Superfluid–Bose-glass transition of disordered bosons with long-range hopping in one dimension

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We study the superfluid–Bose-glass transition in a one-dimensional lattice boson model with power-law decaying hopping amplitude $t_{i-j} \sim 1/|i-j|^\alpha$, using bosonization and the nonperturbative functional renormalization group (FRG). When α is smaller than a critical value $\alpha_c < 3$, the U(1) symmetry is spontaneously broken, which leads to a density mode with nonlinear dispersion and dynamical exponent $z = (\alpha - 1)/2$; the superfluid phase is then stable for sufficiently weak disorder, contrary to the case of short-range hopping where the superfluid phase is destabilized by an infinitesimal disorder when the Luttinger parameter is smaller than 3/2. In the presence of disorder, long-range hopping has however no effect in the infrared limit and the FRG flow eventually becomes similar to that of a boson system with short-range hopping. This implies that the superfluid phase, when stable, exhibits a density mode with linear dispersion ($z = 1$) and the superfluid–Bose-glass transition remains in the Berezinskii-Kosterlitz-Thouless universality class, while the Bose-glass fixed point is insensitive to long-range hopping. We compare our findings with a recent numerical study.

CONTENTS

I. INTRODUCTION.

The ground state of a one-dimensional quantum fluid is generically a Luttinger liquid, i.e. a state characterized by a gapless spectrum and quasi-long-range order [1]. When the particles interact *via* a long-range potential $V(x) \sim 1/|x|^\alpha$ with an exponent $\alpha < 1$, the ground state is not a Luttinger liquid but a Wigner crystal with a spontaneous modulation of the density [2, 3]. On the other hand, the ground state of the spin-$\frac{1}{2}$ XXZ model with long-range hopping amplitude $t_{i-j} \sim 1/|i-j|^\alpha$ spontaneously breaks the U(1) symmetry when the exponent α is smaller than a critical value $\alpha_c < 3$ [4–7].

In a Luttinger liquid, disorder has noticeable effects since it can induce a transition to a localized phase [8, 9], e.g. a Bose glass in the case of a Bose gas. What is the fate of the Luttinger liquid when both disorder and long-range interactions or hopping are present? In Ref. [3], it was shown that the ground state of a Bose gas with long-range interactions is either a Wigner crystal or a Mott glass —a phase intermediate between a Mott insulator and a Bose glass—, depending on the value of the exponent $\alpha < 1$ characterizing the power-law decay of the interaction potential. In the case of long-range hopping, it has been recently suggested using quantum Monte Carlo methods that a one-dimensional boson system is always superfluid (at zero temperature) for a sufficiently weak disorder [10], in contrast to the usual case of short-range hopping where disorder is a relevant perturbation when the Luttinger parameter K is smaller than 3/2 [8, 9]. Furthermore, the superfluid–Bose-glass transition would not be in the Berezinskii-Kosterlitz-Thouless (BKT) universality class —as in the case of short-range hopping [8, 9]— but would behave like a usual continuous phase transition with a power-law divergence of the correlation length [10].

In this paper, we study a one-dimensional disordered lattice boson model with hopping amplitude t_{i-j} decaying with distance as $1/|i-j|^\alpha$. In Sec. II, using bosonization we show that the U(1) symmetry is spontaneously broken when the exponent α is smaller than a critical value $\alpha_c < 3$, in agreement with previous results obtained for the equivalent spin-$\frac{1}{2}$ XXZ model with long-range hopping [4–7]. It is possible to describe this system in the usual Luttinger-liquid formalism provided that we consider a Luttinger parameter $K(q)$ and a velocity $v(q)$ dependent on momentum. The small-q divergence $K(q), v(q) \sim |q|^{(\alpha-3)/2}$ is responsible for the spontaneous breaking of the U(1) symmetry. It also leads to a nonlinear dispersion $\omega \sim |q|^{(\alpha-1)/2}$, with a dynamical exponent $z = (\alpha-1)/2$, of the density mode and to a diverging momentum-dependent superfluid stiffness $\rho_s(q) \sim |q|^{\alpha-3}$ in the limit $q \to 0$. In Sec. III, we study the effect of disorder using the replica formalism and the nonperturbative functional renormalization group (FRG). Contrary to the short-range hopping case, where the superfluid phase is unstable for any nonzero disorder when the Luttinger parameter $K_{\rm LL}$ is smaller than 3/2, we find that the superfluid phase is always stable for sufficient weak disorder. This can be understood by the low-momentum divergence of $K(q)$, which implies that the criterion $K(q) > 3/2$ will always be fulfilled for a sufficient small value of q. The long-range hopping is however not resistant to disorder and in the low-energy limit one always recovers the standard FRG flow equations of a disordered boson system with short-range hopping, i.e. of a disordered Luttinger liquid with a well-defined zero-momentum limit $K(q \to 0)$ of the Luttinger parameter. This implies that the superfluid phase, when stable, exhibits a density mode with a linear dispersion $\omega \sim |q|$, corresponding to a dynamical exponent $z = 1$, and that the superfluid–Bose-glass transition is in the BKT universality class. The Bose-glass fixed point [11, 12] is also insensitive to long-range hopping. We discuss the difference between our findings and those of Ref. [10] in the conclusion.

II. BOSON LATTICE MODEL WITH LONG-RANGE HOPPING

We consider a lattice boson model with Hamiltonian

$$\hat{H} = -\sum_{i<j} t_{i,j}(\hat{\psi}_i^\dagger \hat{\psi}_j + \text{h.c.}) + \frac{U}{2}\sum_i \hat{\psi}_i^\dagger \hat{\psi}_i^\dagger \hat{\psi}_i \hat{\psi}_i, \quad (1)$$

where the intersite hopping amplitude $t_{i,j}$ between sites i and j decays as a power law,

$$t_{i,j} = \frac{t}{|i-j|^\alpha}, \quad (2)$$

with an exponent $\alpha > 1$. This condition ensures energy extensivity in the thermodynamic limit. In Eq. (1), $\hat{\psi}_i^\dagger$ and $\hat{\psi}_i$ denote boson creation and annihilation operators at site i. The Euclidean (imaginary-time) action reads [13]

$$S = \int_0^\beta d\tau \biggl\{ \sum_i \psi_i(\tau)\partial_\tau \psi_i(\tau) + H[\psi^*(\tau), \psi(\tau)] \biggr\}, \quad (3)$$

where $\psi_i(\tau) = \psi_i(\tau+\beta)$ is a complex field satisfying periodic boundary conditions in time. Throughout the paper we set $\hbar$, k_B and the lattice spacing to unity and consider only the zero-temperature limit $\beta = 1/T \to \infty$.

