RETRACTORS IN LOCAL POSITIVE LOGIC

ARTURO RODRÍGUEZ FANLO AND ORI SEGEL

Abstract. We study type spaces and retractors (saturated models) for local positive logic.

Introduction

This paper continues the work initiated by the authors on the development of local positive logic [RFS24] and motivated by [Hru22a, Section 2]. In this paper we study type spaces and saturated models (here called *retractors*) for this new logic.

Types for local positive logic were not really discussed in [Hru22a]. Here, we cover the study of types for local positive logic in Section 2. The right notion of types for local positive logic seems to be *local positive types* (Definition 2.1). For this notion we find in Lemma 2.10 the right version of finite satisfiability. It is worth noting that for pointed variables (for instance, in the one-sort case previously studied by Hrushovski) local positive types can be replaced by positive types (Lemma 2.12). We also study the topology of spaces of local positive types and adapt the definitions of Hausdorffness and semi-Hausdorffness for local positive logic. We conclude giving characterisations of some completeness properties in terms of local positive types (Lemmas 2.35 and 2.36).

In his paper, Hrushovski mostly focuses on the study of *retractor spaces*, being [Hru22a, Proposition 2.1] his main result on local positive logic. We cover his study, going deeper, in Sections 3 to 5. In Section 3, we cover the different equivalent definitions of retractors (Theorem 3.9) and also the first existence and uniqueness results (Theorem 3.16). In Section 4, we prove the Universality Lemma (Theorem 4.4), which is a stronger versions of [Hru22a, Lemma 2.4] and generalises [Seg22, Lemma 2.24]. In Section 5, we recall the equivalence of the existence of retractors and local positive compactness (called *primitive positive compactness*, ppC, in [Hru22a]), and discus consequences of this equivalence under different completeness properties. In Section 6, we discus the relationship between local retractors and non-local homomorphism universal models.

The study of the group of automorphisms of the core space was another fundamental topic in [Hru22b, Hru22a]. We generalise this study to retractors in arbitrary local positive logics in Section 7. In particular, we generalise several results from [Hru22b] that were not studied in [Hru22a].

This is the second of a series of papers which aims to extend several results of [Hru22a]. In subsequent papers we will study definability patterns for local positive logic, extending [Hru22b, Seg22]. Finally, we will apply the definability patterns to study hyperdefinable approximate subgroups and rough approximate subgroups.

2010 *Mathematics Subject Classification.* 03C95, 03B60.

Key words and phrases. Positive logic, local logic, local positive types, retractors, omitting types.

Rodríguez Fanlo is supported by the Israel Academy of Sciences and Humanities & Council for Higher Education Excellence Fellowship Program for International Postdoctoral Researchers and partially supported by STRANO PID2021-122752NB-I0.

Beyond some specific questions, we leave as an open task to adapt more elaborated model theoretic machinery (e.g. classification theory) to local positive logic.

1. Preliminaries in local positive logic

We start by recalling some basic definitions about local positive logic. We will follow the terminology and notation of [RFS24].

Definition 1.1. A *local (first-order) language* is a first-order language $\mathtt{L}$ with the following additional data:

(i) It is a relational language with constants, i.e. it has no function symbols except for constant symbols.

(ii) It has a distinguished partial ordered monoid $\mathtt{D}^s \coloneqq (\mathtt{D}^s, \mathrm{d}^0, *_s, \prec_s)$ of binary relation symbols for each single sort s. The binary relation symbols in $\mathtt{D}^s$ are the *locality relation symbols* on the sort s. The identity d^0 of $\mathtt{D}^s$ is the equality symbol on sort s and it is also the least element for $\prec_s$.

(iii) There is a function $\mathtt{B}$ assigning to any pair $c_1\,c_2$ of constant symbols of the same sort a locality relation symbol $\mathtt{B}_{c_1,c_2}$ on that sort. We call $\mathtt{B}$ the *bound function* and $\mathtt{B}_{c_1,c_2}$ the *bound* for c_1 and c_2.

We call $(\mathtt{D},\mathtt{B})$ the *local signature* of $\mathtt{L}$. A *sort* is a tuple of single sorts; its *arity* is its length. We write $\mathtt{S}$ for the set of single sorts. A *variable* x is a set of single variables; its *arity* is its size. The *sort* of a variable x is the sort $s = (s_{x_i})_{x_i\in x}$ where s_{x_i} is the sort of x_i. We say that a local language $\mathtt{L}$ is *pointed* if it has at least one constant symbol for every sort. We say that a set of parameters (new constants) A is *pointed* for $\mathtt{L}$ if the expansion $\mathtt{L}(A)$ is pointed. We say that a variable x of sort s is *pointed* in $\mathtt{L}$ if there is at least one constant symbol for every sort that does not appear in s. We say that a sort s is *pointed* if there is at least one constant symbol for every sort that does not appear in s.

A formula is *positive* if it is built from atomic formulas by conjunctions, disjunctions and existential quantifiers; let For_+ denote the set of positive formulas. A formula is *negative* if it is the negation of a positive formula; let For_- denote the set of negative formulas. Let φ be a formula, x a single variable, d a locality relation on the sort of x and t a term of the same sort such that x does not appear in t (i.e. a constant or a variable different to x). The *local existential quantification of φ on x in $\mathrm{d}(t)$* is the formula $\exists x{\in}\mathrm{d}(t)\,\varphi \coloneqq \exists x\,\mathrm{d}(x,t)\wedge\varphi$. A formula is *local* if it is built from atomic formulas using negations, conjunctions and local existential quantification; let LFor denote the set of local formulas. A formula is *local positive* if it is built from atomic formulas using conjunctions, disjunctions and local existential quantification; let LFor_+ denote the set of local positive formulas. A formula is *local negative* if it is the negation of a local positive formula.

Definition 1.2. A *local* $\mathtt{L}$*-structure* is an $\mathtt{L}$-structure M satisfying the following locality axioms:

(A1) Every locality relation symbol is interpreted as a symmetric binary relation.

(A2) Any two locality relation symbols d_1 and d_2 on the same sort with $\mathrm{d}_1 \preceq \mathrm{d}_2$ satisfy $\mathrm{d}_1^M \subseteq \mathrm{d}_2^M$.

(A3) Any two locality relation symbols d_1 and d_2 on the same sort satisfy $\mathrm{d}_1^M \circ \mathrm{d}_2^M \subseteq (\mathrm{d}_1 * \mathrm{d}_2)^M$.

(A4) Any two constant symbols c_1 and c_2 of the same sort satisfy $\mathtt{B}_{c_1,c_2}(c_1,c_2)$.

(A5) Any two single elements a and b of the same sort satisfy $\mathrm{d}(a,b)$ for some locality relation d on that sort.

For any element a and locality relation d on its sort, the d-*ball at a* is the set $\mathrm{d}(a) \coloneqq \{b : M \models \mathrm{d}(a,b)\}$.

Expansions and reducts of local structures are defined in the usual way.

Remark 1.3. The first four locality axioms could be easily expressed by first-order universal axiom schemes. We write $\mathtt{T}_{\mathrm{loc}}(\mathtt{L})$ for the universal first-order theory given by axioms A1 to A4 — we omit $\mathtt{L}$ if it is clear from the context.

We restrict satisfaction to local structures: we write $M \models_{\mathtt{L}} \Gamma$ to say that M is an $\mathtt{L}$-local model of Γ. We write $\mathcal{M}(\Gamma/\mathtt{L})$ for the class of $\mathtt{L}$-local models of Γ. We say that Γ is $\mathtt{L}$-*locally satisfiable* if it has an $\mathtt{L}$-local model. We say that Γ is *finitely* $\mathtt{L}$-*locally satisfiable* if every finite subset of Γ is $\mathtt{L}$-locally satisfiable. We typically omit $\mathtt{L}$ if it is clear from the context.

Notation. We write $\models_\star$ for satisfaction for arbitrary structures and $\mathcal{M}_\star(\Gamma)$ for the class of arbitrary models of Γ.

A *negative* $\mathtt{L}$-*local theory* $\mathtt{T}$ is a set of negative sentences closed under local implication. The *associated positive counterpart* $\mathtt{T}_+$ is the set of positive sentences whose negations are not in $\mathtt{T}$. In other words, a positive formula φ is in $\mathtt{T}_+$ if and only if there is a local model $M \models \mathtt{T}$ satisfying φ; we say in that case that M is a *witness of* $\varphi \in \mathtt{T}_+$. We write $\mathtt{T}_\pm \coloneqq \mathtt{T} \wedge \mathtt{T}_+$. We typically omit $\mathtt{L}$ if it is clear from the context. For a local structure M, we write $\mathrm{Th}_-(M)$ for its negative theory, i.e. the set of negative sentences satisfied by M. For a subset A, we write $\mathrm{Th}_-(M/A)$ for the theory of the expansion of M given by adding parameters for A. Note that $\mathrm{Th}_-(M)$ must be a local theory when M is local.

A homomorphism $f\colon A \to B$ between two structures is a *positive embedding* if $B \models_\star \varphi(f(a))$ implies $A \models_\star \varphi(a)$ for any positive formula $\varphi(x)$ and $a \in A^x$; we write $A \precsim^+ B$ if there is a positive embedding from A into B. A *positive substructure* is a substructure $A \leq B$ such that the inclusion $\iota\colon A \to B$ is a positive embedding; we write $A \preceq^+ B$ if A is a positive substructure of B. We say that a structure A is *positively closed* for a class of structures $\mathcal{K}$ if every homomorphism $f\colon A \to B$ with $B \in \mathcal{K}$ is a positive embedding. We say that M is a *local positively closed* model of Γ, written $M \models^{\mathrm{pc}}_{\mathtt{L}} \Gamma$, if M is a local model of Γ positively closed for the class $\mathcal{M}(\Gamma/\mathtt{L})$ of local models of Γ; we write $\mathcal{M}^{\mathrm{pc}}(\Gamma/\mathtt{L})$ for the class of local positively closed models of Γ. We say that M is *local positively closed* if it is a local positively closed model of $\mathrm{Th}_-(M)$. We typically omit $\mathtt{L}$ if it is clear from the context. Given two subsets Γ_1 and Γ_2 of first-order sentences, we write $\Gamma_1 \models^{\mathrm{pc}} \Gamma_2$ if $N \models \Gamma_2$ for any $N \models^{\mathrm{pc}} \Gamma_1$.

Notation. We say that M is a *(non-local) positively closed model of* Γ, written $M \models^{\mathrm{pc}}_\star \Gamma$, if M is a model of Γ and is positively closed for the class $\mathcal{M}_\star(\Gamma)$. We say that $\Gamma_1 \models^{\mathrm{pc}}_\star \Gamma_2$ if $N \models_\star \Gamma_2$ for any $N \models^{\mathrm{pc}}_\star \Gamma_1$

Definition 1.4. Let $\mathtt{T}$ be a locally satisfiable negative local theory:

(I) $\mathtt{T}$ is *irreducible* if, for any two positive formulas $\varphi(x)$ and $\psi(y)$ with x and y disjoint finite variables of the same sort, we have

$$\exists x\, \varphi(x) \in \mathtt{T}_+ \text{ and } \exists y\, \psi(y) \in \mathtt{T}_+ \Rightarrow \exists x, y\ \varphi(x) \wedge \psi(y) \in \mathtt{T}_+.$$

(LJP) $\mathtt{T}$ has the *local joint (homomorphism) property* if for any two local models $A, B \models \mathtt{T}$ there are homomorphisms $f\colon A \to M$ and $g\colon B \to M$ to a common local model $M \models \mathtt{T}$.

(wC) $\mathtt{T}$ is *weakly complete* if $\mathtt{T} = \mathrm{Th}_-(M)$ for some $M \models^{\mathrm{pc}} \mathtt{T}$.

(C) $\mathtt{T}$ is *complete* if $\mathtt{T} = \mathrm{Th}_-(M)$ for every $M \models^{\mathrm{pc}} \mathtt{T}$.

Let φ be a positive formula. A *denial* of φ is a positive formula ψ such that $\mathtt{T} \models \neg\exists x\, \varphi \wedge \psi$. We write $\mathtt{T} \models \varphi \perp \psi$ or $\varphi \perp \psi$ if $\mathtt{T}$ is clear from the context. An *approximation* to φ is a positive formula ψ such that every denial of ψ is a denial of φ; we write $\mathtt{T} \models \varphi \leq \psi$ or $\varphi \leq \psi$ if $\mathtt{T}$ is clear from the context. We say that two positive formulas ψ and φ are *complementary* if φ approximates every denial of ψ; we write $\mathtt{T} \models \varphi \top \psi$ or $\varphi \top \psi$ if $\mathtt{T}$ is clear from the context. A *full system of denials* of φ in $\mathtt{T}$ is a family Ψ of denials of φ such that, for any $M \models^{\mathrm{pc}} \mathtt{T}$ and any $a \in M^x$, either $M \models \varphi(a)$ or $M \models \psi(a)$ for some $\psi \in \Psi$. A *full system of approximations* of φ in $\mathtt{T}$ is a family Ψ of approximations of φ such that, for any $M \models^{\mathrm{pc}} \mathtt{T}$ and any $a \in M^x$, if $M \models \psi(a)$ for all $\psi \in \Psi$, then $M \models \varphi(a)$.

2. Local positive types

From now on, we have fixed a local language $\mathtt{L}$ and a locally satisfiable negative local theory $\mathtt{T}$.

Definition 2.1. Let x be a variable. A *partial local positive type on x of* $\mathtt{T}$ is a subset $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ such that Γ is *locally satisfiable* in $\mathtt{T}$, i.e. there is a local model $M \models \mathtt{T}$ and an element $a \in M^x$ such that $M \models \Gamma(a)$. A *local positive type* of $\mathtt{T}$ is a maximal partial local positive type of $\mathtt{T}$. As usual, we omit x if it is clear from the context.

Let M be a local $\mathtt{L}$-structure, A a subset and $a \in M^x$. The *partial local positive type* of a in M over A is the set

$$\mathrm{ltp}^M_+(a/A) := \{\varphi \in \mathrm{LFor}^x_+(\mathtt{L}(A)) : M \models \varphi(a)\}.$$

We omit M if it is clear from the context, and we omit A if $A = \emptyset$.

Remark 2.2. It is a well known fact in positive logic that, in general, $\mathrm{ltp}^M_+(a)$ is not maximal. Indeed, suppose that there is a homomorphism $h\colon M \to N$ such that $N \models \varphi(h(a))$ with $\varphi \in \mathrm{LFor}_+(\mathtt{L})$ while $M \not\models \varphi(a)$. Then, $\mathrm{ltp}^M_+(a) \subsetneq \mathrm{ltp}^N_+(h(a))$.

Lemma 2.3. *Let $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ be a subset of local positive formulas. Then, Γ is a partial local positive type of $\mathtt{T}$ if and only if there are $M \models^{\mathrm{pc}} \mathtt{T}$ and $a \in M^x$ such that $M \models \Gamma(a)$.*

Proof. Suppose Γ is a partial local positive type. By definition, Γ is locally satisfiable. Then, there is $N \models \mathtt{T}$ and $b \in N^x$ such that $N \models \Gamma(b)$. By [RFS24, Theorem 2.7], there is a homomorphism $f\colon N \to M$ to $M \models^{\mathrm{pc}} \mathtt{T}$. Thus, as homomorphisms preserve satisfaction of positive formulas, we conclude that $M \models \Gamma(a)$ for $a = f(b)$, so Γ is realised in a local positively closed model of $\mathtt{T}$. Q.E.D.

Remark 2.4. Using downwards Löwenheim-Skolem, we can take $|M| \leq |\mathtt{L}| + |x|$ in Lemma 2.3.

Lemma 2.5. *Let $M \models \mathtt{T}$. Then, M is a local positively closed model of $\mathtt{T}$ if and only if $\mathrm{ltp}^M_+(a)$ is a local positive type of $\mathtt{T}$ for every pointed tuple a in M.*

Proof. Suppose that $\mathrm{ltp}^M_+(c)$ is a local positive type of $\mathtt{T}$ for every pointed c in M. Let $h\colon M \to N$ be a homomorphism to a local model $N \models \mathtt{T}$. Suppose $N \models \varphi(h(a))$ for some $\varphi \in \mathrm{For}^x_+(\mathtt{L})$ and $a \in M^x$. We want to show that $M \models \varphi(a)$.

Without loss of generality, $\varphi(x) = \exists y\, \phi(x, y)$ where $\phi(x, y)$ is quantifier free and positive. Then, $N \models \phi(a, c)$ for some $c \in N^y$. Take $b \in M^y$ arbitrary and extend ab to a pointed a' arbitrarily. As N is a local model, we have that $N \models \mathrm{d}(c, h(b))$ for some locality relation symbol d. Consider $\varphi'(x, y) = \exists z {\in} \mathrm{d}(y)\, \phi(x, z) \in \mathrm{LFor}^{x,y}_+(\mathtt{L})$ where z is of the same sort as y. Then, we have that $N \models \varphi'(h(a, b))$, so $\varphi'(x, y) \in \mathrm{ltp}^N_+(h(a'))$.

Since homomorphisms preserve satisfaction of positive formulas, $\mathrm{ltp}^M_+(a') \subseteq \mathrm{ltp}^N_+(h(a'))$. As $\mathrm{ltp}^M_+(a')$ is maximal among partial local positive types of $\mathtt{T}$, it follows that $\mathrm{ltp}^M_+(a, b) = \mathrm{ltp}^N_+(h(a, b))$, so we get $M \models \varphi'(a, b)$. In particular,

$M \models \phi(a, c')$ for some c' in M, so $M \models \varphi(a)$. Since a, h, N and φ are arbitrary, we conclude that M is a local positively closed model of $\mathtt{T}$.

On the other hand, suppose M is a local positively closed model and $a \in M^x$ pointed. Take $\varphi \in \mathrm{LFor}^x_+(\mathtt{L})$ such that $M \not\models \varphi(a)$. By [RFS24, Lemma 2.12], we have $M \models \psi(a)$ for some $\psi \in \mathrm{LFor}^x_+(\mathtt{L})$ such that $\mathtt{T} \models \psi \perp \varphi$. Thus, there is $\psi \in \mathrm{ltp}^M_+(a)$ such that $\{\varphi, \psi\}$ is not locally satisfiable in $\mathtt{T}$. In other words, $\mathrm{ltp}^M_+(a) \wedge \varphi$ is not locally satisfiable in $\mathtt{T}$. As φ is arbitrary, we conclude that $\mathrm{ltp}^M_+(a)$ is a local positive type of $\mathtt{T}$. Q.E.D.

Corollary 2.6. *Let x be a pointed variable and $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ a subset of local positive formulas. Then, Γ is a local positive type of $\mathtt{T}$ if and only if $\Gamma = \mathrm{ltp}^M_+(a)$ for some $M \models^{\mathrm{pc}} \mathtt{T}$ and $a \in M^x$.*

As an immediate consequence of [RFS24, Theorem 2.7] Corollary 2.6, we can show that every partial local positive type on a pointed variable extends to a local positive type. Naturally, the same is still true for arbitrary variables, but we require an alternative argument relying on the notion of *bounded satisfiability*. As Lemma 2.10 shows, this notion is the right replacement of finite satisfiability in local logic. In particular, boundedness is the necessary condition to apply compactness in local logic.

Definition 2.7. Let $x = \{x_i\}_{i\in N}$ be a variable. Let I be the subset of indexes $i \in N$ such that there is a constant symbol on the sort of x_i and J the set of pairs of indexes $(i, j) \in N \times N$ such that x_i and x_j are on the same sort. A *bound* of x in $\mathtt{T}$ is a partial quantifier free type of the form

$$\mathrm{B}(x) \coloneqq \bigwedge_{i\in I} \mathrm{d}_i(x_i, c_i) \wedge \bigwedge_{(i,j)\in J} \mathrm{d}_{i,j}(x_i, x_j)$$

where c_i is some constant on the sort of x_i for $i \in I$, d_i is a locality relation on the sort of x_i for $i \in I$ and $\mathrm{d}_{i,j}$ is a locality relation on the common sort of x_i and x_j for $(i, j) \in J$.

Definition 2.8. A subset of formulas $\Gamma(x) \subseteq \mathrm{For}^x(\mathtt{L})$ is *boundedly satisfiable* in $\mathtt{T}$ if there is a bound $\mathrm{B}(x)$ such that $\Gamma(x) \wedge \mathrm{B}(x)$ is finitely locally satisfiable in $\mathtt{T}$. In that case, we say that B is a *feasible bound* for Γ.

Remark 2.9. Assume $\Gamma \models \mathrm{B}$ for some bound B. Then, Γ is boundedly satisfiable if and only if it is finitely locally satisfiable.

Lemma 2.10. *Let Γ be a subset of Π_1-local formulas of $\mathtt{L}$. Then, Γ is locally satisfiable in $\mathtt{T}$ if and only if it is boundedly satisfiable in $\mathtt{T}$.*

Proof. Obviously, if Γ is locally satisfiable, then it is boundedly satisfiable as any a realising Γ gives us a feasible bound for Γ. On the other hand, suppose that $\Gamma(x)$ is boundedly satisfiable. Pick a feasible bound $\mathrm{B}(x)$ for $\Gamma(x)$. Let $\mathtt{L}(\underline{x})$ be the expansion of $\mathtt{L}$ given by adding new constant symbols for each variable in x: we take $\mathtt{B}_{\underline{x}_i,c_i}$ and $\mathtt{B}_{\underline{x}_i,\underline{x}_j}$ according to B — for any other constant c' in $\mathtt{L}(\underline{x})$, we define $\mathtt{B}_{\underline{x}_i,c'}$ via $\preceq$. Now, consider $\widetilde{\mathtt{T}} = \mathtt{T} \wedge \Gamma(\underline{x})$. By hypothesis, $\widetilde{\mathtt{T}}$ is finitely locally satisfiable. Thus, by [RFS24, Theorem 1.9], there is a local $\mathtt{L}(\underline{x})$-structure $M_{\underline{x}}$ satisfying $\widetilde{\mathtt{T}}$. Hence, the respective $\mathtt{L}$-reduct M is a local model of $\mathtt{T}$ realising Γ. Q.E.D.

Corollary 2.11. *Let $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ be a partial local positive type of $\mathtt{T}$. Then, there is a local positive type extending Γ.*

Proof. As Γ is boundedly satisfiable, pick any feasible bound $\mathrm{B}(x)$. Take $\Gamma'(x) = \Gamma(x) \wedge \mathrm{B}(x)$. Now, consider $\mathcal{K} = \{\Sigma \subseteq \mathrm{LFor}^x_+(\mathtt{L}) : \Gamma' \subseteq \Sigma$ finitely locally satisfiable in $\mathtt{T}\}$, which is partially ordered by $\subseteq$. For any chain $\Omega \subseteq \mathcal{K}$ we have that $\bigcup \Omega$

is an upper bound for Ω in $\mathcal{K}$. Indeed, for any $\Lambda \subseteq \bigcup\Omega$ finite, there is $\Sigma \in \Omega$ such that $\Lambda \subseteq \Sigma$, so $\bigcup\Omega$ is finitely locally satisfiable, concluding $\bigcup\Omega \in \mathcal{K}$. By Zorn's Lemma, there is a maximal element $p \in \mathcal{K}$. We know that $\Gamma \subseteq p$ and p is boundedly satisfiable. On the other hand, if $p \subseteq \Sigma$ with Σ boundedly satisfiable, then $\Sigma \in \mathcal{K}$ and, by maximality of p, $\Sigma = p$. Hence, p is a local positive type extending Γ. Q.E.D.

In general, we consider only partial local positive types, as local compactness only applies to this kind of formulas ([RFS24, Theorem 1.9]). However, when we consider pointed variables, we can instead deal with partial positive types.

