GROMOV–WITTEN INVARIANTS AND MIRROR SYMMETRY FOR NON-FANO VARIETIES USING SCATTERING DIAGRAMS

PER BERGLUND, TIM GRÄFNITZ, AND MICHAEL LATHWOOD

Abstract. Gromov–Witten invariants arise in the topological A-model as counts of worldsheet instantons. On the A-side, these invariants can be computed for a Fano or semi-Fano toric variety using generating functions associated to the toric divisors. On the B-side, the same invariants can be computed from the periods of the mirror. We utilize scattering diagrams (aka wall structures) in the Gross–Siebert mirror symmetry program to extend the calculation of Gromov–Witten invariants to non-Fano toric varieties. Following the work of Carl–Pumperla–Siebert, we compute corrected mirror superpotentials $\vartheta_1(\mathbb{F}_m)$ and their periods for the Hirzebruch surfaces $\mathbb{F}_m$ with $m \geq 2$.

Contents

1. Introduction

In string compactifications on a Calabi-Yau 3-fold Y, D-branes can wrap non-trivial cycles in Y and this gives rise to a class of $\mathcal{N} = 1$ supersymmetric theories in 4 spacetime dimensions [AKV02]. A brane which wraps a space-like cycle in the target spacetime $\mathbb{R}^{1,3} \times Y$ is called an instanton. The worldvolume of such a brane is localized in the time-direction of $\mathbb{R}^{1,3}$ (i.e. only appears for an "instant"), hence the name. Open Gromov–Witten invariants are the counts of open worldsheet instantons, which are described mathematically by holomorphic maps from a disk to X with the boundary ending on the D-brane.

A local Calabi-Yau 3-fold is given by the canonical bundle K_X over a surface X. The zero section of K_X is isomorphic to X and its normal bundle is equal to K_X. When X is Fano, this is negative, so that curves on the zero section can not deform away from the zero section. This suggests a relation between the open Gromov-Witten invariants of K_X and certain relative (or logarithmic) Gromov-Witten invariants of X. This is made precise by the log-local correspondence of

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[vGGR19] combined with the open-closed correspondence of [Cha11]. However, this is no longer true in the non-Fano case, because curves can move away from the zero section. In this paper we will study both, open invariants of K_X and logarithmic invariants of X. We will describe different techniques of computing them, including tropical geometry, and study their corresponding objects under mirror symmetry.

Mirror symmetry is a duality of Calabi-Yau manifolds Y and $\check{Y}$ such that Type IIA string theory compactified on Y is equivalent to Type IIB string theory compactified on $\check{Y}$. Mathematically, the A-model is concerned with symplectic geometry and counts of holomorphic curves, which can be rigorously defined as Gromov-Witten invariants. The B-model is concerned with complex geometry and parameters of the complex structure described by period integrals, pairings of the Kähler form with generators of the group of 1-cycles. Mirror symmetry implies that Gromov-Witten invariants of Y can be computed from the period integrals of $\check{Y}$. The latter are usually easier to compute and hence mirror symmetry gives a powerful tool for computing Gromov-Witten invariants. For example, in [CDGP91] physicists used this method to compute Gromov-Witten invariants of a smooth quintic threefold up to high degree (only limited by computation power). This raised the interest of mathematicians in mirror symmetry, because before these numbers had only be known up to degree 4.

Instead of Calabi-Yau manifolds Y one can consider log Calabi-Yau pairs (X, D), where log Calabi-Yau means that D is an anticanonical divisor of D. We are interested in the case of a smooth divisor D. The "maximal boundary" case with nodal singular divisor has been studied elsewhere [GHK15][BBvG24]. The mirror to a log Calabi-Yau pair is a *Landau-Ginzburg model* $W : \check{X} \to \mathbb{C}$, that is, a non-compact manifold $\check{X}$ and a superpotential function W with compact critical locus. If X is a toric Fano variety, then W is given by the Hori-Vafa potential $W_\Sigma = \sum_{\rho\in\Sigma_1} z^\rho$, a sum of monomials z^ρ corresponding to the rays ρ of the fan Σ of X, with relations among the monomials corresponding to relations among the rays. If X is a non-toric Fano variety, then W is given by the Hori-Vafa potential W_Σ of a toric model X_Σ of X, which means there is a $\mathbb{Q}$-Gorenstein deformation between X and X_Σ. We will see in §6 that in the non-Fano case the superpotential W receives correction terms from the existence of special holomorphic (or tropical) disks. This was noticed in [Aur] and interpreted tropically in [CPS]. In the tropical language, W is the same as the primitive theta function ϑ_1 in the central chamber of a scattering diagram. ϑ_1 is defined as a sum over broken lines, which correspond to 2-marked tropical curves and can be seen as a tropical version of holomorphic disks of Maslov index 2.

The period integrals can be defined as classical periods (Definition 6.14) of the potential $W = \vartheta_1$ or as solutions to Picard-Fuchs type differential equations defined by the Mori vectors (= relations among the rays of the fan). There are three different potential functions one can consider:

(1) The Hori-Vafa potential W_Σ.
(2) The Hori-Vafa potential of the elliptic fibration over X_Σ given by the canonical bundle K_{X_Σ}.
(3) The corrected potential $W = \vartheta_1$ defined by broken lines.

In the toric case they are all equivalent and their periods are, via mirror symmetry, generating functions of relative (or logarithmic) Gromov-Witten invariants of X or, equivalently, up to some factor, of open Gromov-Witten invariants of K_X. In the non-Fano case, they are not equivalent, and the open-log correspondence of Gromov-Witten invariants does not hold. We conjecture that (2) still computes open Gromov-Witten invariants of K_X and show that (3) still computes logarithmic Gromov-Witten invariants of X. Moreover, we show that the corrected potential $W = \vartheta_1$ is equal to the open mirror map after a change of coordinates given by the closed mirror map, and conjecture that the Newton polytope of ϑ_1 gives a toric model of X and we study this (using the language of mutations) for Hirzebruch surfaces $\mathbb{F}_m$. To summarize, our main statements and conjectures are the following.

Definition. A toric variety X_Σ is a toric model of X if there is a $\mathbb{Q}$-Gorenstein deformation between X and X_Σ.

Conjecture 1 (Conjecture 6.11)**.** X has toric model X_Σ, where Σ is the spanning fan of the Newton polytope of ϑ_1, where ϑ_1 is defined in an arbitrary chamber.

Moreover, two (not necessarily Fano) varieties are $\mathbb{Q}$-Gorenstein deformation equivalent if and only if their potentials ϑ_1 are mutation equivalent.

Definition. The open mirror map is $q_0 = z_0 e^{F_W(z)}$, where $F_W(z)$ is the x, y-constant term of $\log(1 - t^{-1}W)$,

$$F_W(z) = \sum_{k>0} \frac{1}{k} \mathrm{coeff}_1(W^k) t^{-k}.$$

The closed mirror map is $q_i = z_i e^{d_i F_W(z)}$, where $d_i = \beta_i \cdot D$. Let $M_W(q) := e^{F_W(z(q))}$ be the open mirror map after inserting the inverse of the closed mirror map.

Definition. The Gross-Siebert slab function is the unique function $h_W(z)$ such that the Gross-Siebert potential

$$W_{\mathrm{GS}} = W + h_W(z)$$

has no constant term. Write $h_W(z(q))$ after insertion of the inverse of the closed mirror map.

Theorem 1 (Theorem 5.15, Theorem 7.4, Proposition 7.10)**.** We have

$$M_{\vartheta_1}(q) = 1 + h_{\vartheta_1}(z(-q)) = \vartheta_1(y)_\infty / y = 1 + \sum_{\beta \in NE(X)} (\beta \cdot D - 1) R_{\beta \cdot D-1,1}(X, \beta) q^\beta,$$

where $R_{\beta \cdot D-1,1}(X, \beta)$ are 2-marked logarithmic Gromov-Witten invariants of (X, D). Here y is related to q by the change of variables $q_i = z_i (t/y)^{d_i}$, with $d_i = \beta_i \cdot D$.

Theorem 2 (Proved in Example 3.2)**.** There are two solutions to the recursion relations for the Picard-Fuchs system for $K_{\mathbb{F}_3}$.

$$F_1(z_1, z_2) = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \geq 3n_2}} (-1)^{n_2} \frac{\Gamma(2n_1 - n_2)}{\Gamma(n_1)\Gamma(n_1 - 3n_2 + 1)\Gamma^2(n_2 + 1)} z_1^{n_1} z_2^{n_2}$$

$$F_2(z_1, z_2) = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \leq 3n_2}} (-1)^{n_1 + n_2} \frac{\Gamma(3n_2 - n_1)}{\Gamma(n_1 + 1)\Gamma(n_2 - 2n_1 + 1)\Gamma^2(n_2 + 1)} z_1^{n_1} z_2^{n_2}$$

The logarithmic solutions are given by $t_a = \log z_a + S_a(z_1, z_2)$ where $S_1 = 2F_1 - F_2$ and $S_2 = -F_1 + 3F_2$. In the language of [Lau], F_1 is the generating function associated to the canonical divisor and F_2 is the generating function associated to negative the exceptional divisor $-E =: \overline{E}$. The open invariants can be obtained from the generating function $S_0 = -F_1$ (displayed in Example 4.3).

We conjecture an analogue of Theorem 1 to hold in the local setting with open Gromov-Witten invariants. For toric Calabi-Yau 3-folds this was shown in [CLL12][CLT13][CCLT16].

Conjecture 2 (Conjecture 7.12)**.** Let Σ' be the fan of K_{X_Σ}, where X_Σ is a toric model of X. Then

$$M_{W_{\Sigma'}}(-q) = 1 + h_{W_{\Sigma'}}(z(q)) = 1 + \sum_{\beta \in NE(X)} N_1(K_X, \beta) q^\beta,$$

where $N_1(K_X, \beta)$ are winding 1 open Gromov-Witten invariants of K_X.

The paper is organized as follows. In §2 we define toric log Calabi-Yau pairs, toric models, toric degenerations, scattering diagrams, and broken lines. We then define a Landau-Ginzburg potential through broken lines. In §3, we describe the open/closed Picard-Fuchs system following [LM]. We then solve this system for $K_{\mathbb{F}_3}$ and remark on the case of an elliptic fibration over $\mathbb{F}_3$. The expectation is that elliptic fibration should reproduce the canonical bundle in the large elliptic fiber limit. In §4, we show that these B-side Picard–Fuchs calculations are consistent with the A-side curve counting calculations. In particular, one needs to take into account a generating function relative to negative the exceptional divisor on the a side, since it will have effective curves for non-Fano $\mathbb{F}_m$. In §5 we use tropical geometry to show that the theta potential ϑ_1 is a generating function for 2-marked logarithmic Gromov-Witten invariants. This is the last equality of Theorem 1. In §6 we study the theta potential ϑ_1 inside a central chamber of the scattering diagram and show that it is a correction of the Hori-Vafa potential W_Σ, with correction terms coming from interntal scatterin. We explicitly compute ϑ_1 for the first Hirzebruch surfaces $\mathbb{F}_m$ and discuss the role of mutations for toric models and more general computations. In §7 we show that the open mirror map $M(q)$ and the Gross-Siebert slab function $h(z(q))$ both equal the theta potential ϑ_1. These are the other equalities of Theorem 1.

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2. Geometric setup

We first set notation and define the geometric objects for which we will compute Gromov–Witten invariants.

2.1. **Toric log Calabi–Yau pairs.** A log Calabi–Yau pair (X, D) consists of a smooth projective variety X and a reduced normal crossing divisor D such that $D + K_X = 0$, i.e., D is an anticanonical divisor. There might be less restrictive definitions, e.g. in [GS21] the authors consider the case where $D+K_X$ is numerically equivalent to an effective $\mathbb{Q}$-divisor supported on D, but for us this will be the setup. Moreover, we will only consider the case where D is smooth. The "maximal boundary" case with nodal singular divisor has been considered e.g. in [GHK15] and [BBvG24].

If X is a toric variety, then D has the same class as the toric boundary $\partial X = \sum_a D_a$, where D_a are the toric divisors. An ample polarization of X gives an embedding into a projective space. This can be described by a polytope Δ and we write $X = \mathbb{P}_\Delta$. We write $n = \dim_{\mathbb{C}} X$ so that $\dim_{\mathbb{C}} \mathbb{P}_\Delta = \dim_{\mathbb{R}} \Delta = n$. When thinking of $\mathbb{P}_\Delta$ as a symplectic manifold, the Newton polytope is the moment polytope, namely it is the image of the moment map $\mu : \mathbb{P}_\Delta \longrightarrow \mathfrak{g}^*$ where $\mathfrak{g}^* \cong \mathbb{R}^n$ is the dual Lie algebra of the dense algebraic torus $G = T := (\mathbb{C}^\times)^n$ in $\mathbb{P}_\Delta$. When Δ is reflexive it is also known as a Delzant polytope.

The combinatorics of Δ and hence $\mathbb{P}_\Delta$ are encoded in the normal fan Σ, which is constructed as follows. The 1-dimensional cones $\rho_i \in \Sigma^{(1)} \subset \Sigma$ are rays normal to[1] the codimension-1 facets of Δ. We write $r = |\Sigma^{(1)}|$ for the number of rays in the fan. One can construct the higher dimensional cones $\sigma_{i_1 \ldots i_k} \in \Sigma^{(k)} \subset \Sigma$ from the $\binom{r}{k}$ combinations of the ρ_i bounding a k-simplex. The primitive vectors that point along the ρ_i form the vertices v_i of the spanning polytope $\Delta^\star$, which is the unique compact top dimensional cone $\Delta^\star \in \Sigma^{(n)}$.

To construct the spanning polytope for the canonical bundle $K_{\mathbb{P}_\Delta}$, we take the cone over $\Delta^\star$ by placing $\Delta^\star$ in a hyperplane of height 1 above the origin in $\mathbb{R}^{n+1}$

$$\operatorname{Cone}(\Delta^\star) = \{(c\nu, c) \,|\, \nu \in \Delta^\star\,,\, c \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^n \times \mathbb{R} \tag{1}$$

Let $v_i \in \Delta^\star$ be the vertices of the spanning polytope and set $\overline{v}_i = (v_i, 1) \in \Delta^\star \times \{1\} \subset \operatorname{Cone}(\Delta^\star)$. Define the vectors $\{\ell^{(a)}\}_{a=1}^{h^{1,1}(\mathbb{P}_\Delta)}$ from the linear relations of the $\overline{v}_i$

$$\sum_{i=0}^{r} \ell_i^{(a)} \overline{v}_i = 0 \tag{2}$$

where $v_0 = (0, \ldots, 0) \in \mathbb{R}^n$. By including v_0 we are restricting to the anticanonical hypersurface $X \hookrightarrow \mathbb{P}_\Delta$, and the Calabi-Yau condition is equivalent to $\sum_i \ell_i^{(a)} = 0$. Then $\ell^{(a)}$ are the generators of the Mori cone, which is dual to the Kähler cone of $\mathbb{P}_\Delta$. The *Mori vectors* $\ell^{(a)}$ allow us to determine several objects associated to $\mathbb{P}_\Delta$, including but not limited to, the Picard–Fuchs operators (see Equation 24), the fundamental period (see Equation 80), and the invariant (algebraic) coordinates on the mirror complex structure moduli space $z_a \in \mathscr{M}_{\mathbb{C}}(\check{X})$ [HKTY95a]

$$z_a = (-1)^{\ell_0^{(a)}} \prod_{i=1}^{r} a_i^{\ell_i^{(a)}}. \tag{3}$$

Here the a_i are coordinates on $\mathscr{M}_{\mathbb{C}}(\check{X})$ that enter the Hori–Vafa superpotential[2] [HV]

$$W = \sum_{i=0}^{r} a_i Y^{v_i}. \tag{4}$$

We can view $W : (\mathbb{C}^*)^n \to \mathbb{C}$ as a torus fibration and compute its period

$$\varpi(W) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_0} \frac{d\log Y_1 \wedge \cdots \wedge d\log Y_n}{a_0 + W} \tag{5}$$

[1]By convention, we choose the inward pointing normals, i.e det $(\nu|m_1|\ldots|m_{n-1}) < 0$ where ν is the normal to the facet spanned by the m_i.

[2]Here $Y^{v_i} = Y_1^{v_i^1} \ldots Y_n^{v_i^n}$ and $v_i = (v_i^1, \ldots, v_i^n) \in \Delta^\star \cap \mathbb{Z}^n$.

One can construct W purely from the vertices v_i of the spanning polytope $\Delta^\star$ associated to the toric variety $\mathbb{P}_\Delta$, and for this reason the additive relations of the v_i are exactly the multiplicative relations of the terms of W. We perform a power series expansion on the integrand so that, due to the form of the differential, the residues are given by the terms that are constant with respect to the Y_i.

$$\begin{aligned}
\varpi(W) &= \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_0} \left(\frac{dY_1 \wedge \dots \wedge dY_n}{Y_1 \dots Y_n}\right) \frac{1}{a_0 + W} \\
&= \sum_{k=0}^{\infty} \left(-\frac{W}{a_0}\right)^k \Bigg|_{Y_i \text{ const.}} \\
&= \sum_{k=0}^{\infty} (-a_0)^{-k} \sum_{\nu_1 + \dots + \nu_r = k} \left(\frac{k!}{\nu_1! \dots \nu_r!} (a_1 Y^{v_1})^{\nu_1} \dots (a_r Y^{v_r})^{\nu_r}\right) \Bigg|_{Y_i \text{ const.}} \\
&= \sum_{k_1 \geq 0} \cdots \sum_{k_s \geq 0} \frac{\prod_j \Gamma(-\sum_a \ell_{0j}^{(a)} k_a + 1)}{\prod_i \Gamma(\sum_a \ell_i^{(a)} k_a + 1)} z_1^{k_1} \dots z_s^{k_s}
\end{aligned} \tag{6}$$

Each vertex $v_i \in \Delta^\star$ corresponds to a prime toric divisor $D_i \subset \mathbb{P}_\Delta$. The anticanonical divisor is given by the sum of all the toric divisors $-K_{\mathbb{P}_\Delta} = \sum_{i=1}^r D_i$. Let s be the number of linearly independent divisors so that the Picard group $\mathrm{Pic}(\mathbb{P}_\Delta)$ has s generators and hence $\mathrm{Pic}(\mathbb{P}_\Delta) \cong \mathbb{Z}^s$. The linear relations among the v_i generate the Stanley–Reisner ideal $\mathfrak{I}$ of the Chow ring $A_*(\mathbb{P}_\Delta)$, and the toric variety can be constructed as the quotient space

$$\mathbb{P}_\Delta = \frac{\mathbb{C}^r \backslash Z_{\mathfrak{I}}}{\mathrm{Hom}(\mathrm{Pic}(\mathbb{P}_\Delta), \mathbb{C}^*)}. \tag{7}$$

Here $Z_{\mathfrak{I}}$ is the variety associated to the Stanley–Reisner ideal $\mathfrak{I}$. One can use $\mathfrak{I}$ to determine the intersection numbers of the D_i.