A. Bosonization

We use a density-phase representation where the boson field

$$\psi_j(\tau) = \sqrt{\rho_j(\tau)}e^{i\theta_j(\tau)} \quad (4)$$

is written in terms of the density $\rho_j = |\psi_j|^2$ (i.e. the number of bosons at site i) and the phase θ_j. This leads to the action $S = S_{\rm loc} + S_{\rm hop}$, where the local part is given by

$$S_{\rm loc} = \int_0^\beta d\tau \biggl(\sum_j i\rho_j \partial_\tau \theta_j + \frac{U}{2}\rho_j^2 \biggr) \quad (5)$$

and the hopping part by

$$S_{\rm hop} = -\sum_{i\neq j} t_{i,j} \int_0^\beta d\tau \sqrt{\rho_i \rho_j}\cos(\theta_i - \theta_j). \quad (6)$$

Expanding $\rho_i = \rho_0 + \delta\rho_i$ about the mean density ρ_0, we rewrite the hopping part of the action as

$$\begin{aligned} S_{\rm hop} = &-\int_0^\beta d\tau \sum_{i\neq j} t_{i,j} \Bigl[\rho_0 + \frac{1}{2}(\delta\rho_i + \delta\rho_j) \\ &- \frac{1}{8\rho_0}(\delta\rho_i - \delta\rho_j)^2 \Bigr] \cos(\theta_i - \theta_j) \end{aligned} \quad (7)$$

to second order in $\delta\rho_i$.

We now consider the continuum limit and follow the standard bosonization rules where the long-wavelength part of the density fluctuation $\delta\rho_i \to \delta\rho(x) = -\frac{1}{\pi}\nabla\varphi(x)$ is written in terms of a phase field φ [1, 14]. Retaining only the leading contribution in derivatives, we obtain the action

$$S_{\rm hop} = -g\int_{x,y,\tau} \frac{\cos(\theta(x,\tau) - \theta(y,\tau))}{|x-y|^\alpha}, \quad (8)$$

where $g = t\rho_0$, and we use the notation $\int_\tau \equiv \int_0^\beta d\tau$ and $\int_x \equiv \int dx$. A short-distance cutoff, of the order of the lattice spacing, is implied. The local part of the action takes the form

$$S_{\rm loc} = \int_{x,\tau} \biggl[\frac{U}{2\pi^2}(\nabla\varphi)^2 - \frac{i}{\pi}\nabla\varphi \partial_\tau \theta \biggr]. \quad (9)$$

The procedure we have followed so far is very crude but allows one to determine the low-energy form of the action $S_{\rm hop}$ associated with long-range hopping. The short-distance physics, which is not included in this naive bosonization, can be taken into account by writing the full action as $S_{\rm hop} + S_{\rm LL}$, where $S_{\rm hop}$ is given by (8) and $S_{\rm LL}$ is the usual action of a Luttinger liquid with Luttinger parameter $K_{\rm LL} \equiv K_{\rm LL}(\alpha)$ and velocity $v_{\rm LL} \equiv v_{\rm LL}(\alpha)$, i.e.

$$S = \int_{x,\tau} \biggl\{ \frac{v_{\rm LL}}{2\pi} \biggl[\frac{1}{K_{\rm LL}}(\nabla\varphi)^2 + K_{\rm LL}(\nabla\theta)^2 \biggr] - \frac{i}{\pi}\nabla\varphi\partial_\tau\theta \biggr\}$$

$$- g \int_{x,y,\tau} \frac{\cos(\theta(x,\tau) - \theta(y,\tau))}{|x-y|^\alpha}. \tag{10}$$

This action must be supplement with a UV momentum cutoff of the order of the inverse lattice spacing. We can check the relevance or irrelevance of g at the Luttinger-liquid fixed point (described by the action $S_{\rm LL}$) by dimensional analysis,

$$[g] = 3 - \alpha - 2[e^{i\theta}]_{\rm LL} = 3 - \alpha - \frac{1}{2K_{\rm LL}}, \tag{11}$$

where the last result follows from $[e^{i\theta}]_{\rm LL} = 1/4K_{\rm LL}$. The cosine term in (10) is therefore relevant if

$$\alpha < 3 - \frac{1}{2K_{\rm LL}}, \tag{12}$$

where we stress that $K_{\rm LL} \equiv K_{\rm LL}(\alpha)$ is a function of α if the low-energy effective action (10) is derived from the lattice boson model (1). In the following, we consider $K_{\rm LL}$ and α as two independent parameters.

When the condition (12) is not fulfilled, the long-range hopping is irrelevant and the system behaves as a (standard) Luttinger liquid in the low-energy limit. In the opposite case, g is a relevant perturbation and one can expand the cosine term to quadratic order, which gives the action

$$S = S_{\rm LL} + \frac{g}{2} \int_{x,y,\tau} \frac{(\theta(x,\tau) - \theta(y,\tau))^2}{|x-y|^\alpha}. \tag{13}$$

In Fourier space, the non-local term in (13) is written as

$$\frac{g}{2} \sum_Q (c_2 q^2 + c_{\alpha-1}|q|^{\alpha-1}) |\theta(Q)|^2 \tag{14}$$

to quadratic order in q, where $Q = (q, i\omega_n)$ and $\omega_n = 2n\pi T$ (n integer) is a bosonic Matsubara frequency. In the zero-temperature limit, ω_n becomes a continuous variable that will be simply denoted by ω. When $1 < \alpha < 3$, the coefficient c_2 is negative and $c_{1-\alpha}$ positive; their exact expression is not important for our discussion. The quadratic term $c_2 q^2$ can be taken into account by a redefinition of $v_{\rm LL}$ and $K_{\rm LL}$: The ratio $v_{\rm LL}/K_{\rm LL}$ is unchanged but $v_{\rm LL} K_{\rm LL}$ is reduced by $\pi g |c_2|$. This leads to

$$\begin{aligned} S = \sum_Q \biggl\{ & \frac{v_{\rm LL}}{2\pi} q^2 \left[\frac{1}{K_{\rm LL}} |\varphi(Q)|^2 + K_{\rm LL} |\theta(Q)|^2 \right] \\ & + \frac{i}{\pi} q\omega \varphi(-Q)\theta(Q) + \frac{1}{2} Z_s |q|^{\alpha-1} |\theta(Q)|^2 \biggr\}, \end{aligned} \tag{15}$$

where $Z_s = g c_{\alpha-1}$. This action can be put in the canonical Luttinger-liquid form,

$$\begin{aligned} S = \sum_Q \biggl\{ & \frac{v(q)}{2\pi} q^2 \left[\frac{1}{K(q)} |\varphi(Q)|^2 + K(q) |\theta(Q)|^2 \right] \\ & + \frac{i}{\pi} q\omega \varphi(-Q)\theta(Q) \biggr\}, \end{aligned} \tag{16}$$

if one introduces a velocity and a Luttinger parameter dependent on q,

$$\begin{aligned} \frac{v(q)}{K(q)} &= \frac{v_{\rm LL}}{K_{\rm LL}}, \\ v(q) K(q) &= v_{\rm LL} K_{\rm LL} + \pi Z_s |q|^{\alpha-3}. \end{aligned} \tag{17}$$