Lemma 2.12. *Let M be a local* $\mathtt{L}$*-structure. Let x be a pointed variable and $a, b \in M^x$. Then, $\mathrm{ltp}^M_+(a) \subseteq \mathrm{ltp}^M_+(b)$ if and only if $\mathrm{tp}^M_+(a) \subseteq \mathrm{tp}^M_+(b)$.*

Proof. Obviously, $\mathrm{tp}_+(a) \subseteq \mathrm{tp}_+(b)$ implies $\mathrm{ltp}_+(a) \subseteq \mathrm{ltp}_+(b)$. Conversely, assume $\mathrm{ltp}_+(a) \subseteq \mathrm{ltp}_+(b)$. Take $\varphi(x) \in \mathrm{For}^x_+(\mathtt{L})$. Without loss of generality, $\varphi(x) = \exists y\, \phi(x, y)$ with ϕ quantifier-free positive. Suppose $M \models \varphi(a)$. Then, there is c such that $M \models \phi(a, c)$. As a is pointed, we can find a term $t(a)$ on the sort of c. As M is a local structure, there is a locality relation d with $\mathrm{d}(t(a), c)$. Thus, $M \models \varphi'(a)$ where $\varphi'(x) = \exists y{\in}\mathrm{d}(t(x))\, \phi(x, y) \in \mathrm{LFor}^x_+(\mathtt{L})$. As $\mathrm{ltp}_+(a) \subseteq \mathrm{ltp}_+(b)$, we conclude that $\models \varphi'(b)$ too. Since φ is arbitrary, we conclude that $\mathrm{tp}_+(a) \subseteq \mathrm{tp}_+(b)$. Q.E.D.

Definition 2.13. Let M be a local $\mathtt{L}$-structure and A a subset. A *(partial) local positive type of M over A* is a (partial) local positive type of $\mathrm{Th}_-(M/A)$.

Lemma 2.14. *Let M be a local* $\mathtt{L}$*-structure and A a subset. Let $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L}(A))$ be a subset of local positive formulas with parameters in A. Assume M has the local joint property over A. Then, Γ is a partial local positive type of M over A if and only if there is an $\mathtt{L}(A)$-homomorphism $f\colon M_A \to N_A$ to $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$ realising Γ. Furthermore, suppose x is pointed over $\mathtt{L}(A)$. Then, $\Gamma(x)$ is a local positive type of M over A if and only if there is an $\mathtt{L}(A)$-homomorphism $f\colon M_A \to N_A$ to some $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$ such that $\Gamma = \mathrm{ltp}^{N_A}_+(a/A)$ for some $a \in N^x$.*

Proof. If there is an $\mathtt{L}(A)$-homomorphism to a local positively closed model of $\mathrm{Th}_-(M/A)$ realising Γ, then Γ is obviously a partial local positive type over A. On the other hand, suppose Γ is a partial local positive type over A. By Lemma 2.3, there are $N'_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$ and $b \in {N'}^x_A$ such that $N'_A \models \Gamma(b)$. By assumption, $\mathrm{Th}_-(M/A)$ satisfies the local joint property, so there are $\mathtt{L}(A)$-homomorphisms $h_0\colon N'_A \to N''_A$ and $h_1\colon M_A \to N''_A$ to a common local model $N''_A \models \mathrm{Th}_-(M/A)$. By [RFS24, Theorem 2.7], there is an $\mathtt{L}(A)$-homomorphism $f'\colon N''_A \to N_A$ to $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$, so $f = f' \circ h_1\colon M_A \to N_A$ is an $\mathtt{L}(A)$-homomorphism with $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$. Since homomorphisms preserve satisfaction of positive formulas, we have that $N_A \models \Gamma(a)$ with $a = f' \circ h_0(b)$, so it realises Γ.

Finally, if Γ is a local positive type of M over A, then $\Gamma = \mathrm{ltp}^N_+(a/A)$ by maximality. On the other hand, by Lemma 2.5, $\mathrm{ltp}^N_+(a/A)$ is a local positive type of $\mathrm{Th}_-(M/A)$ for any $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$ and $a \in N^x_A$ whenever x is pointed. Q.E.D.

Remark 2.15. Applying downwards Löwenheim-Skolem repeatedly we can take $|N| \leq |\mathtt{L}| + |M| + |x|$ in Lemma 2.14.

Corollary 2.16. *Let M be a local* $\mathtt{L}$*-structure and A a subset. Let $p(x)$ be a partial local positive type of M over A on a variable x pointed over $\mathtt{L}(A)$. Then, p is a local positive type of M over A if and only if, for every local positive formula $\varphi(x)$ over A with $\varphi(x) \notin p$, there is a local primitive positive formula $\psi(x)$ over A with $M \models \neg\exists x\, \psi(x) \wedge \varphi(x)$ such that $\psi(x) \in p$.*

Proof. The right-to-left implication is obvious. On the other hand, suppose p is a local positive type over A and $\varphi(x) \notin p$. By Lemma 2.14, there is an $\mathtt{L}(A)$-homomorphism $f\colon M_A \to N_A$ to $N_A \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$ such that $p(x) = \mathrm{ltp}_+^{N_A}(c/A)$ for some $c \in N^x$. As $\varphi(x) \notin p$, we get that $N_A \not\models \varphi(c)$. By [RFS24, Lemma 2.12], there is a local primitive positive formula $\psi(x)$ over A with $\mathrm{Th}_-(M/A) \models \psi(x) \perp \varphi(x)$ such that $N_A \models \psi(c)$, so $\psi(x) \in p$. Q.E.D.

It should be noted that, contrary to what happens in usual positive logic, local positive types might not be preserved under adding parameters. Explicitly, a partial local positive type of M over A is not necessarily a partial local positive type of M over B for $A \subseteq B$. The key issue is that a bound over A is not generally a bound over B. For instance, consider [RFS24, Example 3.7(1)]. Recall that $\mathtt{L} \coloneqq \{P_n, Q_n, \mathrm{d}_n\}_{n\in\mathbb{N}}$ and $\mathtt{T} \coloneqq \mathrm{Th}_-(\mathbb{Z})$ where $\mathbb{Z} \models P_n(x) \Leftrightarrow x > n$, $\mathbb{Z} \models Q_n(x) \Leftrightarrow x < -n$ and $\mathbb{Z} \models \mathrm{d}_n(x,y) \Leftrightarrow |x-y| \leq n$. Then, $\bigwedge_{n\in\mathbb{N}} P_n(x)$ is a partial local positive type over $\emptyset$ but it is not a partial local positive type over 0. Such a counterexample cannot be given when A is pointed:

Lemma 2.17. *Let M be a local $\mathtt{L}$-structure and A a pointed subset. Let $A \subseteq B$. Then, every partial local positive type of M over A is a partial local positive type of M over B.*

Proof. Let $\Gamma(x)$ be a partial local positive type of M over A. By Lemma 2.10, $\Gamma(x)$ is boundedly satisfiable over A. Pick a feasible bound $\mathrm{B}(x)$ of x over A for Γ. Since A is pointed, $\mathrm{B}(x)$ is also a bound over B. Hence, $\Gamma(x)$ is boundedly satisfiable over B with feasible bound B, so a partial local positive type over B by Lemma 2.10. Q.E.D.

It is natural to wonder if we can relax the pointedness hypothesis in the previous lemma. The following lemma shows that, in local positively closed models, the local joint property suffices.

Lemma 2.18. *Let M be a local $\mathtt{L}$-structure, $A \subseteq B$ be subsets of M and $M \models^{\mathrm{pc}} \mathrm{Th}_-(M/A)$. Suppose that M has the local joint property over A. Then, every partial local positive type of M over A is a partial local positive type of M over B. Conversely, if M has the local joint property over B and every partial local positive type of M over A is a partial local positive type of M over B, then M has the local joint property over A.*

Proof. By adding parameters to the language, we may assume that A is empty.

Suppose first that M has the local joint property over $\emptyset$. Let $\Gamma(x)$ be a partial local positive type of M over $\emptyset$. By Lemma 2.14, there is an $\mathtt{L}$-homomorphism $f\colon M \to N$ to $N \models^{\mathrm{pc}} \mathrm{Th}_-(M)$ realising $\Gamma(x)$. Now, f is a positive embedding as M is a local positively closed structure. On the other hand, consider the $\mathtt{L}(B)$-expansion N_B of N given by $b^{N_B} = f(b)$ for $b \in B$. As $f\colon M \to N$ is a positive embedding, $N_B \models \mathrm{Th}_-(M/B)$. Consequently, $\Gamma(x)$ is locally satisfiable in $\mathrm{Th}_-(M/B)$, i.e. it is a partial local positive type of M over B.

Now, assume that M has the local joint property over B and every partial local positive type of M over $\emptyset$ is a partial local positive type of M over B. Let $N \models^{\mathrm{pc}} \mathrm{Th}_-(M)$ and pick c pointed in N. Then, $p(x) \coloneqq \mathrm{ltp}_+^N(c)$ is a local positive type of M over $\emptyset$. By hypothesis, $p(x)$ is a partial local positive type of M over B. By Lemma 2.14, there is an $\mathtt{L}(B)$-homomorphism $f\colon M_B \to M'_B$ to $M'_B \models^{\mathrm{pc}} \mathrm{Th}_-(M/B)$ realising $p(x)$. Pick a realisation c' of $p(x)$. By [RFS24, Remark 2.10], $M_B \models^{\mathrm{pc}} \mathrm{Th}_-(M/B)$, so f is a positive embedding. By maximality, as $p(x)$ is a local positive type over $\emptyset$, we conclude $\mathrm{ltp}_+^{M'}(c') = p(x)$. Let M'_c be the $\mathtt{L}(c)$-expansion of M' given by $c^{M'_c} = c'$. Then, $M'_c \models \mathrm{Th}_-(N/c)$. Indeed, $\varphi(c) \in \mathrm{Th}_-(N/c)$ if and only if $\varphi(x) \notin \mathrm{ltp}^N_+(c/A) = p(x) = \mathrm{ltp}^{M'}_+(c'/A)$, if and only if $M'_c \models \varphi(c)$. Since c is pointed, by [RFS24, Theorem 3.5], there are homomorphisms $g\colon M'_c \to N'_c$ and $h\colon N_c \to N'_c$ to $N'_c \models^{\mathrm{pc}} \mathrm{Th}_-(N/c)$. Then, $N' \models \mathrm{Th}_-(N)$ where $N \models \mathrm{Th}_-(M)$, so $N' \models \mathrm{Th}_-(M)$. Finally, $g \circ f\colon M \to N'$ and $h\colon N \to N'$ are homomorphisms. As N is arbitrary, we conclude that $\mathrm{Th}_-(M)$ has the local joint property. Q.E.D.

However, the following example shows that, for general local structures, assuming only the local joint property is not enough:

Example 2.19. Consider the language $\mathtt{L}$ with two sorts $\mathtt{S} = \{1, 2\}$, locality relations $\mathtt{D}^s = \{\mathrm{d}^s_q : q \in \mathbb{Q}_{\geq 0}\}$ for each $s \in \mathtt{S}$ (and the usual ordered monoid structure induced by the positive rationals), and one binary relation P of sort $(1, 2)$. Let R be the local $\mathtt{L}$-structure whose sorts are two copies of the positive reals, $\mathbb{R}^1_{\geq 0}, \mathbb{R}^2_{\geq 0}$, with interpretations $R \models \mathrm{d}^s_q(x, y) \Leftrightarrow |x-y| \leq q$ and $R \models P(x, y) \Leftrightarrow x \neq 0 \wedge y > {}^1\!/_x$. Consider the sets of parameters $A = \{0^1\}$ and $B = \{0^1, 0^2\}$ and the partial local positive type $p(x, y) = \bigwedge_{q>0} \mathrm{d}^1_q(x, 0^1) \wedge P(x, y)$, where 0^s is the zero of sort s. We claim that R has the local joint property over A and p is a partial local positive type over A, but it is not a partial local positive type over B.

Now, let $\widetilde{R}$ be an $\aleph_1$-saturated elementary extension of R. Since $p(x, y)$ is finitely satisfiable, we find ε^1 of sort 1 and ∞^2 of sort 2 in $\widetilde{R}$ realising p. Consider the substructure I of $\widetilde{R}$ with universe $I^1 = \{\varepsilon^1, 0^1\}$ and $I^2 = \{\infty^2\}$. We get that $I \models \mathrm{Th}_-(R/A)$. Now, for any $M \models \mathrm{Th}_-(R/A)$, consider $f\colon M \to I$ given by $f\colon 0^1 \mapsto 0^1$, $f\colon a \mapsto \varepsilon^1$ for all $a \in M^1$ with $a \neq 0^1$, and $f\colon a \mapsto \infty^2$ for all $a \in M^2$. Trivially, f is a homomorphism. Therefore, $\mathrm{Th}_-(R/A)$ has the local joint property. This also shows that $p(x, y)$ is locally satisfiable over A.

Now, $R \models \neg\exists x\, \mathrm{d}_{1/n}(x, 0^1) \wedge P(x, y) \wedge \mathrm{d}_n(y, 0^2)$ for all $n \in \mathbb{N}$. Therefore, $\mathrm{Th}_-(R/B) \models \mathrm{d}_{1/n}(x, 0^1) \wedge P(x, y) \perp \mathrm{d}_n(y, 0^2)$. Hence, $p(x, y)$ is not boundedly satisfiable over B, so it is not a partial local positive type over B.

Definition 2.20. The *space of local positive types* of $\mathtt{T}$ on x over $\mathtt{L}$, which we denote by $\mathrm{S}^x(\mathtt{T}/\mathtt{L})$, is the set of local positive types of $\mathtt{T}$ on x over $\mathtt{L}$; the *space of local positive types* $\mathrm{S}(\mathtt{T}/\mathtt{L})$ is the respective sorted set. We omit $\mathtt{L}$ if it is clear from the context.

Let $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ be a subset of local positive formulas. The *interpretation* of Γ in $\mathrm{S}^x(\mathtt{T}/\mathtt{L})$ is the set

$$\langle \Gamma(x) \rangle_{\mathrm{S}(\mathtt{T}/\mathtt{L})} \coloneqq \{p \in \mathrm{S}^x(\mathtt{T}/\mathtt{L}) : \Gamma \subseteq p\}$$

As usual, we omit $\mathtt{L}$ and x if they are clear from the context.

The *local positive logic topology* of $\mathrm{S}^x(\mathtt{T})$ is the topology defined by taking $\langle \Gamma \rangle$ as closed for every $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ — we may need to add $\emptyset$ as closed.

Lemma 2.21. *Let x be a variable. The local positive logic topology of $\mathrm{S}^x(\mathtt{T})$ is a well-defined topology.*

Proof. By Corollary 2.11, there is a local positive type extending $x = x$, so $\mathrm{S}^x(\mathtt{T}) \neq \emptyset$. Obviously, $\langle x = x \rangle = \mathrm{S}^x(\mathtt{T})$, so it is closed. By hypothesis, $\emptyset$ is closed.

Let $\{\Gamma_i\}_{i \in I}$ be a family of local positive formulas. Then, $\bigcap \langle \Gamma_i \rangle = \langle \bigwedge \Gamma_i \rangle$, so arbitrary intersection of closed subsets are closed.

Let $\Gamma_1, \Gamma_2 \subseteq \mathrm{LFor}^x_+(\mathtt{L})$ and write $\Gamma_1 \vee \Gamma_2 \coloneqq \{\varphi_1 \vee \varphi_2 : \varphi_1 \in \Gamma_1,\ \varphi_2 \in \Gamma_2\}$. Then, $\langle \Gamma_1 \vee \Gamma_2 \rangle = \langle \Gamma_1 \rangle \cup \langle \Gamma_2 \rangle$. Indeed, trivially $\langle \Gamma_1 \rangle \cup \langle \Gamma_2 \rangle \subseteq \langle \Gamma_1 \vee \Gamma_2 \rangle$. On the other hand, suppose $p \notin \langle \Gamma_1 \rangle \cup \langle \Gamma_2 \rangle$. Then, there are $\varphi_1 \in \Gamma_1$ and $\varphi_2 \in \Gamma_2$ such that $\varphi_1 \notin p$ and $\varphi_2 \notin p$. By Lemma 2.3, take $M \models^{\mathrm{pc}} \mathtt{T}$ an $a \in M^x$ such that $\mathrm{ltp}^M_+(a) = p$. As $M \not\models \varphi_1(a)$ and $M \not\models \varphi_2(a)$, we conclude that $\varphi_1 \vee \varphi_2 \notin p$, so $p \notin \langle \Gamma_1 \vee \Gamma_2 \rangle$. Q.E.D.

Remark 2.22. Note that the local positive logic topology is a T_1 topology (i.e. Fréchet). Indeed, for every local positive type p, we have that $\langle p\rangle$ is closed and, by maximality, only contains p.

As we already noted in [RFS24], compactness is not true in general for local logic. In particular, $\mathrm{S}(\mathtt{T})$ is not necessarily compact. However, compactness is true under boundedness.

Lemma 2.23. *Every bound of x is a compact subset of* $\mathrm{S}^x(\mathtt{T})$.

Proof. Let $\mathrm{B}(x)$ be a bound of x in $\mathtt{T}$. We claim that $\langle \mathrm{B}(x)\rangle$ is a compact subset of $\mathrm{S}^x(\mathtt{T})$. Let $\{\Gamma_i(x)\}_{i\in I}$ be any family of partial local positive types of $\mathtt{T}$ and suppose that $\bigcap_{i\in I_0}\langle\Gamma_i(x)\rangle\cap\langle\mathrm{B}(x)\rangle$ is not empty for every $I_0\subseteq I$ finite. Consider $\Gamma(x)=\bigwedge\Gamma_i(x)\wedge\mathrm{B}(x)$. Then, Γ is finitely locally satisfiable in $\mathtt{T}$ and $\Gamma\models\mathrm{B}$. Therefore, by Lemma 2.10, Γ is locally satisfiable in $\mathtt{T}$. By Corollary 2.11, we conclude that there is $p\in\langle\Gamma\rangle=\bigcap\langle\Gamma_i\rangle\cap\langle\mathrm{B}\rangle$. As $\{\Gamma_i\}_{i\in I}$ is arbitrary, we conclude that $\langle\mathrm{B}\rangle$ is compact. Q.E.D.

Definition 2.24. We say that $\mathtt{T}$ is *Hausdorff* if $\mathrm{S}(\mathtt{T})$ is Hausdorff with the local positive logic topology.

Lemma 2.25. $\mathtt{T}$ *is Hausdorff if and only if, for any two different local positive types $p(x)$ and $q(x)$ on the same variable, there are local positive formulas $\varphi(x)$ and $\psi(x)$ with $\varphi\notin p$, $\psi\notin q$ and $\varphi\top\psi$.*

Proof. Suppose $\mathtt{T}$ is Hausdorff and let $p(x)$ and $q(x)$ be two different local positive types on the same variable. Thus, as $\mathrm{S}^x(\mathtt{T})$ is Hausdorff, there are $\Gamma_1,\Gamma_2\subseteq\mathrm{LFor}^x_+(\mathtt{L})$ such that $\langle\Gamma_1\vee\Gamma_2\rangle=\langle\Gamma_1\rangle\cup\langle\Gamma_2\rangle=\mathrm{S}^x(\mathtt{T})$ and $p\notin\langle\Gamma_1\rangle$ and $q\notin\langle\Gamma_2\rangle$. Thus, there are $\varphi\in\Gamma_1$ and $\psi\in\Gamma_2$ such that $\varphi\notin p$ and $\psi\notin q$. As $\langle\Gamma_1\vee\Gamma_2\rangle=\mathrm{S}^x(\mathtt{T})$, we have that $\langle\varphi\vee\psi\rangle=\mathrm{S}^x(\mathtt{T})$. Thus, for any $M\models^{\mathrm{pc}}\mathtt{T}$ and any $a\in M^x$, we have that $\varphi\vee\psi\in\mathrm{ltp}_+(a)$. In other words, $\mathtt{T}\models^{\mathrm{pc}}\forall x\,\varphi\vee\psi$, i.e. $\varphi\top\psi$.

On the other hand, suppose that for any two different local positive types $p(x)$ and $q(x)$ on the same variable there are $\varphi,\psi\in\mathrm{LFor}^x_+(\mathtt{L})$ such that $\varphi\notin p$, $\psi\notin q$ and $\varphi\top\psi$. Then, $\langle\varphi\rangle\cup\langle\psi\rangle=\langle\varphi\vee\psi\rangle=\mathrm{S}^x(\mathtt{T})$, so $\langle\varphi\rangle^c$ and $\langle\psi\rangle^c$ are open disjoint subsets of $\mathrm{S}^x(\mathtt{T})$ separating p and q. Q.E.D.

Let x and y be two variables. Recall that a *substitution* of x by y is a function $\zeta\colon x\to y$ such that x_i and $\zeta(x_i)$ have the same sort for any $x_i\in x$. Given a formula φ, we write $\zeta_*(\varphi)$ for the formula recursively defined by replacing in φ every free instance of x_i by a free instance of $\zeta(x_i)$ for each $x_i\in x$ at the same time.

In positive logic, a key notion generalizing Hausdorffness is semi-Hausdorffness. The definition of semi-Hausdorffness depends on the fact that pull-backs by substitutions of variables are well-defined continuous maps between the spaces of types. Unfortunately, in local positive logic, for non pointed variables, the pull-back of a local positive type by a substitution of variables is not necessarily maximal, i.e. it may be only a partial local positive type. Therefore, for non pointed variables, pull-backs by substitutions of variables are not necessarily well-defined maps between spaces of types. In particular, the definition of semi-Hausdorffness only makes sense for pointed variables.

Lemma 2.26. *Let x and y be pointed variables and ζ a substitution of x by y. Then:*

(1) $\zeta^*\coloneqq\mathrm{Im}^{-1}\zeta_*\colon\mathrm{S}^y(\mathtt{T})\to\mathrm{S}^x(\mathtt{T})$ *is a well-defined function. Furthermore, for any $M\models\mathtt{T}$ and $a\in M^y$, we have $\zeta^*\mathrm{ltp}_+(a)=\mathrm{ltp}_+(\zeta^*a)$.*

(2) $\mathrm{Im}^{-1}\zeta^*(\langle\Gamma\rangle)=\langle\mathrm{Im}\,\zeta_*(\Gamma)\rangle$ *for any* $\Gamma\subseteq\mathrm{LFor}^x_+(\mathtt{L})$.

Proof. (1) First, recall that $\models_\star \zeta_*\varphi(a)$ if and only if $\models_\star \varphi(\zeta^* a)$. Thus, $\zeta_*\varphi \in \mathrm{ltp}_+(a)$ if and only if $\varphi \in \mathrm{ltp}_+(\zeta^* a)$ for any $\varphi \in \mathrm{LFor}^x_+(\mathtt{L})$, $M \models \mathtt{T}$ and $a \in M^y$. In other words, $\zeta^*\mathrm{ltp}_+(a) = \mathrm{ltp}_+(\zeta^* a)$. In particular, by Corollary 2.6, $\zeta^* \colon \mathrm{S}^y(\mathtt{T}) \to \mathrm{S}^x(\mathtt{T})$ is a well-defined function.

(2) $\mathrm{Im}^{-1}\zeta^*(\langle\Gamma\rangle) = \{q \in \mathrm{S}^y(\mathtt{T}) : \zeta^*(q) \in \langle\Gamma\rangle\} = \{q \in \mathrm{S}^y(\mathtt{T}) : \Gamma \subseteq \mathrm{Im}^{-1}\zeta_*(q)\} = \{q \in \mathrm{S}^y(\mathtt{T}) : \mathrm{Im}\,\zeta_*(\Gamma) \subseteq q\} = \langle \mathrm{Im}\,\zeta_*(\Gamma)\rangle$. Q.E.D.

Corollary 2.27. *Let x and y be pointed variables and ζ a substitution of x by y. Then, $\zeta^* \colon \mathrm{S}^y(\mathtt{T}) \to \mathrm{S}^x(\mathtt{T})$ is a continuous map for the logic topologies.*

Let $x_1, \ldots, x_n$ be pointed variables and write $\overline{x} = (x_1, \ldots, x_n)$. Take $y = x_1 \sqcup \cdots \sqcup x_n$, i.e. y is a variable together with some injective substitutions $\iota_i \colon x_i \to y$ such that $\{\iota_i(x_i)\}_i$ is a partition of y. Then,

$$\begin{array}{rccl} \iota^* : & \mathrm{S}^y(\mathtt{T}) & \to & \mathrm{S}^{\bar{x}}(\mathtt{T}) \coloneqq \prod_i \mathrm{S}^{x_i}(\mathtt{T}) \\ & p & \mapsto & (\iota_1^*(p), \ldots, \iota_n^*(p)) \end{array}$$

is a continuous map.

