A THREE-FIELD MULTISCALE METHOD

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Abstract. *A Three-Field Domain Decomposition Method* is the title of a seminal paper by F. Brezzi and L. D. Marini which introduces a three-field formulation for elliptic partial differential equations. Based on that, we propose the Multiscale-Hybrid-Hybrid Method (MH^{2}M) for the Darcy model, a multiscale finite element method that yields, after a series of formal manipulations, a symmetric positive definite formulation that depends only on the trace of the solution. We show stability and convergence results for a family of finite element spaces and establish relationships with other multiscale finite element methods.

1. Introduction

Simulation of fluid flow in heterogeneous domains, as found in oil reservoirs, transport of contaminants and water resources issues, is a big challenge as fine meshes are needed to capture the multiscale solution. The Darcy equation is the most representative model in the field of porous media, which in its primary formulation is composed of a Poisson equation with a multiscale coefficient. Precisely, the Poisson problem consists of finding the weak solution $u : \Omega \to \mathbb{R}$ of

$$\begin{aligned} -\operatorname{div} \mathcal{A} \boldsymbol{\nabla} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 is an open bounded domain with a polyhedral Lipschitz boundary $\partial\Omega$, and $f \in L^2(\Omega)$. The symmetric tensor $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}_{\text{sym}}$ is uniformly positive definite, and for almost all $\boldsymbol{x} \in \Omega$ let the positive constants $a_{\min}$ and $a_{\max}$ be such that

$$a_{\min}|\boldsymbol{v}|^2 \leq \mathcal{A}(\boldsymbol{x})\, \boldsymbol{v} \cdot \boldsymbol{v} \leq a_{\max}|\boldsymbol{v}|^2 \quad \text{for all } \boldsymbol{v} \in \mathbb{R}^d. \tag{2}$$

The tensor $\mathcal{A}$ may include multiple scales.

The flow velocity, represented by the pressure gradient, is usually the quantity of interest. To approximate it, the mixed version of Darcy's equation is preferable to its elliptic (1) version, for which the classical inf-sup stable pairs of finite elements are the Raviart-Thomas element (RT_k) [39] or the element Brezzi-Douglas-Marini (BDM_k) [11, 14], which prescribes the

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continuity of the normal velocity component combined with discontinuous pressure interpolations. In addition to being stable, these finite elements preserve local mass, a fundamental property in practical engineering applications. However, since such choices produce linear systems that represent saddle point problems, efficient solver are harder to come by [15]. A way to circumvent this problem is by using hybridization [11] techniques, which start with a discontinuous version of a stable element (RT_k or BDM_k), forcing the weak continuity of the normal components through Lagrange multipliers. Velocity and pressure are now fully discontinuous and can be eliminated at the element level, leading to a positive definite symmetrical system where the only unknowns are the Lagrange multipliers [4].

The publication of the seminal work by Franco and Marini [16] was followed by a series of papers investigating, in particular, preconditioning and stabilization techniques that allow for different combinations of space [1, 5, 7–10, 16–19, 38]. Although these concepts are related, they are not the same, specially when the estimates depend on the number of subdomains [7–9]. Note that stabilization is crucial in those papers, an ingredient that we prefer to disconsider. Indeed, in principle, stabilization allows for more flexible interpolation, but it actually either imposes restrictions on the meshes and interpolation spaces, e.g. [5], or involves the inclusion of terms and stabilization parameters of nontrivial computation [10].

Hybridization has also been used to construct and analyze multiscale methods, which have become an attractive option for dealing with primal or mixed weak forms of (1) when the standard Galerkin method based on continuous polynomial interpolation becomes computationally expensive. The multiscale methods share the commonality of having global problems built on the solution of local problems that scale up submesh structures. An attractive feature is that the global problem dimension is multiscale independent, and the basis functions are independent of each other and can be computed in parallel. An hybridization of the primal form of (1), following [40] yields the *Multiscale Hybrid-Mixed* (MHM for short) family of methods [2, 31], where the local problems are of Neumann type. On the contrary, the hybridization of the dual version of (1) leads to the multiscale mortar method [3] and the multiscale version of the HDG method in [23]. Local problems are of the Dirichlet type in these cases. Other alternative multiscale methods in the context of the Darcy model (or Poisson equation) are the VMS method [33], the MsFEM and GMsFEM [25], the PGEM and GEM [28], the HMM [22], the LOD and LSD method [34, 35] and other variants of the MHM method like [21], to name a few.

In this work, we follow the strategy used to build the MHM method, which starts at the continuous level placed on a coarse partition. It consists of decomposing the exact solution into local and global contributions that, when discretized, dissociate local and global problems: the global formulation is defined on the skeleton of the coarse partition, yielding the degrees of freedom; the local problems provide the multiscale basis functions and their computation is embarrassingly parallel as the local problems are independent of each other. However, the MHM method for (1) has a saddle point structure similar to Galerkin's method for the mixed form of (1). Notably, its lower order version generalizes the Galerkin's method with the element RT_0 in simplexes for the case of discretized multiscale coefficient problems with polytopal meshes [6].

This work revisits the three field method of F. Brezzi and D. Marini [16] in the context of the MHM method. Specifically, we relax the continuity of the flow variable λ and weakly enforce it through the action of a second Lagrange multiplier ρ. Then, we decompose the broken infinite dimensional spaces where u and λ belong in such a way that the discretization results in a new multiscale finite element method that

- induces a definite positive symmetric global linear system to compute the degrees of freedom. It differs from the original MHM method, which induces a linear saddle point system;
- approximates the flux variable λ using a multiscale flux basis obtained from a Dirichlet-to-Neumann operator defined from new local Neumann problems driven by the basis functions of ρ. This feature is new when compared to the MHM method strategy where the basis for λ is chosen in an ad-hoc fashion and "ignores" underlying physical properties;
- imposes weak continuity of discrete primal (pressure) and dual (flow) variables on the skeleton of the coarse partition, with a discrete flow that is in local equilibrium with external forces.

In addition to the construction of the new multiscale method, called the Multiscale Hybrid-Hybrid Method (MH^{2}M for short), the main contributions of this work are:

(*i*) demonstrate the well-posedness and the best approximation results for the MH^{2}M under abstract compatibility conditions between interpolation spaces;

(*ii*) propose families of interpolation spaces that fulfill the conditions in (*i*) and for which the optimal convergence of MH^{2}M is demonstrated showing the influence of the different mesh parameters;

(*iii*) bridge the MH^{2}M with other multiscale finite elements, notably the MHM method and MsFEM [24, 25].

This work is summarized as follows: Section 2 presents the functional configuration associated with domain partitions. The process of characterizing the exact solution as a solution to local and global problems is established in Section 3. We introduce the MH^{2}M in Section 4 as well as its connection to MsFEM. Section 5 proposes the mathematical abstraction from which we establish the well-posedness and the best approximation for the MH^{2}M. In Section 6, we introduce families of interpolation spaces that satisfy the conditions presented in the Section 5 and prove optimal convergence under local regularity assumptions. The algorithm associated with the method is the subject of Section 7. Conclusions are drawn in Section 8, and some technical results are presented in the Appendix.

2. Settings

2.1. **Partition, broken spaces and norms.** Let $\mathcal{T}_H$ be a regular mesh, which can be based on different element geometries, and $\partial\mathcal{T}_H$ be the skeleton of $\mathcal{T}_H$. Without loss of generality, we adopt above and in the remainder of the text, the terminology of three-dimensional domains, denoting for instance the boundaries of the elements by faces. For a given element $\tau \in \mathcal{T}_H$, with diameter H_τ, let $\partial\tau$ denote its boundary and $\boldsymbol{n}^\tau$ the unit size normal vector that points outward τ. We denote by $\boldsymbol{n}$ the outward normal vector on $\partial\Omega$. Consider now the following spaces:

$$
\begin{gathered}
H^1(\mathcal{T}_H) := \{v \in L^2(\Omega) : v|_\tau \in H^1(\tau),\, \tau \in \mathcal{T}_H\}, \qquad \Lambda := \prod_{\tau\in\mathcal{T}_H} H^{-1/2}(\partial\tau), \\
H_0^{1/2}(\partial\mathcal{T}_H) := \{v|_{\partial\mathcal{T}_H} : v \in H_0^1(\Omega)\}.
\end{gathered} \tag{3}
$$

For $w,\, v \in L^2(\Omega)$ and $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$, $\mu \in \Lambda$ define

$$
(w, v)_{\mathcal{T}_H} := \sum_{\tau\in\mathcal{T}_H} \int_\tau wv\, d\boldsymbol{x} \quad \text{and} \quad \langle \mu, \rho\rangle_{\partial\mathcal{T}_H} := \sum_{\tau\in\mathcal{T}_H} \langle \mu_\tau, \rho\rangle_{\partial\tau},
$$

where $\langle \cdot, \cdot\rangle_{\partial\tau}$ is the dual product involving $H^{-1/2}(\partial\tau)$ and $H^{1/2}(\partial\tau)$, defined by

$$
\langle \mu, \rho\rangle_{\partial\tau} := \int_\tau \operatorname{div} \boldsymbol{\sigma} v\, d\boldsymbol{x} + \int_\tau \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\, v\, d\boldsymbol{x}
$$

for all $\boldsymbol{\sigma} \in H(\operatorname{div}, \tau)$ such that $\boldsymbol{\sigma} \cdot \boldsymbol{n}^\tau = \mu$, and all $v \in H^1(\tau)$ such that $v|_{\partial\tau} = \rho$. Note that the null space of the operator $\langle \mu, \cdot\rangle_{\partial\mathcal{T}_H}$ is $H_0^{1/2}(\partial\mathcal{T}_H)$, if μ is skeleton trace of the normal component of some function in $H(\operatorname{div}, \Omega)$. We use the same notation for a function in $H_0^{1/2}(\partial\mathcal{T}_H)$ and its restriction to an element boundary $\partial\tau$.

Consider also the following (semi-) norms:

$$
\begin{gathered}
|v|_{H^1_{\mathcal{A}}(\tau)} := \|\mathcal{A}^{1/2}\, \boldsymbol{\nabla}\, v\|_{L^2(\tau)}, \qquad |v|^2_{H^1_{\mathcal{A}}(\Omega)} := \sum_{\tau\in\mathcal{T}_H} |v|^2_{H^1_{\mathcal{A}}(\tau)}, \\
|\xi|_{H^{1/2}(\partial\tau)} := \inf_{\substack{\phi\in H^1(\tau) \\ \phi|_{\partial\tau}=\xi}} |\phi|_{H^1_{\mathcal{A}}(\tau)}, \qquad \|\rho\|_{H^{1/2}(\partial\mathcal{T}_H)} := \inf_{\substack{\phi\in H^1_0(\Omega) \\ \phi|_{\partial\mathcal{T}_H}=\rho}} |\phi|_{H^1_{\mathcal{A}}(\Omega)},
\end{gathered} \tag{4}
$$

where $v \in H^1(\mathcal{T}_H)$, $\xi \in H^{1/2}(\partial\tau)$ and $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$. Also, for $\mu \in \Lambda$ let

$$|\mu|_{H^{-1/2}(\partial\tau)} := \sup_{\widetilde{\phi}\in\widetilde{H}^{1/2}(\partial\tau)} \frac{\langle\mu,\widetilde{\phi}\rangle_{\partial\tau}}{|\widetilde{\phi}|_{H^{1/2}(\partial\tau)}}, \qquad |\mu|_\Lambda^2 := \sum_{\tau\in\mathcal{T}_H} |\mu|^2_{H^{-1/2}(\partial\tau)}, \tag{5}$$

where

$$\widetilde{H}^{1/2}(\partial\tau) := \left\{\xi \in H^{1/2}(\partial\tau) : \int_{\partial\tau} \xi\, d\boldsymbol{x} = 0\right\}. \tag{6}$$

2.2. **Hybridization.** Let $u \in H^1(\mathcal{T}_H)$, $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$, and $\lambda \in \Lambda$ be such that

$$\begin{aligned}
(\mathcal{A}\,\boldsymbol{\nabla}\, u, \boldsymbol{\nabla}\, v)_{\mathcal{T}_H} - \langle\lambda, v\rangle_{\partial\mathcal{T}_H} \qquad\qquad &= (f,v)_{\mathcal{T}_H} &&\text{for all } v \in H^1(\mathcal{T}_H),\\
-\langle\mu, u\rangle_{\partial\mathcal{T}_H} \qquad\qquad + \langle\mu,\rho\rangle_{\partial\mathcal{T}_H} &= 0 &&\text{for all } \mu \in \Lambda,\\
\langle\lambda,\xi\rangle_{\partial\mathcal{T}_H} \qquad\qquad &= 0 &&\text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H).
\end{aligned} \tag{7}$$

This is the formulation proposed in [16,18], and yields a hybrid formulation for (1). Indeed, if u is the weak solution of (1) then $\rho = u|_{\partial\mathcal{T}_H}$ and $\lambda = \mathcal{A}\,\boldsymbol{\nabla}\, u\cdot\boldsymbol{n}^\tau|_{\partial\mathcal{T}_H}$ solve (7). Classical results on Sobolev spaces [11,29] yield the converse statement. If the triplet (u,ρ,λ) solves (7), then, from the third equation, $\lambda = \boldsymbol{\sigma}\cdot\boldsymbol{n}^\tau$ for some $\boldsymbol{\sigma} \in H(\operatorname{div},\Omega)$. We gather from the first equation that $-\operatorname{div}\mathcal{A}\,\boldsymbol{\nabla}\, u = f$ weakly in each element, and that $\lambda = \mathcal{A}\,\boldsymbol{\nabla}\, u\cdot\boldsymbol{n}^\tau|_{\partial\mathcal{T}_H}$. Finally, it follows from the second equation that $\rho = u|_{\partial\mathcal{T}_H}$. Then, $u \in H_0^1(\Omega)$ and (1) holds in the weak sense. We gather from the above arguments that existence and uniqueness for solutions of (7) is immediate.

3. Exact solution decomposition

Consider the decomposition

$$H^1(\mathcal{T}_H) = \mathbb{P}_0(\mathcal{T}_H) \oplus \widetilde{H}^1(\mathcal{T}_H),$$

where $\mathbb{P}_0(\mathcal{T}_H)$ is the space of piecewise constants, and $\widetilde{H}^1(\mathcal{T}_H)$ is the space of functions with zero average within each element border $\partial\tau$, for $\tau \in \mathcal{T}_H$. Note that this decomposition differs from that of [2,20,27,30,31].

