MODIFIED SCATTERING FOR THE CUBIC SCHRÖDINGER EQUATION ON DIOPHANTINE WAVEGUIDES

NICOLAS CAMPS AND GIGLIOLA STAFFILANI

Abstract. We consider the cubic Schrödinger equation posed on a product space subject to a generic Diophantine condition. Our analysis shows that the small-amplitude solutions undergo modified scattering to an effective dynamics governed by some interactions that do not amplify the Sobolev norms. This is in sharp contrast with the infinite energy cascade scenario observed by Hani–Pausader–Tzvetkov–Visciglia in the absence of Diophantine conditions.

Contents

1. Introduction

1.1. **Motivations.** The present work aims at testing the persistence of turbulent mechanisms observed in the long-time dynamics of certain solutions of nonlinear dispersive equations. On compact domains, the interplay between weaker dispersion and nonlinear resonances can lead to energy transfers from low to high frequency scales of oscillations as time evolves. These forward cascade mechanisms manifest themselves through the growth of the high-order Sobolev norms. Bourgain's original formulation in [5] addresses the potential explosion of Sobolev norms in infinite time, known as *the infinite energy cascade*. Despite considerable efforts to elucidate this conjecture, our understanding of the infinite-time behavior of nonlinear dispersive equations on compact domains is still incomplete.

One of the very few results constructing infinite energy cascades is due to Hani–Pausader–Tzvetkov–Visciglia [24]. They consider the cubic defocusing Schrödinger equation posed on the product space $\mathbb{R} \times \mathbb{T}^d$, with $\mathbb{T}^d = \mathbb{R}/\mathbb{Z}^d$:

$$i\partial_t U + \partial_x^2 U + \Delta_y U = |U|^2 U\,, \quad (x, y) \in \mathbb{R} \times \mathbb{T}^d\,. \tag{NLS}$$

The dispersion of the Schrödinger free evolution along the non-compact direction $\mathbb{R}$ serves to dampen and scatter the non-resonant interactions, reducing the effective dynamics to a system governed by the four-waves resonant set of the cubic Schrödinger

Date: 26th April 2024.

2020 *Mathematics Subject Classification.* 35B40 primary.

equation posed on the compact direction $\mathbb{T}^d$. This enables the construction of small amplitude initial data that ignite a forward energy cascade ([24, Corollary 1.4]):

For all $d \in \{2,3,4\}$ and $s > 1$ large enough, there exists $U(0)$ with small amplitude [1] *in $H^s(\mathbb{R} \times \mathbb{T}^d)$ such that the corresponding solution U is global and satisfies*

$$\limsup_{t \to +\infty} \|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} = +\infty\,.$$

In this work, we allow for a more general dispersion relation in the compact direction $\mathbb{T}^d$:

$$i\partial_t U + \partial_x^2 U + \operatorname{div}(\mathrm{A}\nabla_y)U = |U|^2 U\,, \quad (x,y) \in \mathbb{R}\times\mathbb{T}^d\,. \tag{A-NLS}$$

The symmetric positive definite constant matrix A encodes the dispersion relation. Changing the dispersion via A corresponds to rescaling $\mathbb{T}^d$ through a change of variables. For instance when $\mathrm{A} = \mathrm{Id}$, we retrieve the case of the square torus $\mathbb{T}^d$, while rectangular tori correspond to diagonal matrices A. We will focus on generic matrices satisfying an explicit Diophantine condition presented in Definition 1.1. They correspond to generic non-rectangular tori.

A natural question is whether the turbulent mechanisms observed in the case of the square torus [24] persist under generic and non-periodic boundary conditions.

Our main results, stated in Theorem 1.3 and Theorem 1.4, address this question. When (A-NLS) is posed on a waveguide with a generic Diophantine property, we establish that there is no amplification of the Sobolev norms over infinite time for the solutions associated with small amplitude initial data. Hence, the asymptotic behavior of (A-NLS) on the product space $\mathbb{R}\times\mathbb{T}^d$ with $d \geq 2$ is very different when $\mathbb{T}^d$ is replaced by Diophantine tori. This sharp dichotomy that we see very well manifested analytically should also be of warning when one uses periodic boundary conditions to model nonlinear dispersive equations on bounded domains. What we prove here is that any infinitesimal perturbation of a periodic domain may result in a dramatically different behavior of the energy spectrum of the solutions.

1.2. **Context.** In this subsection we recall some previous related works and we stress out further the main motivations.

1.2.1. *Growth of Sobolev norms for nonlinear Schrödinger equations on compact domains and waveguides.* The existence of infinite cascades realized via the growth of Sobolev norms has drawn considerable interest, and a survey of all the existing results goes much beyond the scope of this paper, hence here we recall only the works that are strictly related to it.

A line of research within the study of the infinite energy cascade consists in proving polynomial upper bounds for the growth of Sobolev norms [2] to quantify how fast do the transfer of energy occurs. This was done originally by Bourgain [3], and several other works have appeared in recent years [34, 30, 31, 32, 33]. On rectangular

[1]The precise statement actually requires the smallness of a strong norm, with extra derivatives and space decay in the $\mathbb{R}$-direction

[2]The Cauchy theory and the conservation of the energy give exponential bounds

Diophantine tori [13] improved the polynomials bounds, and [12] obtained polynomial bounds for the energy-critical quintic Schrödinger equation (for the square tori such a result is not known). These results suggest that energy transfers, if they exist, take longer to occur when the boundary conditions are Diophantine rather than periodic.

As mentioned before, much less is known about the lower bounds and the infinite growth of the Sobolev norms. The infinite energy cascade evidenced in [24] is a consequence of the analysis of the Toy Model, a subsystem of the four-waves resonant set, which was introduced in [9] where the equation in (NLS) was considered in a square torus $\mathbb{T}^2$ (see also [8]). In that case, using a stability lemma, one could only partially transfer the growth dynamics of the Toy Model and prove a norm-amplification result in finite time:

For all $s > 1$ and $C > 0$, there exist $T > 0$ and $u(0) \in H^s(\mathbb{T}^2)$ such that the solution u to (NLS) *posed on $\mathbb{T}^2$ satisfies*

$$\|u(T)\|_{H^s(\mathbb{T}^2)} \geq C\|u(0)\|_{H^s(\mathbb{T}^2)} .$$

Along these lines we also recall the work in [20, 21, 25, 22]. Apart from the case of completely resonant equations (the Szegő equation [17], which is the resonant system of the Half-wave equation), this strategy gives norm amplification results in finite time.

1.2.2. *Infinite energy cascade on waveguides.* On the waveguide manifold the dispersion along the $\mathbb{R}$ direction allowed for the full growth of the Toy Model to be harvested in infinite time. The strategy developed in [24] (also explained in [23]) consists in propagating a *weak norm* based on the energy space which is preserved by the effective resonant system, while allowing the stronger norms to slightly grow in time. By doing so, one can both establish modified scattering and construct wave operators from which one can transfer the turbulent dynamics known for the Toy Model. The time-oscillations in the non-resonant interactions are easily [3] converted into decay via a Poincaré–Dulac normal form, while non-stationary phase arguments control space-oscillations, in the spirit of [29].

Subsequent works, such as [38, 7], adopt the same strategy for weakly dispersive equations, resulting in infinite energy cascades [4]. Another notable example is [16], which extracts interesting dynamics with a beating effect from the resonant system of a coupled system of nonlinear Schrödinger equations.

In the spirit of the present work, [19] consider the cubic Schrödinger equation perturbed by a generic convolution potential and prove that there is no norm amplification. For this model, the exact resonances are trivial, and the quasi-resonances are controlled by the third largest frequency so that one can reproduce the strategy

[3]Indeed, the resonant function for NLS on $\mathbb{T}^d$ (periodic boundary conditions) takes integer values. If it does not vanish then it takes integer values greater than 1, unlike in the present work where small divisor problems arise.

[4]In these two works, however, the Cauchy theory at low regularity is not well understood and poses considerable problems. They could achieve the construction of modified wave operators, which is sufficient to prove infinite energy cascade, without proving the modified scattering result.

of [24]. However, our admissibility condition (1.1) on Diophantine tori without potential provides control by the highest frequency, necessitating new approaches that will be discussed in Section 1.4 below. To conclude this subsection, we stress out that on $\mathbb{R}^n \times \mathbb{T}^d$ with $n \geq 2$ there is scattering [36, 37]. Indeed, all the interactions are attenuated by the stronger dispersion on $\mathbb{R}^n$.

1.2.3. *Nonlinear Schrödinger equations posed on Diophantine tori: longer time-scales?* By considering irrational tori instead of the square torus, dispersion is enhanced and the refocusing time needed for the waves to explore the geometry is also longer. This was quantified by [14, 15] who obtained longer time versions of the Strichartz estimates on Diophantine rectangular and non-rectangular tori than in [6].

Moreover, for the Schrödinger equation in (NLS) on irrational tori some exact resonant interactions are eliminated, whereas quasi-resonant interactions emerge: the resonant function is not valued in $\mathbb{Z}$ and can take arbitrarily small values without vanishing. In [18], it was proved that the quasi-resonant interactions can also lead to turbulent mechanisms by extending the norm inflation result of [9] for small amplitude initial data to the case of some rectangular irrational tori in dimension $d = 2$, after exponential time-scale (the Diophantine approximation condition for the length of the torus is detailed in [18, paragraph 2.1]). This result indicates that on irrational tori quasi-resonant interactions have a role to play in the dynamics.

The first author with J. Bernier recently developed in [1] a normal form approach to show stability over polynomial time-scales for the cubic Schrödinger equation posed on Diophantine tori satisfying the Diophantine condition (1.1). In [35], the second author with B. Wilson proved that on rectangular irrational tori the resonant system is much smaller than on $\mathbb{T}^2$ and showed that the energy transfers observed in [8] out of the Toy Model when finitely many modes are excited do not longer occur. Moreover in [28] the second author with A. Hrabski, Y. Pan and B. Wilson also gave some numerical evidence that the high Sobolev norms should not grow as rapidly as the ones on a square torus. For more numerical results one should also consult [10, 27]. Finally, let us point out the result [11] in the context of wave kinetic theory, which goes in the direction of considering generic Diophantine tori in order to bypass the time-scales dictated by the resonant interactions.

1.3. **Set up and main results.** We consider the equation (A-NLS) on the waveguide manifold with the following explicit Diophantine condition on the coefficients of A.

Definition 1.1. *The matrix* A *is admissible with a parameter* $\tau > 0$ *if there exists* $c > 0$ *such that for all* $(a, b) \in (\mathbb{Z}^d \setminus \{0\})^2$,

$$|a^\intercal \mathrm{A} b| \geq \frac{c}{|a|^\tau |b|^\tau}, \tag{1.1}$$

where for $n \in \mathbb{Z}^d$ *we denote by* $|n|$ *the* ℓ^2*-norm.*

Remark 1.2. According to a result from Khinchin, for each $\tau > \frac{d(d+1)}{2}$ the set of admissible matrices with parameter τ is of full Lebesgue measure.

The operator $-\operatorname{div}(\mathrm{A}\nabla)$ has discrete spectrum, with eigenvalues indexed by the points $n \in \mathbb{Z}^d$:

$$-\operatorname{div}(\mathrm{A}\nabla)\, \mathrm{e}^{iny} = \lambda_n^2\, \mathrm{e}^{iny}\,, \quad \text{with} \quad \lambda_n^2 := n^\intercal \mathrm{A} n = \sum_{i,j=1}^{d} n_i n_j \mathrm{A}_{i,j}\,.$$

Since A is symmetric and positive definite, there exist $c, C > 0$ such that for all $n \in \mathbb{Z}^d$,

$$c|n|^2 \le \lambda_n^2 \le C|n|^2\,, \quad \text{where we recall that} \quad |n|^2 := \sum_{i=1}^{d} n_i^2\,. \tag{1.2}$$

The four-waves resonant function is

$$\Omega(\vec{n}) = \lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_n^2\,.$$

Under the zero momentum condition $n = n_1 - n_2 + n_3$ it factorizes to

$$\Omega(\vec{n}) = (n_1 - n_2)^\intercal \mathrm{A}(n_2 - n_3)\,.$$

This factorization motivates Definition 1.1 of the Diophantine admissibility condition for the matrix A. Firstly, it eliminates the exact resonances, as proved in Lemma 2.6 below. Secondly, it introduces quasi-resonant interactions but provides a quantitative polynomial decay of the resonant function. This gives hope the exploit the time-oscillations in $H^s(\mathbb{T}^d)$ despite the presence of small divisors, which appearing in the normal forms method used to convert time-oscillations into decay in time.

Note that rectangular tori are not admissible. If the vector $(\mathrm{A}_{i,j})_{i,j}$ is diophantine then A is admissible, but this is not an equivalence. We refer to the introduction of [1] for more details on the admissibility condition.

We give the precise definitions of the norms in Section 2.2 below, but at this stage we stress that the S-norm, that we will use below in the statement of one of the main theorems, contains s derivatives in the compact direction $\mathbb{T}^d$ and $s+\sigma$ in the $\mathbb{R}$ direction, for $\sigma > 0$ arbitrarily small (which is an improvement compared to [24]). Other spaces that we use in the statements are the Sobolev space $H^s(\mathbb{R}\times\mathbb{T}^d)$ and the space $h^s(\mathbb{T}^d)$, which is equivalent to the homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^d)$ as will be proved also in Section 2.2. We also anticipate that for $\tau > 0$ and $c(d) \in (0,2]$ given in Lemma 2.1, the regularity threshold needed for our results is

$$s_0(\tau, d) := \frac{d}{2} + \frac{4\tau}{c(d)}\,. \tag{1.3}$$

We are now ready to state the main theorems.

Theorem 1.3. *Let $d \ge 2$ and A be admissible with parameter τ. Given $s > s_0(\tau, d)$, there exist $C > 0$ and $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$, if*

$$\|U(0)\|_S \le \varepsilon\,,$$

then the solution U to (A-NLS) with Cauchy data $U(0)$ exists globally in $C(\mathbb{R};S)$. Moreover, for all $t\in\mathbb{R}$,

$$\|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \leq C\varepsilon\,,\quad \|U(t)\|_{L^\infty h^s(\mathbb{R}\times\mathbb{T}^d)} \leq C(1+|t|)^{-\frac{1}{2}}\varepsilon\,. \tag{1.4}$$

Note in particular that (1.4) shows that the solution preserves the dispersion time rate of $(1+|t|)^{-\frac{1}{2}}$ inherited from the absence of boundaries in the $\mathbb{R}$ direction. The proof of Theorem 1.3 is based on the dynamics of an effective system presented below in (3.4) and (3.9), to which the small amplitude solutions undergo modified scattering. The effective system does not only contain the resonant interactions (which are trivial when A is admissible), but also some specific quasi-resonant interactions described in (3.6).

