

Detecting fast vanishing loops in complex-analytic germs (and detecting germs that are inner metrically conical)

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ABSTRACT. Let X be a reduced complex-analytic germ of pure dimension $n \geq 2$, with arbitrary singularities (not necessarily normal or complete intersection). Various homology cycles on $\text{Link}_\epsilon[X]$ vanish at different speeds when $\epsilon \rightarrow 0$. We give a condition ensuring fast vanishing loops on X . The condition is in terms of the discriminant and the covering data for “convenient” coverings $X \rightarrow (\mathbb{C}^n, o)$. No resolution of singularities is involved.

For surface germs ($n = 2$) this condition becomes necessary and sufficient.

A corollary for surface germs that are strictly complete intersections detects fast loops via singularities of the projectivized tangent cone of X .

Fast loops are the simplest obstructions for X to be inner metrically conical. Hence we get simple necessary conditions to the IMC property. For normal surface germs these conditions are also sufficient.

We give numerous classes of non-IMC germs and IMC germs.

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1. INTRODUCTION

1.1. Take a complex-analytic germ $X \subset (\mathbb{C}^N, o)$. The first step in its visualization is Milnor’s Conic structure theorem: X is homeomorphic to the (standard) cone over the link, $\text{Cone}[\text{Link}[X]] \subset (\mathbb{C}^N, o)$. In “most cases” this (non-embedded) homeomorphism cannot be chosen differentiable in any sense.

In some cases this topological equivalence can be strengthened to (non-embedded) bi-Lipschitz equivalence, as follows. The (standard) metric on $(\mathbb{C}^N, o)$ induces the inner metrics on X and $\text{Cone}[\text{Link}[X]]$, by the length of the shortest path between two points. The germ X is called inner-metrically-conical (IMC) if the homeomorphism $X \xrightarrow{\sim} \text{Cone}[\text{Link}[X]]$ can be chosen bi-Lipschitz with respect to these inner metrics. The metrically conical structure is the simplest possible metric structure. E.g. if X metrically conical, then it is bi-Lipschitz equivalent to its triangulation.

Any complex analytic curve germ is IMC. In higher dimensions the situation is more complicated. Initially many complex-analytic surface germs were thought to be IMC. The first weighted-homogeneous non-IMC surface germs were found in [Birbrair-Fernandes.08]. If X is a weighted-homogeneous IMC-germ of quotient type, $\mathbb{C}^2/\mu$, then its two lowest weights are necessarily equal, [B.F.N.08]. The converse statement was verified in [B.F.N.09] for Brieskorn-Pham singularities in $(\mathbb{C}^3, o)$. Finally, [B.N.P.14] established the thick-thin decomposition

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for normal surface germs. In particular, this gave an algorithm to verify the (non)IMC-property via the resolution graph of the singularity. See also [B.F.G.17]. Using this [Okuma.17] proved: a Brieskorn-Pham surface germ that is an isolated complete intersection is IMC iff its two lowest weights are equal. In higher dimensions, among the A_k -types (with equation $\sum_{i=1}^n x_i^2 + x_{n+1}^{k+1} = 0$), the only IMC-case is A_1 , [B.F.N.G.O'S.09].

In all these cases the first reason be non-IMC is the existence of so-called “fast loops”, and their generalizations, “choking horns”, [B.F.G.17] These are topologically non-trivial loops on $Link[X]$ that vanish faster than linearly, as one approaches the base point $o \in X$.

Fast loops (and fast cycles in higher dimensions) are important far beyond the IMC-context. They carry essential information on the (Lipschitz part of) local geometry and topology of X . E.g., each fast loop lies inside a thin zone, a rigid sector of the germ, where the metric is essentially distorted (as compared to the metric on a cone). To detect/forbid these fast loops is in general a complicated task. The only well-studied case is the normal surface singularities, [B.N.P.14], with the fast-loops criteria via the resolution graph of X .

In [Kerner-Mendes.23] we gave sufficient criteria for fast cycles on complete intersection germs of any (co)dimension that are perturbations of weighted-homogeneous germs. In “most cases” if the n lowest weights do not all coincide, then X has a fast cycle.

All these results suggested: IMC-germs are very rare, to the extent that, e.g. complex-analytic IMC's in $(\mathbb{C}^3, o)$ could be possibly classified, up to the ambient topological equivalence.

Now we give simple necessary/sufficient conditions for reduced germs to posses fast loops. These germs are of any dimension and codimension, not necessarily normal or complete intersection, possibly with non-isolated singularities.

- For surface germs the conditions are in terms of the discriminant of the covering. They are necessary and sufficient.
For normal surface germs this allows simple verification of the IMC property. In particular one gets: while IMC's are rare, their classification is in some sense impossible. (Even in the case $X \subset (\mathbb{C}^3, o)$.)
- The higher dimensional case admits (in many cases) a reduction to the surface case. Hence an easy way to detect a fast cycle. (And thus an obstruction to the IMC-property.)

Some of our results (for normal surface germs) could be possibly obtained by the methods of [B.N.P.14]. But our proofs do not use resolution of singularities and are independent of those of [B.N.P.14].

A remark: our results allow to detect the homotopy types of fast loops, and their vanishing speeds. This will be done in the subsequent work.

1.2. The fast cycles and (non)IMC-criteria.

1.2.1. Let X be a reduced, complex-analytic surface germ, of multiplicity p , not necessarily normal, possibly with non-isolated singularity. Present it as a covering space, $\pi : X \xrightarrow{p:1} (\mathbb{C}^2, o)$. (This covering is assumed to be “convenient”.) This map is ramified over the discriminant, $\Delta \subset (\mathbb{C}^2, o)$. We detect fast loops by the explicit numerical condition.

Theorem 3.2. *X has no fast loops iff the inequality $p > r_k - 1 + \sum q_i \cdot \text{mult}(\Delta_{k,i})$ holds for each non-smooth tangential component of Δ .*

The ingredients $\{r_k\}, \{q_i\}, \{\text{mult}(\Delta_{k,i})\}$ form the ramification data of the covering $X \xrightarrow{p:1} (\mathbb{C}^2, o)$, and are defined in §3.1. In many cases this data is easily computable from the equations of X . Thus the theorem is handy in applications. In fact, as the proof shows, $r_k + \sum q_i \cdot \text{mult}(\Delta_{k,i}) - p$ is the lower bound on the number of fast loops “hanging over Δ_k ”.

Corollary 3.3. *A normal surface germ is IMC iff $p > r_k - 1 + \sum q_i \cdot \text{mult}(\Delta_{k,i})$ for each non-smooth tangential component of Δ .*

The absence of fast loops, i.e. this numerical inequality, imposes strong restrictions on the projectivized tangent cone. Let $X \subset (\mathbb{C}^{2+c}, o)$ be an isolated strictly complete intersection

surface germ. Its projectivized tangent cone, $\mathbb{P}T_X \subset \mathbb{P}^{1+c}$, is a curve (singular, possibly non-reduced). Take its degree, $\deg[\mathbb{P}T_X]$, and the degree of its reduced locus, $\deg[\mathbb{P}T_X^{\text{red}}]$. Suppose the germs $(\mathbb{P}T_X, pt_i)$ are reduced and singular for some points $\{pt_i\}$. Take their Milnor numbers, $\mu_i := \mu(\mathbb{P}T_X, pt_i)$.

Theorem 3.10. 1. Suppose these points $\{pt_i\}$ lie in one hyperplane, $\mathbb{P}^c \subset \mathbb{P}^{1+c}$, that is “non-tangent” to $\mathbb{P}T_X$. If $\sum \mu_i \geq 1 + \deg[\mathbb{P}T_X] - \deg[\mathbb{P}T_X^{\text{red}}]$, then X has a fast loop.

2. In particular, if $\mathbb{P}T_X$ is reduced, and X is IMC, then $\mathbb{P}T_X$ must be smooth. (Hence X is an ordinary multiple point.) Additionally, when $\mathbb{P}T_X = \mathbb{P}T_X^{\text{red}}$ any reduced singular point of $\mathbb{P}T_X \cap \mathbb{P}^c$ produces a fast loop.

This is a very simple (and useful) IMC-obstruction. For example:

- The only IMC’s among the super-isolated surface germs are the ordinary multiple points. (Example 3.11)
- The only IMC’s among the surface singularities in $(\mathbb{C}^3, o)$ of right modality ≤ 2 are: A_1, D_4, X_9, P_8 . (Corollary 3.13)

1.2.2. In higher-dimensional case our results are weaker, giving just a sufficient condition for fast loops. Take a convenient covering $\pi : X \rightarrow (\mathbb{C}^n, o)$, with the discriminant $\Delta \subset (\mathbb{C}^n, o)$, and restrict it to a smooth surface germ, $\pi_{\mathcal{S}} : X \cap \pi^{-1}\mathcal{S} \rightarrow \mathcal{S} \cong (\mathbb{C}^2, o)$. Accordingly one gets the discriminantal curve $\Delta_{\mathcal{S}} := \Delta \cap \mathcal{S}$, and the ramification data, $r_k, \{q_i\}, \{\text{mult}(\Delta_{k,i})\}$, as above. Suppose $\mathcal{S}$ is “weakly non-special”, i.e. the same numerical data is obtained for any other surface that is *tangent* to $\mathcal{S}$.

Proposition 3.14. If $p \leq r_k - 1 + \sum q_i \cdot \text{mult}(\Delta_{k,i})$ for some non-smooth tangential component of $\Delta_{\mathcal{S}}$ then X has a fast loop.

This gives a quick check of fast loops/non-IMC property in higher dimensions. E.g. (corollary 3.16): if $\mathbb{P}T_{\Delta}$ has a point of high multiplicity, then X has a fast loop.

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1.4. The contents/structure of the paper.

§2 contains basic definitions/facts on singular germs. They are used in §3 to detect fast loops.

§2.2 recalls the tangent cone, T_X , and its projectivization, $\mathbb{P}T_X$, for complete intersections.

In the later case $\mathbb{P}T_X$ is a complete intersection only if X is a “strictly complete intersection”.

§2.3 defines “convenient” coverings $\pi : X \rightarrow (\mathbb{C}^n, o)$. The critical/discriminantal loci are defined set-theoretically.

§2.4 describes the blowup of the covering π , and the finiteness/flatness of the strict transform $\tilde{\pi}$.

§2.5 recalls the standard scheme structure on the critical locus when X is a complete intersection. Then the critical locus is also a complete intersection. Its image (with the Fitting scheme-structure) is the discriminant, a hypersurface.

§2.6 The covering $\pi : X \rightarrow (\mathbb{C}^n, o)$ is a Lipschitz map. It is far from being locally bi-Lipschitz. Yet, we have: the preimage of a thin set is thin. In particular, any pair of tangent arcs in $(\mathbb{C}^n, o)$ can be lifted to a pair of inner-tangent arcs in X .

§2.7 recalls fast cycles, the basic obstruction to the IMC-property. For normal surface germs the fast loops are the only obstructions. For non-normal surface germs there are various other obstructions.

§3 detects fast loops on X .

§3.1 defines the covering data of $\pi : X \rightarrow (\mathbb{C}^2, o) \supset \Delta$, i.e. the tangential components $\{\Delta_k\}$ of Δ , the total ramification orders $\{q_i\}$, and the special invariant r_k .

§3.2 contains Theorem 3.2: the condition $p \leq r_k - 1 + \sum q_i \cdot \text{mult}(\Delta_{k,i})$ means the existence of a fast loop.

In the proof one studies the covering $X \rightarrow (\mathbb{C}^2, o)$. Suppose the discriminant Δ has a non-smooth tangential component, $\Delta_k \subset \mathbb{C}_{x_1, x_2}^2$. Let its tangent line be $\text{Span}_{\mathbb{C}}(\hat{x}_1)$. The intersection $\Delta_k \cap \{Im(x_1) = 0\}$ consists of several arcs. Connecting them inside the planes $\{Re(x_1) = t\}$ produces a (subanalytic) Hölder polyhedron. (Which is a thin set.) Its lifting, $Höld^X \subset X$, is still a thin set, by §2.5. It remains to verify: one of the components of $Höld^X \setminus o$ is not “linkwise contractible”. Which means: for $\epsilon > 0$ small enough, one of the connected components of $Höld^X \cap V(x_1 - \epsilon)$ is non-contractible. And this is verified via the homology $H_1(Höld^X \cap V(x_1 - \epsilon))$, by computing the Euler characteristic.

Then go explicit examples. We classify all the IMC-surface germs of multiplicity two. Then we classify IMC’s among the surfaces of type $V(z^p + a_1(x, y)z + a_o(x, y)) \subset (\mathbb{C}^3, o)$.

§3.3 gives the condition on the tangent cone imposed by fast-loop, Theorem 3.10. The proof uses Theorem 3.2 and the facts established in §2.3, §2.4.

Now fast loops are detectable immediately and we deduce the applications mentioned in §1.2.1.

§3.4 is the higher dimensional case. We give a sufficient condition for fast loops via the ramification data (proposition 3.14). In some cases even singularities of $\mathbb{P}T_{\Delta}$ that have high multiplicity ensure vanishing cycles (corollary 3.16).

1.5. Notations and conventions.

- i. Through the paper by germs we mean their small representatives.

Everywhere in the paper $X \subset (\mathbb{C}^N, o)$ is a reduced pure-dimensional complex-analytic germ, of dimension $n \geq 2$, and multiplicity p , possibly with non-isolated singularity.

All the other germs are subanalytic and closed.

- ii. All our arcs are (real) subanalytic. Usually we take their length-parametrization, $\gamma(t) = (s_1 \cdot t + o(t), \dots, s_N \cdot t + o(t))$, with $(s_1, \dots, s_N) \neq (0, \dots, 0)$.

