A Constructive Real Projective Plane

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Abstract. The classical theory of plane projective geometry is examined constructively, using both synthetic and analytic methods. The topics include Desargues's Theorem, harmonic conjugates, projectivities, involutions, conics, Pascal's Theorem, poles and polars. The axioms used for the synthetic treatment are constructive versions of the traditional axioms. The analytic construction is used to verify the consistency of the axioms; it is based on the usual model in three-dimensional Euclidean space, using only constructive properties of the real numbers. The methods of strict constructivism, following principles put forward by Errett Bishop, reveal the hidden constructive content of a portion of classical geometry. A number of open problems remain for future studies.

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Introduction

In various forms, the constructivist program goes back to Leopold Kronecker (1823-1891), Henri Poincaré (1854-1912), L. E. J. Brouwer (1881-1966) [Bro08], and many others. The most significant recent work, using the strictest methods, is due to Errett Bishop (1928-1983). A large portion of analysis has been constructivized by Bishop in *Foundations of Constructive Analysis* [B67]; this treatise also serves as a guide for constructive work in other fields. Expositions of constructivist ideas and methods may be found in [B67, BB85]; see also [Sto70, R82, M85].

The initial phase of this program involves the rebuilding of classical theories, using only constructive methods; the entire body of classical mathematics is viewed as a wellspring of theories waiting to be constructivized.

> Every theorem proved with [nonconstructive] methods presents a challenge: to find a constructive version, and to give it a constructive proof.
>
> \- Errett Bishop [B67, p. x]

The present work is based on the classical theory of the real projective plane. The classical theory is highly nonconstructive; it relies heavily, at nearly every turn, on the *Law of Excluded Middle*. For example, it is classically assumed that a given point is either on a given line, or not on the line —although no finite routine is available for making such a determination;

a constructive development must utilize only the finite routines that are specified in the axioms.

Background; classical. Guides to the classical theory that have proven useful include works by O. Veblen, J. W. Young, H. S. M. Coxeter, E. Artin, and G. Pickert [VY10, Cox55, Art57, Pic75]. For a concise historical review, with thorough references, see Cremona's preface [Cre73, pp. v-xii]. An entertaining history of the subject will be found in Lehmer's last chapter [Leh17, pp. 122-143].

Background; constructive. A. Heyting [H28, D90] has developed a portion of the theory, based on axioms for projective space. A plane being thus embedded in a higher dimension, this permits a proof of Desargues's Theorem, and aids the verification of the properties of harmonic conjugates. D. van Dalen [D96] has studied alternative axioms involving the basic relations.

Here we proceed in a different direction; we utilize axioms only for a plane. Since there exist non-Desarguesian projective planes,[1] this means that Desargues's Theorem must be taken as an axiom; it will be used to establish the converse and the essential properties of harmonic conjugates. The theory is developed further, as far as conic sections, Pascal's Theorem, and the theory of polarity. We make full use of duality in establishing some of the fundamental results.

Adhering closely to the methods of strict constructivism, as introduced by Errett Bishop [B67], we eschew additional assumptions, such as those of formal-logic-based intuitionism or recursive function theory. For a full account of the distinctions between these various types of constructivism, see [BR87].

Background; other work in constructive geometry. For the constructive extension of an affine plane to a projective plane, see [H59, D63, M13a, M14]. For the constructive coördinatization of a plane, see [M07].

The constructive geometry of M. Beeson [Bee10] uses Markov's Principle,[2] which is accepted in recursive function theory, but not in the Bishop-type strict constructivism that is followed in the present paper. M. Lombard and R. Vesley [LomVes98] construct axioms for classical and intuitionistic plane geometry, using recursive function theory.

The work of J. von Plato [Pla95, Pla98, Pla10], proceeding within formal logic, is related to type theory, computer implementation, and combinatorial analysis. For constructive plane hyperbolic geometry, also within formal logic, see V. Pambuccian [Pam01]. The Bishop-type constructive mathematics of the present paper works from a position well-nigh opposite that of formal logic; for further comments on this distinction, see [B65, B67, B73, B75].

Synthetic and analytic approaches. We determine the constructive possibilities of synthetic methods, using no axioms of order, in constructing a real projective plane $\mathbb{P}$. The analytic model $\mathbb{P}^2(\mathbb{R})$, constructed in Euclidean space $\mathbb{R}^3$, is used to prove the consistency of the axiom system.

Axioms. In choosing axioms for the projective plane $\mathbb{P}$, we claim to adopt no new axioms, using only constructive versions of the usual classical axioms. These axioms are all constructively valid on the plane $\mathbb{P}^2(\mathbb{R})$, taking note of Bishop's thesis, "All mathematics should have numerical meaning" [B67, p. ix]. No axioms of order are involved here; the

[1] See, for example, [Wei07].

[2] Markov's Principle and other nonconstructive principles will be discussed in section 12.

constructive investigation of ordered projective planes must be left for future studies.[3]

Bishop-type constructivism. We follow the constructivist principles put forward by Errett Bishop in 1967. This variety of constructivism does not form a separate branch of mathematics, nor is it a branch of logic; it is intended as an enhanced approach for all of mathematics. For the distinctive characteristics of Bishop-type constructivism, as opposed to intuitionism or recursive function theory, see [BR87].

Logical setting. This work uses informal intuitionistic logic; it does not operate within a formal logical system. For the origins of modern constructivism, and the disengagement of mathematics from formal logic, see Bishop's "Constructivist Manifesto" [B67, BB85; Chapter 1]. Concerning the source of misunderstandings in the mathematical community as to the methods and philosophy of constructivism, see [B65].

We use intuitionistic logic only so far as to eschew use of the *Law of Excluded Middle* and its corollaries. Intuitionism, in the stronger sense of Brouwer, introduces additional principles which are classically false. In the opposite direction, recursive function theory limits consideration to a restricted class of objects.[4] Constructive mathematics as proposed by Bishop leads down neither of these extreme pathways. No special logical assumptions are made. Avoiding the *Law of Excluded Middle,* constructive mathematics is a generalization of classical mathematics, just as group theory, a generalization of abelian group theory, avoids the commutative law. Every result and proof obtained constructively is also classically valid.

Results. A fair portion of classical projective geometry is found to have a solid constructive content, provided that appropriate axioms, definitions and methods are used.

It is necessary to avoid the classically ubiquitous method of treating separately elements that are, or are not, distinct or separated from other elements; constructively, elements typically arise lacking such information. Thus, harmonic conjugates must be given a single definition for all points on a line, without distinguishing between the base points and other points; projectivities must be shown to exist for any two given ranges, not knowing whether they are identical or distinct; the polar of a point with respect to a conic must be defined uniformly for any point on the plane, without treating points on the conic as special cases. These requirements often reduce the role of the quadrangle in definitions.

Basing the theory only on axioms for a plane, with no axioms of space, means that Desargues's Theorem must be taken as an axiom; the converse is proved as a consequence. The necessity of ensuring that triangles claimed to be perspective have the required distinctness is paid due attention. Similar situations arise in proving the validity of the harmonic conjugate construction, and the other main concepts.

Once the basic definitions and properties are established constructively, the theory proceeds rather smoothly, revisiting results obtained over the centuries — now with constructive methods.

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[3]For a survey of classical axiomatic ordered geometry, see [Pam11]. For a constructive theory of ordered affine geometry, see [Pla98].

[4]For more information concerning these alternative approaches to constructivism, see [BR87].

Part I

Synthetic constructions

From a set of constructively phrased axioms, we derive the basic properties of a real projective plane, including harmonic conjugates, projectivities, conics, Pascal's Theorem, poles, and polars.

1 Constructive methods

One characteristic feature of the constructivist program is meticulous use of the connective "or". To prove "A or B" constructively, it is required that either we prove A, or we prove B; it is not sufficient to prove the contrapositive $\neg(\neg A$ and $\neg B)$. For an illustration of this in geometry, the Brouwerian counterexample given below will show that the statement "Either the point P lies on the line l, or P lies outside l" is constructively invalid.

Constructively invalid statements. To determine the specific nonconstructivities in a classical theory, and thereby to indicate feasible directions for a constructive development, Brouwerian counterexamples are used. The projective plane being not yet constructed here, we give an informal example on the real metric plane, where *P lies on l* means that the distance from the point P to the line l is 0, while *P lies outside l* means that the distance is positive.

Example 1.1. If, on the plane $\mathbb{R}^2$, we have a proof of the statement

Given any point P and any line l, either P lies on l, or P lies outside l,

then we have a method that will either prove the *Goldbach Conjecture,* or construct a counterexample.

Proof. Using a simple finite routine, construct a sequence $\{a_n\}_{n\geq 2}$ such that $a_n = 0$ if $2n$ is the sum of two primes, and $a_n = 1$ if it is not. Now apply the statement in question to the point $P = (0, \Sigma a_n/n^2)$, with the x-axis as the line l. If $P \in l$, then we have proved the Goldbach Conjecture, while if $P \notin l$, then we have constructed a counterexample. $\square$

For this reason, such statements are said to be *constructively invalid*. If the Goldbach question is someday settled, then other famous problems may still be "solved" in this way. These examples demonstrate that use of the *Law of Excluded Middle* inhibits mathematics from attaining its full significance. More information concerning Brouwerian counterexamples will be found below in Section 12.

Many other ordinary geometric statements, especially those involving a disjunction, are also constructively invalid, admitting easily devised Brouwerian counterexamples similar to Example 1.1. The consequence of this Brouwerian analysis is the need for explicit constructive details, in axioms, theorems, and proofs, which from a classical perspective may seem superfluous.

Constructive logic. Following Bishop, we use no system of formal logic. Aside from the need to avoid use of the *Law of Excluded Middle,* and to use the connective "or" only when warranted, no special rules are required. The constructive logic used here is usually called *informal intuitionistic logic*; for more on this subject, see [BV06, Section 1.3].

Certain concepts, such as $x = 0$, for a real number x, are relatively weak, compared to stronger concepts, such as $x \neq 0$. The relation $x \neq 0$ requires the construction of an integer n such that $1/n < |x|$; it then follows that $x = 0$ is equivalent to $\neg(x \neq 0)$, while the statement "$\neg(x = 0)$ implies $x \neq 0$" is constructively invalid.

In geometry, *point outside a line,* $P \notin l$, is the stronger concept, while *point on a line,* $P \in l$, is the weaker.[5] On the constructive real metric plane $\mathbb{R}^2$, the geometric and numeric concepts are directly related; $P \notin l$ if and only $d(P, l) > 0$ [M07, Theorem 10.1]. Thus, while the statement "If $\neg(P \notin l)$, then $P \in l$" will be taken as an axiom, reflecting the constructive properties of the real numbers, the statement "If $\neg(P \in l)$, then $P \notin l$" is constructively invalid.

Further details concerning the constructive properties of the real numbers, and constructively invalid statements, will be found below in Section 12.

2 Axioms

We adopt the usual definitions and axioms for a projective plane, adding the several required to obtain constructive results. The additional axioms are constructively phrased versions of elementary facts that are immediate in classical geometry, when the *Law of Excluded Middle* is used. For a constructive study, these additional facts must be stated explicitly in the axioms, and must be verified whenever one constructs the finite routines for a model.

The model $\mathbb{P}^2(\mathbb{R})$ in Part II will establish the consistency of the axiom system; the question of independence of the individual axioms is left as an open problem. The properties of the model have served to drive the axiom choices for the synthetic theory, taking note of Bishop's thesis, "All mathematics should have numerical meaning" [B67, p. ix].

Definition 2.1. Let $\mathscr{S}$ be a set with an equality relation $=$. An *inequality relation* $\neq$ on $\mathscr{S}$ is called an *tight apartness relation*[6] if, for any x, y, z in $\mathscr{S}$, the following conditions are satisfied.

(i) $\neg(x \neq x)$.

(ii) If $x \neq y$, then $y \neq x$.

(iii) If $x \neq y$, then either $z \neq x$ or $z \neq y$.

(iv) If $\neg(x \neq y)$, then $x = y$.

Property (iii) is known as *cotransitivity,* and (iv) as *tightness.* The implication "$\neg(x = y)$ implies $x \neq y$" is nearly always constructively invalid, the inequality being the stronger of the two conditions.[7] For an example with real numbers, $x \neq 0$ means that there exists an integer n such that $1/n < |x|$, while $x = 0$ means only that it is contradictory that such an integer exists.[8]

[5]For the stronger concept used as the single primitive notion for intuitionistic projective geometry, see [D96].

[6]Early work on apartness relations is due to Heyting; see [H66].

[7]For a comprehensive treatment of constructive inequality relations, see [BR, Section 1.2].

[8]For more details concerning the constructive properties of the real numbers, see [B67, BB85, BV06].

Definition 2.2. A *projective plane* $\mathbb{P}$ consists of a family $\mathscr{P}$ of *points*, and a family $\mathscr{L}$ of *lines,* satisfying the following conditions, and axioms to be specified.
• *Equality* relations, written $=$, are given for both families $\mathscr{P}$ and $\mathscr{L}$.
• *Inequality* relations, written $\neq$, with the properties of tight apartness relations, as specified in Definition 2.1, are given for both families $\mathscr{P}$ and $\mathscr{L}$. When $P \neq Q$, or $l \neq m$, we say that the points P and Q, or the lines l and m, are *distinct*.
• An *incidence* relation $\in$, between the families $\mathscr{P}$ and $\mathscr{L}$, is given; when $P \in l$, we say that the point P *lies on the line* l, and that the line l *passes through the point* P.

Definition 2.3. *Outside* relation. For any point $P \in \mathscr{P}$ and any line $l \in \mathscr{L}$, we say that P *lies outside the line* l, and that l *avoids the point* P, written $P \notin l$, if $P \neq Q$ for all points Q that lie on l.

A constructive definition of distinct lines is at times based on the incidence and outside relations. Here, the relation of distinct lines is internal, referring only to the family $\mathscr{L}$. This is the natural approach for the model $\mathbb{P}^2(\mathbb{R})$ of Part II, and is an instance where the model influences a choice for the synthetic theory. The method here is adapted to the situation where the families $\mathscr{P}$ and $\mathscr{L}$ are independent, as in the model, rather than the situation often seen where lines are sets of points. With the relations of distinct points and distinct lines established internally to the families $\mathscr{P}$ and $\mathscr{L}$, Axiom C5 will then relate the concepts to the incidence and outside relations.

Constructive Axiom Group C. These axioms form the basis for the synthetic theory. The duality of this axiom group will be shown in Theorem 2.10.

In addition, Axiom F (Fano) will be adopted at the end of this section, and will be shown to be self-dual in Theorem 2.15; Axiom D (Desargues) will be adopted in Section 3, where its dual (the converse) will be proved; Axiom E (Extension), to be adopted in Section 5, is self-dual; Axiom T (the crucial component of the *Fundamental Theorem*), to be adopted in Section 6, is also self-dual, as is Axiom P, to be adopted in Section 11 in connection with poles and polars with respect to a conic. The duality of Definition 2.3, for the *outside* relation, will be established in Theorem 2.11.

Thus the duality of the complete set of axioms will be established.

Axiom C1. *There exist a point $P \in \mathscr{P}$, and a line $l \in \mathscr{L}$, such that $P \notin l$.*

Axiom C2. *For any distinct points P and Q, there exists a unique line, denoted PQ, called the* join, *or* connecting line, *of the points, passing through both points.*

Axiom C3. *For any distinct lines l and m, there exists a unique point, denoted $l \cdot m$, called the* meet, *or* point of intersection, *of the lines, lying on both lines.*

Axiom C4. *There exist at least three distinct points lying on any given line.*

Axiom C5. *For any lines l and m, if there exists a point $P \in l$ such that $P \notin m$, then $l \neq m$.*

Axiom C6. *For any point P and any line l, if $\neg(P \notin l)$, then $P \in l$.*

Axiom C7. *If l and m are distinct lines, and P is a point such that $P \neq l \cdot m$, then either $P \notin l$ or $P \notin m$.*

Notes for Axiom Group C.