Example 2.1 ($\mathbb{P}_\Delta = \mathbb{P}^2$)**.** The spanning polytope for 2-dimensional complex projective space is given by the convex hull of the following 3 vertices, which contains the origin v_0 as its unique interior point.

$$\Delta^\star_{\mathbb{P}^2} = \mathrm{Conv}\{(0,0),(1,0),(0,1),(-1,-1)\} = \mathrm{Conv}\{v_i\}_{i=0}^3 \tag{8}$$

The spanning polytope for the canoncial bundle over $\mathbb{P}^2$ is given by the cone over $\Delta^\star_{\mathbb{P}^2}$

$$\Delta^\star_{K_{\mathbb{P}^2}} = \mathrm{Conv}\{(0,0,0), \Delta^\star_{\mathbb{P}^2} \times \{1\}\} \tag{9}$$

The vertices satisfy the linear relation

$$-(v_0,3) + (v_1,1) + (v_2,1) + (v_3,1) = 0 \tag{10}$$

We write this relation in terms of the *Mori vector* $\ell = (-3;1,1,1)$. Since there is only one linearly independent relation among the v_i, we have $s = 1$ and $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ is generated by the class of the line L (hyperplane class). The anticanonical divisor can be thought of as the divisor $-D_0$ corresponding to the unique interior point $(v_0,1) \in \Delta^\star \times \{1\}$. This is given by $-K_{\mathbb{P}^2} = -D_0 = 3L$. By Bézout's theorem $L^2 = 1$ so the degree of $\mathbb{P}^2$ is $(-K_{\mathbb{P}2})^2 = 9$.

Example 2.2 ($\mathbb{P}_\Delta = \mathbb{F}_m$)**.** The Hirzebruch surfaces $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathbb{C})$ are an infinite sequence of toric surfaces which can be thought of as twisted $\mathbb{P}^1$ bundles over $\mathbb{P}^1$, with the parameter m controlling the twisting of the fibers. The spanning polytope $\Delta^\star_m := \Delta^\star_{\mathbb{F}_m}$ for $\mathbb{F}_m$ has the following 4 vertices, see Figure 1.

$$v_1 = (-1, 0)\ ,\ v_2 = (1, 0)\ ,\ v_3 = (0, 1)\ ,\ v_4 = (-m, -1) \tag{11}$$

For the $m > 2$, we find that $\Delta^\star_m$ is not reflexive. The vertices satisfy the linear relations

$$\ell^{(1)} = (\qquad -2;\quad 1,\ 1,\ 0,\ 0) \tag{12}$$

$$\ell^{(2)} = (\ -(2-m);\ -m,\ 0,\ 1,\ 1) \tag{13}$$

so for $m > 2$ we have $\ell^{(2)}_0 > 0$. If E is the exceptional divisor class, F is divisor class corresponding to the fiber, and S is the divisor class corresponding to a section, we have

$$E^2 = -m\ ,\ F^2 = 0\ ,\ S^2 = m \tag{14}$$

in the Chow ring $A_*(\mathbb{F}_m)$. Since there are two linearly independent relations between the vertices of $\Delta^\star_m$, we know $\mathrm{Pic}(\mathbb{F}_m) \cong \mathbb{Z}^2$. We will use both $\{S, F\}$ and $\{E, F\}$ as a basis for $\mathrm{Pic}(\mathbb{F}_m)$, with the two bases being related by $E = S - mF$. The anticanonical divisor is given by

$$-K_{\mathbb{F}_m} = 2S + (2-m)F = 2E + (2+m)F \tag{15}$$

so that the degree of $\mathbb{F}_m$ is $(-K_{\mathbb{F}_m})^2 = 4m + 4(2-m) = 8$, independent of m.

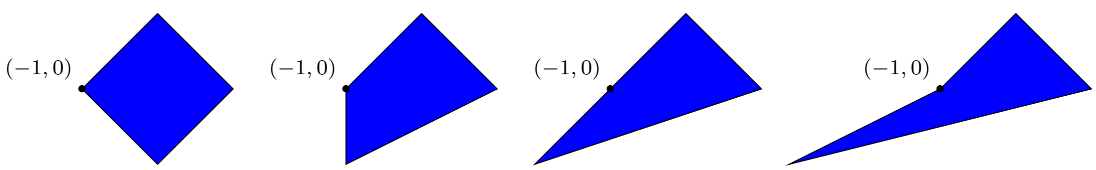

FIGURE 1. The spanning polytopes $\Delta^\star_m$ for the m-twisted Hirzebruch surface $\mathbb{F}_m$ with $m = 0, 1, 2, 3$. The VEX point cases $\Delta^\star_m$ to be non-convex for $m \geq 3$

Example 2.3 ($\mathbb{P}_\Delta = \mathbb{P}^2_{(1,1,q)}$)**.** The weighted projective spaces $\mathbb{P}^2_{(1,1,q)}$ will appear in the central fibers of the toric degenerations that we will consider. As a quotient space,

$$\mathbb{P}^2_{(1,1,q)} = (\mathbb{C}^*)^3 /\!\sim \qquad \text{where} \qquad [x_0, x_1, x_2] \sim [\lambda x_0, \lambda x_1, \lambda^q x_2]\ ,\ \lambda \in \mathbb{C}^*$$

The vertices $\{v_i\}_{i=1}^3$ of the spanning polytope for $\mathbb{P}^2_{(1,1,q)}$ satisfy

$$v_1 + v_2 + q v_3 = 0. \tag{16}$$

2.2. **Toric models.**

Definition 2.4. A toric variety X_Σ is a *toric model* of a variety X if there is a $\mathbb{Q}$-Gorenstein deformation between X and X_Σ.

Example 2.5. A toric model of a cubic surface X is given by the blow up of $\mathbb{P}^2$ in six points that lie on a conic. Such a blow up can be performed torically, resulting in the fan shown in Figure 2.

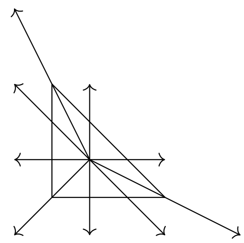

FIGURE 2. The fan and spanning polytope of a toric model for the cubic surface.

Example 2.6. A Hirzebruch surface $\mathbb{F}_m$ is a smooth toric variety, but it is not Fano. A toric model that is Fano is given by $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ if m is even and by $\mathbb{F}_1 = \mathrm{Bl}\,\mathbb{P}^2$ if m is odd. This is clear, because there is a smooth and flat family

$$\mathcal{F} = \{x_0^m y_1 - x_1^m y_0 + t x_0^{m-k} x_1^k y_2 = 0\} \subset \mathbb{P}^1_{x_0,x_1} \times \mathbb{P}^2_{y_0,y_1,y_2} \times \mathbb{A}^1_t,$$

such that the fiber over $t = 0$ is $\mathbb{F}_m$ and the general fiber is $\mathbb{F}_{m-2k}$. The family leaves the anticanonical polarization unchanged (i.e. it is $\mathbb{Q}$-Gorenstein) and acts on curve classes by

$$\begin{aligned} \mathbb{F}_m &\to \mathbb{F}_{m-2k}, \\ F &\mapsto F, \\ E &\mapsto E - kF, \\ S &\mapsto E + kF. \end{aligned}$$

Here F, E and S are the classes of a fiber, the exceptional section, and a general section, respectively.

Remark 2.7. The toric models of a Fano variety are related via mutations of their spanning polytopes, see Proposition 6.9. In §6 we will study a similar relation for non-Fano varieties and mutations of their potentials.

2.3. **Toric degenerations.** In order to construct the mirror $\check{X}$ to a (log) Calabi-Yau X in the Gross–Siebert mirror symmetry program, one must first place X in a toric degeneration $\pi : \mathfrak{X} \longrightarrow \mathbb{A}^1$. This is a family of (log) Calabi-Yaus X_t such that $\pi^{-1}(t) \cong (X, D)$ for $t \neq 0$. The family degenerates to a *central fiber* $X_0 := \pi^{-1}(0)$, which is a union of toric varieties, one for each vertex v_i of the spanning polytope

$$X_0 = \coprod_{v_i} X_i. \tag{17}$$

and the gluing data required to build the central fiber is encoded in Δ.

Definition 2.8. A *toric degeneration* of (X, D) is a flat family $(\mathfrak{X}, \mathfrak{D}) \to T$ over a base T (typically $T = \mathbb{A}^1_t = \mathrm{Spec}\ \mathbb{C}[t]$ or $T = \mathrm{Spec}\ \mathbb{C}[\![t]\!]$) such that the general fiber is isomorphic to (X, D) and the special fiber X_0 is a union of toric varieties glued along toric divisors (and D_0 is a union of toric divisors not involved in the gluing) and such that the family is strictly semistable (i.e. locally of the form $x_1 \cdots x_k = t^l$) away from a codimension 2 subset $Z \subset X_0$. The latter condition is equivalent to $\mathfrak{X} \to T$ being log smooth with respect to the divisorial log structures defined by $X_0 \subset \mathfrak{X}$ and $\{0\} \subset T$. In our case we want to keep track of the curve classes of X, so we work over the base $T = \mathrm{Spec}\ \mathbb{C}[\mathrm{NE}(X)][\![t]\!]$.

Example 2.9 ($\check{\mathfrak{X}}$ for $(\mathbb{P}^2, 3L)$)**.** The following algebraic family serves as a toric degeneration for the log Calabi–Yau pair for $\mathbb{P}^2$.

$$\check{\mathfrak{X}} = \{x_1x_2x_3 = t(y_1 + f_3)\} \tag{18}$$

for $(x_1, x_2, x_3, y_1) \in \mathbb{P}(1, 1, 1, 3)$, $t \in \mathbb{A}^1$, and f_3 is a cubic polynomial that deforms $\check{\mathfrak{X}}$. The zero locus of y_1 defines a family of divisors $\check{\mathfrak{D}} \subset \check{\mathfrak{X}}$. The case $f_3 = 0$ corresponds to the toric boundary divisor $\partial\mathbb{P}^2$, whereas a nontrivial cubic gives the anticannonical family. The central fiber is given by a union of three weighted projective spaces

$$\check{\mathbb{P}}_0 = \mathbb{P}^2(1, 1, 3) \coprod \mathbb{P}^2(1, 1, 3) \coprod \mathbb{P}^2(1, 1, 3) \tag{19}$$

which are glued along toric divisors as prescribed by the combinatorics of Δ.

Remark 2.10. There is a way to construct $\check{\mathfrak{X}}$ as a formal scheme over a certain tropical homology group. Let $\iota : \operatorname{Sing} \check{B} \hookrightarrow \check{B}$ be the inclusion and let Λ be the sheaf of integral tangent vectors. Then we construct the mirror degeneration as

$$\check{\pi} : \check{\mathfrak{X}} \longrightarrow \operatorname{Spf} H^1(\check{B}, \iota_*\check{\Lambda})^* [\![t]\!] \tag{20}$$

There is an isomorphism between $H^1(\check{B}, \iota_*\check{\Lambda}; \mathbb{Z})$ and $\operatorname{Pic}(\mathbb{P}_\Delta)$. The tropicalization of $(\check{\mathfrak{X}}, \check{\mathfrak{D}})$ is exactly the triple $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$. In [Rud21], it is shown that there exists a perfect pairing between $H^1(\check{B}, \iota_*\check{\Lambda})$ and its dual.

In the Fano case the toric degeneration can be constructed globally as a projective variety (in the so called cone or polytope picture), while for the non-Fano case we have to work more locally (in the fan picture).

2.3.1. *The toric Fano case.* For a toric Fano variety $\mathbb{P}_\Delta$, one can construct a toric degeneration of $\mathbb{P}_\Delta$ from its polytope Δ. This is a Fano polytope, i.e., it has a unique interior lattice point. Consider the (unique) convex piecewise linear function $\check{\varphi} : \Delta \to \mathbb{R}$ that takes values 0 at the interior lattice point and 1 along the boundary of Δ. The domains of linearity of $\check{\varphi}$ give the central subdivision of Δ. The upper convex hull of $\check{\varphi}$,

$$\Delta_{\check{\varphi}} = \{(m, h) \in \Delta \times \mathbb{R} \mid h \geq \check{\varphi}(m)\},$$

is an unbounded polytope. It defines a (non-projective) toric variety $\mathfrak{X}_\varphi := X_{\Delta_{\check{\varphi}}}$. Projection to the last factor (which is the unique unbounded direction) defines a map $\mathfrak{X}_{\check{\varphi}} \to \mathbb{A}^1$. This is a flat family of projective toric varieties. The general fiber is isomorphic to $\mathbb{P}_\Delta$ and the central fiber is a union of toric varieties, corresponding to the components of the central subdivision of Δ. One can define a family of divisors $\mathfrak{D} \to \mathbb{A}^1$ by the vanishing of the coordinate corresponding to the interior lattice point. The general fiber gives the toric boundary of $\mathbb{P}_\Delta$. Hence, $\mathfrak{X}_{\check{\varphi}} \to \mathbb{A}^1$ is a toric degeneration of the log Calabi-Yau pair $(\mathbb{P}_\Delta, \partial\mathbb{P}_\Delta)$. The polytope $\check{B} = \Delta$ together with its central subdivison $\check{\mathcal{P}}$ and the PL function $\check{\varphi}$ is a polarized polyhedral affine manifold $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ called the *intersection complex* of $(\mathbb{P}_\Delta, X)$

2.3.2. *The smooth Fano case.* The toric degeneration $\mathfrak{X}_{\check{\varphi}}$ constructed above is a subvariety of $\mathbb{P}^{N-1} \times \mathbb{A}^1$, where N is the number of lattice points of Δ. Its defining equations correspond to relations among the lattice points of Δ, and in general it is not a complete intersection. Now consider a sufficiently general deformation of the defining equations without changing the central fiber. This yields another toric degeneration $\mathfrak{X} \to \mathbb{A}^1$. The general fiber is a pair (X, D), where X is a $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}_\Delta$ and D is a smoothing of $\partial\mathbb{P}_\Delta$. In other words, $(\mathbb{P}_\Delta, \partial\mathbb{P}_\Delta)$ is a *toric model* of (X, D), see §2.2.

The smoothing leads to a modification of the intersection complex. Before, Δ was a polytope, and in particular an affine manifold (a topological manifold with affine linear transition maps). The deformation introduces affine singularities, one on each interior edge of the central subdivision $\mathcal{P}$, such that now $\check{B} = \Delta$ is an affine manifold with singularities. We have to introduce these singularities to ensure that locally at each vertex of Δ the family $\mathfrak{X} \to \mathbb{A}^1$ can still be described by the toric construction $X_{\Delta_{\check{\varphi}}}$ above.

2.3.3. *The fan picture.* Dual to the polytope picture, toric varieties can be described by fans, and a union of toric varieties can be described by gluing together such fans, resulting in a polyhedral complex called the *dual intersection complex.* If X is toric, the total space $\mathfrak{X}$ of the toric degeneration is then simply given by the toric variety obtained by taking the fan over the dual intersection complex. Again, smoothing the general fiber corresponds to changing the affine structure, such that the dual intersection complex becomes an affine manifold with singularities. In this picture, the polarization by the divisor D is given by a multi-valued piecewise linear function $\varphi : B \to \mathbb{Z}^\rho$, where ρ is the Picard rank of X.

The *dual intersection complex* $(B, \mathcal{P}, \varphi)$ can be constructed as the discrete Legendre transform of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$, see [Rud12].

Construction 2.11 (Dual intersection complex $(B, \mathcal{P}, \varphi)$)**.** First, we set $B \cong \mathbb{R}^n$ and construct $\mathcal{P}$ from the normal fan Σ. Let $\mathcal{P}$ be the polyhedral subdivision of B whose codimension-1 cells are the facets of $\Delta^\star$, called *slabs*. The 1-dimensional cells of $\mathcal{P}$ are edges of $\Delta^\star$ and $\rho \setminus \mathrm{Int}\Delta^\star$ for the rays ρ in the normal fan Σ. The top dimensional cells consist of a unique[3] compact chamber $\Delta^\star \in \mathcal{P}$, and several unbounded chambers. We place affine singularities at points $\delta \in B$ along the edges of $\Delta^\star$ and insert codimension-1 monodromy cuts at δ along the adjacent rays ρ_1, ρ_2. Then, the affine structure on B is given by the transformation that maps $\rho_1 \mapsto \rho_2$ and leaves the edge invariant. Lastly, φ is a piecewise linear function that is 0 on $\mathrm{Int}\,\Delta^\star$ and has slope 1 along the 1-dimensional cells of $\mathcal{P}$.

Example 2.12 ($\mathbb{P}^2$)**.** Following Construction 2.11, we plot the dual intersection complex for the anticannonical log Calabi-Yau pair $(\mathbb{P}^2, 3L)$ in Figure 3. The transition maps A_i are given by

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \tag{21}$$

and the monodromy $M = A_3A_2A_1$ is

$$M = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (-K_{\mathbb{P}^2})^2 \\ 0 & 1 \end{bmatrix}. \tag{22}$$

For surfaces, the intersection number $(-K_{\mathbb{P}_\Delta})^2$ is also the volume of the Newton polytope. That is,

$$\mathrm{Vol}(\mathbb{P}_\Delta) = \int_{\mathbb{P}_\Delta} (c_1)^2 = [-K_{\mathbb{P}_\Delta}]^2 = \mathrm{Vol}(\Delta) \tag{23}$$

by the Duistermaat-Heckman theorem [Gui94].