By a foliation we always mean a singular foliation of the germ X by real arcs, all passing through the origin.

- iii. The (inner) tangency order of two arcs $\gamma_1, \gamma_2 \subset X$ is $\text{tord}_X(\gamma_1, \gamma_2) := \text{ord}_t d_X(\gamma_1(t), \gamma_2(t))$. Here $d_X : X \times X \rightarrow \mathbb{R}$ is a subanalytic distance equivalent to the distance given by the infimum of lengths of paths connecting points of X [Kurdyka-Orro.97]. Note that $\text{tord}_X(\gamma_1, \gamma_2)$ can be non-integer.

If $X = (\mathbb{R}^N, o)$ then $d_X = \|\dots\|$ (the usual Euclidean distance) and we write just $\text{tord}(\gamma_1, \gamma_2)$.

By a LNE germ X means a representative where the inner distance d_X and the outer distance $\|\dots\|$ on X are equivalent.

The tangency order of an arc and a subgerm, $\gamma, Y \subset X$, is $\text{tord}_X(\gamma, Y) := \sup_{\gamma_Y \subset Y} \text{tord}_X(\gamma, \gamma_Y) \leq \infty$. The “order of difference” of two subgerms, $Y_1, Y_2 \subset X$, is

$$(1) \quad \text{diff.ord}_X(Y_1, Y_2) := \inf_{\gamma_1 \subset Y_1} \text{tord}_X(\gamma_1, Y_2) = \inf_{\gamma_2 \subset Y_2} \text{tord}_X(\gamma_2, Y_1) \leq \infty.$$

- iv. Take an $\mathbb{R}$ -analytic germ $Z \subset (\mathbb{R}^N, o)$. The tangent cone T_Z can be defined set-theoretically, e.g. as the Whitney cone C_3 , [Whitney, Chapter 7], [Chirka, Chapter 2]. The definition of T_Z is the same for subanalytic germs. The tangent cone is subanalytic as well, with $\dim_{\mathbb{R}} T_Z \leq \dim_{\mathbb{R}} Z$. In particular, if $\dim_{\mathbb{R}} T_Z = 1$, then T_Z is a finite union of lines and half-lines.
- v. Let $Y \subset \mathbb{R}^N$ be a sub-analytic set. Every homology class in $H_*(Y, \mathbb{Z})$ admits a subanalytic representative. Indeed, the singular homology is isomorphic to the simplicial homology. And sub-analytic manifolds admit subanalytic triangulations, [Coste, §4.3].
- vi. The singular locus of a complete intersection $X = V(f_1, \dots, f_c) \subset (\mathbb{C}^{n+c}, o)$ is defined (inside X) by the ideal of maximal minors $I_c[f'_1, \dots, f'_c]$. Here $\{f'_j\}$ are the column gradients.

2. PREPARATIONS

2.1. The (projectivized) tangent cone for complete intersections.

2.1.1. *Hypersurfaces.* Let $X = V(f) \subset (\mathbb{C}^N, o)$, then $T_X = V(l.t.(f)) \subset \mathbb{C}^N$, the lowest-order terms of f . Observe that the polynomial $l.t.(f)$ can be not square-free, in which case T_X is non-reduced.

The projectivized tangent cone is $\mathbb{P}T_X = V(l.t.(f)) \subset \mathbb{P}^{N-1}$. This is a projective hypersurface, possibly reducible, non-reduced. Its (total) degree equals the multiplicity of X , i.e. the Taylor order $ord(f)$.

2.1.2. *Complete intersections.* Let $X = V(\underline{f}) \subset (\mathbb{C}^{n+c}, o)$, where $\underline{f} := f_1, \dots, f_c \in \mathbb{C}\{\underline{x}\}$ is a regular sequence. Then $T_X = V(l.t.(\underline{f}))$, defined by the lowest order terms of all the elements of the ideal $(\underline{f})$. In particular, $l.t.(\underline{f}) \supseteq (l.t.(f_1), \dots, l.t.(f_c))$. This inclusion can be proper.

Example 2.1. For $f_1(x, y) = x^2y + y^4$, $f_2(x, y) = xy^2 + x^4$, one has $l.t.(\underline{f}) = (x^2y, xy^2, x^5 - y^5) \supsetneq (l.t.(f_1), l.t.(f_2))$.

Therefore $T_X \subset \mathbb{C}^{n+c}$ is not necessarily a complete intersection. The germ X is called a *strictly complete intersection* if both X and T_X are complete intersections, [Bennett.77]. In this case:

- There exists a choice of defining equations, $X = V(f_1, \dots, f_c)$ with $T_X = V(l.t.(f_1), \dots, l.t.(f_c))$ and $\deg(\mathbb{P}T_X) = mult X = \prod ord(f_i)$.
- The projectivization $\mathbb{P}T_X \subset \mathbb{P}^{n+c-1}$ is a complete intersection (possibly reducible, non-reduced).
- The strict transform under blowup, $\tilde{X} \subset Bl_o(\mathbb{C}^{n+c}, o)$, is a locally complete intersection at all its points.

2.2. **The convenient covering and its critical locus.** Let X be as in §1.5.i. Take the coordinate plane $\mathbb{C}^n = V(x_{n+1}, \dots, x_N) \subset \mathbb{C}^N$ and the projection $\mathbb{C}^N \ni x \rightarrow (x_1, \dots, x_n) \in \mathbb{C}^n$. We call the restriction of this projection to the germ, $X \xrightarrow{\pi} (\mathbb{C}^n, o)$, *convenient* if:

- π is a (finite) ramified covering;
- The kernel of the projection at o is non-tangent to X , i.e. $T_X \cap V(x_1, \dots, x_n) = o \in \mathbb{C}^n$.

These assumptions hold for generic projections (observe that the condition ii. is enough to get convenience). A projection can be quite non-generic, and yet convenient. E.g. the covering $(\mathbb{C}^3, o) \supset V(z^p - f(x, y)) \rightarrow (\mathbb{C}^2, o)$ is totally ramified over the curve germ $V(f) \subset (\mathbb{C}^2, o)$. This covering is convenient iff $ord f \geq p$. The critical locus of π consists of those points where π is locally not an isomorphism onto its image. Thus $Crit(\pi) \subset X$ is an analytic subgerm, and $\dim_{\mathbb{C}} Crit(\pi) \leq (n-1)$. (By Sard's theorem.)

We remark: $Crit(\pi) \supseteq Sing(X)$.

The discriminant of π is the image of the critical locus, $\Delta := \pi(Crit(\pi))$. As π is a finite morphism, $\Delta \subset (\mathbb{C}^n, o)$ is an analytic subgerm, $\dim_{\mathbb{C}} \Delta \leq n-1$. (By Grauert's direct image theorem.) The inequality can occur here. E.g. let X be the union of two planes (of dimension n) intersecting at one point. Then Δ is just one point, $o \in \mathbb{C}^n$.

When X is a complete intersection, these set-theoretic definitions are replaced by more precise algebraic ones, §2.4, and Δ is well-behaved, e.g. $\dim_{\mathbb{C}} \Delta = n-1$.

Recall the basic properties of convenient projections.

- A small deformation of the projection π preserves the convenience of the covering.
- If π is convenient, then the restriction $\pi_Y = \pi|_{\pi^{-1}Y}$ is a convenient covering for any smooth subvariety $Y \subset (\mathbb{C}^n, o)$, assuming $Y \not\subset \Delta$ and $\pi^{-1}Y$ is reduced.
- A small deformation of the base point does not preserve the convenience. E.g., let $o \neq x_o \in Crit(\pi)$ be a smooth point of X . Then the projection $(X, x_o) \rightarrow (\mathbb{C}^n, \pi(x_o))$ is non-convenient.

$$\begin{array}{ccc} (\mathbb{C}^N, o) \supset X \supset Crit(\pi) & & \\ \searrow \downarrow \pi & & \downarrow \pi| \\ (\mathbb{C}^n, o) \supset \Delta & & \end{array}$$

2.3. The blowup of π . Let X be as in §1.5.i. Suppose the projection $\pi: X \rightarrow (\mathbb{C}^n, o)$ is convenient.

Lemma 2.2. 1. *The blowup of the origin produces the diagram:*

$$\begin{array}{ccccc} (\mathbb{C}^N, o) & \supset X & \leftarrow & \tilde{X} := Bl_o X & \supset \mathbb{P}T_X = \tilde{X} \cap \mathbb{P}^{N-1} \\ & \pi \downarrow & & \tilde{\pi} \downarrow & \downarrow \tilde{\pi}| \\ (\mathbb{C}^n, o) & \leftarrow & Bl_o(\mathbb{C}^n, o) & \supset & \mathbb{P}^{n-1} \end{array}$$

Here the morphisms $\tilde{\pi}, \tilde{\pi}|$ are finite.

2. *If $X \subset (\mathbb{C}^N, o)$ is a strictly complete intersection (§2.1.2), then the morphisms $\tilde{\pi}, \tilde{\pi}|$ are flat.*

Proof.

1. The projection $Bl_o \mathbb{C}^N \dashrightarrow Bl_o \mathbb{C}^n$ is $(x_1, \dots, x_N, [\sigma_1 : \dots : \sigma_N]) \rightarrow (x_1, \dots, x_n, [\sigma_1 : \dots : \sigma_n])$, it is defined only for $(\sigma_1, \dots, \sigma_n) \neq (0, \dots, 0)$. But π is a convenient projection, therefore $\tilde{X} \cap V(\sigma_1, \dots, \sigma_n) = \emptyset$. Therefore the projection $\tilde{\pi}: \tilde{X} \rightarrow Bl_o(\mathbb{C}^n, o)$ is well defined.

All the fibres of $\tilde{X} \setminus \mathbb{P}T_X \rightarrow \mathbb{C}^n \setminus \{o\}$ are finite. And the restriction of the projection to the exceptional divisor, $\tilde{\pi}|: \tilde{X} \cap \mathbb{P}^{N-1} \rightarrow \mathbb{P}^{n-1}$, has finite fibres. Indeed, if a fibre over a point $[\sigma_1 : \dots : \sigma_n] \in \mathbb{P}^{n-1}$ is infinite, then $\tilde{\pi}^{-1}(0, \dots, 0, [\sigma_1 : \dots : \sigma_n]) \subset \mathbb{P}T_X \subset \tilde{X}$ is an algebraic subset of positive dimension. But then $\dim(T_X \cap V(x_1, \dots, x_n)) > 0$, contradicting the convenience of π .

Thus $\tilde{\pi}$ is a projective morphism with finite fibres. Hence $\tilde{\pi}$ finite. And therefore $\tilde{\pi}|$ is finite as well.

2. As X is a strictly complete intersection, both $\tilde{X}$ and $\mathbb{P}T_X$ are locally complete intersections, while $\tilde{\pi}$ and $\tilde{\pi}|$ are finite morphisms with smooth images.

Thus to verify the flatness it is enough to verify: the degree of fibres is constant. For each point $\tilde{x} \in Bl_o(\mathbb{C}^n, o) \setminus \mathbb{P}^{n-1}$ the fibre $\tilde{\pi}^{-1}(\tilde{x})$ is of degree p (=the degree of the covering π). For each point $\tilde{x} \in \mathbb{P}^{n-1}$ we have:

$$(2) \quad \deg \tilde{\pi}^{-1}(\tilde{x}) = \deg \tilde{\pi}|^{-1}(\tilde{x}) = \deg (\mathbb{P}T_X \cap \mathbb{P}^{N-n}) \stackrel{*}{=} \deg(\mathbb{P}T_X) \stackrel{**}{=} \text{mult} X = p.$$

Here (after a $GL(\mathbb{C}^n)$ transformation) $\mathbb{P}^{N-n} = V(\sigma_1, \dots, \sigma_{n-1}) \subset \mathbb{P}^{N-1}$. The equalities $*$, $**$ hold because $X, \mathbb{P}T_X$ have no embedded components of dimension $< n$ (being complete intersections). $\blacksquare$

2.4. The scheme structure on the critical locus and the discriminant. Let X be as in §1.5.i. Let $X \xrightarrow{\pi} (\mathbb{C}^n, o)$ be a (not necessarily convenient) ramified $p:1$ covering.

Lemma 2.3. *If the germ $X \subset (\mathbb{C}^{n+c}, o)$ is a complete intersection then the critical locus and the discriminant (with the scheme structure defined in the proof) have the following properties.*

1. *$\text{Crit}(\pi) \subset (\mathbb{C}^{n+c}, o)$ is a complete intersection of dimension $n-1$, $\Delta \subset (\mathbb{C}^n, o)$ is a hypersurface.*
2. *Let $X = V(\underline{f}) \subset (\mathbb{C}^{n+c}, o)$, for $\underline{f} = (f_1, \dots, f_c)$, the column. Denote $p_i := \text{ord}(f_i)$. Then $\text{mult}(\text{Crit}(\pi)) \geq (\prod p_i) \cdot \text{ord}(\det[\underline{f}']) \geq (\prod p_i) \cdot \sum (p_j - 1)$.*

Here $[\underline{f}'] \in \text{Mat}_{c \times c}$ is the Jacobian matrix of derivatives with respect to the variables $x_{n+1} \dots x_{n+c}$.