1. Axioms C1 thru C4 are the usual classical axioms for incidence and extension. The remaining three axioms are statements that follow immediately when lines are considered as sets of points, and the *Law of Excluded Middle* is used; classically, they need no explicit mention. In this sense, no new axioms are needed for a constructive theory.

2. Axioms C2 and C3 apply only to distinct points and lines. The need for this restriction will follow from Example 14.1, where it is shown that in the model $\mathbb{P}^2(\mathbb{R})$, claiming the existence of a line through two arbitrary points, or a point lying on two arbitrary lines, would be constructively invalid.

3. Axiom C6 would be immediate in a classical setting, when $P \in l$ is used in the sense of set-membership, where $P \notin l$ means $\neg(P \in l)$, and when, applying the Law of Excluded Middle, double negation results in an affirmative statement.

For the constructive treatment here, the situation is quite different. The *outside* relation, $P \notin l$, is given a strong affirmative meaning in Definition 2.3, involving both the inequality relation for points, and the incidence relation which connects the two families. Just as *tightness*, defined by condition (iv) in Definition 2.1, must be assumed in Definition 2.2 for both points and lines, the analogous condition C6, relating the two given families, must be taken as an axiom.

For the metric real plane, with incidence relations as noted in connection with Example 1.1, the condition of Axiom C6 follows from the following constructive property of the real numbers: *For any real number α, if $\neg(\alpha \neq 0)$, then $\alpha = 0$.*[9] For the projective model $\mathbb{P}^2(\mathbb{R})$, which motivates the axiom system, Axiom C6 is verified in Corollary 13.4, using this same constructive property of the reals.

The definitions and axioms of projective geometry may be given a wide variety of different arrangements. For example, in [D96] the relation $P \notin l$ is taken as a primitive notion, and the condition of Axiom C6 becomes the definition of the incidence relation $P \in l$.

4. Axiom C7 is a strongly worded constructive form of the statement that distinct lines have a *unique* common point. Related to this axiom are Heyting's Axiom VI [H28], and van Dalen's Lemma 3(f), obtained using his axiom Ax5 [D96]. Paraphrased to fit the present context, these statements ensure that *If l and m are distinct lines, and P is a point such that $P \neq l \cdot m$ and $P \in l$, then $P \notin m$.* This is a weaker version of Axiom C7; the stronger version will be needed here. It is an open problem to determine whether the two versions are equivalent or constructively distinct, or whether the weaker version would be sufficient. Generally, a condition using the "or" connective is found to be constructively stronger than other versions.

5. An affine form of Axiom C7 is used as Axiom L1 in [M07].

6. Axiom C7 is the only axiom asserting a disjunction. Example 1.1 concerned the constructive invalidity of certain principles found in classical treatments, especially those as-

[9] This property is listed as property (i) in Section 12, where more details are given.

serting a disjunction. In Axiom C7, we have two hypotheses, each being a strong distinctness condition. The verification of this axiom for the model $\mathbb{P}^2(\mathbb{R})$, in Theorem 14.2, will require both these strong hypotheses, other axioms, and other constructive properties of $\mathbb{P}^2(\mathbb{R})$.

7. Axiom C7 may rightly claim a preëminent standing in the axiom system; it will be indispensable for nearly all the constructive proofs.

Proposition 2.4. *Let P, Q, R be distinct points. Then $P \notin QR$ if and only if $PQ \neq PR$.*

Proof. First let $P \notin QR$. From Axiom C5 we have $PR \neq QR$. Since $Q \neq R = PR \cdot QR$, it follows from Axiom C7 that $Q \notin PR$; thus $PQ \neq PR$. Conversely, if $PQ \neq PR$, then from $P \neq Q = PQ \cdot QR$ it follows that $P \notin QR$. □

Proposition 2.5. *If the lines l and m are distinct, then there exists a point $P \in l$ such that $P \notin m$.*

Proof. Set $Q = l \cdot m$, using Axiom C3, and select a point $P \in l$ such that $P \neq Q$, using Axiom C4. It follows from Axiom C7 that either $P \notin l$ or $P \notin m$. The first case is ruled out by Axiom C6; thus $P \notin m$. □

Definition 2.6.
- A set $\mathscr{S}$ of points is *collinear* if $P \in QR$ whenever $P, Q, R \in \mathscr{S}$ with $Q \neq R$.
- A set $\mathscr{S}$ of points is *noncollinear* if there exist distinct points $P, Q, R \in \mathscr{S}$ such that $P \notin QR$.
- A set $\mathscr{T}$ of lines is *concurrent* if $l \cdot m \in n$ whenever $l, m, n \in \mathscr{T}$ with $l \neq m$.
- A set $\mathscr{T}$ of lines is *nonconcurrent* if there exist distinct lines $l, m, n \in \mathscr{T}$ such that $l \cdot m \notin n$.
- The *range of points on a line* l is the set $\bar{l} = \{P \in \mathscr{P} : P \in l\}$.
- The *pencil of lines through a point* Q is the set $Q^* = \{m \in \mathscr{L} : Q \in m\}$.

Example 2.7. A stronger, classically equivalent, alternative definition for *collinear set* is the condition *There exists a line that passes through each point of the set.* The equivalence of the two conditions is constructively invalid for the model $\mathbb{P}^2(\mathbb{R})$. The Brouwerian counterexample given in Example 14.1 will apply; we give a simplified version here, in brief form. Consider two points on the plane $\mathbb{R}^2$, the origin, and a point close to or at the origin. The set formed by these has at most two points, and is collinear according to our definition, yet it is not possible, constructively, to predict what line might contain both points.[10]

Another distinction between the alternative definitions concerns the statement *If $\neg$($\mathscr{S}$ is noncollinear), then $\mathscr{S}$ is collinear.* This statement follows easily from our definition, while under the alternative definition it is seen to be constructively invalid, using the example above.

Proposition 2.8. *If $\mathscr{S}$ is a noncollinear set of points, then for any line l in the plane, there exists a point in $\mathscr{S}$ that lies outside l.*

Proof. Choose distinct points $P, Q, R \in \mathscr{S}$ as in Definition 2.6, with $P \notin QR$. It follows from Proposition 2.4 that $PQ \neq PR$. By cotransitivity for lines, either $l \neq PQ$ or $l \neq PR$. It suffices to consider the first case; set $Y = l \cdot PQ$. Now, either $Y \neq P$ or $Y \neq Q$. In the first subcase, we have $P \neq l \cdot PQ$, so it follows from Axiom C7 that $P \notin l$. Similarly, in the second subcase we find that $Q \notin l$. □

[10] For more on the constructive eccentricities of such sets, see [M13b, Example 2.5].

Proposition 2.9. *If a set $\mathscr{S}$ of three distinct points is noncollinear, then $P \notin QR$, where P, Q, R are the points of $\mathscr{S}$ taken in any order.*

Proof. Given that $P \notin QR$, we have $PR \neq QR$ and $Q \neq R = PR \cdot QR$; thus by Axiom C7 it follows that $Q \notin PR$. By symmetry, we also have $R \notin PQ$. ☐

Given any statement, the *dual statement* is obtained by interchanging the words "point" and "line".

Theorem 2.10. . *The definition of the projective plane $\mathbb{P}$ is self-dual, and the dual of each axiom in Axiom Group C holds on $\mathbb{P}$.*

Proof. Definition 2.2, and Axioms C1, C2/C3, C6 are clearly self-dual.

For the dual of Axiom C4, select a point Q and a line m, with $Q \notin m$, using Axiom C1. Using Axiom C4 select three distinct points R_1, R_2, R_3 on m. Using Definition 2.3, we have $Q \neq R_i$ for each i; set $l_i = QR_i$. Since $Q \notin m$, it follows that $l_i \neq m$ for each i. Since $R_1 \neq R_2 = m \cdot l_2$, it follows from Axiom C7 that $R_1 \notin l_2$, so $l_1 \neq l_2$. By symmetry, the three lines l_i are distinct. This is the desired result for the selected point Q, based on the existence of the line m that avoids Q. Now, given an arbitrary point P, using cotransitivity we may assume that $P \neq R_1$. Since $P \neq R_1 = m \cdot l_1$, it follows that either $P \notin m$ or $P \notin l_1$. In either case, using the same method as for Q and m, we may construct three distinct lines through P.

The dual of Axiom C5 states that *"Points P and Q are distinct, $P \neq Q$, if and only if there exists a line l such that $P \in l$ and $Q \notin l$, or vice-versa"*. First, consider points P and Q with $P \neq Q$, and use Axiom C4(dual) to construct distinct lines l and m through P. Now we have $Q \neq P = l \cdot m$, so by Axiom C7 we may assume that $Q \notin l$. Thus we have a line l through P that avoids Q. Conversely, if for some line l we have $Q \notin l$ and $P \in l$, then $Q \neq P$ by Definition 2.3.

The dual of Axiom C7 states that "*If Q and R are distinct points, and n is a line such that $n \neq QR$, then either $Q \notin n$ or $R \notin n$*". To prove this, set $S = QR \cdot n$, and use cotransitivity to obtain either $S \neq Q$ or $S \neq R$. In the first case, since $Q \neq S = QR \cdot n$, it follows from Axiom C7 that $Q \notin n$. Similarly, in the second case we find that $R \notin n$. Thus Axiom C7 is self-dual. ☐

Theorem 2.11. *Let P be any point, and l any line. Then $P \notin l$ if and only if $l \neq m$* for all lines m that pass through P.

Proof. Let $P \notin l$ and let m be any line through P. Using Axiom C4(dual) and cotransitivity for lines, select a line n passing through P and distinct from m, and select any point $Q \in l$. Then $Q \neq P = m \cdot n$, so by Axiom C7 it follows that either $Q \notin m$, or $Q \notin n$. In the first case, $l \neq m$. In the second case, $l \neq n$; set $R = l \cdot n$. Since $R \in l$, we have $P \neq R$; thus $R \neq P = m \cdot n$, so $R \notin m$, and again $l \neq m$. Thus the dual condition is satisfied.

Now let P and l satisfy the dual condition, and let Q be any point on l. Select a point R on l, distinct from Q. Either $P \neq Q$ or $P \neq R$. In the second case, set $m = PR$; by hypothesis, $l \neq m$. Since $Q \neq R = l \cdot m$, it follows that $Q \notin m$. From Axiom C5(dual), we have $P \neq Q$. Hence $P \notin l$. ☐

Corollary 2.12. *The primary relation adopted in Definition 2.3,* point outside a line*, is self-dual in the context of Axiom Group C.*

From this corollary, and Theorem 2.10, we obtain the *duality principle:*

Theorem 2.13. *On the projective plane $\mathbb{P}$, the dual of any result is immediately valid, with no further proof required.*

Definition 2.14. A *quadrangle* is an ordered set $PQRS$ of four distinct points, the *vertices,* such that each subset of three points is noncollinear. The *sides* are the six lines joining the vertices. The three *diagonal points* are $D_1 = PQ \cdot RS$, $D_2 = PR \cdot QS$, and $D_3 = PS \cdot QR$. A *quadrilateral*, with four sides and six vertices, is the dual configuration.

Note for Definition 2.14. Since $R \notin PQ$, we have $PQ \neq RS$; by symmetry, all six sides are distinct, and the definition of the diagonal points is valid. By cotransitivity, either $D_2 \neq P$ or $D_2 \neq R$. It suffices to consider the first case; thus $D_2 \neq PQ \cdot PR$, so by Axiom C7 we have $D_2 \notin PQ$, and $D_2 \neq D_1$. By symmetry, all three diagonal points are distinct.

It will be convenient to exclude certain finite planes, such as the seven-point "Fano plane" [Fan92], an illustration of which may be found at [VY10, p. 45] or [Wei07, p. 1294]. Thus we adopt the following:

Axiom F. Fano's Axiom. *The diagonal points of any quadrangle are noncollinear.*

Proposition 2.15. *Axiom F is self-dual; the diagonal lines of any quadrilateral are nonconcurrent.*

Proof. Given a quadrilateral $pqrs$, denote four of the six vertices as $P = p \cdot q$, $Q = q \cdot r$, $R = r \cdot s$, $S = s \cdot p$. The diagonal lines of $pqrs$ are then $d_1 = (p \cdot q)(r \cdot s) = PR$, $d_2 = (p \cdot r)(q \cdot s)$, and $d_3 = (p \cdot s)(q \cdot r) = QS$.

To show that $PQRS$ is a quadrangle, we note that since the lines p, q, r are nonconcurrent, we have $P = p \cdot q \notin r = QR$, so the points P, Q, R are noncollinear, and similarly for the other three triads.

The diagonal points of the quadrangle $PQRS$ are $D_1 = PQ \cdot RS = q \cdot s$, $D_2 = PR \cdot QS = d_1 \cdot d_3$, and $D_3 = PS \cdot QR = p \cdot r$; it follows that $d_2 = D_1 D_3$. By Axiom F, we have $D_2 \notin D_1 D_3$; thus $d_1 \cdot d_3 \notin d_2$, and the diagonal lines d_1, d_2, d_3 of the quadrilateral are nonconcurrent. □

3 Desargues's Theorem

We adopt Desargues's Theorem as Axiom D, and then use it to prove the converse, which is its dual.[11]

Definition 3.1.

• A *triangle* is an ordered triad PQR of distinct, noncollinear points. The three points are the *vertices*; the lines PQ,PR,QR are the *sides.*

[11]Desargues's Theorem and the converse are both derived in [H28, §§5-6], using axioms for projective space; here we use only axioms for a plane.

• Triangles PQR and $P'Q'R'$ are *distinct* if corresponding vertices are distinct, and corresponding sides are distinct.
• Distinct triangles are said to be *perspective from the center* O if the three lines joining corresponding vertices are concurrent at O, and O lies outside each of the six sides.
• Distinct triangles are said to be *perspective from the axis* l if the three points of intersection of corresponding sides are collinear on l, and l avoids each of the six vertices.

Axiom D. *If two triangles are perspective from a center, then they are also perspective from an axis.*

Theorem 3.2. *If two triangles are perspective from an axis, then they are also perspective from a center.*

Proof. We are given distinct triangles PQR, $P'Q'R'$, with points $A = QR \cdot Q'R'$, $B = PR \cdot P'R'$, $C = PQ \cdot P'Q'$ collinear on a line l, with $V \notin l$ for all six vertices V.

Since $Q \notin l$, we have $A \neq Q = PQ \cdot QR$; it follows from Axiom C7 that $A \notin PQ$, and thus $A \neq C$. By symmetry, all three points A, B, C are distinct, the points A, Q, Q' are distinct, and the points B, P, P' are distinct. Since $Q \neq A = QR \cdot Q'R'$, we have $Q \notin Q'R' = AQ'$; thus the points A, Q, Q' are noncollinear, and similarly for B, P, P'. Since $P \notin l = AB$, it follows that $AB \neq BP$. Since $A \neq B = AB \cdot BP$, we have $A \notin BP$, so $AQ \neq BP$; similarly, $AQ' \neq BP'$. Since $Q' \neq C = PQ \cdot P'Q'$, it follows that $Q' \notin PQ$; thus $QQ' \neq PQ$, and similarly $PP' \neq PQ$. Since $P \neq Q = PQ \cdot QQ'$, we have $P \notin QQ'$; thus $PP' \neq QQ'$. Set $O = PP' \cdot QQ'$.