Taking a further step, we decompose Λ into a space of "constants" plus "zero-average" functionals over the border of the elements of $\mathcal{T}_H$. For each $\tau_i \in \mathcal{T}_H$, let $\lambda_i^0 \in \Lambda$ be such that

$$\langle\lambda_i^0, v\rangle_{\partial\mathcal{T}_H} := \int_{\partial\tau_i} v\, d\boldsymbol{x} \quad \text{for all } v \in H^1(\mathcal{T}_H). \tag{8}$$

Let N be the number of elements of $\mathcal{T}_H$ and

$$\Lambda^0 := \operatorname{span}\{\lambda_i^0 : i = 1, \dots, N\}, \qquad \widetilde{H}^{-1/2}(\partial\tau) := \{\widetilde{\mu} \in H^{-1/2}(\partial\tau) : \langle \widetilde{\mu}, 1\rangle_{\partial\tau} = 0\},$$
$$\widetilde{\Lambda} := \prod_{\tau\in\mathcal{T}_H} \widetilde{H}^{-1/2}(\partial\tau) = \{\widetilde{\mu} \in \Lambda : \langle \widetilde{\mu}, v^0\rangle_{\partial\mathcal{T}_H} = 0 \text{ for all } v^0 \in \mathbb{P}_0(\mathcal{T}_H)\}.$$

We can now decompose $\Lambda = \Lambda^0 \oplus \widetilde{\Lambda}$ as follows [13]. Given $\mu \in \Lambda$, let $\mu^0 \in \Lambda^0$

$$\langle \mu^0, v^0\rangle_{\partial\mathcal{T}_H} = \langle \mu, v^0\rangle_{\partial\mathcal{T}_H},$$

for all $v^0 \in \mathbb{P}_0(\mathcal{T}_H)$. Observe that μ^0 is well-defined and $\mu^0|_{\partial\tau} = \frac{1}{|\partial\tau|}\langle \mu, 1\rangle_{\partial\tau}$ for all $\tau \in \mathcal{T}_H$. Next, we define uniquely $\widetilde{\mu} := \mu - \mu^0$, i.e.,

$$\langle \widetilde{\mu}, v\rangle_{\partial\mathcal{T}_H} = \langle \mu, v\rangle_{\partial\mathcal{T}_H} - \langle \mu^0, v\rangle_{\partial\mathcal{T}_H},$$

and note that $\widetilde{\mu} \in \widetilde{\Lambda}$ since $\langle \widetilde{\mu}, v^0\rangle_{\partial\mathcal{T}_H} = \langle \mu, v^0\rangle_{\partial\mathcal{T}_H} - \langle \mu^0, v^0\rangle_{\partial\mathcal{T}_H} = 0$.

We then write $u = u^0 + \widetilde{u}$, where $u^0 \in \mathbb{P}_0(\mathcal{T}_H)$ and $\widetilde{u} \in \widetilde{H}^1(\mathcal{T}_H)$, and also $\lambda = \lambda^0 + \widetilde{\lambda}$, for $\lambda^0 \in \Lambda^0$ and $\widetilde{\lambda} \in \widetilde{\Lambda}$. Owing to such decomposition, we can characterize each component of u and λ from data and Lagrange multipliers using (7). To see that, we first test (7) with $(v^0, \mu^0, 0)$, and gather that $\lambda^0 \in \Lambda^0$ and $u^0 \in \mathbb{P}_0(\mathcal{T}_H)$ solve

$$\begin{aligned} \langle \lambda^0, v^0\rangle_{\partial\mathcal{T}_H} &= -\langle f, v^0\rangle_{\mathcal{T}_H} \quad \text{for all } v^0 \in \mathbb{P}_0(\mathcal{T}_H), \\ \langle \mu^0, u^0\rangle_{\partial\mathcal{T}_H} &= \langle \mu^0, \rho\rangle_{\partial\mathcal{T}_H} \quad \text{for all } \mu^0 \in \Lambda^0. \end{aligned} \tag{9}$$

Then, the first equation of (9) defines λ^0. The piecewise constant u^0 is obtained from the second equation of (9), after the computation of ρ. Both can be computed locally as follow

$$\lambda^0|_{\partial\tau} = -\frac{1}{|\partial\tau|}\int_\tau f\, d\boldsymbol{x} \quad \text{and} \quad u^0|_\tau = \frac{1}{|\partial\tau|}\int_{\partial\tau} \rho\, d\boldsymbol{x} \quad \text{for all } \tau \in \mathcal{T}_H. \tag{10}$$

Next, testing (7) with $(\widetilde{v}, \widetilde{\mu}, \xi)$, we gather that $\widetilde{u} \in \widetilde{H}^1(\mathcal{T}_H)$, $\widetilde{\lambda} \in \widetilde{\Lambda}$ and $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$ solve

$$\begin{aligned} (\mathcal{A}\,\boldsymbol{\nabla}\,\widetilde{u}, \boldsymbol{\nabla}\,\widetilde{v})_{\mathcal{T}_H} - \langle \widetilde{\lambda}, \widetilde{v}\rangle_{\partial\mathcal{T}_H} \qquad\qquad &= (f, \widetilde{v})_{\partial\mathcal{T}_H} && \text{for all } \widetilde{v} \in \widetilde{H}^1(\mathcal{T}_H), \\ -\langle \widetilde{\mu}, \widetilde{u}\rangle_{\partial\mathcal{T}_H} \qquad\qquad + \langle \widetilde{\mu}, \rho\rangle_{\partial\mathcal{T}_H} &= 0 && \text{for all } \widetilde{\mu} \in \widetilde{\Lambda}, \\ \langle \widetilde{\lambda}, \xi\rangle_{\partial\mathcal{T}_H} \qquad\qquad &= -\langle \lambda^0, \xi\rangle_{\partial\mathcal{T}_H} && \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H). \end{aligned} \tag{11}$$

The first equation of (11) allows the introduction of local mappings. Let $T : \widetilde{\Lambda} \to \widetilde{H}^1(\mathcal{T}_H)$ and $\widetilde{T} : L^2(\Omega) \to \widetilde{H}^1(\mathcal{T}_H)$ be such that, given $\mu \in \widetilde{\Lambda}$, $q \in L^2(\Omega)$ and $\tau \in \mathcal{T}_H$,

$$\int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T\mu)\cdot\boldsymbol{\nabla}\,\widetilde{v}\, d\boldsymbol{x} = \langle \mu, \widetilde{v}\rangle_{\partial\tau} \quad \text{and} \quad \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(\widetilde{T}q)\cdot\boldsymbol{\nabla}\,\widetilde{v}\, d\boldsymbol{x} = \int_\tau q\widetilde{v}\, d\boldsymbol{x}, \tag{12}$$

for all $\widetilde{v} \in \widetilde{H}^1(\mathcal{T}_H)$. The mappings T and $\widetilde{T}$ are locally well defined as the bilinear forms in (12) are coercive on $\widetilde{H}^1(\tau)$. They are locally bounded as follows

$$|T\widetilde{\mu}|_{H^1_{\mathcal{A}}(\tau)} = |\widetilde{\mu}|_{H^{-1/2}(\partial\tau)} \quad \text{and} \quad |\widetilde{T}q|_{H^1_{\mathcal{A}}(\tau)} \le \frac{C_{\mathcal{P}}}{a_{min}^{1/2}} H_\tau \|q\|_{L^2(\tau)}, \tag{13}$$

for all $\widetilde{\mu} \in \widetilde{\Lambda}$ and $q \in L^2(\Omega)$, where $C_{\mathcal{P}}$ is the positive local Poincaré constant independent of H_τ. The boundedness of T in (13) is a consequence of Lemma 23 (see the Appendix). The second estimate in (13) follows from the definition of $\widetilde{T}$ in (12), the Cauchy-Schwartz and local Poincaré inequalities.

Then $\widetilde{u} = T\widetilde{\lambda} + \tilde{T}f$, and the remaining equations in (11) yield

$$\begin{aligned} -\langle \widetilde{\mu}, T\widetilde{\lambda}\rangle_{\partial\mathcal{T}_H} + \langle \widetilde{\mu}, \rho\rangle_{\partial\mathcal{T}_H} &= \langle \widetilde{\mu}, \widetilde{T}f\rangle_{\partial\mathcal{T}_H} \quad &&\text{for all } \widetilde{\mu} \in \widetilde{\Lambda} \\ \langle \widetilde{\lambda}, \xi\rangle_{\partial\mathcal{T}_H} &= -\langle \lambda^0, \xi\rangle_{\partial\mathcal{T}_H} \quad &&\text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H). \end{aligned} \tag{14}$$

It follows from (13) that the bilinear form $\langle \cdot, T\cdot\rangle_{\partial\mathcal{T}_H}$ is coercive on $\widetilde{\Lambda}$ locally. Then, the first equation (14) also induces a local mapping. We denote it by

$$G : \{v \in L^2(\partial\mathcal{T}_H) : v \in H^{1/2}(\partial\tau),\ \tau \in \mathcal{T}_H\} \to \widetilde{\Lambda}$$

and is such that, given $\phi \in H^{1/2}(\partial\tau)$, we set $\widetilde{\lambda}_\phi := G\phi \in \widetilde{H}^{-1/2}(\partial\tau)$ as the (unique) solution of

$$\int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T\widetilde{\lambda}_\phi)\cdot\boldsymbol{\nabla}\,T\widetilde{\mu}\,d\boldsymbol{x} = \langle \widetilde{\mu}, T\widetilde{\lambda}_\phi\rangle_{\partial\tau} = \langle \widetilde{\mu}, \phi\rangle_{\partial\tau} \quad \text{for all } \widetilde{\mu} \in \widetilde{\Lambda}. \tag{15}$$

The operator G is bounded locally as follow (see Lemma 23 in the Appendix):

$$|G\xi|_{H^{-1/2}(\partial\tau)} = |\xi|_{H^{1/2}(\partial\tau)} \quad \text{for all } \xi \in H^{1/2}(\partial\tau). \tag{16}$$

Next, the first equation in (14) implies that $\widetilde{\lambda} = G(\rho - \widetilde{T}f)$, and then, the last equation of (14) reads

$$\langle G\rho, \xi\rangle_{\partial\mathcal{T}_H} = -\langle \lambda^0, \xi\rangle_{\partial\mathcal{T}_H} + \langle G\widetilde{T}f, \xi\rangle_{\partial\mathcal{T}_H} \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H). \tag{17}$$

Define the bilinear forms $g_\tau : H^{1/2}(\partial\tau) \times H^{1/2}(\partial\tau) \to \mathbb{R}$ for $\tau \in \mathcal{T}_H$, and $g : H_0^{1/2}(\partial\mathcal{T}_H) \times H_0^{1/2}(\partial\mathcal{T}_H) \to \mathbb{R}$ by

$$g_\tau(\xi, \phi) := \langle G\xi, \phi\rangle_{\partial\tau}, \qquad g(\xi, \phi) := \sum_{\tau\in\mathcal{T}_H} g_\tau(\xi, \phi) \quad \text{for } \xi,\ \phi \in H_0^{1/2}(\partial\mathcal{T}_H). \tag{18}$$

Remark 1 (TG local mapping)**.** *Note that both T and G are local operators. Furthermore, we find λ^0 and u^0 locally and trivially from* (10)*, as there are a finite number of unknowns.*

Thus, (17) *is the only global, infinite-dimensional equation depending on* $\mathcal{A}$ *through the operator* G*. For future reference it follows from* (15) *that*

$$(TG\xi)|_{\partial\tau} = \xi + c_\xi, \quad \text{where } c_\xi = -\frac{1}{|\partial\tau|}\int_{\partial\tau} \xi \, d\boldsymbol{x} \tag{19}$$

for all $\xi \in H^{1/2}(\partial\tau)$ *and* $\tau \in \mathcal{T}_H$*.*

Remark 2 (TG equivalent formulation)**.** *Note that* G *is symmetric since, from* (15),

$$\sum_{\tau\in\mathcal{T}_H}\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\, TG\rho\cdot\boldsymbol{\nabla}\, TG\xi = \langle G\rho, TG\xi\rangle_{\partial\mathcal{T}_H} = \langle G\rho, \xi\rangle_{\partial\mathcal{T}_H}, \tag{20}$$

for all $\rho, \xi \in H_0^{1/2}(\partial\mathcal{T}_H)$*. Also, from* (15) *and* (12),

$$\langle G\widetilde{T}f, \xi\rangle_{\partial\mathcal{T}_H} = \langle G\xi, \widetilde{T}f\rangle_{\partial\mathcal{T}_H} = \sum_{\tau\in\mathcal{T}_H}\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\,\widetilde{T}f\cdot\boldsymbol{\nabla}\, TG\xi = \sum_{\tau\in\mathcal{T}_H}\int_\tau fTG\xi.$$

Thus, using (19) *and* (10) *it holds that*

$$\begin{aligned}
&-\langle\lambda^0,\xi\rangle_{\partial\mathcal{T}_H} + \langle G\widetilde{T}f,\xi\rangle_{\partial\mathcal{T}_H} = -\sum_{\tau\in\mathcal{T}_H}\int_{\partial\tau}\lambda^0\xi + \sum_{\tau\in\mathcal{T}_H}\int_\tau fTG\xi \\
&= \sum_{\tau\in\mathcal{T}_H}\int_{\partial\tau}\lambda^0 c_\xi + \sum_{\tau\in\mathcal{T}_H}\int_\tau fTG\xi = -\sum_{\tau\in\mathcal{T}_H}\int_\tau fc_\xi + \sum_{\tau\in\mathcal{T}_H}\int_\tau fTG\xi = \sum_{\tau\in\mathcal{T}_H}\int_\tau f(TG\xi - c_\xi),
\end{aligned}$$

and formulation (17) *is equivalent to*

$$\sum_{\tau\in\mathcal{T}_H}\int_\tau \mathcal{A}\nabla TG\rho\cdot\nabla TG\xi = \sum_{\tau\in\mathcal{T}_H}\int_\tau f(TG\xi - c_\xi) \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H). \tag{21}$$

In the next theorem, we show that $g_\tau(\cdot,\cdot)$ is coercive and continuous, so (17) is well-posed. Furthermore, we collect the results of the constructive approach described above and provide an explicit characterization of u and λ as the solutions to local problems that are brought together via ρ the solution of the global skeletal problem (17).

Theorem 3. *Let* g_τ*,* g *be defined by* (18)*. The following coercivity and continuity results hold:*

$$\begin{aligned}
g_\tau(\widetilde{\xi},\widetilde{\xi}) &= |\widetilde{\xi}|^2_{H^{1/2}(\partial\tau)} \quad \text{for all } \widetilde{\xi}\in \widetilde{H}^{1/2}(\partial\tau),\\
g(\xi,\xi) &= |\xi|^2_{H^{1/2}(\partial\mathcal{T}_H)} \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H),\\
g(\xi,\rho) &\le |\xi|_{H^{1/2}(\partial\mathcal{T}_H)}|\rho|_{H^{1/2}(\partial\mathcal{T}_H)} \quad \text{for all } \xi,\ \rho \in H_0^{1/2}(\partial\mathcal{T}_H).
\end{aligned}$$

Hence, there exists a unique $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$ *such that*

$$g(\rho,\xi) = -\langle\lambda^0,\xi\rangle_{\partial\mathcal{T}_H} + g(\widetilde{T}f,\xi) \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H) \tag{22}$$

and

$$|\rho|_{H^{1/2}(\partial\mathcal{T}_H)} \leq |\lambda^0|_\Lambda + |\widetilde{T}f|_{H^{1/2}(\partial\mathcal{T}_H)}.$$

Moreover, the exact solution u and λ of the hybrid formulation (7) *writes*

$$u = u^0 + TG\rho + (I - TG)\widetilde{T}f \quad \textit{and} \quad \lambda = \lambda^0 + G\rho - G\widetilde{T}f, \tag{23}$$

where u^0 and λ^0 are the piecewise constant functions defined through (10).