Theorem 1.4 (Modified scattering)**.** *Under the same assumptions as in Theorem 1.3, we have that for all $U(0)$ with*

$$\|U(0)\|_S \leq \varepsilon\,,$$

there exists a global solution $G\in C(\mathbb{R};S)$ to the system (3.9) such that $\|G(0)\|_{H^s}\lesssim\varepsilon$ and

$$\lim_{t\to+\infty}\|\,\mathrm{e}^{-it\Delta}\,U(t)-G(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}=0\,.$$

We make some remarks.

Remark 1.5. Infinite energy cascades are excluded, in sharp contrast to the case of the square torus studied in [24]. To the best of the authors' knowledge, Theorem 1.3 is the first result that shows that the dynamics in infinite time is completely different on Diophantine tori from that on the square torus. This indicates that the growth mechanism established in [24] depends on the periodic boundary conditions and is somehow unstable.

Remark 1.6. The Z-norm, that will be defined in Section 2.2 and used for the proofs of the theorems stated above, is the strongest norm preserved by the effective system. Unlike the case of the square torus [24] this norm is not based on the energy space but on $H^s(\mathbb{T}^d)$, for $s>\frac{d}{2}$. As a result, we no longer have low regularity problems and our results cover all the dimensions $d\geq 2$.

Remark 1.7. The case when $d=1$, which is also covered by the proof, is less interesting since there is no quasi-resonant resonant interactions: there is modified scattering to the (trivial) resonant system (3.5), as in the case with the convolution potential [19].

Remark 1.8. The regularity threshold $s>s_0(\tau,d)=\frac{d}{2}+\frac{4\tau}{c(d)}$ depends on the Diophantine properties of the dispersion relation (through the parameter τ) and on the dimension d (through the constant $c(d)$ related to the frequency separation property given in Lemma 2.1). Since $c(d)\leq 2$ we have

$$s_0(\tau,d)\geq\frac{d}{2}+2\tau\,.$$

One can expect to save some of the $\frac{d}{2}$-loss by using the Strichartz estimates from [6]. However, it is not clear how to avoid the loss 2τ due to the quasi-resonant interactions. Since this loss is dominant [5], we prefer not to use the machinery of Strichartz estimates. In the unbounded direction $\mathbb{R}$, we are able to make the loss of regularity arbitrarily small.

1.4. **Idea of the proof.** The proof is inspired by the analysis developed in [24] and also in [29], with two important differences.

– Firstly, the analysis of the time non-resonant interactions is more complicated due to small divisor problems. For some quasi-resonant interactions, we can still perform a Poincaré–Dulac normal form reduction to obtain an integrable decay in time, thanks to a frequency separation property based on a cluster decomposition recalled in Lemma 2.1. The remaining quasi-resonant interactions, which cannot be treated with a normal form method, contribute to the effective system. These non-resonant contributions are precisely the *high* $\times$ *low* $\to$ *high* frequency interactions, where the incoming high frequency and the outgoing frequency are in the same dyadic cluster. We give a precise definition of the effective system in (3.6).

– Secondly, we propose a general strategy leveraging dispersive decay alongside the stability of Sobolev norms to prove modified scattering. Specifically, we make use of the natural dispersive estimate claimed in Theorem 1.3 to simplify the analysis of the space non-resonant interactions, and to propose a concise self-contained proof. This allows us to prove the result in any dimension, and to have an arbiltrarilly small loss of derivatives in the $\mathbb{R}$-direction compared to the strong norm.

1.5. **Organization of the paper.** After introducing the Hamiltonian system in the transverse direction and exploiting the frequency separation property in Section 2, we analyze nonlinear interactions in Section 3, establishing bounds with decay in time. Finally, in Section 4, we establish a priori bounds for the time-dependent norm, as presented in Proposition 4.2. These bounds yield the global existence result and the dispersive decay (1.4) asserted in Theorem 1.3. Additionally, we prove Theorem 1.4 and deduce the boundedness of the H^s-norm, as also claimed in Theorem 1.3.

1.6. **Notations.** For $F \in L^2(\mathbb{R}\times\mathbb{T}^d)$, we define the (partial) Fourier transform: for $y \in \mathbb{T}^d$,

$$\xi \in \mathbb{R} \quad \mapsto \quad \widehat{F}(\xi, y) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{-i\xi x} F(x,y)dx$$

and, for $x \in \mathbb{R}$,

$$n \in \mathbb{Z}^d \quad \mapsto \quad F_n(x) = (2\pi)^{-d}\int_{\mathbb{T}^d} \mathrm{e}^{-in\cdot y} F(x,y)dy\,.$$

The (full) spatial Fourier transform of F is

$$(\xi, n) \in \mathbb{R}\times\mathbb{Z}^d \;\mapsto\; (\mathcal{F}F)(\xi,n) := (2\pi)^{-(d+1)}\int_{\mathbb{R}\times\mathbb{T}^d} \mathrm{e}^{-i(\xi x + n\cdot y)} F(x,y)dxdy\,. \quad (1.5)$$

[5]In order to have a generic condition we need to suppose $2\tau > d(d+1)$

Following [24], we write for $(\xi, n) \in \mathbb{R} \times \mathbb{T}^d$

$$\widehat{F}_n(\xi) := (\mathcal{F}F)(\xi, n)\,, \quad \text{so that} \quad \widehat{F}(\xi, y) = \sum_{n\in\mathbb{Z}^d} \widehat{F}_n(\xi)\, \mathrm{e}^{in\cdot y}\,.$$

Given three incoming waves $(n_1, n_2, n_3) \in (\mathbb{Z}^d)^3$, we define the decreasing rearrangement of $(|n_1|, |n_2|, |n_3|)$ by

$$n_1^* \geq n_2^* \geq n_3^*\,.$$

We denote by $n = n_1 - n_2 + n_3$ the outgoing wave and by $\vec{n} = (n_1, n_2, n_3, n) \in (\mathbb{Z}^d)^4$ the four interacting waves.

Acknowledgments. During the preparation of this work the first author benefited from the support of the Centre Henri Lebesgue ANR-11-LABX-0020-0, the region Pays de la Loire through the project MasCan and the ANR project KEN ANR-22-CE40-0016. The second author was supported by the NSF grant DMS-2052651 and the Simons Foundation Collaboration Grant on Wave Turbulence.

2. Preparations

2.1. **Clustering of the eigenvalues.** Let us recall the frequency separation property, which will play a key role in the present work. This property was proved by Bourgain [4] on $\mathbb{T}^d$ for the flat Laplacian (when $\mathrm{A} = \mathrm{Id}$), and generalized by Berti and Maspero [2] for all A symmetric and positive definite (or, equivalently, for any flat torus).

Lemma 2.1 (Frequency separation, Theorem 2.1 in [2])**.** *There exist $c(d) \in (0, 2]$ and $C(A, d) \geq 2$, and a partition of $\mathbb{Z}^d$ into disjoint clusters $(\mathscr{C}_\alpha)_{\alpha\geq 1}$ satisfying the following:*

(1) The first cluster contains the origin $0 \in \mathscr{C}_0$ and is bounded:

$$\max_{n\in\mathscr{C}_0} |n| \leq C(A, d)\,.$$

(2) The other clusters are dyadic: for all $\alpha \geq 1$,

$$\max_{n\in\mathscr{C}_\alpha} |n| \leq 2 \min_{n\in\mathscr{C}_\alpha} |n|\,.$$

(3) For all $(n_1, n_2) \in \mathscr{C}_{\alpha_1} \times \mathscr{C}_{\alpha_2}$ with $\alpha_1 \neq \alpha_2$,

$$|n_1 - n_2| + |\lambda_{n_1}^2 - \lambda_{n_2}^2| > (|n_1| + |n_2|)^{c(d)}\,. \tag{2.1}$$

The last condition is a separation property that plays a key role in the analysis, to control the quasi-invariant interactions, and we refer to Lemma 2.10.

2.2. **Norms.** We fix the regularity $s > s_0$ (see (1.3)) as in the statement of Theorem 1.3. In our analysis, the relevant Sobolev norms in the compact direction are based on the cluster decomposition. We prove in Lemma 2.9 that they are preserved by the effective system defined in (3.9).

Definition 2.2 (Sobolev norms based on the clusters)**.** *We define*

$$K_0 := 1\,, \quad K_\alpha := \min_{n\in\mathscr{C}_\alpha} |n| \quad \textit{for } \alpha \geq 1\,,$$

and π_α *is the orthogonal projector onto* span $\{e^{in\cdot y}\}_{n\in\mathscr{C}_\alpha}$*: for a function* $u \in \ell^2(\mathbb{T}^d)$*,*

$$\pi_\alpha u(y) = \sum_{n\in\mathscr{C}_\alpha} u_n\, \mathrm{e}^{in\cdot y}\,, \quad u_n = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-in\cdot y}\, u(y)dy\,.$$

For $s \in \mathbb{R}$*, we define the Sobolev norms for functions on* $\mathbb{T}^d$ *and on* $\mathbb{R}\times\mathbb{T}^d$ *by*

$$\|u\|_{h^s(\mathbb{T}^d)} := \Big(\sum_{\alpha\geq 0} K_\alpha^{2s}\|\pi_\alpha u\|^2_{\ell^2(\mathbb{T}^d)}\Big)^{\frac{1}{2}}\,, \tag{2.2}$$

$$\|U\|_{H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d)} := (\|U\|^2_{L^2_x h^s_y(\mathbb{R}\times\mathbb{T}^d)} + \|U\|^2_{H^s_x\ell^2_y(\mathbb{R}\times\mathbb{T}^d)})^{\frac{1}{2}}\,. \tag{2.3}$$

Remark 2.3. Since, for all $n \in \mathbb{Z}^d\setminus\{0\}$ and $\alpha \geq 1$ with $n \in \mathscr{C}_\alpha$, we have

$$K_\alpha \leq |n| \leq 2K_\alpha\,.$$

Therefore, the Sobolev norms (2.2) are equivalent to the standard Sobolev ones.

The motivation for considering this equivalent Sobolev norm $h^s(\mathbb{T}^d)$ is that it does not capture the possible transfers of energy within a same cluster. Hence, we expect this norm to stay bounded under the effective dynamics.

Definition 2.4 (Strong, weak and time-dependent norms)**.** *Let* $s > s_0$ *(see (1.3)) and* $\sigma > 0$ *arbitrarily small. The strong norm is*

$$\|F\|_S := \|(1-\partial_x^2)^{\frac{\sigma}{2}} F\|_{H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d)} + \|xF\|_{L^2_x h^s_y(\mathbb{R}\times\mathbb{T}^d)}\,, \tag{2.4}$$

and the weak norm is

$$\|F\|_Z := \sup_{\xi\in\mathbb{R}} \|\widehat{F}(\xi)\|_{h^s_y(\mathbb{T}^d)}\,. \tag{2.5}$$

For $\delta > 0$ *small enough, the space-time norm is*

$$\|F\|_{X_T} := \sup_{0\leq t\leq T}\big(\|F(t)\|_Z + \langle t\rangle^{-\delta}\|F(t)\|_S + \langle t\rangle^{1-\delta}\|\partial_t F(t)\|_S\big)\,. \tag{2.6}$$

Remark 2.5. Observe that

$$\|F\|_Z = \sup_{\xi\in\mathbb{R}}\|\widehat{F}(\xi)\|_{h^s_y} \lesssim \|\langle x\rangle F\|_{L^2_x h^s_y} \lesssim \|F\|_S\,.$$

2.3. **The Hamiltonian system in the transverse direction.** Let $u \in C^1(\mathbb{R};\mathbb{T}^d)$ be a solution on $\mathbb{T}^d$ of

$$i\partial_t u + \mathrm{div}(\mathrm{A}\nabla)u = |u|^2 u\,,$$

with Fourier coefficients

$$u_n(t) = (2\pi)^{-d}\int_{\mathbb{T}^d} \mathrm{e}^{-in\cdot y}\, u(y)dy\,, \quad n \in \mathbb{Z}^d\,.$$

The vector $a \in C^1(\mathbb{R};\mathbb{C}^{\mathbb{Z}^d})$ defined by

$$a_n(t) := \mathrm{e}^{it\lambda_n^2}\, u_n(t)\,, \quad n \in \mathbb{Z}^d\,,$$

is solution to the interaction system

$$i\partial_t a_n = \sum_{n_1-n_2+n_3=n} \mathrm{e}^{it\Omega(\vec{n})}\, a_{n_1}\overline{a_{n_2}}a_{n_3}\,, \quad n\in\mathbb{Z}^d\,, \tag{2.7}$$

where we recall that for $\vec{n}=(n_1,n_2,n_3,n)\in(\mathbb{Z}^d)^4$, the resonant function reads

$$\Omega(\vec{n}) = \lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_n^2\,.$$

A subsystem of (2.7) is the resonant system:

$$i\partial_t a_n = \sum_{(n_1,n_2,n_3)\,:\,\vec{n}\in\Gamma_0} a_{n_1}\overline{a_{n_2}}a_{n_3}\,, \tag{2.8}$$

where we define the 4 waves resonant set by

$$\Gamma_0 = \big\{\vec{n}\in(\mathbb{Z}^d)^4 \quad : \quad n_1-n_2+n_3-n = \Omega(\vec{n}) = 0\big\}\,. \tag{2.9}$$

Under the zero momentum condition $n_1-n_2+n_3=n$ the resonant function factorizes:

$$\Omega(\vec{n}) = (n_1-n)^\intercal \mathrm{A}(n_2-n_3)\,.$$

Lemma 2.6 (Four-waves resonances)**.** *Under the Diophantine admissibility condition* (1.1) *on* A*, the nonlinear term on the four-waves resonant set is trivial:*

$$\sum_{\vec{n}=(n_1,n_2,n_3,n)\in\Gamma_0} a_{n_1}\overline{a_{n_2}}a_{n_3}\overline{a_n} = 2\Big(\sum_{n\in\mathbb{Z}^d}|a_n|^2\Big)^2 - \sum_{n\in\mathbb{Z}^d}|a_n|^4\,. \tag{2.10}$$

Proof. If we factorize the resonant function as above (using the zero-momentum condition) we obtain from the Diophantine admissibility condition (1.1) on A that

$$(n_1-n_2)^\intercal\mathrm{A}(n_2-n_3)=0 \quad\Longrightarrow\quad n_2=n_1 \quad\text{or}\quad n_2=n_3\,.$$

We conclude from the zero momentum condition that

$$\Gamma_0 = \big\{\vec{n}=(n_1,n_2,n_3,n)\in\mathbb{Z}^4 \;:\; \{n_1,n_3\}=\{n,n_2\}\big\}\,.$$

This proves Lemma 2.6. $\square$

Let us now define the effective quasi-resonant interactions. They consist in the interactions of type *high* $\times$ *low* $\to$ *high*, where the two high frequencies are in a same dyadic cluster, and where the resonant function is smaller than 1. We denote

$$\alpha_0 := \min\{\alpha\geq 1 \;:\; K_\alpha \geq 10^{10}C(\mathrm{A},d)\}\,, \tag{2.11}$$

where we recall that $K_\alpha := \min_{n\in\mathscr{C}_\alpha}|n|$ when $\alpha\geq 1$. We refer to high frequency clusters when $\alpha\geq\alpha_0$.