The action being quadratic in θ, one can integrate out the latter to obtain the φ-only action

$$S = \sum_Q |\varphi(Q)|^2 \frac{v(q)}{2\pi K(q)} \left(q^2 + \frac{\omega^2}{v(q)^2} \right), \tag{18}$$

a form that will be useful when studying the system in presence of disorder.

From Eqs. (17), we deduce

$$K(q) = K_{\rm LL} \left(1 + \frac{\pi Z_s}{K_{\rm LL} v_{\rm LL}} |q|^{\alpha-3} \right)^{1/2}. \tag{19}$$

This expression defines a crossover scale

$$q_x = \left(\frac{\pi Z_s}{K_{\rm LL} v_{\rm LL}} \right)^{1/(3-\alpha)} \tag{20}$$

separating a high-momentum regime where $K(q) \simeq K_{\rm LL}$ from a low-momentum regime dominated by long-range hopping, $K(q) \sim |q|^{(\alpha-3)/2}$. This crossover scale is meaningful only if $q_x \ll \Lambda$. It is difficult to estimate its value in the lattice model defined by (1). On the one hand, one must estimate $K_{\rm LL}$ and $v_{\rm LL}$ (which are functions of α), and then one has to take into account the (negative) coefficient c_2 which reduces the value of $v_{\rm LL} K_{\rm LL}$ (see the preceding discussion). In the following we therefore consider $K_{\rm LL}$, $v_{\rm LL}$ and Z_s, as well as the UV momentum cutoff Λ, as independent parameters. Note that this does affect the global structure of the phase diagram and the nature of the superfluid–Bose-glass transition.

The actions (16) and (18), with the definition (17) of the q-dependent velocity and Luttinger parameter, describe the low-energy limit of a boson system with long-range hopping. Since $v(q), K(q) \sim |q|^{(\alpha-3)/2}$ diverges when $q \to 0$, one expects the low-energy properties to be markedly different from those of a boson system with short-range hopping behaving as a Luttinger liquid in the low-energy limit.

B. Low-energy properties

The actions (16) and (18) being quadratic, they can be straightforwardly diagonalized. One finds a mode with dispersion $\omega = v(q)|q|$ behaving as $|q|^{(\alpha-1)/2}$ in the long-wavelength limit, which gives a dynamical exponent $z = (\alpha - 1)/2$. The compressibility is not affected by long-range hopping,

$$\kappa = \lim_{q\to 0} \frac{K(q)}{\pi v(q)} = \frac{K_{\rm LL}}{\pi v_{\rm LL}}, \tag{21}$$

whereas the q-dependent superfluid stiffness

$$\rho_s(q) = \frac{v(q)K(q)}{\pi} \sim |q|^{\alpha-3} \quad (q \to 0) \tag{22}$$

diverges in the small momentum limit. This latter property can be seen as an enhancement of the superfluid character of the bosons due to long-range hopping. The conductivity

$$\Re[\sigma(q,\omega)] = \frac{D(q)}{2}[\delta(\omega - v(q)q) + \delta(\omega + v(q)q)] \tag{23}$$

exhibits two Dirac peaks at $\omega = \pm v(q)q$, as in a standard Luttinger liquid, but the Drude weight $D(q) = v(q)K(q) = \pi\rho_s(q)$ diverges when $q \to 0$ (see Appendix A for details).

The expectation value $\langle\psi(x,\tau)\rangle \simeq \rho_0\langle e^{i\theta(x,\tau)}\rangle$ of the boson field does not vanish showing that the U(1) symmetry of the boson system is spontaneously broken,

$$\langle e^{i\theta(x,\tau)}\rangle = e^{-\frac{1}{2}\langle\theta(x,\tau)^2\rangle} > 0. \tag{24}$$

The (connected) superfluid correlation function

$$G(x) = \langle e^{i\theta(x,\tau)-i\theta(0,\tau)}\rangle_c \sim 1/|x|^{\frac{3-\alpha}{2}} \tag{25}$$

decays as a power law with an exponent $(3-\alpha)/2$. On the other hand, the density-density correlation function with momentum $q \simeq 2\pi\rho_0$,

$$\chi(x) = \langle e^{2i\varphi(x,\tau)-2i\varphi(0,\tau)}\rangle \sim e^{-\mathrm{const}\times|x|^{(3-\alpha)/2}}, \tag{26}$$

decays as a stretched exponential, to be compared with the power-law decay $\sim 1/|x|^{2K_{\rm LL}}$ in a Luttinger liquid. While in a standard superfluid density fluctuations at wavevectors $q \simeq \pm 2\pi\rho_0$ dominates over superfluid fluctuations when $K_{\rm LL} < 1/2$, the later are always the dominant fluctuations in the presence of long-range hopping. We conclude that long-range hopping enhances the superfluid character of the boson system, leading in particular to a spontaneous breaking of the U(1) symmetry, and tend to suppress density fluctuations at wave vector $q \simeq \pm 2\pi\rho_0$. Our results for the correlation functions are in agreement with those of Refs. [6, 7] once translated in the language of the equivalent spin-$\frac{1}{2}$ models studied in these works.