Proof. For $\widetilde{\xi} \in \widetilde{H}^{1/2}(\partial\tau)$ let $\widetilde{\lambda}_{\widetilde{\xi}} := G\widetilde{\xi}$. The local coercivity holds since, from (19) and Lemma 23-(i) (see appendix),

$$|\widetilde{\xi}|^2_{H^{1/2}(\partial\tau)} = |T\widetilde{\lambda}_{\widetilde{\xi}}|^2_{H^1_{\mathcal{A}}(\tau)} = \langle \widetilde{\lambda}_{\widetilde{\xi}}, \widetilde{\xi} \rangle_{\partial\tau} = g_\tau(\widetilde{\xi}, \widetilde{\xi}).$$

Next, assume that $\xi \in H_0^{1/2}(\partial\mathcal{T}_H)$. Then $\xi = \widetilde{\xi} + \xi^0$, where $\widetilde{\xi}|_{\partial\tau} \in \widetilde{H}^{1/2}(\partial\tau)$ and $\xi^0|_{\partial\tau}$ is constant for each $\tau \in \mathcal{T}_H$. Then, from the definition of mapping G in (15), and from the fact $|\cdot|_{H^{1/2}(\partial\tau)}$ is a semi-norm, we get

$$g(\xi, \xi) = \sum_{\tau\in\mathcal{T}_H} g_\tau(\xi, \xi) = \sum_{\tau\in\mathcal{T}_H} g_\tau(\widetilde{\xi}, \widetilde{\xi}) = \sum_{\tau\in\mathcal{T}_H} |\widetilde{\xi}|^2_{H^{1/2}(\partial\tau)} = \sum_{\tau\in\mathcal{T}_H} |\xi|^2_{H^{1/2}(\partial\tau)} = |\phi_\xi|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}$$

where we used Lemma 22 (see appendix) in the last step.

To show continuity, let ξ, $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$ and set $\widetilde{\lambda}_\phi := G\xi$, $\widetilde{\lambda}_\rho := G\rho$. Then

$$\begin{aligned} g(\xi, \rho) = \sum_{\tau\in\mathcal{T}_H} \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T\widetilde{\lambda}_\xi) \cdot \boldsymbol{\nabla}\, T\widetilde{\lambda}_\rho \, d\boldsymbol{x} &\leq \sum_{\tau\in\mathcal{T}_H} |T\widetilde{\lambda}_\xi|_{H^1_{\mathcal{A}}(\tau)} |T\widetilde{\lambda}_\rho|_{H^1_{\mathcal{A}}(\tau)} \\ &= \sum_{\tau\in\mathcal{T}_H} |T\widetilde{\lambda}_\xi|_{H^{1/2}(\partial\tau)} |T\widetilde{\lambda}_\rho|_{H^{1/2}(\partial\tau)} = \sum_{\tau\in\mathcal{T}_H} |\xi|_{H^{1/2}(\partial\tau)} |\rho|_{H^{1/2}(\partial\tau)}, \end{aligned}$$

where we used Lemma 22 (see appendix), identity (19) and that $|\cdot|_{H^{1/2}(\partial\tau)}$ is a semi-norm. Existence and uniqueness of solution for (22) follows from Lax-Milgram's Lemma. The stability result for ρ follows from the coercivity and continuity of $g(\cdot,\cdot)$ and the definition of the Λ-norm in (5). Finally, the characterization (23) is a straightforward consequence of the decomposition $u = u^0 + \widetilde{u}$ and $\lambda = \lambda^0 + \widetilde{\lambda}$, using

$$\widetilde{\lambda} = G(\rho - \widetilde{T}f) \quad \text{and} \quad \widetilde{u} = T\widetilde{\lambda} + \widetilde{T}f = TG(\rho - \widetilde{T}f) + \widetilde{T}f. \tag{24}$$

□

The structure of exact solutions u and λ in Theorem 3 guides the discretization choices and gives rise to the MH^{2}M as a result of the discrete version of the skeletal variational problem (22). This is addressed next.

4. Discretization

4.1. **The method.** Consider finite dimensional spaces

$$\Gamma_{H_\Gamma} \subset H_0^{1/2}(\partial\mathcal{T}_H), \quad \Lambda_{H_\Lambda} \subset \Lambda \quad \text{and} \quad V_h \subset H^1(\mathcal{T}_H),$$

and

$$\widetilde{\Lambda}_{H_\Lambda} := \Lambda_{H_\Lambda} \cap \widetilde{\Lambda} \quad \text{and} \quad \widetilde{V}_h := V_h \cap \widetilde{H}^1(\mathcal{T}_H),$$

and denote by $\Gamma_{H_\Gamma}(\partial\tau)$, $\Lambda_{H_\Lambda}(\partial\tau)$, $\widetilde{\Lambda}_{H_\Lambda}(\partial\tau)$, $V_h(\tau)$ and $\widetilde{V}_h(\tau)$ their restriction to $\tau \in \mathcal{T}_H$.

For $\widetilde{\mu} \in \widetilde{\Lambda}$ and $q \in L^2(\Omega)$, the discrete versions of mapping T and $\widetilde{T}$, namely $T_h : \widetilde{\Lambda} \to \widetilde{V}_h$ and $\widetilde{T}_h : L^2(\Omega) \to \widetilde{V}_h$, are

$$\int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T_h\widetilde{\mu}) \cdot \boldsymbol{\nabla}\,\widetilde{v}_h\,d\boldsymbol{x} = \langle \widetilde{\mu}, \widetilde{v}_h\rangle_{\partial\tau} \quad \text{and} \quad \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(\widetilde{T}_h q) \cdot \boldsymbol{\nabla}\,\widetilde{v}_h\,d\boldsymbol{x} = \int_\tau q\widetilde{v}_h\,d\boldsymbol{x}, \tag{25}$$

for all $\widetilde{v}_h \in \widetilde{V}_h$. Also, let $G_h : H_0^{1/2}(\partial\mathcal{T}_H) \to \widetilde{\Lambda}_{H_\Lambda}$ be the discrete operator related to G (cf. (15)). For $\phi \in H_0^{1/2}(\partial\mathcal{T}_H)$, define $\widetilde{\lambda}_\phi = G_h\phi$ such that

$$\int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T_h\widetilde{\lambda}_\phi) \cdot \boldsymbol{\nabla}\,T_h\widetilde{\mu}_{H_\Lambda}\,d\boldsymbol{x} = \langle \widetilde{\mu}_{H_\Lambda}, T_h\widetilde{\lambda}_\phi\rangle_{\partial\tau} = \langle \widetilde{\mu}_{H_\Lambda}, \phi\rangle_{\partial\tau} \quad \text{for all } \widetilde{\mu}_{H_\Lambda} \in \widetilde{\Lambda}_{H_\Lambda}. \tag{26}$$

Note that the the same arguments used for G yield that G_h is also symmetric.

Remark 4 (Discrete local mappings)**.** *The operators T_h and $\widetilde{T}_h$ are well defined and bounded. Also, $\widetilde{T}_h$ is bounded as $\widetilde{T}$ given in* (13)*, and*

$$|T_h\widetilde{\mu}|_{H^1_{\mathcal{A}}(\tau)} \le |\widetilde{\mu}|_{H^{-1/2}(\partial\tau)} \quad \textit{for all } \widetilde{\mu} \in \widetilde{\Lambda} \textit{ and } \tau \in \mathcal{T}_H, \tag{27}$$

from (25) *and the norm definitions. On the other hand, note from* (26) *that G_h is a well-defined mapping only if T_h is injective, which does not necessarily hold unless there is some kind of compatibility between spaces $\widetilde{\Lambda}_{H_\Lambda}$ and $\widetilde{V}_h$. The details of such compatibility condition is discussed in Section 5.1.*

Based on Theorem 3, we define the MH^{2}M such that $\rho_{H_\Gamma} \in \Gamma_{H_\Gamma}$ solves

$$\langle G_h\rho_{H_\Gamma}, \xi\rangle_{\partial\mathcal{T}_H} = -\langle \lambda^0, \xi\rangle_{\partial\mathcal{T}_H} + \langle G_h\widetilde{T}_h f, \xi\rangle_{\partial\mathcal{T}_H} \quad \text{for all } \xi \in \Gamma_{H_\Gamma}, \tag{28}$$

where λ^0 is given in (10). Then, the exact solution u and λ are approximate by their discrete counterparts $u_h \in V_h$ and $\lambda_{H_\Lambda} \in \Lambda_{H_\Lambda}$, where

$$u_h := u_h^0 + T_hG_h\rho_{H_\Gamma} + (I - T_hG_h)\widetilde{T}_h f \quad \text{and} \quad \lambda_{H_\Lambda} := \lambda^0 + G_h(\rho_{H_\Gamma} - \widetilde{T}_h f), \tag{29}$$

where $u_h^0 \in \mathbb{P}_0(\mathcal{T}_H)$ is the approximate counterpart of u_0 in (10), i.e.,

$$u_h^0|_\tau = \frac{1}{|\partial\tau|}\int_{\partial\tau} \rho_{H_\Gamma}\,d\boldsymbol{x} \quad \text{for all } \tau \in \mathcal{T}_H. \tag{30}$$

Remark 5 (Conformity of u_h). *Note that, in general, the MH^{2}M is a nonconforming method in $H^1(\Omega)$. It becomes conforming if, for instance, we choose the (impractical) space $\widetilde{\Lambda}_{H_\Lambda} = \widetilde{\Lambda}$ since, in this case,*

$$u_h|_{\partial\tau} = u_h^0|_{\partial\tau} + T_h G \rho_{H_\Gamma}|_{\partial\tau} + (I - T_h G)\widetilde{T}_h f|_{\partial\tau} = \rho_h \quad \text{for all } \tau \in \mathcal{T}_H,$$

from (26) *with T_h replacing T, and* (30).

Remark 6 ($T_h G_h$ equivalent formulation). *From* (26)*, the relationship* (19) *is valid only in a weaker sense when G_h replaces G. However, following the arguments in Remark 2 and using* (25) *and* (26)*, we note that the MH^{2}M* (22) *is equivalent to*

$$\sum_{\tau\in\mathcal{T}_H} \int_\tau \mathcal{A}\nabla T_h G_h \rho \cdot \nabla T_h G_h \xi = \sum_{\tau\in\mathcal{T}_H} \int_\tau f(T_h G_h \xi - c_\xi) \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H), \tag{31}$$

where c_ξ is the constant given in (19).

Before heading to the error analysis, we establish a relationship between the MH^{2}M (22) and the MsFEM [32].

4.2. Bridging the MH^{2}M and MsFEM.

Note that equation (22) that defines our method has some sort of relation with the definition of the MsFEM. In fact, we show bellow that (22) yields the *same trace* as the MsFEM in *some* particular cases. We first consider a "continuous version" of the MsFEM, seeking $\rho \in H_0^{1/2}(\partial\mathcal{T}_H)$ such that

$$\int_\Omega \mathcal{A}\,\boldsymbol{\nabla}\,\mathcal{E}(\rho)\cdot\boldsymbol{\nabla}\,\mathcal{E}(\xi) = \int_\Omega f\,\mathcal{E}(\xi) \quad \text{for all } \xi \in H_0^{1/2}(\partial\mathcal{T}_H), \tag{32}$$

where we denote the $\mathcal{A}$-harmonic extension $\mathcal{E} : H_0^{1/2}(\partial\mathcal{T}_H) \to \mathcal{H} \cap H_0^1(\Omega)$ (the space of piecewise harmonic functions that are also in $H_0^1(\Omega)$).

Note now that, for all $\tau \in \mathcal{T}_H$ and $\xi \in H^{1/2}(\partial\tau)$,

$$\mathcal{E}(\xi) = TG\xi - c_\xi, \tag{33}$$

where c_ξ is as in (19). That the identity above holds on $\partial\tau$ follows immediately from (19). Next, since $G\xi \in \widetilde{\Lambda}$ then $TG\xi - c_\xi$ is $\mathcal{A}$-harmonic, and from uniqueness of the harmonic extension, the identity (33) holds. As a result, we get that solutions ρ of formulations (21) and (32) coincide since from (33) and (32)

$$\sum_{\tau\in\mathcal{T}_H} \int_\tau \mathcal{A}\,\boldsymbol{\nabla}\,TG\rho\cdot\boldsymbol{\nabla}\,TG\xi = \int_\Omega \mathcal{A}\,\boldsymbol{\nabla}\,\mathcal{E}(\rho)\cdot\boldsymbol{\nabla}\,\mathcal{E}(\xi) = \int_\Omega f\,\mathcal{E}(\xi) = \sum_{\tau\in\mathcal{T}_H}\int_\tau f(TG\xi - c_\xi)$$

for all $\rho, \xi \in H_0^{1/2}(\partial\mathcal{T}_H)$, and that is the same as (21).

Discretize (32) using $\Gamma_{H_\Gamma} \subset H_0^{1/2}(\partial\mathcal{T}_H)$ yields the MsFEM (c.f [32]) whose solution is

$$u_{\mathrm{MsFEM}} := \mathcal{E}(\rho_{H_\Gamma})$$

where $\rho_{H_\Gamma} \in \Gamma_{H_\Gamma}$ is the solution of (32) restricted to Γ_{H_Γ}. Note that if $T_h = T$ and $G_h = G$ then the corresponding solution u_h of MH^{2}M relates to u_{MsFEM} as follows

$$u_h = u_{\mathrm{MsFEM}} + (I - TG)\widetilde{T}f = u_{\mathrm{MsFEM}} + \widetilde{T}f - \mathcal{E}(\widetilde{T}f)$$

where we used (29), (33) and (30). It follows in particular that $u_h|_{\partial\tau} = u_{\mathrm{MsFEM}}|_{\partial\tau} = \rho_{H_\Gamma}$ for all $\tau \in \mathcal{T}_H$.

Remark 7 (Polynomial solutions). *If $\Gamma_{H_\Gamma} = \mathbb{P}_1(\partial\mathcal{T}_H) \cap H_0^{1/2}(\partial\mathcal{T}_H)$ and $\mathcal{A} = \alpha\,\mathcal{I}$, with $\alpha \in \mathbb{R}^+$, then*

$$u_h - \widetilde{T}f + \mathcal{E}(\widetilde{T}f) = u_{MsFEM} \in \mathbb{P}_1(\mathcal{T}_H),$$

where $\mathbb{P}_k(D)$ is the space of piecewise polynomial functions of degree up to $k \geq 0$ on the set $D = \mathcal{T}_H$ or $D = \partial\mathcal{T}_H$. In addition, if $f \in \mathbb{P}_0(\mathcal{T}_H)$ then $\widetilde{T}f \in \mathbb{P}_2(\mathcal{T}_H)$ such that $\nabla\widetilde{T}f \in RT_0(\mathcal{T}_H)$ where $RT_0(\mathcal{T}_H)$ stands for the lowest-order Raviart-Thomas space in each $\tau \in \mathcal{T}_H$.

In practice, the discrete mapping G_h is used instead of G and the relation (33) does not hold. So, the solutions of MsFEM and MH^{2}M do not coincide on element boundaries in general.

5. Numerical analysis

This section contains the proof of the well-posedness and best approximation properties of the MH^{2}M given in (17) (see also (22), (28) or (31)).