The above shows that the auxiliary triangles AQQ', BPP' are distinct. The lines AB, PQ, $P'Q'$, joining corresponding vertices, are concurrent at C. Since $C \neq A = l \cdot AQ$, it follows that $C \notin AQ$; similarly, $C \notin AQ'$. Since $P \notin QQ'$, it follows that $CQ = CP \neq QQ'$, and from $C \neq Q = CQ \cdot QQ'$ we have $C \notin QQ'$. Thus C lies outside each side of triangle AQQ', and similarly for triangle BPP'.

Thus the auxiliary triangles AQQ', BPP' are perspective from the center C; it follows from Axiom D that these triangles are perspective from the axis $(AQ \cdot BP)(AQ' \cdot BP') = RR'$. Thus $O \in RR'$, and the axis RR' avoids all six vertices of the auxiliary triangles. This shows that the lines PP', QQ', RR' are concurrent at O. Since $P \notin RR'$, we have $P \neq O$. Since $O \neq P = PP' \cdot PQ$, it follows that $O \notin PQ$. Similarly, O lies outside each side of the triangles PQR, $P'Q'R'$. Hence the original triangles are perspective from the center O. ☐

4 Harmonic conjugates

Harmonic conjugates are often defined using quadrangles or triangles. We must use a less problematic definition; it must apply to every point on the line, including the base points, and any point for which it is not known, constructively, whether or not it coincides with a base point. The simplicity of the definition will facilitate the verification that the joins and intersections used are constructively meaningful, and that the result is independent of the selection of construction elements.

Definition 4.1. Let A and B be distinct points. For any point C on the line AB, select a line l through C, distinct from AB, and select a point R lying outside each of the lines AB and l. Set $P = BR \cdot l$, $Q = AR \cdot l$, and $S = AP \cdot BQ$. Pending verifications in Proposition 4.3 and Theorem 4.7, the point $D = AB \cdot RS$ will be called the *harmonic conjugate of* C *with respect to the points* A, B; we write $D = h(A, B; C)$. The points A, B, C, D are said to form a *harmonic set*, written $h(A, B; C, D)$.

Lemma 4.2. *In Definition 4.1:*
(a) $P \neq A$, $Q \neq B$, $P \neq Q$.
(b) $P \notin AR$, $Q \notin BR$, $A \notin BR$, $B \notin AR$.
(c) $AR \neq BR$, $AP \neq AR$, $AP \neq BR$, $BQ \neq BR$, $BQ \neq AR$.

Proof. Since $A \neq B = AB \cdot BR$, it follows from Axiom C7 that $A \notin BR$; thus $A \neq P$, $AP \neq BR$, and $AR \neq BR$. By symmetry, $B \notin AR$, $B \neq Q$, and $BQ \neq AR$. Since $P \neq R = AR \cdot BR$, we have $P \notin AR$, so $P \neq Q$, and $AP \neq AR$. Similarly, $Q \notin BR$ and $BQ \neq BR$. □

Proposition 4.3. *The construction of a harmonic conjugate, in Definition 4.1, involves valid joins and intersections.*

Proof. Using Lemma 4.2, we need only show that $AP \neq BQ$, and that $R \neq S$. By cotransitivity, we may assume that $C \neq B$. Since $C \neq B = AB \cdot BR$, it follows from Axiom C7 that $C \notin BR$, so $C \neq P$. Since $P \neq C = AB \cdot l$, it follows that $P \notin AB$; thus $AB \neq AP$. Since $B \neq A = AB \cdot AP$, we have $B \notin AP$; thus $AP \neq BQ$. This shows that the definition of S is valid. From $R \neq B = BQ \cdot BR$, it follows that $R \notin BQ$; hence $R \neq S$. This shows that the definition of D is valid. □

The next result is one of the four lemmas required for the proof of Theorem 4.7, which will validate the harmonic conjugate construction, showing that it is independent of the choice of construction elements. This lemma involves the special situation in which the original point is one of the base points.

Lemma 4.4. *Let $A \neq B$. Then $h(A, B; A) = A$ and $h(A, B; B) = B$, for any selection of construction elements l, R in Definition 4.1.*

Proof. When $C = A$, then $Q = AR \cdot l = CR \cdot l = C = A$, so $S = AP \cdot BQ = AP \cdot BA = A$, and thus $D = AB \cdot RS = AB \cdot RA = A$. Similarly when $C = B$. □

Lemma 4.5. *In Definition 4.1:*
(a) *If $C \neq A$, then $Q \notin AB$, $Q \neq S$, $S \neq A$, and $D \neq A$.*
(b) *If $C \neq B$, then $P \notin AB$, $P \neq S$, $S \neq B$, and $D \neq B$.*

Proof. It will suffice to consider (a), since (b) will follow by symmetry. Since $C \neq A = AB \cdot AR$, it follows from Axiom C7 that $C \notin AR$; thus $C \neq Q$. Since $Q \neq C = AB \cdot l$, we have $Q \notin AB$, so $Q \neq A$. Since $A \neq Q = BQ \cdot AR$, it follows that $A \notin BQ$; thus $A \neq S$. Since $S \neq A = AP \cdot AR$, we have $S \notin AR$, so $S \neq Q$, and $AR \neq RS$. Since $A \neq R = AR \cdot RS$, it follows that $A \notin RS$, and thus $A \neq D$. □

The next lemma shows that for a point distinct from both base points, the traditional quadrangle will appear; for a complete statement regarding this configuration, see Corollary 4.8.

Lemma 4.6. *In Definition 4.1 for the construction of a harmonic conjugate, let $C \neq A$ and $C \neq B$. Then the four points P, Q, R, S are distinct and lie outside the line AB, and each subset of three points is noncollinear. Furthermore, $h(A, B; C) \neq C$.*

Proof. Using Lemma 4.5, we see that P, Q, R lie outside AB. By the same lemma, we also have $S \neq B = AB \cdot BQ$, so it follows from Axiom C7 that $S \notin AB$. Thus the four points lie outside AB.

In Definition 4.1, we have $P \neq R$, $Q \neq R$. From Lemma 4.2, we have $P \neq Q$. From Proposition 4.3, we have $R \neq S$. From Lemma 4.5, we have $P \neq S$, $Q \neq S$. Thus the four points are distinct.

For the triads, we use the results of Lemmas 4.2 and 4.5. Since $S \neq A = AP \cdot AR$, it follows that $S \notin AR$, so $AR \neq RS$. Since $Q \neq R = AR \cdot RS$, it follows that $Q \notin RS$; thus the points Q, R, S are noncollinear, and similarly for P, R, S. Since $P \neq B = BQ \cdot BR$, we have $P \notin BQ = QS$; thus the points P, Q, S are noncollinear. Since $P \notin AR = QR$, the points P, Q, R are noncollinear. Thus each triad is noncollinear.

Since two diagonal points of the quadrangle $PQRS$ are $A = PS \cdot QR$ and $B = PR \cdot QS$, Axiom F, Fano's Axiom, asserts that the third diagonal point T lies outside AB. Thus $C \neq T = PQ \cdot RS$; it follows that $C \notin RS$, and hence $C \neq D$. □

The next theorem will validate the harmonic conjugate construction.[12]

Theorem 4.7. *The construction of the harmonic conjugate of an arbitrary point C on a line AB, with respect to the points A,B, using Definition 4.1, results in a point D that is independent of the selections of the line l and the point R used in the construction.*

Proof. Let l', R' be alternative selections, and let D' be the point resulting when the alternatives are used in the construction.

(1) Suppose that $D' \neq D$, and suppose further that $C \neq A$, $C \neq B$, $l' \neq l$, and $R' \neq R$. We will contradict these five assumptions sequentially, in reverse. This process ends in a negation of the first assumption; thus the required conclusion, $D' = D$, will follow from the tightness property of the inequality relation for points, specified in Definition 2.1(iv).

(2) Using only the first four assumptions in (1), we note here a few basic facts. By Lemma 4.6, the points P, Q, R, S are distinct and lie outside the line AB, and each subset of three points is noncollinear. Similarly for the points P', Q', R', S'.

Since $D \neq D' = AB \cdot R'S'$, it follows from Axiom C7 that $D \notin R'S'$; thus $RS \neq R'S'$. From $P \neq C = l \cdot l'$ it follows that $P \notin l'$. Thus $P \neq P'$, and similarly, $Q \neq Q'$. Since $P \neq Q$, we have $PQ = l$; similarly, $P'Q' = l'$. Thus $PQ \neq P'Q'$. From $Q' \neq C = l \cdot l'$ it follows that $Q' \notin l = PQ$, and thus $QQ' \neq PQ$. Since $P \neq Q = QQ' \cdot PQ$, we have $P \notin QQ'$, so $PP' \neq QQ'$. Thus we may define $O = PP' \cdot QQ'$.

[12] A harmonic conjugate construction based on quadrangles is validated in [H28, §7], using axioms for projective space; the construction applies only to points distinct from the base points. Here we use only axioms for a plane, and consider all points on the base line.

(3) Since $R' \neq R = AR \cdot BR$, it follows that either $R' \notin AR$ or $R' \notin BR$. By symmetry, it suffices to consider the second case. Since $PR = BR$, it follows that $PR \neq P'R'$.

(4) Suppose further, in addition to the assumptions in (1), that $R' \notin AR$, $PS \neq P'S'$, $QS \neq Q'S'$, $O \notin RS$, $O \notin R'S'$, and $S \neq S'$.

(5) Consider the triangles PQR, $P'Q'R'$. From (2) and the fifth assumption in (1), the corresponding vertices of the triangles are distinct, and $PQ \neq P'Q'$. Since $R' \notin AR$ by (4), it follows that $QR = AR \neq AR' = Q'R'$. Also, $PR \neq P'R'$ by (3). Thus the triangles are distinct. Since $QR \cdot Q'R' = A$, $PR \cdot P'R' = B$, $PQ \cdot P'Q' = C$, and the six vertices lie outside the line AB, these triangles are perspective from the axis AB. By the converse to Desargues's Theorem, established above as Theorem 3.2, the triangles are perspective from the center $PP' \cdot QQ' = O$.

Thus $O \in RR'$, and O lies outside each of the six sides of the triangles PQR, $P'Q'R'$.

(6) The triangles PQS, $P'Q'S'$ are distinct, using the assumptions in (4), with $PS \cdot P'S' = A$, $QS \cdot Q'S' = B$, $PQ \cdot P'Q' = C$, and vertices outside the line AB. Thus these triangles are perspective from the axis AB; it follows that they are perspective from the center $PP' \cdot QQ' = O$.

Thus $O \in SS'$, and O lies outside each of the six sides of the triangles PQS, $P'Q'S'$.

(7) Now consider the triangles PRS, $P'R'S'$. By (1, 2, 4), the corresponding vertices are distinct. From (2) we have $RS \neq R'S'$, from (3) we have $PR \neq P'R'$, and in (4) we assumed that $PS \neq P'S'$. Thus the triangles are distinct. From (5) we have $O \notin PR$, from (6) we have $O \notin PS$, and from (4) we have $O \notin RS$. Similarly, O lies outside each side of the second triangle. By (5) and (6), the triangles are perspective from the center O; thus by Desargues's Theorem, adopted above as Axiom D, they are also perspective from the axis $(PS \cdot P'S')(PR \cdot P'R') = AB$. Thus the point $E := RS \cdot R'S'$ lies on AB. It follows that $D = E$, and also $D' = E$. This contradiction of the first assumption in (1) negates the last assumption in (4).

Thus $S = S'$.

(8) It follows that $BS = BS'$; i.e., $QS = Q'S'$. This contradiction of the third assumption in (4) negates the fifth; thus $O \in R'S'$.

Similarly, we obtain $PS = P'S'$, a contradiction of the second assumption in (4), negating the fourth; thus $O \in RS$. The condition $QS = Q'S'$ negates the third assumption in (4); hence $QS = Q'S'$, and thus $S \in QQ'$. The second assumption in (4) is negated similarly; thus $PS = P'S'$, and $S \in PP'$.

(9) From (8), it is easily seen (in two ways) that $S = O$. In step (5), the arguments depend only on (1) and the first assumption in (4), not yet negated; thus we may use here the conclusion that $O \in RR'$. It follows that $RS = R'S'$, a contradiction of (2), negating the first assumption in (4).

Thus $R' \in AR$, and $AR = AR'$.

(10) From (8, 9), we have $QS \cdot AR = Q'S' \cdot AR'$, i.e., $Q = Q'$, contradicting (2), and negating the last assumption in (1). Thus $R' = R$. Combining this with (7), we have $RS = R'S'$. This contradiction of (2) negates the fourth assumption in (1). Thus $l' = l$.

(11) By (10), it is evident that $D' = D$, contradicting the first assumption in (1), and negating the third; thus $C = B$. By Lemma 4.4, it follows that $D = B$, and also $D' = B$; this contradicts the first assumption in (1), and negates the second. Thus $C = A$; using the same lemma, this again results in a contradiction, negating the first assumption in (1).

Hence $D' = D$, and this validates the harmonic conjugate construction. $\square$

Corollary 4.8. *Let A, B, C, D be collinear points, with C distinct from both points A and B. Then $D = h(A, B; C)$ if and only if there exists a quadrangle $PQRS$, with vertices outside the line AB, such that $A = PS \cdot QR$, $B = QS \cdot PR$, $C \in PQ$, and $D \in RS$.*

Lemma 4.9. *Let $A \neq B$, and let C and D be any points on the line AB.*
(a) *If $h(A, B; C, D)$, then $h(B, A; C, D)$.*
(b) *If $D = h(A, B; C)$, then $C = h(A, B; D)$.*

Proof. (a) This follows from the symmetry of the construction in Definition 4.1.

(b) By cotransitivity and (a), it suffices to consider the case in which $C \neq A$. Using the notation of Definition 4.1 for the construction of D, from Lemma 4.5 we have $Q \notin AB$, $Q \neq S$, $S \neq A$, and $D \neq A$. Define $l^d = RS$ and $R^d = Q$. Then $D \in l^d$, $l^d \neq AB$, and $R^d \notin AB$. Since $D \neq A = AB \cdot AR$, it follows from Axiom C7 that $D \notin AR$, and thus $AR \neq RS$. Since $Q \neq R = AR \cdot RS$, we have $Q \notin RS$; i.e., $R^d \notin l^d$. Thus the elements l^d, R^d may be used to construct $h(A, B; D)$.

Now $P^d = BR^d \cdot l^d = BQ \cdot RS = S$, and $Q^d = AR^d \cdot l^d = AQ \cdot RS = AR \cdot RS = R$. Thus $S^d = AP^d \cdot BQ^d = AS \cdot BR = AP \cdot BR = P$. It follows that $R^d S^d = QP = l$ and $h(A, B; D) = AB \cdot R^d S^d = AB \cdot l = C$. $\square$

Theorem 4.10. *Let A and B be distinct points in a range $\overline{l}$, and let υ be the mapping of harmonic conjugacy with respect to the base points A, B; i.e., set $X^\upsilon = h(A, B; X)$, for all points X in $\overline{l}$. Then υ is a bijection of the range $\overline{l}$ onto itself, of order 2.*

Proof. Lemma 4.9(b) shows that υ is onto $\overline{l}$, and of order 2. Now let $C_1, C_2 \in AB$, with $C_1 \neq C_2$. To show that the harmonic conjugates $D_1 = h(A, B; C_1)$ and $D_2 = h(A, B; C_2)$ are distinct, we note first that by cotransitivity either $C_1 \neq A$ or $C_1 \neq B$, and similarly for C_2. By symmetry, only two of the four cases need be considered.