Let $\Gamma \subseteq \mathrm{LFor}^y_+(\mathtt{L})$ be a subset of local positive formulas. The *interpretation* of Γ in $\mathrm{S}^{\bar{x}}(\mathtt{T}) = \prod \mathrm{S}^{x_i}(\mathtt{T})$ is the subset $\Gamma(\mathrm{S}^{\overline{x}}(\mathtt{T})) \coloneqq \mathrm{Im}\,\iota^*(\langle\Gamma\rangle)$. We say that a subset V of $\mathrm{S}^{\overline{x}}(\mathtt{T})$ is *local positive $\bigwedge$-definable* if there is $\Gamma \subseteq \mathrm{LFor}^y_+(\mathtt{L})$ such that $V = \Gamma(\mathrm{S}^{\bar{x}}(\mathtt{T}))$. We say that Γ *defines* V if $V = \Gamma(\mathrm{S}^{\bar{x}}(\mathtt{T}))$. For $(p_1, \ldots, p_n) \in \mathrm{S}^{\bar{x}}(\mathtt{T})$, we write $\mathtt{T} \models \Gamma(p_1, \ldots, p_n)$ if $(p_1, \ldots, p_n) \in \Gamma(\mathrm{S}^{\bar{x}}(\mathtt{T}))$.

Remark 2.28. In other words, $\mathtt{T} \models \Gamma(p_1, \ldots, p_n)$ if and only if there are $M \models^{\mathrm{pc}} \mathtt{T}$ and $a_1, \ldots, a_n$ in M such that $M \models \Gamma(a_1, \ldots, a_n)$ and $\mathrm{ltp}^M_+(a_i) = p_i$ for each i.

We say that $\mathtt{T}$ is *semi-Hausdorff* if, for any pointed variable x, the diagonal $\Delta^x(\mathtt{T}) \coloneqq \{(p, q) \in \mathrm{S}^x(\mathtt{T}) : p = q\}$ is local positive $\bigwedge$-definable.

In other words, $\mathtt{T}$ is semi-Hausdorff if and only if, for any pointed variable x, there is a partial local positive type $\Delta(x, y) \subseteq \mathrm{LFor}^{x;y}_+(\mathtt{L})$, where y is disjoint to x and has the same sort, such that, for any $M \models^{\mathrm{pc}} \mathtt{T}$ and any $a, b \in M^x$, $M \models \Delta(a, b)$ if and only if $\mathrm{ltp}^M_+(a) = \mathrm{ltp}^M_+(b)$.

Lemma 2.29. *If $\mathtt{T}$ is Hausdorff, then $\mathtt{T}$ is semi-Hausdorff.*

Proof. Pick x, y disjoint pointed variables of the same sort. Take

$$\Delta(x, y) = \{\varphi(x) \vee \psi(y) : \varphi, \psi \in \mathrm{LFor}^x_+(\mathtt{L}),\ \varphi \top \psi\}.$$

We claim that $\Delta(x, y)$ defines the diagonal of $\mathrm{S}^x(\mathtt{T})$. In other words, we claim that for any $M \models^{\mathrm{pc}} \mathtt{T}$ and $a, b \in M^s$, $M \models \Delta(a, b)$ if and only if $\mathrm{ltp}_+(a) = \mathrm{ltp}_+(b)$.

Say $\mathrm{ltp}_+(a) = \mathrm{ltp}_+(b)$. Take $\varphi, \psi \in \mathrm{LFor}^x_+(\mathtt{L})$ arbitrary with $\varphi \top \psi$. In particular, we have that $M \models \varphi(a) \vee \psi(a)$ and $M \models \varphi(b) \vee \psi(b)$. Without loss of generality, say $M \models \psi(a)$. As $\mathrm{ltp}_+(a) = \mathrm{ltp}_+(b)$, we conclude that $M \models \psi(b)$, so $M \models \varphi(a) \vee \psi(b)$. As φ, ψ are arbitrary, we conclude that $M \models \Delta(a, b)$.

On the other hand, suppose $\mathrm{ltp}_+(a) \neq \mathrm{ltp}_+(b)$. Then, by Hausdorffness, there are $\varphi, \psi \in \mathrm{LFor}^x_+(\mathtt{L})$ such that $\varphi \top \psi$ and $\varphi \notin \mathrm{ltp}_+(a)$ and $\psi \notin \mathrm{ltp}_+(b)$. Hence, $M \not\models \varphi(a) \vee \psi(b)$ with $\varphi(x) \vee \psi(y) \in \Delta(x, y)$, concluding $M \not\models \Delta(a, b)$. Q.E.D.

Remark 2.30. In classic first-order logic, reducts are also continuous functions between the spaces of types. It is a well known fact that, in positive logic, that is no longer the case. The issue is that the reduct of a positive type could be only a partial positive type.

Let M be a local $\mathtt{L}$-structure, x a variable and A a set. The *space of local positive types* $\mathrm{S}^x(M/A)$ of M on x over A is the set of local positive types of M on x over A; the *space of local positive types* $\mathrm{S}(M/A)$ is the respective sorted set. We omit M if it is clear from the context. We never omit A.

Let M be a local $\mathtt{L}$-structure and A a subset. Let $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L}(A))$ be a subset of local positive formulas. We define

$$\langle \Gamma(x)\rangle_{\mathrm{S}(M/A)} \coloneqq \{p \in \mathrm{S}^x(M/A) : \Gamma \subseteq p\}.$$

As usual, we omit M and x if they are clear from the context. Never omit A.

The *local positive logic topology* of $\mathrm{S}^x(A)$ is the topology defined by taking $\langle\Gamma\rangle$ as closed for every $\Gamma \subseteq \mathrm{LFor}^x_+(\mathtt{L}(A))$ — we add $\emptyset$ as closed set if it is required.

Remark 2.31. Tautologically, $\mathrm{S}(A) = \mathrm{S}(\mathrm{Th}_-(M/A))$ as topological spaces.

Let M be a local $\mathtt{L}$-structure and A a subset. We say that M is *Hausdorff over* A if $\mathrm{S}(A)$ is Hausdorff. In other words, M is Hausdorff over A if $\mathrm{Th}_-(M/A)$ is Hausdorff.

We say that M is *semi-Hausdorff over* A if the diagonal $\Delta^x_+(A) \coloneqq \{(p,q) \in \mathrm{S}^x(A) : p = q\}$ is local positive $\bigwedge_A$-definable for any pointed variable x. In other words, M is semi-Hausdorff over A if $\mathrm{Th}_-(M/A)$ is semi-Hausdorff.

Remark 2.32. Note that, as reducts are not continuous, it is very hard to see relations between the spaces of types over different parameters.

Definition 2.33. Let $M \models \mathtt{T}$ and I be a first-order structure. Let $a = (a_i)_{i\in I}$ be a sequence in M^x for some variable x. We say that $(a_i)_{i\in I}$ is *local positive indiscernible* if $\mathrm{ltp}^M_+(a_{\bar\imath}) = \mathrm{ltp}^M_+(a_{\bar\jmath})$ whenever $\mathrm{qftp}^I(\bar\imath) = \mathrm{qftp}^I(\bar\jmath)$.

Lemma 2.34. *Let $M \models \mathtt{T}$ and I be a linearly ordered set*[1]*. Let $a = (a_i)_{i\in I}$ be a sequence in M^x for some variable x. Assume that there is a bound $\mathrm{B}(x)$ of x and a locality relation d on x such that $M \models \mathrm{B}(a_i)$ and $M \models \mathrm{d}(a_i, a_j)$ for any $i, j \in I$. Then, there is a local model $N \models \mathtt{T}$ and a local positive indiscernible sequence $b = (b_i)_{i\in I}$ of N such that, for any local positive formula $\varphi \in \mathrm{LFor}_+(\mathtt{L})$, we have $N \models \varphi(b_{\bar\jmath})$ whenever $M \models \varphi(a_{\bar\imath})$ for every $\bar\imath$ with $\mathrm{qftp}(\bar\imath) = \mathrm{qftp}(\bar\jmath)$.*

Proof. Take $M \preceq \widetilde{N}$ into a (non-local) κ-saturated structure with $\kappa > |\mathtt{L}| + |I|$ — note that $\widetilde{N}$ is not longer a local structure in general. By the Ehrenfeucht-Mostowski's Standard Lemma, we can find an indiscernible sequence $b = (\widetilde{b}_i)_{i\in I}$ in $\widetilde{N}$ such that, for any formula φ, $\widetilde{N} \models_\star \varphi(b_{\bar\jmath})$ whenever $M \models \varphi(a_{\bar\imath})$ for every $\bar\imath$ with $\mathrm{qftp}(\bar\imath) = \mathrm{qftp}(\bar\jmath)$. Pick elements $o = (o_s)$ for each single sort such that $o_s = b_0$ for the sort of x, o_s is a constant if there are constants of sort s and o_s is taken arbitrarily in the rest of cases. Let $\mathtt{D}(o)$ be the local component at o. By the choice of o and the fact that b_0 satisfies a bound, by [RFS24, Lemma 1.8], we conclude that $\mathtt{D}(o) \models \mathtt{T}$ and satisfies every local formula with parameters in $\mathtt{D}(o)$ satisfied by $\widetilde{N}$. As $\widetilde{N} \models \mathrm{d}(b_0, b_i)$ for each $i \in I$, we know that b lies in $\mathtt{D}(o)$. Therefore, $\mathtt{D}(o)$ contains a local positive indiscernible sequence b such that, for any local positive formula $\varphi \in \mathrm{LFor}_+(\mathtt{L})$, we have $\mathtt{D}(o) \models \varphi(b_{\bar\jmath})$ whenever $M \models \varphi(a_{\bar\imath})$ for every $\bar\imath$ with $\mathrm{qftp}(\bar\imath) = \mathrm{qftp}(\bar\jmath)$. Q.E.D.

We say that two subsets $\Gamma_1(x)$ and $\Gamma_2(y)$ of formulas in $\mathtt{L}$ are *(local) compatible* in $\mathtt{T}$ if there is a local model $M \models \mathtt{T}$ realising both of them, i.e. there are $a \in M^x$ and $b \in M^y$ such that $M \models \Gamma_1(a)$ and $M \models \Gamma_2(b)$.

By definition, $\mathtt{T}$ is irreducible if and only if any two locally satisfiable quantifier free positive formulas are compatible. By [RFS24, Lemma 3.11], weak completeness is equivalent to irreducibility in countable languages, which gives a characterisation of weak completeness in terms of compatibility. The following lemma provides a characterisation of completeness in terms of compatibility.

[1]We can consider other Ramsey first-order structures.

Lemma 2.35. *$\mathtt{T}$ is complete if and only if any local positive type p of $\mathtt{T}$ on a finite variable x and any quantifier free positive formula $\varphi(y)$ with $\exists y\, \varphi(y) \in \mathtt{T}_+$ are compatible in $\mathtt{T}$.*

Proof. One direction is obvious. By Lemma 2.3, p is realised in some $M \models^{\mathrm{pc}} \mathtt{T}$. Thus, if $\mathtt{T}$ is complete, since $\mathrm{Th}_-(M) = \mathtt{T}$, $M \models \mathtt{T}_+$, so $M \models \exists y\, \varphi(y)$, concluding that it realises φ.

Conversely assume $\mathtt{T}$ is incomplete. Then, there is some $M \models^{\mathrm{pc}} \mathtt{T}$ such that M is not a model of $\mathtt{T}_+$. Let $\exists y\, \varphi(y) \in \mathtt{T}_+$, with φ quantifier free, such that $M \models \neg\exists y\, \varphi(y)$. Let $s = (s(y_i))_{y_i \in y}$ be the sort of y. For each single sort $s(y_i)$ pick a single variable $x_{s(y_i)}$ disjoint to y of sort $s(y_i)$ and let $x = \{x_{s(y_i)} : y_i \in y\}$. Let z be a variable disjoint to x and y with no repeated single sorts such that xz is pointed. Choose $a \in M^x$ and $b \in M^z$, and set $p \coloneqq \mathrm{ltp}^M_+(a)$ and $q \coloneqq \mathrm{ltp}^M_+(ab)$.

For each $y_i \in y$, choose a locality relation d_{y_i} of sort $s(y_i)$ and set $\mathrm{d}(x, y) = \bigwedge \mathrm{d}_{y_i}(x_{s(y_i)}, y_i)$. Then, the set of formulas

$$\Sigma_{\mathrm{d}}(x, y, z) \coloneqq q(x, z) \wedge \varphi(y) \wedge \mathrm{d}(x, y)$$

is inconsistent with $\mathtt{T}$.

Indeed, for the sake of contradiction, suppose otherwise and take $N \models_\star \mathtt{T}$ with $c \in N^x$, $d \in N^y$ and $e \in N^z$ such that $N \models_\star \Sigma_{\mathrm{d}}(c, d, e)$. Consider $N' \coloneqq \mathtt{D}(ce)$ (and note $d \in N'$ as $N \models_\star \mathrm{d}(c, d)$). By [RFS24, Lemma 1.8], since $\mathtt{T}$ is a negative local theory and Σ_{d} is a set of local positive formulas, $N' \models \Sigma_{\mathrm{d}}(c, d, e)$ and $N' \models \mathtt{T}$. By [RFS24, Theorem 2.7], there is a homomorphism $h\colon N' \to P$ to $P \models^{\mathrm{pc}} \mathtt{T}$. Since homomorphisms preserve satisfaction of positive formulas, we get that $P \models \Sigma_{\mathrm{d}}(h(c, d, e))$. In particular, $\mathrm{ltp}^P_+(h(c, e)) = q$ by maximality of q. Hence, as $\mathrm{ltp}^P_+(h(c, e)) = q$, we get that $P_{ce} \models \psi$ for any $\psi \in \mathrm{LFor}^0_-(\mathtt{L}(ce))$ realised by M_{ce}, where P_{ce} is the $\mathtt{L}(ce)$-expansion of P given by h. As ce is pointed, this implies that $P_{ce} \models \mathrm{Th}_-(M/ce)$. Hence, as $P \models^{\mathrm{pc}} \mathtt{T}$, we get that $P_{ce} \models^{\mathrm{pc}} \mathrm{Th}_-(M/ce)$ and, similarly, $M_{ce} \models^{\mathrm{pc}} \mathrm{Th}_-(M/ce)$. Therefore, by [RFS24, Theorem 3.5] applied to $\mathrm{Th}_-(M/ce)$, we get that $\mathrm{Th}_-(P_{ce}) = \mathrm{Th}_-(M_{ce})$. Therefore, since $P \models \exists y\, \varphi(y)$, we conclude that M satisfies $\exists y\, \varphi(y)$ as well, getting a contradiction.

Thus, for any choice of locality relations d, there is some $\psi_{\mathrm{d}}(x, z) \in q$ such that $\neg\exists x, y, z\, \psi_{\mathrm{d}}(x, z) \wedge \varphi(y) \wedge \mathrm{d}(x, y) \in \mathtt{T}$ — where the existential quantifier ranges only on the variables that actually appears in the formula. Now, note that $\exists z\, \psi_{\mathrm{d}}(x, z) \in p$. Thus, we conclude that p contradicts $\exists y\, \mathrm{d}(x, y) \wedge \varphi(y)$ for any choice of d. In other words, p and $\exists y\, \varphi(y)$ are incompatible. Q.E.D.

On the other hand, we have the following characterisation of the local joint property in terms of compatibility.

Lemma 2.36. *A local negative theory $\mathtt{T}$ has the local joint property if and only if any two local positive types on pointed variables are compatible.*

Proof. One direction is obvious. Let $p(x)$ and $q(y)$ be local positive types on pointed variables. By definition, p and q are realised in some $A, B \models \mathtt{T}$. If $\mathtt{T}$ has the local joint property, there are homomorphisms $f\colon A \to M$ and $g\colon B \to M$ to $M \models \mathtt{T}$. Then, as homomorphisms preserve satisfaction of positive formulas, M realises p and q.

On the other hand, suppose that any two partial local positive types on pointed variables are compatible. Let $A, B \models \mathtt{T}$. Pick $a \in A^x$ and $b \in B^y$ pointed tuples (with x and y disjoint). Then, by assumption, $p(x) \coloneqq \mathrm{ltp}(a)$ and $q(y) \coloneqq \mathrm{ltp}_+(b)$ are compatible, so $p(x) \wedge q(y)$ is boundedly satisfiable. Pick a feasible bound $\mathrm{B}(x, y)$. Let $M^* \models_\star \mathtt{T}$ be (non-local) κ-saturated. Then, $p(x) \wedge q(y) \wedge \mathrm{B}(x, y)$ is realised in M^* by some a' and b'. By saturation, there are homomorphism $f\colon A \to M^*$ and $g\colon B \to M^*$ with $f(a) = a'$ and $g(b) = b'$. Consider the substructure $M \coloneqq \mathtt{D}(a')$.

By [RFS24, Lemma 1.8], $M \models \mathtt{T}$. As $M^* \models \mathrm{B}(a', b')$, we get that b' is in M. Since A and B are local, we get that $f\colon A \to M$ and $g\colon B \to M$, concluding that $\mathtt{T}$ has the local joint property. Q.E.D.

Question 2.37. *On the light of Lemma 2.35, does compatibility of local positive types on finite variables imply the local joint property?*

3. Retractors

Recall that, unless otherwise stated, we have fixed a local language $\mathtt{L}$ and a locally satisfiable negative local theory $\mathtt{T}$.

Definition 3.1. Let $M \models \mathtt{T}$ and κ a cardinal. We say that M is a *local positively κ-ultrahomogeneous* model of $\mathtt{T}$ if, for any positive embedding $f\colon A \to M$ and any homomorphism $g\colon A \to B$ to $B \models \mathtt{T}$ with $|A|, |B| < \kappa$, there is a homomorphism $h\colon B \to M$ such that $f = h \circ g$. We say that M is *local positively absolutely ultrahomogeneous* model of $\mathtt{T}$ if it is local positively κ-ultrahomogeneous for every cardinal κ.

From now on, for the sake of simplicity, we will abbreviate the terminology by writing "ultrahomogeneous" in place of "local positively ultrahomogeneous".

Lemma 3.2. *Let $M \models \mathtt{T}$ and κ a cardinal. If M is κ-ultrahomogeneous for $\mathtt{T}$, then M_A is κ-ultrahomogeneous for $\mathrm{Th}_-(M/A)$ for every subset A with $|A| < \kappa$. Conversely, if $M \models^{\mathrm{pc}} \mathtt{T}$ is κ-ultrahomogeneous for $\mathrm{Th}_-(M)$, then M is κ-ultrahomogeneous for $\mathtt{T}$.*

Proof. Let $f\colon B_A \to M_A$ be a positive embedding and $g\colon B_A \to C_A$ a homomorphism to $C_A \models \mathrm{Th}_-(M/A)$ with $|B|, |C| < \kappa$. In particular, $B, C \models \mathtt{T}$ so, as M is κ-ultrahomogeneous for $\mathtt{T}$, there is an $\mathtt{L}$-homomorphism $h\colon C \to M$ such that $f = h \circ g$. Now, $h(a^{C_A}) = h(g(a^{B_A})) = f(a^{B_A}) = a$ for any $a \in A$, so h is in fact an $\mathtt{L}(A)$-homomorphism. Hence, we conclude that M_A is κ-ultrahomogeneous for $\mathrm{Th}_-(M/A)$.

Conversely, suppose $M \models^{\mathrm{pc}} \mathtt{T}$ and M is κ-ultrahomogeneous for $\mathrm{Th}_-(M)$. Let $f\colon A \to M$ be a positive embedding and $g\colon A \to B$ a homomorphism with $A, B \models \mathtt{T}$ and $|A|, |B| < \kappa$. By [RFS24, Corollary 2.13], as $M \models^{\mathrm{pc}} \mathtt{T}$, we get $A \models^{\mathrm{pc}} \mathtt{T}$, so in particular g is a positive embedding too. As f and g are positive embeddings, $\mathrm{Th}_-(A) = \mathrm{Th}_-(B) = \mathrm{Th}_-(M)$. Since M is κ-ultrahomogeneous for $\mathrm{Th}_-(M)$, we conclude that there is a homomorphism $h\colon B \to M$ with $f = hg$. Hence, we conclude that M is κ-ultrahomogeneous for $\mathtt{T}$. Q.E.D.

Theorem 3.3. *For any $N \models \mathtt{T}$, there is a homomorphism $h\colon N \to M$ to a κ-ultrahomogeneous $M \models^{\mathrm{pc}} \mathtt{T}$.*

Proof. Without loss of generality assume $\kappa \geq |\mathtt{L}|$. By [RFS24, Theorem 2.7], find a homomorphism $f_{-1}\colon N \to M_0$ with $M_0 \models^{\mathrm{pc}} \mathtt{T}$. Take λ cardinal such that $\kappa \leq \mathrm{cf}(\lambda)$. We recursively define a direct sequence $(M_i, f_{i,j})_{i,j\in\lambda, i\geq j}$ of local positively closed models of $\mathtt{T}$ such that, for any $i, j \in \lambda$ with $j \leq i$ and any homomorphism $g\colon A \to B$ from $A \preceq^+ M_j$ to $B \models \mathtt{T}$ with $|A|, |B| < \kappa$, there is a homomorphism $h\colon B \to M_i$ with $f_{ji|A} = h \circ g$.

Let $\alpha \in \lambda$ limit and suppose $(M_i, f_{ij})_{i,j\in\lambda, i\geq j}$ is already defined. We take respective direct limits $M_\alpha = \varinjlim_{i<\alpha} M_i$, $f_{\alpha,j} = \varinjlim_{j\leq i<\alpha} f_{i,j}$ and $f_{\alpha,\alpha} = \mathrm{id}$. By [RFS24, Lemma 2.6], M_α is a local positively closed model of $\mathtt{T}$. On the other hand, for any $j \in \alpha$, given a homomorphism $g\colon A \to B$ from $A \preceq^+ M_j$ to $B \models \mathtt{T}$ with $|A|, |B| < \kappa$, there is $h\colon B \to M_{j+1}$ homomorphism such that $f_{j+1,j|A} = h \circ g$. Thus, $f_{\alpha,j|A} = h' \circ g$ with $h' = f_{\alpha,j+1} \circ h\colon B \to M_\alpha$ homomorphism.

Suppose M_i and $(f_{ij})_{i\geq j}$ are already defined. Let $(A_\eta, B_\eta, g_\eta)_{\eta\in\mu_i}$ be an enumeration of all the possible homomorphisms $g_\eta\colon A_\eta \to B_\eta$ where $A_\eta \preceq^+ M_i$, $B_\eta \models \mathtt{T}$ and $|A_\eta| < \kappa$ and $B_\eta \in \kappa$. We recursively define a directed sequence $(M_i^\eta, f_i^{\eta,\nu})_{\eta,\nu\in\mu,\eta\geq\nu}$ of local positively closed models of $\mathtt{T}$ such that $M_i^0 = M_i$ and, for any $\eta \in \mu_i$ and $\nu \in \eta$, there is a homomorphism $h\colon B_\nu \to M_i^\eta$ such that $f_i^{\eta,0}{}_{|A_\nu} = h \circ g_\nu$. Then, we take $M_{i+1} = \varinjlim_{\eta\in\mu_i} M_i^\eta$, $f_{i+1,i} = \varinjlim_{\eta\in\mu_i} f_i^{\eta,0}$, $f_{i+1,j} = f_{i+1,i} \circ f_{i,j}$ for $j < i$ and $f_{i+1,i+1} = \mathrm{id}$. By [RFS24, Lemma 2.6], M_{i+1} is a local positively closed model of $\mathtt{T}$. Let $A \preceq^+ M_j$ with $j \leq i$ and $g\colon A \to B$ to $B \models \mathtt{T}$ with $|A|, |B| < \kappa$. Consider $A' = f_{i,j}(A)$ and take $\iota\colon B \to B'$ isomorphism with $|B'| \in \kappa$. Then, there is $\eta \in \mu_i$ such that $A' = A_\eta$, $B' = B_\eta$ and $g_\eta = \iota \circ g \circ (f_{i,j|A})^{-1}$. We have then a homomorphism $h\colon B \to M_i^{\eta+1}$ such that $f_i^{\eta,0} \circ f_{i,j|A} = h \circ \iota \circ g$. Thus, $f_{i+1,j|A} = h' \circ g$ with $h' = f_i^\eta \circ h \circ \iota$, where $f_i^\eta = \varinjlim_{\eta<\nu<\mu_i} f_i^{\nu,\eta}$.