Definition 2.7 (Effective quasi-resonant interactions)**.** *For $\alpha\geq\alpha_0$ as above and for all $n\in\mathscr{C}_\alpha$, we let*

$$\Lambda^{(1)}(n) := \big\{(n_1,n_2,n_3)\in(\mathbb{Z}^d)^3 \;:\; n_1-n_2+n_3=n\,,\quad |\Omega(\vec{n})|<1\,, \\ n_1\in\mathscr{C}_\alpha\,,\quad |n_2|,|n_3|<10^{-5}K_\alpha\big\}$$

and

$$\Lambda^{(3)}(n) := \Big\{(n_1,n_2,n_3)\in(\mathbb{Z}^d)^3 \ : \ n_1-n_2+n_3=n\,, \quad |\Omega(\vec{n})|<1\,, \\ n_3\in\mathscr{C}_\alpha\,, \quad |n_2|,|n_1|<10^{-5}K_\alpha\Big\}\,.$$

The effective quasi-resonant interaction equation is as follows:

- *If* $n\in\bigcup_{\alpha\geq\alpha_0}\mathscr{C}_\alpha$ *then*

$$i\partial_t a_n = 2\sum_{(n_1,n_2,n_3)\in\Lambda^{(1)}(n)} \mathrm{e}^{it\Omega(\vec{n})}\, a_{n_1}\overline{a_{n_2}}a_{n_3}\,.$$

- *Otherwise,*

$$i\partial_t a_n = 0\,.$$

Remark 2.8. In the definition of the effective quasi-resonant interaction we implicitly used that n_1 and n_3 play symmetric roles, hence the factor 2 (instead of summing over $\Lambda^{(1)}(n)$ and over $\Lambda^{(3)}(n)$).

We now prove that the effective system preserves the high-frequency super actions based on the dyadic clusters. Recall that high-frequencies refer to the clusters $\mathscr{C}_\alpha$ with $\alpha\geq\alpha_0$ defined in (2.11). Note that if $(n_1,n_2,n_3)\in\Lambda^{(1)}(n)$ then $n_2,n_3\notin\mathscr{C}_\alpha$.

Lemma 2.9. *Suppose that* $a\in C(\mathbb{R};\ell^2(\mathbb{T}^d))$

$$a(t,y)=\sum_{n\in\mathbb{Z}^d} a_n(t)\,\mathrm{e}^{in\cdot y}\,,\quad y\in\mathbb{T}^d\,,$$

where $(a_n(t))_{n\in\mathbb{Z}^d}$ *is solution to the system in Definition* (2.7)*. Then, for all* $\alpha\geq\alpha_0$*, we have*

$$\frac{d}{dt}\|\pi_\alpha a(t)\|^2_{\ell^2(\mathbb{T}^d)}=0\,.$$

Proof. Let $\alpha\geq\alpha_0$. We have

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|\pi_\alpha a(t)\|^2_{\ell^2(\mathbb{T}^d)} &= \operatorname{Im}\Big(\sum_{n\in\mathscr{C}_\alpha}\overline{a_n}\sum_{(n_1,n_2,n_3)\in\Lambda^{(1)}(\vec{n})}\mathbf{1}_{|\Omega(\vec{n})|<1}\,\mathrm{e}^{it\Omega(\vec{n})}\,a_{n_1}\overline{a_{n_2}}a_{n_3}\Big)\\ &= \operatorname{Im}\Big(\sum_{n,n_1\in\mathscr{C}_\alpha}\ \sum_{|n_2|,|n_3|\leq 10^{-5}K_\alpha}\mathbf{1}_{\substack{|\Omega(\vec{n})|<1\\ n_1+n_3=n+n_2}}\,\mathrm{e}^{it\Omega(\vec{n})}\,\overline{a_n}a_{n_1}\overline{a_{n_2}}a_{n_3}\Big)\\ &=0\,.\end{aligned}$$

To see this cancellation, one can take the complex conjugate and interchange the roles of n with n_1 (resp. n_2 with n_3). ∎

In the following Lemma, we use the frequency separation property written in Lemma 2.1 in order to control the small divisors losses that will appear in the Poincaré–Dulac normal form reduction that will be performed in Section 3.2.3. We defined the set $\Lambda^{(1)}(n)$ and $\Lambda^{(2)}(n)$ in the above Definition 2.7, and the high-frequency threshold α_0 in (2.11).

Lemma 2.10. *Let $s \geq 0$. There exists $C > 0$ (depending on d, s and A) such that for all $\alpha \geq \alpha_0$, $n \in \mathscr{C}_\alpha$ and*

$$(n_1, n_2, n_3) \in (\mathbb{Z}^d)^3 \setminus (\Lambda^{(1)}(n) \cup \Lambda^{(3)}(n)), \quad \text{with} \quad n_1 - n_2 + n_3 = n\,, \quad \Omega(\vec{n}) \neq 0\,,$$

we have

$$\frac{1}{|\Omega(\vec{n})|} K_\alpha^s \leq C (n_1^*)^s (n_2^*)^{\frac{4\tau}{c(d)}}\,,$$

where we recall that $n_1^* \geq n_2^* \geq n_3^*$ is the decreasing rearrangement of $|n_1|, |n_2|, |n_3|$.

Proof. Let $\alpha \geq \alpha_0$ and $\vec{n} = (n_1, n_2, n_3, n)$, with $n \in \mathscr{C}_\alpha$ and $n_1 - n_2 + n_3 = n$ (zero momentum condition). In particular,

$$K_\alpha \leq |n| \leq 2K_\alpha\,, \quad \text{and} \quad K_\alpha \leq |n| \leq 3n_1^*\,.$$

When $|\Omega(\vec{n})| \geq 1$, the desired bound follows from the above inequalities between the indices. According to the admissibility condition we can suppose that

$$1 < \frac{1}{|\Omega(\vec{n})|} \lesssim (n_1^*)^{2\tau}\,.$$

Moreover, we suppose without loss of generality that $|n_1| \geq |n_3|$ (they play symmetric roles). Hence, $|n_1| \geq n_2^*$. We separate three cases.

• **Case 1:** $|n_1| = n_2^*$. In this case, we have $n_1^* = |n_2|$ and the condition $|\Omega(\vec{n})| < 1$ together with the coercive bound on the eigenvalues (1.2) yield

$$(n_1^*)^2 \lesssim_{\mathrm{A}} \lambda_{n_2}^2 + \lambda_n^2 < |\Omega(\vec{n})| + \lambda_{n_1}^2 + \lambda_{n_3}^2 \lesssim_{\mathrm{A}} (n_2^*)^2\,,$$

for some implicit constants only depending on A. Therefore,

$$\frac{1}{|\Omega(\vec{n})|} K_\alpha^s \leq (n_1^*)^s (n_1^*)^{2\tau} \lesssim_{\mathrm{A}} (n_1^*)^s (n_2^*)^{2\tau}$$

which is conclusive (we recall that $c(d) \leq 2$, so that $2\tau \leq \frac{4\tau}{c(d)}$).

• **Case 2:** $|n_1| = n_1^*$ and $n_1 \notin \mathscr{C}_\alpha$. The frequency separation property (2.1) gives

$$|\lambda_n^2 - \lambda_{n_1}^2| + |n_2 - n_1| \geq (n_1^*)^{c(d)}\,.$$

We deduce from the condition $|\Omega(\vec{n})| < 1$ and from the zero momentum condition that

$$|\lambda_n^2 - \lambda_{n_3}^2| + |n_2 - n_3| \geq (n_1^*)^{c(d)} - 1 \geq \frac{1}{2}(n_1^*)^{c(d)}\,,$$

which in turn implies that

$$n_2^* = \max(|n_2|, |n_3|) \gtrsim_{\mathrm{A}} (n_1^*)^{\frac{c(d)}{2}}\,.$$

Hence,

$$\frac{1}{|\Omega(\vec{n})|} K_\alpha^s \lesssim_{\mathrm{A}} (n_1^*)^s (n_2^*)^{\frac{4\tau}{c(d)}}\,,$$

which is conclusive.

• **Case 3:** $|n_1| = n_1^*$ and $n_1 \in \mathscr{C}_\alpha$. In particular, $n_1^* \geq |n_1| \geq K_\alpha$. Moreover, since $(n_1, n_2, n_3) \notin \Lambda^{(1)}(n)$ we have from the Definition 2.7 that

$$10^{-5} K_\alpha \leq \max(|n_2|, |n_3|)\,.$$

Therefore, $K_\alpha \lesssim n_2^*$ and we conclude that

$$\frac{1}{|\Omega(\vec{n})|} K_\alpha^s \lesssim (n_1^*)^{2\tau_*} (n_2^*)^s \,.$$

This completes the proof of Lemma 2.10. □

2.4. **Dispersion in the unbounded direction.** Now that we have analyzed the key interaction of the system in the compact direction, we turn to the dispersive properties of the one-dimensional Schrödinger free evolution in the unbounded direction $\mathbb{R}$. In the following lemma we recall the dispersive estimate and its consequence in our context.

Lemma 2.11. *There exists $C > 0$ such that for all $f \in L^2(\mathbb{R}) \cap x^{-1} L^2(\mathbb{R})$ and $t > 0$,*

$$\| \mathrm{e}^{it\partial_x^2} f(x) - \frac{\mathrm{e}^{i\frac{x^2}{4t}}}{(4i\pi t)^{\frac{1}{2}}} \widehat{f}(\frac{x}{2t}) \|_{L^\infty_x(\mathbb{R})} \leq C t^{-\frac{3}{4}} \| x f \|_{L^2_x(\mathbb{R})} \,. \tag{2.12}$$

Considering the profile $F(t) = \mathrm{e}^{-it\Delta} U(t)$, we have that for all $T \geq 1$, $t \in [1, T]$,

$$t^{\frac{1}{2}} \| U(t) \|_{L^\infty_x h^s_y} \lesssim \| F \|_Z + t^{-\frac{1}{4}} \| F \|_S \lesssim \| F \|_{X_T} \,. \tag{2.13}$$

Remark 2.12. The above estimate is a key ingredient in our analysis, especially to obtain the trilinear estimate claimed in Lemma 2.13. Note that such an estimate is not available in the case of the square torus, where the space-time norm X_T provides the dispersive estimate for the $L^\infty_x h^1_y(\mathbb{R} \times \mathbb{T}^d)$-norm only (see [24, Theorem 1.1]).

Proof. The dispersive estimate (2.12) on $\mathbb{R}$ is standard (see for instance Hayachi–Naumkin [26]), and follows from the expression of the semi-group $\mathrm{e}^{it\partial_x^2}$. We write a short proof in Appendix A. Let us now prove (2.13): for all $(x, y) \in \mathbb{R} \times \mathbb{T}^d$, we expand

$$U(t, x, y) = \sum_{n \in \mathbb{Z}^d} \mathrm{e}^{iy\cdot n - it\lambda_n^2} \, \mathrm{e}^{it\partial_x^2} F_n(x) \,.$$

Using the equivalence of the h^s-norms with the standard Sobolev norms on $\mathbb{T}^d$ we have

$$\| U(t) \|_{L^\infty h^s(\mathbb{R} \times \mathbb{T}^d)} \lesssim_s \big\| \big(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} | \mathrm{e}^{it\partial_x^2} F_n(x) |^2 \big)^{\frac{1}{2}} \big\|_{L^\infty(\mathbb{R})} \,.$$

According to the dispersive estimate (2.12), we have that for fixed $x \in \mathbb{R}$, $n \in \mathbb{Z}^d$ and $t > 0$,

$$| \mathrm{e}^{it\partial_x^2} F_n(x) |^2 \lesssim t^{-1} | \widehat{F}_n(\frac{x}{2t}) |^2 + t^{-\frac{3}{4}} \| x F_n \|^2_{L^2_x(\mathbb{R})} \,.$$

When summing over n, the first contribution is bounded by

$$t^{-1} \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} | \widehat{F}_n(\frac{x}{2t}) |^2 = t^{-1} \| \widehat{F}(\frac{x}{2t}) \|^2_{h^s(\mathbb{T}^d)} \leq t^{-1} \sup_{\xi \in \mathbb{R}} \| \widehat{F}(\xi) \|^2_{h^s(\mathbb{T}^d)} \,.$$

For the second contribution, we have

$$\sum_n \langle n \rangle^{2s} \| x F_n \|^2_{L^2_x(\mathbb{R})} = \| x F \|_{L^2_x h^s_y(\mathbb{R} \times \mathbb{T}^d)} \,.$$

This proves that for all $t > 0$,

$$\|U(t)\|_{L^\infty h^s(\mathbb{R}\times\mathbb{T}^d)} \lesssim t^{-\frac{1}{2}} \sup_{\xi\in\mathbb{R}} \|\widehat{F}(\xi)\|_{h^s(\mathbb{T}^d)} + t^{-\frac{1}{2}-\frac{1}{4}}\|xF\|_{L^2h^s(\mathbb{R}\times\mathbb{T}^d)}\,, \tag{2.14}$$

which gives (2.13) provided $\delta < \frac{1}{4}$ in the definition of the X_T-norm. □

2.5. **Profile equation.** We detail the standard preparation in which we pull-back the nonlinear solution U by the linear flow:

$$U(t,x,y) = \mathrm{e}^{it\Delta}\,F(t,x,y)\,.$$

The profile $F(t)$ is solution to the interaction equation

$$i\partial_t F(t) = \mathrm{e}^{-it\Delta}(\mathrm{e}^{it\Delta}\,F(t)\,\mathrm{e}^{-it\Delta}\,\overline{F(t)}\,\mathrm{e}^{it\Delta}\,F(t))\,.$$

We write

$$i\partial_t F(t) = \mathcal{N}^t[F(t),F(t),F(t)]\,, \tag{2.15}$$

where the trilinear form $\mathcal{N}^t$ on $(H^s(\mathbb{R}\times\mathbb{T}^d))^3$ is

$$\mathcal{N}^t[F,G,H] := \mathrm{e}^{-it\Delta}(\mathrm{e}^{it\Delta}\,F\,\mathrm{e}^{-it\Delta}\,\overline{G}\,\mathrm{e}^{it\Delta}\,H)\,.$$

The full spatial Fourier transform (as defined in (1.5)) of the trilinear interaction at $(\xi,n)\in\mathbb{R}\times\mathbb{T}^d$ reads

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,n) := \sum_{n_1-n_2+n_3=n} \mathrm{e}^{it\Omega(\vec{n})} \int_{\mathbb{R}^2} \mathrm{e}^{2it(\eta-\xi)(\sigma-\eta)}\,\widehat{F}_{n_1}(\xi-\eta+\sigma)\overline{\widehat{G}_{n_2}}(\sigma)\widehat{H}_{n_3}(\eta)d\eta d\sigma\,.$$

where the sums run over $(n_1,n_2,n_3)\in(\mathbb{Z}^d)^3$. Once again we denoted

$$\vec{n} = (n_1,n_2,n_3,n)\,,\quad \Omega(\vec{n}) := \lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_n^2\,.$$

Changing variables in the integrals over $(\eta,\sigma)\in\mathbb{R}^2$ we obtain

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,n) := \sum_{n_1-n_2+n_3=n} \mathrm{e}^{it\Omega(\vec{n})} \int_{\mathbb{R}^2} \mathrm{e}^{it2\eta\sigma}\,\widehat{F}_{n_1}(\xi-\sigma)\overline{\widehat{G}_{n_2}}(\xi-\eta-\sigma)\widehat{H}_{n_3}(\xi-\eta)d\eta d\sigma\,. \tag{2.16}$$

2.6. **A priori estimates for the strong norm.** We conclude this preparatory section with the proof of an a priori trilinear bound for the strong norm S, which is a stronger analog of [24, Lemma 2.1]. Indeed, we can readily use the dispersive estimate (2.13), which does not hold in the case of the square torus, to obtain a control of the $L^\infty_{x,y}(\mathbb{R}\times\mathbb{T}^d)$-norm by the X_T-norm with the optimal decay in $t^{-\frac{1}{2}}$.