3. *The diagram of §2.2 is functorial on pullbacks. Namely, for each morphism $\phi: (\mathbb{C}^m, o) \rightarrow (\mathbb{C}^n, o)$ with $\phi(\mathbb{C}^m, o) \not\subset \Delta$, and the corresponding projection $\phi^* \pi: X \times_{(\mathbb{C}^n, o)} (\mathbb{C}^m, o) \rightarrow (\mathbb{C}^m, o)$ one has:*

- $\text{Crit}(\phi^* \pi) = \phi^* \text{Crit}(\pi)$, i.e. $\mathcal{O}_{\text{Crit}(\phi^* \pi)} = \mathcal{O}_{\text{Crit}(\pi)} \otimes_{(\mathbb{C}^n, o)} \mathcal{O}_{(\mathbb{C}^m, o)}$,
- $\Delta_{\phi^* \pi} = \phi^* \Delta$, i.e. $\mathcal{O}_{\Delta_{\phi^* \pi}} = \mathcal{O}_{\Delta} \otimes_{(\mathbb{C}^n, o)} \mathcal{O}_{(\mathbb{C}^m, o)}$.

Here $X \times_{(\mathbb{C}^n, o)} (\mathbb{C}^m, o) := \text{Spec}(\mathcal{O}_X \otimes_{(\mathbb{C}^n, o)} \mathcal{O}_{(\mathbb{C}^m, o)})$ is the pullback of X via ϕ .

Proof. First we define $\text{Crit}(\pi) \subset X$ and $\Delta \subset (\mathbb{C}^n, o)$ as analytic subgerms. Then we deduce the statements.

- a. Recall the standard definitions [Looijenga], [A.G.L.V.II]. When X is smooth, the critical locus of the projection $X \rightarrow (\mathbb{C}^n, o)$ is the degeneracy locus of the induced morphism $T_X \rightarrow T_{(\mathbb{C}^n, o)}$. In the general case the critical locus is defined as the degeneracy locus of the n -form $dx_1 \wedge \cdots \wedge dx_n|_X \in \Omega_X^n$. For the defining equations, $X = V(f_1, \dots, f_c) \subset (\mathbb{C}^{n+c}, o)$, one has $\Omega_X^n = \mathcal{O}_X \otimes \Omega_{(\mathbb{C}^{n+c}, o)}^n / \text{Span}\{df_i\} \wedge \Omega_{(\mathbb{C}^{n+c}, o)}^{n-1}$. Thus the critical locus is the set of points of X where the form $dx_1 \wedge \cdots \wedge dx_n|_X$ belongs to the submodule $\text{Span}\{df_i\} \wedge \Omega_{(\mathbb{C}^{n+c}, o)}^{n-1}|_X$. Equivalently, $dx_1 \wedge \cdots \wedge dx_n \wedge df_1 \wedge \cdots \wedge df_c|_X = 0$. Writing this in coordinates, the critical locus is defined (inside X) by the equation $\det[\partial_{n+1}\underline{f} \dots \partial_{n+c}\underline{f}] = 0$. We get the analytic subgerm $\text{Crit}(\pi) \subset (\mathbb{C}^n, o)$ and its local ring

$$(3) \quad \mathcal{O}_{\text{Crit}(\pi)} = \mathcal{O}_{(\mathbb{C}^{n+c}, o)} / \underline{(f, \det[\partial_{n+1}\underline{f}, \dots, \partial_{n+c}\underline{f}])}.$$

- b. We claim: the subgerm $\text{Crit}(\pi) \subset (\mathbb{C}^{n+c}, o)$ is a complete intersection, i.e. the sequence $f_1, \dots, f_c, \det[\dots]$ is regular in $\mathcal{O}_{(\mathbb{C}^{n+c}, o)}$. Indeed, $\dim(\text{Crit}(\pi)) \geq n-1$, as the critical locus is defined by $(c+1)$ equations. But the projection $X \rightarrow (\mathbb{C}^n, o)$ is generically unramified on every irreducible component of X . Therefore $\dim(\text{Crit}(\pi)) \leq n-1$. Altogether, the sequence $f_1, \dots, f_c, \det[\dots]$ of length $(c+1)$ defines a subgerm $\text{Crit} \subset (\mathbb{C}^{n+c}, o)$ of codimension $(c+1)$. Hence this sequence is regular, [Eisenbud, §17].

Equation (3) gives then Part 2 of the statement.

The discriminant is the image of $\text{Crit}(\pi)$. As π is a finite morphism, Δ is an analytic subset, $\dim(\Delta) = n-1$, i.e. $\Delta \subset (\mathbb{C}^n, o)$ is (set-theoretically) a hypersurface.

The natural scheme structure on Δ is the image structure. The pushforward $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$ is a module over $\mathcal{O}_{(\mathbb{C}^n, o)}$. Its support is $\Delta \subset (\mathbb{C}^n, o)$, of dimension $n-1$. Therefore we define $\Delta \subset (\mathbb{C}^n, o)$ as an analytic subgerm by the zeroth Fitting ideal, $\text{Fitt}_0[\pi_* \mathcal{O}_{\text{Crit}(\pi)}] \subset \mathcal{O}_{(\mathbb{C}^n, o)}$, [Eisenbud, §20]. In detail, take a presentation matrix, $\pi_* \mathcal{O}_{\text{Crit}(\pi)} = \text{coker}[A]$. Below we show that A is a square matrix. Then $\text{Fitt}_0[\pi_* \mathcal{O}_{\text{Crit}(\pi)}] = (\det[A])$.

- c. As $\Delta \subset (\mathbb{C}^n, o)$ is a (set-theoretic) hypersurface germ, we can assume: the intersection $V(x_1, \dots, x_{n-1}) \cap \Delta$ is a point. Thus $x_1, \dots, x_{n-1}$ goes to a regular sequence in the ring $\mathcal{O}_\Delta$.

We claim: the sequence $x_1, \dots, x_{n-1}$ is regular also on the ring $\mathcal{O}_{\text{Crit}(\pi)}$, and hence on the module $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$ as well. First, x_1 is not a zero-divisor on $\mathcal{O}_{\text{Crit}(\pi)}$. Indeed, $\text{Crit}(\pi)$ has no embedded components, being a complete intersection. In addition, $V(x_1)$ intersects $\text{Crit}(\pi)$ properly.

Now restrict onto $V(x_1)$. One has:

$$(4) \quad \mathcal{O}_{\text{Crit}(\pi)} \otimes \mathcal{O}_{(\mathbb{C}^{n+c}, o)} / (x_1) = \mathcal{O}_{(\mathbb{C}^{n+c}, o)} / (x_1, \underline{f}, \det[\dots]) \cong \mathcal{O}_{(\mathbb{C}^{n+c-1}, o)} / (\underline{f}|_{V(x_1)}, \det[\dots|_{V(x_1)}]) = \mathcal{O}_{\text{Crit}(\pi|_{V(x_1)})}.$$

The restricted map $X|_{V(x_1)} \rightarrow (\mathbb{C}^{n-1}, o)$ is still a ramified covering. Therefore the restrictions $(f_1|_{V(x_1)}, \dots, f_c|_{V(x_1)}, \det[\dots]|_{V(x_1)})$ still form a regular sequence and $\mathcal{O}_{\text{Crit}(\pi|_{V(x_1)})}$ is a complete intersection. Repeat the argument for x_2 , and so on.

- d. The sequence $x_1 \dots x_{n-1}$ is regular on the module $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$. Therefore the $\mathcal{O}_{(\mathbb{C}^n, o)}$ -module $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$ is of $\text{depth} = n-1 = \dim(\Delta)$, [Eisenbud, §18]. Hence $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$ is Cohen-Macaulay. By Auslander-Buchsbaum formula, the $\mathcal{O}_{(\mathbb{C}^n, o)}$ -resolution of $\pi_* \mathcal{O}_{\text{Crit}(\pi)}$ (over $\mathcal{O}_{(\mathbb{C}^n, o)}$) is of length one. Thus the presentation matrix, $\pi_* \mathcal{O}_{\text{Crit}(\pi)} = \text{coker}[A]$, is square. Hence $\text{Fitt}_0(\pi_* \mathcal{O}_{\text{Crit}(\pi)}) = (\det[A])$ is a principal ideal. Thus the subscheme $\Delta \subset (\mathbb{C}^n, o)$ is a hypersurface (possibly with multiple components). ■

Example 2.4. i. Let $X := V(z^p - f(x, y)) \subset (\mathbb{C}^3, o)$, where $\text{ord}(f), p \geq 2$. Take the covering $X \rightarrow (\mathbb{C}_{xy}^2, o)$, it is convenient iff $\text{ord}(f) \geq p$. Then the defining ideal of $\text{Crit}(\pi) \subset$

$(\mathbb{C}^3, o)$ is $(z^{p-1}, f(x, y)) \subset \mathcal{O}_{(\mathbb{C}^3, o)}$. Set-theoretically one gets: $\Delta = V(f(x, y)) \subset (\mathbb{C}^2, o)$. To obtain the image scheme structure on Δ we consider $\mathcal{O}_{\text{Crit}(\pi)} = \mathbb{C}\{x, y, z\}/(z^{p-1}, f(x, y))$ as an $\mathbb{C}\{x, y\}$ -module. Its natural generators are $\{1, z, \dots, z^{p-2}\}$, and the module splits as a direct sum. We get the presentation

$$(5) \quad \bigoplus^{p-1} \mathbb{C}\{x, y\} \xrightarrow{\oplus[f]} \bigoplus^{p-1} \mathbb{C}\{x, y\} \rightarrow \mathcal{O}_{\text{Crit}(\pi)} \rightarrow 0.$$

Hence the defining ideal of the discriminant is $I_\Delta = \det[\oplus[f]] = (f)^{p-1} \subset \mathbb{C}\{x, y\}$.

- ii. (Functoriality on restrictions) Take a linear subspace $(\mathbb{C}^m, o) \subset (\mathbb{C}^n, o)$ with $(\mathbb{C}^m, o) \not\subseteq \Delta$. Take the restricted projection $\pi| : \pi^{-1}(\mathbb{C}^m, o) \rightarrow (\mathbb{C}^m, o)$. One gets: $\text{Crit}(\pi|) = \text{Crit}(\pi) \cap \pi^{-1}(\mathbb{C}^m, o)$ and $\Delta(\pi|) = \Delta \cap (\mathbb{C}^m, o)$.

2.4.1. *The discriminant for hypersurface germs.* Take a hypersurface germ $X = V(f) \subset (\mathbb{C}^{n+1}, o)$ of multiplicity p . Suppose the projection $X \rightarrow (\mathbb{C}_{x_1, \dots, x_n}^n, o)$ is convenient. Thus $T_X \cap V(x_1, \dots, x_n) = (o)$, therefore f contains the monomial x_{n+1}^p . By Weierstraß preparation theorem one can present:

$$(6) \quad f(x) = u(x)(x_{n+1}^p + x_{n+1}^{p-1} \cdot a_{p-1}(x_1, \dots, x_n) + \dots + a_0(x_1, \dots, x_n)).$$

Here u is invertible and $a_0, \dots, a_{p-1}$ are analytic power series, with $\text{ord}(a_j) \geq p - j$. Thus we omit u .

This hypersurface is the pullback of the universal hypersurface, $V(z^p + z^{p-1}a_{p-1} + \dots + a_0) \subset \mathbb{C}_{z, a_{p-1}, \dots, a_0}^{1+p}$, under the map $(x_1, \dots, x_n) \rightarrow (a_{p-1}(x), \dots, a_0(x))$. Therefore (by part 3 of lemma 2.3) the discriminant of the projection, $\Delta \subset \mathbb{C}^n$, is obtained as the pullback of the classical discriminant Δ_{class} of the polynomial $z^p + z^{p-1}a_{p-1} + \dots + a_0$. Here $\Delta_{\text{class}} \in \mathbb{Z}[\{a_\bullet\}]$ is a (complicated) polynomial in the coefficients $\{a_\bullet\}$. We use the following properties [G.K.Z.]:

- Assign the weights to the coefficients, $w(a_j) = p - j$, then the polynomial Δ_{class} is weighted-homogeneous, of total weight $p(p-1)$.
- $\Delta_{\text{class}}(\{a_\bullet\}) = (\frac{a_0}{p-1})^{p-1} - (\frac{a_1}{p})^p + g(\{a_\bullet\})$, where $g(\{a_\bullet\}) \in (a_0, a_1) \cdot (a_2 \dots a_{p-1})$.
- For $p \geq 3$ the hypersurface $V(\Delta_{\text{class}}) \subset \mathbb{C}^p$ is singular in codimension 1.

2.4.2. *The polar multiplicity of a curve germ.* Take a reduced complete intersection curve germ, $(C, o) = V(f_1, \dots, f_c) \subset (\mathbb{C}^{1+c}, o)$. Suppose the projection $(\mathbb{C}^{1+c}, o) \supset (C, o) \rightarrow (\mathbb{C}_{x_1+c}^1, o)$ is convenient, §2.2. The critical locus of the projection is $V(f_1, \dots, f_c) \cap V(\det[\partial_i f_j]_{i,j \leq c})$, see equation (3). This is a one-point scheme, a complete intersection. Its degree is the polar multiplicity, $m_1(C, o) = \deg[V(f_1, \dots, f_c) \cap V(\det[\partial_i f_j]_{i,j \leq c})]$, see [Gaffney.93], [Nuñ-Tom.08]. In particular, one has: $\mu(C, o) = m_1(C, o) - \text{mult}(C, o) + 1$, [Nuñ-Tom.08, Theorem 2.6]. (See also [Brieskorn-Greuel.75], [Lê.74].)

Example 2.5. For plane curves ($c = 1$) this one-point scheme is $V(f, \partial_y f) \subset (\mathbb{C}_{xy}^2, o)$. Its degree is the classical kappa invariant, $\kappa(C, o) = \mu(C, o) + \text{mult}(C, o) - 1$, see e.g. [G.L.S., pg.212].

2.5. **Preimage of a thin set is thin (lifting tangent arcs to tangent arcs).** The convenient covering $(\mathbb{C}^N, o) \supset X \xrightarrow{\pi} (\mathbb{C}^n, o)$ is a Lipschitz map. But π is far from being locally bi-Lipschitz. In particular, π can increase the tangency order of arcs. Yet, a weaker property holds.