5.1. **Well-posedness.** Define the bilinear forms $g_{h,\tau} : H^{1/2}(\tau) \times H^{1/2}(\tau) \to \mathbb{R}$ for $\tau \in \mathcal{T}_H$, and $g_h : H_0^{1/2}(\partial\mathcal{T}_H) \times H_0^{1/2}(\partial\mathcal{T}_H) \to \mathbb{R}$ by

$$g_{h,\tau}(\xi,\phi) = \langle G_h\xi, \phi\rangle_{\partial\tau}, \qquad g_h(\xi,\phi) = \sum_{\tau\in\mathcal{T}_H} g_{h,\tau}(\xi,\phi) \tag{34}$$

for ξ, $\phi \in H_0^{1/2}(\partial\mathcal{T}_H)$. Using those notations, the MH^{2}M reads: Find $\rho_{H_\Gamma} \in \Gamma_{H_\Gamma}$ such that

$$g_h(\rho_{H_\Gamma}, \xi) = -\langle\lambda^0, \xi\rangle_{\partial\mathcal{T}_H} + g_h(\widetilde{T}_h f, \xi) \quad \text{for all } \xi \in \Gamma_{H_\Gamma}. \tag{35}$$

We first address the existence and uniqueness of solution for (35). This result is established under the following conditions:

Assumption A: there exists a positive constant β_τ, independent of H_τ, such that

$$|\mu|_{H^{-1/2}(\partial\tau)} \leq \beta_\tau\,|T_h\mu|_{H^1_{\mathcal{A}}(\tau)} \quad \text{for all } \mu \in \widetilde{\Lambda}_{H_\Lambda} \text{ and } \tau \in \mathcal{T}_H;$$

Assumption B: there exists a positive constant α_τ, independent of H_τ, such that

$$|\xi|_{H^{1/2}(\partial\tau)} \leq \alpha_\tau \, |G_h\xi|_{H^{-1/2}(\partial\tau)} \quad \text{for all } \xi \in \Gamma_{H_\Gamma} \cap \widetilde{H}^{1/2}(\partial\tau) \text{ and } \tau \in \mathcal{T}_H.$$

Let $\alpha_{\max} := \max\{\alpha_\tau : \tau \in \mathcal{T}_H\}$ and $\beta_{\max} := \max\{\beta_\tau : \tau \in \mathcal{T}_H\}$.

Remark 8 (Boundeness of G_h)**.** *A first consequence of Assumption A is that the mapping G_h is bounded. Indeed, let $\xi \in H^{1/2}(\partial\tau)$ and select $\mu := G_h\xi \in \widetilde{\Lambda}_{H_\Lambda}$ in Assumption A. Then, from Lemma 23, item (iii)*

$$\begin{aligned}\beta_\tau^{-2}|G_h\xi|^2_{H^{-1/2}(\partial\tau)} &\leq |T_hG_h\xi|^2_{H^1_{\mathcal{A}}(\tau)} = \langle G_h\xi, T_hG_h\xi\rangle_{\partial\tau} \\ &= (G_h\xi, \xi)_{\partial\tau} \leq |G_h\xi|_{H^{-1/2}(\partial\tau)}|\xi|_{H^{1/2}(\partial\tau)}.\end{aligned}$$

Then,

$$|G_h\xi|_{H^{-1/2}(\partial\tau)} \leq \beta_\tau^2|\xi|_{H^{1/2}(\partial\tau)} \quad \textit{and} \quad |G_h\xi|_\Lambda \leq \beta_{\max}^2|\xi|_{H^{1/2}(\partial\mathcal{T}_H)}. \tag{36}$$

The MH^{2}M is well-posed under Assumptions A and B. This follows next.

Theorem 9. *Let $g_{h,\tau}$, g_h be defined by* (34) *and assume that Assumptions A and B hold. Then, we have the following coercivity results:*

$$\begin{aligned}g_{h,\tau}(\widetilde{\xi}_{H_\Gamma}, \widetilde{\xi}_{H_\Gamma}) &\geq (\alpha_\tau\beta_\tau)^{-2}|\widetilde{\xi}_{H_\Gamma}|^2_{H^{1/2}(\partial\tau)} \quad \textit{for all } \widetilde{\xi}_{H_\Gamma} \in \Gamma_{H_\Gamma} \cap \widetilde{H}^{1/2}(\partial\tau), \\ g_h(\xi_{H_\Gamma}, \xi_{H_\Gamma}) &\geq (\alpha_{\max}\beta_{\max})^{-2}|\xi_{H_\Gamma}|^2_{H^{1/2}(\partial\mathcal{T}_H)} \quad \textit{for all } \xi_{H_\Gamma} \in \Gamma_{H_\Gamma}.\end{aligned}$$

Moreover, the following continuity results hold:

$$\begin{aligned}g_{h,\tau}(\xi_{H_\Gamma}, \rho_{H_\Gamma}) &\leq \beta_\tau^2|\xi_{H_\Gamma}|_{H^{1/2}(\partial\tau)}|\rho_{H_\Gamma}|_{H^{1/2}(\partial\tau)}, \\ g_h(\xi_{H_\Gamma}, \rho_{H_\Gamma}) &\leq \beta_{\max}^2|\xi_{H_\Gamma}|_{H^{1/2}(\mathcal{T}_H)}|\rho_{H_\Gamma}|_{H^{1/2}(\mathcal{T}_H)},\end{aligned}$$

for all ξ_{H_Γ}, $\rho_{H_\Gamma} \in \Gamma_{H_\Gamma}$. Then, the method (35) *is well-posed and*

$$|\rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} \leq (\alpha_{\max}\beta_{\max})^2\Big(|\lambda^0|_\Lambda + \beta_{\max}^2|\widetilde{T}_hf|_{H^{1/2}(\partial\mathcal{T}_H)}\Big).$$

Proof. Fix $\widetilde{\xi}_{H_\Gamma} \in \Gamma_{H_\Gamma} \cap \widetilde{H}^{1/2}(\partial\tau)$. The local coercivity holds since, from Assumptions A and B, and the definition of mapping $G_h(\cdot)$ in (26)

$$|\widetilde{\xi}_{H_\Gamma}|^2_{H^{1/2}(\partial\tau)} \leq \alpha_\tau^2\beta_\tau^2|T_hG_h\widetilde{\xi}_{H_\Gamma}|^2_{H^1_{\mathcal{A}}(\tau)} = \alpha_\tau^2\beta_\tau^2\langle G_h\widetilde{\xi}_{H_\Gamma}, T_hG_h\widetilde{\xi}_{H_\Gamma}\rangle_{\partial\tau} = \alpha_\tau^2\beta_\tau^2\, g_{h,\tau}(\widetilde{\xi}_{H_\Gamma}, \widetilde{\xi}_{H_\Gamma}).$$

Next, take $\xi_{H_\Gamma} \in \Gamma_{H_\Gamma}$. Then $\xi_{H_\Gamma} = \widetilde{\xi}_{H_\Gamma} + \xi^0_{H_\Gamma}$, where $\widetilde{\xi}_{H_\Gamma}|_{\partial\tau} \in \widetilde{H}^{1/2}(\partial\tau)$ and $\xi^0_{H_\Gamma}|_{\partial\tau}$ is constant for each $\tau \in \mathcal{T}_H$. So, since $G_h(\cdot)$ is symmetric and its image has zero mean value on $\partial\tau$, it holds that

$$g_h(\xi_{H_\Gamma},\xi_{H_\Gamma}) = \sum_{\tau\in\mathcal{T}_H} g_{h,\tau}(\xi_{H_\Gamma},\xi_{H_\Gamma}) = \sum_{\tau\in\mathcal{T}_H} g_{h,\tau}(\widetilde{\xi}_{H_\Gamma},\widetilde{\xi}_{H_\Gamma})$$
$$\geq \sum_{\tau\in\mathcal{T}_H} (\alpha_\tau\beta_\tau)^{-2}|\widetilde{\xi}_{H_\Gamma}|^2_{H^{1/2}(\tau)} \geq (\alpha_{\max}\beta_{\max})^{-2}|\xi_{H_\Gamma}|^2_{H^{1/2}(\partial\mathcal{T}_H)}.$$

To show continuity, we use Lemma 23, item *(iii)* and (36) to get

$$g_{h,\tau}(\xi_{H_\Gamma},\rho_{H_\Gamma}) \leq |G_h\xi_{H_\Gamma}|_{H^{-1/2}(\partial\tau)}|\rho_{H_\Gamma}|_{H^{1/2}(\partial\tau)} \leq \beta_\tau^2|\xi_{H_\Gamma}|_{H^{1/2}(\partial\tau)}|\rho_{H_\Gamma}|_{H^{1/2}(\partial\tau)}.$$

Finally,

$$g_h(\xi_{H_\Gamma},\rho_{H_\Gamma}) \leq \sum_{\tau\in\mathcal{T}_H} \beta_\tau^2|\xi_{H_\Gamma}|_{H^{1/2}(\partial\tau)}|\rho_{H_\Gamma}|_{H^{1/2}(\partial\tau)} \leq \beta_{\max}^2|\xi_{H_\Gamma}|_{H^{1/2}(\mathcal{T}_H)}|\rho_{H_\Gamma}|_{H^{1/2}(\mathcal{T}_H)}$$

from the Cauchy-Schwartz inequality and Lemma 22. Existence and uniqueness of solution of (35) follow from the Lax-Milgram lemma, and

$$|\rho_{H_\Gamma}|^2_{H^{1/2}(\partial\mathcal{T}_H)} \leq (\alpha_{\max}\beta_{\max})^2 g_h(\rho_{H_\Gamma},\rho_{H_\Gamma}) = (\alpha_{\max}\beta_{\max})^2\Big(-\langle\lambda_0,\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H} + g_h(\widetilde{T}_hf,\rho_{H_\Gamma})_{\partial\mathcal{T}_H}\Big)$$
$$\leq (\alpha_{\max}\beta_{\max})^2\Big(|\lambda^0|_\Lambda + \beta_{\max}^2|\widetilde{T}_hf|_{H^{1/2}(\partial\mathcal{T}_H)}\Big)|\rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)},$$

and the result follows. □

Given the spaces $\widetilde{\Lambda}_{H_\Lambda}$, Γ_{H_Γ} and $\widetilde{V}_h$, it may be difficult to verify directly whether Assumptions A and B are valid, and we propose an alternative based on Fortin operators acting on these finite-dimensional spaces to ease the proof. The upshot is that it clarifies in what sense the spaces Λ_{H_Λ} and V_h, and the spaces Λ_{H_Λ} and Γ_{H_Γ} must be compatible to satisfy Assumptions A and B. We detail this alternative below for both assumptions.

Remark 10 (Assumption A from a Fortin operator)**.** *Assumption A is closely related to a compatibility condition between the finite-dimensional spaces V_h and Λ_{H_Λ}. Specifically, let $V_{h_0} \subset V_h \subset H^1(\mathcal{T}_H)$ be a finite dimensional space such that:*

- *there exists a mapping $\pi_V : H^1(\mathcal{T}_H) \to V_{h_0}$ such that, for all $v \in H^1(\mathcal{T}_H)$ and $\tau \in \mathcal{T}_H$, $\pi_V(v)$ is uniquely defined (up to a bubble) through*

$$\int_{\partial\tau} \mu\,\pi_V(v)\,d\boldsymbol{x} = \int_{\partial\tau} \mu\,v\,d\boldsymbol{x} \quad \text{for all } \mu\in\Lambda_{H_\Lambda}, \qquad |\pi_V(v)|_{H^{1/2}(\partial\tau)} \leq \beta_\tau\,|v|_{H^{1/2}(\partial\tau)}, \tag{37}$$

where β_τ is a positive constant independent of mesh parameters.

To see that (37) *implies Assumption A, take* $\mu \in \widetilde{\Lambda}_{H_\Lambda}$ *and note that* $\pi_V(\widetilde{v}) \in \widetilde{V}_{h_0} := V_{h_0} \cap \widetilde{H}^1(\mathcal{T}_H)$ *for all* $\widetilde{v} \in \widetilde{H}^1(\mathcal{T}_H)$. *Then, from the definition of* $|\cdot|_{H^{-1/2}(\partial\tau)}$ *in* (5), (37), *the definition of* T_h *operator, and Cauchy–Schwartz inequality, we get*

$$|\mu|_{H^{-1/2}(\partial\tau)} = \sup_{\widetilde{\phi}\in\widetilde{H}^{1/2}(\partial\tau)} \frac{\langle \mu, \widetilde{\phi}\rangle_{\partial\tau}}{|\widetilde{v}|_{H^{1/2}(\partial\tau)}} = \sup_{\substack{\widetilde{v}\in\widetilde{H}^1(\tau)\\ \widetilde{v}|_{\partial\tau}=\widetilde{\phi}}} \frac{\langle \mu, \widetilde{v}\rangle_{\partial\tau}}{|\widetilde{v}|_{H^{1/2}(\partial\tau)}} \le \beta_\tau \sup_{\substack{\widetilde{v}\in\widetilde{H}^1(\tau)\\ \widetilde{v}|_{\partial\tau}=\widetilde{\phi}}} \frac{\langle \mu, \pi_V(\widetilde{v})\rangle_{\partial\tau}}{|\pi_V(\widetilde{v})|_{H^{1/2}(\partial\tau)}}$$

$$\le \beta_\tau \sup_{\widetilde{v}_h\in\widetilde{V}_{h_0}(\tau)} \frac{\langle \mu, \widetilde{v}_h\rangle_{\partial\tau}}{|\widetilde{v}_h|_{H^{1/2}(\partial\tau)}} = \beta_\tau \sup_{\widetilde{v}_h\in\widetilde{V}_{h_0}(\tau)} \frac{\int_\tau \mathcal{A}\nabla T_h\mu\cdot\nabla\widetilde{v}_h\, d\boldsymbol{x}}{|\widetilde{v}_h|_{H^{1/2}(\partial\tau)}} \le \beta_\tau\, |T_h\mu|_{H^1_{\mathcal{A}}(\tau)}.$$

Note that Assumptions A holds with the same constant β_τ *if* V_h *replaces* V_{h_0} *in* (37).

The existence of a mapping $\pi_V(\cdot)$ satisfying (37) is sufficient to fullfil Assumption B when $\Gamma_{H_\Gamma}(\partial\tau) \cap \widetilde{H}^{1/2}(\partial\tau) \subseteq \widetilde{V}_{h_0}(\tau)|_{\partial\tau}$ for all $\tau \in \mathcal{T}_H$. This is the subject of the next lemma.

Lemma 11. *If there exists a mapping* $\pi_V(\cdot)$ *satisfying* (37) *and*

$$\Gamma_{H_\Gamma}(\partial\tau) \cap \widetilde{H}^{1/2}(\partial\tau) \subset \widetilde{V}_{h_0}(\tau)|_{\partial\tau} \quad \textit{for all } \tau \in \mathcal{T}_H,$$

then Assumption B holds with $\alpha_\tau = \beta_\tau$.