Lemma 2.13. *Suppose that the profile F is solution to (2.15). Then, for all $T \geq 0$ and $t\in[0,T]$, we have*

$$\|\mathcal{N}^t[F,F,F]\|_S \lesssim \langle t\rangle^{-1+\delta}\|F\|^3_{X_T}\,. \tag{2.17}$$

Proof. Recall that

$$\mathcal{N}^t[F] := \mathcal{N}^t[F,F,F] = \mathrm{e}^{it\Delta}(|U(t)|^2 U(t))\,.$$

The proof when $t \in [0,1]$ is easier (we do not need to gain decay in time), and we only detail the case when $T \geq 1$ and $1 \leq t \leq T$. We have from the product rule of Lemma A.1 and the dispersive bound (2.13) that for $\sigma \geq 0$,

$$\begin{aligned}\|(1-\partial_x^2)^{\frac{\sigma}{2}}\mathcal{N}^t[F]\|_{H^s_{x,y}} &\lesssim \|U(t)\|^2_{L^\infty_x h^s_y}\|(1-\partial_x)^{\frac{\sigma}{2}}U(t)\|_{H^s_{x,y}}\\ &\lesssim t^{-1}\|F\|^2_{X_T}\|(1-\partial_x^2)^{\frac{\sigma}{2}}F(t)\|_{H^s_{x,y}}\,.\end{aligned}$$

We deduce from the definition (2.6) of the X_T-norm that

$$\|(1-\partial_x^2)^{\frac{\sigma}{2}}\mathcal{N}^t[F]\|_{H^s_{x,y}} \lesssim t^{-1+\delta}\|F\|^3_{X_T}\,.$$

As for the second component of the S-norm we have

$$\|x\mathcal{N}^t[F]\|_{L^2_x h^s_y} = \|\partial_\xi \mathcal{F}\mathcal{N}^t[F]\|_{L^2_x h^s_y}\,.$$

For fixed $n \in \mathbb{Z}^d$, following [29], we have from the expression (2.16) that

$$\begin{aligned}\partial_\xi \mathcal{F}\mathcal{N}^t[F](\xi,n) = &\sum_{n_1-n_2+n_3=n} \mathrm{e}^{it\Omega(\vec{n})}\\ &\partial_\xi \int_{\mathbb{R}^2} \mathrm{e}^{2it\eta\sigma}\,\widehat{F}_{n_1}(\xi-\sigma)\overline{\widehat{F}_{n_2}}(\xi-\eta-\sigma)\widehat{F}_{n_3}(\xi-\eta)d\eta d\sigma\,.\end{aligned}$$

Redistributing the phases, we obtain that

$$\|x\mathcal{N}^t(F,F,F)\|_{L^2_x h^s_y} \lesssim \|\,\mathrm{e}^{-it\Delta}(|U(t)|^2 xU(t))\|_{L^2_x h^s_y}\,.$$

We deduce from the dispersive bound (2.13) that

$$\|x\mathcal{N}^t(F,F,F)\|_{L^2_x h^s_y} \lesssim \|U(t)\|^2_{L^\infty_x h^s_y}\|xU(t)\|_{L^2_x h^s_y} \lesssim t^{-1+\delta}\|F\|^3_{X_T}\,,$$

which finishes the proof of Lemma 2.13. ◻

3. Nonlinear interactions

Lemma 2.13 gives an a priori estimate on the strong norm S of the solution by the X_T-norm of the solution, with the appropriate decay in time. To close the a priori estimate for the X_T-norm, it remains to bound the weak norm Z by an integrable time decay.

To this end, we will analyze the trilinear interaction $\mathcal{N}^t$ more carefully. We first isolate the space-resonant interactions (denoted $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$) and we use space oscillations to prove that the contribution of the rest ($\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}$) is acceptable.

Then, we further decompose $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$ into an effective Hamiltonian system (denoted $\mathcal{N}^t_{\mathrm{eff}}$) in the transverse direction $\mathbb{T}^d$ and a remainder interaction (denoted $\mathcal{E}^t$) oscillating sufficiently fast in time.

3.1. **Decomposition of the nonlinearity.** Following [29], we apply the Plancherel's identity in (2.16) and use the explicit expression of the Fourier transform of a Gaussian function to isolate the space-resonant component of the nonlinearity. We obtain (see also the proof of [24, Lemma 3.10]) that for fixed $\xi \in \mathbb{R}$, $t \geq 1$ and $(n_1, n_2, n_3) \in (\mathbb{Z}^d)^3$,

$$\begin{aligned}\int_{\mathbb{R}^2} &\mathrm{e}^{i2t\eta\sigma}\, \widehat{F}_{n_1}(\xi-\sigma)\overline{\widehat{G}_{n_2}}(\xi-\eta-\sigma)\widehat{H}_{n_3}(\xi-\eta)d\eta d\sigma \\ &= \frac{\pi}{t}\int_{\mathbb{R}^3} F_{n_1}(x_1)\overline{G_{n_2}}(x_2)H_{n_3}(x_3)\,\mathrm{e}^{-i\xi(x_1-x_2+x_3)}\,\mathrm{e}^{-i[\frac{1}{2t}(x_1-x_2)(x_2-x_3)]}\,dx_1dx_2dx_3 \\ &= \frac{\pi}{t}\widehat{F}_{n_1}(\xi)\overline{\widehat{G}_{n_2}}(\xi)\widehat{H}_{n_3}(\xi) + \mathcal{O}^t[F_{n_1}, G_{n_2}, H_{n_3}](\xi)\,,\end{aligned}$$

where $\mathcal{O}^t$ contains the space non-resonant interactions: for functions $f, g, h \in H^s_x(\mathbb{R})$,

$$\begin{aligned}\mathcal{O}^t[f,g,h](\xi) := \frac{\pi}{t}\int_{\mathbb{R}^3} f(x_1)\overline{g(x_2)}h(x_3)\,\mathrm{e}^{-i\xi(x_1-x_2+x_3)}& \\ \big(\mathrm{e}^{-i[\frac{1}{2t}(x_1-x_2)(x_2-x_3)]} - 1\big)dx_1dx_2dx_3\,.& \quad (3.1)\end{aligned}$$

One can prove that $\mathcal{O}^t$ has an improved decay in time under some assumptions on the space localization of the functions f, g, h (two of them should be in $\langle x\rangle^{-1}L^2_x(\mathbb{R})$, say). We detail this part of the analysis in Section 3.3.

Recalling the expression (2.16), we have that for $t \geq 1$ and $(\xi, n) \in \mathbb{R}\times\mathbb{Z}^d$,

$$\begin{aligned}\mathcal{F}\mathcal{N}^t[F,G,H](\xi,n) = \frac{\pi}{t}\sum_{n_1-n_2+n_3=n} &\mathrm{e}^{it\Omega(\vec{n})}\,\widehat{F}_{n_1}(\xi)\overline{\widehat{G}_{n_2}}(\xi)\widehat{H}_{n_3}(\xi) \\ &+ \sum_{n_1-n_2+n_3=n}\mathrm{e}^{it\Omega(\vec{n})}\,\mathcal{O}^t[F_{n_1}, G_{n_2}, H_{n_3}](\xi)\,. \quad (3.2)\end{aligned}$$

According to this decomposition we write

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,n) =: \mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F,G,H](\xi,n) + \mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F,G,H](\xi,n)\,. \quad (3.3)$$

We analyze the term $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$ (which regroups the space-resonant interactions) in Section 3.2 and the term $\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}$ in Section 3.3, respectively. First, we further decompose the space-resonant system involving $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$ into two parts: an effective system consisting of resonant interactions ($\mathcal{R}^t$) and of some quasi-resonant interactions preserving the cluster of the outgoing wave ($\mathcal{Q}^t$), and a remainder ($\mathcal{E}^t$):

$$\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F,G,H] := \mathcal{F}\mathcal{R}^t[F,G,H] + \mathcal{F}\mathcal{Q}^t[F,G,H] + \mathcal{F}\mathcal{E}^t[F,G,H]\,.$$

The effective system is

$$\mathcal{N}^t_{\mathrm{eff}} = \mathcal{R}^t + \mathcal{Q}^t\,. \quad (3.4)$$

Recalling definition 2.7 of $\Lambda^{(1)}(n)$, the different interactions are defined as follows:

- *Resonant interaction:*

$$\mathcal{F}\mathcal{R}^t[F,G,H](\xi,n) := \frac{2\pi}{t}\sum_{n_1\in\mathbb{Z}^d}\widehat{F}_{n_1}(\xi)\overline{\widehat{G}_{n_1}}(\xi)\widehat{H}_n(\xi) - \frac{\pi}{t}\widehat{F}_n(\xi)\overline{\widehat{G}_n}(\xi)\widehat{H}_n(\xi)\,, \quad (3.5)$$

- *Effective quasi-resonant interactions:* for $n \in \bigcup_{\alpha \geq \alpha_0} \mathscr{C}_\alpha$,

$$\mathcal{F}\mathcal{Q}^t[F,G,H](\xi,n) = \frac{2\pi}{t} \sum_{(n_1,n_2,n_3)\in\Lambda^{(1)}(n)} \mathrm{e}^{it\Omega(\vec{n})}\, \widehat{F}_{n_1}(\xi)\overline{\widehat{G}_{n_2}}(\xi)\widehat{H}_{n_3}(\xi)\,. \tag{3.6}$$

- When $n \in \bigcup_{\alpha<\alpha_0} \mathscr{C}_\alpha$ is a low frequency,

$$\mathcal{F}\mathcal{Q}^t[F,G,H](\xi,n) = 0\,.$$

- *Remainder:*

$$\mathcal{F}\mathcal{E}^t[F,G,H] := \mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nres}}[F,G,H] - \mathcal{F}\mathcal{R}^t[F,G,H] - \mathcal{F}\mathcal{Q}^t[F,G,H]\,. \tag{3.7}$$

Remark 3.1. In particular, when $n \in \bigcup_{\alpha \geq \alpha_0} \mathscr{C}_\alpha$, we have the expression

$$\mathcal{F}\mathcal{E}^t[F,G,H](\xi,n) := \frac{\pi}{t} \sum_{\substack{(n_1,n_2,n_3)\notin\Lambda^{(1)}(n)\cup\Lambda^{(3)}(n)\\ n_1-n_2+n_3=n}} \mathrm{e}^{it\Omega(\vec{n})}\, \widehat{F}_{n_1}(\xi)\overline{\widehat{G}_{n_2}}(\xi)\widehat{H}_{n_3}(\xi)\,.$$

3.2. **The space resonant interactions.** Let us analyze the system involving $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$, decomposed into two parts. On the one hand, the effective system defined in (3.9) preserves the super-actions based on the frequency cluster decomposition. On the other hand, the remainder $\mathcal{E}^t$ defined in (3.7) oscillates fast enough in time to gain decay.

3.2.1. *Trilinear bounds.* Before analyzing the individual contributions of $\mathcal{R}^t$, $\mathcal{Q}^t$ and $\mathcal{E}^t$ we first prove trilinear bounds for the whole resonant system involving $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$.

Lemma 3.2. *For all $t > 0$,*

$$\begin{aligned}
\|\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F^a,F^b,F^c]\|_Z &\lesssim t^{-1}\|F^a\|_Z\|F^b\|_Z\|F^c\|_Z\,,\\
\|\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F^a,F^b,F^c]\|_S &\lesssim t^{-1}\sum_{\sigma\in\mathfrak{S}_3}\|F^{\sigma(a)}\|_Z\|F^{\sigma(b)}\|_Z\|F^{\sigma(c)}\|_S\,,\\
\|\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F^a,F^b,F^c]\|_{H^s_{x,y}} &\lesssim t^{-1}\sum_{\sigma\in\mathfrak{S}_3}\|F^{\sigma(a)}\|_Z\|F^{\sigma(b)}\|_Z\|F^{\sigma(c)}\|_{H^s_{x,y}}\,,
\end{aligned} \tag{3.8}$$

where $\mathfrak{S}_3$ is the group of permutations of three elements. The same estimates hold for the subsystems involving $\mathcal{R}^t$, $\mathcal{Q}^t$ and $\mathcal{E}^t$.

Proof. For fixed $\xi \in \mathbb{R}$ and $t > 0$ we have from the Young's convolution inequality that

$$\|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F^a,F^b,F^c](\xi)\|_{h^s_y} \lesssim t^{-1}\sum_{\sigma\in\mathfrak{S}_3}\|\widehat{F^{\sigma(a)}}(\xi)\|_{\ell^1_y}\|\widehat{F^{\sigma(b)}}(\xi)\|_{\ell^1_y}\|\widehat{F^{\sigma(c)}}(\xi)\|_{h^s_y}\,,$$

and the bounds follow from the assumption $s > \frac{d}{2}$. $\square$

3.2.2. *The effective system.* We now handle the contribution of the effective interactions defined in (3.4). Here, we use the key property that the effective system preserves the high-frequency super-actions, proved in Lemma 2.9.