Recall, a subanalytic germ Y is called *thin* if $\dim_{\mathbb{R}} Y > \dim T_Y$, see §2.1. The following statement is well known.

Lemma 2.6. *Let $\pi : X \rightarrow (\mathbb{C}^n, o)$ be a convenient covering and $Y \subseteq X$ a subanalytic sub-germ with connected link.*

1. *The tangent cones satisfy: $\dim_{\mathbb{R}} T_Y = \dim_{\mathbb{R}} T_{\pi(Y)}$. In particular, Y is thin iff $(\pi(Y), o)$ is thin.*
2. *Suppose $T_{(\pi(Y), o)}$ is a (real) half-line.*
 - a. *Then the tangent cone $T_Y \subset \mathbb{C}^N$ is a half-line, i.e. any two arcs on Y are outer-tangent.*
 - b. *Moreover, any two arcs on Y are inner-tangent, i.e. $\text{tord}_Y(\gamma_1, \gamma_2) > 1$.*

Remark 2.7. Suppose the arcs $\gamma_1, \gamma_2 \subset Y \subset (\mathbb{R}^N, o)$ are real-analytic, i.e. their length-parametrization is by power series with only integer exponents. Then their outer tangency order (in $\mathbb{R}^N$) is an integer. One might hope that the inner tangency order will be an integer and Part 2.b will give a stronger bound: $\text{tord}_Y(\gamma_1, \gamma_2) \geq 2$. This does not hold. Take a real-analytic germ $Y \subset (\mathbb{R}^3, o)$ that is not LNE, and such that T_Y is a half-line. Suppose the (real-analytic) arcs $\gamma_1, \gamma_2 \subset Y$ realize the non-LNE assumption, i.e. $\text{tord}_Y(\gamma_1, \gamma_2) < \text{tord}_{\mathbb{R}^3}(\gamma_1, \gamma_2) = 2$. Then $1 \leq \text{tord}_Y(\gamma_1, \gamma_2) < 2$.

An explicit example is the set $Y \subset \mathbb{R}^3$ defined by the equation $(x^2 + y^2)^2 = z^3(x^2 - y^2) + \epsilon^2 z^7$, with $0 < \epsilon \ll 1$. (It is instructive to draw its section by $z = 1$.) The tangent cone is the $\hat{z}$ -axis. The two arcs of the intersection $Y \cap \{x = 0, z \geq 0\}$ are real-analytic. Their parametrization is $(0, \frac{\pm 2\epsilon z^2}{\sqrt{1+\sqrt{1+z}}}, z)$. Their outer contact is 2. But their inner contact is $\frac{3}{2}$.

Corollary 2.8. Suppose two arcs $\gamma_1, \gamma_2 \subset (\mathbb{C}^n, o)$ are tangent. For any preimage $\gamma_1^X \subset X \subset (\mathbb{C}^N, o)$ there exists a preimage $\gamma_2^X \subset X$ satisfying $\text{tord}_X(\gamma_1^X, \gamma_2^X) > 1$.

Proof. As the arcs are tangent, we can assume the parameterizations $\gamma_i(t) = (t, \underline{h}_i(t))$, where all the entries of $\underline{h}_i(t)$ are $o(t)$. Build the Hölder triangle on these arcs,

$$(7) \quad \text{Höld}[\gamma_1, \gamma_2] := \cup_{s \in [0,1]} [s\gamma_1 + (1-s)\gamma_2] := \cup_{s \in [0,1]} [t, s \cdot \underline{h}_1(t) + (1-s) \cdot \underline{h}_2(t)] \subset \mathbb{C}^n.$$

By its construction $\text{Höld}[\gamma_1, \gamma_2] \subset \mathbb{C}^n$ is a subanalytic subset (with connected link) whose tangent cone is the (real) half-line. For a fixed lifting $\gamma_1^X \subset X$ we lift the curve $\text{Link}[\text{Höld}[\gamma_1, \gamma_2]]$ (homeomorphically) to a curve $\text{Link}^X[\text{Höld}[\gamma_1, \gamma_2]] \subset X$ that starts at $\text{Link}[\gamma_1^X]$. This defines the lifting $\text{Höld}^X[\gamma_1, \gamma_2] \subset X$. Now apply lemma 2.6, with $Y = \text{Höld}^X[\gamma_1, \gamma_2]$. ■

Remark 2.9. This corollary fails if any of its assumptions are omitted.

- i. (Non-convenient covering) Take the covering $\mathbb{C}_{xyz}^3 \supset X := V(z^p - x) \rightarrow \mathbb{C}_{xy}^2$ and the parameterized arcs $\gamma_1(t) = (0, t)$, $\gamma_2(t) = (t^l, t)$ for $1 < l \leq p$. Then the only liftings are $\gamma_1^X(t) = (0, t, 0)$ and $\gamma_2^X(t) = (t^l, t, \omega \cdot t^{\frac{l}{p}})$, where $\omega^p = 1$. And $1 = \text{tord}_{\mathbb{C}^3}(\gamma_1^X, \gamma_2^X)$. Thus $1 = \text{tord}_X(\gamma_1^X, \gamma_2^X) < \text{tord}_{\mathbb{C}^2}(\gamma_1, \gamma_2) = l$.
- ii. (Non-finite map) Let $\mathbb{C}_{xyz}^3 \supset V(z y) \rightarrow \mathbb{C}_{xy}^2$. Take $\gamma_1(t) = (t, o)$ and $\gamma_2(t) = (t, t^2)$, thus $\text{tord}_{\mathbb{C}^2}(\gamma_1, \gamma_2) = 2$. For the lifting $\gamma_1^X(t) = (t, 0, t)$ there does not exist γ_2^X satisfying: $\text{tord}_{\mathbb{C}^3}(\gamma_1^X, \gamma_2^X) > 1$.
- iii. The conclusion cannot be strengthened to $\text{tord}_X(\gamma_1^X, \gamma_2^X) = \text{tord}_{\mathbb{C}^n}(\gamma_1, \gamma_2)$. For example, let $f(x, y, z) = z^2 + x(x^p + y^p)$, $p > 1$. The projection $\mathbb{C}^3 \supset V(f) \rightarrow \mathbb{C}_{xy}^2 \supset \Delta$ is ramified over $\Delta = V(x(x^p + y^p))$. Take two arcs, $\gamma_1 = (0, t) \subset \Delta$ and $\gamma_2 = (t^q, t)$, with $q > p$. Then

$$\text{tord}_{\mathbb{C}^n}(\gamma_1, \gamma_2) = q, \quad \gamma_1^X = (0, t, 0), \quad \gamma_2^X = (t^q, t, \pm \sqrt{t^{q+p}(1 + t^{q-p})}).$$

Therefore $\text{tord}_X(\gamma_1^X, \gamma_2^X) \leq \frac{q+p}{2} < q$.

2.6. Fast cycles. A fast cycle is (roughly speaking) a thin germ that does not admit a hornic deformation-retraction onto a non-thin germ.

Definition 2.10. Fix a (subanalytic) subgerm $Y \subset X$.

1. A subanalytic neighborhood $Y \subseteq \mathcal{U}(Y) \subseteq X$ is hornic if $\text{diff.ord}_X(Y, \mathcal{U}(Y)) > 1$, see (1).
2. Y is called linkwise retractible to a subgerm $Y' \subset Y$ if there exists a deformation-retraction $Y \rightsquigarrow Y'$ (inside X) that retracts the links, i.e. $\text{Link}_t[Y] \rightsquigarrow \text{Link}_t[Y']$ for all $0 < t \ll 1$.
3. A thin subgerm $Y \subset X$ is called a fast cycle if Y does not admit a hornic neighborhood, $Y \subseteq \mathcal{U}(Y) \subset X$, that linkwise retracts to a subgerm $Y' \subset \mathcal{U}(Y)$ with $\dim_{\mathbb{R}}(Y') = \dim_{\mathbb{R}}(T_{Y'}) \leq \dim_{\mathbb{R}} T_Y$.
4. Let $Y \subset X$ be a fast cycle. Suppose Y admits a hornic linkwise-retraction to $Y' \subset X$, with $\dim_{\mathbb{R}} Y' = 1+l$, and this l is the minimal possible. Then Y is called a fast cycle of dimension l .
A fast cycle whose link is S^1 is called a fast loop.
5. The exponent of a fast cycle is $\text{diff.ord}_X(Y, T_Y)$, see (1).

A remark: these properties depend on the (subanalytic) inner-Lipschitz type of X only, not on a particular embedding $X \hookrightarrow (\mathbb{R}^N, o)$.

Example 2.11. i. If a (subanalytic) neighborhood, $Y \subset \mathcal{U}(Y)$, is hornic, then $T_Y = T_{(\mathcal{U}(Y), o)}$. In particular, if $\dim_{\mathbb{R}} \mathcal{U}(Y) > \dim_{\mathbb{R}} Y$, then $\mathcal{U}(Y)$ is a thin set.

ii. A simple fast loop is the horn $Y = \{x^2 + y^2 = z^{2\beta}, z \geq 0\} \subset \mathbb{R}^3$, with $\beta > 1$, considered inside $X = \text{Ball}_1(o) \setminus \{(0, 0, t) \mid t \geq 0\}$. Note that $\text{Link}[Y]$ is contractible inside $\text{Link}[X]$, but not hornically.

A simple fast cycle with $\dim_{\mathbb{R}} T_Y > 1$ is the cylinder over the horn, $Y = \{x_1^2 + x_2^2 = x_3^{2\beta}, x_3 \geq 0\} \subset \mathbb{R}^n$, considered inside the cylinder $X = \mathbb{R}^{n-3} \times [\text{Ball}_1(o) \setminus \{x_1 = 0 = x_2, x_3 \geq 0\}]$.

iii. Suppose X is a hornic neighborhood of its arc γ , and moreover, X linkwise retracts to γ . Then X has no fast cycles.

For non-hornic neighborhoods X can linkwise retract to γ and yet can have fast cycles. See example ii.

iv. If $\pi_1(\text{Link}[X]) \neq \{1\}$ then one can expect fast loops on X . However, fast loops/cycles exist also for exotic spheres, with $\text{Link}[X] \cong S^{2n-1}$, see Example 4.8 of [Kerner-Mendes.23].

Lemma 2.12. [Kerner-Mendes.23, §2] *An IMC-germ (subanalytic, with closed link) has no fast cycles.*

2.6.1. For normal complex-analytic surface germs the converse of Lemma 2.12 holds:

Theorem 7.5 and Corollary 1.8 of [B.N.P.14]: *If X does not contain fast loops, then X is IMC.*

For non-normal surface germs the fast loops are not the only IMC-obstructions.

Example 2.13. Let $X = V(xz(x - y^2)) \subset (\mathbb{C}^3, o)$. It has no fast loops. But X is not IMC.

Proof. The locus $\text{Sing}(X)$ consists of 2 lines and a parabola. At each point of $\text{Sing}(X)$ the set X is not a topological manifold. If X is inner-Lipschitz equivalent to $\text{Cone}[\text{Link}[X]] \subset (\mathbb{R}^N, o)$, then $\text{Sing}(X)$ must be sent to $\text{Sing}[\text{Cone}[\text{Link}[X]]] = \text{Cone}[\text{Sing}[\text{Link}[X]]]$. But the parabola $V(z, x - y^2) \subset \text{Sing}(X)$ is inner tangent to the line $V(z, x)$. Hence their images in $\text{Cone}[\text{Sing}[\text{Link}[X]]]$ must be tangent as well. This gives the contradiction. ■

3. DETECTING FAST LOOPS VIA THE DISCRIMINANT OF COVERING

3.1. The covering data. Take a reduced complex-analytic germ $X \subset (\mathbb{C}^{n+c}, o)$, possibly with non-isolated singularity, not necessarily a complete intersection.

Take a convenient covering, of degree p . Take the discriminant Δ with its reduced structure. Below $n = 2$, and we assume:

$\Delta \subset (\mathbb{C}^2, o)$ is a curve germ, see §2.2.

$$\begin{array}{ccc} (\mathbb{C}^{n+c}, o) & \supset & X \supset \text{Crit}(\pi) \\ & \searrow \swarrow \pi & \swarrow \pi| \\ (\mathbb{C}^n, o) & \supset & \Delta = \underbrace{\bigcup \Delta_k}_{\text{for } n=2} \end{array}$$

- Take the tangential decomposition $\Delta = \bigcup \Delta_k$. Here T_{Δ_k} is one (possibly multiple) complex line, and $T_{\Delta_k} \cap T_{\Delta_i} = o$ for $i \neq k$. (But each Δ_k can be further reducible.)

Usually we assume that Δ is not an ordinary multiple point. Thus $\text{mult}(\Delta_k) \geq 2$ for some k .

- For each k with $\text{mult}(\Delta_k) \geq 2$ we take the irreducible decomposition into (complex) branches, $\Delta_k = \bigcup \Delta_{k,i}$. They are all tangent, their common tangent (complex) line is T_{Δ_k} .

For each $\Delta_{k,i}$ let q_i be the total ramification index of the covering $\pi : X \rightarrow (\mathbb{C}^2, o)$ over $\Delta_{k,i}$. Namely, $q_i = p - \sharp|\pi^{-1}(x)|$ for $x \in \Delta_{k,i} \setminus o$.

For generic projections, and X with an isolated singularity, the ramification is minimal, $\{q_i = 1\}$. But we often take convenient non-generic projections.