III. LONG-RANGE HOPPING AND DISORDER

In the continuum limit, disorder contributes to the action a term [8, 9]

$$S_{\rm dis} = \int_{x,\tau}\left[-\frac{1}{\pi}\nabla\eta + \rho_2(\xi^* e^{2i\varphi} + \mathrm{c.c.})\right], \tag{27}$$

where $\eta(x)$ (real) and $\xi(x)$ (complex) denote random potentials with Fourier components near 0 and $\pm 2\pi\rho_0$, respectively. ρ_2 is a nonuniversal parameter that depends on microscopic details. $\eta(x)$ can be eliminated by a shift of φ and is not considered in the following. The random potential ξ is assumed to be Gaussian correlated with zero mean and variance

$$\overline{\xi^*(x)\xi(x')} = D\delta(x - x') \tag{28}$$

(all over averages vanish), where the overbar denotes the average over disorder. The full action of the system is thus given by $S + S_{\rm dis}$ where S is the action in the absence of disorder [Eq. (18)]. In the replica formalism, one introduces n copies of the system [9, 12, 15]. Averaging over disorder using (28) and integrating out the θ field then yields the replicated action

$$\begin{aligned} S[\{\varphi_a\}] = &\sum_{a,Q}|\varphi_a(Q)|^2\frac{v(q)}{2\pi K(q)}\left(q^2 + \frac{\omega^2}{v(q)^2}\right) \\ &- \mathcal{D}\sum_{a,b}\int_{x,\tau,\tau'}\cos[2\varphi_a(x,\tau) - 2\varphi_b(x,\tau')], \end{aligned} \tag{29}$$

where $\mathcal{D} = D\rho_2^2$ and $a, b = 1, \dots, n$ are replica indices.

A. FRG formalism

To implement the nonperturbative FRG approach [16–19], we add to the action (29) the infrared regulator term [12]

$$\Delta S_k[\{\varphi_a\}] = \frac{1}{2}\sum_{a,Q}\varphi_a(-Q)R_k(Q)\varphi_a(Q), \tag{30}$$

where k is a (running) momentum scale varying from the UV scale Λ down to zero. The cutoff function $R_k(Q)$ is chosen so that fluctuation modes satisfying $|q|, |\omega|/v_k \ll k$ are suppressed while those with $|q| \gg k$ or $|\omega|/v_k \gg k$ are left unaffected (the k-dependent sound-mode velocity v_k is defined below). Its precise form will be given in the following.

The partition function

$$\begin{aligned} \mathcal{Z}_k[\{J_a\}] = \int\mathcal{D}[\{\varphi_a\}]\exp\Big\{&-S[\{\varphi_a\}] \\ &- \Delta S_k[\{\varphi_a\}] + \sum_a\int_{x,\tau}J_a\varphi_a\Big\} \end{aligned} \tag{31}$$

thus becomes k dependent. The expectation value of the field,

$$\phi_a(x,\tau) = \frac{\delta\ln\mathcal{Z}_k[\{J_f\}]}{\delta J_a(x,\tau)} = \langle\varphi_a(x,\tau)\rangle, \tag{32}$$

is obtained from a functional derivative with respect to the external source J_a (to avoid confusion in the indices we denote by $\{J_f\}$ the n external sources). The scale-dependent effective action

$$\Gamma_k[\{\phi_a\}] = -\ln\mathcal{Z}_k[\{J_a\}] + \sum_a\int_{x,\tau}J_a\phi_a - \Delta S_k[\{\phi_a\}] \tag{33}$$

is defined as a modified Legendre transform which includes the subtraction of $\Delta S_k[\{\phi_a\}]$. Assuming that for $k=\Lambda$ the fluctuations are completely frozen by the term ΔS_Λ, $\Gamma_\Lambda[\{\phi_a\}] = S[\{\phi_a\}]$. On the other hand the effective action of the original model (29) is given by $\Gamma_{k=0}$ provided that $R_{k=0}$ vanishes. The nonperturbative FRG approach aims at determining $\Gamma_{k=0}$ from Γ_Λ using Wetterich's equation [20–22]

$$\partial_t \Gamma_k[\{\phi_a\}] = \frac{1}{2}\mathrm{Tr}\left\{\partial_t R_k\big(\Gamma_k^{(2)}[\{\phi_a\}] + R_k\big)^{-1}\right\}, \quad (34)$$

where $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k and $t=\ln(k/\Lambda)$ a (negative) RG "time". The trace in (34) involves a sum over momenta and frequencies as well as replica indices.

To solve (approximately) the flow equation (34) we consider the following ansatz for the effective action [11, 12],

$$\Gamma_k[\{\phi_a\}] = \sum_a \Gamma_{1,k}[\phi_a] - \frac{1}{2}\sum_{a,b}\Gamma_{2,k}[\phi_a,\phi_b], \quad (35)$$

where

$$\begin{aligned}\Gamma_{1,k}[\phi_a] &= \frac{1}{2}\sum_Q |\phi_a(Q)|^2\big[Z_x q^2 + Z_{\tau,k}(q)\omega^2\big],\\ \Gamma_{2,k}[\phi_a,\phi_b] &= \int_{x,\tau,\tau'} V_k\big(\phi_a(x,\tau)-\phi_b(x,\tau')\big),\end{aligned} \quad (36)$$

with initial conditions

$$Z_x = \frac{v_{\rm LL}}{\pi K_{\rm LL}}, \quad Z_{\tau,\Lambda}(q) = \frac{1}{\pi v(q)K(q)} \quad (37)$$

and

$$V_\Lambda(u) = 2\mathcal{D}\cos(2u). \quad (38)$$

The statistical tilt symmetry, due to the invariance of the disorder part of the action (29) in the time-independent shift $\varphi_a(x,\tau)\to\varphi_a(x,\tau)+w(x)$ with $w(x)$ an arbitrary function of x, implies that Z_x is not renormalized [12]. On the other hand, we shall see that $Z_{\tau,\Lambda}(q)$ is renormalized only by an additive (q-independent) constant,

$$Z_{\tau,k}(q) = Z_{\tau,\Lambda}(q) + Y_k = \frac{1}{\pi v(q)K(q)} + Y_k. \quad (39)$$

It is convenient to write the one-replica effective action as

$$\Gamma_1[\phi_a] = \sum_Q |\phi_a(Q)|^2 \frac{v_k(q)}{2\pi K_k(q)}\left(q^2 + \frac{\omega^2}{v_k(q)^2}\right), \quad (40)$$

where

$$\begin{aligned}\frac{v_k(q)}{K_k(q)} &= \frac{v(q)}{K(q)} = \frac{v_{\rm LL}}{K_{\rm LL}},\\ \frac{1}{v_k(q)K_k(q)} &= \frac{1}{v(q)K(q)} + \pi Y_k.\end{aligned} \quad (41)$$

We further define

$$v_k = v_k(q=k), \quad K_k = K_k(q=k), \quad (42)$$

which represent the characteristic velocity and Luttinger parameter at the running momentum scale k. The velocity v_k is used to define the cutoff function appearing in the infrared regulator term (30),

$$R_k(Q) = Z_x\left(q^2 + \frac{\omega^2}{v_k^2}\right) r\left(\frac{q^2}{k^2} + \frac{\omega^2}{v_k^2 k^2}\right), \quad (43)$$

where $r(y) = \gamma/(e^y-1)$ with γ a constant of order unity.