Lemma 3.3. *Suppose that $G \in C([-T,T];S)$ is a solution to the effective system*

$$i\partial_t G(t) = \mathcal{N}^t_{\mathrm{eff}}[G(t)]\,. \tag{3.9}$$

Then, for all $|t| \leq T$, $\alpha \in \mathbb{N}$ and $\xi \in \mathbb{R}$ we have

$$\|\pi_\alpha \widehat{G}(t,\xi)\|_{\ell^2(\mathbb{T}^d)} = \|\pi_\alpha \widehat{G}(0,\xi)\|_{\ell^2(\mathbb{T}^d)}\,. \tag{3.10}$$

Morevoer,

$$\|G(t)\|_Z = \|G(0)\|_Z\,, \quad \|G(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} = \|G(0)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}\,. \tag{3.11}$$

Proof. Fix $\xi \in \mathbb{R}$, and consider

$$a(\xi;t,y) := \widehat{G}(t,\xi,y) = \sum_{n\in\mathbb{Z}^d} \widehat{G}_n(t,\xi)\,\mathrm{e}^{in\cdot y}\,, \quad a_n(\xi;t) := \widehat{G}_n(t,\xi)\,.$$

In what follows we forget the variable ξ, which is parametric. By the definition (3.4) of the effective system, a is solution to

$$i\partial_t a_n(t) = \mathcal{F}\mathcal{R}[a(t)](n) + \mathcal{F}\mathcal{Q}^t[a(t)](n)\,, \quad n \in \mathbb{Z}^d\,.$$

and therefore for $\alpha \geq 0$ we have

$$\frac{1}{2}\frac{d}{dt}\|\pi_\alpha a\|^2_{\ell^2(\mathbb{T}^d)} = \mathrm{Im}\,\big(\sum_{n\in\mathscr{C}_\alpha} \mathcal{F}\mathcal{R}[a(t)](n)\overline{a_n}(t) + \mathcal{F}\mathcal{Q}^t[a(t)](n)\overline{a_n}(t)\big)\,.$$

According to Lemma 2.9, the quasi-resonant interactions $\mathcal{Q}^t$ do not contribute. We easily prove that the resonant interactions do not contribute either:

$$\mathrm{Im}\,\big(\sum_{n\in\mathscr{C}_\alpha} \mathcal{F}\mathcal{R}[a(t)](n)\overline{a_n}(t)\big) = \mathrm{Im}\,\big(2\sum_{n\in\mathscr{C}_\alpha}\sum_{n_1\in\mathbb{Z}^d} |a_{n_1}(t)|^2|a_n(t)|^2 - \sum_{n\in\mathscr{C}_\alpha}|a_n(t)|^4\big) = 0\,.$$

This proves (3.10). Then, the identities (3.11) follow form Definition 2.2 of the h^s-norm based on the cluster decomposition: for all $\xi \in \mathbb{R}$,

$$\begin{aligned}\|\widehat{G}(t,\xi)\|_{h^s(\mathbb{T}^d)} &= \big(\sum_{\alpha\in\mathbb{N}} K_\alpha^{2s}\|\pi_\alpha\widehat{G}(t,\xi)\|^2_{\ell^2(\mathbb{T}^d)}\big)^{\frac{1}{2}} \\ &= \big(\sum_{\alpha\in\mathbb{N}} K_\alpha^{2s}\|\pi_\alpha\widehat{G}(0,\xi)\|^2_{\ell^2(\mathbb{T}^d)}\big)^{\frac{1}{2}} = \|\widehat{G}(0,\xi)\|_{h^s(\mathbb{T}^d)}\,,\end{aligned}$$

This completes the proof of Lemma 3.3. □

3.2.3. *The time oscillations.* Let us turn to the contributions of $\mathcal{E}^t$ defined in (3.7). First, we prove a trilinear estimate which will allow us to overcome small divisor problems that arise when performing the Poincaré–Dulac normal form. This estimate is a consequence of Lemma 2.10, which was in turn a consequence of the frequency configuration of the interactions contained in $\mathcal{E}^t$.

Lemma 3.4. *For $n \in \bigcup_{\alpha \geq \alpha_0} \mathscr{C}_\alpha$, we let*

$$\mathcal{F}\widetilde{\mathcal{E}}^t[F^a, F^b, F^c](\xi, n) := \sum_{\substack{(n_1,n_2,n_3) \notin \Lambda^{(1)}(n) \cup \Lambda^{(3)}(n) \\ n_1 - n_2 + n_3 = n}} \frac{\mathrm{e}^{it\Omega(\vec{n})}}{\Omega(\vec{n})} \widehat{F}^a_{n_1}(\xi) \overline{\widehat{F}^b_{n_2}}(\xi) \widehat{F}^c_{n_3}(\xi) .$$

Then, for all $t > 0$ and $\xi \in \mathbb{R}$,

$$\|\widetilde{\mathcal{E}}^t[\widehat{F^a, F^b}, F^c](\xi)\|_{h^s(\mathbb{T}^d)} \lesssim \|\widehat{F^a}(\xi)\|_{h^s(\mathbb{T}^d)} \|\widehat{F^b}(\xi)\|_{h^s(\mathbb{T}^d)} \|\widehat{F^c}(\xi)\|_{h^s(\mathbb{T}^d)} .$$

In particular,

$$\begin{aligned} &\|\widetilde{\mathcal{E}}^t[F^a, F^b, F^c]\|_Z \lesssim \|F^a\|_Z \|F^b\|_Z \|F^c\|_Z . \\ &\|\widetilde{\mathcal{E}}^t[F^a, F^b, F^c]\|_S \lesssim \sum_{\sigma \in \mathfrak{S}_3} \|F^{\sigma(a)}\|_S \|F^{\sigma(b)}\|_Z \|F^{\sigma(c)}\|_Z . \end{aligned}$$

Proof. For fixed $\xi \in \mathbb{R}$, we have by duality that

$$\|\widetilde{\mathcal{E}}^t[F^a, F^b, F^c](\xi)\|_{h^s(\mathbb{T}^d)} \lesssim \sup_{\|v\|_{\ell^2} \leq 1} \sum_{\alpha \geq \alpha_0} \sum_{n \in \mathscr{C}_\alpha} K^s_\alpha |\mathcal{F}\widetilde{\mathcal{E}}^t[F^a, F^b, F^c](\xi, n)| |v_n| .$$

According to Lemma 2.10, we have that for all $n \in \bigcup_{\alpha \geq \alpha_0} \mathscr{C}_\alpha$ and $(n_1, n_2, n_3) \in (\mathbb{Z}^d)^3 \setminus (\Lambda^{(1)}(n) \cup \Lambda^3(n))$ with $n = n_1 - n_2 + n_3$,

$$\frac{1}{|\Omega(\vec{n})|} K^s_\alpha \lesssim (n_1^*)^s (n_2^*)^{\frac{4\tau}{c(d)}}$$

We deduce that

$$|\mathcal{F}\widetilde{\mathcal{E}}^t[F^a, F^b, F^c](\xi, n)| \leq \sum_{n_1 - n_2 + n_3 = n} (n_1^*)^s (n_2^*)^{\frac{4\tau}{c(d)}} |\widehat{F}^a_{n_1}(\xi) \widehat{F}^b_{n_2}(\xi) \widehat{F}^c_{n_3}(\xi)| .$$

We conclude with the Young's convolution inequality that for all $\xi \in \mathbb{R}$ and from the Sobolev embedding that

$$\begin{aligned} \|\mathcal{F}\widetilde{\mathcal{E}}^t[F^a, F^b, F^c](\xi)\|_{h^s(\mathbb{T}^d)} &\lesssim \sum_{\sigma \in \mathfrak{S}_3} \|\langle n \rangle^s \widehat{F_n^{\sigma(a)}}(\xi)\|_{\ell^2_n} \|\langle n \rangle^{\frac{4\tau}{c(d)}} \widehat{F_n^{\sigma(b)}}(\xi)\|_{\ell^1_n} \|\widehat{F_n^{\sigma(c)}}(\xi)\|_{\ell^1_n} , \\ &\lesssim_\eta \sum_{\sigma \in \mathfrak{S}_3} \|\langle n \rangle^s \widehat{F_n^{\sigma(a)}}(\xi)\|_{\ell^2_n} \|\langle n \rangle^{\frac{4\tau}{c(d)} + \frac{d}{2} + \eta} \widehat{F_n^{\sigma(b)}}(\xi)\|_{\ell^2_n} \|\langle n \rangle^{\frac{d}{2} + \eta} \widehat{F_n^{\sigma(c)}}(\xi)\|_{\ell^2_n} , \end{aligned}$$

for any $\eta > 0$, which is conclusive under our assumption that $s > \frac{d}{2} + \frac{4\tau}{c(d)}$. ◻

Lemma 3.5 (Poincaré–Dulac normal form)**.** *Suppose that $T \geq 1$ and $F \in X_T$. Then, for all $t \in [1, T]$ and $\xi \in \mathbb{R}$*

$$\sum_{\alpha \in \mathbb{N}} N_\alpha^{2s} \Big| \sum_{n \in \mathscr{C}_\alpha} \int_1^t \mathcal{F}\mathcal{E}^\tau[F, F, F](\xi, n) \overline{\widehat{F}_n}(\tau, \xi) d\tau \Big| \lesssim \|F\|^4_{X_T} . \tag{3.12}$$

Moreover,

$$\Big\| \int_{\frac{t}{4}}^t \mathcal{E}^t[F, F, F] d\tau \Big\|_{H^s(\mathbb{R} \times \mathbb{T}^d)} \lesssim t^{-1+4\delta} \|F\|^3_{X_T} . \tag{3.13}$$

Proof. Recalling the expression (3.7) of $\mathcal{E}^t$, we have that for fixed $\xi \in \mathbb{R}$, for $n \in \bigcup_{\alpha \geq \alpha_0} \mathscr{C}_\alpha$

$$\mathcal{F}\mathcal{E}^\tau[F,F,F](\xi,n) = \frac{\pi}{\tau} \sum_{\substack{(n_1,n_3,n_3)\notin(\Lambda^{(1)}(n)\cup\Lambda^{(3)}(n))\\ n_1-n_2+n_3=n}} \mathrm{e}^{i\tau\Omega(\vec{n})}\, \widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi)\,,$$

and for $n \in \bigcup_{\alpha < \alpha_0} \mathscr{C}_\alpha$,

$$\mathcal{F}\mathcal{E}^\tau[F,F,F](\xi,n) = \frac{\pi}{\tau} \sum_{\substack{n_1-n_2+n_3=n\\ \Omega(\vec{n})\neq 0}} \mathrm{e}^{i\tau\Omega(\vec{n})}\, \widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi)\,.$$

Since the terms contributing to $\mathcal{E}^t$ are not resonant, $\Omega(\vec{n}) \neq 0$ and the Leibniz rule gives

$$\begin{aligned}\frac{\pi}{\tau}\mathrm{e}^{i\tau\Omega(\vec{n})}\, \widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi) =& \frac{d}{d\tau}\Big(\frac{\pi}{\tau}\frac{\mathrm{e}^{i\tau\Omega(\vec{n})}}{i\Omega(\vec{n})}\widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi)\Big)\\ &+ \frac{\pi}{\tau^2}\frac{\mathrm{e}^{i\tau\Omega(\vec{n})}}{i\Omega(\vec{n})}\widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi)\\ &- \frac{\pi}{\tau}\frac{\mathrm{e}^{i\tau\Omega(\vec{n})}}{i\Omega(\vec{n})}\frac{d}{dt}\Big(\widehat{F}_{n_1}(\tau,\xi)\overline{\widehat{F}_{n_2}}(\tau,\xi)\widehat{F}_{n_3}(\tau,\xi)\Big)\,.\end{aligned}$$

Hence, we can gain decay in time at the price of a small divisor $|\Omega(\vec{n})|^{-1}$. For the contributions to low-frequencies $n \in \bigcup_{\alpha<\alpha_0} \mathscr{C}_\alpha$, this loss is easy to overcome in the energy estimates (3.12) and (3.13): indeed, we can bound

$$K^s_\alpha \leq K^s_{\alpha(0)} := C(s,d,\mathrm{A})\,,$$

and the loss $|\Omega(\vec{n})|^{-1} \lesssim (n_1^*)^{2\tau}$ can be absorbed (since $s > 2\tau$).

For this reason, we only give details for the terms contributing to the high frequencies $n \in \bigcup_{\alpha\geq\alpha_0}\mathscr{C}_{\alpha_0}$. We will exploit Lemma 3.4, in which we proved how to overcome the small divisor loss. Defining the trilinear interaction $\widetilde{\mathcal{E}}^t$ by

$$\mathcal{F}\widetilde{\mathcal{E}}^\tau[F^a,F^b,F^c](\xi,n) := \sum_{\substack{(n_1,n_3,n_3)\notin\Lambda(n)\\ n_1-n_2+n_3=n}} \frac{\pi}{\tau}\frac{\mathrm{e}^{i\tau\Omega(\vec{n})}}{\Omega(\vec{n})}\widehat{F}^a_{n_1}(\xi)\overline{\widehat{F}^b_{n_2}}(\xi)\widehat{F}^c_{n_3}(\xi)\,,$$

we need to bound for $n \in \bigcup_{\alpha\geq\alpha_0}\mathscr{C}_{\alpha_0}$

$$\begin{aligned}&\frac{d}{d\tau}\Big(\frac{\pi}{\tau}\mathcal{F}\widetilde{\mathcal{E}}^\tau[F,F,F](\xi,n)\Big) + \frac{\pi}{\tau^2}\mathcal{F}\widetilde{\mathcal{E}}^\tau[F,F,F](\xi,n)\\ &\qquad - \frac{\pi}{\tau}\Big(\mathcal{F}\widetilde{\mathcal{E}}^\tau[\partial_\tau F,F,F] + \mathcal{F}\widetilde{\mathcal{E}}^\tau[F,\partial_\tau F,F] + \mathcal{F}\widetilde{\mathcal{E}}^\tau[F,F,\partial_\tau F]\Big)(\xi,n)\,. \end{aligned} \tag{3.14}$$

Let us first prove the energy estimate (3.12). We integrate by parts, and control the first contribution in (3.14):

$$\begin{aligned}\int_1^t \frac{d}{d\tau}\Big(\frac{\pi}{\tau}\mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F](\xi,n)\Big)\overline{\widehat{F}_n}(\xi,\tau)d\tau =& -\int_1^t \frac{\pi}{\tau}\mathcal{F}\widetilde{\mathcal{E}}^\tau[F,F,F](\xi,n)\partial_\tau\overline{\widehat{F}_n}(\xi,\tau)d\tau\\ &+ \mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F](\xi,n)\overline{\widehat{F}_n}(\xi,t) - \mathcal{F}\widetilde{\mathcal{E}}^1[F,F,F](\xi,n)\overline{\widehat{F}_n}(\xi,1)\,.\end{aligned}$$

Then, we deduce from the definition of the Z-norm that

$$\sum_{\alpha\in\mathbb{N}} N_\alpha^{2s} \Big| \sum_{n\in\mathscr{C}_\alpha} \int_1^t \frac{d}{d\tau}\big(\frac{\pi}{\tau}\mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F](\xi,n)\big)\overline{\widehat{F}_n}(\xi,\tau)d\tau\Big|$$

$$\lesssim \int_1^t \|\mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F]\|_Z \|\partial_\tau \widehat{F}\|_Z \frac{d\tau}{\tau} + \|\mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F]\|_Z \|\widehat{F}_n(t)\|_Z + \|\mathcal{F}\widetilde{\mathcal{E}}^1[F,F,F]\|_Z \|\widehat{F}_n(1)\|_Z$$

$$\lesssim \int_1^t \|F(\tau)\|_Z^3 \|\partial_\tau F(\tau)\|_Z \frac{d\tau}{\tau} + \|F(t)\|_Z^4 + \|F(1)\|_Z^4 .$$