- Blowup the origin, take the strict transforms and exceptional divisors, see the diagram.
 Note that the curve germs $\{\tilde{\Delta}_j\}$ are now disjoint.
 Fix some Δ_k , its tangent cone T_{Δ_k} is just one line.
 Thus the intersection $\tilde{\Delta}_k \cap \mathbb{P}^1$ is just one point.
 Denote it by o_k . The fibre $\tilde{\pi}^{-1}(o_k) \subset \tilde{X}$ is a finite set of points, of cardinality $\#\tilde{\pi}^{-1}(o_k) \leq p$.
 Some of them can be not on a critical locus.
- Accordingly $(\tilde{X}, \tilde{\pi}^{-1}(o_k))$ is a (reduced) multi-germ. The number of its connected components is just $\#\tilde{\pi}^{-1}(o_k)$. Each connected component of $(\tilde{X}, \tilde{\pi}^{-1}(o_k))$ can be further reducible (even if X is irreducible), see Example 3.1. Denote the total number of the connected components of the multi-germ $(\tilde{X}, \tilde{\pi}^{-1}(o_k)) \setminus \mathbb{P}T_X$ by r_k . Thus $1 \leq \#\tilde{\pi}^{-1}(o_k) \leq r_k \leq p$.
 In Step 2 of the proof of theorem 3.2 we identify r_k also as the number of conected components of $\pi^{-1}Ball$, where $Ball \subset \mathbb{C}^2$ is a small section transversal to some Δ_k .

$$\begin{array}{ccc} \mathbb{P}^{1+c} \subset Bl_o(\mathbb{C}^{2+k}, o) \supset \tilde{X} \supset \tilde{\pi}^{-1}(o_k) & & \\ & \nwarrow \tilde{\pi} & \searrow \\ \mathbb{P}^1 \subset Bl_o(\mathbb{C}^2, o) \supset \tilde{\Delta} = \coprod \tilde{\Delta}_j \ni o_k & & \end{array}$$

Example 3.1. i. If X has an isolated singularity, then $\tilde{X} \setminus \mathbb{P}T_X$ is smooth. Then r_k coincides with the total number of irreducible components of the multi-germ $(\tilde{X}, \tilde{\pi}^{-1}(o_k)) \setminus \mathbb{P}T_X$.
 ii. Cases with $r_k > \#\tilde{\pi}^{-1}(o_k)$ occur often, even if X is irreducible, with an isolated singularity. As the simplest case consider the surface germ $X = V(z^2 - f(x, y)) \subset (\mathbb{C}^3, o)$. Suppose $ord(f) = 2d \geq 4$ and moreover $f(x, y)$ contains the monomial x^{2d} . Blowup the origin, the space $Bl_o(\mathbb{C}^3) \subset \mathbb{C}^3 \times \mathbb{P}^2$ is defined by the condition $(x, y, z) \sim [\sigma_x : \sigma_y : \sigma_z]$. Take the chart with $\sigma_x = 1$, its coordinates are x, σ_y, σ_z . (Thus $y = x\sigma_y$, $z = x\sigma_z$.) The strict transform $\tilde{X}$ is then defined by $\sigma_z^2 + x^{2d-2}(1 + g(x, \sigma_y))$, where $g(x, \sigma_y) \in (x, \sigma_y)$. This surface germ is reducible, hence $r_k = 2 > \#\tilde{\pi}^{-1}(o_k) = 1$.

3.2. Fast loop criterion for surface germs. (Keeping the notations of §3.1.)

Theorem 3.2. *The germ X has no fast loops iff one the following holds:*

- either Δ is one point or a curve germ that is an ordinary multiple point (i.e. all its branches are smooth and pairwise non-tangent);
- or Δ is a curve germ and each non-smooth tangential component $\Delta_k \subset \Delta$ (with $mult(\Delta_k) \geq 2$ and the data $\Delta_k = \cup_i \Delta_{k,i}$, q_i , r_k , as above) satisfies: $p > r_k - 1 + \sum_i q_i \cdot mult(\Delta_{k,i})$.

Proof. We realize $Link[X]$ as $X \cap V(|x_1|^2 + |x_2|^2 - \epsilon)$, $0 < \epsilon \ll 1$, by the convenience of the covering.

A fast loop (if any) must lie in a hornic neighborhood of the critical locus, $Crit(\pi) \subset X$. Therefore in the case “ Δ is one point” (and hence $Crit(\pi)$ is a finite set) there can be no fast loops.

Below we assume that the critical locus is a curve germ, and hence also Δ . Then the image of a fast loop must lie in a hornic neighborhood $\mathcal{U}(\Delta_k)$ for some tangential component $\Delta_k \subset \Delta$. Moreover, this image cannot lie in $\mathcal{U}(\Delta_k)$ for a smooth component Δ_k . Indeed, if $mult(\Delta_k) = 1$, then each component of X lying over $\mathcal{U}(\Delta_k)$ is linkwise-contractible, see definition 2.10. Therefore below we consider only tangential components with $ord \Delta_k \geq 2$.

- In Step 1 we show: $Link[X]$ contains a non-contractible loop “over $\mathcal{U}(\Delta_k)$ ” iff X has a fast loop in a hornic neighborhood of $\pi^{-1}\Delta_k$. Here the direction $\Leftarrow$ is immediate.
- In Step 2 we translate the condition “ $Link[X]$ contains a non-contractible loop over $\mathcal{U}(\Delta_k)$ ” into “ $p \leq r_k - 1 + \sum_i q_i \cdot mult(\Delta_{k,i})$ ”.

Step 1. Take the $\hat{x}_1$ -axis as the tangent of $\Delta_k \subset \mathbb{C}_{x_1 x_2}^2$. We work hornically near Δ_k . For each $0 < t_o \ll 1$ take the section $V(x_1 - t_o) \cap \Delta_k$. These are several points, their number is $mult(\Delta_k)$.

As the parameter $t \in \mathbb{R}_{>0}$ goes to zero these points draw the subanalytic arcs, $\{\gamma_i\}_i \subset \Delta_k \subset \mathbb{C}^2$. Each of these arcs has the length-parametrization $(t, g_i(t))$, here $g_i(t) = o(t)$ because $T_{\Delta_k} = Span_{\mathbb{C}}(\hat{x}_1)$. In particular, these arcs $\{\gamma_i\}$ are all tangent

in $(\mathbb{C}^2, o)$. For each pair of these arcs we take the Hölder triangle, as in (7),

$$(8) \quad \text{Höld}[\gamma_i, \gamma_j] := \cup_{s \in [0,1]} [s \cdot \gamma_i + (1-s) \cdot \gamma_j] := \cup_{s \in [0,1]} [t, s \cdot g_i(t) + (1-s) \cdot g_j(t)] \subset \mathbb{C}^2.$$

This defines the tangent foliation on each $\text{Höld}[\gamma_i, \gamma_j]$. Indeed, any two arcs $\gamma_s, \gamma_{s'} \subset \text{Höld}[\gamma_i, \gamma_j]$ intersect at o only and are tangent. (Because $g_i(t) = g_j(t)$ occurs when either $i = j$ or $t = 0$.)

The union $\mathbb{C}^2 \supset \text{Höld} := \cup_{i \neq j} \text{Höld}[\gamma_i, \gamma_j]$ is the “Hölder polyhedron”. This is a subanalytic set and its tangent cone is the real line $\text{Span}_{\mathbb{R}}(\hat{x}_1)$. Its link is connected and can be realized as $\text{Höld} \cap V(x_1 - t_o)$.

The lifted Hölder polyhedron is the full pre-image:

$$(9) \quad \text{Höld}^X := \pi^{-1} \text{Höld} = \cup_{i \neq j} \pi^{-1}(\text{Höld}[\gamma_i, \gamma_j]) \subset X.$$

Puncturing at o we get several connected components, $\text{Höld}^X \setminus o = \coprod_l (\text{Höld}_l^X \setminus o)$. Each Höld_l^X is a (real) subanalytic surface, foliated by the preimages of the foliations on $\text{Höld}[\gamma_i, \gamma_j]$ for all i, j . And $\text{Link}[\text{Höld}_l^X]$ is connected. Lemma 2.6 gives: the tangent cone of each Höld_l^X is a real line, and on each Höld_l^X the arcs of our foliation are inner-tangent.

Inside the complex line $V(x_1 - t_o) \subset \mathbb{C}^2$ take an open ball Ball satisfying:

$$(10) \quad \Delta_k \cap V(x_1 - t_o) \subset \text{Ball} \quad \text{and} \quad (\Delta \setminus \Delta_k) \cap \text{Ball} = \emptyset.$$

Such a ball exists for $0 < t_o \ll 1$ as every branch of $\Delta \setminus \Delta_k$ is non-tangent to Δ_k .

Suppose the full preimage $\pi^{-1} \text{Ball} \subset X \cap V(x_1 - t_o)$ contains a (compact, connected) non-contractible loop Z . By deforming/shrinking $Z \subset \pi^{-1} \text{Ball}$ we can assume $\pi(Z) \subset \text{Höld} \cap \text{Ball}$. Indeed, fix a simply-connected path $\eta \subset \text{Höld} \cap \text{Ball}$ that visits all the points of $\Delta_k \cap V(x_1 - t_o)$. The retraction $\text{Ball} \rightsquigarrow \eta$ lifts to $\pi^{-1} \text{Ball} \rightsquigarrow \pi^{-1} \eta$.

Therefore Z lies inside $\text{Höld}^X \cap V(x_1 - t_o)$. Then it lies inside a connected component, $Z \subset \text{Höld}_l^X \cap V(x_1 - t_o)$ for some l . Finally, one can assume: Z is subanalytic, §1.5.v.

Using the (inner-tangent) foliation on that component Höld_l^X we get the subanalytic surface germ

$$(11) \quad \mathbb{R}_{>0} Z := \cup_{z \in Z} \gamma_z \subset \text{Höld}_l^X \subset X.$$

Its tangent cone is a real half-line, it lies inside the tangent cone of Höld_l^X . Thus $\mathbb{R}_{>0} Z$ is a thin germ. By our construction the germ $\mathbb{R}_{>0} Z$ is not linkwise-retractible inside $\pi^{-1} \text{Cone}[\text{Ball}]$, see §2.6. Hence it is not linkwise retractible inside any hornic neighborhood of $T_{\mathbb{R}_{>0} Z}$. Therefore $\mathbb{R}_{>0} Z$ is the promised fast loop.

Step 2. It remains to check whether a non-contractible (connected, compact) loop $Z \subset \pi^{-1} \text{Ball}$ exists. Namely (in terms of the first homology), whether $h^1(\pi^{-1} \text{Ball}) > 0$. Therefore it remains to show the equivalence:

$$(12) \quad h^1(\pi^{-1} \text{Ball}) > 0 \quad \text{if and only if} \quad p \leq r_k - 1 + \sum_i q_i \cdot \text{mult}(\Delta_{k,i}).$$

Thus we compute the Euler characteristic $\chi(\pi^{-1} \text{Ball})$ and the number of connected components $h^0(\pi^{-1} \text{Ball})$.

- The map $X \supset \pi^{-1} \text{Ball} \xrightarrow{p:1} \text{Ball} \subset V(x_1 - t_o) \subset \mathbb{C}^2$ is a covering of complex analytic curves. If X has a non-isolated singularity then the complex-analytic curve $\pi^{-1}(\text{Ball})$ can be singular. The covering is ramified over the discriminantal points $\Delta_k \cap V(x_1 - t_o)$. For the branch decomposition $\Delta_k = \cup \Delta_{k,i}$ we get $\#|\Delta_{k,i} \cap V(x_1 - t_o)| = \text{mult}(\Delta_{k,i})$, and these points are of ramification index q_i . Riemann-Hurwitz formula applied to this covering gives:

$$(13) \quad \chi(\pi^{-1} \text{Ball}) = p \cdot \chi(\text{Ball}) - \sum q_j = p - \sum q_i \cdot \text{mult}(\Delta_{k,i}).$$

- We compute the number of connected components: $h^0(\pi^{-1}Ball) = r_k$. Blowup the origin to get the diagram, as in §3.1. Take the intersection point $o_k := \tilde{\Delta} \cap \mathbb{P}^1 \cap \widehat{Cone[Ball]} = \tilde{\Delta}_k \cap \mathbb{P}^1$. Take the pre-images $\tilde{\pi}^{-1}(o_k) \subset \mathbb{P}T_X \subset \tilde{X}$.

We get the multi-germ $(\tilde{X}, \tilde{\pi}^{-1}(o_k))$. By definition, r_k is the number of connected components of $(\tilde{X}, \tilde{\pi}^{-1}(o_k)) \setminus \mathbb{P}T_X =: \coprod \tilde{Y}_l$. Here each $\tilde{Y}_l$ is a connected (locally-closed) complex-analytic surface.

Starting from $Ball \subset V(x_1 - t_o) \subset \mathbb{C}^2$, we get $(\sigma \circ \tilde{\pi})^{-1}Ball \subset V(\widetilde{x_1 - t_o}) \cap \tilde{X}$. As we work with the multi-germ $(\tilde{X}, \tilde{\pi}^{-1}(o_k))$ we can assume: $(\sigma \circ \tilde{\pi})[V(x_1 - t_o) \cap (\coprod \tilde{Y}_l)] \subset Ball$. Therefore we can ignore all the components of $\Delta \setminus \Delta_k$, i.e. we can assume $\Delta = \Delta_k$.