Let $\alpha \in \mu_i$ limit and suppose $(M_i^\eta, f_i^{\eta,\nu})_{\eta,\nu\in\alpha,\eta\geq\nu}$ is already defined. We take the respective direct limits $M_i^\alpha = \varinjlim_{\eta<\alpha} M_i^\eta$, $f_i^{\alpha,\nu} = \varinjlim_{\nu\leq\eta<\alpha} f_i^{\eta,\nu}$ and $f_i^{\alpha,\alpha} = \mathrm{id}$. By [RFS24, Lemma 2.6], we know that M_i^α is a local positively closed model of $\mathtt{T}$. On the other hand, for any $\eta \in \alpha$, there is a homomorphism $h\colon B_\eta \to M_i^{\eta+1}$ such that $f_i^{\eta+1,0}{}_{|A_\eta} = h \circ g_\eta$. Hence, $f_i^{\alpha,0}{}_{|A_\eta} = h' \circ g_\eta$ with $h' = f_i^{\alpha,\eta+1} \circ h$ homomorphism.

Suppose M_i^η and $(f_i^{\eta,\nu})_{\eta\geq\nu}$ are already defined. Consider the negative local $\mathtt{L}(A_\eta)$-theory $\mathtt{T}_i^\eta \coloneqq \mathrm{Th}_-(M_i^\eta/A_\eta)$ where we interpret $a^{M_i^\eta} = f_i^{\eta,0}(a)$ for each $a \in A_\eta$. As $A_\eta \precsim^+ M_i^\eta$ via $f_i^{\eta,0}$, we have that $\mathrm{Th}_-(A_\eta/A) = \mathtt{T}_i^\eta$. By [RFS24, Corollary 2.13], as M_i^η is local positively closed for $\mathtt{T}$, we have that $A_\eta \models^{\mathrm{pc}} \mathtt{T}$. By [RFS24, Lemma 2.9], we conclude that $B_\eta \models \mathtt{T}_i^\eta$ with the interpretation $a^{B_\eta} = g_\eta(a)$ for $a \in A_\eta$. As A_η is pointed, by [RFS24, Theorem 3.5], we conclude that $\mathtt{T}_i^\eta$ has the local joint property. Therefore, using [RFS24, Theorem 2.7], there is a local model $M_i^{\eta+1} \models^{\mathrm{pc}} \mathtt{T}_i^\eta$ and $\mathtt{L}(A_\eta)$-homomorphisms $f_i^{\eta+1,\eta}\colon M_i^\eta \to M_i^{\eta+1}$ and $h\colon B_\eta \to M_i^{\eta+1}$. We take $M_i^{\eta+1}$ to be the $\mathtt{L}$-reduct, $f_i^{\eta+1,\nu} = f_i^{\eta+1,\eta} \circ f_i^{\eta,\nu}$ for $\nu \leq \eta$ and $f_i^{\eta+1,\eta+1} = \mathrm{id}$. As $f_i^{\eta+1,\eta}$ and h are $\mathtt{L}(A_\eta)$-homomorphisms, we have that $f_i^{\eta+1,0}(a) = f_i^{\eta+1,\eta} \circ f_i^{\eta,0}(a) = h \circ g_\eta(a)$ for every $a \in A_\eta$. Finally, note that, by [RFS24, Lemma 2.9], $M_i^{\eta+1} \models^{\mathrm{pc}} \mathtt{T}$.

Take $M = \varinjlim_{i\in\lambda} M_i$, $f_i = \varinjlim_{i\leq j<\lambda} f_{j,i}$ and $f = f_0 \circ f_{-1}\colon N \to M$. By [RFS24, Lemma 2.6], M is a local positively closed model of $\mathtt{T}$. Let $f\colon A \to M$ be a positive embedding and $g\colon A \to B$ a homomorphism with $|A|, |B| < \kappa$. As $\kappa < \mathrm{cf}(\lambda)$, we get $\mathrm{Im}\, f \subseteq \mathrm{Im}\, f_i$ for some $i \in \lambda$. As f and f_i are positive embeddings, there is a positive embedding $f'\colon A \to M_i$ with $f = f_i \circ f'$. By construction, there is $h\colon B \to M_{i+1}$ such that $f' = h \circ g$, so $f = h' \circ g$ with $h' = f_i \circ h$. Hence, M is κ-ultrahomogeneous for $\mathtt{T}$. Q.E.D.

Remark 3.4. Using Löwenheim-Skolem Theorem we can add a bound on the size of M in Theorem 3.3. Indeed, suppose $|N| \leq \kappa$, then, taking $\lambda = 2^\kappa$ and using Löwenheim-Skolem Theorem at each step, we can add that $|M| \leq 2^\kappa$. Explicitly, we can take $|M_i| \leq 2^\kappa$, $\mu_i \leq 2^\kappa$ and $|M_i^\eta| \leq 2^\kappa$ for each $i, \eta \in 2^\kappa$.

Definition 3.5. Let M be a local $\mathtt{L}$-structure and κ a cardinal. We say that M is *local positively κ-saturated* if it realises every local positive type on x over A for any variable $|x| < \kappa$ and any subset A with $|A| < \kappa$ such that M satisfies the local joint property over A. We say that M is *local positively absolutely saturated* if it is local positively κ-saturated for every κ.

From now on, for the sake of simplicity, we will abbreviate the terminology by writing "saturated" in place of "local positively saturated".

Remark 3.6. Note that, by [RFS24, Theorem 3.5], every structure has the local joint property over any pointed set of parameters. Thus, every κ-saturated model realises every local positive type over any pointed set of parameters of size less than κ.

Lemma 3.7. *Assume $\kappa > |\mathtt{L}|$. Every κ-ultrahomogeneous model for $\mathtt{T}$ is κ-saturated. Conversely, every κ-saturated local positively closed model of $\mathtt{T}$ is κ-ultrahomogeneous for $\mathtt{T}$.*

Proof. Let M be κ-ultrahomogeneous for $\mathtt{T}$ and $p \in \mathrm{S}^x(A)$ with $|x| < \kappa$ and A a subset with $|A| < \kappa$ such that M satisfies the local joint property over A. By downwards Löwenheim-Skolem, there is $B_A \preceq M_A$ with $|B| \leq |\mathtt{L}| + |A| < \kappa$. As $B_A \preceq M_A$, we have that $\mathrm{Th}_{-}(B_A) = \mathrm{Th}_{-}(M/A)$. Then, by Lemma 2.14, there is an $\mathtt{L}(A)$-homomorphism $g\colon B_A \to N_A$ to $N_A \models^{\mathrm{pc}} \mathrm{Th}_{-}(M/A)$ such that $N_A \models p(b)$ for some b. Furthermore, we can take $|N| \leq |\mathtt{L}| + |B| + |x| < \kappa$ by using downwards Löwenheim-Skolem Theorem. As M is κ-ultrahomogeneous for $\mathtt{T}$, we find a homomorphism $h\colon N \to M$ such that $\mathrm{id} = h \circ g\colon B \to M$ is the natural inclusion. In particular, $h(a^N) = h(g(a)) = a$, so h is an $\mathtt{L}(A)$-homomorphism too. As homomorphisms preserve satisfaction of positive formulas, we conclude that $h(b)$ realises p in M.

Conversely, suppose $M \models^{\mathrm{pc}} \mathtt{T}$ is κ-saturated. Let $f\colon A \to M$ be a positive embedding and $g\colon A \to B$ a homomorphism with $A, B \models \mathtt{T}$ and $|A|, |B| < \kappa$. Since M is local positively closed for $\mathtt{T}$, so is A by [RFS24, Corollary 2.13]. Hence, g is a positive embedding. As f and g are positive embeddings, we conclude that $\mathrm{Th}_{-}(B_A) = \mathrm{Th}_{-}(A/A) = \mathrm{Th}_{-}(M/A)$ where B_A and M_A are the natural $\mathtt{L}(A)$-expansions given by g and f respectively. Let $b = (b_i)_{i\in I}$ be an enumeration of B and let p be the local positive type of b in B_A. Then, p is a local positive type of $\mathrm{Th}_{-}(M/A)$, i.e. a local positive type of M over $g(A)$. By saturation of M, there is $c = (c_i)_{i\in I}$ in M realising p. Consider the map $h\colon B \to M$ given by $h\colon b_i \mapsto c_i$. Obviously, $f = h \circ g$. Also, as b and c have the same local positive type, they have in particular the same quantifier free positive type, so h is a homomorphism. Q.E.D.

Definition 3.8. Let $f\colon A \to B$ be a homomorphism. A *(local) retraction* of f is a homomorphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}$. A *retractor of* $\mathtt{T}$ is a local model $\mathfrak{C} \models \mathtt{T}$ such that every homomorphism $f\colon \mathfrak{C} \to N$ to $N \models \mathtt{T}$ has a retraction.

Theorem 3.9. *The following are equivalent:*

(i) *$\mathfrak{C}$ is an absolutely ultrahomogeneous model of $\mathtt{T}$.*
(ii) *$\mathfrak{C}$ is a retractor of $\mathtt{T}$.*
(iii) *$\mathfrak{C}$ is an absolutely saturated local positively closed model of $\mathtt{T}$.*

Proof. (i)$\Rightarrow$(ii) Let $f\colon \mathfrak{C} \to A$ be a homomorphism with $A \models \mathtt{T}$. Applying ultrahomogeneity with $\mathrm{id}\colon \mathfrak{C} \to \mathfrak{C}$, there is a homomorphism $g\colon A \to \mathfrak{C}$ such that $\mathrm{id} = g \circ f$, so g is a retraction of f.

(ii)$\Rightarrow$(iii) Take a homomorphism $h\colon \mathfrak{C} \to M$ to $M \models \mathtt{T}$. Since $\mathfrak{C}$ is a retractor, there is $g\colon M \to \mathfrak{C}$ such that $\mathrm{id} = g \circ h$. Take $\varphi \in \mathrm{For}^x_+(\mathtt{L})$ and $a \in \mathfrak{C}^x$. Suppose $M \models \varphi(h(a))$. Then, $\mathfrak{C} \models \varphi(gh(a))$, so $\mathfrak{C} \models \varphi(a)$ as $gh(a) = a$. Hence, we conclude that $\mathfrak{C}$ is local positively closed.

Let A be a subset such that $\mathfrak{C}$ satisfies the local joint property over A and let $p(x)$ be a local positive type of $\mathfrak{C}$ over A. By Lemma 2.14, there is an $\mathtt{L}(A)$-homomorphism $f\colon \mathfrak{C}_A \to N_A$ with $N_A \models^{\mathrm{pc}} \mathrm{Th}_{-}(\mathfrak{C}/A)$ and $b \in N^x$ such that $N_A \models p(b)$. Since $\mathfrak{C}$ is a retractor, there is a homomorphism $g\colon N \to \mathfrak{C}$ such that $\mathrm{id} = g \circ f$. As f is an $\mathtt{L}(A)$-homomorphism, we conclude that g is an $\mathtt{L}(A)$-homomorphism. Thus, $\mathfrak{C}_A \models p(g(b))$, concluding that $\mathfrak{C}$ realises p.

(iii)$\Rightarrow$(i) By Lemma 3.7. Q.E.D.

The following corollary is [Hru22a, Proposition 2.1(3)]. We give the proof for the sake of completeness.

Corollary 3.10. *Let $\mathfrak{C}$ be a retractor of* $\mathtt{T}$. *Then, every endomorphism of $\mathfrak{C}$ is an automorphism.*

Proof. Let $f\colon \mathfrak{C}\to\mathfrak{C}$ be a homomorphism. Since $\mathfrak{C}$ is a retractor, there is $g\colon\mathfrak{C}\to\mathfrak{C}$ such that $\mathrm{id}=g\circ f$. In particular, g is surjective. By Theorem 3.9, $\mathfrak{C}\models^{\mathrm{pc}}\mathtt{T}$, so g is a positive embedding. In particular, g is injective. Hence, g is an automorphism and, *a posteriori*, $f=g^{-1}$ is an automorphism too. Q.E.D.

The following corollary is [Hru22a, Proposition 2.1(4)]. We provide an alternative proof of it.

Corollary 3.11. *Let $\mathfrak{C}$ be a retractor of* $\mathtt{T}$. *Let A be a subset of $\mathfrak{C}$ and x a variable pointed over $\mathtt{L}(A)$. Take $a,b\in\mathfrak{C}^x$. Then, $\mathrm{ltp}^{\mathfrak{C}}_+(a/A)=\mathrm{ltp}^{\mathfrak{C}}_+(b/A)$ if and only if there is an automorphism $\sigma\in\mathrm{Aut}(\mathfrak{C}/A)$ such that $\sigma(a)=b$. In particular, $\mathrm{ltp}^{\mathfrak{C}}_+(a/A)=\mathrm{ltp}^{\mathfrak{C}}_+(b/A)$ if and only if $\mathrm{tp}(a/A)=\mathrm{tp}(b/A)$.*

Proof. The backward direction is obvious. On the other hand, consider the family $\mathcal{K}$ of partial homomorphisms $f\colon D\subseteq\mathfrak{C}$ fixing A and mapping $f(a)=b$. Assuming $\mathrm{ltp}^{\mathfrak{C}}_+(a/A)=\mathrm{ltp}^{\mathfrak{C}}_+(b/A)$, we have that $\mathcal{K}\neq\emptyset$. Obviously, $\mathcal{K}$ is partially ordered by $\subseteq$. Also, for any chain $\Omega\subseteq\mathcal{K}$, we have that $\bigcup\Omega\in\mathcal{K}$. Therefore, by Zorn's Lemma, there is a maximal element $\sigma\in\mathcal{K}$. Write $D\coloneqq\mathrm{Dom}\,\sigma$ and take an arbitrary single element $c\in\mathfrak{C}$. As a,A is pointed, D is pointed. By saturation of $\mathfrak{C}$, we conclude that there is an element d such that $\mathrm{ltp}^{\mathfrak{C}}_+(c,D)=\mathrm{ltp}^{\mathfrak{C}}_+(d,\sigma(D))$. Therefore, $\sigma\cup\{(c,d)\}\in\mathcal{K}$, concluding by maximality that $c\in D$. As c is arbitrary, we conclude that $\sigma\colon\mathfrak{C}\to\mathfrak{C}$ is an endomorphism. By Corollary 3.10, we conclude that $\sigma\in\mathrm{Aut}(\mathfrak{C}/A)$. Q.E.D.

Definition 3.12. Let $M\models\mathtt{T}$ and κ a cardinal. We say that M is *local homomorphism κ-universal for* $\mathtt{T}$ if for every $N\models\mathtt{T}$ with $|N|<\kappa$ there is a homomorphism $f\colon N\to M$. We say that it is *local homomorphism absolutely universal for* $\mathtt{T}$ if it is local homomorphism κ-universal for every cardinal κ.

From now on, for the sake of simplicity, we will abbreviate the terminology by writing "universal" in place of "local homomorphism universal".

The following lemma is [Hru22a, Proposition 2.1(6)]. We give the proof for the sake of completeness.

Lemma 3.13. *Assume* $\mathtt{T}$ *has the local joint property and $\kappa>|\mathtt{L}|$. Then, every κ-ultrahomogeneous model of* $\mathtt{T}$ *is κ-universal for* $\mathtt{T}$. *In particular, when* $\mathtt{T}$ *satisfies the local joint property, there is at most one retractor of* $\mathtt{T}$ *up to automorphism and it is absolutely universal for* $\mathtt{T}$.

Proof. Let M be κ-ultrahomogeneous for $\mathtt{T}$. Take $B\models\mathtt{T}$ with $|B|<\kappa$. By downwards Löwenheim-Skolem Theorem, there is $A\preceq M$ with $|A|<\kappa$. By the local joint property, there are homomorphisms $f\colon A\to N$ and $g\colon B\to N$ to a common $N\models\mathtt{T}$. By downwards Löwenheim-Skolem Theorem again, we can find N with $|N|<\kappa$. Thus, by ultrahomogeneity, there is $h\colon N\to M$ such that $\mathrm{id}=h\circ f$. In particular, we have a homomorphism $h\circ g\colon B\to M$. As B is arbitrary, we conclude that M is κ-universal for $\mathtt{T}$.

Suppose $\mathfrak{C},\mathfrak{C}'$ are retractors of $\mathtt{T}$. Then, by universality of $\mathfrak{C}$, we have a homomorphism $f\colon\mathfrak{C}'\to\mathfrak{C}$. Now, we have the homomorphism $\mathrm{id}\colon\mathfrak{C}'\to\mathfrak{C}'$. By ultrahomogeneity of $\mathfrak{C}'$, there is a homomorphism $g\colon\mathfrak{C}\to\mathfrak{C}'$ such that $\mathrm{id}=g\circ f$. On the other hand, we have the homomorphism $\mathrm{id}\colon\mathfrak{C}\to\mathfrak{C}$. By ultrahomogeneity of $\mathfrak{C}$, there is $f'\colon\mathfrak{C}'\to\mathfrak{C}$ such that $\mathrm{id}=f'\circ g$. Hence, $g\colon\mathfrak{C}\to\mathfrak{C}'$ is an isomorphism and, *a posteriori*, $f=f'=g^{-1}$. Q.E.D.

Without the local joint property, retractors are not necessarily unique. However, we can still classify them in a natural way, getting also a bound on the number of retractors. In [Hru22a], Hrushovski suggested a way of doing so; he suggested considering all the possible local positive types and when they can be realised at the same time. This method seems a bit complicated in the many sorted case. We provide here a rephrasing of that idea that seems to work better in the many sorted case.

Definition 3.14. We say that two local models A and B of $\mathtt{T}$ are *compatible in* $\mathtt{T}$ if there are homomorphisms $f\colon A \to M$ and $g\colon B \to M$ to a common local model $M \models \mathtt{T}$; write $A \frown_{\mathtt{T}} B$. For a local model $A \models \mathtt{T}$, we write $\mathrm{Com}_{\mathtt{T}}(A) \coloneqq \{B \models \mathtt{T} : A \frown_{\mathtt{T}} B\}$ and $\mathrm{Com}^{\mathrm{pc}}_{\mathtt{T}}(A) \coloneqq \{B \in \mathrm{Com}_{\mathtt{T}}(A) : B \models^{\mathrm{pc}} \mathtt{T}\}$. As usual, we omit $\mathtt{T}$ if it is clear from the context.

Lemma 3.15. *Let $M \models^{\mathrm{pc}} \mathtt{T}$. Then, for any $A_1, A_2 \in \mathrm{Com}(M)$, $A_1 \frown A_2$. In particular, $\frown$ is an equivalence relation in the class $\mathcal{M}^{\mathrm{pc}}(\mathtt{T})$ and $\mathrm{Com}^{\mathrm{pc}}(M)$ is the equivalence class of M for every $M \models^{\mathrm{pc}} \mathtt{T}$.*

Proof. As $A_1 \frown M$ and $A_2 \frown M$. There are then homomorphisms $f_1\colon A_1 \to B_1$, $f_2\colon M \to B_1$, $g_1\colon M \to B_2$ and $g_2\colon A_2 \to B_2$ with $B_1, B_2 \models \mathtt{T}$. As $M \models^{\mathrm{pc}} \mathtt{T}$, we have that f_2 and g_1 are positive embeddings, so $B_1, B_2 \models \mathrm{Th}_-(M/M)$. As M is pointed, by [RFS24, Theorem 3.5], there are $\mathtt{L}(M)$-homomorphisms $h_1\colon B_1 \to N$ and $h_2\colon B_2 \to N$ to a common local model $N \models \mathrm{Th}_-(M/M)$. In particular, $h_1 \circ f_1\colon A_1 \to N$ and $h_2 \circ g_2\colon A_2 \to N$ are homomorphisms with $N \models \mathtt{T}$. Therefore, $A_1 \frown A_2$. Q.E.D.

Theorem 3.16. *Let $\mathcal{E}$ be an equivalence class of compatibility in $\mathcal{M}^{\mathrm{pc}}(\mathtt{T})$. Then:*

(1) *$\mathfrak{C} \in \mathcal{E}$ is a retractor of $\mathtt{T}$ if and only if it is universal for $\mathcal{E} = \mathrm{Com}^{\mathrm{pc}}(\mathfrak{C})$.*
(2) *There is at most one retractor in $\mathcal{E}$ up to isomorphism.*
(3) *There is a retractor in $\mathcal{E}$ if and only if $\mathcal{E}$ is bounded, i.e. there is a cardinal κ such that every element of $\mathcal{E}$ has cardinality at most κ.*

Proof. (1) Pick $A \in \mathcal{E}$. As $A \frown \mathfrak{C}$, there are homomorphisms $f\colon A \to B$ and $g\colon \mathfrak{C} \to B$ to $B \models \mathtt{T}$. Now, there is a retraction $h\colon B \to \mathfrak{C}$ of g. In particular, $h \circ f\colon A \to \mathfrak{C}$ is a homomorphism. Since A is arbitrary, we conclude that $\mathfrak{C}$ is universal for $\mathcal{E}$.

Now, suppose $\mathfrak{C}$ is universal for $\mathcal{E}$. By Theorem 3.3, there is a homomorphism $h\colon \mathfrak{C} \to \mathfrak{C}'$ where $\mathfrak{C}'$ is a λ-ultrahomogeneous local positively closed model of $\mathtt{T}$ for any $\lambda > |\mathfrak{C}|$. As $\mathfrak{C}' \in \mathcal{E}$, we conclude that there is a positive embedding $g\colon \mathfrak{C}' \to \mathfrak{C}$. In particular, we have that $|\mathfrak{C}'| \leq |\mathfrak{C}| < \lambda$. By λ-ultrahomogeneity of $\mathfrak{C}'$, there is a homomorphism $f\colon \mathfrak{C} \to \mathfrak{C}'$ such that $\mathrm{id} = f \circ g$, in particular, f is surjective. As $\mathfrak{C}$ is local positively closed for $\mathtt{T}$, f is a positive embedding. Hence, f is a surjective embedding, that is an isomorphism. Hence, $\mathfrak{C}$ is λ-ultrahomogeneous for $\mathtt{T}$. As λ is arbitrary, we conclude that it is a retractor of $\mathtt{T}$.

(2) Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be compatible retractors of $\mathtt{T}$. By (1), there is a homomorphism $f\colon \mathfrak{C}_1 \to \mathfrak{C}_2$. Now, considering $\mathrm{id}\colon \mathfrak{C}_1 \to \mathfrak{C}_1$, by ultrahomogeneity of $\mathfrak{C}_1$, there is a homomorphism $g\colon \mathfrak{C}_2 \to \mathfrak{C}_1$ such that $\mathrm{id} = g \circ f$. Again, considering $\mathrm{id}\colon \mathfrak{C}_2 \to \mathfrak{C}_2$, by ultrahomogeneity of $\mathfrak{C}_2$, there is a homomorphism $f'\colon \mathfrak{C}_1 \to \mathfrak{C}_2$ such that $\mathrm{id} = f' \circ g$. Hence, g is an isomorphism, $\mathfrak{C}_1 \cong \mathfrak{C}_2$ and, *a posteriori*, $f = f' = g^{-1}$.