In the last estimate, we applied the trilinear bound obtained in Lemma 3.4. By definition (2.6) of the X_T-norm we obtain that for all $t \in [1,T]$,

$$\sum_{\alpha\in\mathbb{N}} N_\alpha^{2s} \Big| \int_1^t \frac{d}{d\tau}\big(\frac{\pi}{\tau}\mathcal{F}\widetilde{\mathcal{E}}^t[F,F,F](\xi,n)\big)\overline{\widehat{F}_n}(\xi,\tau)d\tau\Big| \lesssim \|F\|_{X_T}^4 \big(1+\int_1^t \frac{d\tau}{\tau^{2(1-\delta)}}\big) \lesssim \|F\|_{X_T}^4 .$$

The control of the other contributions follows in a very similar way. We now turn to the proof of the bound (3.13). We have

$$\Big\| \int_{\frac{t}{4}}^t \mathcal{E}^t[F,F,F]d\tau \Big\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}^2 = \int_{\mathbb{R}} \langle\xi\rangle^{2s} \Big\| \int_{\frac{t}{4}}^t \mathcal{E}^\tau\widehat{[F,F,F]}(\xi)d\tau\Big\|_{h^s}^2 d\xi$$

Considering only the contributions to the high frequencies, it remains to control

$$\int_{\mathbb{R}} \langle\xi\rangle^{2s} \Big(\Big\| \frac{1}{t}\widetilde{\mathcal{E}}^t\widehat{[F,F,F]}(\xi) - \frac{4}{t}\widetilde{\mathcal{E}}^{\frac{t}{4}}\widehat{[F,F,F]}(\xi)\Big\|_{h^s}^2$$

$$+ \Big\| \int_{\frac{t}{4}}^t \big(\frac{1}{\tau}\widetilde{\mathcal{E}}^\tau\widehat{[F,F,F]}(\xi) - \widetilde{\mathcal{E}}^\tau\widehat{[\partial_\tau F,F,F]}(\xi) - \widetilde{\mathcal{E}}^\tau\widehat{[F,\partial_\tau F,F]}(\xi) - \widetilde{\mathcal{E}}^\tau\widehat{[F,F,\partial_\tau F]}(\xi)\big)\frac{d\tau}{\tau}\Big\|_{h^s}^2\Big) d\xi .$$

To bound the first contribution we apply Lemma 3.4:

$$\int_{\mathbb{R}} \langle\xi\rangle^{2s} \Big(\Big\| \frac{1}{t}\widetilde{\mathcal{E}}^t\widehat{[F,F,F]}(\xi) - \frac{4}{t}\widetilde{\mathcal{E}}^{\frac{t}{4}}\widehat{[F,F,F]}(\xi)\Big\|_{h^s}^2\Big) d\xi$$

$$\lesssim \frac{1}{t}\int_{\mathbb{R}} \langle\xi\rangle^{2s} \big(\|\widehat{F}(\xi)\|_{h^s}^2 \|\widehat{F}(\xi)\|_{h^s}^2 \|\widehat{F}(\xi)\|_{h^s}^2\big) d\xi \lesssim \frac{1}{t}\|F\|_{H^s}^3 \lesssim \frac{1}{t^{1-3\delta}}\|F\|_{X_T}^3 .$$

The bounds for the other contributions follow similarly, using the rapid decay of $\partial_t F$ encoded by the X_T-norm. This concludes the proof of Lemma 3.5. □

3.3. **The space oscillations.** We now control the contribution of the space non-resonant interactions $\mathcal{N}_{\mathbb{R},\mathrm{nr}}^t$ defined in (3.2). For this purpose, we adapt the argument from Kato–Pusateri [29] (see also [24, Lemma 3.10]). Here, we do not use the oscillations in the transverse direction $\mathbb{T}^d$.

Lemma 3.6 (Space oscillations). *For all $T \geq 1$, $t \in [1,T]$, and any $\gamma \in (0,\frac{1}{4})$ we have*

$$\|\mathcal{N}_{\mathbb{R},\mathrm{nr}}^t[F^a,F^b,F^c]\|_Z \lesssim_\gamma t^{-1-\gamma+3\delta}\|F^a\|_{X_T}\|F^b\|_{X_T}\|F^c\|_{X_T} . \tag{3.15}$$

Moreover, for $\delta > 0$ sufficiently small (depending on s, σ, d and γ), we have

$$\|\mathcal{N}_{\mathbb{R},\mathrm{nr}}^t[F^a,F^b,F^c]\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \lesssim t^{-1-2\delta}\|F^a\|_{X_T}\|F^b\|_{X_T}\|F^c\|_{X_T} . \tag{3.16}$$

In both cases we obtain an integrable decay in time provided that $3\delta < \gamma$. The second bound (3.16) will be useful to prove the modified scattering result in the topology of $H^s(\mathbb{R}\times\mathbb{T}^d)$, from which we deduce the stability in the space $H^s(\mathbb{R}\times\mathbb{T}^d)$ claimed in the main Theorem 1.3.

Proof. We first prove (3.15). We show that for all $\gamma\in(0,\frac{1}{4})$,

$$\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z \lesssim_\gamma t^{-1-\gamma}\|F^a\|_S\|F^b\|_S\|F^c\|_S\,. \tag{3.17}$$

Following [29] we have that for all $\gamma\in(0,\frac{1}{4}]$, $t>0$, and f^a,f^b,f^c,

$$\begin{aligned}\sup_{\xi\in\mathbb{R}}\ |\mathcal{O}^t[f^a,f^b,f^c](\xi)| &\lesssim t^{-1-\gamma}\int_{\mathbb{R}^3}|\eta|^\gamma|\sigma|^\gamma|f^a(t,x-\eta)||f^b(t,x)||f^c(t,x-\sigma)|dxd\eta d\sigma\\ &\lesssim t^{-1-\gamma}\max_{\sigma\in\mathfrak{S}_3}\|f^{\sigma(a)}\|_{L^1_x(\mathbb{R})}\||x|^{2\gamma}f^{\sigma(b)}\|_{L^1_x(\mathbb{R})}\||x|^{2\gamma}f^{\sigma(c)}\|_{L^1_x(\mathbb{R})}\\ &\lesssim t^{-1-\gamma}\|\langle x\rangle^{\frac12+2\gamma+0}f^a\|_{L^2_x(\mathbb{R})}\|\langle x\rangle^{\frac12+2\gamma+0}f^b\|_{L^2_x(\mathbb{R})}\|\langle x\rangle^{\frac12+2\gamma+0}f^c\|_{L^2_x(\mathbb{R})}\,,\end{aligned}$$

where we denoted by $\mathfrak{S}_3$ the group of permutations of three elements. We obtain that for fixed (n_1,n_2,n_3) and $\gamma\in(0,\frac{1}{4})$,

$$\sup_{\xi\in\mathbb{R}}|\mathcal{O}^t[F^a_{n_1},F^b_{n_2},F^c_{n_3}](\xi)|\lesssim_\gamma t^{-1-\gamma}\mathbf{f}^a_{n_1}\mathbf{f}^b_{n_2}\mathbf{f}^c_{n_3}\,,$$

where we denote for $n\in\mathbb{Z}^d$

$$\mathbf{f}^a_n := \|\langle x\rangle F^a_n\|_{L^2_x(\mathbb{R})}\,,\quad \mathbf{f}^b_n := \|\langle x\rangle F^b_n\|_{L^2_x(\mathbb{R})}\,,\quad \mathbf{f}^c_n := \|\langle x\rangle F^c_n\|_{L^2_x(\mathbb{R})}\,.$$

Observe that, since $s>\frac{d}{2}$,

$$\|\mathbf{f}^a_n\|_{\ell^1_n(\mathbb{Z}^d)}\lesssim\|\langle n\rangle^{2s}\,\mathbf{f}^a_n\|_{\ell^2_n(\mathbb{Z}^d)}\lesssim\|\langle x\rangle F^a\|_{L^2h^s(\mathbb{R}\times\mathbb{T}^d)}\lesssim\|F^a\|_S\,. \tag{3.18}$$

We obtain

$$\begin{aligned}\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z &= \sup_{\xi\in\mathbb{R}}\|\sum_{n_1-n_2+n_3=n}\mathrm{e}^{it\Omega(\vec{n})}\,\mathcal{O}^t[F^a_{n_1},F^b_{n_2},F^c_{n_3}](\xi)\|_{h^s(\mathbb{Z}^d)}\\ &\lesssim_\gamma t^{-1-\gamma}\|\sum_{n_1-n_2+n_3=n}\mathbf{f}^a_{n_1}\mathbf{f}^b_{n_2}\mathbf{f}^c_{n_3}\|_{h^s(\mathbb{Z}^d)}\\ &\lesssim_\gamma t^{-1-\gamma}\sum_{\sigma\in\mathfrak{S}_3}\|\mathbf{f}^{\sigma(a)}_n\|_{\ell^1_n(\mathbb{Z}^d)}\|\mathbf{f}^{\sigma(b)}_n\|_{\ell^1_n(\mathbb{Z}^d)}\|\langle n\rangle^{2s}\,\mathbf{f}^{\sigma(c)}_n\|_{\ell^2(\mathbb{Z}^d)}\,,\end{aligned}$$

and the observation (3.18) concludes the proof of estimate (3.17).

We now derive the bound (3.16). For this purpose, we use the slightly non-integrable [6] bound in the S-norm already proved for the whole system involving $\mathcal{N}^t$ and for the space-resonant system with $\mathcal{N}^t_{\mathbb{R},\mathrm{res}}$, and somehow interpolate with the decay estimate (3.17). For all $\alpha\in(0,1)$,

$$\begin{aligned}\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_{L^2_xh^s_y(\mathbb{R}\times\mathbb{T}^d)} &= \Big(\int_{\mathbb{R}}\|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^2_{h^s(\mathbb{T}^d)}d\xi\Big)^{\frac12}\\ &\le\sup_{\xi\in\mathbb{R}}\|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^\alpha_{h^s(\mathbb{T}^d)}\Big(\int_{\mathbb{R}}\langle\xi\rangle^{-2\alpha}\langle\xi\rangle^{2\alpha}\|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^{2(1-\alpha)}_{h^s(\mathbb{T}^d)}d\xi\Big)^{\frac12}\,.\end{aligned}$$

[6] Namely with time decay $t^{-1+\delta}$, with $\delta>0$.

By Hölder's inequality, and assuming that $\frac{\alpha}{1-\alpha} < \sigma$, we obtain

$$\lesssim \|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z^\alpha \Big(\int_{\mathbb{R}} \langle \xi \rangle^{\frac{2\alpha}{1-\alpha}} \|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^2_{h^s(\mathbb{T}^d)} d\xi \Big)^{\frac{1-\alpha}{2}}$$
$$\lesssim \|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z^\alpha \|(1-\partial_x^2)^{\frac{\sigma}{2}} \mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|^{1-\alpha}_{H^s_x \ell^2_y(\mathbb{R}\times\mathbb{T}^d)}.$$

Similarly, we have that for all $\alpha \in (0,1)$,

$$\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_{H^s_x \ell^2_y(\mathbb{R}\times\mathbb{T}^d)} = \Big(\int_{\mathbb{R}} \langle \xi \rangle^{2s} \|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^2_{h^s(\mathbb{T}^d)} d\xi \Big)^{\frac{1}{2}}$$
$$\leq \sup_{\xi\in\mathbb{R}} \|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^\alpha_{h^s(\mathbb{T}^d)} \Big(\int_{\mathbb{R}} \langle \xi \rangle^{-2\alpha} \langle \xi \rangle^{2(s+\alpha)} \|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^{2(1-\alpha)}_{h^s(\mathbb{T}^d)} d\xi \Big)^{\frac{1}{2}}$$

We deduce from Hölder's inequality in the integral over ξ that

$$\lesssim \|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z^\alpha \Big(\int_{\mathbb{R}} \langle \xi \rangle^{\frac{2(s+\alpha)}{1-\alpha}} \|\mathcal{F}\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c](\xi)\|^2_{h^s(\mathbb{T}^d)} d\xi \Big)^{\frac{1-\alpha}{2}}.$$

By choosing $\alpha > 0$ sufficiently small (only depending on s and σ from the definition of the S-norm) to ensure that

$$\frac{s+\alpha}{1-\alpha} < s+\sigma\,,$$

we conclude that

$$\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_{H^s_{x,y}} \lesssim \|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_Z^\alpha \; \|(1-\partial_x^2)^{\frac{\sigma}{2}} \mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|^{1-\alpha}_{H^s_{x,y}}\,.$$

We have from the trilinear bounds (2.17) and (3.8)

$$\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_{H^s_{x,y}} \lesssim \|\mathcal{N}^t[F^a,F^b,F^c]\|_{H^s_{x,y}} + \|\mathcal{N}^t_{\mathbb{R},\mathrm{res}}[F^a,F^b,F^c]\|_{H^s_{x,y}}$$
$$\lesssim \langle t \rangle^{-1+\delta} \|F^a\|_{X_T} \|F^b\|_{X_T} \|F^c\|_{X_T}\,.$$

Hence, we deduce from the first bound (3.17) (in the Z-norm) that for all $t \geq 1$,

$$\|\mathcal{N}^t_{\mathbb{R},\mathrm{nr}}[F^a,F^b,F^c]\|_{H^s_{x,y}} \lesssim t^{-1-\alpha\gamma+(1-\alpha)\delta} \|F^a\|_{X_T} \|F^b\|_{X_T} \|F^c\|_{X_T}\,,$$

which is conclusive when

$$-\alpha\gamma + (1-\alpha)\delta < -2\delta\,,$$

satisfied for $\delta > 0$ small enough (for fixed α, γ). This proves of Lemma 3.6. □

4. A priori bounds and asymptotic behavior

In this section we bring together the multilinear bounds collected in the previous sections and prove the main results, namely Theorem 1.3 and Theorem 1.4.

4.1. **A priori bounds.** We already obtained the slightly divergent bounds for the strong norm S in Lemma 2.13. It remains to prove the a priori bounds for the weak norm Z.