It is enough to prove: $\tilde{Y}_l \cap ((\sigma \circ \tilde{\pi})^{-1}Ball)$ is connected for each l . This will imply: the number of connected components of $\coprod \tilde{Y}_l \cap ((\sigma \circ \tilde{\pi})^{-1}Ball)$ coincides with the number of connected components of $\coprod \tilde{Y}_l$. And therefore:

$$(14) \quad h^0(\pi^{-1}Ball) = h^0((\sigma \circ \tilde{\pi})^{-1}Ball) = h^0((\coprod \tilde{Y}_l) \cap ((\sigma \circ \tilde{\pi})^{-1}Ball)) = h^0(\coprod \tilde{Y}_l) = r_k.$$

Fix two points, $\tilde{p}_1, \tilde{p}_2 \in \tilde{Y}_l \cap (\sigma \circ \tilde{\pi})^{-1}Ball$. We can assume (w.l.o.g.) $\tilde{p}_1, \tilde{p}_2 \notin Crit(\tilde{\pi})$. Take a path $[\tilde{p}_1 \xrightarrow{\tilde{\eta}} \tilde{p}_2] \subset \tilde{Y}_l$. We want to deformation-retract $\tilde{\eta}$ into a path inside $\tilde{Y}_l \cap (\sigma \circ \tilde{\pi})^{-1}Ball$. Take the image $\eta := (\sigma \circ \tilde{\pi})(\tilde{\eta}) \subset \mathbb{C}^2 \setminus \{o\}$. It connects the points $\sigma(\tilde{\pi}(\tilde{p}_1)), \sigma(\tilde{\pi}(\tilde{p}_2)) \in Ball$.

- i. Suppose $\tilde{\eta} \cap Crit(\tilde{\pi}) = \emptyset$. Then we can assume $\eta \cap \Delta = \emptyset$, i.e. $\eta \subset \mathbb{C}^2 \setminus \Delta$. Then η can be deformation-retracted (inside $\mathbb{C}^2 \setminus \Delta$, and preserving the end-points) to a path inside $Ball \setminus (Ball \cap \Delta)$.

This deformation lifts uniquely to the deformation of $\tilde{\eta}$ inside $\tilde{X} \setminus Crit(\tilde{\pi})$, while preserving the points $\tilde{p}_1, \tilde{p}_2$. The deformed path, $[\tilde{p}_1 \xrightarrow{\tilde{\eta}_1} \tilde{p}_2]$ lies fully inside $\tilde{Y}_l \cap (\sigma \circ \tilde{\pi})^{-1}Ball$.

- ii. The path $\tilde{\eta}$ can happen to intersect the critical locus $Crit(\tilde{\pi})$. Moreover, $\tilde{\eta}$ can be non-deformable off $Crit(\tilde{\pi})$. E.g. let Y_l be connected but reducible, with $\tilde{p}_1, \tilde{p}_2$ lying in distinct components. Then $\tilde{\eta}$ inevitably intersects the locus $Sing(Y_l) \subseteq Crit(\tilde{\pi})$.

We claim: $\tilde{\eta}$ can be deformed (inside $\tilde{Y}_l$) to a path $[\tilde{p}_1 \xrightarrow{\tilde{\eta}_1} \tilde{p}_2]$ satisfying: $\tilde{\eta}_1 \cap Crit(\tilde{\pi}) \subseteq (\sigma \circ \pi)^{-1}Ball \cap \tilde{Y}_l$. Indeed, we can assume (w.l.o.g) that $\tilde{\eta}_1 \cap Crit(\tilde{\pi})$ is a finite number of points. Therefore the set $\eta \cap \Delta$ is finite. Over each such intersection point we can slightly deform $\tilde{\eta}$ to get: the intersection $\eta \cap \Delta$ is transverse.

Now we slide-move all the points of $\eta \cap \Delta$ (along Δ) into $Ball \cap \Delta$. This deformation lifts uniquely to a deformation of $\tilde{\eta}$. (Because $\Delta \setminus o$ is a smooth curve and $\eta \pitchfork \Delta$.)

Finally, having reached the path with $\tilde{\eta}_1 \cap Crit(\tilde{\pi}) \subseteq (\sigma \circ \pi)^{-1}Ball \cap \tilde{Y}_l$, one applies part i. to all the connected components of $\tilde{\eta}_1 \setminus (\tilde{\eta}_1 \cap Crit(\tilde{\pi}))$.

We have proved: $h^1(\pi^{-1}Ball) = r_k - p + \sum q_i \cdot mult(\Delta_{k,i})$. This gives equation (12) Hence the statement. ■

Combining this theorem with §2.6.1, we get:

Corollary 3.3. *Suppose X is a normal surface germ and Δ is a curve germ. Then X is IMC iff all the non-smooth tangential components $\Delta_k \subset \Delta$ (with $mult(\Delta_k) \geq 2$ and the data $\Delta_k = \cup_i \Delta_{k,i}$, q_i , r_k , as above) satisfy: $p > r_k - 1 + \sum_i q_i \cdot mult(\Delta_{k,i})$.*

3.2.1. *An example: $X \subset (\mathbb{C}^{2+c}, o)$ is weighted-homogeneous, with weights $\omega_1 \leq \omega_2 \leq \dots \leq \omega_{2+c}$.*

Corollary 3.4. *Suppose $\omega_1 = \omega_2$ and the projection $X \rightarrow (\mathbb{C}_{x_1x_2}^2, o)$ is convenient. Then X has no fast loops. If, moreover, X is a normal surface germ then X is IMC.*

A remark, if $\omega_1 < \omega_2$, then X necessarily has a fast loop, [Kerner-Mendes.23, §4].

Proof. The discriminant $\Delta \subset (\mathbb{C}^2, o)$ of the projection $X \rightarrow (\mathbb{C}_{x_1x_2}^2, o)$ is weighted-homogeneous with weights ω_1, ω_2 . Hence it is homogeneous, i.e. an ordinary multiple point. Now invoke (trivially) theorem 3.2. ■

3.2.2. *An example: surface germs of multiplicity 2.* Such a germ is necessarily a surface in $(\mathbb{C}^3, o)$. Thus we start from $X = V(z^2 - a_o(x, y)) \subset (\mathbb{C}^3, o)$, where $\text{ord}(a_o) \geq 2$.

Corollary 3.5. *X has no fast loops iff the curve germ $V(a_o(x, y)) \subset (\mathbb{C}^2, o)$ is set-theoretically an ordinary multiple point.*

For X with an isolated singularity this gives the IMC criterion.

Proof. The covering $X \rightarrow (\mathbb{C}^2, o)$ is totally ramified over the discriminantal curve, $\Delta := V(a_o(x, y)) \subset (\mathbb{C}^2, o)$, taken set-theoretically. If Δ is an ordinary multiple point, then there is nothing to check in theorem 3.2. Otherwise Δ contains a non-smooth tangential component, Δ_k , with $\text{mult}(\Delta_k) \geq 2$. Then $p = 2 \leq \text{mult}(\Delta_k) \cdot (2 - 1) + (r_k - 1)$. Hence X has a fast loop. ■

Example 3.6. (IMC's of multiplicity 2 and right-modality ≤ 2)

- Among the simple types (i.e. ADE's) the only IMC's are: A_1 ($x^2 + y^2 + z^2$) and D_4 ($z^2 + x^3 + y^3$). This is well known, see e.g. [B.F.N.08].
- The only non-simple IMC germ of multiplicity 2 and right-modality ≤ 2 is X_9 ($z^2 + x^4 + y^4 + a \cdot x^2y^2$), see the tables in [A.G.L.V.I, pg.24]. The next case, $z^2 + x^5 + y^5 + a_{3,2}x^3y^2 + a_{2,3}x^2y^3 + a_{3,3}x^3y^3$, has modality ≥ 3 .

3.2.3. *An example: germs of type $V(z^p + za_1(x, y) + a_0(x, y)) \subset (\mathbb{C}^3, o)$, $p \geq 3$.* Here $\text{ord}(a_o) \geq p$, $\text{ord}(a_1) \geq p - 1$. By §2.4 we get a simple expression for the discriminant, $\Delta = (\frac{a_0(x, y)}{p-1})^{p-1} - (\frac{a_1(x, y)}{p})^p \in \mathbb{C}\{x, y\}$. This can be not square-free. Denote its reduced version by Δ_{red} . Take the lowest order terms, $\text{l.t.}(\Delta)$, resp. $\text{l.t.}(\Delta_{\text{red}})$. These homogeneous polynomials define the tangent-cone of $V(\Delta)$, resp. of $V(\Delta_{\text{red}})$. Blowup the origin, $\mathbb{P}^2 \subset \text{Bl}(\mathbb{C}^3, o) \rightarrow (\mathbb{C}^3, o)$, and take the strict transform $\tilde{X} \rightarrow X$.

Corollary 3.7. 1. *Suppose $a_1 = 0$. Then X has no fast loops iff $V(a_o) \subset (\mathbb{C}^2, o)$ is set-theoretically an ordinary multiple point.*

2. *Suppose $a_1 \neq 0$. Factorize the lowest order terms, $\text{l.t.}(\Delta) = \prod l_k(x, y)^{m_k}$, resp. $\text{l.t.}(\Delta_{\text{red}}) = \prod l_k(x, y)^{m_{k,\text{red}}}$. Then X has no fast loops iff for each k with $m_{k,\text{red}} \geq 2$ the following conditions hold:*

- i. $m_k \leq p$.
- ii. *The total number of irreducible components of the analytic germ $\tilde{X}$ at all the points of $\tilde{X} \cap \mathbb{P}^2 \cap \widehat{V(l_k)}$ is at most $p - m_k$.*

A remark: in most cases already the condition 2.i. is violated, i.e. one has $m_k > p$.

Proof. In both cases we should verify the bound $p > r_k - 1 + \sum_i q_i \cdot \text{mult}(\Delta_{k,i})$ of theorem 3.2.

1. If $a_1 = 0$ then the covering $X \rightarrow (\mathbb{C}^2, o)$ is totally ramified over the curve $V(a_o(x, y)) \subset (\mathbb{C}^2, o)$. The total ramification index is then $q_i = p - 1$, and $r_k \leq p$. Then the bound holds iff $\text{mult}(\Delta_k) = 1$ for all k .
2. In this case $m_{k,\text{red}} = \text{mult}(\Delta_k)$ and $\sum_i q_i \cdot \text{mult}(\Delta_{k,i}) = m_k$. Thus the necessary and sufficient condition to be IMC, i.e. $p > r_k - 1 + m_k$, means exactly the conditions i. and ii. ■

Already in this (very particular) case we get a vast amount of IMC's.

Example 3.8. Let $X = V(z^p + za_1(x, y) + a_0(x, y)) \subset (\mathbb{C}^3, o)$, with $p \geq 3$, and an isolated singularity. Suppose the power series $a_0(x, y), a_1(x, y)$ are co-prime.

- Suppose $\frac{\text{ord}(a_1(x, y))}{p-1} > \frac{\text{ord}(a_0(x, y))}{p}$. Then $l.t.(\Delta) = l.t.(a_0(x, y)^{p-1})$. Thus X is IMC iff the (set-theoretic) curve germ $V(a_0(x, y)) \subset (\mathbb{C}^2, o)$ is an ordinary multiple point.
- Suppose $\frac{\text{ord}(a_1(x, y))}{p-1} < \frac{\text{ord}(a_0(x, y))}{p}$. Then $l.t.(\Delta) = l.t.(a_1(x, y)^p)$. Thus X is IMC iff the (set-theoretic) curve germ $V(a_1(x, y)) \subset (\mathbb{C}^2, o)$ is an ordinary multiple point.

3.3. Fast loops imposed by singular points of the tangent cone. Suppose a surface germ $X \subset (\mathbb{C}^{2+c}, o)$ is a strictly complete intersection with *isolated* singularity, §2.1.2. The projectivized tangent cone is then a (globally) complete intersection curve, $\mathbb{P}T_X \subset \mathbb{P}^{1+c} \subset Bl_o(\mathbb{C}^{2+c}, o)$. It can be reducible, non-reduced, i.e. can have multiple components. Occasionally we take the underlying set-germ (with reduced structure), $\mathbb{P}T_X^{red}$. One has the total degree, $\deg[\mathbb{P}T_X] = p$, and the set-theoretic degree, $\deg[\mathbb{P}T_X^{red}] \leq p$.

Call a point $pt \in \mathbb{P}T_X$ *reduced* if the curve germ $(\mathbb{P}T_X, pt)$ is reduced. Thus $(\mathbb{P}T_X, pt)$ is an ICIS, and its Milnor number, $\mu(\mathbb{P}T_X, pt)$, is well defined.

Call a hyperplane $\mathbb{P}^c \subset \mathbb{P}^{1+c} \subset Bl_o(\mathbb{C}^{2+c}, o)$ *non-tangent* to the curve $\mathbb{P}T_X$ if $\mathbb{P}^c$ is transverse to the tangent cone of the germ $(\mathbb{P}T_X, pt)$ at each intersection point $pt \in \mathbb{P}^c \cap \mathbb{P}T_X$.

- Example 3.9.** i. The generic hyperplane $\mathbb{P}^c \subset \mathbb{P}^{1+c}$ is non-tangent. It intersects $\mathbb{P}T_X^{red}$ transversally and only at smooth points of $\mathbb{P}T_X^{red}$.
- ii. Suppose the germ $(\mathbb{P}T_X, pt)$ is reduced and singular. Among all the hyperplanes through pt , take the generic one. It is non-tangent to $\mathbb{P}T_X$.
- iii. More generally, take some reduced singular points, $pt_1, \dots, pt_j \in \text{Sing}(\mathbb{P}T_X)$, with $j \leq c$. The linear system of hyperplanes through these points has dimension $\geq c + 1 - j$. And the generic member of this family is non-tangent to $\mathbb{P}T_X$ at the points $pt_1, \dots, pt_j$, unless these points are in degenerate position with respect to the curve $\mathbb{P}T_X$.

Theorem 3.10. 1. If X is IMC and the curve $\mathbb{P}T_X$ is reduced then $\mathbb{P}T_X$ is smooth. (I.e. T_X has an isolated singularity, i.e. X is a “generalized ordinary multiple point”.)