[3] $\Delta^\star$ need not be the unique compact chamber when $\mathbb{P}_\Delta$ is non-Fano. As we will show, this is the case for $\mathbb{F}_m$ with $m > 2$.

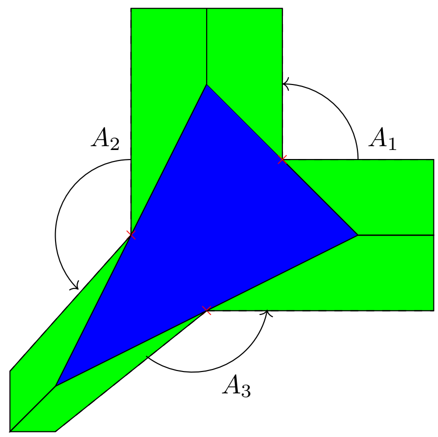

FIGURE 3. The dual intersection complex $(B, \mathcal{P})$ for $(\mathbb{P}^2, 3L)$.

Now, given $(B, \mathcal{P}, \varphi)$ one can construct a mirror pair of central fibers $X_0, \check{X}_0$. The Gross–Siebert mirror symmetry construction associates a mirror toric degeneration $\check{\mathfrak{X}} \to \mathbb{A}^1$. The central fiber of $\check{\mathfrak{X}}$ plays an important role as the uncorrected mirror to (X, D)

Example 2.13. Figure 4 shows the intersection complex of $\mathbb{F}_1$, and the dual intersection complex in two different charts. Note that $\mathbb{F}_1$ is smooth, so the deformation only changes the divisor from the toric boundary to a smooth anticanonical divisor.

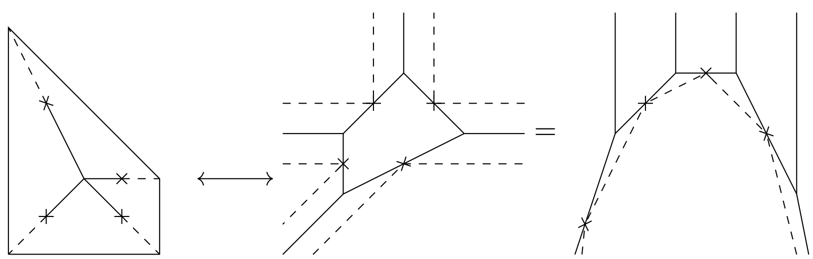

FIGURE 4. The intersection complex of $\mathbb{F}_1$ (left) and the dual intersection complex of $\mathbb{F}_1$ in two different charts (middle and right).

2.3.4. *The non-Fano case.* In the non-Fano case, we cannot work in the polytope picture, because polytopes correspond to toric varieties with an ample polarization. This means we cannot construct a toric degeneration globally as a projective variety. However, we can still work in the fan picture and construct the toric degeneration by gluing together affine pieces described by the gluing of fans. As before the asymptotic rays are the rays of a toric model of X. But now the spanning polytope $\Delta^\star$ is non-convex. In order for all fans to be complete, we need to subdivide $\Delta^\star$. Geometrically this corresponds to a blow up of a torus fixed point. Since the point lies on the central fiber, this does not change the general fiber. The blow up introduces a new component of the central fiber, corresponding to the new vertex. On the other components it acts like to blow up at a point. While $\Delta^\star$ was non-convex, the cells of the new subdivision are all convex.

Note that $\mathfrak{X}_\varphi$ is not defined as a subvariety of $\mathbb{P}^{N-1} \times \mathbb{A}^1$, so we cannot simply deform its defining equations. But $\mathfrak{X}_\varphi$ is defined by gluing toric varieties along toric divisors, and we can perturb this gluing. This is just the same as in the Fano case, and again we introduce affine singularities on the interior edges of the dual intersection complex. So while we don't have an intersection complex for $\mathfrak{X} \to \mathbb{A}^1$, we indeed have a dual intersection complex.

Example 2.14. Figure 5 shows the intersection complex of $\mathbb{F}_3$, and the dual intersection complex in two different charts. The spanning polytope $\Delta^\star$ is non-convex and subdivided.

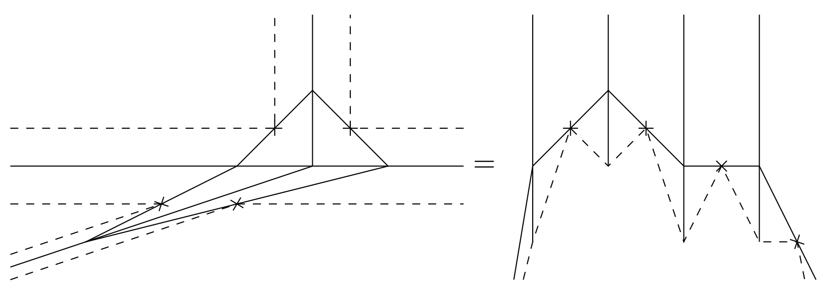

FIGURE 5. The dual intersection complex of $\mathbb{F}_3$ in two different charts.

2.3.5. *Multi-polytopes.* If one tries to find a intersection complex, i.e. polytope picture, for a non-Fano variety, one ends up with the notion of a multi-polytope.

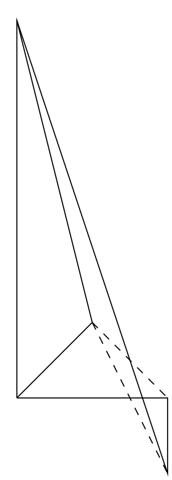

FIGURE 6. A subdivision of a multi-polytope can be seen as the intersection complex of $\mathbb{F}_3$.

2.4. **Scattering and broken lines.** In this section we introduce the concepts of tropical curves, scattering diagrams and broken lines, and show that they are all related to each other. Then we show that they give certain relative (or logarithmic) Gromov-Witten invariants, which basically follows from the degeneration formula for logarithmic Gromov-Witten invariants.

2.4.1. *Scattering diagrams.* Let B be an affine manifold with singularities and let φ be a piecewise affine function on B. In general we will assume φ to be multi-valued to distinguish curve classes, but taking the total degree (intersection with the divisor D) we obtain a single valued function. Let $\iota : B^\circ \hookrightarrow B$ be the complement of the singular locus and let $\Lambda_B = \iota_\star \Lambda_{B^\circ}$ be the pushforward of the sheaf of integral tangent vectors on B°. For simplicity we restrict to the 2-dimensional case.

Definition 2.15. A *ray* $\mathfrak{d}$ on B consists of a base $b_\mathfrak{d} \in B$, a direction $m_\mathfrak{d} \in \Lambda_{B,b_\mathfrak{d}}$ and a function $f_\mathfrak{d} \in \mathbb{C}[z^{m_\mathfrak{d}}][\![t]\!]$. Via parallel transport this defines a section of the sheaf $\mathcal{R}$ whose stalk at a point $x \in B$ is given by $R_x = \varprojlim \mathbb{C}[P_x]/(t^k)$, where

$$P_x = \{p = (m,h) \in \Lambda_{B,x} \oplus \mathbb{Z} \mid h \leq \varphi_x(m)\}.$$

Here φ_x is the linear part of φ locally at x and $t = z^{(0,1)}$, such that the t-order is given by $\varphi_x(m) - h$. Note that the t-order of a ray can increase at it propagates.

We demand the following properties:

- If $b_\mathfrak{d} \in \mathrm{Sing}(B)$, then $f_\mathfrak{d} = 1 + z_\mathfrak{d}^m$. (This is sometimes called a *slab*.) Note that in this case $\Lambda_{B,x}$ is only 1-dimensional, so $m_\mathfrak{d}$ is defined up to sign.
- If $b_\mathfrak{d} \notin \mathrm{Sing}(B)$, then $f_\mathfrak{d} \equiv 1 \bmod (t)$. (This is sometimes called a *wall*.)

Note that we use different sign conventions than many other treatments of this topic. This is to avoid negative signs. For example, [GPS10] uses the convention $f_\mathfrak{d} \in \mathbb{C}[z^{-m_\mathfrak{d}}][\![t]\!]$ for ("outgoing") rays.

Definition 2.16. A *scattering diagram* $\mathscr{S}$ on B is a collection of rays such that for each $k \geq 0$ there are only finitely many rays with $f_\mathfrak{d} \not\equiv 1 \bmod (t^k)$. The *support* $|\mathscr{S}|$ of $\mathscr{S}$ is the union of its rays, considered as subsets of B. A *chamber* $\mathfrak{u}$ of $\mathscr{S}$ is a connected component of $B \setminus |\mathscr{S}|$.

Definition 2.17. An affine manifold with singularities B defines a scattering diagram $\mathscr{S}_0(B)$ by taking all possible slabs (2 for each affine singularity).

Definition 2.18. A point $x \in B$ where two or more rays intersect is called a *vertex*. (This is sometimes called a *joint*.) Locally at a vertex x we produce new rays with base x by the following *scattering procedure*. For each ray $\mathfrak{d}$ containing x consider the complement $\mathfrak{d} \setminus \{x\}$. This consists of one component (if $x = m_\mathfrak{d}$) or two components (otherwise). We cyclically order all such components (with respect to a simple loop around x that is disjoint from $\mathrm{Sing}(B)$) to obtain a sequence $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$. Each $\mathfrak{d}_i$ defines a $\mathbb{C}[\![t]\!]$-automorphism $\theta_{\mathfrak{d}_i}$ of the localized ring $(R_x)_{\prod f_{\mathfrak{d}_i}}$ by

$$\theta_{\mathfrak{d}_i}(z^m) = f_{\mathfrak{d}_i}^{\langle n_i, m\rangle},$$

where n_i is the primitive normal vector to $m_{\mathfrak{d}_i}$, positive with respect to the chosen ordering (i.e. the orientation coming from the loop around x). Now, for $k \geq 0$, define

$$\theta^k = \theta_{\mathfrak{d}_s} \circ \ldots \circ \theta_{\mathfrak{d}_1} \bmod (t^k).$$

It turns out that this acts on z^m by multiplication with a polynomial. Hence, to make θ^k the identity, we have to add finitely many rays. Doing this iteratively for all $k \geq 0$, we obtain scattering diagrams $\mathscr{S}_0, \mathscr{S}_1, \ldots$ that are *consistent to order* k. Taking the formal limit we get a consistent scattering diagram $\mathscr{S}_\infty$.

2.4.2. *Broken lines.*

Definition 2.19. A *broken line* for a wall structure $\mathscr{S}$ on B is a proper continuous map $\mathfrak{b} : (-\infty, 0] \to B_0$ with image disjoint from any vertices of $\mathscr{S}$, along with a sequence $-\infty = t_0 < t_1 < \cdots < t_r = 0$ for some $r \geq 1$ with $\mathfrak{b}(t_i) \in |\mathscr{S}|$ for $i \leq r-1$, and for each $i = 1, \ldots, r$ an expression $a_i z^{m_i}$ with $a_i \in \mathbb{C} \setminus \{0\}$, $m_i \in \Lambda_{\mathfrak{b}(t)}$ for any $t \in (t_{i-1}, t_i)$, defined at all points of $\mathfrak{b}([t_{i-1}, t_i])$, and subject to the following conditions:

- $\mathfrak{b}|_{(t_{i-1},t_i)}$ is a non-constant affine map with image contained in a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}$, and $\mathfrak{b}'(t) = -m_i$ for all $t \in (t_{i-1}, t_i)$.
- For each $i = 1, \ldots, r-1$ the expression $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$, i.e., is a term in the expansion of $\theta_{\mathfrak{d}}(a_i z^{m_i})$, where $\mathfrak{d}$ is the ray that contains the intersection of closures $\bar{\mathfrak{u}_i} \cap \bar{\mathfrak{u}_{i+1}}$.
- $a_1 = 1$ and (m_1, h) has t-order zero at $\mathfrak{b}(t_1)$, i.e., $h = \varphi(m_1)$.

Write $a_{\mathfrak{b}} z^{m_{\mathfrak{b}}}$ for the *ending monomial* $a_r z^{m_r}$.

Example 2.20. Figure 7 shows the scattering diagram and some broken lines for $\mathbb{F}_3$, in two different charts.

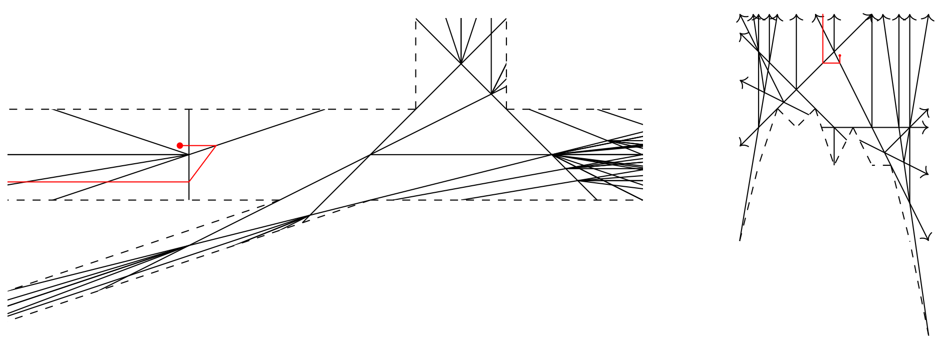

FIGURE 7. Scattering diagram and a broken line for $\mathbb{F}_3$.

The idea of the Gross-Siebert program is that the dual intersection complex of X after scattering, i.e. $\mathscr{S}_\infty(B)$, is the intersection complex of its mirror $\check{X}$. One can construct $\check{X}$ from B by gluing thickenings of affine models [GS12]. The consistency of $\mathscr{S}_\infty(B)$ ensures that the gluing does not depend on the affine chart.

2.5. **Landau–Ginzburg models via theta functions.** Supersymmetric sigma models with Fano target X are mirror to Landau–Ginzburg models $W : (\mathbb{C}^*)^n \to \mathbb{C}$ [HV].

The superpotential W is given by the sum of ending monomials of all broken lines on $\mathscr{S}_\infty(B)$ [CPS]. This is made precise with the definition of theta functions [GHS19].

Theta functions are sections of bundles over toric degenerations; in our case we would like to consider theta functions for $\check{\mathfrak{X}}$.

Definition 2.21. Let X be a variety with dual intersection complex (of a toric model) B. Let m be an asymptotic direction on B. Let $\mathfrak{B}_m(B)_P$ be the set of broken lines $\mathfrak{b}$ on $\mathscr{S}_\infty(B)$ with initial monomial $a_1 z^{m_1} = z^m$ and endpoint $\mathfrak{b}(0) = P$. Define the *theta function*

$$\vartheta_m(X)_P = \sum_{\mathfrak{b} \in \mathfrak{B}_m(B)_P} a_{\mathfrak{b}} z^{m_{\mathfrak{b}}}.$$

In our case, with smooth divisor D, the dual intersection complex B has exactly one unbounded direction m_{out}, so asymptotic directions on B are just multiples of m_{out}. We write $\vartheta_q(P)$ for $\vartheta_{q \cdot m_{\text{out}}}(P)$ with $q \in \mathbb{N}$.

The proper Landau–Ginzburg superpotential is given by the primitive theta function $W = \vartheta_1(x)$, and, as shown in [GRZ], this gives the open mirror map.

Consistency of the scattering diagram $\mathscr{S}_\infty(X)$ (see §2.4) guarantees that the local description $\vartheta_p(x)$ for theta functions patch together to give global functions on $\check{\mathfrak{X}}$, i.e. give an element of $\Gamma(\check{\mathfrak{X}}, \mathcal{O}_{\check{\mathfrak{X}}})$. In other words, the ϑ_p form coordinates on the toric degeneration $\check{\mathfrak{X}}$ and we can form the coordinate ring. The central idea of intrinsic mirror symmetry [GS19], is that the mirror is the spectrum of this ring.

Proposition 2.22 ([GHS19], Theorem 3.24, [GS19], Theorem 1.9)**.** Theta functions generate a commutative ring with unit ϑ_0 by the multiplication rule

$$\vartheta_p(X)_P \cdot \vartheta_q(X)_P = \sum_{r=0}^{\infty} \alpha^r_{p,q} \vartheta_r(X)_P$$

with structure constants

$$\alpha^r_{p,q} = \sum_{\substack{(\mathfrak{b}_1,\mathfrak{b}_2) \in \mathfrak{B}_p(B)_P \times \mathfrak{B}_q(B)_P \\ m_{\mathfrak{b}_1} + m_{\mathfrak{b}_2} = r}} a_{\mathfrak{b}_1} a_{\mathfrak{b}_2}.$$

The theta functions $\vartheta_p(X)_P$ only depend on the chamber containing P and the structure constants $\alpha^r_{p,q}$ do not depend on P.

The structure constants α^r_{pq} are equal to certain punctured Gromov-Witten invariants [ACGS21], which can be seen as an algebraic analogue of holomorphic disks. The idea of *intrinsic mirror symmetry* [GS19] is that one can circumvent the gluing procedure above and directly define the mirror to X as the spectrum of the ring of theta functions, with structure constants α^r_{pq} given punctured Gromov-Witten invariants.

3. B-model: periods and Picard–Fuchs equations

Curve counting invariants of a Calabi–Yau manifold X are encoded in the periods of its mirror $\check{X}$ [CDGP91]. The periods satisfy a system of differential equations of generalized hypergeometric type called the *Picard–Fuchs equations*. The Picard–Fuchs operators $\mathcal{L}_a$ are determined by the Mori vectors $\ell^{(a)}$ as follows

$$\mathcal{L}_a = \prod_{\ell_i^{(a)}>0} \prod_{j=0}^{\ell_i^{(a)}-1} \left(\sum_{b=1}^{s} \ell_i^{(b)} \theta_b - j \right) - z_a \prod_{\ell_i^{(a)}<0} \prod_{j=0}^{-\ell_i^{(a)}-1} \left(\sum_{b=1}^{s} \ell_i^{(b)} \theta_b - j \right) \tag{24}$$

where $\theta_a = z_a \partial_{z_a}$ is the logarithmic derivative with respect to z_a.