Proof. Let $\xi \in \Gamma_{H_\Gamma}(\tau) \cap \widetilde{H}^{1/2}(\partial\tau)$ and $v \in \widetilde{V}_{h_0}(\tau)$ be such that $\xi|_{\partial\tau} = v|_{\partial\tau}$ for all $\tau \in \mathcal{T}_H$. Note that from the definition of $\pi_V(\cdot)$ in (37) and (26) (with T instead of T_h), for all $\mu \in \widetilde{\Lambda}_{H_\Lambda}$, it holds that

$$\int_{\partial\tau} \mu\,\xi\, d\boldsymbol{x} = \int_{\partial\tau} \mu\, v\, d\boldsymbol{x} = \int_{\partial\tau} \mu\, T G_h v\, d\boldsymbol{x} = \int_{\partial\tau} \mu\, \pi_V(T G_h v)\, d\boldsymbol{x}\,.$$

Then, the above equality is also valid for all $\mu \in \Lambda_{H_\Lambda}$ and (37) yields

$$\int_{\partial\tau} \mu\,(\xi - \pi_V(TG_h v))\, d\boldsymbol{x} = 0 \;\Rightarrow\; v|_{\partial\tau} = \pi_V(TG_h v)|_{\partial\tau} \in \widetilde{V}_{h_0}(\tau) \quad \text{for all } \tau \in \mathcal{T}_H. \tag{38}$$

Consequently, we get from (38), the stability of $\pi_V(\cdot)$ in (37), Lemma 23$-(i)$ and the stability of T in (13)

$$|\xi|_{H^{1/2}(\partial\tau)} = |\pi_V(TG_h v)|_{H^{1/2}(\partial\tau)} = \beta_\tau\, |G_h v|_{H^{-1/2}(\partial\tau)} = \beta_\tau\, |G_h \xi|_{H^{-1/2}(\partial\tau)},$$

which corresponds to Assumption B with $\alpha_\tau = \beta_\tau$. □

Remark 12 (Assumption B from a Fortin operator)**.** *Assumption B is related to a compatibility condition between the spaces* Λ_{H_Λ} *and* Γ_{H_Γ}. *Specifically, let* $\Lambda_{H_{\Lambda_0}} \subset \Lambda_{H_\Lambda} \subset \Lambda$ *be a finite-dimensional space such that*

– *there exists a mapping $\pi_\Lambda : \Lambda \to \Lambda_{H_{\Lambda_0}}$ and a positive constant α_τ, independent of mesh parameters, such that for all $\mu \in \Lambda$ and $\tau \in \mathcal{T}_H$, it follows that*

$$\begin{aligned}\int_{\partial\tau} \pi_\Lambda(\mu)\, \xi \, d\boldsymbol{x} = \langle \mu, \xi \rangle_{\partial\tau} \quad \text{for all } \xi \in \Gamma_{H_\Gamma}, \\ |\pi_\Lambda(\mu)|_{H^{-1/2}(\partial\tau)} \le \alpha_\tau \, |\mu|_{H^{-1/2}(\partial\tau)}.\end{aligned} \tag{39}$$

Assumption B follows from (39)*. Indeed, take $\xi \in \Gamma_{H_\Gamma}(\tau) \cap \widetilde{H}^{1/2}(\partial\tau)$ for all $\tau \in \mathcal{T}_H$ and note that $\pi_\Lambda(\widetilde{\mu}) \in \widetilde{\Lambda}_{H_{\Lambda_0}} := \Lambda_{H_{\Lambda_0}} \cap \widetilde{\Lambda}$ for all $\widetilde{\mu} \in \widetilde{\Lambda}$. Next, use the characterization $|\cdot|_{H^{1/2}(\partial\tau)}$ in* (61)*,* (39)*, the definition of T and its stability* (13)*, Cauchy-Schwartz inequality, and Lemma 23–(i), to obtain*

$$\begin{aligned}|\xi|_{H^{1/2}(\partial\tau)} &= \sup_{\widetilde{\mu} \in \widetilde{H}^{-1/2}(\partial\tau)} \frac{\langle \widetilde{\mu}, \xi \rangle_{\partial\tau}}{|\widetilde{\mu}|_{H^{-1/2}(\partial\tau)}} \le \alpha_\tau \sup_{\widetilde{\mu} \in \widetilde{H}^{-1/2}(\partial\tau)} \frac{\langle \pi_\Lambda(\widetilde{\mu}), \xi \rangle_{\partial\tau}}{|\pi_\Lambda(\widetilde{\mu})|_{H^{-1/2}(\partial\tau)}} \le \alpha_\tau \sup_{\widetilde{\mu}_{H_\Gamma} \in \widetilde{\Lambda}_{H_{\Lambda_0}}(\partial\tau)} \frac{\langle \widetilde{\mu}_{H_\Gamma}, \xi \rangle_{\partial\tau}}{|\widetilde{\mu}_{H_\Gamma}|_{H^{-1/2}(\partial\tau)}} \\ &= \alpha_\tau \sup_{\widetilde{\mu}_{H_\Gamma} \in \widetilde{\Lambda}_{H_{\Lambda_0}}(\partial\tau)} \frac{\int_\tau \mathcal{A} \nabla T G_h \xi \cdot \nabla T \widetilde{\mu}_{H_\Gamma} \, d\boldsymbol{x}}{|\widetilde{\mu}_{H_\Gamma}|_{H^{-1/2}(\partial\tau)}} \le \alpha_\tau \, |T G_h \xi|_{H^1_{\mathcal{A}}(\tau)} = \alpha_\tau \, |G_h \xi|_{H^{-1/2}(\tau)}.\end{aligned}$$

In addition, Assumptions B holds with the same constant α_τ if Λ_{H_Λ} replaces $\Lambda_{H_{\Lambda_0}}$ in (39)*.*

5.2. **Best approximation.** We start by estimating the consistency error $G - G_h$ based on the first Strang Lemma [26]. In what follows, we denote by C positive constants independent of the mesh parameters, which can change at each occurrence.

Lemma 13. *Under the assumptions of Theorem 9, for all $\phi \in H^{1/2}(\partial\mathcal{T}_H)$, it holds that*

$$|G\phi - G_h\phi|_\Lambda \le E(G\phi),$$

where

$$E(G\phi) := (1 + \beta_{\max}^2) \inf_{\widetilde{\mu}_{H_\Lambda} \in \widetilde{\Lambda}_{H_\Lambda}} |G\phi - \widetilde{\mu}_{H_\Lambda}|_\Lambda + \beta_{\max}^2 |(T - T_h) G\phi|_{H^{1/2}(\partial\mathcal{T}_H)}. \tag{40}$$

Proof. Given $\phi \in H^{1/2}(\partial\mathcal{T}_H)$, let $\widetilde{\lambda} := G\phi$ and $\widetilde{\lambda}_{H_\Lambda} := G_h\phi$. Then, by definition of mapping G and G_h, for $\widetilde{\mu}_{H_\Lambda} \in \widetilde{\Lambda}_{H_\Lambda}$,

$$\langle \widetilde{\mu}_{H_\Lambda}, T\widetilde{\lambda} \rangle_{\partial\mathcal{T}_H} = \langle \widetilde{\mu}_{H_\Lambda}, T_h \widetilde{\lambda}_{H_\Lambda} \rangle_{\partial\mathcal{T}_H} = \langle \widetilde{\mu}_{H_\Lambda}, \phi \rangle_{\partial\mathcal{T}_H}. \tag{41}$$

Hence, from Assumption A and (41)

$$\begin{aligned}\beta_{\max}^{-2} |\widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda^2 &\le \langle \widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}, T_h(\widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}) \rangle_{\partial\mathcal{T}_H} \\ &= \langle \widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}, T_h(\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}) \rangle_{\partial\mathcal{T}_H} + \langle \widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}, T_h(\widetilde{\lambda}_{H_\Lambda} - \widetilde{\lambda}) \rangle_{\partial\mathcal{T}_H} \\ &= \langle \widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}, T_h(\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}) \rangle_{\partial\mathcal{T}_H} + \langle \widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}, (T - T_h)\widetilde{\lambda} \rangle_{\partial\mathcal{T}_H} \\ &\le |\widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda \Big(|\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda + |(T - T_h)\widetilde{\lambda}|_{H^{1/2}(\partial\mathcal{T}_H)} \Big)\end{aligned}$$

where we also used Lemma 23$-(iii, iv)$ (with T replaced by T_h) and (27). From the triangle inequality and inequality above we get

$$|\widetilde{\lambda} - \widetilde{\lambda}_{H_\Lambda}|_\Lambda \le |\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda + |\widetilde{\lambda}_{H_\Lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda \le \Big(1 + \beta^2_{\max}\Big)|\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda + \beta^2_{\max}|(T - T_h)\widetilde{\lambda}|_{H^{1/2}(\partial\mathcal{T}_H)}$$

and the result follows. □

Owing to the previous results, the next theorem shows that the method yields best approximation results.

Theorem 14. *Assume that the conditions A and B are valid, and let $(u, \rho, \lambda) \in H^1(\mathcal{T}_H) \times H_0^{1/2}(\mathcal{T}_H) \times \Lambda$ solve (7) and $\rho_{H_\Gamma} \in \Lambda_{H_\Lambda}$ solves (28), and $(u_h, \lambda_{H_\Lambda}) \in V_h \times \Lambda_{H_\Lambda}$ be given in (29). Then,*

$$|\rho - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} \le C\Big(\inf_{\phi_{H_\Gamma}\in\Gamma_{H_\Gamma}} |\rho - \phi_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + |(\widetilde{T} - \widetilde{T}_h)f|_{H^{1/2}(\partial\mathcal{T}_H)} + E(\widetilde{\lambda})\Big), \tag{42}$$

and

$$\begin{aligned} |\lambda - \lambda_{H_\Lambda}|_\Lambda &\le C\Big(|\rho - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + |(\widetilde{T} - \widetilde{T}_h)f|_{H^{1/2}(\partial\mathcal{T}_H)} + E(\widetilde{\lambda})\Big), \\ |u - u_h|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &\le |\lambda - \lambda_{H_\Lambda}|_\Lambda + |(T - T_h)\widetilde{\lambda} + (\widetilde{T} - \widetilde{T}_h)f|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}. \end{aligned} \tag{43}$$

Moreover, the following weak continuity holds:

$$\langle \mu_{H_\Lambda}, u_h - \rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H} = 0 \quad \textit{for all } \mu_{H_\Lambda} \in \Lambda_{H_\Lambda}, \tag{44}$$

and the discrete flux λ_{H_Λ} respects the local equilibrium constraint

$$\int_{\partial\tau} \lambda_{H_\Lambda}\, d\boldsymbol{x} = \int_\tau f\, d\boldsymbol{x} \quad \textit{for all } \tau \in \mathcal{T}_H. \tag{45}$$

Proof. We gather from the triangle inequality that, for an arbitrary $\phi_{H_\Gamma} \in \Gamma_{H_\Gamma}$,

$$|\rho - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} \le |\rho - \phi_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + |\phi_{H_\Gamma} - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)}, \tag{46}$$

and from Theorem 9, (22), Lemma (13) and (36),

$$
\begin{aligned}
&(\alpha_{max}\beta_{max})^{-2}|\phi_{H_\Gamma}-\rho_{H_\Gamma}|^2_{H^{1/2}(\partial\mathcal{T}_H)}\\
&\qquad\le g_h(\phi_{H_\Gamma}-\rho_{H_\Gamma},\phi_{H_\Gamma}-\rho_{H_\Gamma})\\
&\qquad= g_h(\phi_{H_\Gamma}-\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma})-g_h(\rho_{H_\Gamma},\phi_{H_\Gamma}-\rho_{H_\Gamma})+g_h(\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma})\\
&\qquad\quad+g(\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma})-g(\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma})\\
&\qquad=\langle G_h(\phi_{H_\Gamma}-\rho),\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}+\langle (G_h-G)\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}\\
&\qquad\quad+\langle (G\widetilde{T}-G_h\widetilde{T}_h)f,\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}\\
&\qquad=\langle G_h(\phi_{H_\Gamma}-\rho),\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}+\langle (G_h-G)\rho,\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}\\
&\qquad\quad+\langle (G-G_h)\widetilde{T}f,\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}+\langle G_h(\widetilde{T}-\widetilde{T}_h)f,\phi_{H_\Gamma}-\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}\\
&\qquad\le\Big[\beta_{max}^2\Big(|\rho-\phi_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)}+|(\widetilde{T}-\widetilde{T}_h)f|_{H^{1/2}(\partial\mathcal{T}_H)}\Big)+E(G(\rho-\widetilde{T}f))\Big]|\phi_{H_\Gamma}-\rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)}
\end{aligned}
$$

and the result (42) follows from (46) and the above inequality recalling that $\widetilde{\lambda}=G(\rho-\widetilde{T}f)$.

Also, from Lemma 13 and (36)

(47)
$$
\begin{aligned}
|(G\rho-G_h\rho_{H_\Lambda}-(G\widetilde{T}-G_h\widetilde{T}_h)f|_\Lambda&\le|(G-G_h)(\rho-\widetilde{T}f)|_\Lambda+|G_h(\widetilde{T}_hf-\widetilde{T}f)|_\Lambda+|G_h(\rho-\rho_{H_\Lambda})|_\Lambda\\
&\le E(\widetilde{\lambda})+\beta_{\max}^2\Big(|(\widetilde{T}-\widetilde{T}_h)f|_{H^{1/2}(\partial\mathcal{T}_H)}+|\rho-\rho_{H_\Lambda}|_{H^{1/2}(\partial\mathcal{T}_H)}\Big),
\end{aligned}
$$

and the first estimate in (43) follows since

$$\lambda-\lambda_{H_\Lambda}=\widetilde{\lambda}-\widetilde{\lambda}_{H_\Lambda}=G\rho-G_h\rho_{H_\Lambda}-(G\widetilde{T}-G_h\widetilde{T}_h)f. \tag{48}$$

Next, using

$$u-u_h=u^0-u_h^0+\widetilde{u}-\widetilde{u}_h, \tag{49}$$

the second estimate in (43) follows from

$$\tilde{u}-\tilde{u}_h=T\widetilde{\lambda}-T_h\widetilde{\lambda}_{H_\Lambda}+\widetilde{T}f-\widetilde{T}_hf=T_h(\widetilde{\lambda}-\widetilde{\lambda}_{H_\Lambda})+(T-T_h)\widetilde{\lambda}+(\widetilde{T}-\widetilde{T}_h)f$$

and using the stability of T_h in (27) to get

$$|T_h(\lambda-\lambda_{H_\Lambda})|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}=|T_h(\widetilde{\lambda}-\widetilde{\lambda}_{H_\Lambda})|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\le|\widetilde{\lambda}-\widetilde{\lambda}_{H_\Gamma}|_\Lambda=|\lambda-\lambda_{H_\Gamma}|_\Lambda.$$

To show (44), for $\widetilde{\mu}_{H_\Lambda}\in\widetilde{\Lambda}_{H_\Lambda}$ and using the definition of the mapping G_h in (26), we get

$$\langle\widetilde{\mu}_{H_\Lambda},u_h\rangle_{\partial\mathcal{T}_H}=\langle\widetilde{\mu}_{H_\Lambda},\widetilde{u}_h\rangle_{\partial\mathcal{T}_H}=\big\langle\widetilde{\mu}_{H_\Lambda},T_hG_h\rho_{H_\Gamma}+(I-T_hG_h)\widetilde{T}_hf\big\rangle_{\partial\mathcal{T}_H}=\langle\widetilde{\mu}_{H_\Lambda},\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H}.$$

Next, for $\mu_{H_\Lambda}\in\Lambda^0$ it follows from (30),

$$\langle\mu_{H_\Lambda},u_h\rangle_{\partial\mathcal{T}_H}=\langle\mu_{H_\Lambda},u_h^0\rangle_{\partial\mathcal{T}_H}=\langle\mu_{H_\Lambda},\rho_{H_\Gamma}\rangle_{\partial\mathcal{T}_H},$$

and then (44) holds. Finally, the equilibrium condition (45) follows from (10) since

$$\int_{\partial\tau} \lambda_{H_\Lambda}\, d\boldsymbol{x} = \int_{\partial\tau} \lambda^0\, d\boldsymbol{x} = \int_\tau f\, d\boldsymbol{x} \quad \text{for all } \tau \in \mathcal{T}_H.$$

$\square$

6. Compatible finite element spaces

This section provides examples of compatible two dimensional finite element spaces satisfying Assumptions A and B. To this end, we introduce partitions of the elements and faces, which could be different for different elements.