In the absence of disorder, the scale-dependent Luttinger parameter $K_k^{(\rm pure)} = K(q=k) = K_{\rm LL} v_k^{(\rm pure)}/v_{\rm LL}$ is obtained from (19), i.e.

$$K_k^{(\rm pure)} = K_{\rm LL}\left(1 + \frac{\pi Z_s}{K_{\rm LL}v_{\rm LL}}k^{\alpha-3}\right)^{1/2}. \quad (44)$$

It reduces to $K_{\rm LL}$ when $q_x \ll k \le \Lambda$ and is fully determined by long-range hopping, i.e. $K_k^{(\rm pure)} \sim k^{(\alpha-3)/2}$, when $k \ll q_x$.

B. Flow equations

The flow equations are best written in terms of the dimensionless two-replica potential [12]

$$\tilde V_k(\phi_a - \phi_b) = \frac{K_{\rm LL}^2}{v_{\rm LL}^2}\frac{V_k(\phi_a-\phi_b)}{k^3}. \quad (45)$$

Furthermore we introduce the k-dependent dynamical exponent z_k defined by

$$z_k - 1 = \partial_t \ln v_k, \quad (46)$$

as well as the exponent θ_k,

$$\theta_k = \partial_t \ln K_k = z_k - 1, \quad (47)$$

with the last result coming from $v_k/K_k = v_{\rm LL}/K_{\rm LL}$ and (46). The derivation of the RG equation of the two-replica potential is standard [12] and gives

$$\begin{aligned}\partial_t\delta_k(u) = &-3\delta_k(u) - K_k l_1 \delta_k''(u)\\ &+ \pi \bar l_2\big\{\delta_k''(u)[\delta_k(u)-\delta_k(0)] + \delta_k'(u)^2\big\},\end{aligned} \quad (48)$$

where $\delta_k(u) = -\tilde V_k''(u)$ and l_1 and $\bar l_2$ are k-dependent threshold functions defined in Appendix B. The function $\delta_k(u)$ is π-periodic and can be expanded in circular harmonics,

$$\delta_k(u) = \sum_{n=1}^{\infty}\delta_{n,k}\cos(2nu), \quad (49)$$

a form which is useful for the numerical solution of the flow equations (with an upper cutoff $n_{\max}$ on the number of harmonics).

As for the flow of $Z_{\tau,k}(q)$ [Eq. (39)], we find

$$\partial_t Y_k = -\frac{1}{v_k K_k}\delta_k''(0)\bar{m}_\tau, \tag{50}$$

where $\bar{m}_\tau$ is another (k-dependent) threshold function, which leads to

$$\begin{aligned}\theta_k = {}& \frac{\alpha-3}{2}(1-\pi v_k K_k Y_k)\left[1-\frac{K_{\rm LL}^2}{K_k^2}(1-\pi v_k K_k Y_k)\right] \\ &+ \frac{\pi}{2}\delta_k''(0)\bar{m}_\tau.\end{aligned} \tag{51}$$

In the absence of disorder, $Y_k = 0$ and $K_k = K_k^{\rm (pure)}$, one obtains $\theta_k = (\alpha-3)/2$ for $k \to 0$ in agreement with $K_k^{\rm (pure)} \sim k^{(\alpha-3)/2}$.

C. Qualitative analysis

The stability of the superfluid phase in the limit of an infinitesimal disorder is usually studied by considering the scaling dimension of the disorder variance $\mathcal{D}$ in (29). In a Luttinger liquid with dynamical exponent $z=1$ and Luttinger parameter $K_{\rm LL}$, $[e^{2i\varphi}] = K_{\rm LL}$, so that $[\mathcal{D}] = 1+2z-2[e^{2i\varphi}] = 3-2K_{\rm LL}$, which indicates that the superfluid phase is stable for $K_{\rm LL} > 3/2$ but unstable when $K_{\rm LL} < 3/2$ [8, 9, 12]. A superfluid with long-range hopping is described by a scale-dependent Luttinger parameter $K(q) \sim |q|^{(\alpha-3)/2}$ —or, equivalently, $K_k^{\rm (pure)} \sim k^{(\alpha-3)/2}$— which diverges in the long-wavelength limit. The (scale-dependent) scaling dimension $[\mathcal{D}] = 3-2K_k^{\rm (pure)}$ will therefore always become negative in the small-k limit thus making the disorder irrelevant. We conclude that, for sufficiently weak disorder, the superfluid phase is stable.

How is this conclusion modified when disorder is not infinitesimal? In that case it is not possible to ignore the term Y_k in $Z_{\tau,k}$ [Eq. (39)], and the q-dependent velocity $v_k(q)$ at scale k is given by

$$\frac{1}{v_k(q)^2} = \frac{1}{v(q)^2} + \frac{\pi K_{\rm LL}}{v_{\rm LL}}Y_k. \tag{52}$$

Since $v(q) \sim |q|^{(\alpha-3)/2}$ diverges, the velocity $v_k(q) \to (v_{\rm LL}/\pi K_{\rm LL} Y_k)^{1/2}$ takes a finite value for $q \to 0$, which is independent of the long-range hopping encoded in $v(q)$ and $K(q)$. In other words, long-range hopping (and the associated enhanced superfluid character) is not resistant to disorder, and at sufficiently small scale k, one recovers the flow of a standard disordered Luttinger liquid with a scale-dependent Luttinger parameter $K_k = (K_{\rm LL}/\pi v_{\rm LL} Y_k)^{1/2}$. This implies, as will be further discussed in the following section, that long-range hopping cannot modify the universality class of the superfluid–Bose-glass transition.

To fully characterize the flow, it is not sufficient to consider the running Luttinger parameter K_k. One must also identify the crossover between the short-distance regime where disorder is not important —the system then becomes dominated by long-range hopping when $k \lesssim q_x$— and the long-distance regime where disorder effectively suppresses long-range hopping. This can be achieved by considering the ratio $K_k/K_k^{\rm (pure)}$, where $K_k^{\rm (pure)}$ is the (k-dependent) Luttinger parameter in the pure system [Eq. (44)].

D. Numerical solution of the flow equations

1. *Stability of the superfluid phase*

In Fig. 1, we show the flow of the Luttinger parameter K_k, the dynamical exponent z_k, the amplitude $\delta_{1,k}$ of the first harmonic of $\delta_k(u)$, and the ratio $K_k/K_k^{\rm (pure)}$ obtained for $\alpha = 2$, a very weak value of the disorder and various values of $K_{\rm LL}$ in the range 0.2–1.6. All figures are obtained for $\Lambda \simeq 20$, $v_{\rm LL} = 1$ and $Z_s = 0.1$. These (arbitrary) values ensure that the momentum crossover scale q_x is much smaller than Λ so that the early stages of the flow is always dominated by short-range hopping.