The solutions of the closed Picard–Fuchs system $L_c = \{\mathcal{L}_a\}_{a=1}^s$ are given by the periods of $\check{X}$. There is a solution $\varpi^{(0)}(z)$ to L_c called the *fundamental period*, see Equation 80. In the compact case, all of the solutions to the Picard–Fuchs system can be obtained from appropriate linear combinations of derivatives of $\varpi^{(0)}$ [HKTY95b]. This allows us to construct the *period vector*, which for a Calabi–Yau 3-fold is of the form

$$\Pi = \begin{bmatrix} \varpi^{(0)} \\ \varpi_a^{(1)} \\ \varpi_a^{(2)} \\ \varpi^{(3)} \end{bmatrix} \tag{25}$$

where $\varpi_a^{(1)}, \varpi_a^{(2)}, \varpi^{(3)}$ are obtained using the Frobenius method. Since the roots of the $\mathcal{L}_a$ are separated by integers, the additional solutions will have $(\log z_a)^k$ terms for $0 \leq k \leq 3$, and a power series term. If κ_{ijk} is the intersection matrix of X, then

$$\begin{aligned} \varpi_a^{(1)} &= \partial_{n_a} \varpi^{(0)} \\ \varpi_a^{(2)} &= \frac{1}{2} \kappa_{ajk} \partial_{n_j} \partial_{n_k} \varpi^{(0)} \\ \varpi^{(3)} &= -\frac{1}{6} \kappa_{ijk} \partial_{n_i} \partial_{n_j} \partial_{n_k} \varpi^{(0)} \end{aligned} \tag{26}$$

The closed mirror maps $t_a(z)$ are given by the ratio of the log periods to the fundamental period.

$$t_a(z) = \frac{\varpi_a^{(1)}(z)}{\varpi^{(0)}(z)} = \log(z_a) + \sum_n S_a(z) \tag{27}$$

Define the closed string symplectic parameters

$$q_a = e^{t_a} = z_a e^{S_a(z)}. \tag{28}$$

There is a linear combination $\varpi_{\mathcal{W}}^{(2)}$ of the $\varpi_a^{(2)}$ solutions which, after inverting the mirror map to obtain $z_a(t_a)$, can be used to define a 4-dimensional $\mathscr{N} = 1$ superpotential [May]

$$\mathcal{W}(t) = \frac{\varpi_{\mathcal{W}}^{(2)}(z(t))}{\varpi^{(0)}(z(t))} \tag{29}$$

The associated Yukawa couplings can be computed by differentiating the $\varpi_a^{(2)}$ solutions

$$Y_{ijk} = \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \frac{\varpi_k^{(2)}(z(t))}{\varpi^{(0)}(z(t))} \tag{30}$$

The Yukawa couplings can be expanded to obtain the counts of rational curves on the anticanonical Calabi–Yau hypersurface $X \hookrightarrow \mathbb{P}_\Delta$.

Example 3.1 (Elliptic fibrations over $\mathbb{F}_m$)**.** Elliptic fibrations over toric surfaces can be constructed as a hypersurface in weighted projective space $\mathbb{P}^4(w_1, w_2, w_3, w_4, 1)$ [HKTY95a]. The weights are chosen so that one can obtain a Weierstrass model over the chosen toric base. Here we construct an elliptic fibrations over the Hirzebruch surfaces $\mathbb{F}_m$ for $m \leq 3$. Consider the spanning polytope $\Delta^\star \subset \mathbb{R}^4$ with the vertices

$$
\begin{aligned}
v_1 = (1,0,0,0),\ v_2 = (0,1,0,0),\ v_3 = (0,0,1,0),\ v_4 = (0,0,0,1) \\
v_5 = (-w_1,-w_2,-w_3,-w_4),\ v_6 = (-3,-2,0,0), \\
v_7 = (-3(2+m), -2(2+m), -m, -1)
\end{aligned} \tag{31}
$$

For $m \leq 2$, the Mori vectors can be written directly by using the Mori vectors from Example 2.2

$$
\begin{aligned}
\ell^{(1)} &= (\ -6;\ 2,\ 3,\ \ 1,\ \ 0,\ 0,\ 0,\ 0) \\
\ell^{(2)} &= (\ \ 0;\ 0,\ 0,\ -2,\ \ 1,\ 1,\ 0,\ 0) \\
\ell^{(3)} &= (\ \ 0;\ 0,\ 0,\ m-2,\ -m,\ 0,\ 1,\ 1)
\end{aligned} \tag{32}
$$

For $m = 3$, the Mori vectors are

$$
\begin{aligned}
\ell^{(1)} &= (\ -6;\ \ 2,\ \ 3,\ \ 1,\ \ 0,\ 0,\ 0,\ 0) \\
\ell^{(2)} &= (\ \ 0;\ \ 0,\ \ 0,\ -2,\ \ 1,\ 1,\ 0,\ 0) \\
\ell^{(3)} &= (\ \ 6;\ -2,\ -3,\ \ 0,\ -3,\ 0,\ 1,\ 1)
\end{aligned} \tag{33}
$$

and the associated Picard–Fuchs operators are

$$\mathcal{L}_1 = (\theta_1 - \theta_3)(\theta_1 - 2\theta_2)\theta_1 - 12z_1\theta_1(6(\theta_3 - \theta_1) + 1)(6(\theta_3 - \theta_1) + 5) \tag{34}$$

$$\mathcal{L}_2 = (\theta_2 - 3\theta_3)\theta_2 - z_2(\theta_1 - 2\theta_2 - 1)(\theta_1 - 2\theta_2) \tag{35}$$

$$\mathcal{L}_3 = \theta_3^2(6(\theta_3 - \theta_1) + 1)(6(\theta_3 - \theta_1) + 5) - \frac{1}{12}z_3(\theta_1 - \theta_3)\prod_{j=0}^{2}(\theta_2 - 3\theta_3 - j) \tag{36}$$

The fundamental period is given by

$$\varpi^{(0)} = \sum_{n_1 \geq 2n_2 \geq 6n_3} \frac{\Gamma(6(n_1 - n_3) + 1)}{\Gamma_e(n_1, n_2, n_3)\Gamma_b(n_2, n_3)} z_1^{n_1} z_2^{n_2} z_3^{n_3} \tag{37}$$

where

$$\Gamma_e(n_1, n_2, n_3) = \Gamma(2(n_1 - n_3) + 1)\Gamma(3(n_1 - n_3) + 1)\Gamma(n_1 - 2n_2 + 1) \tag{38}$$

$$\Gamma_b(n_2, n_3) = \Gamma(n_2 - 3n_3 + 1)\Gamma(n_2 + 1)\Gamma^2(n_3 + 1) \tag{39}$$

are Gamma function factors coming from the elliptic fibers and the base, respectively. By taking the large elliptic fiber limit $z_1 \to 0$ (and hence $\theta_1 \to 0$), one can obtain the Picard–Fuchs operators for the canonical bundle $K_{\mathbb{F}_3}$

$$\widetilde{\mathcal{L}}_2 = (\theta_2 - 3\theta_3)\theta_2 - 4z_2\theta_2\left(\theta_2 + \frac{1}{2}\right) \tag{40}$$

$$\widetilde{\mathcal{L}}_3 = \theta_3^2(6\theta_3 - 1)(6\theta_3 - 5) + \frac{1}{12}z_3\theta_3\prod_{j=0}^{2}(\theta_2 - 3\theta_3 - j) \tag{41}$$

In the non-compact case, the fundamental period becomes a constant which we normalize to 1. This can be seen in the above example: to obtain the non-compact space we take $z_1 \to 0$ and the only surviving term in the sum 37 is $n_1 = n_2 = n_3 = 0$, so $\varpi^{(0)} = 1$. Hence we cannot obtain the log and the log squared solutions by differentiating $\varpi^{(0)}$. We illustrate our approach in the non-compact case with the following example.

Example 3.2 (Closed mirror map for $K_{\mathbb{F}_3}$)**.** From Example 2.2, we have the Mori vectors $\ell^{(1)}, \ell^{(2)}$ for $K_{\mathbb{F}_3}$. We use these to construct the Picard–Fuchs operators following Equation 24

$$\mathcal{L}_1 = \theta_1(\theta_1 - 3\theta_2) - z_1(-2\theta_1 + \theta_2)(-2\theta_1 + \theta_2 - 1) \tag{42}$$

$$\mathcal{L}_2 = (-2\theta_1 + \theta_2)\theta_2 - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2) \tag{43}$$

The fundamental period is again given by the constant solution $\varpi^{(0)} = 1$, and hence we look for a solution of the form in Equation 27. Let

$$F_a(z_1, z_2) = \sum_{n_1, n_2 \geq 0} c^{(a)}_{n_1,n_2} z_1^{n_1} z_2^{n_2} \tag{44}$$

be power series in z_1 and z_2. The power series terms S_a of the closed mirror maps $t_a = \log z_a + S_a(z_1, z_2)$ will be written as linear combinations of the F_a. By imposing $\mathcal{L}_b F_a = 0$, we get recurrence relations on the coefficients $c^{(a)}_{n_1,n_2}$.

$$\mathcal{L}_1 F_a = 0 \implies c^{(a)}_{n_1+1,n_2} = \frac{(2n_1 - n_2)(2n_1 - n_2 + 1)}{(n_1 + 1)(n_1 - 3n_2 + 1)} c^{(a)}_{n_1,n_2} \tag{45}$$

$$\mathcal{L}_2 F_a = 0 \implies c^{(a)}_{n_1,n_2+1} = -\frac{(n_1 - 3n_2)(n_1 - 3n_2 - 1)(n_1 - 3n_2 - 2)}{(n_2 + 1)^2(2n_1 - n_2 - 1)} c^{(a)}_{n_1,n_2} \tag{46}$$

For the first solution, we solve Equation 45 with $n_2 = 0$

$$c^{(1)}_{n_1+1,0} = \frac{2n_1(2n_1 + 1)}{(n_1 + 1)^2} c^{(1)}_{n_1,0} \implies c^{(1)}_{n_1,0} = \frac{(2n_1 - 1)!}{(n_1!)^2}$$

With similar factorial recursion identities (coming from the properties of the Gamma function), one can substitute $c^{(1)}_{n_1,0}$ into Equation 46 to solve for $c^{(1)}_{n_1,n_2}$.

$$\begin{aligned} c^{(1)}_{n_1,n_2} &= (-1)^{n_2} \frac{(2n_1 - n_2 - 1)!}{(2n_1 - 1)!} \frac{n_1!}{(n_1 - 3n_2)!} \frac{1}{(n_2!)^2} c^{(1)}_{n_1,0} \\ &= (-1)^{n_2} \frac{(2n_1 - n_2 - 1)!}{n_1!(n_1 - 3n_2)!(n_2!)^2}. \end{aligned}$$

For the second solution, we solve Equation 46 with $n_1 = 0$

$$c^{(2)}_{0,n_2+1} = -\frac{3n_2(3n_2 + 1)(3n_2 + 2)}{(n_2 + 1)^3} c^{(2)}_{0,n_2} \implies c^{(2)}_{0,n_2} = (-1)^{n_2} \frac{(3n_2 - 1)!}{(n_2!)^3}.$$

After factoring out a minus sign to apply more factorial identities, one can substitute $c^{(2)}_{0,n_2}$ into Equation 45 to solve for $c^{(2)}_{n_1,n_2}$.

$$\begin{aligned} c^{(2)}_{n_1,n_2} &= (-1)^{n_1} \frac{(3n_2 - n_1 - 1)!}{(3n_2 - 1)!} \frac{n_2!}{(n_2 - 2n_1)!} \frac{1}{n_1!} c^{(2)}_{0,n_2} \\ &= (-1)^{n_1+n_2} \frac{(3n_2 - n_1 - 1)!}{n_1!(n_2 - 2n_1)!(n_2!)^2} \end{aligned}$$

One can check that $c^{(1)}_{n_1,n_2}$ and $c^{(2)}_{n_1,n_2}$ solve the recurrence relations 45 and 46. Now to obtain the log solutions we note that $\mathcal{L}_a t_b = \mathcal{L}_a(\log z_b + S_b) = 0$ and hence $\mathcal{L}_a S_b = -\mathcal{L}_a \log z_b$. We calculate

$$\begin{aligned} \mathcal{L}_1 S_1 &= 2z_1 & \mathcal{L}_1 S_2 &= -z_1 \\ \mathcal{L}_2 S_1 &= 2z_2 & \mathcal{L}_2 S_2 &= -6z_2 \end{aligned}$$

Looking at the lowest order terms of F_1 and F_2 we have

$$c^{(1)}_{1,0} z_1 = z_1 \qquad c^{(1)}_{0,1} z_2 = 0$$
$$c^{(2)}_{1,0} z_1 = 0 \qquad c^{(2)}_{0,1} z_2 = -2z_2.$$

This fixes S_1 and S_2

$$S_1 = 2F_1 - F_2 \qquad S_2 = -F_1 + 3F_2 \tag{47}$$

where

$$F_1(z_1, z_2) = \sum_{\substack{n_1,n_2 \geq 0 \\ n_1 \geq 3n_2}} (-1)^{n_2} \frac{\Gamma(2n_1 - n_2)}{\Gamma(n_1)\Gamma(n_1 - 3n_2 + 1)\Gamma^2(n_2 + 1)} z_1^{n_1} z_2^{n_2} \tag{48}$$

$$F_2(z_1, z_2) = \sum_{\substack{n_1,n_2 \geq 0 \\ n_1 \leq 3n_2}} (-1)^{n_1+n_2} \frac{\Gamma(3n_2 - n_1)}{\Gamma(n_1 + 1)\Gamma(n_2 - 2n_1 + 1)\Gamma^2(n_2 + 1)} z_1^{n_1} z_2^{n_2} \tag{49}$$

The z_1 parameter is mirror to the fiber class parameter

$$q_1 = z_1 e^{S_1} \tag{50}$$

and the z_2 parameter is mirror to the exceptional class parameter

$$q_2 = z_2 e^{S_2} \tag{51}$$

The work of Lerche and Mayr [LM, May] studies the $\mathcal{N} = 1$ moduli space for the B-model with D-branes by deriving a system of differential equations that relates the open and closed moduli spaces $L_{oc} = \{\mathcal{L}^{(\mathrm{oc})}_a\}^s_{a=0}$. The operators in L_{oc} involve logarithmic derivatives $\theta_0 = z_0 \partial_{z_0}$ with respect to an additional open string parameter z_0. We have $L_c = L_{oc}|_{\theta_0 \to 0}$, so the solutions to the closed system with no z_0 dependence are also solutions to the open-closed system. The parameters z_0 together with $z_1, \ldots, z_s$ define flat coordinates on the open string moduli space $\mathscr{M}^o_{\mathbb{C}}(\check{X})$. We now explain how to obtain the additional Mori vector $\ell^{(0)}$ with an explicit D-brane construction, following [AV00, AKV02], and give an example for $K_{\mathbb{F}_3}$.

The open-string sector of the B-model consists of D5-branes wrapped on a 2-cycle $C \in H_2(X, \mathbb{Z})$. These are mirror to D6-branes on the mirror manifold $\check{X}$ with a non-perturbative superpotential W. The superpotential for the D5-brane on C can been calculated by integrating over a 3-chain $\Gamma(z_0)$ with $\partial\Gamma(z_0) = C(z_0) - C_*$

$$W(z_0) = \int_{\Gamma(z_0)} \Omega = \int_{C(z_0) - C_*} \omega \tag{52}$$

Define the $\mathbb{C}^*$ coordinates $\widetilde{y}_i = Y^{v_i}$ and form the conical bundle

$$\{W(z) + uv = 0\} \tag{53}$$

and use this defining equation to write $\widetilde{y}_i = \widetilde{y}_i(z_0)$ so that the curve C is defined by the equations

$$\widetilde{y}_i(0) = z_0 \widetilde{y}_j(0) \tag{54}$$

This yields an additional Mori vector

$$\ell^{(0)} = (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0) \tag{55}$$

where 1 is in the ith spot and -1 is in the jth spot. Since the v_i and hence the $\widetilde{y}_i$ correspond to a choice of divisor, the choice of $\ell^{(0)}$ specifies the D-brane that an open string starts and ends on.

Example 3.3 (Open mirror map for $K_{\mathbb{F}_3}$)**.** We include the additional Mori vector $\ell^{(0)}$ to obtain

$$\begin{aligned}\ell^{(1)} &= (\quad -2, \quad 1, 1, \quad 0, 0; \quad 0, 0)\\ \ell^{(2)} &= (-(2-m), -m, 0, \quad 1, 1; \quad 0, 0)\\ \ell^{(0)} &= (\quad 1, \quad 0, 0, -1, 0; -1, 1)\end{aligned}$$

Now the Picard–Fuchs operators are modified

$$\mathcal{L}_1 = \theta_1(\theta_1 - 3\theta_2) - z_1(-2\theta_1 + \theta_2 + \theta_0)(-2\theta_1 + \theta_2 + \theta_0 - 1) \tag{56}$$

$$\mathcal{L}_2 = (-2\theta_1 + \theta_2 - \theta_0)(\theta_2 - \theta_0) - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2) \tag{57}$$

$$\mathcal{L}_0 = (\theta_0 - 2\theta_1 + \theta_2)\theta_0 + z_0(\theta_2 - \theta_0)\theta_0 \tag{58}$$

The log solutions t_1, t_2 still solve this open-closed Picard–Fuchs system since S_1, S_2 are only functions of the closed string moduli and hence $\theta_0 t_a = 0$. However, we have an additional solution $t_0 = \log z_0 + S_0(z_1, z_2)$, and we solve for S_0 in terms of F_1 and F_2 the same way we solved for S_1 and S_2. Namely, we compute $\mathcal{L}_b S_0 = -\mathcal{L}_b \log z_0$ and choose the appropriate linear combination which reproduces this lowest order term.

$$\left.\begin{aligned}\mathcal{L}_1 S_0 &= -\mathcal{L}_1 \log z_0 = -z_1\\ \mathcal{L}_2 S_0 &= -\mathcal{L}_2 \log z_0 = 0\end{aligned}\right\} \implies S_0 = -F_1$$

Hence we can compute the open mirror map

$$t_0 = z_0 e^{S_0} \tag{59}$$

4. A-model: methods for computing Gromov–Witten invariants

Here we describe four methods for computing Gromov–Witten invariants of log Calabi-Yau pairs (X, D), using $\mathbb{P}^2$ and $\mathbb{F}_m$ as the running examples. The toric data that we will need to do these computations is given in Examples 2.1 and 2.2, respectively. We will later compare the methods in §4.1 and §4.4 for the Fano and semi-Fano examples. The method in §4.4 utilizes intrinsic mirror symmetry which is not dependent on the ampleness of $-K_{\mathbb{P}_\Delta}$, so we solely employ this method in the non-Fano cases.