2. Suppose a hyperplane $\mathbb{P}^c \subset \mathbb{P}^{1+c}$ is non-tangent to $\mathbb{P}T_X$, and contains reduced singular points $\{pt_i\}$ of the curve $\mathbb{P}T_X$. If their Milnor numbers satisfy $\sum \mu(\mathbb{P}T_X, pt_i) \geq 1 + \deg[\mathbb{P}T_X] - \deg[\mathbb{P}T_X^{red}]$, then X has a fast loop.

In particular, if $\mathbb{P}T_X = \mathbb{P}T_X^{red}$, any (reduced) singular point of $\mathbb{P}T_X \cap \mathbb{P}^c$ produces a fast loop.

The converse to part 1 is Example 3.8 in [Kerner-Mendes.23].

Proof. Part 1 follows straight from Part 2. Part 2 is proved in steps.

- Take the (convenient) projection $\mathbb{C}^{2+c} \rightarrow \mathbb{C}^2$ associated to $\mathbb{P}^c$. Then $\mathbb{P}^c$ is sent to a point $o_k \in \mathbb{P}^1 \subset Bl_o(\mathbb{C}^2, o)$. Take the corresponding tangential component of the discriminant, $\Delta_k \subseteq \Delta$. To use theorem 3.2 we verify the condition $p \leq r_k - 1 + \sum q_i \cdot \text{mult}(\Delta_{k,i})$. First we bound $r_k \geq \sharp \tilde{\pi}^{-1}(o_k)$.
- We express $\sum q_i \cdot \text{mult}(\Delta_{k,i})$ as the multiplicity of the polar curve, and use the formula of §2.4.2.

Step 1. The projection $(\mathbb{C}^{2+c}, o) \xrightarrow{\pi} (\mathbb{C}_{x_1 x_2}^2, o)$ lifts to $Bl_o(\mathbb{C}^{2+c}, o) \supset \mathbb{P}^{1+c} \setminus \widetilde{V(x_1, x_2)} \rightarrow \mathbb{P}^1 \subset Bl_o(\mathbb{C}^2, o)$. By a preliminary $GL(\mathbb{C}^{2+c})$ transformation one can assume:

$$(15) \quad \mathbb{P}^c = \widetilde{V(x_1)} \cap \mathbb{P}^{1+c}, \quad \widetilde{V(x_1, x_2)} \cap \mathbb{P}T_X = \emptyset,$$

the intersection $\widetilde{V(a_1 x_1 + a_2 x_2)} \cap \mathbb{P}T_X$ is finite for all $[a_1 : a_2] \in \mathbb{P}^1$.

Indeed, once the first condition is satisfied, one uses the remaining freedom $x_i \rightarrow x_i + l_i(x_1 \dots x_{2+c})$ for $i = 2, \dots, 2 + c$, and generic linear forms $\{l_i\}$.

Thus π sends $\mathbb{P}^c \setminus \widetilde{V(x_1, x_2)}$ to the point $o_k := [0 : 1] \in \mathbb{P}^1$. We get the diagram:

$$\begin{array}{ccccccc}
(\mathbb{C}^{2+c}, o) & \leftarrow & Bl_o(\mathbb{C}^{2+c}, o) & \supset & \mathbb{P}^{1+c} \\
\cup & & \cup & & \cup \\
(16) \quad Crit(\pi) & \subset & X & \leftarrow & \tilde{X} & \supset & \mathbb{P}T_X \supset \tilde{\pi}^{-1}(o_k) \supseteq \{pt_i\} \\
\downarrow & & \pi \downarrow & & \tilde{\pi} \downarrow & & \downarrow \tilde{\pi}| \\
\Delta & \subset & (\mathbb{C}^2, o) & \leftarrow & Bl_o(\mathbb{C}^2, o) & \supset & \mathbb{P}^1 \ni o_k
\end{array}$$

Here are the needed properties of this diagram.

- i. The projection π is convenient, as $T_X \cap V(x_1, x_2) = (o)$, as $\widetilde{V(x_1, x_2)} \cap \mathbb{P}T_X = \emptyset$.
- ii. The projection $\tilde{\pi}$ is finite, as π is convenient. (By part 1 of Lemma 2.2.)
- iii. The curve $\mathbb{P}T_X \subset \mathbb{P}^{1+c}$ is a complete intersection of the (total) degree p . Therefore, $\tilde{\pi}|$ is a (flat, ramified) $p : 1$ covering. (Part 2 of Lemma 2.2.)
- iv. The fibre $\tilde{\pi}^{-1}(o_k) \subset \mathbb{P}T_X$ contains the reduced singular points $\{pt_i\}$. Its cardinality is bounded, $\sharp \tilde{\pi}^{-1}(o_k) \leq p - \sum_{\tilde{o}_j \in \tilde{\pi}^{-1}(o_k)} (mult(\mathbb{P}T_X, \tilde{o}_j) - 1)$. As the hyperplane $\mathbb{P}^c$ is non-tangent to $\mathbb{P}T_X$, this bound is an equality.

Altogether, we get:

$$\begin{aligned}
(17) \quad r_k \geq \sharp \tilde{\pi}^{-1}(o_k) &= p - \sum_{\tilde{o}_j \in \tilde{\pi}^{-1}(o_k)} (mult(\mathbb{P}T_X, \tilde{o}_j) - 1) = \\
&= p - \sum (mult(\mathbb{P}T_X, pt_i) - 1) - deg[\mathbb{P}T_X] + deg[\mathbb{P}T_X^{red}].
\end{aligned}$$

- v. The projection $Crit(\pi) \rightarrow \Delta$ lifts to the (finite) projection $Crit(\tilde{\pi}) \rightarrow \tilde{\Delta} \cup \mathbb{P}^1$. Here $\tilde{X}$ is locally-complete intersection, thus $Crit(\tilde{\pi})$ as well (part 1 of Lemma 2.3).

We claim: $Crit(\tilde{\pi}) = \widetilde{Crit(\pi)} \cup Non.Red(\mathbb{P}T_X)$, the non-reduced components of $\mathbb{P}T_X$. Indeed, $\tilde{\pi}$ is not a local isomorphism at any point of $Non.Red(\mathbb{P}T_X)$. (Because the local ramification index of $\tilde{\pi}$ at each point of $Non.Red(\mathbb{P}T_X)$ is the multiplicity of $Non.Red(\mathbb{P}T_X)$, hence ≥ 2 .) Vice versa, $Crit(\tilde{\pi})$ is a locally complete intersection curve, thus $\overline{Crit(\tilde{\pi}) \setminus Non.Red(\mathbb{P}T_X)}$ is of pure dimension 1.

Hence, besides $\widetilde{Crit(\pi)}$, it can consist only of some irreducible components of $\mathbb{P}T_X$. But the restriction of $\tilde{\pi}$ to each irreducible, reduced component of $\mathbb{P}T_X$ is a local isomorphism (over $\mathbb{P}^1$) except at a finite number of points.

Step 2. Take the image $\tilde{\Delta}_k := \tilde{\pi}(Crit(\tilde{\pi}), \pi^{-1}(o_k)) \subset (Bl_o(\mathbb{C}^2, o), o_k)$. This is the strict transform of $\Delta_k \subset (\mathbb{C}^2, o)$, a tangential component of Δ .

Take the coordinate change $\Phi : x \rightarrow x + h(x)$, where the terms $h(x) = (h_1(x), \dots, h_{2+c}(x))$ of order two and generic. This lifts to an automorphism $\tilde{\Phi} \circ Bl_o(\mathbb{C}^{2+c}, o)$, with $\tilde{\Phi}|_{\mathbb{P}^{1+c}} = Id$. (Because $\Phi'|_o = Id$.) Therefore $\tilde{\Phi}$ preserves o_k , and all the properties of Step 1 are preserved. And applying Φ the covering $X \rightarrow (\mathbb{C}^2, o)$ becomes generic in the following sense: $q_i = 1$ for all i .

Therefore $\sum_i q_i \cdot mult(\Delta_{k,i}) = mult(\Delta_k, o)$, and we compute this multiplicity.

Pass from $(\mathbb{C}^2, o)$ to $Bl_o(\mathbb{C}^2, o)$ by $mult(\Delta_k) = \langle \tilde{\Delta}_k, \mathbb{P}^1 \rangle$. By our choice of coordinates, $o_k = [0 : 1] \in \mathbb{P}^1$, i.e. the tangent line of Δ_k is $Span(\hat{x}_2)$. Thus at any point of $\tilde{\pi}^{-1}(o_k)$ the exceptional divisor is defined locally by $x_2 = 0$. Then $\langle \tilde{\Delta}_k, \mathbb{P}^1 \rangle = \langle \tilde{\Delta}_k, V(x_2 - t_o) \rangle$, as $\tilde{\Delta}_k \subset Bl_o(\mathbb{C}^2, o)$ is a hypersurface. Take the multi-germ $Crit_k(\tilde{\pi}) := (Crit(\tilde{\pi}), \tilde{\pi}^{-1}(o_k))$. Then

$$(18) \quad mult(\Delta_k) = \langle Crit_k(\tilde{\pi}), \widetilde{V(x_2 - t_o)} \rangle \geq \langle \overline{Crit_k(\tilde{\pi}) \setminus \mathbb{P}T_X}, \mathbb{P}^{1+c} \rangle \geq \sum \langle (Crit(\tilde{\pi}), pt_i), \mathbb{P}^{1+c} \rangle.$$

Both inequalities hold here as $Crit(\tilde{\pi})$ (with its scheme structure) is a locally complete intersection (lemma 2.3).

At the chosen reduced singular points, $pt_i \in \mathbb{P}T_X \cap \mathbb{P}^c$, one has: $\langle Crit(\tilde{\pi}), \mathbb{P}^{1+c} \rangle_{pt_i} = \mu_i + mult(\mathbb{P}T_X, pt_i) - 1$. (Apply §2.4.2 to the reduced curve germ with $(\mathbb{P}T_X, pt)$.)

Altogether we have: $\text{mult}(\Delta_k) \geq \sum (\mu_i + \text{mult}(\mathbb{P}T_X, pt_i) - 1)$. In particular $\text{mult}(\Delta_k) \geq 2$.

Combine this with (17) to get: $r_k + \text{mult}(\Delta_k) - p \geq \sum \mu_i - \deg[\mathbb{P}T_X] + \deg[\mathbb{P}T_X^{\text{red}}]$. Together with the assumption $\sum \mu_i \geq 1 + \deg[\mathbb{P}T_X] - \deg[\mathbb{P}T_X^{\text{red}}]$ we get: $r_k + \sum q_i \cdot \text{mult}(\Delta_{k,i}) \geq p + 1$. Hence the statement. $\blacksquare$

Example 3.11. i. For any super-isolated surface singularity in $(\mathbb{C}^3, o)$ the projectivized tangent cone is reduced. In this case X is IMC iff it is an ordinary multiple point. This was obtained in [B.N.P.14], example 15.1.

ii. Suppose a strictly complete intersection surface germ (with isolated singularity) is both LNE and IMC. Then X is an ordinary multiple point. In particular, $X \cong (T_X, o)$.

Indeed, T_X of an LNE is reduced, [Fernandes.Sampaio.19]. Thus $\mathbb{P}T_X$ is smooth.

Remark 3.12. Part 2 of this theorem does not hold for germs with non-isolated singularities. E.g. let $X \subset (\mathbb{C}^3, o)$ be an arrangement of planes (through the origin). Then X is an IMC and LNE, and $\mathbb{P}T_X$ is reduced but non-smooth.

3.3.1. *An example: IMC's of right modality ≤ 2 .* (Continuing example 3.6.)

Corollary 3.13. *Among the surface singularities in $(\mathbb{C}^3, o)$ of right modality ≤ 2 the only IMC's are: A_1, D_4, P_8 ($x^3 + y^3 + z^3 + a \cdot xyz$), and X_9 ($z^2 + x^4 + y^4 + a \cdot x^2 y^2$).*

Proof. The case of corank=2 was treated in §3.2.2. Thus we consider only the cases of corank=3. Their tables are listed e.g. in [A.G.L.V.I, pg. 24]. In particular, the only ordinary multiple points of modality ≤ 2 are D_4, P_8 .

Suppose X is IMC and not an ordinary multiple point. Then the curve $\mathbb{P}T_X$ must have multiple components. But all the corank=3 surface singularities of modality ≤ 2 have reduced tangent cone. (By the direct check of those tables.) $\blacksquare$

3.4. **Detecting fast loops in higher dimensions, $n \geq 3$.** Let $X \subset (\mathbb{C}^n, o)$ be as in §1.5.i, with an arbitrary singularity. Take a convenient covering, $(\mathbb{C}^n, o) \supset X \rightarrow (\mathbb{C}^n, o) \supset \Delta$. We assume $\dim_{\mathbb{C}} \Delta = n - 1$, e.g. this holds if X is a complete intersection.

Fix a smooth surface germ $\mathcal{S} \subset (\mathbb{C}^n, o)$, and (set-theoretically) restrict the covering, $X \cap \pi^{-1}\mathcal{S} \xrightarrow{\pi|_{\mathcal{S}}} \mathcal{S} \subset (\mathbb{C}^n, o)$. Then $X \cap \pi^{-1}\mathcal{S}$ is a (pure dimensional, possibly reducible) surface, and the projection $\pi|_{\mathcal{S}}$ is still a convenient covering (see §2.2), of degree p . Now $\pi|_{\mathcal{S}}$ is ramified over the curve germ $\Delta|_{\mathcal{S}} := \Delta \cap \mathcal{S}$. As in §3.1 we take a tangential component, $\Delta|_{\mathcal{S}} \rightsquigarrow \Delta_k = \cup(\Delta_{k,i}, o)$, and the corresponding covering data $r_k, \{q_i\}, \{\text{mult}(\Delta_{k,i})\}$.

