a = a(t)$ being the cosmological scale factor). Two comments are in order at this point:

(i) According to Eq. (1a), when peculiar motions are present, the cosmic medium cannot be treated as perfect. The "imperfection" appears as a nonzero *peculiar flux* vector (q_a), solely triggered by the moving matter (e.g. see § 5.2.1 in [10] for related comments and also for the generalised linear version of (1a)).

(ii) Expression (1b) ensures that peculiar velocities are driven by the 4-acceleration. Indeed, when $A_a = 0$, linear peculiar velocities either vanish identically, or decay (as $v \propto 1/a$) with the universal expansion. Since a decaying v_a-field is at direct odds with the plethora of bulk-flow reports, the 4-acceleration is the key to the evolution of linear peculiar velocities.

Newtonian peculiar velocities are driven by the acceleration, which (for zero pressure) is the gradient of the gravitational potential (e.g. see [11]-[13]). The quasi-Newtonian 4-acceleration is also given by the gradient of an effective scalar potential [14, 15]. The latter, however, is arbitrary and (for all practical purposes) identical to its Newtonian analogue. Moreover, this is achieved after imposing strict constraints on the host spacetime, which compromise its relativistic nature (see § 6.8.2 in [10] and also § 3 here). In contrast, the proper relativistic *peculiar 4-acceleration* follows naturally from the energy and the momentum conservation laws (e.g. see Eqs. (1.3.17), (1.3.18) in [9], or (5.11), (5.12) in [10]). On our Einstein-de Sitter background these linearise to

$$\dot{\rho} = -\Theta\rho - \mathrm{D}^a q_a \qquad \text{and} \qquad \rho A_a = -\dot{q}_a - 4Hq_a\,, \tag{2}$$

respectively. Note that $q_a = \rho v_a$ is the peculiar flux, $\Theta = \mathrm{D}^a u_a > 0$ is the universal expansion scalar and D_a is the 3-D (covariant) derivative operator. The flux terms seen on the right-hand sides of the above reflect the aforementioned purely general-relativistic contribution of the peculiar flux to the (perturbed) energy-momentum tensor. This feeds into the Einstein equations and leads to flux-related terms in the linear conservation laws of the energy and the momentum.

Expressions (2) also provide the linear form of the relativistic peculiar 4-acceleration, which makes all the difference. Indeed, taking the gradient of (2a) and keeping in mind that $\mathrm{D}_a\dot{\rho} = (\mathrm{D}_a\rho)^{\cdot} - \dot{\rho}A_a + H\mathrm{D}_a\rho$ to first order, leads to the linear evolution formula

$$\dot{\Delta}_a = -\mathcal{Z}_a - 3aHA_a - a\mathrm{D}_a\vartheta\,, \tag{3}$$

of the density inhomogeneities ($\Delta_a = (a/\rho)\mathrm{D}_a\rho$).[2] Also, $Z_a = a\mathrm{D}_a\Theta$ and $\vartheta = \mathrm{D}^a v_a$ by definition. Finally, solving the above for the 4-acceleration, immediately gives

$$A_a = -\frac{1}{3H}\,\mathrm{D}_a\vartheta - \frac{1}{3aH}\left(\dot{\Delta}_a + \mathcal{Z}_a\right)\,. \tag{4}$$

This is the 4-acceleration that drives the linear peculiar velocities in a proper relativistic study. No constraints have been imposed and no scalar potential has been used. Following Eq. (4), linear overdensities/underdensities in the matter distribution and inhomogeneities in the universal expansion (represented by Δ_a and $\mathcal{Z}_a$ respectively) imply a nonzero 4-acceleration, which in turn drives the peculiar motion of the matter (see expression (1b)).