4.1. Symplectic geometry.

In [Lau], it is shown that the corrections to the slab function f_0 associated to the canonical divisor $D_0 := [K_{\mathbb{P}_\Delta}] = -\sum_i D_i$ in a (semi-Fano) toric variety $\mathbb{P}_\Delta$ are given by the wall-crossing generating function of open Gromov–Witten invariants $\exp(g_0)$ for the canonical bundle $K_{\mathbb{P}_\Delta}$ over $\mathbb{P}_\Delta$. One can compute $\exp(g_0)$ explicitly in terms of the intersection data of $\mathbb{P}_\Delta$ as follows.

(1) To each toric divisor D_i one can associate a hypergeometric series

$$g_j(z) = \sum_{\alpha \in \mathrm{NE}_j(\mathbb{P}_\Delta)} (-1)^{-D_j\cdot\alpha - 1} \frac{\Gamma(-D_j\cdot\alpha)}{\prod_{i\neq j}\Gamma(D_i\cdot\alpha+1)} z^\alpha \tag{60}$$

where the sum is over all numerically effective curve classes $\alpha \in \mathrm{NE}_j(\mathbb{P}_\Delta) = H_2(\mathbb{P}_\Delta, D_j; \mathbb{Z})_{\text{eff}}$ relative to D_j

$$H_2(\mathbb{P}_\Delta, D_j; \mathbb{Z})_{\text{eff}} = \{\alpha \in H_2(\mathbb{P}_\Delta) \mid \alpha\cdot D_j < 0 \text{ and } \alpha\cdot D_i \geq 0 \ \forall i \neq j\} \tag{61}$$

and the intersection products are taken in the Chow ring $A_*(\mathbb{P}_\Delta)$ by using the divisor class dual to α in an appropriate basis of effective curves $\{\ell_1, \dots, \ell_s\}$ in $\mathrm{NE}_j(\mathbb{P}_\Delta)$. These effective curves are dual to the divisors $\{D_1, \dots, D_s\}$ that generate the Picard group $\mathrm{Pic}(\mathbb{P}_\Delta)$, so the divisor classes will be used to denote effective curves. The complex structure moduli $z = (z_1, \dots, z_s) = (z^{\ell_1}, \dots, z^{\ell_s})$ form algebraic coordinates[4] on the complex structure moduli space of the mirror $\mathscr{M}_{\mathbb{C}}(\check{X})$.

(2) Under the *mirror map* $M : \mathscr{M}_{\mathbb{C}}(\check{X}) \to \mathscr{M}_{\mathrm{K}}(X)$, the image of the algebraic coordinates are the flat coordinates $q_a = M(z_a)$ on the Kähler moduli space of X. The mirror map takes the form

$$q_a = z_a \exp\left(\sum_{i=0}^{s} Q_{ia}\, g_i(z)\right) \tag{62}$$

where Q_{ia} is the $U(1)^s$ charge matrix of a gauged linear sigma model (GLSM) for which $\mathbb{P}_\Delta$ is the space of supersymmetric ground states[5]. One can invert this power series to obtain the *inverse mirror map* M^{-1} : $\mathscr{M}_{\mathrm{K}}(X) \to \mathscr{M}_{\mathbb{C}}(\check{X})$ so that $M^{-1}(q_a) = z_a$.

(3) The open Gromov–Witten invariants $n_{\beta_j+\alpha}$ are then given by writing g_0 in the flat coordinates and taking an exponential series

$$\exp\big(g_j(M^{-1}(z))\big) = \sum_{\alpha \in \mathrm{NE}_j(\mathbb{P}_\Delta)} n_{\beta_j+\alpha} q^\alpha \tag{63}$$

Each toric prime divisor D_i corresponds to a basic disc class $\beta_i \in \pi_2(\mathbb{P}_\Delta, T)$ bounded by a Lagrangian torus fiber $T \in \mathbb{P}_\Delta$. We write

$$n_{\beta_0+k_1\ell_1+k_2\ell_2+\dots+k_s\ell_s} = n_{k_1k_2\dots k_s} \tag{64}$$

Example 4.1 ($\mathbb{P}^n$)**.** The spanning polytope for n-dimensional complex projective space is given by the convex hull of the following $n+1$ vertices.

$$\Delta^\star = \mathrm{Conv}\{(1,0,\dots,0),(0,1,\dots,0),\dots,(0,0,\dots,1),(-1,-1,\dots,-1)\} \tag{65}$$

Therefore $\ell = (-(n+1); 1, 1, \dots, 1)$ and hence by Equation 60, we have

$$g_0(z) = \sum_{k\geq 0} (-1)^{(n+1)k} \frac{((n+1)k-1)!}{(k!)^{n+1}} z^k \tag{66}$$

Example 4.2 ($\mathbb{P}^2$)**.** The effective curve classes relative to $D_0 = -3L$ are given by multiples of the hyperplane class

$$\mathrm{NE}_0(\mathbb{P}^2) = \{kL \mid k \in \mathbb{Z}_{>0}\}. \tag{67}$$

From Equation 60, we have the generating function

$$g_0(z) = -\sum_{k\geq 0} (-1)^{3k} \frac{\Gamma(3k)}{\Gamma^3(k+1)} z^k \tag{68}$$

Then the open Gromov–Witten invariants are given by Equation 63.

$$\exp\big(g_0(M^{-1}(z))\big) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots \tag{69}$$

We tabulate these below

[4]The number of Kähler moduli of a Calabi-Yau n-fold X is the same as the number of complex structure moduli of its mirror, i.e. $s = h^{1,1}(X) = h^{n-1,1}(\check{X})$

[5]The Q_{ia} can be obtained by writing the toric divisors in the $\{\ell_i\}$ basis $D_a = \sum_{i=1}^{s} Q_{ia}\ell_i$

k	0	1	2	3	4	5	6	7	8
n_{β_0+kL}	1	-2	5	-32	286	-3038	35870	-454880	6073311

TABLE 1. The open Gromov–Witten invariants n_{β_0+kL} for $K_{\mathbb{P}^2}$.

Example 4.3 ($\mathbb{F}_m$)**.** We choose the basis $\text{span}_{\mathbb{Z}}\{E,F\} \cong H_2(\mathbb{F}_m) \subset A_*(\mathbb{F}_m)$ so that an effective curve may be written $\beta = n_F F + n_E E$ for some integers n_F, n_E. The effective curves relative to $D_0 = -2S-(m-2)F = -2E-(2+m)F$ are given by imposing constraints on n_F and n_E from Equation 61

$$\text{NE}_0(\mathbb{F}_m) = \{n_F F + n_E E \,|\, n_F, n_E \geq 0\,,\, n_F \geq m n_E\} \tag{70}$$

Using the intersection numbers in Example 2.2, we have the generating function associated to the canonical divisor.

$$g_0(z) = \sum_{n_F \geq m n_E} (-1)^{m n_E} \frac{\Gamma(2n_F + (2-m)n_E)}{\Gamma(n_F+1)\Gamma(n_F - m n_E + 1)\Gamma^2(n_E+1)} z_F^{n_F} z_E^{n_E} \tag{71}$$

and we can also write down the generating function associated to the fiber class

$$g_F(z) = \sum_{n_E \geq 0} \frac{\Gamma(2n_E)}{\Gamma^2(n_E+1)} z_E^{n_E} \tag{72}$$

Since $(-E)\cdot(-K_{\mathbb{F}_m}) = m-2$, there are numerically effective curves relative to $-E =: \overline{E}$ for $m > 2$, and hence in the non-Fano case we obtain an additional generating function

$$g_{\overline{E}}(z) = \sum_{n_F \leq m n_E} (-1)^{n_F+n_E} \frac{\Gamma(m n_E - n_F)}{\Gamma(n_F+1)\Gamma((m-2)n_E - 2n_F + 1)\Gamma^2(n_E+1)} z_F^{n_F} z_E^{n_E} \tag{73}$$

By applying Equation 62, we can obtain the closed mirror maps. For $m = 0, 1$, we have

$$q_F = z_F e^{Q_{F0} g_0} = z_F e^{2g_0} \tag{74}$$

$$q_E = z_E e^{Q_{E0} g_0} = z_E e^{(2-m)g_0}. \tag{75}$$

For $m = 2$, the charge on the exceptional divisor is zero $Q_{E0} = 0$, so we have

$$q_F = z_F e^{2g_0 - g_F} \tag{76}$$

$$q_E = z_E e^{2g_F}. \tag{77}$$

Lastly, for $m > 2$ we include the generating function $g_{\overline{E}}$ associated to negative the exceptional divisor, where $Q_{\widetilde{E}0} = 2-m$ and $Q_{\overline{E}E} = m$.

$$q_F = z_F e^{2g_0 + (2-m)g_{\overline{E}}} \tag{78}$$

$$q_E = z_E e^{(2-m)g_0 + m g_{\overline{E}}} \tag{79}$$

Again, the open Gromov–Witten invariants are given by Equation 63. Below we tabulate these invariants for $\mathbb{F}_m$ with $m \leq 3$. This matches the results obtained from the Picard-Fuchs method described in Example 3.2.

$n_{\beta_0+k_F F+k_E E}$	0	1	2	3	4
0	1	1	0	0	0
1	1	3	5	7	9
2	0	5	35	135	385
3	0	7	135	1100	5772
4	0	9	385	5772	50250

TABLE 2. The open Gromov–Witten invariants $n_{\beta_0+k_F F+k_E E}$ for $K_{\mathbb{F}_0} = K_{\mathbb{P}^1\times\mathbb{P}^1}$. Here k_F runs horizontally and k_E runs vertically.

$\{F, S\}$ basis

$n_{\beta_0+k_F F+k_S S}$	0	1	2	3	4
0	1	-1	1	-1	1
1	2	4	6	8	10
2	-9	-65	-810	-2035	-4459
3	88	1136	7888	38960	152968
4	-1125	-21196	-209988	-1450977	-7835334

$\{F, E\}$ basis

$n_{\beta_0+k_F F+k_E E}$	0	1	2	3	4
0	1	1	0	0	0
1	0	-2	-4	-6	-8
2	0	0	5	35	1135
3	0	0	0	-32	-400
4	0	0	0	0	286

TABLE 3. The open Gromov–Witten invariants $n_{\beta_0+k_F k_F+k_2\ell_2}$ for $K_{\mathbb{F}_1}$ with $\ell_2 = E$ or S. Here k_F runs horizontally and k_2 runs vertically.

n_{k_1,k_2}	0	1	2	3	4
0	1	1	0	0	0
1	0	1	3	5	7
2	0	0	0	5	35
3	0	0	0	0	7
4	0	0	0	0	0

TABLE 4. The open Gromov–Witten invariants $n_{\beta_0+k_F F+k_E E}$ for $K_{\mathbb{F}_2}$. Here k_F runs horizontally and k_E runs vertically.

4.2. **Mirror Landau–Ginzburg models.** In the previous section we discussed how the open Gromov–Witten invariants of can be calculated by using the intersection data encoded in the spanning polytope $\Delta^\star$, which for reflexive polytopes is the polar dual of the polar polytope $\Delta = (\Delta^\star)^\circ$. By Batryev–Borisov mirror symmetry [Bat], the mirror to a Calabi–Yau hypersurface in $\mathbb{P}_\Delta$ is given by

n_{k_1,k_2}	0	1	2	3	4	5	6	7	8	9
0	1	1	0	0	0	0	0	0	0	0
1	0	−2	−2	−4	−6	−8	−10	−12	−14	−16
2	0	5	8	9	20	56	162	418	948	3621
3	0	−32	−70	−96	−140	−300	−768	−2220	−6756	−20440

TABLE 5. The open Gromov–Witten invariants $n_{\beta_0+k_F F+k_E E}$ for $K_{\mathbb{F}_3}$. Here k_F runs horizontally and k_E runs vertically.

a Calabi-Yau hypersurface in $\mathbb{P}_{\Delta^\star}$. Therefore g_0 is closely related to the fundamental period $\varpi^{(0)}(\check{X})$ of the mirror family $\check{X}_z$ to the anticanonical hypersurface $X \in [-K_{\mathbb{P}_\Delta}] \hookrightarrow \mathbb{P}_\Delta$ (see [HKTY95b]).

$$\varpi^{(0)}(\check{X}_z) = \int_{\gamma_0} \check{\Omega}_z = \sum_{\alpha \in \mathrm{NE}_0(\mathbb{P}_\Delta)} \left(\frac{\Gamma(-D_0 \cdot \alpha + 1)}{\prod_i \Gamma(D_i \cdot \alpha + 1)} \right) z^\alpha \tag{80}$$

The holomorphic volume form $\check{\Omega}_z \in H^{n,0}_{\bar{\partial}}(\check{X}_z)$ is given by

$$\check{\Omega}_z = \frac{dx_1 \wedge \cdots \wedge dx_n}{dW_z(x)} \tag{81}$$

The mirror family can be thought of as being fibered over $\mathscr{M}_{\mathbb{C}}(\check{X})$ near the large complex structure limit $z_a \to 0$ with the fiber over $z = (z_1, \ldots, z_s)$ given by

$$\check{X}_z = W_z^{-1}(1) \tag{82}$$

where $W_z : (\mathbb{C}^*)^{n+1} \to \mathbb{C}$ is the Landau–Ginzburg superpotential mirror to the ambient toric variety $\mathbb{P}_\Delta$.

4.3. **Slab function normalization.** The slab function normalization condition states that any slab function cannot contain pure q powers in its log expansion. Applying this condition to the slab function associated to the canonical divisor f_0 allows one to compute g_0 order-by-order since f_0 is given by

$$f_0 = e^{g_0(q)} + \sum_{v_i \in \Delta^\star \cap \mathbb{Z}^n} z^{\ell_i} x^{v_{i,x}} y^{v_{i,y}} \tag{83}$$

where $v_i = (v_{i,x}, v_{i,y}) \in \Delta^\star$ are the integral points of $\Delta^\star$. We illustrate this with the following examples.

Example 4.4 ($\mathbb{P}^2$, following [Lau])**.** The slab function for two dimensional complex projective space is given by

$$f_0 = e^{g_0} + x + y + \frac{z}{xy} \tag{84}$$

Taking the log, we have

$$\log f_0 = g_0 + \log\left(\widetilde{x} + \widetilde{y} + \frac{q}{\widetilde{x}\widetilde{y}} \right) \tag{85}$$

where $\widetilde{x} = e^{-g_0}x$, $\widetilde{y} = e^{-g_0}y$, and $q = ze^{-3g_0}$. Therefore we can expand the log and the constant terms (with respect to $\widetilde{x}, \widetilde{y}$) will give us g_0. Since $\ell_0 = 3$, constant terms can only occur at in the expansion at orders that are a multiple of 3

$$\log\left(\widetilde{x}+\widetilde{y}+\frac{q}{\widetilde{x}\widetilde{y}}\right)\Big|_{\widetilde{x},\widetilde{y}\,\text{const.}} = \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{k}\left(\widetilde{x}+\widetilde{y}+\frac{q}{\widetilde{x}\widetilde{y}}\right)^{k}\Big|_{\widetilde{x},\widetilde{y}\,\text{const.}}$$
$$= \sum_{k=0}^{\infty}\frac{(3k-1)!}{(k!)^3}$$

This is the same result we obtained previously.

Example 4.5 ($\mathbb{P}^1\times\mathbb{P}^1$)**.** The slab function normalization condition was applied to $\mathbb{P}^1\times\mathbb{P}^1$ in [Wan21, Example 3.3.2]. This reproduces our result from Example 4.3.

4.4. **Scattering diagrams with `scattering.sage`.** Scattering diagrams (see §2.4) are an important tool to calculate the instanton corrections to the central fiber $\check{X}_0$ of the mirror toric degeneration $\pi : \check{\mathfrak{X}} \longrightarrow \mathbb{A}^s$. In particular, scattering diagrams carry the invariants that allow us to "smoothen" $\check{X}_0$ into a mirror family where a general fiber is the mirror to (X, D).

The second author developed a code to compute scattering diagrams and broken lines for log Calabi–Yau pairs (X, D), see §2.4 for definitions. When the cited review paper was published, this was only done for very ample log Calabi–Yau pairs i.e. when X is a surface with $K_X^2 \geq 3$. However, we have since developed the code to include non-Fano cases of interest.

Remark 4.6. A wall structure consistent to all orders, called the canonical wall structure $\mathscr{S}_\infty(X)$, was constructed by Gross and Siebert [GS21]. Our code computes the canonical wall structure $\mathscr{S}_k(X)$ to finite order k if given enough time and memory.

Example 4.7 ($\mathbb{P}^2$)**.** The *initial scattering diagram* $\mathscr{S}_0(X)$ is given by defining pairs of antiparallel outgoing walls $\mathfrak{d}_\pm$ at each affine singularity. One can then iterate the consistency condition to obtain the higher order scattering diagrams $\mathscr{S}_k(X)$. See below for a plot produced by the code for $\mathbb{P}^2$.