Proposition 4.1 (Energy estimate)**.** *Suppose that $T \geq 1$ and that F is a profile solution to the interaction system (2.15) on $[0,T]$ satisfying*

$$\|F\|_{X_T} \lesssim 1\,.$$

Then, for all $t \in [1,T]$, we have

$$\|F(t) - F(1)\|_Z \leq C\|F\|_{X_T}^2\,.$$

Proof. For fixed $\xi \in \mathbb{R}$, we have

$$\|\widehat{F}(t,\xi) - \widehat{F}(1,\xi)\|_{h^s(\mathbb{T}^d)}^2 \leq \sum_{\alpha\in\mathbb{N}} N_\alpha^{2s}\Big|\int_1^t \frac{d}{d\tau}\|\pi_\alpha \widehat{F}(\xi,\tau)\|_{\ell^2(\mathbb{T}^d)}^2 d\tau\Big|\,.$$

According to the decomposition (3.3) we have that for all $\alpha \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}&\|\pi_\alpha \widehat{F}(\tau,\xi)\|_{\ell^2(\mathbb{Z})}^2 \\ &= \operatorname{Im}\Big(\sum_{n\in\mathscr{C}_\alpha}\overline{\widehat{F}_n}(\tau,\xi)\big(\mathcal{F}\mathcal{N}_{\text{eff}}^\tau[F](\xi,n) + \mathcal{F}\mathcal{E}^\tau[F](\xi,n) + \mathcal{F}\mathcal{N}_{\mathbb{R},\text{nr}}^\tau[F](\xi,n)\big)\Big)\,.\end{aligned}$$

We proved in Lemma 3.3 that the effective system involving $\mathcal{N}_{\text{eff}}$ does not contribute to the above energy estimate. This gives

$$\frac{1}{2}\frac{d}{d\tau}\|\pi_\alpha \widehat{F}(\xi,\tau)\|_{\ell^2(\mathbb{Z})}^2 = \operatorname{Im}\Big(\sum_{n\in\mathscr{C}_\alpha}\overline{\widehat{F}_n}(\xi,\tau)\big(\mathcal{F}\mathcal{E}^\tau[F](\xi,n) + \mathcal{F}\mathcal{N}_{\mathbb{R},\text{nr}}^\tau[F](\xi,n)\big)\Big)\,,$$

and we obtain

$$\begin{aligned}\|\widehat{F}(t,\xi) - \widehat{F}(1,\xi)\|_{h^s}^2 \leq \sum_{\alpha\in\mathbb{N}} N_\alpha^{2s}\Big|\sum_{n\in\mathscr{C}_\alpha}\int_1^t \mathcal{F}\mathcal{E}^\tau[F](\xi,n)\overline{\widehat{F}_n}(\tau,\xi)d\tau\Big| \\ + \int_1^t \|\mathcal{N}_{\mathbb{R},\text{nr}}^\tau[F](\xi)\|_{h^s}\|\widehat{F}(\tau,\xi)\|_{h^d}d\tau\,.\end{aligned}$$

A Poincaré–Dulac normal form performed in Lemma 3.5 allowed us to control the contributions of $\mathcal{E}^\tau$: we have from (3.12) that

$$\|\widehat{F}(t,\xi) - \widehat{F}(1,\xi)\|_{h^s}^2 \lesssim \|F\|_{X_T}^4 + \int_1^t \|\mathcal{N}_{\mathbb{R},\text{nr}}^\tau[F](\xi)\|_{h^s}\|\widehat{F}(\tau,\xi)\|_{h^s}d\tau\,.$$

Then, we apply (3.15) from Lemma 3.6 to control the space non-resonant interactions:

$$\|\widehat{F}(t,\xi) - \widehat{F}(1,\xi)\|_{h^s}^2 \lesssim \|F\|_{X_T}^4 + \|F\|_{X_T}^3\int_1^t \tau^{-1-\gamma+3\delta}\|F(\tau)\|_Z d\tau \lesssim \|F\|_{X_T}^4\,.$$

provided $3\delta < \gamma$. This concludes the proof of Proposition 4.1. ∎

Proposition 4.2 (A priori bounds)**.** *There exists $C > 0$ such that for all $\varepsilon > 0$ and $U(0) \in S$ with*

$$\|U(0)\|_S \leq \varepsilon\,, \tag{4.1}$$

the solution U to (A-NLS) is global in $C(\mathbb{R}; S)$ and satisfies for all $T \geq 0$,

$$\|F\|_{X_T} \leq C\|U(0)\|_S\,, \tag{4.2}$$

where $F(t) := \mathrm{e}^{-it\Delta}\, U(t)$.

Proof. Let $T \geq 0$ and $0 \leq t \leq T$. It follows from Lemma 2.13 that when $0 \leq t \leq 1$,

$$\|F(t) - F(0)\|_S \leq \sup_{\tau\in[0,t]} \|\mathcal{N}^\tau[F]\|_S \lesssim \|F\|^3_{X_T}\,.$$

Recalling that the S norm controls the Z norm (remark 2.5), we deduce from the definition (2.6) of the X_T-norm that it remains to show that for all $T \geq 1$ and $1 \leq t \leq T$,

$$\|F(t) - F(1)\|_Z + t^{-\delta}\|F(t) - F(1)\|_S + t^{1-\delta}\|\partial_t F(t)\|_S \lesssim \|F\|^2_{X_T} + \|F\|^3_{X_T}\,. \tag{4.3}$$

We proved in Proposition 4.1 that

$$\|F(t) - F(1)\|_Z \lesssim \|F\|^2_{X_T}\,,$$

In Lemma 2.13 we proved that for all $t \in [1, T]$,

$$\|\mathcal{N}^t[F]\|_S \lesssim t^{-1+\delta}\|F\|^3_{X_T}\,.$$

This implies that

$$\|\partial_t F(t)\|_S = \|\mathcal{N}^t[F]\|_S \lesssim t^{-1+\delta}\|F\|^3_{X_T}\,,$$

and

$$\|F(t) - F(1)\|_S \leq \int_1^t \|\mathcal{N}^\tau[F]\|_S d\tau \lesssim t^\delta \|F\|^3_{X_T}\,.$$

Hence,

$$t^{-\delta}\|F - F(1)\|_S + t^{1-\delta}\|\partial_t F\|_S \lesssim \|F\|^3_{X_T}\,.$$

This gives (4.3) and completes the proof of Proposition 4.2. □

4.2. **Proof of the main Theorems.** We have all the ingredients to prove our main Theorems.

Proof that Theorem 1.4 $\Longrightarrow$ Theorem 1.3. The global existence and the dispersive estimate (second inequality in (1.4)) follow from the a priori bound (4.2) proved in Proposition 4.2 and from Lemma 2.11. Then, let G be the global solution to the effective system (3.9) given by Theorem 1.4. The uniform bound on the H^s-norm of U (first inequality in (1.4)) is a consequence of the conservation of the H^s-norm of G from Lemma 3.3, from the modified scattering result in H^s and from the initial bound $\|G(0)\|_{H^s} \lesssim \varepsilon$. □

It remains to prove Theorem 1.4. The proof follows the same line as the proof of [24, Theorem 6.1] but we write it in our context for completeness.

Proof of Theorem 1.4. Given, $U(0)$ satisfying the smallness condition (4.1) in the S-norm, we first construct the initial state $G_\infty(0)$ such that the global solution U to (A-NLS) with data $U(0)$ scatters to G, the global solution to the effective system (3.9) with data $G(0) = G_\infty(0)$.

Let $n \in \mathbb{N}$ and $T_n := \mathrm{e}^n$. We let $(G_n)_n$ be the sequence of global solutions in S to (3.9) with Cauchy data

$$G_n(T_n) := F(T_n)\,, \quad \text{where} \quad F(t) = \mathrm{e}^{-it\Delta}U(t)\,.$$

According to (3.11) proved in Lemma 3.3 we have that for all $t \in \mathbb{R}$,

$$\|G_n(t)\|_Z = \|G_n(T_n)\|_Z = \|F(T_n)\|_Z \lesssim \varepsilon\,, \tag{4.4}$$

where we used the global bound (4.2) on the X_T-norm of F given by Proposition 4.2. In addition, we have from Lemma 3.2 that for $t \geq T_n$,

$$\|\partial_t G_n(t)\|_{H^s_{x,y}} \lesssim t^{-1}\|G_n(t)\|_Z^2 \|G_n(t)\|_{H^s_{x,y}} \lesssim t^{-1}\varepsilon^2 \|G_n(t)\|_{H^s_{x,y}}\,. \tag{4.5}$$

Hence, using also the control on $G_n(T_n) = F(T_n)$ in $H^s_{x,y}$ given by (4.2), we deduce from the Gronwall's inequality that for all $t \geq T_n$,

$$\|G_n(t)\|_{H^s_{x,y}} \lesssim \varepsilon t^\delta\,.$$

We now prove that for all $T_n \leq t \leq 4T_n$,

$$\|F(t) - G_n(t)\|_{H^s_{x,y}} \lesssim \varepsilon^3 T_n^{-\delta}\,. \tag{4.6}$$

For this, we use the equation

$$F(t) - G_n(t) = -i\int_{T_n}^t (\mathcal{E}^\tau[F] + \mathcal{N}^\tau_{\mathbb{R},\mathrm{nr}}[F])d\tau + \int_{T_n}^t (\mathcal{N}^\tau_{\mathrm{eff}}[F] - \mathcal{N}^\tau_{\mathrm{eff}}[G_n])d\tau\,. \tag{4.7}$$

To control the source term (namely the first integral term in (4.7)), we use Lemma 3.5 and Lemma 3.6 together with the a priori bound (4.2) on F. This gives that for $T_n \leq t \leq 4T_n$,

$$\Big\|\int_{T_n}^t (\mathcal{E}^\tau[F] + \mathcal{N}^\tau_{\mathbb{R},\mathrm{nr}}[F])d\tau\Big\|_Z \lesssim T_n^{-1+4\delta}\varepsilon^3 + \varepsilon^3\int_{T_n}^t \frac{d\tau}{\tau^{-1-2\delta}} \lesssim T_n^{-2\delta}\varepsilon^3\,.$$

Factorizing

$$\mathcal{N}^t_{\mathrm{eff}}[F] - \mathcal{N}^t_{\mathrm{eff}}[G_n] = \mathcal{N}^t_{\mathrm{eff}}[F - G_n, F, F] + \mathcal{N}^t_{\mathrm{eff}}[G_n, F - G_n, F] + \mathcal{N}^t_{\mathrm{eff}}[G_n, G_n, F - G_n]\,,$$

we deduce from the a priori bound (4.4) that

$$\begin{aligned}\|F(t) - G_n(t)\|_Z &\lesssim \varepsilon^3 T_n^{-2\delta} + \int_{T_n}^t (\|G_n(\tau)\|_Z^2 + \|F(\tau)\|_Z^2)\|F(\tau) - G(\tau)\|_Z \frac{d\tau}{\tau} \\ &\lesssim \varepsilon^3 T_n^{-2\delta} + \varepsilon^2\int_{T_n}^t \|F(\tau) - G_n(\tau)\|_Z \frac{d\tau}{\tau}\,.\end{aligned}$$

Gronwall's inequality yields that for all $T_n \leq t \leq 4T_n$,

$$\|F(t) - G_n(t)\|_Z \lesssim \varepsilon^3 T_n^{-2\delta}\,.$$

By proceeding similarly, and using also the a-priori bounds (4.5), this in turn implies (4.6) since

$$\|F(t)-G_n(t)\|_{H^s_{x,y}} \lesssim \varepsilon^3 T_n^{-2\delta} + \varepsilon^2 \int_{T_n}^t \|F(\tau)-G_n(\tau)\|_{H^s_{x,y}} \frac{d\tau}{\tau} + \varepsilon \int_{T_n}^t (\|G_n(\tau)\|_{H^s_{x,y}} + \|F(\tau)\|_{H^s_{x,y}})\|F(\tau)-G_n(\tau)\|_Z \frac{d\tau}{\tau}\,.$$

The a priori bound (4.6) implies that if ε is small enough (depending on $\delta^{\frac{1}{2}}$) then for all $n \in \mathbb{N}$

$$\|G_n(0)-G_n(0)\|_{H^s_{x,y}} \lesssim \varepsilon^3 \,\mathrm{e}^{-\frac{n\delta}{2}}\ .$$

Indeed, since $T_{n+1} \leq 4T_n$ we proved that

$$\|G_n(T_{n+1})-G_{n+1}(T_{n+1})\|_{H^s_{x,y}} = \|G_n(T_{n+1})-F(T_{n+1})\|_{H^s_{x,y}} \lesssim \varepsilon^3 T_n^{-\delta}\,,$$

and Gronwall's inequality for the effective system (3.9) (that is solved both by G_n and G_{n+1}) gives

$$\|G_n(0)-G_{n+1}(0)\|_{H^s_{x,y}} \lesssim \varepsilon^3 T_n^{-\delta}\,\mathrm{e}^{C\varepsilon^2\log(T_{n+1})} \lesssim \varepsilon^3\,\mathrm{e}^{-\delta n + C\varepsilon^2(n+1)} \lesssim \varepsilon^3\,\mathrm{e}^{-\frac{n\delta}{2}}$$

In particular $(G_n(0))$ converges in $H^s_{x,y}$ to a limit $G_\infty(0)$, which satisfies

$$\|G_\infty(0)\|_Z \lesssim \varepsilon\,, \quad \|G_n(0)-G_\infty(0)\|_{H^s_{x,y}} \lesssim \varepsilon^3\,\mathrm{e}^{-\frac{\delta n}{2}}\ .$$

Using that for $n=1$, $\|G_1(0)-F(1)\|_{H^s} \lesssim \varepsilon$ and $\|F(1)\|_{H^s} \lesssim \varepsilon$ we obtain that

$$\|G_\infty(0)\|_{H^s} \lesssim \varepsilon\,.$$

Considering the global solution G with Cauchy data $G(0)=G_\infty(0)$, another Gronwall's inequality gives

$$\sup_{[0,T_{n+2}]} \|G(t)-G_n(t)\|_{H^s_{x,y}} \lesssim \varepsilon^3\,\mathrm{e}^{-\frac{\delta n}{4}}\,,$$

and we conclude from (4.6) that

$$\sup_{T_n\leq t\leq T_{n+1}} \|F(t)-G(t)\|_{H^s_{x,y}} \lesssim \sup_{T_n\leq t\leq T_{n+1}} \|F(t)-G_n(t)\|_{H^s_{x,y}} + \sup_{T_n\leq t\leq T_{n+1}} \|G(t)-G_n(t)\|_{H^s_{x,y}} \lesssim \varepsilon^3\,\mathrm{e}^{-\frac{\delta n}{4}}\,,$$

which completes the proof of Theorem 1.4. □

Appendix A. Some elementary Lemmas

In order to make the exposition self contained, we now recall the proof of the dispersive estimate for the free Schrödinger evolution on $\mathbb{R}$.