3 Relativistic vs quasi-Newtonian studies

As we already mentioned, the quasi-Newtonian study adopts a (reference) frame with zero linear shear and vorticity, which means that there are no gravitational waves either. However, all this imposes strict constraints upon the perturbed spacetime, which compromise its relativistic nature and lead to Newtonian-like equations and results (see § 6.8.2 in [10] for "warning" comments). The apparent "advantage" is mathematical simplicity. Technically, without shear and vorticity, one can appeal to a scalar potential (φ) and write the 4-acceleration as the gradient

$$A_a = \mathrm{D}_a\varphi\,, \tag{5}$$

instead of involving the relativistic conservation laws (2). As a result, the gravitational input of the peculiar flux is inadvertently bypassed and the physics is severely compromised. Moreover, the potential is arbitrary and its time evolution follows from the ansatz $\dot{\varphi} = -\Theta/3$, which is not uniquely determined either. Combining (1a) and (5), the quasi-Newtonian linear peculiar velocities are governed by [14, 15]

$$\ddot{v}_a + \frac{2}{t}\,\dot{v}_a - \frac{4}{9t^2}\,v_a = 0\,, \tag{6}$$

[2] Alternatively, one can obtain Eq. (3) by linearising around the Einstein-de Sitter background the nonlinear expression (2.3.1) of [9], or Eq. (10.101) of [10], while keeping in mind that the 4-acceleration is given by (2b).

which simply recovers the Newtonian $v \propto t^{1/3}$ growth-rate reported in [11]-[13]. Before proceeding to compare with the proper relativistic analysis, it is important to note that the "damaging" zero shear and zero vorticity constraints of the quasi-Newtonian treatment, as well as the introduction of the scalar potential (φ) in Eq. (5), are not necessary, at least when studying the linear evolution of peculiar velocities (see below). Thus, from the relativistic viewpoint, the quasi-Newtonian studies are mere mathematical exercises without real physical substance.

The proper relativistic analysis of peculiar velocities does not set the linear shear and vorticity to zero, which are also treated as linear perturbations [16, 17]. There is no need to do so because both of these variables are irrelevant, since neither is involved in the linear calculations leading to Eqs. (1)-(4). Hence, there is no need for introducing an ad hoc scalar potential either. All this relieves the (perturbed) spacetime from any a priori restrictions that compromise its relativistic nature, reduce generality and eventually lead to spurious Newtonian results.

Instead of the quasi-Newtonian expression (5), the purely relativistic peculiar 4-acceleration is given by (4), which accounts for the gravitational input of the peculiar flux. The profound differences between these two formulae foreshadow a clear difference in the results of the corresponding approaches. Indeed, differentiating (1b) and (4) with respect to time, using the linear commutation law $(\mathrm{D}_a\vartheta)^{\cdot} = \mathrm{D}_a\dot{\vartheta} - H\mathrm{D}_a\vartheta$, and then combining the resulting expressions leads to

$$\ddot{v}_a + \frac{1}{2}\,H\dot{v}_a - 2H^2v_a = \frac{1}{3H}\,\mathrm{D}_a\dot{\vartheta} + \frac{1}{3aH}\left(\ddot{\Delta}_a + \dot{\mathcal{Z}}_a\right) . \tag{7}$$

This is the relativistic formula for the linear evolution of peculiar velocities in a perturbed, tilted Einstein-de Sitter universe. Expression (7) is an inhomogeneous differential equation without analytic solution. We will therefore confine to the homogeneous part of (7), namely to[3]

$$\ddot{v}_a + \frac{1}{3t}\,\dot{v}_a - \frac{8}{9t^2}\,v_a = 0\,, \tag{8}$$

since $a \propto t^{2/3}$ and $H = 2/3t$ in the Einstein-de Sitter background. The above, which is identical to the one obtained in [16, 17], accepts the power-law solution

$$v = \mathcal{C}_1 t^{4/3} + \mathcal{C}_2 t^{-2/3}\,, \tag{9}$$

with a considerably stronger linear growing mode ($v \propto t^{4/3}$) for the peculiar velocity field (recall that $v \propto t^{1/3}$ in the Newtonian/quasi-Newtonian studies).[4]

Solution (9) holds between decoupling and the accelerated phase, namely when the universe is believed to be close to the Einstein-de Sitter model. During that period, the results of the relativistic analysis clearly support surveys reporting peculiar velocities faster than it is generally anticipated [2]-[8]. This in turn can provide the theoretical basis for a simple and physically motivated answer to the bulk-flow puzzle. An additional point of interest is that, once cosmic acceleration starts, the evolution of the v_a-field can change drastically. Assuming that the

[3]The terms inhomogenous and homogeneous refer to the nature of the differential equation and not to the inhomogeneity/homogeneity of the host spacetime.