(3) By (1), if $\mathfrak{C}$ is a retractor in $\mathcal{E}$, then $\mathfrak{C}$ is universal for $\mathcal{E}$, so every element of $\mathcal{E}$ has cardinality at most $|C|$. On the other hand, suppose $\mathcal{E}$ is bounded. Since it is bounded, we can take a set $\mathcal{E}_0$ of representatives of $\mathcal{E}$ such that every element of $\mathcal{E}$ is isomorphic to exactly one element of $\mathcal{E}_0$. Now, $\mathcal{E}_0$ is partially ordered by $\lesssim$, where $A \lesssim B$ if there is an embedding $f\colon A \to B$. Furthermore, note that $\lesssim$ is a

directed order since any $A_1, A_2 \in \mathcal{E}_0$ are compatible. Now, take a chain $\{A_i\}_{i\in I}$ in $\mathcal{E}_0$. By [RFS24, Lemma 2.6], $A \coloneqq \varinjlim A_i$ is a local positively closed model of $\mathtt{T}$ and $A_i \lesssim A$ for every $i \in I$. Therefore, by Zorn's Lemma, there are maximal elements in $\mathcal{E}_0$. As $\lesssim$ is directed, we conclude that, in fact, $\mathcal{E}_0$ has a maximum $\mathfrak{C}$. Thus, $\mathfrak{C}$ is universal for $\mathrm{Com}^{\mathrm{pc}}(\mathfrak{C}) = \mathcal{E}_0$ and, by (1), it is a retractor. Q.E.D.

Remark 3.17. By downwards Löwenheim-Skolem Theorem, using [RFS24, Corollary 2.13], every equivalence class of compatibility in $\mathcal{M}^{\mathrm{pc}}(\mathtt{T})$ contains at least one element A with $|A| \leq |\mathtt{L}|$. Therefore, there are at most $2^{|\mathtt{L}|}$ possible equivalence classes. In particular, $\mathtt{T}$ has at most $2^{|\mathtt{L}|}$ different retractors.

Example 3.18. The bound $2^{|\mathtt{L}|}$ on the number of retractors is sharp:

Let I be a set and $\mathtt{L}$ the one-sorted local language with unary predicates $\{P_i, Q_i\}_{i\in I}$ and locality relations $\mathtt{D} = \{\mathrm{d}_n\}_{n\in\mathbb{N}}$ (with the usual ordered monoid structure induced by the natural numbers). Consider the $\mathtt{L}$-structure M with universe 2^I and interpretations $M \models \mathrm{d}_n(\eta, \zeta) \Leftrightarrow |\{i : \eta(i) \neq \zeta(i)\}| \leq n$ for $n \in \mathbb{N}$, $M \models P_i(\eta) \Leftrightarrow \eta(i) = 1$ and $M \models Q_i(\eta) \Leftrightarrow \eta(i) = 0$ for each $i \in I$. Let $\mathtt{T} = \mathrm{Th}_-(M)$ be the negative theory of M. Since $M \models \mathtt{T}_{\mathrm{loc}}$, $\mathtt{T}$ is locally satisfiable by [RFS24, Theorem 1.9]. For $\eta \in M$, write $M_\eta \coloneqq \mathtt{D}(\eta)$.

Now, we look at M as no-local structure, i.e. as $\mathtt{L}_\star$-structure. Note that the positive logic topology in M is coarser than the Tychonoff topology, so M is compact with the positive logic topology. Thus, by [Seg22, Lemma 2.24], we get that M is universal for $\mathcal{M}_\star(\mathtt{T})$. In particular, by Lemma 6.1, $\mathtt{T}$ has retractors.

Let $\eta \in M$ and suppose $M \models \varphi$ with φ a positive formula in $\mathtt{L}$. For $I_0 \subseteq I$ finite, let $\mathtt{L}_{I_0}$ be the reduct of $\mathtt{L}$ consisting of $\{\mathrm{d}_k\}_{k\in\mathbb{N}} \cup \{P_i, Q_i : i \in I_0\}$ and take I_0 such that φ is a formula in $\mathtt{L}_{I_0}$. Consider the map $f\colon M \to M_\eta$ given by $f(\xi)(i) = \xi(i)$ for $i \in I_0$ and $f(\xi)(i) = \eta(i)$ for $i \notin I_0$. Obviously, $P_i(\xi)$ implies $P_i(f(\xi))$ and $Q_i(\xi)$ implies $Q_i(f(\xi))$ for any $i \in I_0$. On the other hand, $|\{i : f(\xi)(i) \neq f(\zeta)(i)\}| \leq |\{i : \xi(i) \neq \zeta(i)\}|$, so $\mathrm{d}_k(\xi, \zeta)$ implies $\mathrm{d}_k(f(\xi), f(\zeta))$ for any k. Thus, f is an $\mathtt{L}_{I_0}$-homomorphism. As homomorphisms preserve satisfaction, we conclude that $M_\eta \models \varphi$. As φ is arbitrary, we conclude that $\mathrm{Th}_-(M_\eta) = \mathtt{T}$ for any $\eta \in M$. Hence, $\mathtt{T}$ is a local theory.

Note that $M_\eta \not\perp M_\zeta$ for $\eta, \zeta \in M$ with $\{i : \zeta(i) \neq \eta(i)\}$ infinite. In order to reach a contradiction, suppose that there are homomorphisms $f\colon M_\zeta \to N$ and $g\colon M_\zeta \to N$ with $N \models \mathtt{T}$. Then, $\bigwedge_{i\in I} R_i(f(\eta), g(\zeta))$ where $R_i(x, y) = (P_i(x) \wedge Q_i(y)) \vee (Q_i(x) \wedge P_i(y))$ and $I = \{i : \zeta(i) \neq \eta(i)\}$. On the other hand, for any finite subset $J \subseteq I$, $\mathtt{T} \models \neg\exists x, y \bigwedge_{i\in J} R_i(x, y) \wedge \mathrm{d}_{|J|-1}(x, y)$. Thus, we get $N \not\models \mathrm{d}_n(f(\eta), g(\zeta))$ for every $n \in \mathbb{N}$, contradicting that N is a local structure.

The following easy lemma provides a variation of Theorem 3.16(1) which is also a converse to Corollary 3.10.

Lemma 3.19. *Let $\mathfrak{C} \models \mathtt{T}$. Suppose that $\mathfrak{C}$ is universal for* $\mathrm{Com}(\mathfrak{C})$ *and every endomorphism of $\mathfrak{C}$ is an automorphism. Then, $\mathfrak{C}$ is a retractor of* $\mathtt{T}$.

Proof. Let $f\colon \mathfrak{C} \to M$ be a homomorphism to $M \models \mathtt{T}$. By universality of $\mathfrak{C}$ for $\mathrm{Com}(\mathfrak{C})$, there is a homomorphism $g\colon M \to \mathfrak{C}$. Thus, $h \coloneqq g \circ f$ is an endomorphism, so an automorphism. Therefore, $h^{-1} \circ g\colon M \to \mathfrak{C}$ is a retraction of f, concluding that $\mathfrak{C}$ is a retractor. Q.E.D.

4. Local positive logic topologies

Recall that, unless otherwise stated, we have fixed a local language $\mathtt{L}$ and a locally satisfiable negative local theory $\mathtt{T}$.

Let M be a local $\mathtt{L}$-structure, x a variable and A a subset. We say that $D \subseteq M^x$ is *local positive A-definable* if there is a local positive formula $\underline{D}(x)$ with parameters in A defining it, i.e. $D = \{a \in M^x : M \models \underline{D}(a)\}$. We say that D is *local positive $\bigwedge_A$-definable* if there is $\underline{D}(x) \subseteq \mathrm{LFor}^x_+(\mathtt{L}(A))$ defining it, i.e. $D = \{a \in M^x : M \models \underline{D}(a)\}$. The *local positive A-logic topology* of M^x is the topology on M^x given by taking as closed sets the local positive $\bigwedge_A$-definable subsets — we may need to add the empty set. The local positive M-logic topology is simply called the *local positive logic topology.*

Lemma 4.1. *Let M be a local $\mathtt{L}$-structure, x, y two disjoint variables and A a subset. Then, the local positive A-logic topology in $M^x \times M^y$ is finer than the product topology from the local positive A-logic topologies of M^x and M^y.*

Proof. It suffices to show that the Cartesian projections are continuous. By symmetry, we just show that $\mathrm{proj}_x\colon M^x \times M^y \to M^x$ is continuous. Let $V \subseteq M^x$ be locally positive $\bigwedge_A$-definable. Then, $\mathrm{proj}_x^{-1}(V)$ is trivially locally positive $\bigwedge_A$-definable, being defined by the same partial type $\underline{V}$ as V but with extra dummy variable y. Q.E.D.

The main aim of this subsection is to prove the following two fundamental results, one being the converse of the other. The first one was already proved in [Hru22a, Lemma 2.3]. We give here the proof for the sake of completeness.

Theorem 4.2. *Let $\mathfrak{C}$ be a retractor of $\mathtt{T}$. Then, every ball of $\mathfrak{C}$ is compact in the local positive logic topology.*

Proof. Consider a ball $\mathrm{d}(a)$ in $\mathfrak{C}$. Let $\{V_i\}_{i\in I}$ be a family of local positive $\bigwedge_{\mathfrak{C}}$-definable subsets of $\mathfrak{C}$ such that $\bigcap_{i\in I_0} V_i \cap \mathrm{d}(a) \neq \emptyset$ for any finite $I_0 \subseteq I$. Consider $\underline{V}(x) = \bigwedge_{i\in I} \underline{V}_i(x) \wedge \mathrm{d}(x, a)$. Note that $\mathrm{d}(x, a)$ is a bound of x. By hypothesis, every finite subset of $\underline{V}$ is satisfiable in $\mathfrak{C}$, so $\underline{V}$ is boundedly satisfiable. By Lemma 2.10, we conclude that $\underline{V}$ is locally satisfiable in $\mathrm{Th}_-(\mathfrak{C}/\mathfrak{C})$. By saturation of $\mathfrak{C}$ (Theorem 3.9), we conclude that there is $b \in V = \bigcap V_i \cap \mathrm{d}(a)$. As $\{V_i\}_{i\in I}$ is arbitrary, we conclude that $\mathrm{d}(a)$ is compact. Q.E.D.

The next theorem is a stronger version of [Hru22a, Lemma 2.4], closer to the original version for positive logic.

Theorem 4.3. *Let $M \models^{\mathrm{pc}} \mathtt{T}$. Suppose that every ball in a single sort of M is compact in the local positive logic topology. Then, M is a retractor of $\mathtt{T}$.*

Proof. Let $f\colon A \to M$ be a positive embedding and $g\colon A \to B$ a homomorphism with $A, B \models \mathtt{T}$. Consider the family $\mathcal{K}$ of partial homomorphisms $h\colon D \subseteq B \to M$ with $f = h \circ g$ and preserving satisfaction of local positive formulas (i.e. if $B \models \varphi(d)$ with $d \in D$ and φ local positive, then $M \models \varphi(h(d))$). By [RFS24, Corollary 2.13], as $A \precsim^+ M \models^{\mathrm{pc}} \mathtt{T}$, $A \models^{\mathrm{pc}} \mathtt{T}$. Therefore, $g\colon A \to B$ is a positive embedding, so $h_0 = f \circ g^{-1}\colon \mathrm{Im}\, g \to M$ is a partial homomorphism from B to M with $f = h_0 \circ g$. If $B \models \varphi(g(a))$ with φ local positive, then $A \models \varphi(a)$ as g is a positive embedding, so $M \models \varphi(f(a))$, so h preserves satisfaction of local positive formulas. In particular, $\mathcal{K} \neq \emptyset$. Obviously, $\mathcal{K}$ is partially ordered by $\subseteq$ and, for any chain $\Omega \subseteq \mathcal{K}$, $\bigcup \Omega \in \mathcal{K}$. By Zorn's Lemma, there is a maximal element $h \in \mathcal{K}$. Write $D \coloneqq \mathrm{Dom}\, h$ and take an arbitrary single element $b \in B$. For some $a \in A$ and locality relation d, we have $b \in \mathrm{d}(g(a))$. Consider $\Gamma(x, D) = \mathrm{ltp}^B_+(b/D)$. Then, for any $\Delta(x, D) \subseteq \Gamma(x, D)$ finite, we have that $B \models \exists x{\in}\mathrm{d}(g(a)) \bigwedge \Delta(x, D)$. Since h preserves satisfaction of local positive formulas, we have that $M \models \exists x{\in}\mathrm{d}(a) \bigwedge \Delta(x, h(D))$. Therefore, $\Delta(M, h(D)) \cap \mathrm{d}(a) \neq \emptyset$ for any $\Delta \subseteq \Gamma$ finite. By compactness of $\mathrm{d}(a)$ in the local positive logic topology, we conclude that there is $c \in \mathrm{d}(a)$ such that $M \models \Gamma(c, h(D))$. Thus, $h \cup \{(b, c)\} \in \mathcal{K}$, concluding by maximality that $b \in D$. As b is arbitrary, we conclude that $h\colon B \to M$ is a homomorphism with $f = h \circ g$. Q.E.D.

Without local positive closedness, M is not necessarily a retractor. Indeed, the local positive logic topology of a reduct is always coarser than the local positive logic topology on the original language, so every reduct of a retractor still satisfies that balls are compact on the local positive logic topology. However, we know that the reduct of a local positively closed structure might be no local positively closed, so the reduct of a retractor is not necessarily a retractor.

However, it is worth noting that, in the proof of Theorem 4.3, the fact that M is local positively closed for $\mathtt{T}$ is only used to conclude that A is a local positively closed model of $\mathtt{T}$. Therefore, dropping out this hypothesis, we can get the following useful variation.

Theorem 4.4 (Universality Lemma)**.** *Let $M \models \mathtt{T}$. Suppose that every ball in a single sort in M is compact in the local positive logic topology. Then, M_A is universal for* $\mathrm{Th}_-(M/A)$ *for any $A \leq M$. Thus, if $A \models^{\mathrm{pc}} \mathtt{T}$, M is universal for* $\mathrm{Com}^{\mathrm{pc}}(A)$*, and so there is a retractor of $\mathtt{T}$ at A.*

Proof. Let $B_A \models \mathrm{Th}_-(M/A)$. Write $f\colon A \to M$ for the inclusion and let $g\colon A \to B$ be the map given by $a \mapsto a^{B_A}$. Consider the family $\mathcal{K}$ of partial homomorphisms $h\colon D \subseteq B \to M$ preserving satisfaction of local positive formulas such that $f = h \circ g$.

First, note that $\mathcal{K} \neq \emptyset$. If $a_1 \neq a_2$, then $\neg a_1 = a_2 \in \mathrm{Th}_-(M/A)$, so $a_1^{B_A} \neq a_2^{B_A}$. Hence, g is injective. On the other hand, since $B_A \models \mathrm{Th}_-(M/A)$, g preserves satisfaction of local negative formulas, so $g^{-1}\colon \mathrm{Im}\, g \to A$ preserves satisfaction of local positive formulas. Therefore, $h_0 = f \circ g^{-1}\colon \mathrm{Im}\, g \to M$ is a partial homomorphism from B to M with $f = h_0 \circ g$ preserving satisfaction of local positive formulas, concluding that $h_0 \in \mathcal{K}$.

Obviously, $\mathcal{K}$ is partially ordered by $\subseteq$ and, for any chain $\Omega \subseteq \mathcal{K}$, $\bigcup \Omega \in \mathcal{K}$. By Zorn's Lemma, there is a maximal element $h \in \mathcal{K}$. As in the proof of Theorem 4.3, we conclude that $\mathrm{Dom}\, h = B$, so, in other words, $h\colon B_A \to M_A$ is an $\mathtt{L}(A)$-homomorphism.

Suppose now that $A \models^{\mathrm{pc}} \mathtt{T}$. Then, we have that $B_A \models \mathrm{Th}_-(M/A)$ if and only if $B \models \mathtt{T}$ and $A \lesssim B$ — where B is the $\mathtt{L}$-reduct of B_A and B_A is the $\mathtt{L}(A)$-expansion of B given by the embedding $A \lesssim B$. Indeed, if $B_A \models \mathrm{Th}_-(M/A)$, obviously $B \models \mathtt{T}$ and $g\colon a \mapsto a^B$ defines a homomorphism from A to B. On the other hand, as $A \models^{\mathrm{pc}} \mathtt{T}$, if $B \models \mathtt{T}$ and there is a homomorphism $g\colon A \to B$, then $B_A \models \mathrm{Th}_-(A/A) = \mathrm{Th}_-(M/A)$ via $a^{B_A} = g(a)$.

Now, let $B \models \mathtt{T}$ with $A \frown B$. There is $N \models \mathtt{T}$ such that $A \lesssim N$ and $B \lesssim N$. It follows that $N_A \models \mathrm{Th}_-(M/A)$, so $N \lesssim M$ by universality of M for $\mathrm{Th}_-(M/A)$. In particular, $B \lesssim M$. As $B \in \mathrm{Com}(A)$ is arbitrary, we conclude that M is universal for $\mathrm{Com}(A)$. In particular, $|N| \leq |M|$ for every $N \in \mathrm{Com}^{\mathrm{pc}}(A)$, so there is a retractor of $\mathtt{T}$ at A by Theorem 3.16(3). Q.E.D.

Definition 4.5. Let d be a locality relation. An *entourage in* d is a pair $\psi(x,y)$ and $\varphi(x,y)$ of local positive formulas such that $\psi(x,y) \leq \mathrm{d}(x,y)$, $\varphi(x,y) \perp x = y$ and $\psi(x,y) \top \varphi(x,y)$. We say that $\mathtt{T}$ has *entourages* if it has entourages in all sorts.

Remark 4.6. If (ψ,φ) is an entourage, then $= \,\leq \psi$ by definition of $\psi \top \varphi$.

Lemma 4.7. *Let* d *and* d' *be locality relations.*

(1) *Let (ψ,φ) and (ψ',φ') be entourages in* d*. Then, $(\psi \wedge \psi', \varphi \vee \varphi')$ is an entourage in* d.

(2) *Let (ψ,φ) be an entourage in* d*. Let ψ_0 and φ_0 be local positive formulas with $\psi \leq \psi_0 \leq \mathrm{d}$ and $\varphi \leq \varphi_0$ with $\varphi_0 \perp\, =$. Then, (ψ_0,φ_0) is an entourage in* d.

(3) *Suppose* d *and* d' *are on the same sort. Let (ψ,φ) be an entourage in* d*. If* $\mathrm{d} \leq \mathrm{d}'$*, then it is an entourage in* d'.

(4) *Let* $(\psi(x,y),\varphi(x,y))$ *be an entourage in* $\mathrm{d}(x,y)$ *and* $(\psi'(x',y'),\varphi'(x',y'))$ *an entourage in* $\mathrm{d}'(x',y')$. *Then,* $(\psi(x,y)\wedge\psi'(x',y'),\varphi(x,y)\vee\varphi'(x',y'))$ *is an entourage in* $\mathrm{d}_1\times\mathrm{d}_2$.

Proof. Elementary by [RFS24, Lemma 2.14]. Q.E.D.

Lemma 4.8. *Let* $M\models^{\mathrm{pc}}\mathtt{T}$. *Let* d *be a locality relation and* (ψ,φ) *an entourage in* d. *Then, every* d*-ball is a closed neighbourhood of its centre in the local positive logic topology of* M. *Furthermore,* $\psi(M,a)$ *is a closed neighbourhood of* a.

Proof. Obviously, $\psi(M,a)$ and $\varphi(M,a)$ are closed subsets of M^x in the local positive logic topology. As $\varphi(x,y)\perp x=y$, $a\notin\varphi(M,a)$. On the other hand, as $\psi\top\varphi$ and $M\models^{\mathrm{pc}}\mathtt{T}$, we get that $\varphi(M,a)\cup\psi(M,a)=M^x$ by [RFS24, Lemma 2.14(2)]. Thus, $a\in\neg\varphi(M,a)\subseteq\psi(M,a)$, concluding that $\psi(M,a)$ is a closed neighbourhood of a. Finally, as $\psi\leq\mathrm{d}$ and $M\models^{\mathrm{pc}}\mathtt{T}$, by [RFS24, Lemma 2.14(2)], we get $\psi(M,a)\subseteq\mathrm{d}(a)$, so $\mathrm{d}(a)$ is a closed neighbourhood of a as well. Q.E.D.

Corollary 4.9. *Let* $\mathfrak{C}$ *be a retractor of* $\mathtt{T}$. *Suppose* d *has an entourage. Then, for every* a*, the* d*-ball at* a *is a compact neighbourhood of* a *in the local positive logic topology. In particular,* $\mathfrak{C}$ *is weakly locally closed compact in the local positive logic topology whenever* $\mathtt{T}$ *has entourages.*

Proof. By Theorem 4.2 and Lemma 4.8. Q.E.D.

Definition 4.10. Let d be a locality relation. A *full system of entourages* is a family $\{(\psi_i,\varphi_i)\}_{i\in I}$ of entourages in d such that $\{\psi_i\}_{i\in I}$ is a full system of approximations of $=$. We say that $\mathtt{T}$ has *full systems of entourages* if it has a full system of entourages for every sort.

Corollary 4.11. *Let* $\mathfrak{C}$ *be a retractor of* $\mathtt{T}$. *Let* $\mathrm{d}(x,y)$ *be a locality relation and* $\{(\psi_i,\varphi_i)\}_{i\in I}$ *a full system of entourages in* d. *Then, for every* $a\in\mathfrak{C}^y$*, the family* $\{\bigwedge_{i\in I_0}\psi_i(\mathfrak{C},a): I_0\subseteq I \text{ finite}\}$ *is a local base of closed compact neighbourhoods of* a *in the local positive logic topology. In particular,* $\mathfrak{C}$ *is locally closed compact in the local positive logic topology whenever* $\mathtt{T}$ *has full systems of entourages.*

Proof. By Lemma 4.7(1), $(\bigwedge_{i\in I_0}\psi_i,\bigvee_{i\in I_0}\varphi_i)$ is an entourage of d for any $I_0\subseteq I$ finite. Therefore, by Corollary 4.9, $\{\bigwedge_{i\in I_0}\psi_i(\mathfrak{C},a) : I_0\subseteq I \text{ finite}\}$ is family of compact neighbourhoods of a in the local positive logic topology. Let $\phi(x)$ be a local positive formula with $\mathfrak{C}\not\models\phi(a)$. Since $\{\psi_i\}_{i\in I}$ is a full system of approximations, $\bigcap_{i\in I}\psi_i(\mathfrak{C},a)=\{a\}$. Thus, $\bigcap_{i\in I}\psi_i(\mathfrak{C},a)\cap\phi(\mathfrak{C})=\emptyset$. By compactness of $\mathrm{d}(a)$ (Theorem 4.2), we conclude that there is $I_0\subseteq I$ finite with $\bigcap_{i\in I_0}\psi_i(\mathfrak{C},a)\cap\phi(\mathfrak{C})=\emptyset$, so $\bigwedge_{i\in I_0}\psi_i(\mathfrak{C},a)\subseteq\neg\phi(\mathfrak{C})$. Q.E.D.

5. Local positive compactness

Recall that, unless otherwise stated, we have fixed a local language $\mathtt{L}$ and a locally satisfiable negative local theory $\mathtt{T}$.

We say that $\mathtt{T}$ is *local positive compact*[2] if for any local positive formula $\varphi(x,y,z)$ with $\exists z\,\varphi(x,y,z)\perp x=y$, any bound $\mathrm{B}(x,z)$ and any locality relation symbol $\mathrm{d}(x,y)$, where x and y are different single variables on the same sort and z is an arbitrary finite variable, there is a number $k\in\mathbb{N}_{>0}$ such that

$$\mathtt{T}\models\neg\exists x_1,\ldots,x_k,z\bigwedge_{i\neq j}\mathrm{B}(x_i,z)\wedge\mathrm{d}(x_i,x_j)\wedge\varphi(x_i,x_j,z).$$

[2]Since Hrushovski preferably works with (local) primitive positive formulas, this property is called "(local) primitive positive compactness", abbreviated (ppC), in [Hru22a]. In his paper, Hrushovski attributes the origin of this property to Walter Taylor.