Case 1; $C_1 \neq A$ and $C_2 \neq A$. Select a point R with $R \notin AB$, and select a point $Q \in AR$ with $Q \neq A$ and $Q \neq R$. Since $Q \neq A = AB \cdot AR$, it follows from Axiom C7 that $Q \notin AB$; thus $Q \neq C_1$ and $Q \neq C_2$. Set $l_1 = C_1 Q$ and $l_2 = C_2 Q$; thus $l_1 \neq AB$ and $l_2 \neq AB$. Since $A \neq C_1 = AB \cdot l_1$, it follows that $A \notin l_1$, so $AR \neq l_1$; similarly, $AR \neq l_2$. Since $C_1 \neq C_2 = AB \cdot l_2$, we have $C_1 \notin l_2$, so $l_1 \neq l_2$. Since $R \neq Q = AR \cdot l_1$, it follows that $R \notin l_1$; similarly, $R \notin l_2$. Thus l_1, R and l_2, R may be used in Definition 4.1 to construct the harmonic conjugates D_1 and D_2.

Clearly, $Q_1 = Q_2 = Q$. Since $P_1 \neq Q = l_1 \cdot l_2$, it follows that $P_1 \notin l_2$, and thus $P_1 \neq P_2$. Since $P_2 \neq P_1 = AP_1 \cdot BR$, we have $P_2 \notin AP_1$, so $AP_1 \neq AP_2$. Since $S_1 \neq A = AP_1 \cdot AP_2$, it follows that $S_1 \notin AP_2$, and thus $S_1 \neq S_2$. Since $R \neq Q = AR \cdot BQ$, we have $R \notin BQ$, so $BQ \neq RS_2$. Since $S_1 \neq S_2 = BQ \cdot RS_2$, it follows that $S_1 \notin RS_2$, and thus $RS_1 \neq RS_2$. Since $D_1 \neq R = RS_1 \cdot RS_2$, we have $D_1 \notin RS_2$, and hence, finally, $D_1 \neq D_2$.

Case 2; $C_1 \neq A$ and $C_2 \neq B$. By Lemma 4.5 it follows that $D_1 \neq A$. From cotransitivity it follows that either $D_2 \neq D_1$ or $D_2 \neq A$, so we may assume that $D_2 \neq A$. From Lemmas 4.5 and 4.9(b), we have $C_2 \neq A$, and now Case 1 applies. $\square$

5 Projectivities

We use the Poncelet [Pon22] definition of *projectivity.* Theorem 5.3 will show that every Poncelet projectivity is a von Staudt [Sta47] projectivity.

Definition 5.1.
• The *projection* $\rho : \bar{l} \to \overline{m}$, of a range $\bar{l}$ of points onto a range $\overline{m}$, from the *center* T, where the point T lies outside both lines l and m, is the bijection defined by $X^\rho = TX \cdot m$, for all points X in the range $\bar{l}$. We write $\rho = \rho(T; l, m)$.
• The *projection* $\rho : P^* \to Q^*$, of a pencil P^* of lines onto a pencil Q^*, by the *axis* n, where the line n avoids both points P and Q, is the bijection defined by $l^\rho = (n \cdot l)Q$, for all lines l in the pencil P^*. We write $\rho = \rho(n; P, Q)$.
• The *section* of a pencil P^*, by a line m that avoids the point P, is the bijection $\rho : P^* \to \overline{m}$ defined by $l^\rho = l \cdot m$, for all lines l in the pencil P^*. The dual and inverse $\rho^{-1} : \overline{m} \to P^*$, defined by $X \to PX$, for all points X in the range $\overline{m}$, is also called a *section.*
• Any projection or section is said to be a *perspectivity.*

Definition 5.2.
• A *projectivity* is a finite product of perspectivities. These mappings are often called *Poncelet projectivities.*
• When, for example, a projectivity π maps the points A, B, C into the points D, E, F, in the order written, we write $ABC\langle\pi\rangle DEF$.
• We write $\pi \neq \iota$, where ι is the identity, when, for example, there exists a point A in the range such that $A^\pi \neq A$.

Theorem 5.3. *A projectivity preserves harmonic conjugates. Thus every Poncelet projectivity is a von Staudt projectivity.*

Proof. It will suffice to prove that a harmonic set of points in a range $\overline{r}$ projects onto a harmonic set of lines in a pencil P^*, where $P \notin r$. Given points A, B, C, D on r, with $D = h(A, B; C)$, and $a = PA$, $b = PB$, $c = PC$, $d = PD$, it is required to show that $d = h(a, b; c)$. Since $A \neq B$, it follows from Proposition 2.4 that $a \neq b$. By cotransitivity, it suffices to consider the case in which $C \neq A$.

Suppose that $h(a, b; c) \neq d$, and suppose further that $C \neq B$. Thus $c \neq b$. Setting $l = c$, we have $l \neq AB$. Select a point $R \in b$ so that $R \neq B$ and $R \neq P$. Since $R \neq B = AB \cdot b$, it follows from Axiom C7 that $R \notin AB$. Since $R \neq P = b \cdot c$, we have $R \notin l$. Thus the elements l, R may be used in Definition 4.1 to construct the harmonic conjugate $h(A, B; C)$; it follows that $D = AB \cdot RS$, where $S = AP \cdot BQ$, $P = BR \cdot l$, and $Q = AR \cdot l$.

To construct the harmonic conjugate $h(a, b; c)$, we first set $L = Q$; then $L \in c$. By Lemma 4.2, $L \neq P$, and by Lemma 4.5, $L \notin r$. Thus we may use the elements L, r to construct the line $h(a, b; c)$ using Definition 4.1(dual). It follows that $h(a, b; c) = (a \cdot b)(r \cdot s)$, where $s = (a \cdot p)(b \cdot q)$, $p = (b \cdot r)L$, and $q = (a \cdot r)L$. Now $s = (AP \cdot BQ)(BP \cdot AQ) = SR$, and thus $h(a, b; c) = P(AB \cdot RS) = PD = d$, contradicting the first assumption above, and negating the second; hence $C = B$.

It follows that $c = b$. By Lemma 4.4 we have $D = B$, so $d = b$. By the dual of the same lemma, $h(a, b; c) = b$; thus $h(a, b; c) = d$, contradicting the first assumption above. Hence $h(a, b; c) = d$. □

The existence of projectivities between ranges will be established in Theorem 5.6 for the general situation where it is not known, constructively, whether or not the two ranges coincide, or, if distinct, whether the common point coincides with one of the points specified to be mapped. We first consider two lemmas concerning special situations in which some of this information is available.

Lemma 5.4. *Let l and m be distinct lines, with common point A. If Q, R are distinct points on l, and Q', R' are distinct points on m, with all four points distinct from A, then there exists a projection $\rho : \overline{l} \to \overline{m}$ such that $AQR\langle\rho\rangle AQ'R'$.*

Proof. Since $Q' \neq A = l \cdot m$, it follows from Axiom C7 that $Q' \notin l$, so $Q' \neq Q$, and $QQ' \neq l$; similarly, $RR' \neq l$. Since $R \neq Q = QQ' \cdot l$, it follows that $R \notin QQ'$, so $RR' \neq QQ'$. Set $S = QQ' \cdot RR'$; then $R \neq S$. Now $S \neq R = RR' \cdot l$, so $S \notin l$, and by symmetry, $S \notin m$. Thus we may define $\rho = \rho(S; l, m)$; it is clear that $AQR\langle\rho\rangle AQ'R'$. □

Lemma 5.5. *Let l and m be distinct lines, with common point O. If P, Q, R are distinct points on l, and P', Q', R' are distinct points on m, with all six points distinct from O, then there exists a projectivity $\pi : \overline{l} \to \overline{m}$ such that $PQR\langle\pi\rangle P'Q'R'$.*

Proof. Since $Q \neq O = l \cdot m$, it follows that $Q \notin m$; thus $Q \neq P'$, and similarly for all six points. Set $n = P'Q$; thus $l \cdot n = Q$. Since $Q \neq R = RR' \cdot l$, it follows from Axiom C7 that $Q \notin RR'$, so $RR' \neq n$. Thus we may define $R_0 = RR' \cdot n$. Then $Q \neq R_0$, and by symmetry, $P' \neq R_0$. By Lemma 5.4, there exists a projection $\rho_1 : \overline{l} \to \overline{n}$ such that $PQR\langle\rho_1\rangle P'QR_0$. Also, $n \cdot m = P'$, so by the same lemma there exists a projection $\rho_2 : \overline{n} \to \overline{m}$ such that $P'QR_0\langle\rho_2\rangle P'Q'R'$. Setting $\pi = \rho_2\rho_1$, we obtain $PQR\langle\pi\rangle P'Q'R'$. □

For a constructive proof of Theorem 5.6, and also for Lemma 8.6, which will be needed for Pascal's Theorem, we require more points on a line than has been assumed in Axiom C4. Thus we adopt an additional axiom here. The determination of the exact number of required points remains an open problem.

Axiom E. Extension. *There exist at least six distinct points lying on any given line.*

The dual statement is easily verified.

Theorem 5.6. *Given any three distinct points P, Q, R in a range $\overline{l}$, and any three distinct points P', Q', R' in a range $\overline{m}$, there exists a projectivity $\pi : \overline{l} \to \overline{m}$ such that $PQR\langle\pi\rangle P'Q'R'$. Similar projectivities exist for other pairs of ranges or pencils.*

Proof. Select a point $O_1 \in l$, distinct from the three given points in the range $\overline{l}$, and a line l' through O_1, distinct from l. Select a point $O_2 \in m$, distinct from the three given points in $\overline{m}$, and a line m' through O_2, distinct from both m and l'. Set $O_3 = l' \cdot m'$, select distinct points $P_1, Q_1, R_1 \in l'$ distinct from both O_1 and O_3, and select distinct points $P_2, Q_2, R_2 \in m'$ distinct from both O_2 and O_3. Now Lemma 5.5 constructs projectivities such that $PQR \to P_1Q_1R_1 \to P_2Q_2R_2 \to P'Q'R'$. □

Classically, at most three perspectivities are needed for Theorem 5.6. The constructive proof here uses three applications of Lemma 5.5, each of which requires two projections, for a total of six. The determination of the minimum number of perspectivities for a constructive proof is an open problem.

Definition 5.7. A projectivity π, between two ranges or two pencils, is said to be *nonperspective* if $x^\pi \neq x$ for every element x in the domain.

The following lemma will be needed for Corollary 6.3, and at several places in the study of conics, which will be constructed using nonperspective projectivities.

Lemma 5.8. *Let l and m be distinct lines with common point O, let A, B, C be distinct points on l, let A', B', C' be distinct points on m, with all six points distinct from O, and define*

$$R = AA' \cdot BB', \quad S = BB' \cdot CC', \quad n = A'C,$$

$$\rho_1 = \rho(R; l, n), \quad \rho_2 = \rho(S; n, m), \quad \pi = \rho_2\rho_1.$$

Then $ABC\langle\pi\rangle A'B'C'$, and the following conditions are equivalent:

(a) *The projectivity $\pi : \overline{l} \to \overline{m}$ is nonperspective.*

(b) *$O^\pi \neq O$.*

(c) *The lines AA', BB', CC' are nonconcurrent.*

(d) *$R \neq S$.*

Proof. Except for a change in notation, π is the projectivity constructed in Lemma 5.5; thus the definitions are valid, and $ABC\langle\pi\rangle A'B'C'$. That (a) implies (b) requires no proof. The equivalence of (c) and (d) is the dual of Proposition 2.4. Setting $O_1 = O^{\rho_1}$, we have $O^\pi = O_1^{\rho_2} = SO_1 \cdot m$.

(b) implies (a). Given that $O^\pi \neq O$, consider any point X in the range $\overline{l}$. By cotransitivity, either $X^\pi \neq O$ or $X^\pi \neq O^\pi$. In the first case, $X^\pi \neq l \cdot m$; by Axiom C7 it follows that $X^\pi \notin l$, and $X^\pi \neq X$. In the second case, $X \neq O = l \cdot m$, so $X \notin m$, and $X \neq X^\pi$.

(b) implies (d). Since $O^\pi \neq O = RO \cdot m$, it follows that $O^\pi \notin RO$, so $SO_1 \neq RO = RO_1$. Since $R \neq O_1 = RO_1 \cdot SO_1$, we have $R \notin SO_1$, and $R \neq S$.

(d) implies (b). Given that $R \neq S$, with $R, S \in BB'$, we have $BB' = RS$; thus $B \in RS$. Since $B \neq O = RO \cdot l$, it follows that $B \notin RO = RO_1$, so $RS \neq RO_1$. Since $S \neq R = RS \cdot RO_1$, we have $S \notin RO_1$; thus $SO_1 \neq RO_1$. Since $A \neq O$, we have $A' = A^{\rho_1} \neq O^{\rho_1} = O_1$. Since $O_1 \neq A' = m \cdot n$, it follows that $O_1 \notin m$, and thus $O_1 \neq O^\pi$. Since $O^\pi \neq O_1 = RO_1 \cdot SO_1$, we have $O^\pi \notin RO_1$, and hence $O^\pi \neq O$. □

6 The Fundamental Theorem

The *Fundamental Theorem of Projective Geometry* is the basis for many results, including Pascal's Theorem, the goal of the present work. The crucial component must be derived from an axiom.

Axiom T. *If a projectivity of a range or pencil onto itself has three distinct fixed elements, it is the identity.*

Classically, Axiom T has the equivalent form, *If a projectivity π from a range onto itself has distinct fixed points M and N, with $\pi \neq \iota$, and Q is a point of the range distinct from both M and N, then $Q^\pi \neq Q$.* Constructively, this appears to be a stronger statement, since the implication "$\neg(Q^\pi = Q)$ implies $Q^\pi \neq Q$" is constructively invalid. To give a proof of the stronger statement, or to give a Brouwerian counterexample using the analytic model $\mathbb{P}^2(\mathbb{R})$, remains an open problem.

Theorem 6.1. Fundamental Theorem. *Given any three distinct points P, Q, R in a range $\overline{l}$, and any three distinct points P', Q', R' in a range $\overline{m}$, there exists a unique projectivity $\pi : \overline{l} \to \overline{m}$ such that $PQR\langle\pi\rangle P'Q'R'$. Similar properties hold also for other types of projectivity.*

Proof. The required projectivity was constructed in Theorem 5.6; uniqueness follows from Axiom T. □

Corollary 6.2. *If a projectivity from a range to a distinct range has a fixed point, then it is a perspectivity.*

Proof. If A is a fixed point of the projectivity π, then it is the point common to the two ranges. Choose distinct points Q, R in the domain, distinct from A, denoting the images Q', R'; thus $AQR\langle\pi\rangle AQ'R'$. Use Lemma 5.4 to construct a projection ρ that agrees with π at these three distinct points; it follows that $\pi = \rho$. □

Corollary 6.3. *Let l and m be distinct lines with common point O, let A, B, C be distinct points on l, let A', B', C' be distinct points on m, with all six points distinct from O, and let π be the projectivity from $\overline{l}$ to $\overline{m}$ such that $ABC\langle\pi\rangle A'B'C'$. Then π is nonperspective if and only if the lines AA', BB', CC' are nonconcurrent.*

Proof. Lemma 5.8 constructs a projectivity that agrees with π at three distinct points. □

Definition 6.4. Let $\pi : \overline{l} \to \overline{m}$ be a nonperspective projectivity between distinct ranges $\overline{l}$ and $\overline{m}$. Set $O = l \cdot m$, $V = O^\pi$, and $U = O^{\pi^{-1}}$. Pending verification below, the line UV will be called the *axis of homology for π*.