The initial conditions of the flow correspond to $K_\Lambda = K_\Lambda^{\rm (pure)} \simeq K_{\rm LL}$ and $z_\Lambda \simeq 1$. In the beginning of the flow, the disorder plays essentially no role and $K_k \simeq K_k^{\rm (pure)}$ increases as k decreases. Disorder then starts to play an important role; the amplitude $\delta_{1,k}$ increases, z_k decreases and K_k begins to deviate from $K_k^{\rm (pure)}$. In all cases however, the Luttinger parameter ends up being larger than $3/2$ and $\delta_{1,k}$ decreases towards zero. Thus, for the chosen value of the disorder, the system is in the superfluid phase for all values $0.2 \leq K_{\rm LL} \leq 1.6$. Regardless of the value of $K_{\rm LL}$, it is always possible to choose a sufficiently small value of the disorder strength for the system to be in the superfluid phase. However, we note that in the superfluid phase $z_k \to 1$ when $k \to 0$ for any nonzero value of the disorder; the low-energy mode with dispersion $\omega \sim |q|^{(\alpha-1)/2}$ of the pure system is suppressed and replaced by the usual sound mode with linear dispersion $\omega \sim |q|$.

2. *BKT transition*

Figure 2 shows some RG trajectories in the three-dimensional space defined by K_k, $K_k/K_k^{\rm (pure)}$ and $\sqrt{\delta_{1,k}}$ ($\alpha = 2$). The initial condition corresponds to $K_{\rm LL} = 1$ and various values of disorder near the critical value so that all trajectories pass close to the BKT point defined by $K = 3/2$, $K/K^{\rm (pure)} = 0$ and $\delta(u) = 0$. The ratio $K_k/K_k^{\rm (pure)}$ rapidly decreases and long-range hopping plays no role at low energy since $K_k/K_k^{\rm (pure)} \to 0$ for $k \to 0$; the nearly critical trajectories are ultimately located in the plane $(K_k, \sqrt{\delta_{1,k}})$.

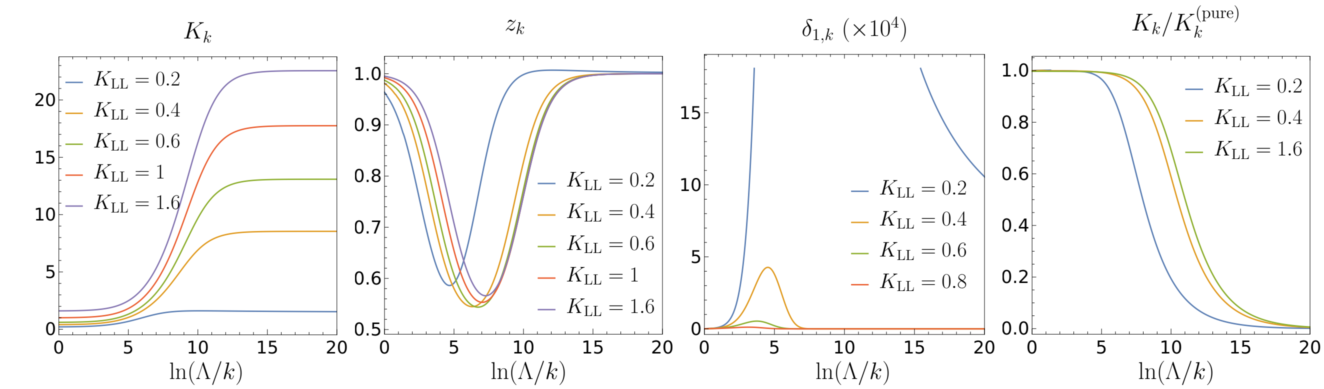

Figure 1. Luttinger parameter K_k, dynamical exponent z_k, amplitude $\delta_{1,k}$ of the first harmonic of $\delta_k(u)$, and ratio $K_k/K_k^{(\mathrm{pure})}$ vs $\ln(\Lambda/k)$ for a very weak value of the disorder —corresponding to $\delta_{1,\Lambda}=10^{-6}$— and various values of $K_{\rm LL}$ in the range 0.2–1.6 ($\alpha=2$).

Figure 3 shows the trajectories projected onto the plane $(K_k,\sqrt{\delta_{1,k}})$. The initial increase of K_k is due to the effect of long-range hopping and occurs in a regime where $K_k/K_k^{(\mathrm{pure})}$ is still of order unity (Fig. 2). To analyze the flow near the BKT point, one can retain only the first harmonic $\delta_{1,k}$, which is the only one being relevant when K_k is near $3/2$. The flow equations then read

$$\begin{aligned}\partial_t\delta_{1,k}&=(-3+4K_kl_1)\delta_{1,k}+4\pi\bar{l}_2\delta_{1,k}^2,\\ \partial_tK_k&=-2\pi\bar{m}_\tau\delta_{1,k}K_k.\end{aligned}\tag{53}$$

The relevant variables for the study of the nearly critical trajectories are $x=K_k-3/2$ and $y=\sqrt{\delta_{1,k}}$. To obtain the flow equations to second order in these variables, it is sufficient to evaluate the threshold functions l_1 and $\bar{m}_\tau$ for $\theta_k=\partial_t\ln K_k=0$, which gives the universal value $l_1=1/2$ (independent of the choice of the function r in (43)). This leads to the standard equations of the BKT transition [23, 24],

$$\begin{aligned}\partial_ty&=xy,\\ \partial_tx&=-3\pi\bar{m}_\tau y^2,\end{aligned}\tag{54}$$

where $\bar{m}_\tau\equiv\bar{m}_\tau(\theta_k=0)<0$ is a nonuniversal number that depends on the choice of the cutoff function R_k. The critical trajectory corresponds to $y=\pm x/\sqrt{3\pi|\bar{m}_\tau|}$, in very good agreement with the numerical solution of the full flow equations (Fig. 3).