In [Hru22a, Proposition 2.1(1)], Hrushovski proved that every local positively compact theory has retractors. Furthermore, in [Hru22a], Hrushovski briefly argues (assuming the local joint property) that local positive compactness, existence of retractors and boundedness of local positively closed models are equivalent properties — these equivalences are true under weak completeness. Here, in this subsection, we provide the full proof of these equivalences and show how they vary under different completeness conditions.

We start reproving that local positive compactness implies the existence of retractors. Although this was already proved in [Hru22a, Proposition 2.1(1)], we provide the full proof for the sake of completeness.

Theorem 5.1. *Suppose* $\mathtt{T}$ *is local positively compact. Then, every* $|\mathtt{L}|^+$*-ultrahomogeneous local positively closed model of* $\mathtt{T}$ *is a retractor. In particular,* $\mathtt{T}$ *has retractors.*

Proof. Let $\mathfrak{C}$ be an $|\mathtt{L}|^+$-ultrahomogeneous local positively closed model of $\mathtt{T}$. By Theorem 3.3, find a homomorphism $f\colon \mathfrak{C}\to\mathfrak{C}'$ to a κ-ultrahomogeneous local positively closed model $\mathfrak{C}'$ of $\mathtt{T}$ with $\kappa\geq|\mathtt{L}|^+$. As $\mathfrak{C}$ is local positively closed, we know that f is a positive embedding. To simplify the notation, we identify $\mathfrak{C}$ with its image, so $\mathfrak{C}\preceq^+\mathfrak{C}'$. We claim that $\mathfrak{C}=\mathfrak{C}'$ — in other words, f is an isomorphism. As κ is arbitrary, this concludes that $\mathfrak{C}$ is a retractor of $\mathtt{T}$.

In order to reach a contradiction, suppose falsity of our claim. There is then some (single) element b in $\mathfrak{C}'$ outside $\mathfrak{C}$. Pick A_0 pointed subset of $\mathfrak{C}$ such that $|A_0|\leq|\mathtt{L}|$. We define by recursion a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathfrak{C}$ such that $\mathrm{ltp}^{\mathfrak{C}'}_+(a_i/A_i)=\mathrm{ltp}^{\mathfrak{C}'}_+(b/A_i)$ for every $i\in\mathbb{N}$, where $A_i=A_0\cup\{a_0,\ldots,a_{i-1}\}$. Indeed, suppose $a_0,\ldots,a_{i-1}$ are already defined and denote $\mathfrak{C}_i\coloneqq\mathfrak{C}_{A_i}$ and $\mathfrak{C}'_i\coloneqq\mathfrak{C}'_{A_i}$. Consider the local positive type $p(x)=\mathrm{ltp}^{\mathfrak{C}'_i}_+(b/A_i)$ of $\mathfrak{C}'$ over A_i. As $\mathfrak{C}\preceq^+\mathfrak{C}'$, $\mathrm{Th}_-(\mathfrak{C}'_i)=\mathrm{Th}_-(\mathfrak{C}_i)$. Then, $p(x)$ is a local positive type of $\mathfrak{C}$ over A_i. By saturation of $\mathfrak{C}$ (Lemma 3.7), we conclude that there is a_i in $\mathfrak{C}$ realising p. Hence, $\mathrm{ltp}^{\mathfrak{C}}_+(a_i/A_i)=\mathrm{ltp}^{\mathfrak{C}'}_+(b/A_i)$. As $\mathfrak{C}\preceq^+\mathfrak{C}'$, we have $\mathrm{ltp}^{\mathfrak{C}'}_+(a_i/A_i)=\mathrm{ltp}^{\mathfrak{C}}_+(a_i/A_i)=\mathrm{ltp}^{\mathfrak{C}'}_+(b/A_i)$.

Let $\mathrm{B}(x,c)$ be any bound realised by b over A_0. Also, as $\mathfrak{C}'$ is a local structure, pick a locality relation with $\mathfrak{C}'\models\mathrm{d}(a_0,b)$. It follows that $\mathfrak{C}'\models\mathrm{B}(a_i,c)$ and $\mathfrak{C}'\models\mathrm{d}(a_i,a_j)$ for any $i,j\in\mathbb{N}$. By Lemma 2.34, and using saturation of $\mathfrak{C}$, we can find a local positive A_0-indiscernible sequence $(b_i)_{i\in\mathbb{N}}$ in $\mathfrak{C}$ such that $\mathrm{ltp}^{\mathfrak{C}}_+(b_i/A_0,b_0,\ldots,b_{i-1})=\mathrm{ltp}^{\mathfrak{C}'}_+(b/A_0,b_0,\ldots,b_{i-1})$ for any $i\in\mathbb{N}$. Again, as $\mathfrak{C}\preceq^+\mathfrak{C}'$, we conclude that $(b_i)_{i\in\mathbb{N}}$ is local positive A_0-indiscernible in $\mathfrak{C}'$ and $\mathrm{ltp}^{\mathfrak{C}'}_+(b_i/A_0,b_0,\ldots,b_{i-1})=\mathrm{ltp}^{\mathfrak{C}'}_+(b/A_0,b_0,\ldots,b_{i-1})$ for any $i\in\mathbb{N}$.

By [RFS24, Lemma 2.12], as $\mathfrak{C}'$ is local positively closed and $b_0\neq b$, there is a positive formula $\varphi_0(x,y)\in\mathrm{LFor}^{x,y}_+(\mathtt{L})$ such that $\varphi_0(x,y)\perp x=y$ and $\mathfrak{C}'\models\varphi_0(b,b_0)$. Say $\varphi_0=\exists w\,\phi_0(x,y,w)$ and take c in A_0 on the sort of w. There is a locality relation d_1 such that $\mathfrak{C}'\models\varphi_1(b,b_0,c)$ with $\varphi_1(x,y,z)=\exists w{\in}\mathrm{d}_1(z)\,\phi_0(x,y,w)$. Take $\varphi(x,y,z)=\varphi_1(x,y,z)\vee\varphi_1(y,x,z)$ and note that $\varphi(x,y,z)$ is a local positive formula with $\exists z\,\varphi(x,y,z)\perp x=y$ and $\mathfrak{C}'\models\varphi(b,b_0,c)$. Hence, $\mathfrak{C}'\models\varphi(b_i,b_0,c)$ for any $i\in\mathbb{N}_{>0}$. As $(b_i)_{i\in\mathbb{N}}$ is local positive A_0-indiscernible, we conclude that $\mathfrak{C}'\models\varphi(b_i,b_j,c)$ for any $i\neq j$. Hence, $\mathfrak{C}'\models\bigwedge_{i\neq j}\mathrm{B}(b_i,c)\wedge\mathrm{d}(b_i,b_j)\wedge\varphi(b_i,b_j,c)$. Thus, we have $\exists x_1,\ldots,x_k,z\bigwedge_{i\neq j}\mathrm{B}(x_i,z)\wedge\mathrm{d}(x_i,x_j)\wedge\varphi(x_i,x_j,z)\in\mathtt{T}_+$ for every $k\in\mathbb{N}$ as witnessed by $\mathfrak{C}'$. In particular, $\mathtt{T}$ is not local positively compact, contradicting our initial assumption. Q.E.D.

We now list all the conditions equivalent to existence of retractors. We start with a version that does not require irreducibility conditions:

Theorem 5.2. *Let* $M\models^{\mathrm{pc}}\mathtt{T}$*. Then, the following are equivalent:*

(1) *There is a homomorphism* $h\colon M\to N$ *with* $N\models \mathtt{T}$ *whose balls in single sorts are compact in the local positive logic topology.*

(2) $\mathtt{T}$ *has a retractor at* M*.*

(3) $\mathrm{Com}^{\mathrm{pc}}(M)$ *is bounded.*

(4) $\mathrm{Th}_-(M/M)$ *is local positively compact (in* $\mathtt{L}(M)$*).*

(5) *Every* $|\mathtt{L}(M)|^+$*-ultrahomogeneous local positively closed model* N *of* $\mathtt{T}$ *with* $M\precsim^+ N$ *is a retractor of* $\mathtt{T}$*.*

Proof. (1)⇔(2) By Theorems 4.2 and 4.4.

(2)⇔(3) By Theorem 3.16(3).

(2)⇒(4) Let x be a single variable and z a finite variable. Take a bound $\mathrm{B}(x,z)$ of x in $\mathrm{Th}_-(M/M)$, a local positive formula $\varphi(x,y,z)\in \mathrm{LFor}_+(\mathtt{L}(M))$ with $\mathrm{Th}_-(M/M)\models \exists z\,\varphi(x,y,z)\perp x=y$ and a locality relation $\mathrm{d}(x,y)$. In order to reach a contradiction, suppose that

$$M\models \exists x_1,\ldots,x_k,z\bigwedge_{i\neq j}\mathrm{B}(x_i,z)\wedge \mathrm{d}(x_i,x_j)\wedge\varphi(x_i,x_j,z)$$

for every $k\in\mathbb{N}$. Consider the partial local positive type

$$\Gamma(\bar x,z)=\bigwedge_{i\neq j}\mathrm{B}(x_i,z)\wedge \mathrm{d}(x_i,x_j)\wedge\varphi(x_i,x_j,z)$$

with $\bar x=\{x_i\}_{i\in\mathbb{N}}$ countable set of single variables. By supposition, $\Gamma(\bar x,z)$ is finitely locally satisfiable in $\mathrm{Th}_-(M/M)$. As $\Gamma(\bar x,z)$ implies a bound of $\bar x,z$, by Lemma 2.10, we conclude that $\Gamma(\bar x,z)$ is locally satisfiable in $\mathrm{Th}_-(M/M)$. Let $\mathfrak{C}$ be a retractor of $\mathtt{T}$ for M. As M is local positively closed, $\mathrm{Th}_-(\mathfrak{C}/M)=\mathrm{Th}_-(M/M)$. By Lemma 3.2, $\mathfrak{C}_M$ is a retractor of $\mathrm{Th}_-(M/M)$. By Theorem 3.9(iii), $\mathfrak{C}$ satisfies $\Gamma(\bar x,z)$. Therefore, there are $(a_i)_{i\in\mathbb{N}}$ and c in $\mathfrak{C}$ with $a_i\in\mathrm{d}(a_0)$ for each $i\in\mathbb{N}$ and $\mathfrak{C}\models\varphi(a_i,a_j,c)$ for each $i\neq j$ — in particular, $a_i\neq a_j$ for $i\neq j$. Applying Zorn's Lemma, find $(a_i)_{i\in\lambda}$ maximal such that $a_i\in\mathrm{d}(a_0)$ for every $i\in\lambda$ and $\mathfrak{C}\models\varphi(a_i,a_j,c)$ for each $i\neq j$ — in particular, $a_i\neq a_j$ for $i\neq j$, so $\lambda\leq|C|$. Now, consider $U_i=\neg\varphi(\mathfrak{C},a_i,c)\cup\neg\varphi(a_i,\mathfrak{C},c)$. By maximality of $(a_i)_{i\in\lambda}$, it follows that $\mathrm{d}(a_0)\subseteq\bigcup_{i\in\lambda}U_i$. By Theorem 4.2, $\mathrm{d}(a_0)$ is compact in the local positive logic topology. As $\{U_i\}_{i\in\lambda}$ is an open covering, there is $I_0\subseteq\lambda$ finite such that $\mathrm{d}(a_0)\subseteq\bigcup_{i\in I_0}U_i$. As I_0 is finite, there is $n\in\mathbb{N}$ with $n\notin I_0$. Then, by choice of $(a_i)_{i\in\lambda}$, we have that $a_n\in\mathrm{d}(a_0)$ and $a_n\notin\bigcup_{i\in I_0}U_i$, getting a contradiction.

(4)⇒(5) By Theorem 5.1.

(5)⇒(2) By Theorem 3.3. Q.E.D.

Corollary 5.3. *We have* $|\mathfrak{C}|\leq 2^{|\mathtt{L}|}$ *for every retractor* $\mathfrak{C}$ *of* $\mathtt{T}$*.*

Proof. By Löwenheim-Skolem, take $M\preceq\mathfrak{C}$ with $|M|\leq|\mathtt{L}|$. Using Theorem 3.3 and Remark 3.4, find an $|\mathtt{L}|^+$-ultrahomogeneous $\mathfrak{C}_0$ model of $\mathtt{T}$ with $|\mathfrak{C}_0|\leq 2^{|\mathtt{L}|}$ and $M\leq\mathfrak{C}_0$. Since $M\preceq\mathfrak{C}$ and $\mathfrak{C}\models^{\mathrm{pc}}\mathtt{T}$, we have $M\models^{\mathrm{pc}}\mathtt{T}$ by [RFS24, Corollary 2.13]. Thus, $M\preceq^+\mathfrak{C}_0$. As $\mathfrak{C}$ is a retractor of $\mathtt{T}$ at M, by Theorem 5.2, $\mathrm{Th}_-(M)$ is local positively compact. As $M\preceq^+\mathfrak{C}_0$, $\mathrm{Th}_-(\mathfrak{C}_0)=\mathrm{Th}_-(M)$. By Lemma 3.2, $\mathfrak{C}_0$ is $|\mathtt{L}|^+$-ultrahomogeneous for $\mathrm{Th}_-(M)$. By Theorem 5.1, we conclude that $\mathfrak{C}_0$ is a retractor of $\mathrm{Th}_-(M)$. By Lemma 3.2, as $\mathfrak{C}_0\models^{\mathrm{pc}}\mathtt{T}$, we conclude that it is a retractor of $\mathtt{T}$ too. Note that $\mathfrak{C}\frown M\frown\mathfrak{C}_0$ and $M\models^{\mathrm{pc}}\mathtt{T}$. By Lemma 3.15, $\mathfrak{C}\frown\mathfrak{C}_0$. By Theorem 3.16(2), we conclude that $\mathfrak{C}\cong\mathfrak{C}_0$. In particular, $|\mathfrak{C}|\leq 2^{|\mathtt{L}|}$. Q.E.D.

Corollary 5.4. *The following are equivalent:*

(1) $\mathtt{T}$ *has retractors.*

(2) *There is some cardinal* κ *such that* $|M|<\kappa$ *for every* $M\models^{\mathrm{pc}}\mathtt{T}$*.*

(3) *For every* $M\models^{\mathrm{pc}}\mathtt{T}$*,* $|M|\leq 2^{|\mathtt{L}|}$*.*

We now state the equivalences under irreducibility assumptions.

Corollary 5.5. *Assume* $\mathtt{T}$ *is weakly complete. Then,* $\mathtt{T}$ *has retractors if and only if it is local positively compact.*

Proof. By Theorem 5.1, if $\mathtt{T}$ is local positively compact, then $\mathtt{T}$ has retractors. On the other hand, suppose that $\mathtt{T}$ is weakly complete and has retractors. Let $\mathfrak{C}$ be a retractor at M with $M \models^{\mathrm{pc}} \mathtt{T}_{\pm}$. As homomorphisms preserve satisfaction, we get $\mathrm{Th}_-(\mathfrak{C}) = \mathtt{T}$. By Theorem 5.2, $\mathrm{Th}_-(\mathfrak{C}/\mathfrak{C})$ is local positively compact so, in particular, $\mathtt{T} = \mathrm{Th}_-(\mathfrak{C})$ is local positively compact. Q.E.D.

Corollary 5.6. *Assume* $\mathtt{T}$ *is complete. Then,* $\mathtt{T}$ *has retractors if and only if it has at least one retractor.*

Proof. Suppose $\mathtt{T}$ is complete and has at least one retractor $\mathfrak{C}$. By Theorem 5.2, we conclude that $\mathrm{Th}_-(\mathfrak{C}/\mathfrak{C})$ is local positively compact. In particular, $\mathrm{Th}_-(\mathfrak{C})$ is local positively compact. By completeness, $\mathtt{T} = \mathrm{Th}_-(\mathfrak{C})$ is local positively compact. Then, by Theorem 5.1, $\mathtt{T}$ has retractors. Q.E.D.

Example 5.7. Corollary 5.6 may not hold if $\mathtt{T}$ is merely weakly complete. Indeed, take the one-sorted local language $\mathtt{L}$ with locality relations $\mathtt{D} = \{\mathrm{d}_n\}_{n\in\mathbb{N}}$ (with the usual ordered monoid structure induced by the natural numbers), unary predicates $\{P_n, Q_n\}_{n\in\mathbb{N}}$ and a binary relation $\prec$.

Consider the local structure in $\mathbb{R}$ given by $P_n(x) \Leftrightarrow x \geq n$, $Q_n(x) \Leftrightarrow x \leq -n$, $x \prec y \Leftrightarrow 0 < x < y$ and $\mathrm{d}_n(x, y) \Leftrightarrow |x - y| \leq n$. Let $\mathtt{T} = \mathrm{Th}_-(\mathbb{R})$. Then, $\mathtt{T}$ is not local positive compact. Indeed, we can take $\varphi(x, y) \coloneqq x \prec y \vee y \prec x$, that satisfies $\varphi(x, y) \perp x = y$, and the locality relation d_1. In that case, we find that $S = \{1/i\}_{i\in\mathbb{N}_+}$ satisfies that, for any two distinct i, j, we have $\varphi(i, j) \wedge \mathrm{d}_1(i, j)$. Thus, by Corollary 5.5, $\mathtt{T}$ does not have retractors (that is, there is some models of $\mathtt{T}$ incompatible with all retractors).

On the other hand, let $\widetilde{\mathbb{R}}$ be an ω-saturated elementary extension of $\mathbb{R}$. Then, since $\bigwedge_n Q_n(x)$ is finitely satisfiable, there is some $a \in \widetilde{\mathbb{R}}$ realising it. Consider the substructure A with universe $\{a\}$. Since $\widetilde{\mathbb{R}} \cong \mathbb{R}$, we have $\widetilde{\mathbb{R}} \models_{\star} \mathtt{T}$, so $A \models \mathtt{T}$. Furthermore, A is a retractor as, if $g\colon A \to M$ is a homomorphism with $M \models \mathtt{T}$, then the constant function $f\colon M \to A$ mapping everything to a is homomorphism too (so a retractor). Indeed, if there is a homomorphism $g\colon A \to M$, we must have that $P_n(M) = \prec (M) = \emptyset$, since $M \models Q_{m+1}(h(a))$ while $\neg\exists x, y\, Q_{m+1}(x) \wedge P_n(y) \wedge \mathrm{d}_{m+1}(x, y) \in \mathtt{T}$ and $\neg\exists x, y, z\, Q_{m+1}(x) \wedge y \prec z \wedge \mathrm{d}_m(x, y) \in \mathtt{T}$ for all $m \in \mathbb{N}$. On the other hand, for any $n \in \mathbb{N}$, we have $A \models Q_n(a)$ and $A \models \mathrm{d}_n(a, a)$, thus f is a homomorphism.

6. Local retractors and non-local universal models

As covered in [Hru22b, Subsection 2.5], negative theories have a non-local (homomorphism) universal positively closed model if and only if there is a bound on the cardinality of their non-local positively closed models. Since every negative local theory is, in particular, a negative theory, it is natural to ask whether there is a connection between the existence of local retractors and the existence of non-local universal positively closed models. One direction is straightforward:

Lemma 6.1. *Assume* $\mathtt{T}$ *is a negative local theory that has a non-local universal model* $\mathfrak{U}$*. Then,* $\mathtt{T}$ *has local retractors, and every local retractor embeds into* $\mathfrak{U}$*.*

Proof. Let $M \models^{\mathrm{pc}} \mathtt{T}$. Then, since $M \models_{\star} \mathtt{T}$, there is some homomorphism $h\colon M \to \mathfrak{U}$. Since $h(M) \models \mathtt{T}$, we conclude that h is a positive embedding into $h(M)$ and, in particular, injective. We get that $|M| \leq |\mathfrak{U}|$ for all $M \models^{\mathrm{pc}} \mathtt{T}$. Thus, by Theorem 3.16(3), $\mathtt{T}$ has retractors. Q.E.D.

Unfortunately, this is the only connection that holds — a theory with local retractors does not need to have a non-local universal model.

Example 6.2. Consider the one-sorted local language $\mathtt{L} = \{0, 1, +, \cdot\}$ with locality relations $\mathtt{D} = \{\mathrm{d}_n\}_{n\in\mathbb{N}}$ (with the usual ordered monoid structure induced by the natural numbers), where $0, 1$ are constants and $+, \cdot$ are ternary relations.

Consider the structure $\mathbb{R}$ with the usual field interpretation ($+, \cdot$ are the graphs of addition and multiplication respectively) and metric interpretation of the locality relations. By Theorem 4.4, $\mathbb{R}$ is local universal for $\mathtt{T}_{\mathbb{R}} = \mathrm{Th}_-(\mathbb{R})$. Since every field endomorphism of $\mathbb{R}$ is the identity, it is in particular an automorphism thus $\mathbb{R}$ is a retractor by Lemma 3.19. Since $\mathbb{R}$ is local universal, it is the unique retractor (up to isomorphism).

However, as a negative theory, $\mathtt{T}_{\mathbb{R}}$ is unbounded. Indeed, we have the positive formula $x < y$, which is expressed by $\exists z, w\, z\cdot w = 1 \wedge y = x + z^2$. Now, $x < y$ denials equality in $\mathtt{T}_{\mathbb{R}}$. Since $\mathbb{R}$ has infinite ordered sequences, by compactness, there is a positively closed model of $\mathtt{T}_{\mathbb{R}}$ of any cardinality.

More generally, in fields we can use the formula $\exists z, w\, z \cdot w = 1 \wedge y = x + z$, which positively defines inequality in every field. Hence, by compactness, for any infinite field K, the negative theory $\mathtt{T}_-(K)$ must have positively closed models of arbitrarily large cardinality. Thus, for instance, we can consider $\mathbb{C}$ or $\mathbb{Q}_p$. Indeed:

The p-adic numbers $\mathbb{Q}_p$ as $\mathtt{L}$-structure where the locality predicates are interpreted according to the p-adic metric is again local universal in their own theory $\mathtt{T}_p = \mathrm{Th}_-(\mathbb{Q}_p)$ by Theorem 4.4. Since every field endomorphism of $\mathbb{Q}_p$ is also the identity, $\mathbb{Q}_p$ is the unique retractor of $\mathtt{T}_p$ by Lemma 3.19. However, it has arbitrarily large (non-local) positively closed models as inequality is positively definable.

Similarly, we can consider $\mathbb{C}$ as $\mathtt{L}$-structure and observe that, again, $\mathtt{T}_{\mathbb{C}} = \mathrm{Th}_-(\mathbb{C})$ has arbitrarily large (non-local) positively closed models, and $\mathbb{C}$ is local universal for $\mathtt{T}_{\mathbb{C}}$ by Theorem 4.4. If $h\colon \mathbb{C} \to \mathbb{C}$ is an $\mathtt{L}$-endomorphism, it fixes $\mathbb{Q}$ (since it is in particular a field endomorphism). For any $a, b \in \mathbb{C}$, we have that $\mathbb{C} \models \mathrm{d}_m(na, nb)$ if and only if $\|a - b\| \leq {}^m\!/n$. Consequently, h is continuous. Since $\mathbb{R}$ is the closure of $\mathbb{Q}$, h setwise fixes $\mathbb{R}$, so $h_{|\mathbb{R}}$ is an endomorphism of $\mathbb{R}$, so it pointwise fixes $\mathbb{R}$. Thus, it is either the identity or complex conjugation, and in any case an automorphism. We thus get here as well that $\mathbb{C}$ is the unique retractor for $\mathtt{T}_{\mathbb{C}}$ by Lemma 3.19.