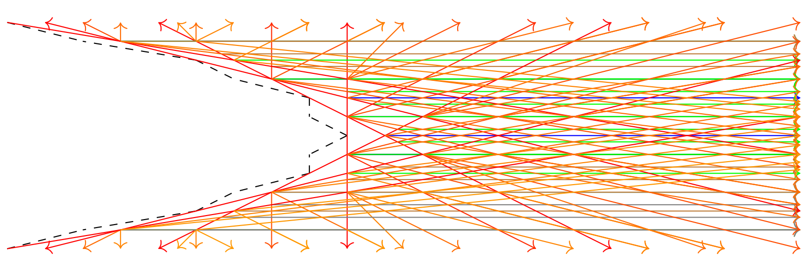

FIGURE 8. The sixth order scattering diagram for $\mathbb{P}^2$, denoted $\mathscr{S}_6(\mathbb{P}^2)$.

5. Tropical curves and tropical correspondence

5.1. **Tropical curves.** Tropical curves live on the dual intersection complex, so this doesn't depend on the Fano property.

Definition 5.1. A *tropical curve* on an affine manifold with singularities B is a map $h : \Gamma \to B$ from a graph Γ, without bivalent vertices but possibly with some legs (non-compact edges with only 1 vertex), together with

(1) a non-negative integer g_V (*genus*) for each vertex V;
(2) a non-negative integer ℓ_E (*length*) for each compact edge E;
(3) a weight vector $u_{(V,E)} \in \Lambda_{B,h(V)}$ for each flag (V, E);

such that h respects the graph structure of Γ and

(i) if E is a compact edge with vertices V_1, V_2, then h maps E affine linearly to the line segment connecting $h(V_1)$ and $h(V_2)$, and $h(V_2) - h(V_1) = \ell_E u_{(V_1,E)}$. In particular, $u_{(V_1,E)} = -u_{(V_2,E)}$. Here the affine structure on Γ is given by the lengths ℓ_E of its edges;
(ii) if E is a leg with vertex V, then h maps E affine linearly either to the ray $h(V) + \mathbb{R}_{\geq 0} u_{(V,E)}$ or to the line segment $[h(V), \delta)$ for δ an affine singularity of B such that $\delta - h(V) \in \mathbb{R}_{>0} u_{(V,E)}$, i.e., $u_{(V,E)}$ points from $h(V)$ to δ. Note that in the latter case the direction of $u_{(V,E)}$ is determined by $\Lambda_{B,\delta} \simeq \mathbb{Z}$.
(iii) at each vertex V we have the *balancing condition*

$$\sum_{E \ni V} u_{(V,E)} = 0.$$

We write the set of compact edges of Γ as $E(\Gamma)$, the set of legs as $L(\Gamma)$, the set of legs mapped to a ray (*unbounded legs*) as $L_\infty(\Gamma)$ and the set of legs mapped to an open line segment (*bounded legs*) as $L_\Delta(\Gamma)$ (since such edges end at the singular locus Δ of B).

Remark 5.2. All our tropical curves will be genus 0, so that $g_V = 0$ for all vertices and Γ is a tree.

Definition 5.3. Let $h : \Gamma \to B$ be a tropical curve. For a trivalent vertex $V \in V(\Gamma)$ define $m_V = |u_{(V,E_1)} \wedge u_{(V,E_2)}| = |\det(u_{(V,E_1)}|u_{(V,E_2)})|$, where E_1, E_2 are any two edges adjacent to V. For a vertex $V \in V(\Gamma)$ of valency $\nu_V > 3$ let h_V be the one-vertex tropical curve describing h locally at V and let h'_V be a deformation of h_V to a trivalent tropical curve. It has $\nu_V - 2$ vertices. Define $m_V = \prod_{V' \in V(h'_V)} m_{V'}$. For a bounded leg $E \in L_\Delta(\Gamma)$ define $m_E = (-1)^{w_E+1}/w_E^2$. Then define the *multiplicity* of h to be

$$\mathrm{Mult}(h) = \frac{1}{|\mathrm{Aut}(h)|} \cdot \prod_V m_V \cdot \prod_{E \in L_\Delta(\Gamma)} m_E.$$

5.2. **Tropical disks, scattering and broken lines.**

Definition 5.4. A *tropical disk* $h^\circ : \Gamma \to B$ is a tropical curve with the choice of univalent vertex V_∞, adjacent to a unique edge E_∞, such that the balancing condition (Definition 5.1, (iii)) only holds for vertices $V \neq V_\infty$. Define the multiplicity $m_{V_\infty} = 1$, such that $\mathrm{Mult}(h^\circ)$ is the multiplicity of the balanced tropical curve obtained by forgetting V_∞.

Definition 5.5. Let $\mathfrak{d} \in \mathscr{S}_\infty(B)$ be a ray and choose a point $P \in \mathrm{Int}(\mathfrak{d})$.

Define $\mathfrak{H}^\circ_{\mathfrak{p},w}(B)$ to be the set of all tropical disks $h^\circ : \Gamma \to B$ with no unbounded legs and such that $h^\circ(V_\infty) = P$ and $u_{(V_\infty,E_\infty)} = -w \cdot m_{\mathfrak{d}}$.

Proposition 5.6 ([Gra], Proposition 5.9)**.** For a ray $\mathfrak{d} \in \mathscr{S}_\infty(B)$ we have

$$\log f_{\mathfrak{d}} = \sum_{w=1}^{\infty} \sum_{h^\circ \in \mathfrak{H}^\circ_{\mathfrak{p},w}(B)} w\mathrm{Mult}(h^\circ) t^w z^{wm_{\mathfrak{p}}}.$$

Definition 5.7. Let $\mathfrak{H}^\circ_q(B)_P$ be the set of tropical disks $h^\circ : \Gamma \to B$ with one unbounded leg E_{out} and such that $h^\circ(V_\infty) = P$ and $u_{(V_{\mathrm{out}},E_{\mathrm{out}})} = qm_{\mathrm{out}}$ and $u_{(V_\infty,E_\infty)} = -pm_{\mathrm{out}}$.

Lemma 5.8. Let $\mathfrak{b}$ be a broken line in $\mathfrak{B}^{(k)}_q(B)_P$. If P lies in an unbounded chamber of $\mathscr{S}_k$, then $\bar{m}_{\mathfrak{b}}$ is parallel to m_{out}.

Proof. This is [GRZ], Proposition 3.5, or [Gra22], Proposition 2.3. $\square$

Proposition 5.9. There is a surjective map with finite preimages

$$\mu : \mathfrak{H}^\circ_q(B)_P \to \mathfrak{B}_q(B)_P$$

defined by taking the path from V_∞ to V_{out}. We say a tropical disk in $\mu^{-1}(\mathfrak{b})$ is obtained from $\mathfrak{b}$ by disk completion. We have

$$a_{\mathfrak{b}} = \sum_{h^\circ \in \mu^{-1}(\mathfrak{b})} \mathrm{Mult}(h^\circ).$$

Proof. This is [CPS], Lemma 6.4 and Proposition 6.15, with some more details given in [Gra22], Proposition 3.2 and Proposition 4.18. $\square$

5.2.1. *Leg extension.* If h° is a tropical disk with $u_{(V_\infty,E_\infty)} = -wm_{\mathrm{out}}$, then we can forget V_∞ and extend E_∞ to an unbounded leg E_{out} with $u_{(V_{\mathrm{out}},E_{\mathrm{out}})} = wm_{\mathrm{out}}$. Hence, the above statements for tropical disks can be translated to tropical curves as follows.

Definition 5.10. Let $\mathfrak{H}_w(B)$ be the set of tropical curves on B with one unbounded leg E_{out} of weight w. This implies $u_{(V_{\mathrm{out}},E_{\mathrm{out}})} = wm_{\mathrm{out}}$. Further, let $\mathfrak{H}_{p,q}(B)_P$ be the set of tropical curves on B having two unbounded legs, of weights p and q, and such that the image of the unbounded leg of weight p contains P.

Corollary 5.11. We have

$$\log \prod_{\mathfrak{d}:m_{\mathfrak{d}}=wm_{\mathrm{out}}} f_{\mathfrak{d}} = \sum_{w=1}^{\infty} \sum_{h \in \mathfrak{H}_w(B)} w\mathrm{Mult}(h) z^{(wm_{\mathfrak{p}},0)}.$$

Corollary 5.12. Let $P \in B$ be a point in an unbounded chamber of $\mathscr{S}_{p+q}(B)$. Then there is a surjective map with finite preimages

$$\mu : \mathfrak{H}_{p,q}(B) \to \mathfrak{B}_{p,q}(B)_P,$$

such that

$$a_{\mathfrak{b}} = \sum_{h \in \mu^{-1}(\mathfrak{b})} \mathrm{Mult}(h).$$

A preimage is given by disk completion and leg extension.

5.3. **Tropical curve classes.**

Definition 5.13. The curve class of a tropical curve is defined by its intersection with tropical cocycles defined by *elementary tropical curves*. For more details see [Gra22], §3, and [GRZ], §4.2.

Write $\mathfrak{H}_w(B,\beta)$ and $\mathfrak{H}_{p,q}(B,\beta)_P$ for the set of tropical curves of class β in $\mathfrak{H}_w(B)$ and $\mathfrak{H}_{p,q}(B)_P$, respectively.

Example 5.14. Figure 9 shows some tropical curves on $\mathbb{F}_3$ of class $2F+E$:

5.4. **The degeneration formula and tropical correspondence.** The idea is that tropical curves describe the components of the curve on the central fiber of the toric degeneration. Formally we have to work with a log smooth degeneration. This is obtained by successively blowing up components on the central fiber of the degeneration. The multiplicities m_L of bounded legs (Definition 5.3) account for components mapping onto or intersecting the exceptional locus. The degeneration formula works locally, so the Fano condition doesn't matter.

Theorem 5.15. Let B be the dual intersection complex of a toric model of X. Then

$$R_w(X,\beta) = \sum_{h\in\mathfrak{H}_w(B,\beta)} \mathrm{Mult}(h),$$

and, for any point $P \in B_0$,

$$R_{p,q}(X,\beta) = \frac{1}{p}\sum_{h\in\mathfrak{H}_{p,q}(B,\beta)_P} \mathrm{Mult}(h).$$

Corollary 5.16. At infinity we have

$$\vartheta_q(X)_\infty = y^q + \sum_{p\geq 1}\sum_{\beta:\beta.E=p+q} pR_{p,q}(X,\beta)s^\beta t^{p+1}y^{-p}.$$

Example 5.17. Figure 9 shows some tropical curves contributing to $R_3(\mathbb{F}_3, 2F+E) = 9$ and $R_{2,1}(\mathbb{F}_3, 2F+E) = 1$.

5.5. **Relating scattering diagrams of Hirzebruch surfaces.** We know that the Hirzebruch surfaces of even/odd parity have the same relative Gromov-Witten invariants. So we expect their scattering diagrams to be isomorphic. Figure 10 shows how the $\mathbb{F}_1$-diagram and the $\mathbb{F}_3$-diagram are related. The tropical curves of class $F+E$ in $\mathbb{F}_1$, i.e. class $2F+E$ in $\mathbb{F}_3$, are shown in red and blue.

6. The corrected potential and mutations

6.1. **The potential.**

Definition 6.1. The *theta potential* of X is the first theta function $\vartheta_1(X) = \vartheta_{m_{\text{out}}}(X)$. It depends on the chamber of the scattering diagram $\mathscr{S}_\infty(X)$ in which it is defined. We write $\vartheta_1(X)_0$ for the potential close to the origin, i.e. in some chamber inside the (possibly non-convex) polytope $\Delta^\star$, which is the spanning polytope of the fan of a toric model of X. We write $\vartheta_(X)_\infty$ for the potential at infinity, i.e. in an unbounded chamber. Note that $\vartheta(X)_\infty$ only depends on one variable y, while $\vartheta_1(X)_0$ depends on several variables.

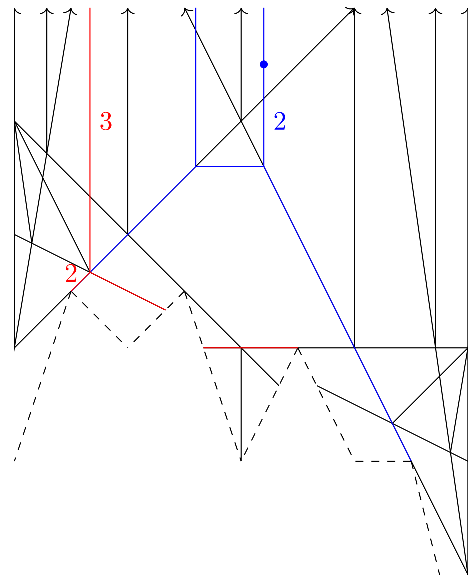

FIGURE 9. Some tropical curves for $\mathbb{F}_3$ of class $2F + E$.

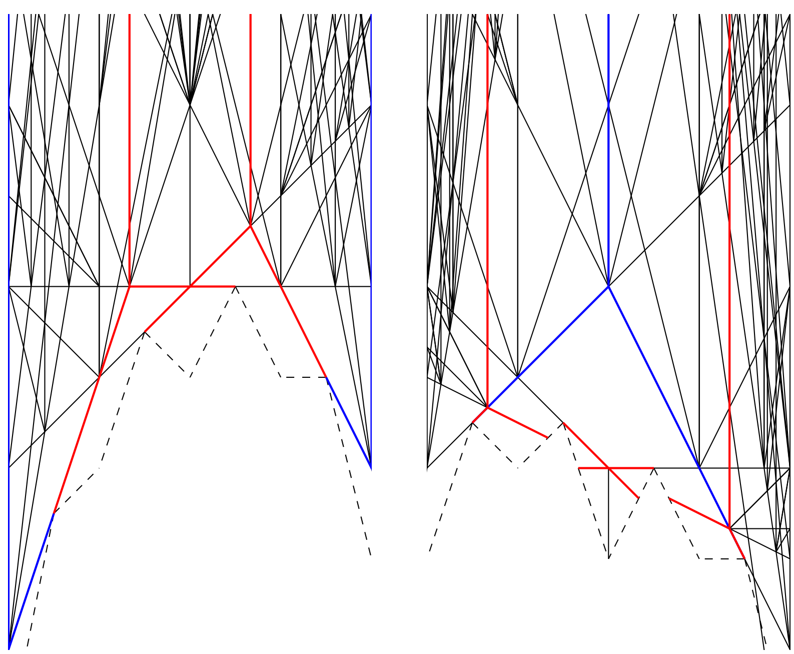

FIGURE 10. The $\mathbb{F}_1$-diagram (left) and the $\mathbb{F}_3$-diagram (right).

Example 6.2. In the Fano case, $\vartheta_1(X)_0$ is given by the toric potential

$$\vartheta_1(X)_0 = W_\Sigma = t \cdot \sum_{\rho \in \Sigma^{[1]}} z^{m_\rho}.$$

Here Σ is the fan of any toric model of X, see Definition 2.4.

It was noticed in [FOOO10], Theorems 4.5 and 4.6, that in the non-Fano case there are additional correction terms. For the Hirzebruch surfaces $\mathbb{F}_2$ and $\mathbb{F}_3$ they have been computed explicitly in [Aur] using wall crossing (in the cone picture) and in [CPS] using broken lines (in the fan picture). Here we explain the latter approach for $\mathbb{F}_2$, $\mathbb{F}_3$ and additionally for $\mathbb{F}_4$, $\mathbb{F}_4$, and make some remarks about the t-order grading. In [CPS] this grading differs from the other approaches due to a different choice of polarization. We argue that one can indeed take the anticanonical polarization to obtain the same t-grading as in [Aur], and we explain how to recognize curves classes from the broken lines. Moreover, we show that the potentials for $\mathbb{F}_0$, $\mathbb{F}_2$, $\mathbb{F}_4$, and for $\mathbb{F}_1$, $\mathbb{F}_3$, indeed have the same periods under the change of curve classes $F \mapsto F, S \mapsto S - F$.

Proposition 6.3. We have

$$\vartheta_1(X)_0 = W_\Sigma + W',$$

where W_Σ is the toric potential of a toric model of X and W' is a correction term that comes from scattering inside the convex hull $\mathrm{Conv}(\Delta^\star)$. In particular, if X is Fano, then $W' = 0$.

Example 6.4. We compute the potentials $\vartheta_1(\mathbb{F}_m)$ of Hirzebruch surfaces $\mathbb{F}_m$ in the chamber below the non-convex vertex of $\Delta^\star$. To do so, we can choose the point $P = (-1, -\epsilon)$ with $\epsilon > 0$ sufficiently small. This gives the following potentials, where $z_1 = z^F$ and $z_2 = z^E$,

$$\begin{aligned}
\vartheta_1(\mathbb{F}_0)_{(-1,-\epsilon)} &= t \cdot \left(x + y + \frac{z_1}{x} + \frac{z_2}{y} \right) \\
\vartheta_1(\mathbb{F}_1)_{(-1,-\epsilon)} &= t \cdot \left(x + y + \frac{z_1}{x} + \frac{z_1 z_2}{xy} \right) \\
\vartheta_1(\mathbb{F}_2)_{(-1,-\epsilon)} &= t \cdot \left(x + y + \frac{z_1}{x} + \frac{z_1 z_2}{x} \left(1 + \frac{z_1}{xy} \right) \right) \\
\vartheta_1(\mathbb{F}_3)_{(-1,-\epsilon)} &= t \cdot \left(x + y + \frac{z_1}{x} + \frac{z_1 z_2 y}{x} \left(1 + \frac{z_1}{xy} \right)^2 \right) \\
\vartheta_1(\mathbb{F}_4)_{(-1,-\epsilon)} &= t \cdot \left(x + y + \frac{z_1}{x} + z_1 z_2 y \left(1 + \frac{z_1}{xy} \right) + \frac{z_1 z_2 y^2}{x} \left(1 + \frac{z_1}{xy} \right)^3 \right)
\end{aligned}$$

Note that $\mathbb{F}_0$ and $\mathbb{F}_1$ are Fano, so there are no correction terms and the potential is simply the toric one. For $\mathbb{F}_2$ and $\mathbb{F}_3$ the potential was computed in [Aur] using symplectic methods and in [CPS] using broken lines. Note the different conventions. In [CPS] they don't give classes and have the wrong t-order. Moreover, they have one broken line that actually isn't there. Instead, one broken line appears with coefficient 2.