Theorem 6.5. *Let l and m be distinct lines with common point O, and let $\pi : \overline{l} \to \overline{m}$ be a nonperspective projectivity. If A and B are distinct points on l, each distinct from O, then $A \neq B^\pi$, $B \neq A^\pi$, $AB^\pi \neq BA^\pi$, and the point $AB^\pi \cdot BA^\pi$ lies on the axis of homology for π.*

Proof. We use the notation of Definition 6.4. Since π is nonperspective, we have $V \neq O = l \cdot m$, so it follows from Axiom C7 that $V \notin l$ and $V \neq U$. Thus the definition of the axis of homology UV is valid. Since $A \neq O = l \cdot m$, we have $A \notin m$, so $A \neq B^\pi$; similarly, $B \neq A^\pi$. By cotransitivity, either $A \neq U$ or $B \neq U$. In the first case, we have $A^\pi \neq O = l \cdot m$, so $A^\pi \notin l$, $OA^\pi = m$, and $BA^\pi \neq l$. Since $A \neq B = BA^\pi \cdot l$, it follows that $A \notin BA^\pi$, so $AB^\pi \neq BA^\pi$. By symmetry, we obtain the same result in the second case. Thus we may set $Q = AB^\pi \cdot BA^\pi$.

Since $A^\pi \notin l$, we have a projection $\rho_1 : \overline{l} \to A^{\pi *}$. Since $A \neq O$, it follows that $A \notin m$, so we also have the section $\rho_2 : A^* \to \overline{m}$. Define $\pi_1 = \rho_1 \pi^{-1} \rho_2$, a projectivity from A^* to $A^{\pi *}$. Since AA^π is a fixed line, Corollary 6.2(dual) shows that π_1 is a perspectivity.

Now $(AU)^{\pi_1} = A^\pi U$, $(AV)^{\pi_1} = m$, and $(AB^\pi)^{\pi_1} = A^\pi B$. Thus the axis of the perspectivity π_1 passes through all three of the points $AU \cdot A^\pi U = U$, $AV \cdot m = V$, and Q. Hence Q lies on UV. □

Corollary 6.6. *Let l and m be distinct lines with common point O, let $\pi : \overline{l} \to \overline{m}$ be a nonperspective projectivity, let h be the axis of homology for π, and let A be a point on l with $A \neq O$ and $A^\pi \neq O$. If B is a point on l with $B \neq O$ and $B \neq A$, then $B^\pi = A(BA^\pi \cdot h) \cdot m$.*

Proof. We have $h = UV$, where $V = O^\pi$ and $U = O^{\pi^{-1}}$, according to Definition 6.4. Since π is nonperspective, $O^\pi \neq O$. Applying π^{-1}, we have $O \neq O^{\pi^{-1}} = U$. Since $U \neq O = l \cdot m$, it follows from Axiom C7 that $U \notin m$, and thus $m \neq h$. Similarly, from $V \neq O$ we obtain $l \neq h$. From $A^\pi \neq O$, applying π^{-1}, we have $A \neq U = l \cdot h$; thus $A \notin h$. From $A \neq O$, it follows that $A^\pi \neq O^\pi = V = m \cdot h$; thus $A^\pi \notin h$, so $BA^\pi \neq h$. Now we may set $E = BA^\pi \cdot h$; since $A \notin h$, we have $A \neq E$. By the theorem, $E \in AB^\pi$, so $B^\pi \in AE$. Thus $B^\pi = AE \cdot m = A(BA^\pi \cdot h) \cdot m$. □

Notes for Theorem 6.5 and Corollary 6.6. The restrictions on the points, or some such, are necessary. For example, if $A = O$ and $B = U$, then the expression AB^π in the theorem takes the meaningless form OO. In the corollary, if the conditions are all satisfied except that $A^\pi = O$, then the expression $A(BA^\pi \cdot h)$ reduces to UU.

The concept of projectivity may be extended to the entire plane. A *collineation* of the plane $\mathbb{P}$ is a bijection of the family $\mathscr{P}$ of points, onto itself, that preserves collinearity and noncollinearity. A collineation σ induces an analogous bijection σ' of the family $\mathscr{L}$ of lines. A collineation is *projective* if it induces a projectivity on every range and pencil.

Proposition 6.7. *A projective collineation with four distinct fixed points, each three of which are noncollinear, is the identity.*

Proof. Let the collineation σ have the fixed points P, Q, R, S as specified; thus the three distinct lines PQ, PR, PS are fixed. The mapping σ' induces a projectivity on the pencil P^*; by the *Fundamental Theorem* this projectivity is the identity. Thus every line through P is fixed under σ'; similarly, the same is true for the other three points.

Now let X be any point on the plane $\mathbb{P}$. By three successive applications of cotransitivity for points, we may assume that X is distinct from each of the points P, Q, R. Since $PQ \neq PR$ by Proposition 2.4, using cotransitivity for lines we may assume that $XP \neq PQ$. Since $Q \neq P = XP \cdot PQ$, it follows from Axiom C7 that $Q \notin XP$; thus $XP \neq XQ$. Since $X = XP \cdot XQ$, and the lines XP and XQ are fixed under σ', it follows that $\sigma X = X$. □

The construction of a collineation of the plane mapping any set of four distinct points, each three of which are noncollinear, onto any similar set of corresponding points, using a constructive synthetic theory, is an open problem.

7 Involutions

The property of the harmonic conjugate construction, that the process applied again to the resulting point produces the original point, Lemma 4.9(b), admits a generalization.

Definition 7.1. An *involution* is a projectivity, from a range or pencil to itself, of order 2.

Theorem 7.2. *Let A and B be distinct points in a range $\bar{l}$, and let υ be the mapping of harmonic conjugacy with respect to the base points A, B; i.e., set $X^\upsilon = h(A, B; X)$, for all points X in the range $\bar{l}$. Then υ is an involution.*

Proof. Theorem 4.10 shows that υ is a bijection of the range $\bar{l}$ onto itself, of order 2.

To show that υ is a projectivity, we use Definition 4.1 and the notation adopted there. Select a point R outside AB, and select a point P on BR distinct from both B and R. Construct the perspectivities $\rho_1(P; AB, AR)$, $\rho_2(B; AR, AP)$, $\rho_3(R; AP, AB)$, and the projectivity $\pi_B = \rho_3\rho_2\rho_1$.

Let X be any point in the range $\bar{l}$, with $X \neq B$. Since $P \neq B = BR \cdot AB$, it follows from Axiom C7 that $P \notin AB$, so $PX \neq AB$; set $l_X = PX$. Since $B \neq X = AB \cdot PX$, we have $B \notin PX$; thus $BR \neq PX$. Now $R \neq P = BR \cdot PX$, so $R \notin l_X$. Thus, to construct the harmonic conjugate X^υ, we may use the point R and the line l_X in the definition. We find that $X^{\pi_B} = R(B(PX \cdot AR) \cdot AP) \cdot AB = R(B(l_X \cdot AR) \cdot AP) \cdot AB = R(BQ_X \cdot AP) \cdot AB = RS_X \cdot AB = X^\upsilon$. Thus π_B agrees with υ for all points in the range $\bar{l}$ that are distinct from B.

Similarly, construct the projectivity π_A; it will agree with υ for all points in $\bar{l}$ that are distinct from A. Choose any three distinct points on l, each distinct from both A and B. Since π_A and π_B agree at these three points, by the *Fundamental Theorem* they are the same projectivity; call it π. By cotransitivity, each point in the range $\bar{l}$ is either distinct from A, or distinct from B. Hence $\upsilon = \pi$. □

Lemma 7.3. *Given any four distinct points A, B, C, D in a range $\bar{l}$, there exists a projectivity π from $\bar{l}$ to itself such that $ABCD\langle\pi\rangle BADC$.*

Proof. Select a line m through D, distinct from l, and select a point Q outside both lines l and m. Set $\rho_1 = \rho(Q; l, m)$. Since $A \neq D = l \cdot m$, it follows from Axiom C7 that $A \notin m$, and thus $AQ \neq m$. Similarly, both B and C lie outside m, and BQ and CQ are both distinct from m. Since $A \neq C = CQ \cdot l$, we have $A \notin CQ$. Similarly, $B \notin CQ$, so $BQ \neq CQ$. Set $\rho_2 = \rho(A; m, CQ)$, $R = AQ \cdot m$, $S = BQ \cdot m$, and $T = CQ \cdot m$. Since $S \neq Q = BQ \cdot CQ$, it follows that $S \notin CQ$ and $AS \neq CQ$; set $U = AS \cdot CQ$. Since $D \neq B = BQ \cdot l$, we have $D \notin BQ$, so $D \neq S$. Since $S \neq D = l \cdot m$, we have $S \notin l$, so $S \neq A$. Finally, set $\rho_3 = \rho(S; CQ, l)$. It is clear that $ABCD\langle\rho_1\rangle RSTD\langle\rho_2\rangle QUTC\langle\rho_3\rangle BADC$; set $\pi = \rho_3\rho_2\rho_1$. □

Theorem 7.4. *A projectivity from a range to itself, that interchanges two distinct elements, is an involution.*

Proof. Let $AB\langle\pi\rangle BA$, where $A \neq B$. Let X be any point in the range, and set $Y = X^\pi$. By symmetry and cotransitivity, it suffices to consider the case in which $X \neq A$. Suppose that $Y^\pi \neq X$, and suppose further that $Y \neq X$ and $X \neq B$.

Applying π, it follows that $Y \neq A$ and $Y \neq B$. Using Lemma 7.3, construct a projectivity π_1 with $ABXY\langle\pi_1\rangle BAYX$. Since the projectivity π agrees with π_1 at three distinct points, by the *Fundamental Theorem* we have $\pi = \pi_1$, so $Y^\pi = X$. This contradicts the first assumption above, and negates the last.

Thus $X = B$; applying π here, we have $Y = A$. Thus $Y^\pi = B = X$, contradicting the first assumption and negating the second. Thus $Y = X$, and it follows that $Y^\pi = X^\pi = Y = X$, contradicting and negating the first assumption. Hence $Y^\pi = X$. □

Theorem 7.5. *If an involution π on a range $\bar{l}$ has a fixed point M, then it has a second, distinct, fixed point N, and π is the mapping of harmonic conjugacy with respect to these points; thus $X^\pi = h(M, N; X)$, for all points X in $\bar{l}$.*

Proof. Select a point A in $\bar{l}$ so that $A^\pi \neq A$. Either $M \neq A$ or $M \neq A^\pi$; in either case, applying π we find that M is distinct from both A and A^π. Set $N = h(A, A^\pi; M)$; by Lemma 4.6, $N \neq M$, and by Lemma 4.9(a), we have $N = h(A^\pi, A; M)$. Applying the projectivity π, we have $N^\pi = h(A, A^\pi; M) = N$. Thus N is a second fixed point.

If an alternative selection of the point A results in the second fixed point $N_1 \neq M$, then, by the *Fundamental Theorem,* $N_1 = N$, since $N_1 \neq N$ would mean that we have three distinct fixed points. Thus, for any point X in the range $\bar{l}$, if $X^\pi \neq X$, then $h(X, X^\pi; M, N)$.

Now consider any point X in the range $\bar{l}$. By cotransitivity, we may assume that $X \neq M$. Suppose that $X^\pi \neq h(M, N; X)$, and suppose further that $X \neq N$ and $X \neq X^\pi$. We have $h(X, X^\pi; M, N)$ from above. By Theorem 5.3 and Lemma 7.3, it follows that $h(M, N; X, X^\pi)$, contradicting the first assumption, and negating the last; thus $X = X^\pi$. Now we have three distinct fixed points; this contradiction negates the second assumption. Thus $X = N$, a contradiction negating the first assumption. Hence $X^\pi = h(M, N; X)$. □

8 Conics

We define conics by means of projectivities, using the method of Steiner [Ste32].

Definition 8.1. Let $\pi : U^* \to V^*$ be a nonperspective projectivity between distinct pencils of lines. The *conic* $\kappa = \kappa(\pi; U, V)$ *defined by* π is the locus of points $\{l \cdot l^\pi : l \in U^*\}$.[13] For any point X, we say that X *lies outside* κ, written $X \notin \kappa$, if $X \neq Y$ for all points Y on κ. At times, the locus κ may be called a *point-conic*; the dual locus is a *line-conic*.

Proposition 8.2. *Let $\kappa = \kappa(\pi; U, V)$ be a conic.*

(a) The base points U and V are points of κ.

(b) Any three distinct points on κ are noncollinear.

(c) For any point P on κ, the line $l \in U^*$, such that $P = l \cdot l^\pi$, is unique. If $P \neq U$, then $l = UP$, while if $P \neq V$, then $l = (VP)^{\pi^{-1}}$.

(d) For any point X, if $\neg(X \notin \kappa)$, then $X \in \kappa$.

Proof. (a) The line $o = UV$ in the pencil U^* has a corresponding line o^π in V^*; this pair of lines determines the point $o \cdot o^\pi = V$ of κ. Similarly for U.

(b) This follows from Corollary 6.3(dual).

(c) By cotransitivity, either $P \neq U$ or $P \neq V$. In the first case, both U and P lie on l, so $l = UP$. Similarly, in the second case we have $l^\pi = VP$.

(d) Let X be a point on the plane such that $\neg(X \notin \kappa)$. By cotransitivity, we may assume that $X \neq U$; set $x = UX$ and $Z = x \cdot x^\pi$. Suppose that $X \neq Z$. We will show that $X \neq Y$ for any point Y of κ. Either $Y \neq X$ or $Y \neq U$. We need to consider only the second case; set $y = UY$. By (c), we have $Y = y \cdot y^\pi$. Either $Y \neq X$ or $Y \neq Z$; again we need to

[13] With π being nonperspective, this is usually called a *non-singular conic*; we leave the constructive study of the singular conics for a later time.

consider only the second case. Since $Y \neq x \cdot x^\pi$, it follows from Axiom C7 that either $Y \notin x$ or $Y \notin x^\pi$. In the first subcase, $y \neq x$. In the second subcase, $y^\pi \neq x^\pi$, and since π is a bijection we again have $y \neq x$. Since $X \neq U = x \cdot y$, it follows that $X \notin y$, and $X \neq Y$. This shows that $X \notin \kappa$, a contradiction. Hence $X = Z$, and now we have $X \in \kappa$. □

Proposition 8.3. *Given any five distinct points U,V,A,B,C, each three of which are noncollinear, there exists a unique conic, with base points U and V, containing all five points.*

Proof. This follows from Corollary 6.3(dual). □

The following three lemmas are required for Pascal's Theorem. The first is a special case; two vertices of the hexagon are the base points of a projectivity that defines the conic.

Lemma 8.4. *Let $\kappa = \kappa(\pi; U, V)$ be any conic, and A, B, C, X points of κ, with all six points distinct. Then the three points $O = UA \cdot VC$, $Y = UX \cdot BC$, $Z = VX \cdot AB$ are distinct and collinear.*

Proof. That the three points in question are properly defined follows from Proposition 8.2(b). Set $D = UA \cdot BC$ and $E = VC \cdot AB$; thus $D \neq A$, so $DA = UA$, and $E \neq C$, so $CE = VC$. Consider the sections $\rho_1 : U^* \to \overline{BC}$ and $\rho_2 : V^* \to \overline{AB}$; thus $UA, UB, UC, UX\langle\rho_1\rangle DBCY$ and $VA, VB, VC, VX\langle\rho_2\rangle ABEZ$. Setting $\pi_1 = \rho_2\pi\rho_1^{-1}$, we obtain a projectivity $\pi_1 : \overline{BC} \to \overline{AB}$, with $DBCY\langle\pi_1\rangle ABEZ$. Since B is a fixed point, it follows from Corollary 6.2 that π_1 is a projection; the center is $DA \cdot CE = UA \cdot VC = O$. The center of a projection lies outside each of the ranges it maps; thus the three points are distinct. Also, $Z = Y^{\pi_1} = OY \cdot AB$, and hence $Z \in OY$. □

Lemma 8.5. *Let U, A, B, C, V, X be distinct points, each three of which are noncollinear. If the three points $O = UA \cdot VC$, $Y = UX \cdot BC$, $Z = VX \cdot AB$ are collinear, then there exists a nonperspective projectivity $\pi : U^* \to V^*$ such that all six points are on the conic $\kappa(\pi; U, V)$.*

Proof. Set $a = UA$, $b = UB$, $c = UC$, $x = UX$, $a' = VA$, $b' = VB$, $c' = VC$, $x' = VX$, $l = BC$, $m = AB$, $D = a \cdot l$, and $E = c' \cdot m$; thus $O = a \cdot c'$, $Y = x \cdot l$, and $Z = x' \cdot m$.