If there is a non-local universal positively closed model $\mathfrak{U}$, one might expect that the local retractors are precisely the maximal local substructures of $\mathfrak{U}$ (i.e. the local components containing the constants). This is the case for instance in [RFS24, Example 3.7(3)]. However, the following example shows that this is not true in general:

Example 6.3. Consider the one-sorted local language $\mathtt{L}$ with locality predicates $\mathtt{D} = \{\mathrm{d}_n\}_{n\in\mathbb{N}}$ (with the usual ordered monoid structure induced by the natural numbers) and binary predicates $\{S\} \cup \{E_n, U_n\}_{n\in\mathbb{N}}$. Consider the $\mathtt{L}$-structure M with universe 2^ω and interpretations $M \models \mathrm{d}_n(\eta, \xi) \Leftrightarrow |\{i : \eta(i) \neq \xi(i)\}| \leq n$, $M \models E_n(\eta, \xi) \Leftrightarrow \eta(n) = \xi(n)$, $M \models U_n(\eta, \xi) \Leftrightarrow \eta(n) \neq \xi(n)$ and $M \models S(\eta, \xi) \Leftrightarrow$ η, ξ have no common zeros. Let $\mathtt{T} = \mathrm{Th}_-(M)$ be the negative theory of M. Since $M \models \mathtt{T}_{\mathrm{loc}}$, $\mathtt{T}$ is locally satisfiable by [RFS24, Theorem 1.9]. For $\eta \in M$, write $M_\eta \coloneqq \mathtt{D}(\eta)$. Let $1 \in M$ be the constant sequence 1.

Now, we look at M as non-local structure, i.e. as $\mathtt{L}_*$-structure. Note that the positive logic topology in M is coarser than the Tychonoff topology, so M is compact with the positive logic topology. Thus, by [Seg22, Lemma 2.24], we get that M is universal for $\mathcal{M}_*(\mathtt{T})$. Furthermore, note that, in M, the unique element η satisfying $M \models S(\eta, \eta)$ is precisely $\eta = 1$. Thus, $\eta(n) = 1 \Leftrightarrow M \models \exists x\ S(x, x) \wedge E_n(x, \eta)$ and $\eta(n) = 0 \Leftrightarrow M \models \exists x\ S(x, x) \wedge U_n(x, \eta)$. Consequently, the only endomorphism of M is the identity. By Lemma 3.19 applied to $\mathtt{L}_\star$-structures, M is a non-local universal positively closed model of $\mathtt{T}$.

Suppose $M \models \varphi$ with φ a positive formula in $\mathtt{L}$. For $n \in \mathbb{N}$, let $\mathtt{L}_n$ be the reduct of $\mathtt{L}$ consisting of $\{S\} \cup \{\mathrm{d}_k\}_{k\in\mathbb{N}} \cup \{E_k, U_k\}_{k\leq n}$ and take n such that φ is a formula in $\mathtt{L}_n$. Consider the map $f\colon M \to M_1$ given by $f(\xi)(k) = \xi(k)$ for $k \leq n$ and $f(\xi)(k) = 1$ for $k > n$. Obviously, $E_k(\xi,\zeta)$ implies $E_k(f(\xi), f(\zeta))$ and $U_k(\xi,\zeta)$ implies $U_k(f(\xi), f(\zeta))$ for any $k \leq n$. Also, $\{i : f(\xi)(i) = f(\zeta)(i) = 0\} \subseteq \{i : \xi(i) = \zeta(i) = 0\}$, so $S(\xi,\zeta)$ implies $S(f(\xi), f(\zeta))$. On the other hand, $|\{m : f(\xi)(m) \neq f(\zeta)(m)\}| \leq |\{m : \xi(m) \neq \zeta(m)\}|$, so $\mathrm{d}_k(\xi,\zeta)$ implies $\mathrm{d}_k(f(\xi), f(\zeta))$ for any k. Thus, f is an $\mathtt{L}_n$-homomorphism. Since homomorphisms preserve satisfaction, we conclude that $M_1 \models \varphi$. As φ is arbitrary, we conclude that $\mathrm{Th}_-(M_1) = \mathtt{T}$. Hence, $\mathtt{T}$ is a weakly complete local theory.

Pick $\eta \in M \setminus M_1$. We claim that M_η is not a retractor of $\mathtt{T}$. In particular, we claim that M_η is not local positively closed. Consider the map $f\colon M_\eta \to M_1$ given by $f(\xi)(i) = 1$ if $x(i) = \eta(i)$ and $f(\xi)(i) = 0$ if $\xi(i) \neq \eta(i)$. We claim that f is a homomorphism from M_η to M_1 which is not a positive embedding. Obviously,

$$\begin{aligned}\{i : \zeta(i) = \xi(i)\} &= \{i : \zeta(i) = \xi(i) = \eta(i)\} \cup \{i : \zeta(i) = \xi(i) \neq \eta(i)\}\\ &= \{i : f(\zeta)(i) = f(\xi)(i) = 1\} \cup \{i : f(\zeta)(i) = f(\xi)(i) = 0\}\\ &= \{i : f(\zeta)(i) = f(\xi)(i)\},\end{aligned}$$

so f preserves E_n, U_n, d_n for all n. As $\eta \notin M_1$, for any $\zeta, \xi \in M_\eta$, we have $\neq S(\zeta,\xi)$. Hence, f is a homomorphism. However, $M_1 \models S(1,1)$ and $M_\eta \not\models S(\eta,\eta)$, so f is not a positive embedding.

On the other hand, in Example 6.3, there is one local component which is local positively closed, so a local retractor. In fact, it turns out that Example 6.3 has the local joint property, so this is the only local retractor.

Question 6.4. *Suppose* $\mathtt{T}$ *has a non-local universal positively closed model* $\mathfrak{U}$*. Is there always at least one local component of* $\mathfrak{U}$ *which is a local retractor of* $\mathtt{T}$*? For instance, consider the pointed case: Is the local component at the constants of* $\mathfrak{U}$ *a local retractor?*

Example 6.5. Without assuming $\mathtt{T}$ is a local theory (i.e. closed under local implications), Question 6.4 is obviously false.

Consider the single sorted language $\mathtt{L} = \{S, R\}$ where S, R are binary relations, and with $\mathtt{D} = \{=, \mathrm{d}_1\}$ (with trivial ordered monoid structure). Consider N with universe $\{a,b,c,d\}$ and interpretations $R = \{(a,d),(b,c),(d,a),(c,b)\}$, $S = \{(a,b),(b,a),(c,d),(d,c)\}$, and $\mathrm{d}_1 = \Delta_N \cup \{(a,c),(c,a),(b,d),(d,b)\}$, where Δ_N is the diagonal in N.

Then, N has two maximal local substructures — $\{a,c\}$ and $\{b,d\}$ — but neither is a local positively closed model of $\mathrm{Th}_-(N)$. In fact, the unique (local) retractor for N is a singleton with both R and S empty.

7. Automorphisms of retractors

Recall that, unless otherwise stated, we have fixed a local language $\mathtt{L}$ and a locally satisfiable negative local theory $\mathtt{T}$. From now on, unless otherwise stated, we also fix a retractor $\mathfrak{C}$ of $\mathtt{T}$.

The *pointwise topology* on $\mathrm{Aut}(\mathfrak{C})$ is the initial topology from the local positive logic topology given by the evaluation maps $\mathrm{ev}_a\colon \mathrm{Aut}(\mathfrak{C}) \to \mathfrak{C}^x$ for $a \in \mathfrak{C}^x$ with x finite. In other words, a base of open sets is given by the sets

$$\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(a))\}$$

where $\varphi(x) \in \mathrm{LFor}_-(\mathtt{L}(\mathfrak{C}))$ and $a \in \mathfrak{C}^x$.

Remark 7.1. Let x be a variable and $a \in \mathfrak{C}^x$. Then, $\mathrm{ev}_a \colon \mathrm{Aut}(\mathfrak{C}) \to \mathfrak{C}^x$ is continuous. For x finite this is precisely the definition of the pointwise topology. For x infinite it follows from the fact that the local positive logic topology of $\mathfrak{C}^x$ is the initial topology given by the projections $\mathrm{proj}_{x'} \colon \mathfrak{C}^x \to \mathfrak{C}^{x'}$ for $x' \subseteq x$ finite.

Let $\mathrm{d} = (\mathrm{d}_s)_{s\in \mathtt{S}}$ be a choice of locality relations and $a = (a_s)_{s\in\mathtt{S}}$ a choice of points for each sort. The d-*ball at* a *in* $\mathrm{Aut}(\mathfrak{C})$ is the subset $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a) := \{f : f(a_s) \in \mathrm{d}_s(a_s) \text{ for each } s \in \mathtt{S}\}$.

We now prove the main result of this section. In the one-sort case, Theorem 7.2(4) was proved in [Hru22a, Lemma 2.6] by Hrushovski. His proof works without modifications in the many sorts case as well. In any case, we recall it here for the sake of completeness. The trivial statements Theorem 7.2(3,5) were already noted in [Hru22a] too. The easy fact Theorem 7.2(1) is original. Finally, Theorem 7.2(2) corresponds to [Hru22b, Remark 3.27] in the non-local case, but was not discussed in [Hru22a].

Theorem 7.2. *The following hold:*

(1) $\mathrm{Aut}(\mathfrak{C})$ *is a closed subspace of the set of all function on* $\mathfrak{C}$ *with the* pointwise topology, *i.e. the initial topology given by the evaluation maps on tuples.*

(2) *Let* B *be a ball in* $\mathfrak{C}^x$ *and* F *a closed subset of* B *and* U *an open subset. Then,* $\{\sigma : \sigma(F) \subseteq U\}$ *is open. In particular, if there is an entourage on* $\mathfrak{C}^x$*, then* $\{\sigma : \sigma(F) \subseteq U\}$ *is open for every closed compact subset* F *and open set* U*.*

(3) $\mathrm{Aut}(\mathfrak{C})$ *is* T_1*, i.e. Fréchet.*

(4) *Every ball of* $\mathrm{Aut}(\mathfrak{C})$ *is compact.*

(5) *The inversion map* $g \mapsto g^{-1}$ *is continuous and left and right translations are continuous, i.e.* $\mathrm{Aut}(\mathfrak{C})$ *is a semi-topological group.*

Proof. (1) Note that the set of endomorphisms of $\mathfrak{C}$ is given by

$$\mathrm{End}(\mathfrak{C}) = \bigcap_a \mathrm{ev}_a^{-1}(\mathrm{qftp}_+(a)(\mathfrak{C})),$$

so it is a closed subset in the pointwise topology. Since $\mathrm{Aut}(\mathfrak{C}) = \mathrm{End}(\mathfrak{C})$ by Corollary 3.10, we conclude.

(2) Let $\underline{F}$ be the partial local positive type defining F and $\underline{U}$ the set of local negative formulas such that $\neg\underline{U}$ defines $\mathfrak{C}^x \setminus U$. Assume $\underline{F}$ is closed under finite conjunctions and $\underline{U}$ is closed under finite disjuctions. By compactness of B (Theorem 4.2), $\{\sigma : \sigma(F) \subseteq U\} = \bigcup_{\varphi\in\underline{F},\psi\in\underline{U}}\{\sigma : \sigma(\varphi(\mathfrak{C})) \subseteq \psi(\mathfrak{C})\}$. Hence, it suffices to show that $\{\sigma : \sigma(\varphi(\mathfrak{C})) \subseteq \psi(\mathfrak{C})\}$ is open for ψ local negative and φ local positive with $\varphi(\mathfrak{C}) \subseteq B$.

Say $B = \mathrm{d}(b)$ and c are the parameters of φ and ψ. Then, $\sigma(\varphi(\mathfrak{C}, c)) \subseteq \psi(\mathfrak{C}, c)$ if and only if $\mathfrak{C} \models \varphi(a, c)$ implies $\mathfrak{C} \models \psi(\sigma(a), c)$. Now, since $\sigma \in \mathrm{Aut}(\mathfrak{C})$, it follows that $\mathfrak{C} \models \psi(\sigma(a), c)$ if and only if $\mathfrak{C} \models \psi(a, \sigma^{-1}(c))$. Therefore, $\sigma(\varphi(\mathfrak{C}, c)) \subseteq \psi(\mathfrak{C}, c)$ if and only if $\mathfrak{C} \models \forall x\, \varphi(x, c) \to \psi(x, \sigma^{-1}(c))$. Now, since $\varphi(\mathfrak{C}, c) \subseteq B$, we have that $\forall x\, \varphi(x, c) \to \psi(x, \sigma^{-1}(c))$ is equivalent to $\phi(b, c, \sigma^{-1}(c)) := \neg\exists x{\in}\mathrm{d}(b)\, \varphi(x, c) \wedge \neg\psi(x, \sigma^{-1}(c))$. Hence,

$$\begin{aligned}\{\sigma : \sigma(\varphi(\mathfrak{C}, c)) \subseteq \psi(\mathfrak{C}, c)\} &= \{\sigma : \mathfrak{C} \models \phi(b, c, \sigma^{-1}(c))\}\\ &= \{\sigma : \mathfrak{C} \models \phi(\sigma(b), \sigma(c), c)\} = \mathrm{ev}_{b,c}^{-1}(\phi(\mathfrak{C}, \mathfrak{C}, c)).\end{aligned}$$

If there is an entourage on $\mathfrak{C}^x$, then every compact subset of $\mathfrak{C}^x$ is contained in some ball. Indeed, if F is compact and d is an entourage on $\mathfrak{C}^x$, as $F \subseteq \bigcup_{a\in F} \mathrm{d}(a)$, we find a finite subset $a_0, \ldots, a_{n-1} \in F$ such that $F \subseteq \bigcup_{i<n} \mathrm{d}(a_i)$ by compactness of F and Lemma 4.8. By the locality axiom A5, there are d_i for $0 < i < n$ such that $a_i \in \mathrm{d}_i(a_0)$. Hence, we get that $F \subseteq \mathrm{d}'(a_0)$ for $\mathrm{d}' := \mathrm{d} * \mathrm{d}_1 * \cdots * \mathrm{d}_{n-1}$.

(3) Obvious as $\mathfrak{C}$ is T_1. Indeed, we have $\{f\} = \bigcap_a \mathrm{ev}_a^{-1}(\{f(a)\})$ for any $f \in \mathrm{Aut}(\mathfrak{C})$.

(4) Consider a ball $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$ in $\mathrm{Aut}(\mathfrak{C})$. It suffices to show that $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$ is closed under ultralimits.

Pick a sequence $(g_i)_{i\in I}$ in $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$ and $\mathfrak{u}$ an ultrafilter in I. Let $\mathfrak{C}^{\mathfrak{u}}$ be the corresponding (non-local) ultrapower. Consider the diagonal inclusion $\iota\colon \mathfrak{C} \to \mathfrak{C}^{\mathfrak{u}}$ and $\mathtt{D}(a)$ the local substructure of $\mathfrak{C}^{\mathfrak{u}}$ at $a \coloneqq \iota(a)$. Note that $\iota\colon \mathfrak{C} \to \mathtt{D}(a)$. By Łoś's Theorem, $\mathfrak{C}^{\mathfrak{u}} \models_\star \mathtt{T}$. By [RFS24, Lemma 1.8], $\mathtt{D}(a) \models \mathtt{T}$. Consider the map $g_{\mathfrak{u}}\colon x \mapsto [g_i(x)]_{\mathfrak{u}}$. By Łoś's Theorem, $g_{\mathfrak{u}}$ is a homomorphism. As $\mathfrak{C} \models \mathrm{d}(g_i(a), a)$ for all $i \in I$ (and $\mathfrak{C}$ is local), $g_{\mathfrak{u}}\colon \mathfrak{C} \to \mathtt{D}(a)$. On the other hand, since $\mathfrak{C}$ is a retractor, there is $r\colon \mathtt{D}(a) \to \mathfrak{C}$ such that $\mathrm{id} = r \circ \iota$. Set $g \coloneqq r \circ g_{\mathfrak{u}}\colon \mathfrak{C} \to \mathfrak{C}$.

By Corollary 3.10, $g \in \mathrm{Aut}(\mathfrak{C}, a)$. Since $(g_i)_{i\in I}$ is in $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$, we get that $\mathfrak{C} \models \mathrm{d}_s(g_i(a_s), a_s)$ for all $i \in I$ and all sort s. Thus, by Łoś's Theorem, $\mathfrak{C}^{\mathfrak{u}} \models \mathrm{d}_s(g_{\mathfrak{u}}(a_s), a_s)$ for all s, concluding that $\mathfrak{C} \models \mathrm{d}_s(g(a_s), a_s)$. Therefore, $g \in \mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$.

Let $\varphi(x) \in \mathrm{LFor}^x_-(\mathtt{L}(\mathfrak{C}))$ and $b \in \mathfrak{C}^x$ such that $\mathfrak{C} \models \varphi(g(b))$. Then, $\mathtt{D}(a) \models \varphi(g_{\mathfrak{u}}(b))$. By [RFS24, Lemma 1.8], as φ is local, we conclude that $\mathfrak{C}^{\mathfrak{u}} \models \varphi(g_{\mathfrak{u}}(b))$. Hence, by Łoś's Theorem, $\{i : \mathfrak{C} \models \varphi(g_i(b))\} \in \mathfrak{u}$. Therefore, as φ and b are arbitrary, we conclude that g is the $\mathfrak{u}$-limit of $(g_i)_{i\in I}$.

(5) $\mathrm{ev}_a^{-1}(\varphi(\mathfrak{C}, b))^{-1} = \{\sigma^{-1} : \mathfrak{C} \models \varphi(\sigma(a), b)\} = \{\sigma^{-1} : \mathfrak{C} \models \varphi(a, \sigma^{-1}(b))\} = \mathrm{ev}_b^{-1}(a, \mathfrak{C})$. Thus, the inverse map is a homeomorphism. On the other hand, pick $\alpha \in \mathrm{Aut}(\mathfrak{C})$. Then, $\mathrm{ev}_a^{-1}(\varphi(\mathfrak{C})) \circ \alpha = \{\sigma \circ \alpha : \mathfrak{C} \models \varphi(\sigma(a))\} = \{\sigma \circ \alpha : \mathfrak{C} \models \varphi(\sigma \circ \alpha(\alpha^{-1}(a)))\} = \mathrm{ev}_{\alpha^{-1}(a)}(\varphi(\mathfrak{C}))$. Thus, right translations are homeomorphism. As the inverse map and right translations are continuous, left translations are continuous too. Q.E.D.

Remark 7.3. By Theorem 7.2(1), it seems natural to try to apply Tychonoff's Theorem to prove Theorem 7.2(4). Unfortunately, the pointwise topology in the space of function of $\mathfrak{C}$ to itself is not the usual Tychonoff topology, as we are considering valuations under arbitrary finite tuples and not only under single elements. Still, it is obvious from the definition that the space of functions of $\mathfrak{C}$ to itself with the pointwise topology can be canonically identified with the topological subspace

$$\Big\{f \in \prod \mathfrak{C}^a : \text{There is } g\colon \mathfrak{C} \to \mathfrak{C} \text{ with } f_a = g(a) \text{ for all } a\Big\}$$

of $\prod \mathfrak{C}^a$ with the Tychonoff topology, where $\mathfrak{C}^a$ is denoting a copy of $\mathfrak{C}^s$ for each $a \in \mathfrak{C}^s$ of finite sort s. Unfortunately, it seems that the space of functions from $\mathfrak{C}$ to itself is not a closed subset of $\prod \mathfrak{C}^a$, as in general $\prod \mathfrak{C}^a$ is not Hausdorff.

For the rest of the section, we adapt to the local case several results from [Hru22b] which where not discussed at all in [Hru22a]. We start with [Hru22b, Lemma 3.23].

Lemma 7.4. *Any evaluation on a pointed tuple is a continuous closed map.*

Proof. Continuity was already noted in Remark 7.1. Consider a non-empty basic closed subset D of $\mathrm{Aut}(\mathfrak{C})$, so $D = \mathrm{ev}_b^{-1}\varphi(\mathfrak{C})$ where $\varphi(y)$ is a local positive formula with parameters in $\mathfrak{C}$ and $b \in \mathfrak{C}^y$. We now show that $\mathrm{ev}_a(D)$ is closed. By Corollary 3.11, $a' \in \mathrm{ev}_a(D)$ if and only if there is $b' \in \mathfrak{C}^y$ with $\mathfrak{C} \models \varphi(b')$ such that $\mathrm{ltp}_+(a', b') = \mathrm{ltp}_+(a, b)$. Write $p(x, y) \coloneqq \mathrm{ltp}_+(a, b)$. In other words, we have

$$\mathrm{ev}_a(D) = \{a' : \text{There is } y \text{ such that } \varphi(y, c) \wedge p(a', y)\}.$$

Consider $\Sigma(x) = \{\exists y\, \varphi(y) \wedge \psi(x, y) : \psi \in p\} \subseteq \mathrm{LFor}^x_+(\mathtt{L}(\mathfrak{C}))$. We conclude that $\Sigma(\mathfrak{C}) = \mathrm{ev}_a(D)$. Indeed, if $a' \in \mathrm{ev}_a(D)$, then there is b' such that $\mathfrak{C} \models \varphi(b') \wedge p(a', b')$, so, in particular, a' satisfies Σ. On the other hand, if a' satisfies Σ, then $\varphi(y) \wedge p(a', y)$ is finitely locally satisfiable. As a' is pointed, $p(a', y)$ implies some bound of y with parameters in a'. Thus, $\varphi(y, c) \wedge p(a', y)$ is boundedly satisfiable in $\mathrm{Th}_-(\mathfrak{C}/\mathfrak{C})$, so it is locally satisfiable by Lemma 2.10. Hence, by saturation of $\mathfrak{C}$ (Theorem 3.9), we get that $\varphi(y) \wedge p(a', y)$ is realised by some b' in $\mathfrak{C}$, concluding that $a' \in \mathrm{ev}_a(D)$. In particular, $\mathrm{ev}_a(D)$ is closed. As D is arbitrary, we conclude that the image of every basic closed subset of $\mathrm{Aut}(\mathfrak{C})$ is closed.

Let F be an arbitrary closed subset of $\mathrm{Aut}(\mathfrak{C})$. Then, $F = \bigcap_{i\in I} D_i$ with D_i basic closed sets. Suppose a' is in the closure of $\mathrm{ev}_a(F)$. Then, a' is in the closure of $\mathrm{ev}_a(\bigcap_{i\in I_0} D_i)$ for all $I_0 \subseteq I$. Since the basic closed subsets are closed under finite intersections, we get that $\mathrm{ev}_a(\bigcap_{i\in I_0} D_i)$ is closed for $I_0 \subseteq I$ finite. Hence, $\mathrm{ev}_a^{-1}(a') \cap \bigcap_{i\in I_0} D_i \neq \emptyset$ for any $I_0 \subseteq I$ finite. Now, pick $\mathrm{d} = (\mathrm{d}_s)_{s\in \mathrm{S}}$ such that $\mathrm{d}(a, a')$. Then, $\mathrm{ev}_a^{-1}(a') \subseteq \mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a)$. By Theorem 7.2(4), we get that $\mathrm{ev}_a^{-1}(a')$ is compact. Therefore, $\mathrm{ev}_a^{-1}(a') \cap F \neq \emptyset$, concluding that $a' \in \mathrm{ev}_a(F)$. As a' is arbitrary, we get that $\mathrm{ev}_a(F)$ is closed. Q.E.D.

Lemma 7.5. *Suppose* $\mathtt{T}$ *has full systems of entourages. Then,* $\mathrm{Aut}(\mathfrak{C})$ *is a Hausdorff topological group and evaluation maps on finite sorts are jointly continuous.*

Proof. Let us show first that composition in $\mathrm{Aut}(\mathfrak{C})$ is continuous. By Theorem 7.2(2,3), we have the topological base

$$\{B(F, U) : U \text{ open and } F \text{ closed compact in the same sort of } \mathfrak{C}\},$$

where $B(F, U) \coloneqq \{\sigma : \sigma(F) \subseteq U\}$. Thus, it suffices to show that $\{(f, g) : f \circ g \in B(F, U)\}$ is open for every closed compact F and open U.