For $\mathbb{F}_3$, $\mathbb{F}_4$, the computation is shown in Figures 11, 12, respectively. For $m \geq 4$ there is infinite scattering inside $\Delta^\star$. All rays have t-order zero inside $\Delta^\star$, but the t-order increases as they leave $\Delta^\star$, so only a very limited number of them (actually only the initial rays) are involved in the broken line calculation.

One can easily check that the classical periods of $\vartheta_1(\mathbb{F}_m)_0$ agree for m even/odd, as expected.

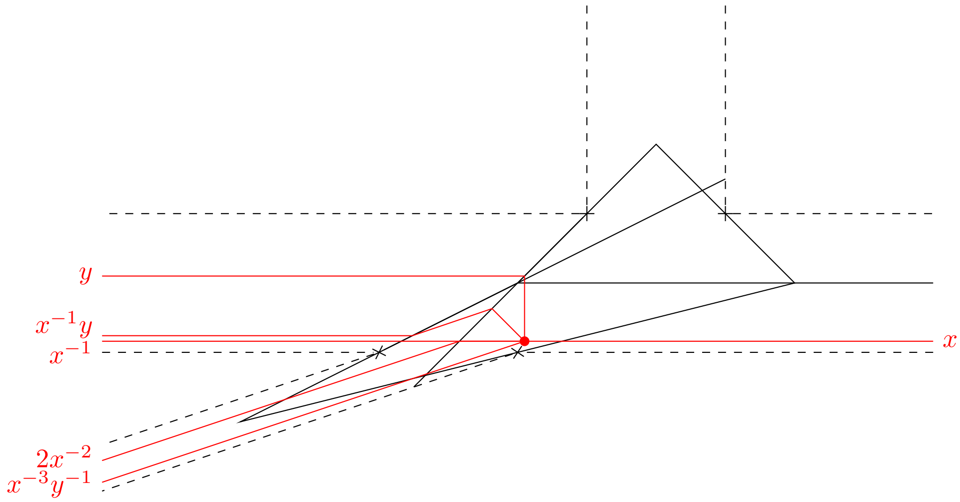

FIGURE 11. The potential $\vartheta_1(\mathbb{F}_3)_{(-1,-\epsilon)}$ inside $\Delta^\star$.

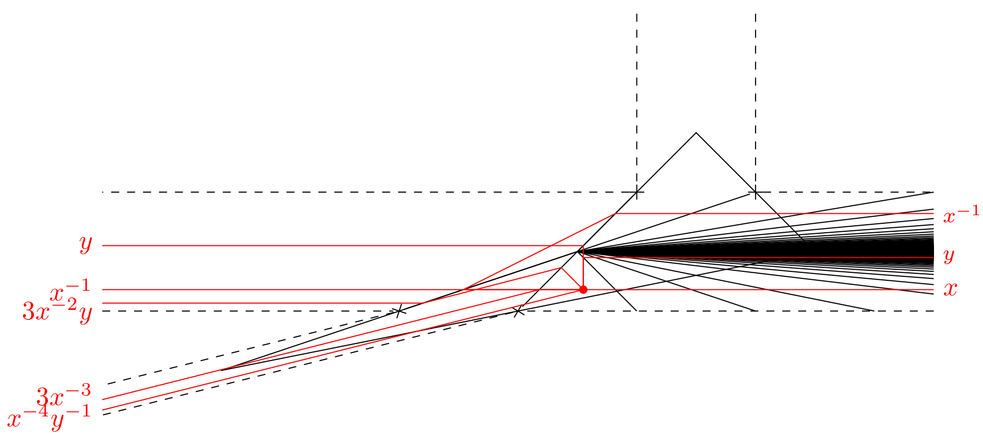

FIGURE 12. The potential $\vartheta_1(\mathbb{F}_4)_{(-1,-\epsilon)}$ inside $\Delta^\star$.

Example 6.5. We conjecture that

$$\vartheta_1(\mathbb{F}_5)_{(-1,-\epsilon)} = t\cdot\left(x + y + \frac{z_1}{x} + z_1z_2xy + z_1z_2y^2\left(1+\frac{z_1}{xy}\right)^2 + \frac{z_1z_2y^3}{x}\left(1+\frac{z_1}{xy}\right)^4\right).$$

This is what we get from the mutations in the next section, and we verified that this gives the same period as $\vartheta_1(\mathbb{F}_3)_{(-1,-\epsilon)}$ after $z_2 \mapsto z_1z_2$ and as $\vartheta_1(\mathbb{F}_1)_{(-1,-\epsilon)}$ after $z_2 \mapsto z_1^2z_2$. However, we could not find the broken line for the monomial z_1z_2xy. See Figure 13 for the other broken lines. The shape of a broken line with positive x-coordinate has to be different, because all rays in the interior have positive x-coordinate, so can only increase the x-coordinate. For $\vartheta_1(\mathbb{F}_7)_{(-1,-\epsilon)}$ we could find a broken line with positive x-coordinate, as shown in Figure 14.

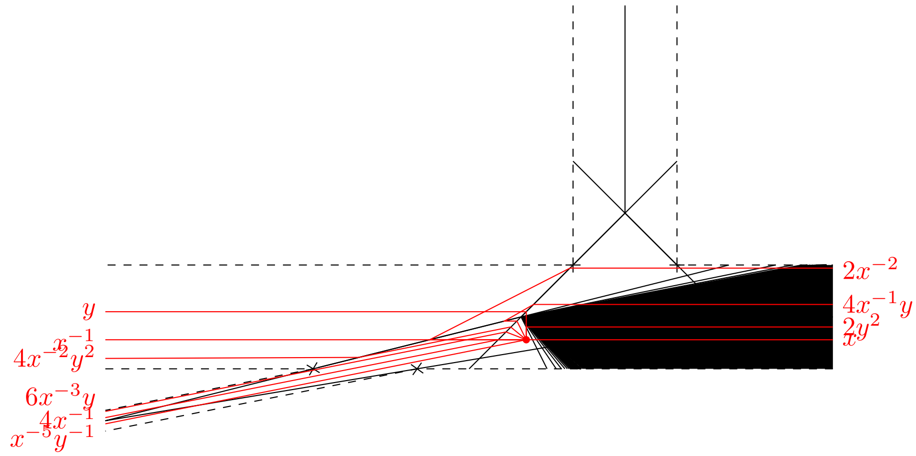

FIGURE 13. The potential $\vartheta_1(\mathbb{F}_5)_{(-1,-\epsilon)}$ inside $\Delta^\star$.

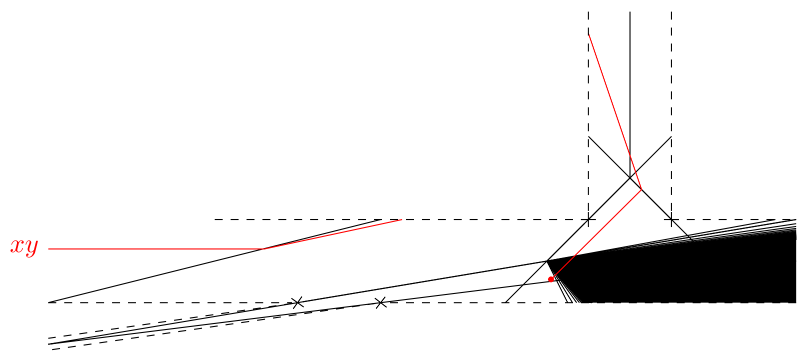

FIGURE 14. A broken line for $\vartheta_1(\mathbb{F}_7)_{(-1,-\epsilon)}$ with positive x-coordinate.

Remark 6.6. In the calculations above we have omitted the variables z_i, i.e. the curve classes. One can find the curve class corresponding to a broken line by shifting the endpoint to infinity and completing the broken lines to a tropical curve, as in Proposition 5.9. Then one can calculate the curve class of the tropical curve by intersection with tropical cocycles.

See Figure 15 for an example calculation.

6.2. **Mutations.** Mutations of Laurent polynomials were introduced in [FZ01][GU10][Akh12] and related to mirror symmetry in [CCG$^+$].

Definition 6.7. Let $N \simeq \mathbb{Z}^2$ be a lattice and let $w \in M$ be a primitive vector in the dual lattice. Then w induces a grading of $\mathbb{C}[N]$. Let $a \in \mathbb{C}[w^\perp \cap N]$ be a Laurent polynomial in the zeroth piece of $\mathbb{C}[N]$, where $w^\perp \cap N = \{v \in N \mid w(v) = 0\}$. The pair (w, a) defines an automorphism of the Laurent polynomial ring $\mathbb{C}(N)$ by

$$\mu_{w,a} : \mathbb{C}(N) \to \mathbb{C}(N), \quad x^v \mapsto x^v a^{w(v)}.$$

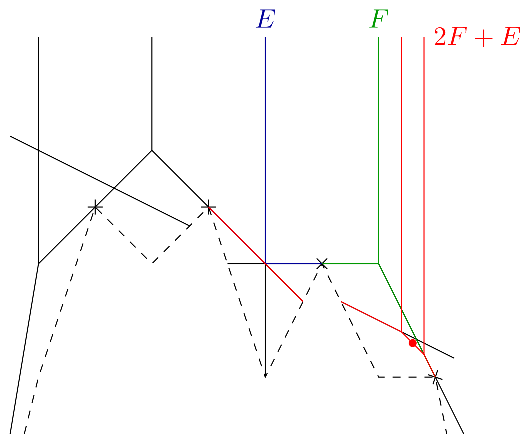

FIGURE 15. Finding the class $2F+E$ of the $\frac{z_1^2 z_2}{x^2}$-term in $\vartheta_1(\mathbb{F}_3)_{(-1,-\epsilon)}$ by completion to a tropical curve.

Let $f \in \mathbb{C}[N]$. We say f is *mutable with respect to* (w,a) if $\mathrm{mut}_{w,a}(f) \in \mathbb{C}[N]$, in which case we call $\mathrm{mut}_{w,a}(f)$ a *mutation* of f.

Remark 6.8. The wall crossing morphisms $\theta_{\mathfrak{d}}$ are equal to mutations $\mathrm{mut}_{w,a}$ with $w = n_{\mathfrak{d}}$ and $a = f_{\mathfrak{d}}$. The *Laurent phenomenon* of [Akh12] in our case means that in any chamber inside $\Delta^\star$ the potential $\vartheta_1(X)$ is a Laurent polynomial and mutable with respect to its boundary walls, so that it stays a Laurent polynomial after mutation.

Proposition 6.9 ([Ilt12], Theorem 1.3)**.** If $\Delta_1^\star$ and $\Delta_2^\star$ are related by a mutation of Laurent polynomials supported on them, then they are $\mathbb{Q}$-Gorenstein deformation equivalent. In particular, mutation exchanges toric models of a variety X.

Example 6.10. For a Hirzebruch surface $\mathbb{F}_m$, the potential $\vartheta_1(\mathbb{F}_m)$ is a mutation of $\vartheta_1(\mathbb{F}_{m-2})$ with respect to $w=(-1,1)$ and $a = 1+\frac{z_1}{xy}$ under the change of variables

$$\begin{aligned} \vartheta_1(\mathbb{F}_m) &\xrightarrow{\sim} \mathrm{mut}_{(-1,1),1+\frac{z_1}{xy}}\vartheta_1(\mathbb{F}_{m-2}) \\ x &\mapsto y^{-1} \\ y &\mapsto x \\ z_1 &\mapsto z_1 \\ z_2 &\mapsto z_1^{-1}z_2 \end{aligned}$$

The map $(x,y)\mapsto(y^{-1},x)$ rotates the Newton polytope of $\vartheta_1(X)_0$, but doesn't change the potential. The map $(z_1,z_2)\mapsto(z_1,z_1^{-1}z_2)$, gives the change of curve classes $F,E \mapsto F, E-F$, as in 2.6. For $m \leq 5$ the mutations are shown in Figures 16, 17, and from this the potentials for $m \leq 9$ are anticipated in Figures 18, 19.

6.3. **Toric models of non-Fano varieties: a conjecture.**

Conjecture 6.11. X has toric model X_Σ, where Σ is the spanning fan of the Newton polytope of $\vartheta_1(X)_0$. In particular, two (not necessarily Fano) varieties X

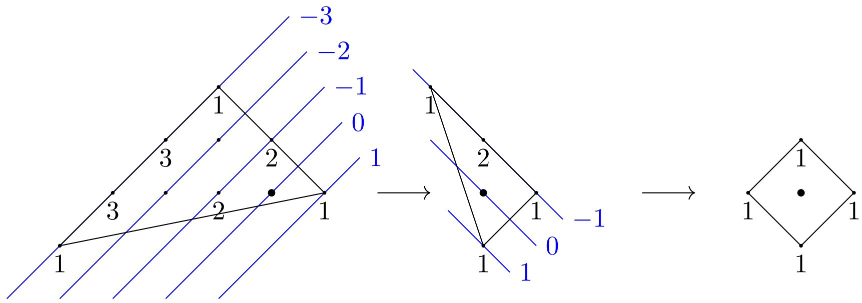

FIGURE 16. Mutations $\vartheta_1(\mathbb{F}_4)_{(-1,-\epsilon)} \to \vartheta_1(\mathbb{F}_2)_{(-1,-\epsilon)} \to \vartheta_1(\mathbb{F}_0)_{(-1,-\epsilon)}$.

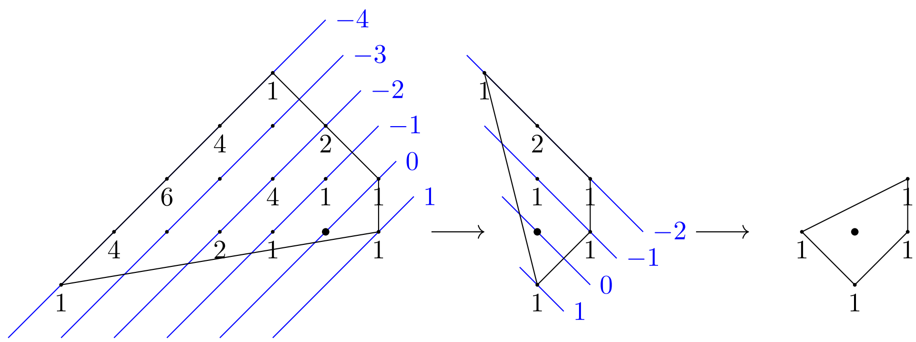

FIGURE 17. Mutations $\vartheta_1(\mathbb{F}_5)_{(-1,-\epsilon)} \to \vartheta_1(\mathbb{F}_3)_{(-1,-\epsilon)} \to \vartheta_1(\mathbb{F}_1)_{(-1,-\epsilon)}$.

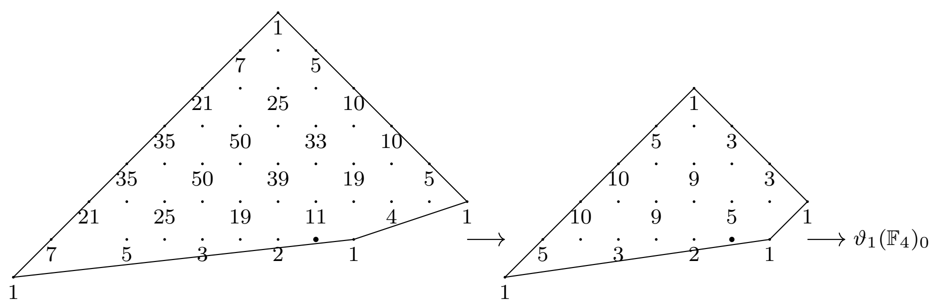

FIGURE 18. $\vartheta_1(\mathbb{F}_8)_{(-1,-\epsilon)}$ and $\vartheta_1(\mathbb{F}_6)_{(-1,-\epsilon)}$ anticipated from mutation.

and Y are $\mathbb{Q}$-deformation equivalent if and only if their potentials ϑ_1 are mutation equivalent.

Remark 6.12. Note that if the conjecture is true, then X has a Fano toric model if and only if $\vartheta_1(X)_0$ is mutation equivalent to a Laurent polynomial whose Newton polytope is reflexive, as in the case of Hirzebruch surfaces above.

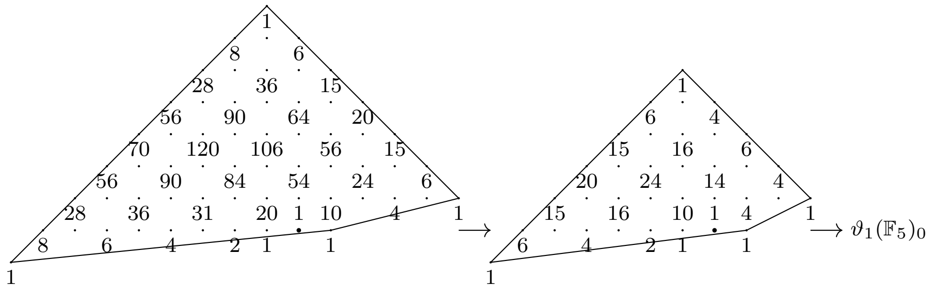

FIGURE 19. $\vartheta_1(\mathbb{F}_9)_{(-1,-\epsilon)}$ and $\vartheta_1(\mathbb{F}_7)_{(-1,-\epsilon)}$ anticipated from mutation.

Proposition 6.13. If two (not necessarily Fano) varieties X and Y have mutation equivalent potentials $\vartheta_1(X)_0$ and $\vartheta_1(Y)_0$, then their periods, disk invariants, and mirror maps agree.

6.4. **Mirror symmetry and rigid MMLPs.** It was proposed in [CCG$^+$] that the mirror partner of a Fano variety X is a Laurent polynomial f such that the classical period of f equals the regularized quantum period of X.