6.1. **Element and face partitions.** Consider the following

- $\mathcal{T}_{h_0}(\tau)$ (baseline partition): conforming triangulation of $\tau \in \mathcal{T}_H$;
- $\mathcal{T}_h(\tau)$: conforming triangulation of $\tau \in \mathcal{T}_H$, with diameter h, obtained by refining $\mathcal{T}_{h_0}(\tau)$;
- $\mathcal{T}_{h_0}$: union of all $\mathcal{T}_{h_0}(\tau)$ yielding a triangulation of Ω (not globally conforming in general)
$$\mathcal{T}_{h_0} := \cup_{\tau\in\mathcal{T}_H}\mathcal{T}_{h_0}(\tau);$$
- $\mathcal{T}_h$: union of all $\mathcal{T}_h(\tau)$ yielding a triangulation of Ω (not globally conforming in general)
$$\mathcal{T}_h := \cup_{\tau\in\mathcal{T}_H}\mathcal{T}_h(\tau) \quad \text{with diameter } h;$$
- $\mathcal{E}(\partial\tau)$: set of faces associated with $\tau \in \mathcal{T}_H$
$$\mathcal{E}(\partial\tau) := \{F \subset \partial\tau : F \text{ is a face of } \tau\};$$
- $\mathcal{E}$: set of faces associated with the partition $\mathcal{T}_H$
$$\mathcal{E} := \cup_{\tau\in\mathcal{T}_H}\mathcal{E}(\partial\tau);$$
- $\mathcal{E}_{H_{\Lambda_0}}(\partial\tau)$ (baseline partition): conforming partition of $\partial\tau$. Moreover, if two elements share a face, then the corresponding face triangulation is identical;
- $\mathcal{E}_{H_\Lambda}(\partial\tau)$: conforming partition of $\partial\tau$, obtained by refining $\mathcal{E}_{H_{\Lambda_0}}(\partial\tau)$;
- $\mathcal{E}_{H_{\Lambda_0}}$ (baseline partition): union of all baseline face partitions
$$\mathcal{E}_{H_{\Lambda_0}} := \cup_{\tau\in\mathcal{T}_H}\mathcal{E}_{H_{\Lambda_0}}(\partial\tau);$$
- $\mathcal{E}_{H_\Lambda}$: union of all $\mathcal{E}_{H_\Lambda}(\partial\tau)$ yielding a partition of $\mathcal{E}$
$$\mathcal{E}_{H_\Lambda} := \cup_{\tau\in\mathcal{T}_H}\mathcal{E}_{H_\Lambda}(\partial\tau) \quad \text{with diameter } H_\Lambda;$$
- $\mathcal{E}_{H_\Gamma} := \mathcal{E}_{H_{\Lambda_0}}$ with diameter H_Γ.

We assume that the triangulations of faces and elements are general, but related to each other. Specifically, given $\tau \in \mathcal{T}_H$, the baseline element partition $\mathcal{T}_{h_0}(\tau)$ is such that it satisfies:

(M1) Given $\tau \in \mathcal{T}_H$ and $F \in \mathcal{E}_{H_\Lambda}(\partial\tau)$, there exist two elements $\kappa_1, \kappa_2 \in \mathcal{T}_{h_0}(\tau)$ such that $(\partial\kappa_1 \cap \partial\tau) \cup (\partial\kappa_2 \cap \partial\tau) = F$.

Given $\tau \in \mathcal{T}_H$, a practical way to construct these meshes is:

(1) Define $\mathcal{E}_{H_{\Lambda_0}}(\partial\tau)$;
(2) Construct $\mathcal{E}_{H_\Lambda}(\partial\tau)$ refining each face of $\mathcal{E}_{H_{\Lambda_0}}(\partial\tau)$;
(3) Define $\mathcal{T}_{h_0}(\tau)$ such that condition $(M1)$ is valid, and $\mathcal{T}_h(\tau)$ as a refinement of $\mathcal{T}_{h_0}(\tau)$.

We next present some stable finite dimensional spaces.

6.2. **The $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$ element.** Let $k \geq 0$ and Γ_{H_Γ} be the space of continuous piecewise polynomial functions on faces of degree up to $k+1$, i.e.,

$$\Gamma_{H_\Gamma} := \left\{ \xi \in H_0^{1/2}(\partial\mathcal{T}_H) \,:\, \xi|_F \in \mathbb{P}_{k+1}(F),\, F \in \mathcal{E}_{H_\Gamma} \right\}, \tag{50}$$

and Λ_{H_Λ}, $\Lambda_{H_{\Lambda_0}}$ and $\widetilde{\Lambda}_{H_\Lambda}$, $\widetilde{\Lambda}_{H_{\Lambda_0}}$ be the following spaces of discontinuous piecewise polynomial functions on faces of degree up to k

$$\begin{aligned} \Lambda_{H_\Lambda} &:= \{\mu \in \Lambda :\, \mu|_F \in \mathbb{P}_k(F),\, F \in \mathcal{E}_{H_\Lambda}\}, \qquad \widetilde{\Lambda}_{H_\Lambda} := \Lambda_{H_\Lambda} \cap \widetilde{\Lambda}, \\ \Lambda_{H_{\Lambda_0}} &:= \left\{\mu \in \Lambda :\, \mu|_F \in \mathbb{P}_k(F),\, F \in \mathcal{E}_{H_{\Lambda_0}}\right\}, \qquad \widetilde{\Lambda}_{H_{\Lambda_0}} := \Lambda_{H_{\Lambda_0}} \cap \widetilde{\Lambda}. \end{aligned} \tag{51}$$

In addition, we chose the following functional spaces of degree up to $k+1$ inside the elements i.e.,

$$\begin{aligned} V_{h_0} &:= \left\{v_h \in H^1(\mathcal{T}_H) :\, v_h|_\tau \in \mathbb{P}_{k+1}(\tau),\, \tau \in \mathcal{T}_{h_0}\right\}, \qquad \widetilde{V}_{h_0} := V_{h_0} \cap \widetilde{H}^1(\mathcal{T}_H), \\ V_h &:= \left\{v_h \in H^1(\mathcal{T}_H) :\, v_h|_\tau \in \mathbb{P}_{k+1}(\tau),\, \tau \in \mathcal{T}_h\right\}, \qquad \widetilde{V}_h := V_h \cap \widetilde{H}^1(\mathcal{T}_H). \end{aligned} \tag{52}$$

From condition (M1) there exists a Fortin operator $\Pi_V : H^1(\mathcal{T}_H) \to V_{h_0}$ (see [6, Lemma 2]) fully defined by

$$\begin{aligned} \int_{\partial\tau} \mu\, \Pi_V(v)\, d\boldsymbol{x} &= \int_{\partial\tau} \mu\, v\, d\boldsymbol{x} \quad \text{for all } \mu \in \Lambda_{H_\Lambda}, \\ \|\Pi_V(v)\|_{H^1(\tau)} &\leq C_\tau\, \|v\|_{H^1(\tau)} \quad \text{for all } v \in V \text{ and } \tau \in \mathcal{T}_H, \end{aligned} \tag{53}$$

where the positive constant C_τ is independent of mesh parameters. Then, we define $\pi_V(\cdot)$ as the operator $\Pi_V(\cdot)$ and the first equation in (37) is valid. Also, for all $v \in \widetilde{H}^1(\mathcal{T}_H)$ and

$\tau \in \mathcal{T}_H$, from (53) and the Poincaré inequality it holds that

$$\begin{aligned}
|\pi_V(v)|_{H^{1/2}(\partial\tau)} = |\Pi_V(v)|_{H^{1/2}(\partial\tau)} &\leq |\Pi_V(v)|_{H^1_{\mathcal{A}}(\tau)} \\
&\leq a_{max}^{1/2} \|\Pi_V(v)\|_{H^1(\tau)} \\
&\leq C_\tau\, a_{max}^{1/2} \|v\|_{H^1(\tau)} \\
&\leq C_\tau\, a_{max}^{1/2} (1 + C_P H_\Lambda) |v|_{H^1(\tau)} \\
&\leq \frac{C_\tau\, a_{max}^{1/2} (1 + C_P H_\Lambda)}{a_{min}^{1/2}} |v|_{H^1_{\mathcal{A}}(\tau)},
\end{aligned}$$

and then $|\pi_V(v)|_{H^{1/2}(\partial\tau)} \leq \beta_\tau |v|_{H^{1/2}(\partial\tau)}$ with $\beta_\tau = \frac{C_\tau\, a_{max}^{1/2}(1+C_P H_\Lambda)}{a_{min}^{1/2}}$. Assumption A follows from Remark 10 using the mapping $\pi_V(\cdot) := \Pi_V(\cdot)$ defined above.

Next, as a consequence of the way partitions were defined in Section 6.1, condition (M1) implies $\Gamma_{H_\Gamma}(\partial\tau) \cap \widetilde{H}^{1/2}(\partial\tau) \subset \widetilde{V}_{h_0}(\tau)|_{\partial\tau}$ for all $\tau \in \mathcal{T}_H$. Then, Assumption B holds true from Lemma 11 and we have the following result.

Theorem 15 (Well-posedness and best approximation)**.** *Assume that* $(M1)$ *holds. Let* Γ_{H_Γ} *be given in* (50)*, and* Λ_{H_Λ} *and* V_{h_0} *be such that* $\Lambda_{H_{\Lambda_0}} \subset \Lambda_{H_\Lambda}$ *and* $V_{h_0} \subset V_h$*, where* $\Lambda_{H_{\Lambda_0}}$ *is given in* (51) *and* V_{h_0} *in* (52)*. Then, the MH²M* (28) *is well-posed and* (29) *satisfies the estimates in Theorem 14.*

Proof. Since Assumptions A and B are valid, the result follows from Theorems 9 and 14. ☐

Remark 16 (Relaxing assumption $(M1)$)**.** *The condition* $(M1)$ *is sufficient to establish the well-posedness of MH²M* (28)*, but it is not necessary in general. For example, let* $k \geq 0$ *be even and consider the finite element* $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$ *supported in partitions* $\mathcal{E}_{H_\Gamma}(\partial\tau) = \mathcal{E}_{H_\Lambda}(\partial\tau)$ *with one element per face, and in the partition* $\mathcal{T}_{h_0}(\tau)$ *with one element per simplex* $\tau \in \mathcal{T}_H$ *(see Figure 1 for an illustration). Thus, adapting the arguments from the proof of* [39, Lemma 10] *to our case, we conclude the existence of a bounded mapping* $\pi_V(\cdot)$ *satisfying* (37)*. Specifically, from* [39, Lemma 10]*, there exists a positive constant* C_τ *independent of mesh parameters, such that for all* $\mu \in \widetilde{\Lambda}_{H_\Lambda}$

$$\sup_{v \in H^1(\tau)} \frac{\langle \mu, v \rangle_{\partial\tau}}{\|v\|_{H^1(\tau)}} \leq C_\tau \sup_{v_h \in V_{h_0}(\tau)} \frac{\langle \mu, v_h \rangle_{\partial\tau}}{\|v_h\|_{H^1(\tau)}}.$$

Then,

$$
\begin{aligned}
|\mu|_{H^{-1/2}(\partial\tau)} &= \sup_{\widetilde{\phi}\in\widetilde{H}^{1/2}(\partial\tau)} \frac{\langle \mu, \widetilde{\phi}\rangle_{\partial\tau}}{|\widetilde{\phi}|_{H^{1/2}(\partial\tau)}} = \sup_{\substack{\widetilde{v}\in\widetilde{H}^1(\tau)\\ \widetilde{v}|_{\partial\tau}=\widetilde{\phi}}} \frac{\langle \mu, \widetilde{v}\rangle_{\partial\tau}}{|\widetilde{v}|_{H^1(\tau)}} \leq (1+C_P H_\tau) \sup_{\substack{\widetilde{v}\in\widetilde{H}^1(\tau)\\ \widetilde{v}|_{\partial\tau}=\widetilde{\phi}}} \frac{\langle \mu, \widetilde{v}\rangle_{\partial\tau}}{\|\widetilde{v}\|_{H^1(\tau)}} \\
&\leq (1+C_P H_\tau) \sup_{v\in H^1(\tau)} \frac{\langle \mu, v\rangle_{\partial\tau}}{\|v\|_{H^1(\tau)}} \leq C_\tau\,(1+C_P H_\tau) \sup_{v_h\in V_{h_0}(\tau)} \frac{\langle \mu, v_h\rangle_{\partial\tau}}{\|v_h\|_{H^1(\tau)}} \\
&\leq C_\tau\,(1+C_P H_\tau) \sup_{\widetilde{v}_h\in\widetilde{V}_{h_0}(\tau)} \frac{\langle \mu, \widetilde{v}_h\rangle_{\partial\tau}}{\|\widetilde{v}_h\|_{H^1(\tau)}} \leq C_\tau\,(1+C_P H_\tau) a_{\max}^{1/2} \sup_{\widetilde{v}_h\in\widetilde{V}_{h_0}(\tau)} \frac{\langle \mu, \widetilde{v}_h\rangle_{\partial\tau}}{|\widetilde{v}_h|_{H^1_{\mathcal{A}}(\tau)}} \\
&= C_\tau\,(1+C_P H_\tau) a_{\max}^{1/2} |T_h\mu|_{H^1_{\mathcal{A}}(\tau)}
\end{aligned}
$$

which corresponds to Assumption A with $\beta_\tau = C_\tau\,(1+C_P H_\tau) a_{\max}^{1/2}$*. Thus, as Assumption B follows from Lemma 11 the statements of Theorem 15 are also valid for this family of elements.*

Remark 17 (Comparison with Marini-Brezzi's three-field method)**.** *The element* $\mathbb{P}_1(F)\times\mathbb{P}_0(F)\times\mathbb{P}_1(\tau)$ *was first adopted in the three-field numerical method* [16] *for which it proved to be well-posed and to achieve optimal convergence in* [18]*. This was made possible by adding regularization terms to the standard Galerkin method which discretizes the three-field continuous weak formulation* (7)*. Note that MH*^{2}*M* (28)-(30) *is also stable and optimally convergent with the* $\mathbb{P}_1(F)\times\mathbb{P}_0(F)\times\mathbb{P}_1(\tau)$ *element. Interestingly, such properties are obtained without the need for the additional stabilization included in the three-field method, which is replaced simply by assuming the* $(M1)$ *condition on the sub-meshes.*

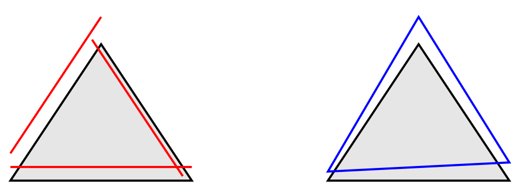

FIGURE 1. A piecewise constant function representative of the space Λ_{H_Λ} and a continuous piecewise linear function of the space Γ_{H_Γ} with faces and element discretized with one element.