Since $C \notin a$, we have $C \neq O$. Since $O \neq C = c' \cdot l$, it follows from Axiom C7 that $O \notin l$. Similarly, $O \notin m$, so we may construct the projection $\rho = \rho(O; l, m)$. We have $D^\rho = OD \cdot m = a \cdot m = A$, and $C^\rho = OC \cdot m = c' \cdot m = E$. By hypothesis, $Z \in OY$, so $Y^\rho = OY \cdot m = Z$. Thus $DBCY\langle\rho\rangle ABEZ$. We also have sections of the pencils, $\rho_1 : U^* \to \overline{l}$ and $\rho_2 : V^* \to \overline{m}$, such that $abcx\langle\rho_1\rangle DBCY$ and $a'b'c'x'\langle\rho_2\rangle ABEZ$. Setting $\pi = \rho_2^{-1}\rho\rho_1$, we obtain a projectivity $\pi : U^* \to V^*$, with $abcx\langle\pi\rangle a'b'c'x'$. Since the points $a \cdot a'$, $b \cdot b'$, $c \cdot c'$ are noncollinear, it follows from Corollary 6.3(dual) that π is nonperspective. The conic $\kappa(\pi; U, V)$ clearly includes all six points. □

Lemma 8.6. (Steiner) *Given any conic $\kappa = \kappa(\pi; U, V)$, and any distinct points U_1, V_1 on κ, there exists a unique nonperspective projectivity $\pi_1 : U_1^* \to V_1^*$ such that $\kappa = \kappa(\pi_1; U_1, V_1)$.*

Proof. (a) *Special case; $U_1 = U$ and $V_1 = W$, where W is any point of κ that is distinct from both U and V.* Select points A, B on κ such that the points U, A, W, B, V are distinct, and let X be any point of κ distinct from these five. Applying Lemma 8.4 to the points U, A, W, B, V, X, we obtain three collinear points of interest. A cyclic permutation of these six points results in W, B, V, X, U, A, with the same three points of interest. Thus Lemma 8.5 applies, and we obtain a conic $\kappa_1 = \kappa(\pi_1; U, W)$ containing all six points. By Proposition 8.3, this conic, with base points U, W, is independent of the choice of X.

Let Y be any point of κ, and suppose that $Y \notin \kappa_1$. Now Y is distinct from each of the points U, V, W, A, B, so the construction of κ_1 may be repeated with Y in place of X; thus Y is on κ_1, a contradiction. From Proposition 8.2(d), it follows that $Y \in \kappa_1$; thus $\kappa \subset \kappa_1$.

Applying the same construction method, now to κ_1, we obtain a conic $\kappa_2 = \kappa(\pi_2; U, V)$, with $\kappa_1 \subset \kappa_2$. By Proposition 8.3, $\kappa = \kappa_2$, and hence $\kappa = \kappa_1$.

(b) *General case.* Using Axiom E, select distinct points W_1, W_2 on κ, each distinct from all four points U, V, U_1, V_1. Four applications of special case (a) result in the sequence (U, V), (U, W_2), (W_1, W_2), (U_1, W_2), (U_1, V_1) of base-point changes.

(c) The uniqueness follows from the *Fundamental Theorem*. □

Theorem 8.7. *There exists a unique conic containing any given five distinct points, each three of which are noncollinear.*

Proof. The existence follows from Theorem 5.6 and Corollary 6.3(dual), the uniqueness from Proposition 8.3 and Lemma 8.6. □

9 Pascal's Theorem

For information concerning Blaise Pascal (1623-62), see [Kli72, p. 295-299].

Definition 9.1. A *simple hexagon* is a set $ABCDEF$ of six distinct points, in cyclic order, each three of which are noncollinear. The six points are the *vertices*; the six lines joining successive points are the *sides*. The pairs of sides (AB, DE), (BC, EF), (CD, FA) are said to be *opposite*.

Theorem 9.2. (Pascal) *If a simple hexagon is inscribed in a conic, the three points of intersection of the pairs of opposite sides are distinct and collinear.*

Proof. Label the inscribed hexagon as $UABCVX$, and apply Lemma 8.6 to view the conic as $\kappa = \kappa(\pi; U, V)$. Now Lemma 8.4 yields the result. □

According to legend, Pascal gave in addition some four hundred corollaries. Here we have only one; it recalls a traditional construction method for drawing a conic "point by point".[14]

Corollary 9.3. *Let A, B, C, D, E be five distinct points of a conic κ, and let l be a line through E that avoids A, B, C, D. If l passes through a distinct sixth point F of κ, then $F = l \cdot A(CD \cdot (AB \cdot DE)(BC \cdot l))$.*

Proof. The *Pascal line* p of the hexagon $ABCDEF$ passes through the three points $X = AB \cdot DE$, $Y = BC \cdot EF$, and $Z = CD \cdot AF$. Since $A \notin CD$, we have $A \neq Z$, so $AF = AZ$. Since $B \notin CD$, we have $BC \neq CD$, so by cotransitivity for lines either $p \neq BC$ or $p \neq CD$. In the first case, since $C \notin EF$, we have $C \neq Y = BC \cdot p$, so it follows from Axiom C7 that $C \notin p$. Thus in both cases we have $CD \neq p$, and $Z = CD \cdot p$. Now $F = EF \cdot AF = l \cdot AZ = l \cdot A(CD \cdot p) = l \cdot A(CD \cdot XY) = l \cdot A(CD \cdot (AB \cdot DE)(BC \cdot l)$. □

[14] For example, as in [You30, p. 68].

10 Tangents and secants

The construction of poles and polars with respect to a conic, in Section 11, will involve the properties of tangents and secants.

Definition 10.1. Let κ be a conic, and P a point on κ. A line t that passes through P is said to be *tangent to κ at P* if P is the unique point of κ that lies on t. The dual concept is a *point of contact* L of a line l that belongs to a line-conic λ.

Proposition 10.2. *Let κ be a conic, P a point on κ, and t a line passing through P. The following are equivalent.*

(a) The line t is tangent to κ at P.

(b) For any point Q of κ, if $Q \neq P$ and π is the nonperspective projectivity such that $\kappa = \kappa(\pi; Q, P)$, then $t = (QP)^\pi$.

(c) There exists a point Q of κ with $Q \neq P$, and corresponding nonperspective projectivity π, with $\kappa = \kappa(\pi; Q, P)$, such that $t = (QP)^\pi$.

Proof. Given (a), and a point Q as specified in (b), set $u = t^{\pi^{-1}}$. Since $u \cdot t$ is a point of κ which lies on t, it must be P; thus $u = QP$, so $t = u^\pi = (QP)^\pi$.

Given (c), and a point Q as specified, let R be a point on κ with $R \in t$. Suppose that $R \neq P$; thus $t = PR$. From Proposition 8.2(c), applied to $\pi^{-1} : P^* \to Q^*$, it follows that $R = t \cdot t^{\pi^{-1}} = t \cdot QP = P$, a contradiction; hence $R = P$, and this proves (a). □

Corollary 10.3. *Let κ be a conic, and P any point on κ.*

(a) There exists a unique line t that is tangent to κ at P.

(b) Let t be the tangent to κ at P. If Q is any point on κ, with $Q \neq P$, then $Q \notin t$.

Proof. (a) follows directly from Lemma 8.6 and Proposition 10.2. For (b), select any point Q as specified in (b) of the same proposition; thus $t = (QP)^\pi$. Since π is nonperspective, we have $t \neq QP$. Since $Q \neq P = t \cdot QP$, it follows from Axiom C7 that $Q \notin t$. □

Theorem 10.4. *Let κ be any conic, and let $UABCV$ be a pentagon inscribed in κ, with five distinct vertices. The point of intersection of the tangent u at U, with the side opposite, is collinear with the points of intersection of the other two pairs of nonadjacent sides. That is, the three points $O = UA \cdot VC$, $Z = UV \cdot AB$, $Y = u \cdot BC$ are collinear.*

Proof. Using Lemma 8.6, construct the projectivity π so that $\kappa = \kappa(\pi; U, V)$. That the three points in question are properly defined follows from Proposition 8.2(b) and Proposition 10.3(b). Set $D = UA \cdot BC$, and $E = VC \cdot AB$. Since $A \notin BC$, we have $A \neq D$, so $DA = UA$. Since $C \notin AB$, we have $C \neq E$, so $CE = VC$.

Construct the sections $\rho_1 : U^* \to \overline{BC}$ and $\rho_2 : V^* \to \overline{AB}$; clearly, $UA, UB, UC, u \langle \rho_1 \rangle DBCY$ and $VA, VB, VC, VU \langle \rho_2 \rangle ABEZ$. Setting $\pi_1 = \rho_2 \pi \rho_1^{-1}$, we obtain a projectivity $\pi_1 : \overline{BC} \to \overline{AB}$, with $DBCY \langle \pi_1 \rangle ABEZ$. Since B is a fixed point, it follows from Corollary 6.2 that π_1 is a projection; the center is $DA \cdot CE = UA \cdot VC = O$. Thus $Z = Y^{\pi_1} \in OY$. □

Theorem 10.5. *Let κ be any conic, and let $UABV$ be a quadrangle inscribed in κ. The point of intersection of the tangent at U with the side VB, the point of intersection of the tangent at V with the side UA, and the diagonal point lying on UV, are collinear.*

Proof. Let π be the projectivity such that $\kappa = \kappa(\pi; U, V)$, and denote the tangents at U and V by u and v. The three points in question are then $Y = u \cdot VB$, $O = v \cdot UA$, and $Z = UV \cdot AB$. Set $D = UA \cdot VB$, and $E = v \cdot AB$. Since $A \notin VB$, we have $A \neq D$, so $DA = UA$. Since $V \notin AB$, we have $V \neq E$, so $VE = v$.

Construct the sections $\rho_1 : U^* \to \overline{VB}$ and $\rho_2 : V^* \to \overline{AB}$; it is clear that $UA, UB, UV, u \langle \rho_1 \rangle DBVY$ and $VA, VB, v, VU \langle \rho_2 \rangle ABEZ$. Setting $\pi_1 = \rho_2 \pi \rho_1^{-1}$, we obtain a projectivity $\pi_1 : \overline{VB} \to \overline{AB}$, with $DBVY \langle \pi_1 \rangle ABEZ$. Since B is a fixed point, π_1 is a projection with center $DA \cdot VE = UA \cdot v = O$. Thus $Z = Y^{\pi_1} \in OY$. □

Theorem 10.6. *Let κ be any conic, and let $UABV$ be a quadrangle inscribed in κ. The point of intersection of the tangents at U and V, and the two diagonal points not lying on UV, are distinct and collinear.*

Proof. Construct the projectivity π so that $\kappa = \kappa(\pi; U, V)$. Denote the tangents at U and V by u and v; the three points in question are then $O = u \cdot v$, $D_1 = UA \cdot VB$, $D_2 = UB \cdot VA$. Set $E = u \cdot VB$ and $F = v \cdot UB$. Since $U \notin VB$, we have $U \neq E$, so $EU = u$; by symmetry, $FV = v$.

Construct the sections $\rho_1 : U^* \to \overline{VB}$ and $\rho_2 : V^* \to \overline{UB}$; clearly, $u, UB, UV, UA \langle \rho_1 \rangle EBVD_1$ and $VU, VB, v, VA \langle \rho_2 \rangle UBFD_2$. Setting $\pi_1 = \rho_2 \pi \rho_1^{-1}$, we obtain a projectivity $\pi_1 : \overline{VB} \to \overline{UB}$, with $EBVD_1 \langle \pi_1 \rangle UBFD_2$. Since B is a fixed point, π_1 is a projection; the center is $EU \cdot FV = u \cdot v = O$. The center of a projection lies outside each of the ranges it maps; thus the three points are distinct. Since $D_2 = D_1^{\pi_1} \in OD_1$, the points are collinear. □

Corollary 10.7. *Given any conic $\kappa = \kappa(\pi; U, V)$, the center of homology of the projectivity π is the intersection $u \cdot v$ of the tangents to κ at U and V.*

The following theorem is related to the existence of secants, to be constructed in Theorem 10.10.

Theorem 10.8. *The following two statements are equivalent:*

(a) The tangents at any three distinct points of a point-conic are nonconcurrent; the points of contact of any three distinct lines of a line-conic are noncollinear.

(b) The family of all tangents to a point-conic is a line-conic; the family of all points of contact of a line-conic is a point-conic.

Proof. Since (a) follows directly from (b) and Proposition 8.2(b), with its dual, it only remains to prove that (a) implies (b).

Given a point-conic $\kappa = \kappa(\pi; A, B)$, select a point $C \in \kappa$, distinct from both A and B, and let a, b, c be the tangents at these three points. Since $A \neq B$, by Proposition 10.3(b) we have $A \notin b$, and thus $a \neq b$; similarly for the other points and tangents. Set $E = a \cdot b$, $F = b \cdot c$, and $G = a \cdot c$. It follows from (a) that the points A, E, G are distinct, as are E, B, F. Thus we may construct the projectivity $\varphi : \overline{a} \to \overline{b}$ such that $AEG \langle \varphi \rangle EBF$. Since $E^\varphi \neq E$, it follows from Lemma 5.8 that φ is nonperspective, so the family of lines $\lambda = \lambda(\varphi; a, b) = \{QQ^\varphi : Q \in a\}$ is a line-conic; the axis of homology for φ is clearly $h = AB$.

(1) *If P is any point of the point-conic κ with $P \neq A, B, C$, then the tangent p to κ at P is a line of the line-conic λ.* To prove this, denote the diagonals of the quadrangle $ABCP$

by $D_1 = AC \cdot BP$, $D_2 = AB \cdot CP$, and $D_3 = AP \cdot BC$. Also, set $S = a \cdot p$ and $T = b \cdot p$. By Theorem 10.6, the points F, D_1, D_2 are distinct and collinear, as are the points S, D_1, D_2, the points G, D_1, D_2 and the points T, D_1, D_2.

Since $F \neq E = a \cdot b$, it follows from Axiom C7 that $F \notin a$, so $F \neq S$ and $SF = SD_2$. From (a), we have $G \neq E$, $F \neq E$, $S \neq E$, and $S \neq G$, so we may apply Corollary 6.6 and the axis of homology. Thus $S^\varphi = G(SF \cdot h) \cdot b = G(SD_2 \cdot AB) \cdot b = GD_2 \cdot b = T$, and $SS^\varphi = ST = (a \cdot p)(b \cdot p) = p$; hence $p \in \lambda$.

(2) *The tangents a, b, c of the point-conic κ are each lines of the line-conic λ.* For a and b, this follows from Proposition 8.2(a)(dual). For c, it suffices to note that $GF = c$.