Observe that the embedded topological type of the planar curve germ $\Delta|_{\mathcal{S}} \subset \mathcal{S} \cong (\mathbb{C}^2, o)$ can vary when the section $\mathcal{S} \subset (\mathbb{C}^n, o)$ varies.

Proposition 3.14. *Suppose the section $\Delta|_{\mathcal{S}}$ has a tangential component Δ_k with $\text{mult}(\Delta_k) \geq 2$. Suppose any deformation $\mathcal{S}(t)$ of the surface $\mathcal{S} = \mathcal{S}(o)$ that preserves the tangent plane $T_{\mathcal{S}} \subset \mathbb{C}^n$ results in an equisingular family of the planar curve germs, $\Delta_k(t) \subset \mathcal{S}(t)$. If $p \leq r_k - 1 + \sum_i q_i \cdot \text{mult}(\Delta_{k,i})$ is satisfied for $\mathcal{S}(0)$, then X has a fast loop.*

Here $t \in \mathbb{R}^1$ in the family $\mathcal{S}(t)$, i.e. t is not necessarily small.

Proof. Restrict the covering as in the assumption, $X \supset X_{\mathcal{S}} \rightarrow \mathcal{S} \supset \Delta|_{\mathcal{S}}$. This is still a convenient covering. As in the proof of theorem 3.2 we get a fast loop $\mathbb{R}_{>0}Z \subset X_{\mathcal{S}}$. We show that it is a fast loop also inside X .

Any deformation of $\mathcal{S}$ that preserves the tangent plane $T_{\mathcal{S}}$ induces an equisingular family $\Delta_k(t)$. Thus it induces a family of fast loops, $\mathbb{R}_{>0}Z(t) \subset X_{\mathcal{S}(t)}$, with no hornic link-wise retraction, §2.6. Thus $\mathbb{R}_{>0}Z$ could be possibly link-wise retractible inside X only in a family with varying tangent planes. But such a retraction is non-hornic. Therefore $\mathbb{R}_{>0}Z$ is a fast loop also

inside X , by §2.6. ■

Remark 3.15. The assumption of equisingularity of $\Delta_k(t) \cap \mathcal{S}(t)$ is important. Without it one could start, e.g. from a homogeneous $\Delta \subset (\mathbb{C}^n, o)$, and take an intersection with a tangent smooth surface, to get a singular branch of high multiplicity. This would ensure a germ that is a fast cycle inside $X \cap \pi^{-1}\mathcal{S}$, but admits a hornic linkwise retraction inside X .

As a corollary we get: points of high multiplicity of $\mathbb{P}T_\Delta \subset \mathbb{P}^{n-1} \subset Bl_o(\mathbb{C}^n, o)$ ensure fast loops in X .

Corollary 3.16. *Let $X \subset (\mathbb{C}^{n+c}, o)$ be a complete intersection of dimension $n \geq 3$ and multiplicity p . Take a convenient covering $X \rightarrow (\mathbb{C}^n, o) \supset \Delta$ and suppose the total ramification index over each point of Δ is at least q , see §3.1. Suppose a point $o_k \in \mathbb{P}T_\Delta$ satisfies: $\text{mult}(\mathbb{P}T_\Delta, o_k) \geq \frac{p}{q}$ and $\text{mult}(\mathbb{P}T_\Delta, o_k) > \text{mult}(\tilde{\Delta}, o_k)$. Then X has a fast loop.*

Proof. To apply proposition 3.14 we blowup $Bl_o(\mathbb{C}^n, o) \xrightarrow{\sigma} (\mathbb{C}^n, o)$ and fix a smooth ($\mathbb{C}$ -analytic) surface germ $\mathcal{S} \subset (\mathbb{C}^n, o)$ satisfying:

$$o_k \in \tilde{\mathcal{S}} \cap \mathbb{P}^{n-1} \subset Bl_o(\mathbb{C}^n, o) \quad \text{but } \mathcal{S} \text{ is generic otherwise.}$$

Then $\mathcal{S} \cap \Delta$ is a reduced plane curve germ. One of its tangential component is $\Delta_k := \sigma(\tilde{\mathcal{S}} \cap \tilde{\Delta}, o_k)$. We bound its multiplicity, $\text{mult}(\Delta_k) = \langle \tilde{\Delta}_k, \mathbb{P}^{n-1} \rangle = \text{mult}(\mathbb{P}T_\Delta, o_k) \geq \frac{p}{q}$. Therefore we have the assumption of proposition 3.14:

$$(19) \quad p = \frac{p}{q} \cdot q \leq (r_k - 1) + q \cdot \text{mult}(\Delta_k) \leq (r_k - 1) + \sum_i q_i \cdot \text{mult}(\Delta_{k,i}).$$

It remains to verify: any deformation $\mathcal{S}(t)$ that preserves the tangent plane, $T_{\mathcal{S}(t)} = T_{\mathcal{S}(0)}$, induces an equisingular family of planar curve germs, $\Delta_k(t) \subset \mathcal{S}(t)$. This is done by explicit computation. Denote $\text{mult}(\Delta) = d$ and $\text{mult}(\mathbb{P}T_\Delta, o_k) = m$. Choose the coordinates in $(\mathbb{C}^n, o)$ such that $o_k = [0 : \dots : 0 : 1] \in \mathbb{P}T_\Delta$, but generically otherwise. Then the defining equation of Δ belongs to the ideal $(x_1, \dots, x_{n-1})^m \cdot (x_1, \dots, x_n)^{d-m} + (x_n)^{d+1} \subset \mathbb{C}\{x\}$. Moreover, by the genericity of the coordinate choice, the defining equation of Δ contains the monomials $x_1^d, \dots, x_{n-1}^d, x_1^m x_n^{d-m}, \dots, x_{n-1}^m x_n^{d-m}, x_n^{d+m-1}$. (The last monomial is ensured by the condition $\text{mult}(\mathbb{P}T_\Delta, o_k) > \text{mult}(\tilde{\Delta}, o_k)$.) After a $GL(n-1)$ -transformation on the coordinates $x_1, \dots, x_{n-1}$, the surface $\mathcal{S}$ is defined by the ideal $\langle x_2 - a_2(x_1, x_n), \dots, x_{n-1} - a_{n-1}(x_1, x_n) \rangle$, with $a_i \in (x_1) + (x_n)^2$. Thus the curve germ $\Delta \cap \mathcal{S}$ is defined by: $x_n^{d+m-k} + x_1^m x_n^{d-m} + x_1^d + (h.o.t.)$, for some $m > k > 0$, and with some coefficients. Any deformation $\mathcal{S}(t)$, i.e. a deformation of $a_2(x_1, x_n), \dots, a_{n-1}(x_1, x_n)$ by terms in $(x_1, x_n)^2$, results in monomials strictly above the Newton diagram. Hence it gives an equisingular family $\Delta_k(t)$. Thus both assumptions of proposition 3.14 are satisfied. ■

Example 3.17. Consider hypersurface germ $X = V(x_{n+1}^p - f(x_1, \dots, x_n)) \subset (\mathbb{C}^{n+1}, o)$. Suppose $\text{ord}(f) \geq p \geq 2$ and f is square-free. Then the covering $X \rightarrow (\mathbb{C}_{x_1 \dots x_n}^n, o)$ is convenient, and its discriminant, $\Delta = V(f) \subset (\mathbb{C}^n, o)$, is reduced. The total ramification index over each point of Δ equals $(p-1)$. Therefore the ratio in corollary 3.16 is $\frac{p}{q} = \frac{p}{p-1} \leq 2$. Suppose Δ is not an ordinary multiple point, i.e. $\mathbb{P}T_\Delta$ has a singular point, say $o_k \in \mathbb{P}^{n-1}$. If $\text{mult}(\mathbb{P}T_\Delta, o_k) > \text{mult}(\tilde{\Delta}, o_k)$, then X has a fast loop.

As a particular case, let $X = V(x_{n+1}^p - \sum_{i=1}^n x_i^{d_i}) \subset (\mathbb{C}^{n+1}, o)$, with $d_1 \geq d_2 \geq \dots \geq d_n \geq p \geq 2$. The discriminant is $\Delta = V(\sum_{i=1}^n x_i^{d_i}) \subset (\mathbb{C}^n, o)$. Suppose $d_1 > d_n$, then $\mathbb{P}T_\Delta$ has multiplicity d_n at the point $o_k := [1 : 0 : \dots : 0] \in \mathbb{P}^{n-1} \subset Bl_o(\mathbb{C}^n, o)$. Define the local coordinates at this point via $x_1 = \tilde{x}_1, x_2 = \tilde{x}_2 \cdot \tilde{x}_1, \dots, x_n = \tilde{x}_n \cdot \tilde{x}_1$. Take the strict transform $\tilde{\Delta} = V(\tilde{x}_1^{d_1-d_n} + \dots + \tilde{x}_n^{d_n}) \subset Bl_o(\mathbb{C}^n, o)$. Then $\text{mult}(\mathbb{P}T_\Delta, o_k) > \text{mult}(\tilde{\Delta}, o_k)$ iff $d_1 < 2d_n$. Altogether, if $d_n < d_1 < 2d_n$ then X has a fast loop. In fact, a sufficient condition for a fast loop is just $d_n < d_1$, see [Kerner-Mendes.23].

REFERENCES

- [A.G.L.V.I] V. I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev, *Singularity theory. I*. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993; MR1230637]. Springer-Verlag, Berlin, 1998. iv+245 pp
- [A.G.L.V.II] V. I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev, *Singularity theory. II. Classification and applications*. Encyclopaedia of Mathematical Sciences. 39. Berlin: Springer-Verlag. 235 p.
- [Bennett.77] B. Bennett, *Normally flat deformations*. Trans. Amer. Math. Soc. 225 (1977), 1–57
- [Birbrair-Fernandes.08] L. Birbrair, A. Fernandes, *Inner metric geometry of complex algebraic surfaces with isolated singularities*. Communications on Pure and Applied Mathematics, v. 61, p. 1483-1494, 2008.
- [B.F.G.17] L. Birbrair, A. Fernandes, V. Grandjean, *Thin-thick decomposition for real definable isolated singularities*. Indiana Univ. Math. J. 66, No. 2, 547–557 (2017).
- [B.F.N.08] L. Birbrair, A. Fernandes, W. D. Neumann, *Bi-Lipschitz geometry of weighted homogeneous surface singularities*. Mathematische Annalen, v. 342, p. 139-144, 2008.
- [B.F.N.09] L. Birbrair, A. Fernandes, W. D. Neumann, *Bi-Lipschitz geometry of complex surface singularities*. Geometriae Dedicata, v. 139, p. 259-267, 2009.
- [B.F.N.G.O'S.09] L. Birbrair, A. Fernandes, V. Grandjean, D. O'Shea, *Choking horns in Lipschitz geometry of complex algebraic varieties*. Appendix by Walter D. Neumann. J. Geom. Anal. 24, No. 4, 1971–1981 (2014).
- [B.N.P.14] L. Birbrair, W. D. Neumann, A. Pichon, *The thick-thin decomposition and the bilipschitz classification of normal surface singularities*. Acta Mathematica, v. 212, p. 199-256, 2014.
- [Brieskorn-Greuel.75] E. Brieskorn, G.-M. Greuel, *Singularities of complete intersections*. Manifolds Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), pp. 123-129 Published for the Mathematical Society of Japan by University of Tokyo Press, Tokyo, 1975.
- [Chirka] E. M. Chirka, *Complex analytic sets*. Kluwer Academic Publishers, 1989.
- [Coste] M. Coste, *An introduction to O-minimal geometry*, lecture notes.
- [Eisenbud] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- [Fernandes.Sampaio.19] A. Fernandes, J. E. Sampaio, *Tangent cones of Lipschitz normally embedded sets are Lipschitz normally embedded*. Appendix by Anne Pichon and Walter D. Neumann. Int. Math. Res. Not. IMRN(2019), no.15, 4880-4897.
- [Gaffney.93] T. Gaffney, *Polar multiplicities and equisingularity of map germs*, Topology 32 (1993) 185-223.
- [G.K.Z.] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008.
- [Greuel.77] G.-M. Greuel, *Cohomologie des singularités non isolées*. C. R. Acad. Sci., Paris, Sér. A 284, 321-322 (1977).
- [G.L.S.] G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007. xii+471 pp.
- [Kerner-Mendes.23] D. Kerner, R. Mendes, *Fast vanishing cycles on perturbations of (complex) weighted-homogeneous germs*, preprint.
- [Kurdyka-Orro.97] K. Kurdyka, P. Orro, *Geodesic distance on a subanalytic set*. Rev.Mat.Univ. Complutense Madr.10, Spec.Iss, 173-182 (1997).
- [Lê.74] D. T. Lê, *Computation of the Milnor number of an isolated singularity of a complete intersection*. Funkcional. Anal. i Priložen.8(1974), no.2, 45-49.
- [Looijenga] E. J. N. Looijenga, *Isolated singular points on complete intersections*. 2nd revised ed. Surveys of Modern Mathematics 5. Somerville, MA: International Press; Beijing: Higher Education Press.
- [Nuñ-Tom.08] J. J. Nuño-Ballesteros, J. N. Tomazella, *The Milnor number of a function on a space curve germ*. Bull. Lond. Math. Soc. 40, No. 1, 129–138 (2008).
- [Okuma.17] T. Okuma, *A remark on metrically conical surface singularities of Brieskorn type*, Saitama Math. J. 31 (2017), 137-139.
- [Pham-Teissier.69] F. Pham, B. Teissier, *Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski*, Centre de Mathématiques de l'Ecole Polytechnique (Paris), June 1969.
- [Whitney] H. Whitney, *Complex analytic varieties*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972. xii+399 pp.

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