Definition 6.14. The *classical period* of a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is defined as

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^n \int_{|x_1|=\ldots=|x_n|=1} \frac{1}{1-tf}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}, \quad t \in \mathbb{C}, |t| \ll \infty.$$

This has Taylor expansion

$$\pi_f(t) = \sum_{k=0}^{\infty} \mathrm{coeff}_1(f^k)t^k.$$

Proposition 6.15 ([CPS])**.** The period $\pi_{\vartheta_1(X)}$ does not depend on the chamber of $\mathscr{S}_\infty(X)$ in which $\vartheta_1(X)$ is defined.

Definition 6.16. The *regularized quantum period* of a Fano manifold X is defined as

$$\hat{G}_X(t) = 1 + \sum_{k=2}^{\infty} k! c_k t^k,$$

where $c_0 = 1$, $c_1 = 1$, and for $k \geq 2$,

$$c_k = \int_{\mathscr{M}_{0,1,(k)}} \psi^{k-2}\mathrm{ev}^\star[\mathrm{pt}]$$

are the *descendant Gromov-Witten invariants* of X.

Definition 6.17. A Laurent polynomial f is a *mirror partner* for X if $\pi_f = \hat{G}_X$.

It was conjectured in [CKPT21] that Fano manifolds correspond under mirror symmetry to mutation equivalence classes of *rigid maximally mutable Laurent polynomials* (rigid MMLPs). This leads us to the following conjecture.

Definition 6.18. A Laurent polynomial f is *maximally mutable* if it is maximal with respect to the partial ordering defined by label-preserving injections of mutation graphs. Then it is called *rigid* if it is uniquely determined by its Newton polytope, mutation graph and the coefficient of its constant term being 0.

Conjecture 6.19. The potential $\vartheta_1(X)_0$ is a rigid maximally mutable Laurent polynomial.

Example 6.20. Conjecture 6.19 is true for Fano surfaces and Fano threefolds. In this case $\vartheta_1(X)_0$ is the toric potential and the conjecture of [CKPT21] is known for these cases.

Example 6.21. If Conjecture 6.11 is true, then Conjecture 6.19 is true for all surfaces and threefolds that admit a Fano toric model, for the same reasons.

Example 6.22. Conjecture 6.19 is true for all Hirzebruch surfaces $\mathbb{F}_m$, since they have a Fano toric model $\mathbb{F}_0$ or $\mathbb{F}_1$.

7. Mirror symmetry

7.1. The LG potential equals the open mirror map.

Definition 7.1. The open mirror map of X with respect to a potential function W is defined as $q_i = z_i e^{F_W(z)}$, where $F_0(z)$ is the x,y-constant term of $-\log(1-W)$, that is,

$$F_W(z) = \sum_{k>0} \frac{1}{k} \mathrm{coeff}_1(W^k).$$

Write $M_W(q) := e^{F_W(z(q))}$ for the open mirror map after inserting the inverses of the closed mirror maps $q_i = z_i e^{d_i F_W(z)}$, where $d_i = \beta_i \cdot D$.

Remark 7.2. Note that $\mathrm{coeff}_1(\vartheta_1^k)$ has a t^k-factor, so $M_{t^{-1}\vartheta_1}(q)$ does not depend on t. Further note that by Proposition 6.15 $M_{t^{-1}\vartheta_1}(q)$ does not depend on the chamber of $\mathscr{S}_\infty(X)$ in which ϑ_1 is defined.

Conjecture 7.3. Let Σ' be the fan of K_{X_Σ}, where X_Σ is a toric model of X. Then $\log(z_0) + F_{W_{\Sigma'}}(z)$ is a solution to the Picard-Fuchs equation defined by the Mori vectors coming from K_{X_Σ}.

Theorem 7.4. Under the change of variables $q_i = z_i(t/y)^{d_i}$, with $d_i = \beta_i \cdot D$, we have

$$\vartheta_1(y)_\infty = y M_{t^{-1}\vartheta_1}(q).$$

To prove Theorem 7.4 we first we need two lemmas.

Definition 7.5. For a Laurent series $f(x) = \sum_{k \geq k_{\min}} f_k x^k$ write $[x^k]f = f_k$.

Lemma 7.6 (Lagrange inversion for Laurent series)**.** Let $f(y) = y^{-1} + \sum_{k\geq 0} f_k y^k$ be a Laurent series with only simple poles. Then the composition inverse $g(z)$ satisfying $f(g(z)) = z$ is a power series in z^{-1} given by

$$g(z^{-1}) = \sum_{k>0} \frac{z^{-k}}{k}[x^{-1}]f^k.$$

Proof. Look at the multiplicative inverse

$$\frac{1}{f(y)} = \frac{y}{h(y)}, \quad h(y) = yf(y) = 1 + \sum_{k\geq 0} f_k y^{k+1}.$$

This is a power series with nonzero linear term, and by the Laurent inversion theorem we have $1/f(g(z)) = z$ for

$$g(z) = \sum_{k>0} \frac{z^k}{k}[x^{k-1}]h^k = \sum_{k>0} \frac{z^k}{k}[x^{-1}]f^k.$$

The equation $1/f(g(z)) = z$ is equivalent to $f(g(z)) = z^{-1}$ and to $f(g(z^{-1})) = z$. So the inverse of $f(y)$ is $g(z^{-1})$ as claimed. □

Remark 7.7. Note that the Laurent series $f(y)$ is not bijective, so there is no global inverse. The inverse is around $z = 0$ for which $1/f(g(z))$ is finite.

Lemma 7.8. Let $f(x) = 1 + \sum_{k\geq 1} f_k x^k$ be a power series. Then

$$\exp\left(\sum_{k>0} \frac{1}{k}[x^k]f^k z^k\right) = \sum_{k>0} \frac{1}{k}[x^{k-1}]f^k z^{k-1}.$$

Proof. This is proved in [GRZZ] as a consquence of certain Bell polynomial identities. □

Proof of Theorem 7.4. Note that the closed mirror map is $q_i = z_i M_{t^{-1}\vartheta_1}(q)^{d_i}$, so that the change of variables $q_i = z_i(t/y)^{d_i}$ means $t/y = M_{t^{-1}\vartheta_1}(q)$ or

$$y(t) = tM_{t^{-1}\vartheta_1}(q)^{-1} = te^{-F_0(z)} = te^{-\sum_{k>0}\frac{1}{k}[y^0]\vartheta_1^k t^{-k}}.$$

Note that $[y^0]\vartheta_y^k$ has a t^k-factor and $F_0(z)$ does not depend on t. Now the claimed equation is

$$\vartheta_1(te^{-F_0(z)}) = t.$$

Consider the Laurent series $f(y) = \vartheta_1(y^{-1})$. By Lemma 7.6, the function $g(t)$ satisfying $f(g(t)) = t$ is given by

$$g(t) = \sum_{k>0} \frac{1}{k}[y^{-1}]f^k t^{-k}.$$

The claimed formula is equivalent to $y(t) = g(t)^{-1}$, hence to $y(t^{-1})^{-1} = g(t^{-1})$, which is

$$t\exp\left(\sum_{k>0} \frac{1}{k}[y^0]f^k t^k\right) = \sum_{k>0} \frac{1}{k}[y^{-1}]f^k t^k.$$

This follows from Lemma 7.8 applied to the power series $yf(y)$. □

Corollary 7.9. We have

$$M_{t^{-1}\vartheta_1}(q) = 1 + \sum_{\beta\in NE(X)} (\beta\cdot D - 1)R_{\beta\cdot D-1,1}(X,\beta)q^\beta.$$

Proof. Combine Theorem 7.4 and Corollary 5.16. □

7.2. **Local mirror symmetry and slab functions.** The following idea is taken from [Wan22, §4]. Let W be a potential, e.g. the Hori-Vafa potential

$$W_\Sigma = 1 + \sum_{i=1}^n x_i + \sum_{i=1}^\rho z_i x^{v_i}$$

The Gross-Siebert potential [GS14] is the correction

$$W_{\mathrm{GS}} = 1 + \sum_{i=1}^n x_i + \sum_{s=1}^\rho z_i x^{v_i} + h(z),$$

where $h(z)$ is defined by the normalization condition that $\log(W_{\mathrm{GS}})$ has no x_i-constant terms. Let $\check{x}_i = e^{F_W(z)} x_i$ be the open mirror map. The closed mirror map is $q_i(z) = z_i e^{d_i F_W(z)}$, hence

$$z_i(q) = q_i e^{-d_i F_W(z(q))} = q_i e^{(\sum_{j=1}^n v_{i,j} - 1) F_W(z(q))}$$

Then the Gross-Siebert potential becomes

$$\begin{aligned} W_{\mathrm{GS}} &= 1 + h_W(z(q)) + \sum_{i=1}^n x_i + \sum_{i=1}^\rho e^{-F_W(z(q))} q_i (e^{F_W(z(q))} x)^{v_i} \\ &= e^{-F_W(z)} \left((1 + h_W(z(q))) e^{F_W(z(q))} + \sum_{i=1}^n e^{F_W(z(q))} x_i + \sum_{i=1}^\rho q_i (e^{F_W(z(q))} x)^{v_i} \right). \end{aligned}$$

By the normalization condition the constant term of the logarithm of the term in the large bracket must be $F_W(z)$, but this means $(1 + h_W(z(q))) e^{F_W(z(q))} = 1$. Combining this with Theorem 7.4 we see that the slab function $h_W(q)$ is (up to sign) equal to the open mirror map. Note that we could have started with another potential W instead of the Hori-Vafa potential W_Σ. Hence, we have the following.

Proposition 7.10 ([Wan22], §4)**.** We have

$$1 + h_W(z(q)) = M_W(-q).$$

7.3. **The open mirror map and disk counting.** It was shown in [Lau] that in the Fano case the Gross-Siebert slab function $h(z(q))$ is a generating function for open Gromov-Witten invariants on the local Calabi-Yau K_X, the total space of the canonical bundle of X. The discussion in §7.2 together with Corollary 5.16 implies that $h(z(q))$ is a generating function for 2-marked logarithmic Gromov-Witten invariants (up to sign and some factor). This is consistent with the log-open correspondence (see e.g. [GRZ]).

In the non Fano case, if one uses the corrected potential $\vartheta_1(X)_0$ instead of the toric potential W_Σ to define $h(z(q))$, then $h(z(q))$ is still a generating function for 2-marked logarithmic Gromov-Witten invariants for the same reason. This gives the following.

Corollary 7.11. We have $h_{t^{-1}\vartheta_1}(z(-q)) = \vartheta_1(y)_\infty$ under the change of variables $q_i = z_i (t/y)^{d_i}$, hence

$$h_{t^{-1}\vartheta_1}(z(q)) = \sum_{\beta \in NE(X)} (-1)^{\beta \cdot D} (\beta \cdot D - 1) R_{\beta \cdot D - 1, 1}(X, \beta) q^\beta.$$

The log-open correspondence is not valid in the non-Fano case, because curves can move away from the zero section of K_X. We conjecture, that the result of [Lau] is still valid in the non-Fano case if one uses the Hori-Vafa potential $W_{\Sigma'}$, where Σ' is the fan of K_{X_Σ} for some toric model X_Σ of X.

Conjecture 7.12. We have

$$h_{W_{\Sigma'}}(z(q)) = \sum_{\beta \in NE(X)} N_1(X,\beta) q^\beta,$$

where $N_1(X,\beta)$ is the open Gromov-Witten invariant of K_X with winding one.

References

[ACGS21] Dan Abramovich, Qile Chen, Mark Gross, and Bernd Siebert. Punctured logarithmic maps, 2021, 2009.07720.

[Akh12] Mohammad Akhtar. Minkowski polynomials and mutations. *Symmetry, Integrability and Geometry: Methods and Applications*, December 2012.

[AKV02] Mina Aganagic, Albrecht Klemm, and Cumrun Vafa. Disk instantons, mirror symmetry and the duality web. *Zeitschrift für Naturforschung A*, 57(1-2):1–28, feb 2002.

[Aur] D. Auroux. Special Lagrangian fibrations, wall-crossing, and mirror symmetry. 0902.1595.

[AV00] Mina Aganagic and Cumrun Vafa. Mirror Symmetry, D-Branes and Counting Holomorphic Discs. 2000, hep-th/0012041.

[Bat] Victor V. Batyrev. Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties. alg-geom/9310003.

[BBvG24] Pierrick Bousseau, Andrea Brini, and Michel van Garrel. Stable maps to Looijenga pairs. *Geometry & Topology*, 28(1):393–496, February 2024.

[CCG+] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander Kasprzyk. *Mirror Symmetry and Fano Manifolds*, page 285–300. European Mathematical Society Publishing House.

[CCLT16] Kwokwai Chan, Cheol-Hyun Cho, Siu-Cheong Lau, and Hsian-Hua Tseng. Gross fibrations, SYZ mirror symmetry, and open Gromov–Witten invariants for toric Calabi–Yau orbifolds. *Journal of Differential Geometry*, 103(2), jun 2016.

[CDGP91] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359(1):21–74, 1991.

[Cha11] Kwokwai Chan. A formula equating open and closed gromov–witten invariants and its applications to mirror symmetry. *Pacific Journal of Mathematics*, 254(2):275–293, December 2011.

[CKPT21] Tom Coates, Alexander M. Kasprzyk, Giuseppe Pitton, and Ketil Tveiten. Maximally mutable laurent polynomials. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 477(2254), October 2021.

[CLL12] Kwokwai Chan, Siu-Cheong Lau, and Naichung Conan Leung. SYZ mirror symmetry for toric Calabi-Yau manifolds. *Journal of Differential Geometry*, 90(2), February 2012.

[CLT13] Kwokwai Chan, Siu-Cheong Lau, and Hsian-Hua Tseng. Enumerative meaning of mirror maps for toric calabi–yau manifolds. *Advances in Mathematics*, 244:605–625, September 2013.

[CPS] Michael Carl, Max Pumperla, and Bernd Siebert. A tropical view on Landau-Ginzburg models. 2205.07753.

[FOOO10] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian Floer theory on compact toric manifolds, I. *Duke Mathematical Journal*, 151(1), jan 2010.

[FZ01] Sergey Fomin and Andrei Zelevinsky. Cluster algebras I: Foundations, 2001, math/0104151.

[GHK15] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log calabi-yau surfaces i, 2015, 1106.4977.

[GHS19] Mark Gross, Paul Hacking, and Bernd Siebert. Theta functions on varieties with effective anti-canonical class, 2019, 1601.07081.

[GPS10] Mark Gross, Rahul Pandharipande, and Bernd Siebert. The tropical vertex. *Duke Mathematical Journal*, 153(2), June 2010.

[Gra] Tim Graefnitz. Counting (tropical) curves via scattering.sage. 2210.10455.
[Gra22] Tim Graefnitz. Theta functions, broken lines and 2-marked log gromov-witten invariants, 2022, 2204.12257.
[GRZ] Tim Gräfnitz, Helge Ruddat, and Eric Zaslow. The proper landau-ginzburg potential is the open mirror map. 2204.12249.
[GRZZ] Tim Gräfnitz, Helge Ruddat, Eric Zaslow, and Ben Zhou. Quantum theta functions from open mirror symmetry. in preparation.
[GS12] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. 2012, math/0703822.
[GS14] Mark Gross and Bernd Siebert. Local mirror symmetry in the tropics, 2014, 1404.3585.
[GS19] Mark Gross and Bernd Siebert. Intrinsic mirror symmetry. 2019, 1909.07649.
[GS21] Mark Gross and Bernd Siebert. The canonical wall structure and intrinsic mirror symmetry, 2021, 2105.02502.
[GU10] Sergey Galkin and Alexandr Usnich. Cluster algebras i: Foundations, 2010, IPMU preprint.
[Gui94] Victor Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian Tn–spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser, Boston, USA, 1994.
[HKTY95a] S. Hosono, A. Klemm, S. Theisen, and S-T Yau. Mirror symmetry, mirror map and applications to calabi-yau hypersurfaces. *Communications in Mathematical Physics*, 167(2):301–350, feb 1995.
[HKTY95b] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau. Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces. *Nuclear Physics B*, 433(3):501–552, jan 1995.
[HV] Kentaro Hori and Cumrun Vafa. Mirror symmetry. hep-th/0002222.
[Ilt12] Nathan Owen Ilten. Mutations of laurent polynomials and flat families with toric fibers. *Symmetry, Integrability and Geometry: Methods and Applications*, July 2012.
[Lau] Siu-Cheong Lau. Gross–Siebert's slab functions and open GW invariants for toric Calabi-Yau manifolds. 1405.3863.
[LM] W. Lerche and P. Mayr. On N=1 Mirror Symmetry for Open Type II Strings. hep-th/0111113.
[May] P. Mayr. N=1 Mirror Symmetry and Open/Closed String Duality. hep-th/0108229.
[Rud12] Helge Ruddat. Mirror duality of Landau-Ginzburg models via Discrete Legendre Transforms. 2012, 1204.5611.
[Rud21] Helge Ruddat. A homology theory for tropical cycles on integral affine manifolds and a perfect pairing. *Geometry and Topology*, 25(6):3079–3132, nov 2021.
[vGGR19] Michel van Garrel, Tom Graber, and Helge Ruddat. Local gromov-witten invariants are log invariants. *Advances in Mathematics*, 350:860–876, July 2019.
[Wan21] Junxiao Wang. *The Gamma Conjecture for Local Mirror Symmetry: a Tropical Approach via the Gross-Siebert Program*. Phd thesis, Northwestern University, Evanston, IL, September 2021.
[Wan22] Junxiao Wang. The Gamma Conjecture for Tropical Curves in Local Mirror Symmetry, 2022, 2011.01729.

University of New Hampshire, Department of Physics and Astronomy, 105 Main St, Durham, NH 03824, USA
Email address: per.berglund@unh.edu

Leibniz-Universität Hannover, Institut für Algebraische Geometrie, Welfengarten 1, 30167 Hannover, Germany
Email address: graefnitz@math.uni-hannover.de

University of New Hampshire, Department of Physics and Astronomy, 105 Main St, Durham, NH 03824, USA
Email address: michael.lathwood@unh.edu