(3) *Each tangent to the point-conic κ is a line of the line-conic λ.* Let P be any point of κ, with tangent p, and suppose that $p \notin \lambda$. Suppose further, in succession, that $P \neq A$, $P \neq B$, $P \neq C$. Now we have $p \in \lambda$ by (1), a contradiction. Thus $P = C$, so $p = c$, and $p \in \lambda$ by (2), a contradiction. Thus $P = B$. Continuing this way, we arrive at a final contradiction. Thus $\neg(p \notin \lambda)$, and it follows from Proposition 8.2(d)(dual) that $p \in \lambda$.

(4) *Each point of contact of λ is a point of κ.* Apply the dual of the method in (1) to the line-conic λ, using the second part of condition (a). The result is a point-conic $\kappa_1 = \kappa(\psi : A, B)$, where $aeg\langle\psi\rangle ebf$, and $e = AB$, $f = BC$, $g = AC$. It follows from the dual of (3) that every point of contact of λ is a point of κ_1. We have $a, AB, AC\langle\pi\rangle BA, b, BC$, and these six lines, in order, are identical to those just noted for ψ. Thus, by the *Fundamental Theorem*, $\psi = \pi$ and $\kappa_1 = \kappa$.

(5) *Each line of the line-conic λ is a tangent of the point-conic κ.* Let l be a line of λ, with point of contact L. By Definition 10.1, l is the unique line of λ passing through L, and by (4), $L \in \kappa$. Let t denote the tangent to κ at L. By (3), t is a line of λ; hence $t = l$. $\square$

A line that passes through two distinct points of a conic κ is a *secant of* κ.

Lemma 10.9. *Let κ be a conic, P a point on κ, and t the tangent to κ at P. If l is a line through P, and $l \neq t$, then l passes through a second point R of κ, distinct from P; thus l is a secant of κ.*

Proof. Select Q and π as in Proposition 10.2(c). Thus $\kappa = \kappa(\pi; Q, P)$ and $t = (QP)^\pi$, so $P \in t^{\pi^{-1}}$. Set $R = l \cdot l^{\pi^{-1}}$. Since $l \neq t$ and $P \neq Q = l^{\pi^{-1}} \cdot t^{\pi^{-1}}$, it follows from Axiom C7 that $P \notin l^{\pi^{-1}}$, and hence $P \neq R$. $\square$

For any conic, the following theorem will provide, through an arbitrary point of the plane, the one secant needed to construct polars in Lemma 11.1, and the two distinct secants needed for Corollary 11.4, relating polars to inscribed quadrangles. The need for this theorem contrasts with complex geometry, where every line meets every conic.

Theorem 10.10. *Let κ be a conic, and assume statement (a) of Theorem 10.8.*

(a) Through any given point of the plane, we may construct at least two distinct secants of κ.

(b) On any given line of the plane, we may construct at least two distinct points, through each of which pass two tangents of κ.

Proof. (a) Let P and κ be given, and select distinct points A, B, C on κ, with tangents a, b, c. By hypothesis, these tangents are nonconcurrent; thus the points $E = a \cdot b$ and $F = b \cdot c$ are distinct. Either $P \neq E$ or $P \neq F$; it suffices to consider the first case. Then, by Axiom C7, either $P \notin a$ or $P \notin b$. It suffices to consider the first subcase; thus $P \neq A$ and $PA \neq a$. Now it follows from Lemma 10.9 that PA is a secant.

Denote the second point of PA that lies on κ by R, and choose distinct points A', B', C' on κ, each distinct from both A and R. Using these three points, construct a secant through P with the above method; we may assume that it is PA'. By Proposition 8.2(b), $A' \notin AR = PA$; hence $PA' \neq PA$.

(b) This now follows from the dual of (a) and Theorem 10.8(b). ☐

11 Poles and polars

For this section only, we adopt an additional axiom, asserting statement (a) of Theorem 10.8. Under Axiom P we are enabled to use Theorem 10.10 to construct, through any point of the plane, a secant to any conic. It remains an open problem to determine whether this axiom may be derived from the others.

Axiom P. The tangents at any three distinct points of a point-conic are nonconcurrent; the points of contact of any three distinct lines of a line-conic are noncollinear.

The traditional method for defining a polar using a quadrangle, considering separately points either on or not on a conic, is precluded, since we cannot always decide, constructively, which case applies to a given point.

Theorem 11.1. Construction of a polar. *Let κ be a conic, and let P any point on the plane. Through the point P, construct a secant q of κ, using* Axiom P *and Theorem 10.10. Denote the intersections of q with κ by Q_1 and Q_2, let the tangents at these points be denoted q_1 and q_2, and set $Q = q_1 \cdot q_2$. Set $Q' = h(Q_1, Q_2; P)$, the harmonic conjugate of P with respect to the points Q_1, Q_2. Then the line $p = QQ'$ is independent of the choice of the secant q.*

Proof. Since $Q_1 \neq Q_2$, by cotransitivity we may assume that $P \neq Q_2$. From Proposition 10.3(b) it follows that $Q_1 \notin q_2$; thus $Q_1Q_2 \neq q_2$. From Lemma 4.5 we have $Q' \neq Q_2 = Q_1Q_2 \cdot q_2$, so by Axiom C7 it follows that $Q' \notin q_2$; thus $Q' \neq Q$, and the line p is properly defined.

Now let r be any secant of κ, through P, with $R_1, R_2, r_1, r_2, R, R'$ defined similarly, and set $s = RR'$. We must show that $s = p$.

(a) *Special case; $P \in \kappa$.* In this case, $Q_1 = P$, so $q_1 = t$, the tangent at P; thus $Q \in t$. Also, by Lemma 4.4, we have $Q' = P$, so $Q' \in t$. Thus $p = t$; similarly, $s = t$.

(b) *Special case; $P \notin \kappa$ and $r \neq q$.* Since $Q_1 \neq P = q \cdot r$, it follows that $Q_1 \notin r$, so $Q_1 \neq R_1$. Similarly, all four points Q_1, Q_2, R_1, R_2 are distinct. By Theorem 10.6, applied to the quadrangle $Q_1Q_2R_1R_2$, the point Q is collinear with the diagonals $D_1 = Q_1R_2 \cdot Q_2R_1$ and $D_2 = Q_1R_1 \cdot Q_2R_2$, so we have $Q \in D_1D_2$.

The harmonic conjugate of P with respect to Q_1, Q_2 is given by Definition 4.1. Corresponding to the configuration C, A, B, l, R, P, Q, S in the definition, where $h(A, B; C) = AB \cdot RS$, here we have the configuration $P, Q_1, Q_2, PR_2, D_2, R_2, R_1, D_1$. By Theorem 4.7, harmonic conjugates are independent of the choice of construction elements; thus $Q' = Q_1Q_2 \cdot D_1D_2$, so $p = D_1D_2$. Similarly, $s = D_1D_2$.

(c) *General case.* Suppose that $s \neq p$, and suppose further that $P \notin \kappa$, and $r \neq q$. This contradicts (b), negating the third assumption, so $r = q$, and now it is evident that $s = p$. This contradicts the first assumption, negating the second; thus $P \in \kappa$. This contradicts (a), negating the first assumption; hence $s = p$. □

Definition 11.2. Let κ be a conic, and P any point on the plane. The line p obtained in Theorem 11.1 is called the *polar of P with respect to κ.*

Corollary 11.3. *Let κ be a conic, P any point on the plane, and p the polar of P. Then p passes through:*

(i) the harmonic conjugate of P with respect to the points of intersection of any secant of κ that passes through P;

(ii) the point of intersection of the tangents to κ at the points of intersection of any secant of κ that passes through P.

Corollary 11.4. *Let κ be a conic, and let P be any point outside κ. Inscribe a quadrangle in κ with P as one diagonal point, using Theorem 10.10. Then the polar of P is the line joining the other two diagonal points.*

Definition 11.5. Let κ be a conic, and l any line on the plane. A construction analogous to that of Theorem 11.1 results in a point L, called the *pole of l with respect to κ.*

The dual of Theorem 10.10 yields a point E on l, which might be called a *dual-secant*. Through E pass two distinct lines e_1, e_2 of the line-conic λ. By Theorem 10.8, λ is the family of all tangents to κ. Joining the points of contact E_1, E_2 of the lines e_1, e_2, we obtain a line $e = E_1E_2$. We also construct a line $e' = h(e_1, e_2; l)$, the harmonic conjugate of l, in the pencil E^*, with respect to the base lines e_1, e_2. The pole of l is thus the point $L = e \cdot e'$. Corollaries analogous to those above also apply to poles.

Theorem 11.6. *Let κ be a conic.*

(a) If a line p is the polar of a point P, then P is the pole of p, and conversely.

(b) If a point P is on κ, then the polar of P is the tangent to κ at P.

Proof. Statement (a) follows from Corollary 11.3 and its analog for poles. For (b), we note that in Theorem 11.1 for the construction of the polar p, we now have $Q_1 = P$, by Proposition 8.2(b). Thus $q_1 = t$, the tangent at P, and $Q \in t$. Also, by Lemma 4.4, we have $Q' = P$, so $Q' \in t$. Thus $p = QQ' = t$. □

It remains an open problem to construct correlations and polarities using the axioms adopted here, to develop the theory of conics constructively using the von Staudt [Sta47] definition, whereby a conic is a locus of points defined by a polarity, and to prove that von Staudt conics are equivalent to the Steiner [Ste32] conics considered here.

Part II

Analytic constructions

A projective plane $\mathbb{P}^2(\mathbb{R})$ is built from subspaces of the linear space $\mathbb{R}^3$, using only constructive properties of the real numbers. This model will establish the consistency of the axiom system adopted in Part I. The properties of $\mathbb{P}^2(\mathbb{R})$ have guided the choice of axioms, taking note of Bishop's thesis, "All mathematics should have numerical meaning" [B67, p. ix].

12 Real numbers

To clarify the methods used here, we give examples of familiar properties of the real numbers that are constructively *invalid*, and also properties that are constructively *valid*.

The following classical properties of a real number α are constructively invalid: *Either* $\alpha < 0$ *or* $\alpha = 0$ *or* $\alpha > 0$, and *If* $\neg(\alpha = 0)$, *then* $\alpha \neq 0$. Constructively invalid statements in classical metric geometry result when the condition *P lies outside l,* written $P \notin l$, is taken to mean that the distance $d(P, l)$ is positive. For examples of constructively invalid statements for the metric plane $\mathbb{R}^2$, we have the statement *Either the point P lies on the line l or P lies outside l,* considered in Example 1.1, and the statement *If* $\neg(P \in l)$, *then* $P \notin l$.

Bishop determined the constructive properties of the real numbers, using Cauchy sequences of rationals, while referring to no axiom system of formal logic, but only a presupposition of the positive integers [B67, p.2]. A notable resulting feature is that the relation $\alpha \neq 0$ does not refer to negation, but is given a strong affirmative definition; one must construct an integer n such that $1/n < |\alpha|$. Among the resulting constructive properties of the reals are the following:

(i) *For any real number* α, *if* $\neg(\alpha \neq 0)$, *then* $\alpha = 0$.

(ii) *For any real numbers* α *and* β, *if* $\alpha\beta \neq 0$, *then* $\alpha \neq 0$ *and* $\beta \neq 0$.

(iii) *Given any real numbers* α and β *with* $\alpha < \beta$, *for any real number* x, *either* $x > \alpha$ *or* $x < \beta$.

Property (iii) serves as a constructive substitute for the *Trichotomy property* of classical analysis, which is constructively invalid. For more details, and other constructive properties of the real number system, see [B67, BB85, BV06]. For a constructive axiomatic study of the reals, with applications to formal systems of computable analysis, see [Bri99]. For axioms for the real numbers, and a construction of the reals without using the axiom of countable choice, see [R08].

Brouwerian counterexamples. To determine the specific nonconstructivities in a classical theory, and thereby to indicate feasible directions for constructive work, Brouwerian counterexamples are used, in conjunction with omniscience principles. A *Brouwerian counterexample* is a proof that a given statement implies an omniscience principle. In turn, an *omniscience principle* would imply solutions or significant information for a large number of well-known unsolved problems. This method was introduced by L. E. J. Brouwer [Bro08] to demonstrate that use of the *Law of Excluded Middle* inhibits mathematics from attaining its full significance.

Omniscience principles are primarily formulated in terms of binary sequences, at times called *decision sequences*; the zeros and ones may represent the results of a search for a solution to a specific problem, as in Example 1.1. These omniscience principles have equivalent statements in terms of real numbers; the following are those most often used in connection with Brouwerian counterexamples.

Limited principle of omniscience (LPO). *For any real number α, either $\alpha = 0$ or $\alpha \neq 0$.*

Lesser limited principle of omniscience (LLPO). *For any real number α, either $\alpha \leq 0$ or $\alpha \geq 0$.*[15]

Markov's principle (MP). *For any real number α, if $\neg(\alpha = 0)$, then $\alpha \neq 0$.*

A statement is considered *constructively invalid* if it implies an omniscience principle. The statement considered in Example 1.1, *Either $P \in l$, or $P \notin l$,* implies LPO; thus it is constructively invalid. For more information concerning Brouwerian counterexamples, and other omniscience principles, see [B67, BB85, M83, M88, M89, R02].

13 The model $\mathbb{P}^2(\mathbb{R})$ in Euclidean space

The model will be built following well-known classical methods, adding constructive refinements to the definitions and proofs.

Definition 13.1. The plane $\mathbb{P}^2(\mathbb{R})$ consists of a family $\mathscr{P}_2$ of points, and a family $\mathscr{L}_2$ of lines.
• A *point* P in $\mathscr{P}_2$ is a subspace of the linear space $\mathbb{R}^3$, of dimension 1. When the vector $p = (p_1, p_2, p_3)$ spans P, we write $P = \langle p \rangle = \langle p_1, p_2, p_3 \rangle$.
• A *line* λ in $\mathscr{L}_2$ is a subspace of $\mathbb{R}^3$, of dimension 2. When the vectors u, v span λ, and $l = u \times v$, we write $\lambda = [l] = [l_1, l_2, l_3]$, and $\lambda = UV$, where $U = \langle u \rangle$ and $V = \langle v \rangle$.
• Points $P = \langle p \rangle$ and $Q = \langle q \rangle$ are *equal*, written $P = Q$, if $p \times q = 0$; they are *distinct*, written $P \neq Q$, if $p \times q \neq 0$.
• Lines $\lambda = [l]$ and $\mu = [m]$ are *equal*, written $\lambda = \mu$, if $l \times m = 0$; they are *distinct*, written $\lambda \neq \mu$, if $l \times m \neq 0$.
• Incidence relation. Let $P = \langle p \rangle$ be a point and $\lambda = [l]$ a line. We say that P *lies on* λ, and that λ *passes through* P, written $P \in \lambda$, if $p \cdot l = 0$.
• Outside relation. For any point P and any line λ, we say that P *lies outside the line* λ, and that λ *avoids the point* P, written $P \notin \lambda$, if $P \neq Q$ for all points Q on λ.

Notes for Definition 13.1.

1. The definitions are independent of the choice of vectors spanning the respective subspaces. That they are in accord with Definitions 2.2 and 2.3 for a projective plane is evident, except for cotransitivity of the inequality relations, which will be verified in Theorem 13.8.

[15] The omniscience principle LLPO was introduced by Bishop [B73].

2. The constructive properties of the real numbers will carry over to vectors in $\mathbb{R}^3$. For example, $v \neq 0$ means that at least one of the components of the vector v is constructively nonzero.

3. The equality, inequality, incidence, and outside relations are invariant under a change of basis.

4. We avoid interpreting the conditions $P \in \lambda$ and $P \notin \lambda$ using the relations of set-membership and set-inclusion in the classical sense. Theorem 13.2 and Example 13.7 will confirm that the primary relation $P \notin \lambda$, *point outside a line,* is constructively stronger than the condition $\neg(P \in \lambda)$.