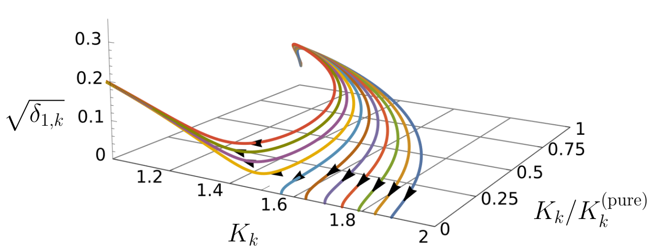

Figure 2. Flow trajectories projected onto the three-dimensional space $(K_k,K_k/K_k^{(\mathrm{pure})},\sqrt{\delta_{1,k}})$, obtained for $K_{\rm LL}=1$ and $\alpha=2$.

3. Flow diagram

In Fig. 4, we show several flow trajectories in the three-dimensional space $(K_k,K_k/K_k^{(\mathrm{pure})},\sqrt{\delta_{1,k}})$, corresponding to $K_{\rm LL}=1$ and $K_{\rm LL}=2$ and various values of the disorder intensity ($\alpha=2$). In both cases, the trajectories flow to the superfluid phase if the disorder is sufficiently weak, but end up in the Bose-glass phase for strong enough disorder. For $K_{\rm LL}=2$, a very strong value of disorder is necessary to destabilize the superfluid phase. Long-range hopping plays no role, i.e. $K_k/K_k^{(\mathrm{pure})}\simeq 0$, near the BKT point and up to the Bose-glass fixed point. The latter is characterized by a vanishing Luttinger parameter $K=0$, associated with pinning of the φ field, and a singular disorder correlator $\delta^*(\phi_a-\phi_b)$ whose functional dependence assumes a cuspy form related to the existence of metastable states. At nonzero momentum

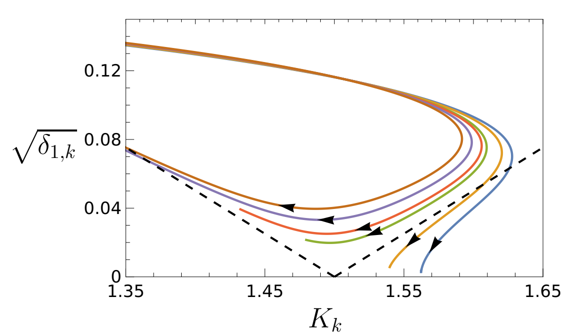

Figure 3. Flow trajectories passing close to the BKT point obtained for $K_{\rm LL}=1$ and various values of the disorder strength. The flow diagram is projected onto the plane $(K_k,\sqrt{\delta_{1,k}})$ and the dashed lines show the two separatrix of the BKT flow (54).

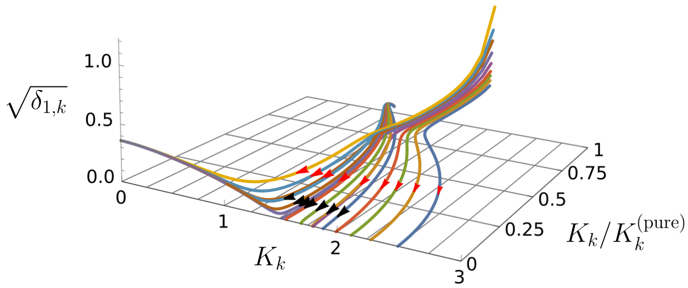

Figure 4. Flow trajectories in the three-dimensional space $(K_k, K_k/K_k^{(\rm pure)}, \sqrt{\delta_{1,k}})$, obtained for $K_{\rm LL} = 1$ and $K_{\rm LL} = 2$ with $\alpha = 2$. The black (red) arrows correspond to flow trajectories obtained for $K_{\rm LL} = 1$ (2). The attractive fixed point located at $(K = 0, K/K^{(\rm pure)} = 0, \sqrt{\delta_1^*})$ describes the Bose-glass phase.

scale k, quantum tunneling between the ground state and low-lying metastable states leads to a rounding of the cusp singularity into a quantum boundary layer. The latter controls the low-energy dynamics and yields a (dissipative) conductivity vanishing as ω^2 in the low-frequency limit. We refer to Refs. [11, 12] for a more detailed discussion.

IV. CONCLUSION

A boson system with long-range hopping is characterized by an enhanced superfluid character, which manifests itself by a spontaneous breaking of the U(1) symmetry and a superfluid stiffness diverging in the small-momentum limit. Consequently, a minimum disorder strength is required to destabilize the superfluid phase in favor of a Bose-glass phase. Nevertheless, this is the only property of the system that differentiates it from a boson system with short-range hopping. Long-range hopping is indeed not resistant to disorder and plays no role in the low-energy limit of the FRG flow equations. This implies that the superfluid phase, when stable, is characterized by a sound mode with linear dispersion $\omega \sim |q|$ for $q \to 0$ and the superfluid–Bose-glass transition remains in the BKT universality class.

Except for the stability of the superfluid phase at weak disorder, our conclusions disagree with a recent quantum Monte Carlo study of a lattice hard-core boson model with long-range hopping [10]. This numerical study seems to indicate that the pure system behaves as a regular Luttinger liquid with no spontaneous breaking of the U(1) symmetry and power-law decaying superfluid correlations (characterized by an effective Luttinger parameter), which contradicts results obtained from bosonization, density-matrix renormalization group and spin-wave analysis [6, 7] in the equivalent spin-$\frac{1}{2}$ XXZ model. In addition, the authors of Ref. [10] find that the superfluid–Bose-glass transition is not in the BKT universality class but rather corresponds to a usual second-order phase transition with a power-law diverging correlation length. This conclusion contradicts the analysis reported in the present manuscript, and also seems incompatible with the authors' own conclusion that a boson system with long-range hopping would behave as a Luttinger liquid in the absence of disorder. The failure of the numerical analysis to detect the spontaneous U(1) symmetry breaking of the pure system could be due to the crossover length $1/q_x$, above which long-range hopping dominates, being larger than the system size, but this would not explain why the superfluid–Bose-glass transition is not found in the BKT universality class if the pure system appears to behave as a regular Luttinger liquid.

ACKNOWLEDGMENT

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Appendix A: Correlation functions of the pure system

Correlation functions of the pure system can be obtained from the propagator of the φ and θ fields derived from the action (16),

$$\begin{aligned} G_{\varphi\varphi}(Q) &= \frac{\pi v(q) K(q)}{\omega^2 + v(q)^2 q^2}, \\ G_{\theta\theta}(Q) &= \frac{\pi v(q)/K(q)}{\omega^2 + v(q)^2 q^2}. \end{aligned} \tag{A1}$$

The long-wavelength part of the density-density correlation function is given by

$$\chi_{\rho\rho}(Q) = \frac{q^2}{\pi^2} G_{\varphi\varphi}(Q) = \frac{q^2}{\pi} \frac{v(q) K(q)}{\omega^2 + v(q)^2 q^2}, \tag{A2}$$

which gives the compressibility

$$\kappa = \lim_{q\to 0} \chi_{\rho\rho}(q, i\omega = 0) = \lim_{q \to 0} \frac{K(q)}{\pi v(q)}. \tag{A3}$$

The conductivity

$$\sigma(q,\omega) = \frac{\omega + i0^+}{iq^2} \chi_{\rho\rho}(q, \omega + i0^+) \tag{A4}$$

can be obtained from $\chi_{\rho\rho}$ using gauge invariance [13], and its real part is given by (23).