[4]According to the theory of differential equations, the full solution of the inhomogeneous equation (7) is formed by the general solution of its homogeneous component, namely by (9), plus one partial solution of the inhomogeneous equation. Therefore, solving (7) in full, will make physical difference only if the partial solution grows faster than the fastest growing mode of the homogeneous solution.

accelerated expansion is driven by dark energy with an effective $p = -\rho$ equation of state, there are no sources of linear velocity perturbations and no peculiar fluxes to gravitate (see [17] for further discussion and technical details). As a result, no new velocity perturbations are generated and the existing ones decay as $v \propto a^{-1}$ [17]. Nevertheless, given that the accelerated phase started relatively recently, the residual peculiar velocity field should be strong enough to explain the fast bulk flows reported in recent surveys. With these in mind, one should expect to measure bulk velocities faster than it is generally expected, but also to see them decline at low redshifts. Qualitatively speaking, such a peculiar-velocity profile closely resembles the one recently reported in [8]. Testing this theoretical prediction quantitatively goes beyond the scope of the present work, since it requires the initial conditions of the peculiar-velocity field and the equation of state of the accelerating universe, as well as more bulk-flow data.

4 The fundamental role of the peculiar flux

At first, the quasi-Newtonian and the relativistic studies proceed in parallel and they both arrive at the same (intermediate) linear formula for the evolution of the peculiar velocity field (see § 2 and compare (1b) to Eq. (39) in [14]). However, when the gravitational contribution of the peculiar flux comes into play the two approaches start to deviate and lead to completely different expressions for the 4-acceleration (see § 3 earlier). The latter is the key physical agent because it drives the peculiar-velocity field in both studies.

Without accounting for the gravitational input of the peculiar flux, the quasi-Newtonian equations reproduced the purely Newtonian ($v \propto t^{1/3}$) growth-rate for the linear-peculiar velocity field (see solution (6) above). However, this was achieved after imposing strict constraints upon the host spacetime and after adopting the Newtonian-like expression (5) for the 4-acceleration (see § 3 previously). In contrast, without imposing any constraints and by accounting for the gravitational contribution of the peculiar flux, the relativistic analysis provides the profoundly different expression (4) for the 4-acceleration (see § 2 earlier). The latter gives the considerably stronger growth ($v \propto t^{4/3}$) for linear peculiar velocities. Consequently, while the Newtonian/quasi-Newtonian studies find it hard to explain the observed fast bulk flows, the relativistic analysis can do so, since it clearly favours peculiar motions faster than anticipated.

In the literature, there are more examples of studies involving peculiar motions that start relativistically but end up Newtonian, when the gravitational effect of the peculiar flux is bypassed for one reason or another. The relativistic approach to the Zeldovich approximation of [18], in particular, reproduced the Newtonian "pancake" attractor, once the quasi-Newtonian frame was adopted and the 4-acceleration was replaced by the gradient of a potential, identical to that of [14, 15]. Another example is the relativistic treatment of the Meszaros "stagnation" effect, where density perturbations "freeze" in the late radiation era [19]. This time no quasi-Newtonian frame was introduced, the gravitational input of the peculiar flux was accounted for and all the flux-related terms (as seen in (2) and (3) here) were initially included in the linear equations. In the process, however, the analysis was switched to the Landau-Lifshitz (or energy) frame, where the flux vanishes by default (see § 3.3.3 and § 3.3.4 in [9], or § 10.4.3 in [10]). Without the flux input, the "relativistic" result was essentially identical to the Newtonian solution of [19]. It would be interesting to see what happens when the peculiar flux is properly accounted for in both of the aforementioned cases.