5. The triad $\langle p_1, p_2, p_3 \rangle$ is traditionally referred to as *homogeneous coördinates* for the point P, and the triad $[l_1, l_2, l_3]$ as *line coördinates* for the line λ.

Theorem 13.2. *Let $P = \langle p \rangle$ be a point of $\mathbb{P}^2(\mathbb{R})$, and $\lambda = [l]$ a line. Then $P \notin \lambda$ if and only if $p \cdot l \neq 0$.*

Proof. By a change of basis, we may assume that $l = (0, 0, 1)$.

First let $P \notin \lambda$. In the case $p_3 \neq 0$ we have $p \cdot l \neq 0$ directly. In the other two cases, we may set $Q = \langle q \rangle = \langle p_1, p_2, 0 \rangle$; then $Q \in \lambda$, so $P \neq Q$. Thus $p \times q \neq 0$; i.e., $(-p_3 p_2, p_3 p_1, 0) \neq 0$, and hence $p \cdot l = p_3 \neq 0$.

For the converse, let $p \cdot l \neq 0$; thus $p_3 \neq 0$. For any point $Q = \langle q \rangle$ on λ, we have $q_3 = 0$, so either $q_1 \neq 0$ or $q_2 \neq 0$. It follows that $p \times q = (-p_3 q_2, p_3 q_1, p_1 q_2 - p_2 q_1) \neq 0$, and hence $P \neq Q$. ∎

Corollary 13.3. *Let $P = \langle p \rangle$, $Q = \langle q \rangle$, $R = \langle r \rangle$ be points of $\mathbb{P}^2(\mathbb{R})$, with $Q \neq R$. Then $P \notin QR$ if and only if the vectors* p, q, r are independent.

Corollary 13.4. *Let $P = \langle p \rangle$ be any point of $\mathbb{P}^2(\mathbb{R})$, and $\lambda = [u \times v]$ any line. The following conditions are equivalent:*

(a) $\neg(P \notin \lambda)$.

(b) $P \in \lambda$.

(c) *The vector p is in the span of the vectors u, v.*

Corollary 13.5. *Let $Q = \langle q \rangle$, and $R = \langle r \rangle$ be points on $\mathbb{P}^2(\mathbb{R})$, with $Q \neq R$, and let P be a point on QR. If $P \neq R$, then there exists a unique real number α such that $P = \langle q + \alpha r \rangle$.*

Corollary 13.6. *Definition 13.1, for the plane $\mathbb{P}^2(\mathbb{R})$, is self-dual.*

Example 13.7. For the plane $\mathbb{P}^2(\mathbb{R})$, the following statements are constructively invalid.

(a) *If P and Q are any points, then either $P = Q$ or $P \neq Q$.*

(b) *If λ and μ are any lines, then either $\lambda = \mu$ or $\lambda \neq \mu$.*

(c) *If P is any point, and λ any line, then either $P \in \lambda$ or $P \notin \lambda$.*

(d) *If λ is any line, and P is a point such that $\neg(P \in \lambda)$, then $P \notin \lambda$.*

Proof. Let α be any real number. For (a), set $P = \langle p \rangle = \langle 0, 0, 1 \rangle$ and $Q = \langle q \rangle = \langle \alpha, 0, 1 \rangle$. Then $p \times q = (0, \alpha, 0)$, so the statement implies LPO. A similar counterexample serves for (b). For (c), set $P = \langle 0, \alpha, 0 \rangle$ and $\lambda = [0, 1, 0]$; the statement implies LPO. For (d), assume also that $\neg(\alpha = 0)$, with P and λ as in the example for (c); now the statement implies MP. ∎

Theorem 13.8. Cotransitivity. *If P and Q are points on the plane $\mathbb{P}^2(\mathbb{R})$, with $P \neq Q$, then for any point R, either $R \neq P$, or $R \neq Q$.*

Proof. By a change of basis, it suffices to consider the situation in which $P = \langle p \rangle = \langle 1, 0, 0 \rangle$, $Q = \langle q \rangle = \langle 0, 1, 0 \rangle$, and $R = \langle r \rangle = \langle r_1, r_2, r_3 \rangle$; then $r \times p = (0, r_3, -r_2)$ and $r \times q = (-r_3, 0, r_1)$. In the case $r_1 \neq 0$, we have $r \times q \neq 0$, so $R \neq Q$. In the other two cases we obtain $R \neq P$. □

Lemma 13.9. *On the plane $\mathbb{P}^2(\mathbb{R})$, for any projection $\rho : \overline{\lambda} \to \overline{\mu}$ of a range of points $\overline{\lambda}$ onto a range $\overline{\mu}$, there exists a non-singular linear transformation τ of $\mathbb{R}^3$ that induces ρ; i.e., $X^\rho = \langle \tau x \rangle$, for all points $X = \langle x \rangle$ in the range $\overline{\lambda}$.*

Proof. We adapt the proof found in [Art57, p. 94]. Select vectors m and t so that $\mu = [m]$ and the center of ρ is $T = \langle t \rangle$. Select distinct points $U_1 = \langle u_1 \rangle$ and $U_2 = \langle u_2 \rangle$ in $\overline{\lambda}$, select vectors v_i such that $U_i^\rho = \langle v_i \rangle$, and construct a non-singular linear transformation $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\tau u_i = v_i$. Since $\langle v_1 \rangle \in TU_1$, and $\langle v_1 \rangle \neq T$, it follows from Corollary 13.5 that $\langle v_1 \rangle = \langle u_1 + \alpha t \rangle$ for some scalar α, and thus $v_1 = \beta u_1 + t_1$ for a nonzero scalar β and a vector t_1 in T; we may assume that $\beta = 1$. Similarly, we have $v_2 = u_2 + t_2$, for some $t_2 \in T$.

Let $X = \langle x \rangle$ be any point of $\overline{\lambda}$, with $x = \alpha_1 u_1 + \alpha_2 u_2$. Then $\tau x = \alpha_1 v_1 + \alpha_2 v_2$, so $\langle \tau x \rangle \in \mu$. Also, $\tau x = \alpha_1 u_1 + \alpha_2 u_2 + t_3 = x + t_3$, where $t_3 \in T$. Now set $y = t \times x$, so $TX = [y]$. Then $y = t \times (\tau x - t_3) = t \times \tau x$, and $\tau x \cdot y = \tau x \cdot t \times \tau x = 0$, so $\langle \tau x \rangle \in TX$. Hence $\langle \tau x \rangle = TX \cdot \mu = X^\rho$. □

Theorem 13.10. *For any projectivity π of the plane $\mathbb{P}^2(\mathbb{R})$, there exists a non-singular linear transformation τ of $\mathbb{R}^3$ that induces π.*

Corollary 13.11. *Any projectivity of a range or pencil of $\mathbb{P}^2(\mathbb{R})$, onto itself, with three distinct fixed elements, is the identity.*

14 Axioms verified for the plane $\mathbb{P}^2(\mathbb{R})$

This verification will establish the consistency of the axiom system adopted in Part I. The following example shows that for Axiom C3, *Distinct lines have a common point,* the condition of distinctness is essential.

Example 14.1. On the plane $\mathbb{P}^2(\mathbb{R})$, the following statements are constructively invalid.

(a) *Given any points P and Q, there exists a line that passes through both points.*

(b) *Given any lines λ and μ, there exists a point that lies on both lines.*

Proof. It will suffice to consider the second statement. For a Brouwerian counterexample, let α be any real number, and set $\alpha^+ = \max\{\alpha, 0\}$ and $\alpha^- = \max\{-\alpha, 0\}$. Define lines $\lambda = [\alpha^+, 0, 1]$ and $\mu = [0, \alpha^-, 1]$. By hypothesis, we have a point $R = \langle r \rangle = \langle r_1, r_2, r_3 \rangle$ that lies on both lines. Thus $\alpha^+ r_1 + r_3 = 0$, and $\alpha^- r_2 + r_3 = 0$. If $r_3 \neq 0$, then we have both $\alpha^+ \neq 0$ and $\alpha^- \neq 0$, an absurdity; thus $r_3 = 0$. This leaves two cases. If $r_1 \neq 0$, then $\alpha^+ = 0$, so $\alpha \leq 0$, while if $r_2 \neq 0$, then $\alpha^- = 0$, so $\alpha \geq 0$. Hence LLPO results. □

Notes for Example 14.1.

1. The counterexample for the first statement, the dual of that given for the second, is easier to visualize. On $\mathbb{R}^2$, thought of as a portion of $\mathbb{P}^2(\mathbb{R})$, consider two finite points which are extremely near or at the origin: P on the x-axis, and Q on the y-axis. If P is very slightly off the origin, and Q is at the origin, then the x-axis is the required line λ. But in the opposite situation we would need the y-axis. Such a large jump in the output, resulting from a miniscule variation of the input, would be a severe discontinuity in a proposed constructive routine, and is a very strong indication that a solution would be constructively invalid.

2. *Note on the Heyting extension* [H59]. The above example is essentially the same as that used in [M13a] to show that in the Heyting extension the *strong common point property* (i.e., for all lines, not only distinct lines) is constructively invalid. In [M13a, Note, p. 113] it was claimed that a constructive projective plane ought to have the strong property. However, it is now seen that various versions of a constructive real projective plane are possible. The question of the common point property for the Heyting extension remains an interesting open problem. For comments concerning this issue in the classical literature, see [Pic75, Section 1.2].

3. *Note on the projective extension of* [M14]. The strong common point property was obtained for this plane, but the *cotransitivity* property was found to be constructively invalid [M14, pp. 704-5]. The results of the various studies tend to indicate the incompatibility of the two properties, strong common point and cotransitivity, in any constructive projective plane; making this idea precise is an open problem.

Theorem 14.2. *Axiom Group C, and Axioms F, D, E, T, are valid on* $\mathbb{P}^2(\mathbb{R})$.

Proof. Axioms C1 and C4 of Section 2, and Axiom E of Section 5, are evident. Given distinct points $P = \langle p \rangle$ and $Q = \langle q \rangle$, set $l = p \times q$, and $\lambda = [l]$. Then $p \cdot l = 0$, so $P \in \lambda$, and similarly $Q \in \lambda$. Similarly, if lines $\lambda = [l]$ and $\mu = [m]$ are distinct, then the point $P = \langle l \times m \rangle$ lies on both lines. Thus Axioms C2 and C3 are verified. Axiom C6 was verified in Corollary 13.4

For Axiom C5, consider lines $\lambda = [l]$ and $\mu = [m] = [u \times v]$, and let $P = \langle p \rangle$ be a point on λ that is outside μ. Thus $p \cdot u \times v \neq 0$, and the vectors p, u, v are independent. By a change of basis, we may assume that u, v, p is the standard basis; thus $m = e_3$. Since $P \in \lambda$, we have $p \cdot l = 0$, so $l_3 = 0$. Now $l \times m = (l_2, -l_1, 0) \neq 0$, and hence $\lambda \neq \mu$.

The converse to Axiom C5 is also valid: *If the lines λ and μ are distinct, then there exists a point $P \in \lambda$ such that $P \notin \mu$.* To prove this, let $\lambda = [l]$ and $\mu = [m]$ be distinct lines. By a change of basis, we may assume that $l = e_3$. Since $l \times m \neq 0$, we have $(-m_2, m_1, 0) \neq 0$; thus either $m_1 \neq 0$ or $m_2 \neq 0$. In the first case, set $P = \langle e_1 \rangle$. Then $p \cdot l = 0$, so $P \in \lambda$, while $p \cdot m = m_1 \neq 0$, so $P \notin \mu$. The second case is similar.

For Axiom C7, let λ and μ be distinct lines, and let $P \neq Q = \lambda \cdot \mu$. Select a point $R \in \lambda$ so that $R \neq Q$; thus $\lambda = QR$. By the converse to Axiom C5, verified in the preceding paragraph, we may select a point $S \in \mu$ such that $S \notin \lambda$. Thus $S \neq Q$, so $\mu = QS$, and the points Q, R, S are noncollinear. By a change of basis, we may assume that $R = \langle e_1 \rangle$, $S = \langle e_2 \rangle$, $Q = \langle e_3 \rangle$, and $P = \langle p \rangle = \langle p_1, p_2, p_3 \rangle$. Then $\lambda = [e_2]$, and $\mu = [e_1]$. Since $P \neq Q$, we have $p \times e_3 \neq 0$, so $(p_2, -p_1, 0) \neq 0$, and thus either $p_2 \neq 0$, or $p_1 \neq 0$. In the first case, we have $P \notin \lambda$, while in the second case we find that $P \notin \mu$.

For Axiom F of Section 2, Fano's Axiom, by a change of basis we may assume that the quadrangle $PQRS$ has vertices $\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle$, and $\langle e \rangle = \langle 1, 1, 1 \rangle$. The six sides are then $PQ = [0,0,1]$, $PR = [0,1,0]$, $PS = [0,1,-1]$, $QR = [1,0,0]$, $QS = [1,0,-1]$, and $RS = [1,-1,0]$. The diagonal points are $D_1 = PQ \cdot RS = \langle 1,1,0 \rangle$, $D_2 = PR \cdot QS = \langle 1,0,1 \rangle$, and $D_3 = \langle d \rangle = PS \cdot QR = \langle 0,1,1 \rangle$. Thus $D_1D_2 = [m] = [1,-1,-1]$. Since $d \cdot m \neq 0$, we have $D_3 \notin D_1D_2$. Thus the diagonal points are noncollinear.

For Axiom D of Section 3, Desargues's Theorem, consider triangles PQR and $P'Q'R'$, perspective from a center $O = \langle o \rangle$. By a change of basis, we may assume that $P, Q, R = \langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle$; thus $QR = [e_1]$, $RP = [e_2]$, and $PQ = [e_3]$. Since $O \notin PQ$, we have $o_3 \neq 0$, and similarly for o_1 and o_2; thus we may assume that $O = \langle 1,1,1 \rangle$. Now $OP = [0,1,-1]$. Since $P' \in OP$, and $P' \neq P$, it follows from Corollary 13.5 that $P' = \langle \alpha, 1, 1 \rangle$ for some scalar α; similarly, $Q' = \langle 1, \beta, 1 \rangle$, and $R' = \langle 1, 1, \gamma \rangle$. It follows that $P'Q' = [1-\beta, 1-\alpha, \alpha\beta - 1]$, $Q'R' = [\beta\gamma - 1, 1-\gamma, 1-\beta]$, and $R'P' = [1-\gamma, \gamma\alpha - 1, 1-\alpha]$. Now the points in question are $A = \langle a \rangle = PQ \cdot P'Q' = \langle \alpha - 1, 1 - \beta, 0 \rangle$, $B = \langle b \rangle = QR \cdot Q'R' = \langle 0, \beta - 1, 1 - \gamma \rangle$, and $C = \langle c \rangle = RP \cdot R'P' = \langle \alpha - 1, 0, 1 - \gamma \rangle$. Since $c \cdot a \times b$ works out to 0, it follows that $C \in AB$. Thus the points A, B, C are collinear. To show that the line AB avoids each of the six vertices, we first note that since the center O lies outside each of the six sides, we have $O \neq Q'$; it follows that $\beta \neq 1$, and similarly, $\gamma \neq 1$. Thus $e_1 \cdot a \times b = (1-\beta)(1-\gamma) \neq 0$, and we have $P \notin AB$. By symmetry of the vertices of the triangle, Q and R also lie outside AB. By symmetry of the two triangles, the points P', Q', R' lie outside AB. Hence the triangles are perspective from the axis AB.

Axiom T of Section 6, the uniqueness portion of the *Fundamental Theorem*, was verified in Corollary 13.11. □

It remains an open problem to develop the analytic theory of conics constructively, and to determine the constructive validity of Axiom P of Section 11 in an analytic setting.

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