6.3. **The** $\mathbb{P}_{k+1}(F)\times\mathbb{P}_{k+1}(F)\times\mathbb{P}_{k+1}(\tau)$ **element.** Here we use a richer space to approximate the flux variable λ than in the previous case. Nonetheless, the strategy of proving well-posedness and approximation properties for the MH^{2}M with the element $\mathbb{P}_{k+1}(F)\times\mathbb{P}_{k+1}(F)\times\mathbb{P}_{k+1}(\tau)$ also follows from the existence of a bounded mapping that satisfies (53). To do so, we must consider two scenarios:

- First, if $k \geq 2$ then it is sufficient to assume (M1) to guarantee the existence of a mapping $\Pi_V(\cdot)$ that satisfies (53) (c.f. [6, Section 3.2]) and then Assumption A is satisfied using the same argument used for the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$. Furthermore, the Assumption B holds from Lemma 11 and then the analogue of Theorem 15 is valid for the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(\tau)$;
- On the other hand, when $k = 0$ and $k = 1$, the condition (M1) must be replaced by:
  Given $F_\Lambda \in \mathcal{E}_{H_\Lambda}(\partial\tau)$, there are three elements $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{T}_{h_0}(\tau)$ such that $(\partial\kappa_1 \cap \partial\tau) \cup (\partial\kappa_2 \cap \partial\tau) \cup (\partial\kappa_3 \cap \partial\tau) = F_\Lambda$.
  This condition guarantees the existence of $\Pi_V(\cdot)$ that satisfies (53) (c.f. [6, Section 3.2]) and the well-posedness and best approximation results established in Theorem 15 are also valid using again the proof strategy used for the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$.

6.4. **The one-level $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F)$ and $\mathbb{P}_k(F) \times \mathbb{P}_{k+1}(F)$.** The method nomenclature indicates that we replace V_{h_0} by $H^1(\mathcal{T}_H)$ (i.e., $T_h = T$ and $\widetilde{T}_h = \widetilde{T}$), which corresponds to assuming that the multiscale basis functions driven by the operators T and $\widetilde{T}$ have a close formula (see algorithm in Section 7 for details). Notably, the one-level MH^{2}M corresponds to find $\rho_{H_\Gamma}^{one} \in \Gamma_{H_\Gamma}$ such that

$$\langle G_h \rho_{H_\Gamma}^{one}, \xi \rangle_{\partial\mathcal{T}_H} = -\langle \lambda^0, \xi \rangle_{\partial\mathcal{T}_H} + \langle G_h \widetilde{T} f, \xi \rangle_{\partial\mathcal{T}_H} \quad \text{for all } \xi \in \Gamma_{H_\Gamma}, \tag{54}$$

where λ^0 is given in (10). The approximate solution u_h^{one} and λ_h^{one} are

$$u_h^{one} := u_h^{one,0} + T G_h \rho_{H_\Gamma}^{one} + (I - T G_h)\widetilde{T} f \quad \text{and} \quad \lambda_{H_\Lambda}^{one} := \lambda^0 + G_h(\rho_{H_\Gamma}^{one} - \widetilde{T} f), \tag{55}$$

where $u_h^{one,0}$ satisfies (30) using $\rho_{H_\Gamma}^{one}$ in place. The function u_h^{one} is finite dimensional although the space spam by the image of mappings T and $\widetilde{T}$ are not expected to be polynomial. Recalling that in the one-level case $T_h = T$, note that

$$|\mu|_{H^{-1/2}(\partial\tau)} = \sup_{\widetilde{\phi} \in \widetilde{H}^{1/2}(\partial\tau)} \frac{\langle \mu, \widetilde{\phi} \rangle_{\partial\tau}}{|\widetilde{\phi}|_{H^{1/2}(\partial\tau)}} = \sup_{\substack{\widetilde{v} \in \widetilde{H}^1(\tau) \\ \widetilde{v}|_{\partial\tau} = \widetilde{\phi}}} \frac{\int_\tau \mathcal{A} \nabla T\mu \cdot \nabla \widetilde{v} \, d\boldsymbol{x}}{|\widetilde{v}|_{H^{1/2}(\partial\tau)}} = |T\mu|_{H^1_{\mathcal{A}}(\tau)},$$

for all $\mu \in \widetilde{\Lambda}_{H_\Lambda}$, and then the Assumption A is valid without the need for Condition $(M1)$.

As for Assumption B, we first remember the existence of a mapping $\pi_V : H^1(\mathcal{T}_H) \to V_{h_0}$, where V_{h_0} is the polynomial space in (52). Then, observe that $\Gamma_{H_\Gamma}(\tau) \cap \widetilde{H}^{1/2}(\partial\tau) \subset \widetilde{V}_{h_0}(\tau)|_{\partial\tau} \subset \widetilde{H}^1(\mathcal{T}_H)|_{\partial\tau}$ for all $\tau \in \mathcal{T}_H$ using the way the meshes were constructed (c.f. Section 6.1). As a result, Assumption B also holds and thus, the one-level MH^{2}M in (54) is well-posed from Theorem 9 with the unique solution being (55), and the best approximation properties in Theorem 14 simplify to

$$|\rho - \rho^{one}_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} \le C\Big(\inf_{\phi_{H_\Gamma}\in\Gamma_{H_\Gamma}} |\rho - \phi_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + E(\widetilde{\lambda})\Big),$$

$$|\lambda - \lambda^{one}_{H_\Lambda}|_\Lambda \le C\Big(|\rho - \rho^{one}_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + E(\widetilde{\lambda})\Big) \quad \text{and} \quad |u - u^{one}_h|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le |\lambda - \lambda^{one}_{H_\Lambda}|_\Lambda,$$

where $E(\widetilde{\lambda}) = (1+\beta^2_{\max})\inf_{\widetilde{\mu}_{H_\Lambda}\in\widetilde{\Lambda}_{H_\Lambda}} |\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda$.

6.5. **Convergence.** We establish error estimates for the two-level MH^{2}M (35) using the $\mathbb{P}_{k+1}(F)\times\mathbb{P}_k(F)\times\mathbb{P}_{k+1}(\tau)$ element of Section 6.2. First, we recall some interpolation operators with optimal properties. Let $w \in H^{k+2}(\mathcal{T}_H)$ and $\mathcal{A}\nabla w \in H^{k+1}(\mathcal{T}_H)\cap H(\mathrm{div},\Omega)$. Then, we have:

There exist $\phi_{H_\Gamma}\in\Gamma_{H_\Gamma}$ and $\widetilde{\mu}_{H_\Lambda}\in\widetilde{\Lambda}_{H_\Lambda}$ such that

$$|\xi - \phi_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} \le C\, H_\Gamma^{k+1}|w|_{H^{k+2}(\mathcal{T}_H)} \quad \text{and} \quad |\widetilde{\mu} - \widetilde{\mu}_{H_\Lambda}|_\Lambda \le C\, H_\Lambda^{k+1}|\mathcal{A}\nabla w|_{H^{k+1}(\mathcal{T}_H)} \tag{56}$$

where $\xi := w\,|_{\partial\tau}$ and $\mu\,|_{\partial\tau} := \mathcal{A}\nabla w\cdot\boldsymbol{n}^\tau\,|_{\partial\tau}$ for all $\tau\in\mathcal{T}_H$. For example, we may choose ϕ_{H_Γ} as the standard Lagrange interpolator on $\mathbb{P}_k(\mathcal{T}_H)$ [26] and $\widetilde{\mu}_{H_\Lambda}$ given in [6, Lemma 3]. In fact, from [6, Lemma 3], there exists $\mu_{H_\Lambda}\in\Lambda_{H_\Lambda}$ and a positive constant C, independent of mesh parameters, such that

$$\sup_{v\in H^1(\mathcal{T}_H)} \frac{\langle \mu - \mu_{H_\Lambda}\,, v\rangle_{\partial\mathcal{T}_H}}{\|v\|_{H^1(\mathcal{T}_H)}} \le C\, H_\Lambda^{k+1}|\mathcal{A}\nabla w|_{H^{k+1}(\mathcal{T}_H)}.$$

Then, using the definition of the norm $|\cdot|_\Lambda$, Poincaré inequality and estimate above, it holds that

$$\begin{aligned}
|\widetilde{\mu} - \widetilde{\mu}_{H_\Lambda}|_\Lambda &= \sup_{\widetilde{\phi}\in\widetilde{H}^{1/2}(\partial\mathcal{T}_H)} \frac{\langle \widetilde{\mu} - \widetilde{\mu}_{H_\Lambda}, \widetilde{\phi}\rangle_{\partial\mathcal{T}_H}}{|\widetilde{\phi}|_{H^{1/2}(\partial\mathcal{T}_H)}} = \sup_{\substack{\widetilde{v}\in\widetilde{H}^1(\mathcal{T}_H)\\ \widetilde{v}|_{\partial\mathcal{T}_H}=\widetilde{\phi}}} \frac{\langle \widetilde{\mu} - \widetilde{\mu}_{H_\Lambda}, \widetilde{v}\rangle_{\partial\mathcal{T}_H}}{|\widetilde{v}|_{H^1(\mathcal{T}_H)}} \\
&= \sup_{\widetilde{v}\in\widetilde{H}^1(\mathcal{T}_H)} \frac{\langle \mu - \mu_{H_\Lambda}, \widetilde{v}\rangle_{\partial\mathcal{T}_H}}{|\widetilde{v}|_{H^1(\mathcal{T}_H)}} \le C \sup_{\widetilde{v}\in\widetilde{H}^1(\mathcal{T}_H)} \frac{\langle \mu - \mu_{H_\Lambda}, \widetilde{v}\rangle_{\partial\mathcal{T}_H}}{\|\widetilde{v}\|_{H^1(\mathcal{T}_H)}} \\
&\le C \sup_{v\in H^1(\mathcal{T}_H)} \frac{\langle \mu - \mu_{H_\Lambda}, v\rangle_{\partial\mathcal{T}_H}}{\|v\|_{H^1(\mathcal{T}_H)}} \le C\, H_\Lambda^{k+1}|\mathcal{A}\nabla w|_{H^{k+1}(\mathcal{T}_H)}.
\end{aligned}$$

Also, we recall the approximation error associated with the Galerkin approximation at the second level

$$|(T-T_h)\widetilde{\mu}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le C\, h^{k+1}|T\widetilde{\mu}|_{H^{k+2}(\mathcal{T}_H)} \text{ and } |(\widetilde{T}-\widetilde{T}_h)q|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le C\, h^{k+1}|\widetilde{T}q|_{H^{k+2}(\mathcal{T}_H)} \tag{57}$$

where $q \in L^2(\Omega)$ and $\widetilde{\mu} \in \widetilde{\Lambda}$, under the assumption that $T\widetilde{\mu}$, $\widetilde{T}q \in H^{k+2}(\mathcal{T}_H)$. Therefore, using (24) and setting $\widetilde{\lambda}\,|_{\partial\tau} = \mathcal{A}\nabla u \cdot \boldsymbol{n}^\tau\,|_{\partial\tau}$ it holds that

$$
\begin{aligned}
E(\widetilde{\lambda}) &= (1+\beta_{\max}^2) \inf_{\widetilde{\mu}_{H_\Lambda} \in \widetilde{\Lambda}_{H_\Lambda}} |\widetilde{\lambda} - \widetilde{\mu}_{H_\Lambda}|_\Lambda + \beta_{\max}^2 |(T - T_h)\widetilde{\lambda}|_{H^{1/2}(\partial\mathcal{T}_H)} \\
&\leq C \left(H_\Lambda^{k+1} |\mathcal{A}\nabla u|_{H^{k+1}(\mathcal{T}_H)} + h^{k+1} |T\widetilde{\lambda}|_{H^{k+2}(\mathcal{T}_H)} \right) \qquad (58) \\
&\leq C \left[H_\Lambda^{k+1} |\mathcal{A}\nabla u|_{H^{k+1}(\mathcal{T}_H)} + h^{k+1} \Big(|u|_{H^{k+2}(\mathcal{T}_H)} + |\widetilde{T}f|_{H^{k+2}(\mathcal{T}_H)} \Big) \right],
\end{aligned}
$$

where $E(\cdot)$ was defined in (40).

Owing to the interpolation estimates (56) with u replacing w and (57) with $\widetilde{\mu} = \widetilde{\lambda}$ and $q = f$, we estimate MH^{2}M error using the $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$ element as follows.

Theorem 18. *Let (ρ, λ, u) be the exact solution of (7) and $(\rho_{H_\Gamma}, \lambda_{H_\Lambda}, u_h)$ be the MH^{2}M solution where $\rho_{H_\Gamma} \in \Lambda_{H_\Lambda}$ solves (28), and $(u_h, \lambda_{H_\Lambda}) \in V_h \times \Lambda_{H_\Lambda}$ is given in (29). Under the assumptions of Theorem 14, and assuming $u \in H^{k+2}(\mathcal{T}_H)$, $\mathcal{A}\nabla u \in H^{k+1}(\mathcal{T}_H) \cap H(\mathrm{div}, \Omega)$ and $\widetilde{T}f \in H^{k+2}(\mathcal{T}_H)$, with $k \geq 0$, it holds that*

$$
\begin{aligned}
|\rho - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + |\lambda - \lambda_{H_\Lambda}|_\Lambda + |u - u_h|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq C \Big[\Big(H_\Gamma^{k+1} + h^{k+1} \Big) |u|_{H^{k+2}(\mathcal{T}_H)} \\
+ H_\Lambda^{k+1} |\mathcal{A}\nabla u|_{H^{k+1}(\mathcal{T}_H)} + h^{k+1} |\widetilde{T}f|_{H^{k+2}(\mathcal{T}_H)} \Big].
\end{aligned}
$$

Proof. The result is a direct consequence of Theorem 14 and estimates (56)-(58). ☐

Remark 19 (The $\mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(\tau)$ case)**.** *Under the assumptions of Theorem 18 with the additional regularity $\mathcal{A}\nabla u \in H^{k+2}(\mathcal{T}_H)$ and following the arguments in the proof of Theorem 18, the MH^{2}M solution with the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(\tau)$ satisfies*

$$
\begin{aligned}
&|\rho - \rho_{H_\Gamma}|_{H^{1/2}(\partial\mathcal{T}_H)} + |\lambda - \lambda_{H_\Lambda}|_\Lambda + |u - u_h|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \\
&\qquad \leq C \Big[\Big(H_\Gamma^{k+1} + h^{k+1} \Big) |u|_{H^{k+2}(\mathcal{T}_H)} + H_\Lambda^{k+2} |\mathcal{A}\nabla u|_{H^{k+2}(\mathcal{T}_H)} + h^{k+1} |\widetilde{T}f|_{H^{k+2}(\mathcal{T}_H)} \Big].
\end{aligned}
$$

Note that, since $H_\Lambda \leq H_\Gamma$ the leading error is $O(H_\Gamma^{k+1} + h^{k+1})$ showing the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(\tau)$ provides no gain over the cheaper element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$.