Pick $\alpha \circ \beta \in B(F, U)$. Then, $\beta(F) \subseteq \alpha^{-1}(U)$. Since $\mathfrak{C}$ is locally closed compact by Corollary 4.11, there are V open and K closed compact such that $\beta(F) \subseteq V \subseteq K \subseteq \alpha^{-1}(U)$. Hence, $(\alpha, \beta) \in B(K, U) \times B(F, V) \subseteq \{(f, g) : f \circ g \in B(F, U)\}$. Indeed, if $f \in B(K, U)$ and $g \in B(F, V)$, then $g(F) \subseteq V \subseteq K \subseteq f^{-1}(U)$, concluding that $f \circ g \in B(F, U)$. Since α, β and $B(F, U)$ are arbitrary, we conclude that composition is continuous. Since we already know that the inverse map is continuous by Theorem 7.2(5), we conclude that $\mathrm{Aut}(\mathfrak{C})$ is a topological group. Since $\mathrm{Aut}(\mathfrak{C})$ is Fréchet by Theorem 7.2(3), it is also Hausdorff by [Hus66, §21 Theorem 4].

Let us now show that evaluations are jointly continuous. Pick a finite variable x, $a \in \mathfrak{C}^x$, $\sigma \in \mathrm{Aut}(\mathfrak{C})$ and U open subset of $\mathfrak{C}^x$ such that $\sigma(a) \in U$. Then, $a \in \sigma^{-1}(U)$. By local closed compactness of $\mathfrak{C}^x$, find an open set V and a closed compact set F such that $a \in V \subseteq F \subseteq \sigma^{-1}(U)$. Then, $(\sigma, a) \in B(F, U) \times V \subseteq \{(f, b) : f(b) \in U\}$. Indeed, if $f \in B(F, U)$ and $b \in V$, then $b \in V \subseteq F \subseteq f^{-1}(U)$, concluding that $f(b) \in U$. Since σ, a and U are arbitrary, we conclude that evaluations are jointly continuous. Q.E.D.

Lemma 7.6. *Suppose that* $\mathtt{L}$ *only has finitely many single sorts and set an enumeration* s *of all of them. Let* d *be a locality relation of sort* s*.*

(1) *Suppose* (ψ, φ) *is an entourage in* d*. Then,* $\mathrm{ev}_a^{-1}\psi(\mathfrak{C}, a) = \{\sigma : \mathfrak{C} \models \psi(\sigma(a), a)\}$ *is a compact neighbourhood of the identity for any* $a \in \mathfrak{C}^s$*. In particular,* $\mathrm{Aut}(\mathfrak{C})$ *is weakly locally compact if* $\mathfrak{C}$ *has entourages.*

(2) *Suppose* $\{(\psi_i, \varphi_i)\}_{i\in I}$ *is a full system of entourages in* d*. Then,*

$$\left\{ \bigcap_{a\in A,\, i\in I_0} \mathrm{ev}_a^{-1}\psi_i(\mathfrak{C}, a) : I_0 \subseteq I,\ A \subseteq \mathfrak{C}^s \text{ both finite} \right\}$$

is a base of compact neighbourhoods of the identity. In particular, $\mathrm{Aut}(\mathfrak{C})$ *is locally compact if* $\mathfrak{C}$ *has full systems of entourages.*

Proof. (1) By Corollary 4.9, $\psi(\mathfrak{C}, a) \subseteq \mathrm{d}(a)$ is a closed neighbourhood of $a = \mathrm{id}(a)$. Since ev_a is continuous, we conclude that $\mathrm{ev}_a^{-1}\psi(\mathfrak{C}, a)$ is a closed neighbourhood of id. By Theorem 7.2(4), $\mathrm{d}(\mathrm{Aut}(\mathfrak{C}), a) = \mathrm{ev}_a^{-1}\mathrm{d}(a)$ is compact, concluding that

$\mathrm{ev}_a^{-1}\psi(\mathfrak{C},a)$ is a closed compact neighbourhood of id. Since translations are continuous, we get that $\mathrm{Aut}(\mathfrak{C})$ is weakly locally closed compact.

(2) By compactness and the fact that translations are continuous, it suffices to show that $\{\mathrm{id}\}=\bigcap_{a\in\mathfrak{C}^s,\, i\in I}\mathrm{ev}_a^{-1}\psi_i(\mathfrak{C},a)$. This is obvious since $\bigcap_{i\in I}\mathrm{ev}_a^{-1}\psi_i(\mathfrak{C},a)=\{\sigma:\sigma(a)=a\}$ when $\{\psi_i\}$ a full system of approximations of $=$. Q.E.D.

As Lemmas 7.5 and 7.6 show, the existence of full systems of entourages is a very strong hypothesis and, even in the non-local context, there is a plenty of examples that do not satisfy it. In its absence, one considers Hausdorff quotients of $\mathrm{Aut}(\mathfrak{C})$. We conclude this subsection adapting the study of these Hausdorff quotients from [Hru22b].

Let x be a variable and X a local positive $\bigwedge_0$-definable subset of $\mathfrak{C}^x$. Consider

$$\mathfrak{g}_X\coloneqq\{\sigma:\sigma(U)\cap U\neq\emptyset \text{ for every nonempty open subset } U\subseteq X\},$$

In other words, according to [Hru22b, Appendix C], $\mathfrak{g}_X$ is the infinitesimal subgroup of the evaluation action of $\mathrm{Aut}(\mathfrak{C})$ on X. In the case $X=\mathfrak{C}^s$, write $\mathfrak{g}_s\coloneqq\mathfrak{g}_{\mathfrak{C}^s}$. Finally, let $\mathfrak{g}\coloneqq\bigcap\mathfrak{g}_s$ and $\mathfrak{g}_!\coloneqq\bigcap_X\mathfrak{g}_X$. So, according to [Hru22b, Appendix C], $\mathfrak{g}$ is the infinitesimal subgroup of the evaluation action on $\bigsqcup\mathfrak{C}^s$ and $\mathfrak{g}_!$ is the infinitesimal subgroup of the evaluation action on $\bigsqcup X$.

By [Hru22b, Lemma C.1], we know that $\mathfrak{g}$, $\mathfrak{g}_X$ and $\mathfrak{g}_!$ are closed normal subgroups of $\mathrm{Aut}(\mathfrak{C})$. Let $\mathcal{G}_X\coloneqq{}^{\mathrm{Aut}(\mathfrak{C})}/_{\mathfrak{g}_X}$, $\mathcal{G}_s\coloneqq{}^{\mathrm{Aut}(\mathfrak{C})}/_{\mathfrak{g}_s}$, $\mathcal{G}\coloneqq{}^{\mathrm{Aut}(\mathfrak{C})}/_{\mathfrak{g}}$ and $\mathcal{G}_!\coloneqq{}^{\mathrm{Aut}(\mathfrak{C})}/_{\mathfrak{g}_!}$. Then, by [Hru22b, Lemma C.1], [AT08, Theorem 1.5.3] and Theorem 7.2(5), $\mathcal{G}_X$, $\mathcal{G}_s$, $\mathcal{G}$ and $\mathcal{G}_!$ are Hausdorff semitopological groups. Furthermore, by [AT08, Theorem 1.5.3], the quotient homomorphisms from $\mathrm{Aut}(\mathfrak{C})$ are continuous and open.

Lemma 7.7. *The following hold:*

(1) *Let X and Y be local positive $\bigwedge_0$-definable with $Y\subseteq X$. Then, $\mathfrak{g}_Y\subseteq\mathfrak{g}_X$. In particular, the quotient homomorphism $\pi\colon\mathcal{G}_Y\to\mathcal{G}_X$ is a surjective continuous open map.*

(2) *Let $X\subseteq\mathfrak{C}^x$ and $Y\subseteq\mathfrak{C}^y$ be local positive $\bigwedge_0$-definable with $x\subseteq y$. Suppose* $\mathrm{proj}(Y)\subseteq X$, *where* proj *is the restriction of the projection of $\mathfrak{C}^y$ to $\mathfrak{C}^x$. Then, $\mathfrak{g}_Y\subseteq\mathfrak{g}_X$. In particular, the quotient homomorphism $\pi\colon\mathcal{G}_Y\to\mathcal{G}_X$ is a surjective continuous open map.*

(3) *$\mathfrak{g}_!\subseteq\mathfrak{g}$. In particular, the quotient homomorphism $\pi\colon\mathcal{G}_!\to\mathcal{G}$ is a surjective continuous open map.*

Proof. (1) By definition, the inclusion $\iota\colon Y\to X$. is continuous. Also, note that the evaluation action of $\mathrm{Aut}(\mathfrak{C})$ commutes with ι, i.e. $\iota(\sigma(a))=\sigma(\iota(a))$. Suppose $\sigma\in\mathfrak{g}_Y$. Pick $U\subseteq X$ open. As $\sigma\in\mathfrak{g}_X$, there is $a\in\sigma(\iota^{-1}(U))\cap\iota^{-1}(U)$. Hence, $\iota(a)\in U$ and $\sigma^{-1}(\iota(a))=\iota(\sigma^{-1}(a))\in U$, concluding $\iota(a)\in\sigma(U)\cap U$. As U and σ are arbitrary, $\mathfrak{g}_Y\subseteq\mathfrak{g}_X$. By [AT08, Theorem 1.5.3], the quotient homomorphism $\pi\colon\mathcal{G}_Y\to\mathcal{G}_X$ is continuous and open.

(2) By Lemma 4.1, proj is continuous. Also, note that the evaluation action of $\mathrm{Aut}(\mathfrak{C})$ commutes with proj, i.e. $\mathrm{proj}(\sigma(a))=\sigma(\mathrm{proj}(a))$. Suppose $\sigma\in\mathfrak{g}_Y$. Pick $U\subseteq X$ open. As $\sigma\in\mathfrak{g}_Y$, there is $a\in\sigma(\mathrm{proj}^{-1}(U))\cap\mathrm{proj}^{-1}(U)$. Hence, $\mathrm{proj}(a)\in U$ and $\sigma^{-1}(\mathrm{proj}(a))=\mathrm{proj}(\sigma^{-1}(a))\in U$, concluding $\mathrm{proj}(a)\in\sigma(U)\cap U$. As U and σ are arbitrary, $\mathfrak{g}_Y\subseteq\mathfrak{g}_X$. By [AT08, Theorem 1.5.3], the quotient homomorphism $\pi\colon\mathcal{G}_Y\to\mathcal{G}_X$ is continuous and open.

(3) By (1), $\mathfrak{g}_!\subseteq\mathfrak{g}$. By [AT08, Theorem 1.5.3], the quotient homomorphism $\pi\colon\mathcal{G}_!\to\mathcal{G}$ is continuous and open. Q.E.D.

Lemma 7.8. *Suppose that* `L` *only has finitely many single sorts and* `T` *has entourages. Then, $\mathcal{G}_X$ is a locally compact Hausdorff topological group for any local* *positive $\bigwedge_0$-definable set X. Similarly, $\mathcal{G}$ and $\mathcal{G}_!$ are locally compact Hausdorff topological groups too.*

Proof. By Lemma 7.6(1), $\mathrm{Aut}(\mathfrak{C})$ is weakly compact. Then, since $\pi\colon \mathrm{Aut}(\mathfrak{C}) \to \mathcal{G}_X$ is continuous and open, we get that $\mathcal{G}_X$ is weakly locally compact. Hence, by Hausdorffness, it is locally compact. As it is a semitopological group, by Ellis' Theorem [Ell57, Theorem 2], we conclude that it is a locally compact Hausdorff topological group. Similar for $\mathcal{G}_!$ and $\mathcal{G}$. Q.E.D.

Let X be a local positive $\bigwedge_0$-definable subset. Let $X/\mathfrak{g}_X$ denote the quotient space of $\mathfrak{g}_X$-orbits of X. In [Hru22b], it was proved that this quotient space is Hausdorff and, if X is defined by a local positive type, the $\mathfrak{g}_X$-conjugacy relation in X is local positive $\bigwedge_0$-definable — see [Hru22b, Propositions 3.24]. We adapt here these results.

Lemma 7.9. *Let X be a local positive $\bigwedge_0$-definable subset of $\mathfrak{C}^x$ on a pointed variable x. Write $\underline{X}(x)$ for a partial local positive type without parameters defining it. Then:*

(1) *$X/\mathfrak{g}_X = X/\mathfrak{g}_!$ is Hausdorff.*

(2) *Suppose $\underline{X}$ is a local positive type. Then, the $\mathfrak{g}_X$-conjugacy equivalence relation on X is local positive $\bigwedge_0$-definable.*

Proof. (1) By [Hru22b, Lemma C.1], $\mathcal{G}_!$ and $\mathcal{G}_X$ are Hausdorff. By Lemma 7.4, $\mathrm{Aut}(\mathfrak{C})$ acts by closed maps. By [Hru22b, Remark C.3], it follows that $X/\mathfrak{g}_! = X/\mathfrak{g}_X$. By [Hru22b, Lemma C.2(1)], $X/\mathfrak{g}_X$ is Hausdorff.

(2) By (1), $X/\mathfrak{g}_X$ is Hausdorff, so the diagonal is closed. By Lemma 4.1, we conclude that the $\mathfrak{g}_X$-conjugacy equivalence relation E is closed in $X \times X$. Let $\Sigma(x, y, c)$ be a partial local positive type with parameters c defining $\mathfrak{g}_X$-conjugacy in X. Without loss of generality, take c pointed. Then, $\Sigma(a, b, c)$ if and only if $\Sigma(a, b, c')$ for any c' such that $\mathrm{ltp}_+(c) = \mathrm{ltp}_+(c')$. One direction is obvious. On the other hand, by Corollary 3.11, since $\mathrm{ltp}_+(c) = \mathrm{ltp}(c')$, there is $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\sigma(c') = c$. Then, as $\Sigma(a, b, c')$, we get $\Sigma(\sigma(a), \sigma(b), c)$, so there is $g \in \mathfrak{g}_X$ with $g\sigma(a) = \sigma(b)$, so $\sigma^{-1}g\sigma(a) = b$. By normality of $\mathfrak{g}_X$, we get $\sigma^{-1}g\sigma \in \mathfrak{g}_X$, so $\Sigma(a, b, c)$ by definition of E.

Let $p(z) \coloneqq \mathrm{ltp}_+(c)$. Picking any $a \in X$, by A5, there is some bound $\mathrm{B}(x, z)$ such that $\mathfrak{C} \models \mathrm{B}(a, c)$. Then, by Corollary 3.11, for any $a' \in X$, there is $c' \in p(\mathfrak{C})$ with $\mathfrak{C} \models \mathrm{B}(a', c')$. Hence, we conclude that $\mathfrak{g}_X$-conjugacy is defined by

$$\begin{aligned}\underline{E}(x, y) \coloneqq{}& \exists z\, \Sigma(x, y, z) \wedge p(z) \wedge \mathrm{B}(x, z)\\ \coloneqq{}& \left\{\exists z\, \varphi(x, y, z) \wedge \psi(z) \wedge \bigwedge B_0(x, z) : \varphi \in \Sigma,\ \psi \in p,\ B_0 \subseteq B \text{ finite}\right\}.\end{aligned}$$

Indeed, if a and b are $\mathfrak{g}_X$-conjugates, then $\mathfrak{C} \models \Sigma(a, b, c')$ for any $c' \in p(\mathfrak{C})$. Now, there is $c' \in p(\mathfrak{C})$ such that $\mathfrak{C} \models \mathrm{B}(a, c')$, so we get that $\mathfrak{C} \models \underline{E}(a, b)$ witnessed by $z = c'$. On the other hand, suppose a and b satisfy $\underline{E}$. Then, the partial local positive type $\Sigma(a, b, z) \wedge p(z) \wedge \mathrm{B}(a, z)$ is finitely locally satisfiable. Now, $\mathrm{B}(a, z)$ is a bound, so we get by Lemma 2.10, that there is c' realising it. In other words, $\mathrm{ltp}_+(c') = \mathrm{ltp}_+(c)$ with $\Sigma(a, b, c')$, so a and b are $\mathfrak{g}_X$-conjugates.

Therefore, we conclude that the $\mathfrak{g}_X$-conjugacy equivalence relation is local positive $\bigwedge_0$-definable. Q.E.D.

We finish with [Hru22b, Proposition 3.25 and Corollary 3.26].

Lemma 7.10. *Let $p(x)$ be a local positive type without parameters of $\mathfrak{C}$ on a pointed variable x. Slightly abusing of the notation, write $p \coloneqq p(\mathfrak{C})$. Then, $p/\mathfrak{g}_!$ has a dense subset of size $|\mathrm{L}|^{|x|}$. In particular, $|\mathrm{Aut}\,(p/\mathfrak{g}_!)| \leq 2^{|\mathrm{L}|^{|x|}}$.*

Proof. Pick $c \in P$ arbitrary. By A5, $p = p \cap \mathtt{D}(c)$. As there are at most $|\mathtt{L}|^{|x|}$ balls centred at c, it suffices to show that ${}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$ has a dense subset of size $|\mathtt{L}|$ for any ball $\mathrm{d}(c)$, where $\mathrm{d}(p,c) \coloneqq p \cap \mathrm{d}(c)$.

Let E be the equivalence relation of $\mathfrak{g}_!$-conjugacy in $\mathrm{d}(p,c)$. By Lemma 7.9(2), E is local positive $\bigwedge_c$-definable. Thus, by [RFS24, Lemma 2.12], as $\mathfrak{C} \models^{\mathrm{pc}} \mathtt{T}$, if $(a,b) \notin E$, there is a local positive c-definable subset D such that $(a,b) \in D$ and $D \cap E = \emptyset$. Since there are at most $|\mathtt{L}|$ local positive formulas over c, we have a family $\{D_i\}_{i<|\mathtt{L}|}$ of local positive c-definable subsets disjoint to E such that, for any $(a,b) \notin E$, there is $i < |\mathtt{L}|$ with $(a,b) \in D_i$.

For each $i < |\mathtt{L}|$, consider

$$\widetilde{D}_i \coloneqq \{(a',b') : \text{There are } (a,b) \in D_i \text{ with } (a,a'),(b,b') \in E\}.$$

Let φ_i be a local positive formula over c defining D_i and Ψ be a partial local positive 0-type closed under finite conjunctions defining E. Then, $\widetilde{D}_i$ is defined by the partial local positive 0-type

$$\Phi_i(x,y) \coloneqq \{\exists x',y'\, \varphi_i(x',y') \wedge \psi(x,x') \wedge \psi(y,y') \wedge \mathrm{d}(x',c) \wedge \mathrm{d}(y',c) : \psi \in \Psi\}.$$

Indeed, obviously $\widetilde{D}_i \subseteq \Phi_i(\mathfrak{C})$. On the other hand, if $\mathfrak{C} \models \Phi_i(a,b)$, then $\{\varphi_i(x',y') \wedge \psi(a,x') \wedge \psi(b,y') \wedge \mathrm{d}(x',c) \wedge \mathrm{d}(y',c) : \psi \in \Psi\}$ is finitely satisfiable. As it implies a bound, we conclude that it is boundedly satisfiable, so there are $(a',b') \in D_i$ with $(a,a'),(b,b') \in E$ by saturation of $\mathfrak{C}$. In other words, $(a,b) \in \widetilde{D}_i$.

Since E is an equivalence relation and $D_i \cap E = \emptyset$, we conclude that $\widetilde{D}_i \cap E = \emptyset$. Also, note that $\widetilde{D}_i = \bar{\pi}^{-1}\bar{\pi}(D_i)$ where $\bar{\pi} = (\pi,\pi)$ is the quotient map $\pi \colon \mathrm{d}(p,c) \to {}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$. Hence, $\bar{\pi}^{-1}\bar{\pi}(\mathrm{d}(p,c) \setminus \widetilde{D}_i) = \mathrm{d}(p,c) \setminus \widetilde{D}_i$, so $\pi(\mathrm{d}(p,c) \setminus \widetilde{D}_i)$ is open in the quotient topology. Now, ${}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$ is Hausdorff, and the quotient topology in ${}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!} \times {}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$ is compact and finer than the product topology by Lemma 4.1. Therefore, the quotient and product topologies agree, concluding that the diagonal in ${}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$ is the intersection of $|\mathtt{L}|$ open sets in the product topology. Thus, by compactness, ${}^{\mathrm{d}(p,c)}/_{\mathfrak{g}_!}$ has a dense subset of size $|\mathtt{L}|$.

As ${}^{p}/_{\mathfrak{g}_!}$ is Hausdorff, any automorphism fixing a dense subset is the identity. Hence, as ${}^{p}/_{\mathfrak{g}_!}$ has a dense subset of size $|\mathtt{L}|^{|x|}$, we conclude that $|\mathrm{Aut}\,({}^{p}/_{\mathfrak{g}_!})| \leq 2^{|\mathtt{L}|^{|x|}}$. Q.E.D.

Corollary 7.11. *Let $p(x)$ be a local positive type without parameters of $\mathfrak{C}$ on a pointed variable x. Then, $|\mathcal{G}_p| \leq 2^{|\mathtt{L}|^{|x|}}$. Consequently, $|\mathcal{G}_!| \leq 2^{|\mathtt{L}|^{\lambda}}$ where λ is the number of sorts of* $\mathtt{L}$.

Proof. Slightly abusing of the notation, write $p \coloneqq p(\mathfrak{C})$. Consider the group homomorphism $f \colon \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Aut}\left({}^{p}/_{\mathfrak{g}_p}\right)$ given by $f(\sigma)([a]) = [\sigma(a)]$, where $[\bullet]$ denotes the $\mathfrak{g}_p$-conjugacy class. By Lemma 7.9(2), $\mathfrak{g}_p$-conjugacy in p is local positive $\bigwedge_0$-definable, so $[\sigma(a)] = [\sigma(b)]$ when $[a] = [b]$. In other words, f is well-defined.

By Lemma 7.10, $\left|\mathrm{Aut}\left({}^{p}/_{\mathfrak{g}_p}\right)\right| \leq 2^{|\mathtt{L}|^{|x|}}$. Thus, by the isomorphism theorem, it suffices to show that $\ker(f) = \mathfrak{g}_p$. Trivially, $\mathfrak{g}_p \subseteq \ker(f)$.

Pick $\sigma \in \ker(f)$ and $U \subseteq p$ an arbitrary open subset of p in the local positive logic topology. By [Hru22b, Appendix C, eq. 2], we need to show that $\sigma(U) \subseteq \mathrm{cl}(U)$, where $\mathrm{cl}(U)$ is the closure of U in the local positive logic topology. Let $a \in U$ arbitrary. Since $\sigma \in \ker(f)$, $\sigma(a)$ and a are $\mathfrak{g}_p$-conjugates. Thus, there is $\sigma' \in \mathfrak{g}_p$ such that $\sigma(a) = \sigma'(a)$. Hence, $\sigma(a) = \sigma'(a) \in \sigma'(U) \subseteq \mathrm{cl}(U)$. As $a \in U$ is arbitrary, we conclude that $\sigma(U) \subseteq \mathrm{cl}(U)$. Q.E.D.

References

[AT08] Alexander Arhangel'skii and Mikhail Tkachenko. *Topological groups and related structures*, volume 1 of *Atlantis Studies in Mathematics*. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. ISBN:9789078677062. doi:10.2991/978-94-91216-35-0.

[Ell57] Robert Ellis. Locally compact transformation groups. *Duke Math. J.*, 24(2):119–125, 1957. doi:10.1215/S0012-7094-57-02417-1.

[Hru22a] Ehud Hrushovski. Beyond the lascar group, 2022. arXiv:2011.12009v3.

[Hru22b] Ehud Hrushovski. Definability patterns and their symmetries, 2022. arXiv:1911.01129v4.

[Hus66] Taqdir Husain. *Introduction to topological groups*. W. B. Saunders Co., Philadelphia, Pa.-London, 1966. ISBN:9780721648552.

[RFS24] Arturo Rodriguez Fanlo and Ori Segel. Completeness in local positive logic, 2024. arXiv:2401.03260v2.

[Seg22] Ori Segel. Positive definability patterns, 2022. arXiv:2207.12449v2.

Einstein Institute of Mathematics, Hebrew University of Jerusalem, 91904, Jerusalem, Israel.

Email address, A. Rodríguez Fanlo: Arturo.Rodriguez@mail.huji.ac.il

Einstein Institute of Mathematics, Hebrew University of Jerusalem, 91904, Jerusalem, Israel.

Email address, O. Segel: Ori.Segel@mail.huji.ac.il