Phase fluctuations are bounded,

$$\begin{aligned} \langle \theta(x,\tau)^2 \rangle &= \int_{q,\omega} G_{\theta\theta}(Q) = \frac{\pi v_{\rm LL}}{2K_{\rm LL}} \int_q \frac{1}{v(q)|q|} \\ &\sim \int_0 dq \frac{1}{|q|^{\frac{\alpha-1}{2}}} \end{aligned} \tag{A5}$$

(we use the notation $\int_\omega = \int \frac{d\omega}{2\pi}$ and $\int_q = \int \frac{dq}{2\pi}$), since the momentum integral is infrared convergent when $\alpha < 3$ (a UV momentum cut-off is implicit). This implies the spontaneous breaking of the U(1) invariance [Eq. (24)]. The long-distance behavior of the connected superfluid correlation function [Eq. (25)]

$$G(x) = e^{G_{\theta\theta}(x,0)-G_{\theta\theta}(0,0)} - e^{-G_{\theta\theta}(0,0)} \tag{A6}$$

can be computed using

$$\begin{aligned} G_{\theta\theta}(x,0) - G_{\theta\theta}(0,0) &= \int_{q,\omega} [\cos(qx)-1] G_{\theta\theta}(Q) \\ &= \frac{\pi v_{\rm LL}}{2K_{\rm LL}} \int_q \frac{\cos(qx)-1}{v(q)|q|} \\ &\simeq -C|x|^{\frac{\alpha-3}{2}} + C', \end{aligned} \tag{A7}$$

where C and C' are constants (with $C>0$).

Finally, using

$$\begin{aligned} G_{\varphi\varphi}(x,0) - G_{\varphi\varphi}(0,0) &= \int_{q,\omega} [\cos(qx)-1] G_{\varphi\varphi}(Q) \\ &= \frac{\pi}{2} \int_q \frac{K(q)}{|q|} [\cos(qx)-1] \\ &\simeq -C|x|^{\frac{3-\alpha}{2}}, \end{aligned} \tag{A8}$$

we obtain the long-distance behavior of the $q = 2\pi\rho_0$ density-density correlation function

$$\chi(x) = e^{4G_{\varphi\varphi}(x,0)-4G_{\varphi\varphi}(0,0)} \tag{A9}$$

given in Eq. (26).

Appendix B: Threshold functions

The threshold functions involved in the RG equation of the two-replica potential are defined by

$$\begin{aligned} l_1 &= \pi \int_{\tilde Q} \tilde P_k(\tilde Q)^2 \partial_t \tilde R_k(\tilde Q), \\ \bar l_2 &= 2\pi \int_{\tilde q} \tilde P_k(\tilde q, 0)^3 \partial_t \tilde R_k(\tilde q, 0), \end{aligned} \tag{B1}$$

where

$$\tilde P_k(\tilde Q) = \left\{ \tilde q^2 [1 + r(\tilde Q^2)] + \tilde\omega^2 \left[\frac{v_k^2}{v_k(q)^2} + r(\tilde Q^2) \right] \right\}^{-1} \tag{B2}$$

is the dimensionless propagator obtained from $\Gamma_1[\phi_a]$, with $\tilde Q = (\tilde q, i\tilde\omega)$, $\tilde Q^2 = \tilde q^2 + \tilde\omega^2$, $\tilde q = q/k$ and $\tilde\omega = \omega/v_k k$. The dimensionless cutoff function and its time derivative are given by

$$\begin{aligned} \tilde R_k(\tilde Q) &= \tilde Q^2 r(\tilde Q^2), \\ \partial_t \tilde R_k(\tilde Q) &= -2\big\{ [r(\tilde Q^2) + \tilde Q^2 r'(\tilde Q^2)](z_k - 1)\tilde\omega^2 \\ &\quad + \tilde Q^4 r'(\tilde Q^2) \big\}. \end{aligned} \tag{B3}$$

The threshold function l_1 is not defined for $Y_k = 0$, i.e. when $k = \Lambda$, since the integral over $\tilde Q$ is infrared divergent. A nonzero Y_k appearing for $k < \Lambda$, we assume a very small initial value $Y_\Lambda \sim 10^{-3}$, which makes l_1 finite for $k = \Lambda$. The flow is essentially independent of the chosen value for Y_Λ.

The threshold function $\bar m_\tau$ appearing in the expression (51) of θ_k is defined by

$$\begin{aligned} \bar m_\tau &= \partial_{\tilde\omega^2} \bar l_1(i\tilde\omega)\big|_{\tilde\omega = 0}, \\ \bar l_1(i\tilde\omega) &= \pi \int_{\tilde q} \tilde P_k(\tilde Q)^2 \partial_t \tilde R_k(\tilde Q). \end{aligned} \tag{B4}$$

It can be written in the form

$$\begin{aligned} \bar m_\tau = \pi \int_{\tilde q} \big\{ & 2\tilde P_k(\tilde Q) [\partial_{\tilde\omega^2} \tilde P_k(\tilde Q)] \partial_t \tilde R_k(\tilde Q) \\ & + \tilde P_k(\tilde Q)^2 \partial_{\tilde\omega^2} \partial_t \tilde R_k(\tilde Q) \big\}_{\tilde\omega = 0}, \end{aligned} \tag{B5}$$

where

$$\begin{aligned} \partial_{\tilde\omega^2} \tilde P_k(\tilde Q) &= -\left[\frac{v_k^2}{v_k(q)^2} + r(\tilde Q^2) + \tilde Q^2 r'(\tilde Q^2) \right] \tilde P_k(\tilde Q)^2, \\ \partial_{\tilde\omega^2} \partial_t \tilde R_k(\tilde Q)\big|_{\tilde\omega=0} &= -2\big\{ (z_k - 1)[r(\tilde q^2) + \tilde q^2 r'(\tilde q^2)] \\ &\quad + \tilde q^2 [2r'(\tilde q^2) + \tilde q^2 r''(\tilde q^2)] \big\}. \end{aligned} \tag{B6}$$

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