These characteristic examples clearly demonstrate that, when the gravitational input of the peculiar flux is (for one reason or another) unaccounted for, studies that may have started as relativistic reduce to Newtonian for all practical purposes.

5 Discussion

In general relativity, moving matter has an additional input to the gravitational field, since its flux also contributes to the energy-momentum tensor. As a result, relativistic studies that bypass the above effect are in danger of reducing to Newtonian.

An example of matter in motion, on cosmological scales, are the observed bulk peculiar flows. These are believed to have started as weak velocity perturbations at recombination, which were subsequently amplified by structure formation. There are problems, however, because several recent surveys have reported bulk velocities well in excess of those expected. Having said that, the available theoretical studies are still few and sparse and they are almost all Newtonian. There are also few quasi-Newtonian works that have the external appearance of a proper relativistic analysis. Nevertheless, the quasi-Newtonian framework is so restrictive that it eventually leads to Newtonian-like equations and results (e.g. see § 1.4.2 here and also § 6.8.2 in [10]).

The key difference between the Newtonian/quasi-Newtonian studies of peculiar motions and the relativistic ones are in the adopted form of the acceleration/4-acceleration. The latter is critical because it drives the linear peculiar-velocity field in all these studies. In Newtonian physics, the acceleration is the gradient of the gravitational potential and leads to a growth-rate of $v \propto t^{1/3}$ between recombination and the onset of the accelerated expansion (e.g. see [11]-[13]). This growth is too weak to explain the reported fast bulk flows, without introducing new parameters. In the quasi-Newtonian studies, the 4-acceleration is also given by the gradient of a Newtonian-like scalar potential. It is therefore not surprising that the outcome is the same mediocre ($v \propto t^{1/3}$) growth-rate for the peculiar-velocity field [14, 15].

However, the agreement between the Newtonian and the quasi-Newtonian results is misleading because it simply reflects the fact that the latter analysis is also Newtonian (for all practical purposes). This happens because both approaches bypass (for different reasons) the gravitational input of the peculiar flux. In Newtonian physics this is unavoidable, since only the density of the matter gravitates. In the quasi-Newtonian studies, however, the effect of the peculiar flux is not accounted for because of the entirely unnecessary zero shear and vorticity constraints and the subsequent introduction of an (also unnecessary) scalar potential for the 4-acceleration. All this blurs the physics and diverts the attention from the key role of the peculiar flux, so that its contribution to the relativistic gravitational field and subsequently to the linear evolution of peculiar velocities are inadvertently bypassed. As a result, the driving force of the peculiar velocity field is practically identical to its purely Newtonian counterpart, which explains why the two studies arrive at the same result.

Accounting for the gravitational contribution of the peculiar flux changes the picture drastically. The flux input to the energy-momentum tensor feeds to the relativistic conservation laws and then emerges in the linear evolution formula of the peculiar-velocity field, which differs profoundly from its Newtonian/quasi-Newtonian counterpart. The solution reveals a considerably stronger growth-rate ($v \propto t^{4/3}$) for linear peculiar velocities between recombination and the dark-energy epoch, which could explain the observed bulk peculiar flows [16, 17]. No restrictions have been imposed and no ad hoc assumptions have been made. It is pure general relativity and it could provide a simple, as well as physically motivated, answer to the bulk-flow question.

The general problems of the quasi-Newtonian studies, namely of adopting a reference frame with zero linear shear and vorticity (and the rest of the severe constraints that follow), have been known and they are stated in § 6.8.2 of [10]. However, the extent and the depth of the problem was not realised because there was no direct comparison with a proper relativistic treatment that did not employ the quasi-Newtonian frame. Here, we made this comparison in the study of linear peculiar velocities, by imposing no restrictions on our reference frame. In so doing, we identified the key role of the peculiar flux and then showed how deeply damaging it can be to bypass it. Without accounting for the gravitational input of the flux, peculiar-motion studies that may appear relativistic reduce to Newtonian. Therefore, incorporating the peculiar-flux input to gravity is what distinguishes the relativistic treatments from the rest and in so doing it can also provide a solution to the bulk-flow puzzle.

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