Remark 20 (The one-level MH^{2}M case)**.** *Under the assumptions of Theorem 18, that $\widetilde{V}_h = \widetilde{H}^1(\mathcal{T}_H)$, $T_h = T$ and $\widetilde{T}_h = \widetilde{T}$, the error estimate in Theorem 18 becomes*

$$
|\rho - \rho_{H_\Gamma}^{one}|_{H^{1/2}(\partial\mathcal{T}_H)} + |\lambda - \lambda_{H_\Lambda}^{one}|_\Lambda + |u - u_h^{one}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq C \Big[H_\Gamma^{k+1} |u|_{H^{k+2}(\mathcal{T}_H)} + H_\Lambda^{k+1} |\mathcal{A}\nabla u|_{H^{k+1}(\mathcal{T}_H)} \Big]
$$

when we use the MH^{2}M with the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_k(F) \times \mathbb{P}_{k+1}(\tau)$, and

$$
|\rho - \rho_{H_\Gamma}^{one}|_{H^{1/2}(\partial\mathcal{T}_H)} + |\lambda - \lambda_{H_\Lambda}^{one}|_\Lambda + |u - u_h^{one}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq C \Big(H_\Gamma^{k+1} |u|_{H^{k+2}(\mathcal{T}_H)} + H_\Lambda^{k+2} |\mathcal{A}\nabla u|_{H^{k+2}(\mathcal{T}_H)} \Big),
$$

with the element $\mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(F) \times \mathbb{P}_{k+1}(\tau)$.

7. Multiscale basis, degrees of freedom and algorithm

Let be $\{\rho_1,\ldots,\rho_N\}$ be a base for Γ_{H_Γ} and $\{\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{N'}\}$ for $\widetilde{\Lambda}_{H_\Lambda}$, where N, $N'\in\mathbb{N}^+$, and $\{\widetilde{\psi}_1,\ldots,\widetilde{\psi}_M\}$ a base for $\widetilde{V}_h(\tau)$, with $\tau\in\mathcal{T}_H$ and $M\in\mathbb{N}^+$. Let $\widetilde{\Gamma}_{H_\Gamma}:=\Gamma_{H_\Gamma}\cap\widetilde{H}^{1/2}(\partial\mathcal{T}_H)$ be the space of functions in Γ_{H_Γ} with zero average in each element boundary.

The MH^{2}M computational algorithm writes:

i) Compute λ^0 from (10);

ii) Compute $T_h\widetilde{\lambda}_k =: \eta_k^{ms}$, for $k=1,\ldots,N'$, and $\widetilde{T}_h f =: \eta_f^{ms}$ from (25), i.e.,

$$\sum_{j=1}^{M} v_j^k \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(\widetilde{\psi}_i)\cdot\boldsymbol{\nabla}\,\widetilde{\psi}_j\,d\boldsymbol{x} = \langle\widetilde{\lambda}_k,\widetilde{\psi}_i\rangle_{\partial\tau}\quad\text{for all } i=1,\ldots,M$$

and

$$\sum_{j=1}^{M} q_j \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(\widetilde{\psi}_i)\cdot\boldsymbol{\nabla}\,\widetilde{\psi}_j\,d\boldsymbol{x} = \int_\tau f\widetilde{\psi}_i\,d\boldsymbol{x}\quad\text{for all } i=1,\ldots,M.$$

Set

$$\eta_k^{ms} := \sum_{j=1}^{M} v_j^k\widetilde{\psi}_j\quad\text{and}\quad\eta_f^{ms} := \sum_{j=1}^{M} q_j\widetilde{\psi}_j$$

iii) Compute $G_h\rho_k =: \widetilde{\lambda}_k^{ms}$, for $k=1,\ldots,N$, and $G_h\eta_f^{ms} =: \widetilde{\lambda}_f^{ms}$, using η_j^{ms} from (26), i.e.,

$$\sum_{j=1}^{N'} c_j^k\langle\widetilde{\lambda}_i,\eta_j^{ms}\rangle_{\partial\tau} = \langle\widetilde{\lambda}_i,\rho_k\rangle_{\partial\tau}\quad\text{for all } i=1,\ldots,N'$$

and

$$\sum_{j=1}^{N'} d_j\langle\widetilde{\lambda}_i,\eta_j^{ms}\rangle_{\partial\tau} = \langle\widetilde{\lambda}_i,\eta_f^{ms}\rangle_{\partial\tau}\quad\text{for all } i=1,\ldots,N'.$$

Set

$$\widetilde{\lambda}_j^{ms} := \sum_{k=1}^{N'} c_j^k\widetilde{\lambda}_k\quad\text{and}\quad\widetilde{\lambda}_f^{ms} := \sum_{k=1}^{N'} d_k\widetilde{\lambda}_k\quad\text{with } j=1,\ldots,N$$

iv) Solve (17), i.e., compute $\alpha=\{\alpha_j\}_{j=1}^N$

$$\sum_{j=1}^{N}\alpha_j\langle\widetilde{\lambda}_j^{ms},\rho_i\rangle_{\partial\mathcal{T}_H} = -\langle\lambda^0,\rho_i\rangle_{\partial\mathcal{T}_H} + \langle\widetilde{\lambda}_f^{ms},\rho_i\rangle_{\partial\mathcal{T}_H}\quad\text{with } i=1,\ldots,N \tag{59}$$

v) Compute ρ_{H_Γ}, $\widetilde{\lambda}_{H_\Lambda}$ and $\widetilde{u}_h$

$$\rho_{H_\Gamma} := \sum_{i=1}^{N}\alpha_i\rho_i,\quad \widetilde{\lambda}_{H_\Lambda} := \sum_{i=1}^{N}\alpha_i\widetilde{\lambda}_i^{ms} - \widetilde{\lambda}_f^{ms},\quad \widetilde{u}_h := \sum_{i=1}^{N}\alpha_i T_h\widetilde{\lambda}_i^{ms} - T_h\widetilde{\lambda}_f^{ms} + \eta_f^{ms}$$

vi) Compute u_h^0 from (10)

vii) Compute u_h and λ_{H_Λ}

$$u_h := u_h^0 + \sum_{i=1}^{N} \alpha_i T_h \widetilde{\lambda}_i^{ms} - T_h \widetilde{\lambda}_f^{ms} + \eta_f^{ms} \quad \text{and} \quad \lambda_{H_\Lambda} := \lambda^0 + \sum_{i=1}^{N} \alpha_i \widetilde{\lambda}_i^{ms} - \widetilde{\lambda}_f^{ms}$$

or equivalently,

$$u_h := u_h^0 + \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N'} c_j^i \sum_{k=1}^{M} v_k^j \widetilde{\psi}_k - \sum_{j=1}^{N'} d_j \sum_{k=1}^{M} v_k^j \widetilde{\psi}_k + \sum_{k=1}^{M} q_k \widetilde{\psi}_k$$

and

$$\lambda_{H_\Lambda} := \lambda^0 + \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N'} c_j^i \widetilde{\lambda}_j - \sum_{j=1}^{N} d_j \widetilde{\lambda}_j.$$

Remark 21. *Some points are worth mentioning:*

- *Although the solutions u_h and λ_{H_Λ} belong to the space V_h and Λ_{H_Λ} initially, they belong to much smaller subspaces of dimension $N := \dim(\Gamma_{H_\Lambda})$. In fact, by construction, $\widetilde{u}_h \in \operatorname{span}\left\{\eta_1^{ms}, \ldots, \eta_N^{ms}, \eta_f^{ms}\right\} \subset \widetilde{V}_h$ and $\widetilde{\lambda}_{H_\Lambda} \in \operatorname{span}\left\{\widetilde{\lambda}_1^{ms}, \ldots, \widetilde{\lambda}_N^{ms}, \widetilde{\lambda}_f^{ms}\right\} \subset \widetilde{\Lambda}_{H_\Lambda}$, respectively. Such a basis accounts for the multiscale characteristics of the problem, and are numerically approximated at the local level through steps $(i) - (iii)$ as the approximate formula for the base functions are not generally available. The MH^{2}M may be seen then as a member of reduced basis methods [12].*
- *Steps $(i) - (iii)$ can be done in an offline stage. Furthermore, they are "embarrassingly" parallel in the sense that the problems are local and entirely independent of each other. The global formulation (iv) is the only global formulation because its degrees of freedom are nodal and associated with the coarse mesh skeleton;*
- *All linear systems in steps (ii)-(iv) are symmetric and positive definite. A closer look shows that two types of local linear systems must be assembled to solve the problems in (ii)-(iv). The associated matrices can then be factored once and for all in an offline stage and then applied to the various right sides. This is particularly attractive when it comes to troubleshooting multiple queries as found in time marching process or in varying the source data f in stochastic problems.*

8. Conclusion

Brezzi and Marini's three-field method [16] provided the background for new families of multiscale finite elements for the Poisson (Darcy) problem. The MH^{2}M incorporates a multiscale basis for the Lagrange multiplier associated with the flow variable, obtained from the multiscale basis of the original MHM (multiscale hybrid-mixed) method [2]. Thanks to a new spatial decomposition, these multiscale bases are computed locally from positive definite algebraic systems and are ready to leverage parallel computational architectures. Unlike the MHM method, the global problem associated with MH^{2}M is coercive and the degrees of freedom are associated with the trace of the primal variable. Such similarity led us to establish a connection with MsFEM [25].

We proved that the MH^{2}M converges optimally with respect to the characteristic mesh lengths under reasonable conditions of compatibility between approximation spaces and under assumptions of local regularity (as usual in hybrid methods). Furthermore, we highlighted the interaction between the face mesh diameters (H_Γ and H_Λ) and the second-level mesh parameter (h) on convergence. The sensitivity of the method to resonance errors to approximate oscillatory solutions is outside the scope of the present work, which could be addressed following the strategy proposed in [36, 37].

9. Appendix

Lemma 22. *Let $\tau \in \mathcal{T}_H$ and $\xi \in H^{1/2}(\partial\tau)$. Then $|\xi|_{H^{1/2}(\partial\tau)} = |\phi_\xi|_{H^1_{\mathcal{A}}(\tau)}$, where ϕ_ξ weakly solves*

$$-\operatorname{div} \mathcal{A}\, \nabla\, \phi_\xi = 0 \quad \text{in } \tau, \qquad \phi_\xi = \xi \quad \text{on } \partial\tau. \tag{60}$$

Moreover, if $\xi \in H_0^{1/2}(\partial\mathcal{T}_H)$ and (60) holds for all τ, then $\phi_\xi \in H^1_0(\Omega)$ and

$$|\xi|^2_{H^{1/2}(\partial\mathcal{T}_H)} = |\phi_\xi|^2_{H^1_{\mathcal{A}}(\Omega)} = \sum_{\tau\in\mathcal{T}_H} |\xi|^2_{H^{1/2}(\partial\tau)}.$$

Proof. Given $\xi \in H^{1/2}(\partial\tau)$, it is immediate that $|\phi_\xi|_{H^1_{\mathcal{A}}(\tau)}$ minimizes $|\cdot|_{H^1_{\mathcal{A}}(\tau)}$ over the space of $H^1(\tau)$ functions with trace ξ on $\partial\tau$. Similarly, if $\xi \in H^1_0(\Omega)$, then (60) holds in each element. But then $\phi_\xi \in H^1(\mathcal{T}_H)$ and $\phi_\xi|_{\partial\mathcal{T}_H} \in H_0^{1/2}(\partial\mathcal{T}_H)$ implies that $\phi_\xi \in H^1_0(\Omega)$. Then

$$\sum_{\tau\in\mathcal{T}_H} |\xi|^2_{H^{1/2}(\partial\tau)} = \sum_{\tau\in\mathcal{T}_H} |\phi_\xi|^2_{H^1_{\mathcal{A}}(\tau)} = |\phi_\xi|^2_{H^1_{\mathcal{A}}(\Omega)}.$$

□

In the following lemma, we collect some useful technical results.

Lemma 23. *For a fixed $\tau \in \mathcal{T}_H$ the following results hold:*

(i) $|T\widetilde{\mu}|_{H^{1/2}(\partial\tau)} = |T\widetilde{\mu}|_{H^1_{\mathcal{A}}(\tau)} = |\widetilde{\mu}|_{H^{-1/2}(\partial\tau)}$ *for all* $\widetilde{\mu} \in \widetilde{\Lambda}$;

(ii) $|G\xi|_{H^{-1/2}(\partial\tau)} = |\xi|_{H^{1/2}(\partial\tau)}$ *for all* $\xi \in H^{1/2}(\partial\tau)$;

(iii) $\langle \widetilde{\mu}, \xi \rangle_{\partial\tau} \le |\widetilde{\mu}|_{H^{-1/2}(\partial\tau)} |\xi|_{H^{1/2}(\partial\tau)}$ *for all* $\widetilde{\mu} \in \widetilde{\Lambda}$ *and* $\xi \in H^{1/2}(\partial\tau)$*. Also, for all* $\xi \in H^{1/2}(\partial\tau)$,

$$\sup_{\widetilde{\mu}\in\widetilde{H}^{-1/2}(\partial\tau)} \frac{\langle \widetilde{\mu}, \xi \rangle_{\partial\tau}}{|\widetilde{\mu}|_{H^{-1/2}(\partial\tau)}} = |\xi|_{H^{1/2}(\partial\tau)}; \tag{61}$$

(iv) $\langle \widetilde{\mu}, T\widetilde{\eta} \rangle_{\partial\mathcal{T}_H} \le |\widetilde{\mu}|_\Lambda |\widetilde{\eta}|_\Lambda$ *for all* $\widetilde{\mu}$, $\widetilde{\eta} \in \widetilde{\Lambda}$.

Proof. The first identity in (i) follows from the definition of T and Lemma 22. To the second identity of (i) , we have:

$$|\widetilde{\mu}|_{H^{-1/2}(\partial\tau)} = \sup_{\psi\in\widetilde{H}^{1/2}(\partial\tau)} \frac{\langle \widetilde{\mu}, \psi \rangle_{\partial\tau}}{|\psi|_{H^{1/2}(\partial\tau)}} = \sup_{v\in\widetilde{H}^1(\tau)} \frac{\langle \widetilde{\mu}, v \rangle_{\partial\tau}}{|v|_{H^1_{\mathcal{A}}(\tau)}} = \sup_{v\in\widetilde{H}^1(\tau)} \frac{\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\, T\widetilde{\mu} \cdot \boldsymbol{\nabla}\, v\, d\boldsymbol{x}}{|v|_{H^1_{\mathcal{A}}(\tau)}} = |T\widetilde{\mu}|_{H^1_{\mathcal{A}}(\tau)}.$$

Next, (ii) follows from (i) with $\widetilde{\mu} = G\xi$ and (19). Item (iii) follows from the definition of the semi-norms $|\cdot|_{H^{1/2}(\partial\tau)}$ and $|\cdot|_{H^{-1/2}(\partial\tau)}$. To show (61), first denote by ϕ_ξ the $\mathcal{A}$-harmonic extension of ξ, i.e., the solution of (60). From (i), we gather that

$$\frac{\langle \widetilde{\mu}, \xi \rangle_{\partial\tau}}{|\widetilde{\mu}|_{H^{-1/2}(\partial\tau)}} = \frac{\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\, T\widetilde{\mu} \cdot \boldsymbol{\nabla}\, \phi_\xi\, d\boldsymbol{x}}{|T\widetilde{\mu}|_{H^1_{\mathcal{A}}(\tau)}} \le |\phi_\xi|_{H^1_{\mathcal{A}}(\tau)} = |\xi|_{H^{1/2}(\partial\tau)}$$

where we used Lemma 22 at the last step. The identity with the supremum follows immediately. Finally, identity (iv) follows from (iii) with $\xi = T\widetilde{\eta}$ and (i), and the definition of the semi-norm $|\cdot|_\Lambda$ in (5). □

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