Proof of (2.12). We have

$$e^{it\partial_x^2} f(x) = \frac{1}{(4i\pi t)^{\frac12}} \int_{\mathbb{R}} e^{\frac{i(x-z)^2}{4t}} f(z)dz = \frac{e^{\frac{ix^2}{4t}}}{(4i\pi t)^{\frac12}} \int_{\mathbb{R}} e^{-\frac{ixz}{2t}} e^{\frac{iz^2}{4t}} f(z)dz$$
$$= \frac{e^{\frac{ix^2}{4t}}}{(2it)^{\frac12}} \big[\widehat{f}(\frac{x}{2t}) + r[f](t,x)\big]\,,$$

where

$$r[f](t,x) := \frac{1}{(2\pi)^{\frac12}} \int_{\mathbb{R}} e^{-\frac{ixz}{2t}} (e^{\frac{iz^2}{4t}} - 1) f(z)dz\,.$$

Proving (2.12) amounts to get the desired bound for $r[f]$. For this, we split the integral into two different regions: $r_1[f](t,x)$ (resp. $r_2[f](t,x)$) corresponds to the region $|z| \leq \sqrt{t}$ (resp. $|z| > \sqrt{t}$). On the one hand, for $x \in \mathbb{R}$ and $|t| \geq 1$,

$$|r_1[f](t,x)| \lesssim \frac{1}{t} \int_{|z|\leq\sqrt{t}} z^2 |f(z)|dz \lesssim \frac{1}{t} \Big(\int_{|z|\leq\sqrt{t}} z^2 dz\Big)^{\frac12} \|zf(z)\|_{L^2(\mathbb{R})}$$
$$\lesssim \frac{1}{t^{\frac14}} \|zf(z)\|_{L^2(\mathbb{R})}\,.$$

On the other hand,

$$|r_2[f](t,x)| \lesssim \int_{|z|\geq\sqrt{t}} |f(z)|dz \lesssim \Big(\int_{|z|\geq\sqrt{t}} \frac{1}{|z|^2} dz\Big)^{\frac12} \|zf(z)\|_{L^2(\mathbb{R})}$$
$$\lesssim \frac{1}{t^{\frac14}} \|zf(z)\|_{L^2(\mathbb{R})}\,.$$

This completes the proof of the dispersive estimate (2.12). □

The product rule stated in the following Lemma was useful in the trilinear estimates on the strong norm of Lemma 2.13.

Lemma A.1 (Product rule)**.** *For all $s > \frac{d}{2}$ and $\sigma \geq 0$, there exists $C > 0$ such that for every functions $F, G \in (1-\partial_x^2)^{-\sigma} H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d) \cap L^\infty_x h^s_y(\mathbb{R}\times\mathbb{T}^d)$,*

$$\|(1-\partial_x^2)^{\frac{\sigma}{2}}(FG)\|_{H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d)} \lesssim \|(1-\partial_x^2)^{\frac{\sigma}{2}} F\|_{H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d)} \|G\|_{L^\infty_x h^s_y(\mathbb{R}\times\mathbb{T}^d)}$$
$$+ \|F\|_{L^\infty_x h^s_y(\mathbb{R}\times\mathbb{T}^d)} \|(1-\partial_x^2)^{\frac{\sigma}{2}} G\|_{H^s_{x,y}(\mathbb{R}\times\mathbb{T}^d)}\,.$$

Proof. The bound on the norm $H^s_x\ell^2_y$ follows from the product rule on $\mathbb{R}$:

$$\|FG\|_{H^s_x\ell^2_y} \lesssim \big\|\|F\|_{H^s_x}\|G\|_{L^\infty_x}\big\|_{\ell^2_y} + \big\|\|F\|_{L^\infty_x}\|G\|_{H^s_x}\big\|_{\ell^2_y}$$
$$\lesssim \|F\|_{H^s_x\ell^2_y}\|G\|_{L^\infty_x\ell^\infty_y} + \|F\|_{L^\infty_x\ell^\infty_y}\|G\|_{H^s_x\ell^2_y}\,.$$

It remains to prove the bound for the norm $L^2_x h^s_y$.

- **Case** $\sigma = 0$: we have from the assumption $s > \frac{d}{2}$ that

$$\|FG\|_{L^2_x h^s_y} \lesssim \big\|\|F\|_{h^s_y}\|G\|_{h^s_y}\big\|_{L^2_x} \lesssim \|F\|_{L^\infty_x h^s_y}\|G\|_{L^2_x h^s_y}\,.$$

- **Case** $\sigma > 0$: we employ Littlewood–Paley decomposition in the x-variable only, and use that $h^s(\mathbb{T}^d)$ is an algebra when $s > \frac{d}{2}$. Given φ a bump function on $\mathbb{R}$, supported in $[-2,2]$ and equals to 1 on $[-1,1]$, we set for $N>1$ dyadic, $\xi\in\mathbb{R}$ and $y\in\mathbb{T}^d$

$$\widehat{P_N F}(\xi,y)=\big(\varphi(\frac{\xi}{N})-\varphi(\frac{\xi}{2N})\big)\widehat{F}(\xi,y)\,,$$

and

$$\widehat{P_{\leq 1}F}(\xi,y)=\varphi(\xi)\widehat{F}(\xi,y)\,.$$

In particular, for all fixed $y\in\mathbb{T}^d$,

$$\|(1-\partial_x^2)^{\frac{\sigma}{2}}F(y)\|_{L^2_x(\mathbb{R})}\sim\big(\|P_{\leq 1}F(y)\|^2_{L^2_x(\mathbb{R})}+\sum_{N>1}N^{2\sigma}\|P_N F(y)\|^2_{L^2_x(\mathbb{R})}\big)^{\frac12}\,.$$

In particular, we obtain from the Fubini–Tonelli's theorem that

$$\|(1-\partial_x^2)^{\frac{\sigma}{2}}F\|_{L^2_x h^s_y(\mathbb{R}\times\mathbb{T}^d)}\sim\big(\|P_{\leq 1}F\|^2_{L^2_x h^s(\mathbb{R}\times\mathbb{T}^d)}+\sum_{N>1}N^{2\sigma}\|P_N F\|^2_{L^2_x h^s_y(\mathbb{R}\times\mathbb{T}^d)}\big)^{\frac12}\,.$$

For a fixed dyadic N we decompose

$$\|P_N(FG)\|_{L^2_x h^s_y}\lesssim\|P_{<\frac{N}{8}}FP_{\sim N}G\|_{L^2_x h^s_y}+\sum_{M\geq\frac{N}{8}}\|P_M FG\|_{L^2_x h^s_y}\,,$$

where we denoted

$$P_{\sim N}=\sum_{\frac{N}{8}<M<8N}P_M\,.$$

To handle the first contribution, we observe that

$$\|P_{<\frac{N}{8}}FP_{\sim N}G\|_{L^2_x h^s_y}\lesssim\|P_{<\frac{N}{8}}F\|_{L^\infty_x h^s_y}\|P_{\sim N}G\|_{L^2_x h^s_y}\,.$$

Since

$$\|P_{<\frac{N}{8}}F\|_{L^\infty h^s}=\|\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(x')F(\cdot+\frac{x'}{N})dx'\|_{L^\infty_x h^s_y}\lesssim\|\widehat{\varphi}\|_{L^1(\mathbb{R})}\|F\|_{L^\infty_x h^s_y}\,,$$

we conclude that

$$\|P_{<\frac{N}{8}}FP_{\sim N}G\|_{L^2_x h^s_y}\lesssim\|F\|_{L^\infty_x h^s_y}\|P_{\sim N}G\|_{L^2_x h^s_y}\,,$$

which gives an acceptable contribution when summing over the frequency scales N. As for the second contribution, we write

$$\sum_{M\geq\frac{N}{8}}\|P_M FG\|_{L^2 h^s}\lesssim\big(\sum_{M\geq\frac{N}{8}}M^{-\sigma}\|(1-\partial_x^2)^{\frac{\sigma}{2}}P_M F\|_{L^2_x h^s_y}\big)\|G\|_{L^\infty_x h^s_y}\,.$$

Using Cauchy–Schwarz and the fact that $\sum_{M\geq\frac{N}{8}}(NM^{-1})^\sigma=\mathcal{O}(1)$ when $\sigma>0$ we obtain

$$\begin{aligned}N^{2\sigma}\big(&\sum_{M\geq\frac{N}{8}}M^{-\sigma}\|(1-\partial_x^2)^{\frac{\sigma}{2}}P_M F\|_{L^2_x h^s_y}\big)^2\\&\lesssim\sum_{M\geq\frac{N}{8}}(NM^{-1})^\sigma\sum_{M\geq\frac{N}{8}}(NM^{-1})^\sigma\|(1-\partial_x^2)^{\frac{\sigma}{2}}P_M F\|^2_{L^2_x h^s_y}\\&\lesssim\sum_{M\geq\frac{N}{8}}(NM^{-1})^\sigma\|(1-\partial_x^2)^{\frac{\sigma}{2}}P_M F\|^2_{L^2_x h^s_y}\,.\end{aligned}$$

Since $\sigma > 0$ we can then sum over N without loosing derivatives for fixed M, and then sum over M to conclude that this second contribution is also acceptable. $\square$

References

[1] J. Bernier and N. Camps. Long time stability for cubic nonlinear Schrödinger equations on non-rectangular flat tori. *arXiv:2402.04122*, 2024.

[2] M. Berti and A. Maspero. Long time dynamics of Schrödinger and wave equations on flat tori. *J. Differ. Equations*, 267(2):1167–1200, 2019.

[3] J. Bourgain. On the growth in time of higher Sobolev norms of smooth solutions of hamiltonian pde. *International Mathematics Research Notices*, 1996(6):277?304, 1996.

[4] J. Bourgain. On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential. *J. Anal. Math.*, 77:315–348, 1999.

[5] J. Bourgain. Problems in Hamiltonian PDE's. In *GAFA 2000. Visions in mathematics–Towards 2000. Proceedings of a meeting, Tel Aviv, Israel, August 25–September 3, 1999. Part I.*

[6] J. Bourgain and C. Demeter. The proof of the l^2 decoupling conjecture. *Ann. Math. (2)*, 182(1):351–389, 2015.

[7] X. Chen. Existence of modified wave operators and infinite cascade result for a half wave schrödinger equation on the plane. *arXiv:2302.12067*, 2023.

[8] R. Carles and E. Faou. Energy cascades for NLS on the torus. *Discrete Contin. Dyn. Syst.*, 32(6):2063–2077, 2012.

[9] J. E. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1):39–113, 2010.

[10] J. Colliander, G. Simpson, and C. Sulem. Numerical simulations of the energy-supercritical nonlinear Schrödinger equation. *J. Hyperbolic Differ. Equ.*, 7(2), 279–296, 2010.

[11] C. Collot and P. Germain. Derivation of the homogeneous kinetic wave equation: longer time scales. *arXiv:2007.03508*, 2020.

[12] Y. Deng. On growth of Sobolev norms for energy critical NLS on irrational tori: small energy case. *Commun. Pure Appl. Math.*, 72(4):801–834, 2019.

[13] Y. Deng and P. Germain. Growth of solutions to NLS on irrational tori. *Int. Math. Res. Not.*, 2019(9):2919–2950, 2019.

[14] Y. Deng, P. Germain, and L. Guth. Strichartz estimates for the Schrödinger equation on irrational tori. *J. Funct. Anal.*, 273(9):2846–2869, 2017.

[15] Y. Deng, P. Germain, L. Guth, and S. L. R. Myerson. Strichartz estimates for the Schrödinger equation on non-rectangular two-dimensional tori. *Am. J. Math.*, 144(3):701–745, 2022.

[16] V. Vilaça Da Rocha. Modified scattering and beating effect for coupled Schrödinger systems on product spaces with small initial data. *Trans. Am. Math. Soc.*, 371(7):4743–4768, 2019.

[17] P. Gérard and S. Grellier. The cubic Szegő equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(5):761–810, 2010.

[18] F. Giuliani and M. Guardia. Sobolev norms explosion for the cubic NLS on irrational tori. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 220:25, 2022. Id/No 112865.

[19] B. Grébert, É. Paturel, and L. Thomann. Modified scattering for the cubic Schrödinger equation on product spaces: the nonresonant case. *Math. Res. Lett.*, 23(3):841–861, 2016.

[20] M. Guardia, Z. Hani, E. Haus, A. Maspero, and M. Procesi. A note on growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 30(4), 865–880, 2019.

[21] M. Guardia, E. Haus, and M. Procesi. Growth of Sobolev norms for the analytic NLS on $\mathbb{T}^2$. *Adv. Math.*, 301, 615–692, 2016.

[22] M. Guardia, and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation. *J. Eur. Math. Soc. (JEMS)* 17, no. 1, 71?149, 2015..

[23] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Growing Sobolev norms for the cubic defocusing Schrödinger equation. *Sémin. Laurent Schwartz, EDP Appl.*, 2013-2014:ex, 2014.
[24] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Modified scattering for the cubic Schrödinger equation on product spaces and applications. *Forum Math. Pi*, 3:63, 2015. Id/No e4.
[25] E. Haus and M. Procesi. Growth of Sobolev norms for the quintic NLS on $\mathbb{T}^2$. *Anal. PDE*, 8(4), 883–922, 2015.
[26] N. Hayashi and P. I. Naumkin. Asymptotic for large time of solutions to the nonlinear Schrödinger and Hartree equations. *Am. J. Math.*, 120(2):369–389, 1998.
[27] A. Hrabski and Y. Pan. Effect of discrete resonant manifold structure on discrete wave turbulence. *Physical Review E*, 102(4), 041101, 2020.
[28] A. Hrabski, Y. Pan, G. Staffilani, and B. Wilson. Energy transfer for solutions to the nonlinear schrödinger equation on irrational tori. *arXiv:2107.01459*, 2022.
[29] J. Kato and F. G. Pusateri. A new proof of long-range scattering for critical nonlinear schrödinger equations. *Differential and Integral Equations*, 24(9-10):923–940, 2011.
[30] F. Planchon, N. Tzvetkov, and N. Visciglia. On the growth of Sobolev norms for NLS on 2 and 3 dimensional manifolds. *Analysis & PDE*, 10(5), 1123–1147, 2017.
[31] V. Sohinger. Bounds on the growth of high Sobolev norms of solutions to nonlinear Schrödinger equations on $\mathbb{R}$. *Indiana Univ. Math. J.*, 60(5), 1487–1516, 2011.
[32] V. Sohinger. Bounds on the growth of high Sobolev norms of solutions to nonlinear Schrödinger equations on S^1. *Differential Integral Equations*, 24(7-8), 653–718, 2011.
[33] V. Sohinger. Bounds on the growth of high Sobolev norms of solutions to 2D Hartree equations. *Discrete Contin. Dyn. Syst.*, 32(10), 3733–3771, 2012.
[34] G. Staffilani. On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations. *Duke Math. J.*, 86(1):109–142, 1997.
[35] G. Staffilani and B. Wilson. Stability of the cubic nonlinear Schrodinger equation on an irrational torus. *SIAM J. Math. Anal.*, 52(2):1318–1342, 2020.
[36] N. Tzvetkov and N. Visciglia. Small data scattering for the nonlinear Schrödinger equation on product spaces. *Commun. Partial Differ. Equations*, 37(1-3):125–135, 2012.
[37] N. Tzvetkov and N. Visciglia. Well-posedness and scattering for nonlinear Schrödinger equations on $\mathbb{R}^d \times \mathbb{T}$ in the energy space. *Rev. Mat. Iberoam.*, 32(4):1163–1188, 2016.
[38] H. Xu. Unbounded Sobolev trajectories and modified scattering theory for a wave guide nonlinear Schrödinger equation. *Math. Z.*, 286(1-2):443–489, 2017.

Nantes Université, CNRS, Laboratoire de Mathématiques Jean Leray, LMJL, F-44000 Nantes, France

Email address: nicolas.camps@ens-paris-saclay.fr

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307 USA.

Email address: gigliola@math.mit.edu.